AN ABSTRACT OF THE DISSERTATION OF

Erica S. Rode for the degree of Doctor of Philosophy in Mathematics presented February 25, 2013.

Title: Gaussian Random Fields Related to Lévy's Brownian Motion: Representations and Expansions

Abstract approved:

Mina E. Ossiander

This dissertation examines properties and representations of several isotropic Gaussian random fields in the unit ball in d-dimensional Euclidean space. First we consider Lévy's Brownian motion. We use an integral representation for the covariance function to find a new expansion for Lévy's Brownian motion as an infinite linear combination of independent standard Gaussian random variables and orthogonal polynomials.

Next we introduce a new family of isotropic Gaussian random fields, called the p-processes, of which Lévy's Brownian motion is a special case. Except for Lévy's Brownian motion the p-processes are not locally stationary. All p-processes also have a representation as an infinite linear combination of independent standard Gaussian random variables.

We use these expansions of the random fields to simulate Lévy's Brownian motion and the p-processes along a ray from the origin using the Cholesky factorization of the covariance matrix.

Gaussian Random Fields Related to Lévy's Brownian Motion: Representations and Expansions

by
Erica S. Rode

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented February 25, 2013
Commencement June 2013

Doctor of Philosophy dissertation of Erica S. Rode presented on February 25, 2013.

APPROVED:

____

Major Professor, representing Mathematics

____

Chair of the Department of Mathematics

____

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

____

Erica S. Rode, Author

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

Page

LIST OF FIGURES

Gaussian Random Fields Related to Lévy's Brownian Motion: Representations and Expansions

Erica Rode

February 25, 2013

Chapter 1 Introduction

The Central Limit Theorem states that the normalized average of a sequence of independent random variables with finite mean and variance converges to a standard Gaussian random variable. Because of this, Gaussian random variables and random fields are used extensively as models in fields such as physics, finance and medical imaging. In particular they appear in models with fluctuations that can be thought of as Gaussian random variables. Models can be adapted for different applications by modifying the covariance function and mean vector.

Standard Brownian motion is a Gaussian random field with independent increments. Arguably, it first appeared as a model of the random motion of a particle suspended in a fluid. Paul Lévy introduced Lévy's Brownian motion as a way to generalize standard Brownian motion to multiple dimensions. Lévy's Brownian motion is an isotropic locally stationary random field with mean zero. It has since been studied extensively. In [12] McKean found an expansion for Lévy's Brownian motion in $\mathbb{R}^d$ which he used to draw conclusions about the Markov properties of the random field. In [14] Noda presented the Karhunen-Loéve expansion for Lévy's Brownian motion on the surface of the unit sphere.

In order to use random fields in computational settings it is necessary to be able to simulate them. This can be accomplished using expansions in terms of basis functions and standardized random variables. Orthogonal polynomials are integral to expansions of isotropic random fields. Lévy's Brownian motion has an integral representation given in [15] which we use to find a new representation for the covariance function. This thesis presents a new expansion for the Lévy's Brownian motion as an infinite linear combination of independent standard Gaussian random variables. This expansion is not the Karhunen-Loéve expansion, but shares some desirable properties of the Karhunen-Loéve expansion.

We also introduce a new family of Gaussian random fields, called the p-processes, of which Lévy's Brownian motion is a special case. If we let A_s be the ball in $\mathbb{R}^d$ with the vector s as the diameter and ν_p be the measure with density function

$$d\nu_p = c_d|u|^p \, du$$

then the p-process is defined to be a mean zero Gaussian process in the unit ball in $\mathbb{R}^d$ with covariance function

$$K_p(s,t) = \nu_p(A_s \cap A_t).$$

The case $p + d = 1$ is Lévy's Brownian motion. Lévy's Brownian motion is a locally stationary process, though non-stationary. The p-process are isotropic but, except for Lévy's Brownian motion, not locally stationary. We also find an expansion for the p-processes in terms of spherical harmonics and standard Gaussian random variables. Using an orthonormal basis for $L^2(B^d)$ we show that for $t \in B^d$ and $p + d > 0$ the p-processes have the expansion

$$X_{t,p} = c_{p,d} \sum_{m,k=0}^{\infty} \Lambda_{m,k,p}(|t|) \sum_{j=1}^{h(m)} \varphi_{m,j}(t/|t|) Z_{m,k,j}$$

where $\Lambda_{m,k,p}$ is a function of orthogonal polynomials on $[-1, 1]$, $\{\varphi_{m,j}\}$ is an orthonormal basis of spherical harmonics for $L^2(S^{d-1})$ and $\{Z_{m,k,j}\}$ is an array of independent standard Gaussian random variables.

These expansions can be used to simulate Lévy's Brownian motion and the p-processes. We use expansions for the covariance functions to approximate the covariance matrix and we present simulations along a ray from the origin using the Cholesky factorization of the approximated covariance matrix.

Chapter 2 covers the necessary background in probability and random fields. There we introduce an integral representation for Lévy's Brownian motion and an orthonormal basis for $L^2(B^d)$ that will be needed to find expansions for the random fields. In Chapter 3 the Brownian sheet, which is typically defined only on $\mathbb{R}^d_+$, is generalized to all of $\mathbb{R}^d$. This is used to define a white noise integral using a constructive approach. In Chapter 4 we find a decomposition for Levy's Brownian motion and its covariance function in terms of spherical harmonics and independent standard Gaussian random variables. In chapter 5 we define the p-processes and find an expansion for the p-processes in terms of spherical harmonics and independent standard Gaussian random variables. Chapter 6 is devoted to simulations of Lévy's Brownian motion and the p-processes using the expansions found in Chapters 4 and 5.

Chapter 2 Preliminaries

This chapter will cover the background material in random fields and orthogonal polynomials necessary to obtain the expansions for Lévy's Brownian motion and the p-processes.

2.1 Random Variables

A probability space is often considered in the context of an experiment where the outcome is not known, but the set of possible outcomes is known as well as a way to assign probabilities to these outcomes.

Definition 2.1 (Probability Space) *[3] A probability space is an ordered triple* $(\Omega, \mathcal{A}, P)$ *where*

- Ω *is a collection of outcomes, called the sample space*
- $\mathcal{A}$ *is a* σ*-field of subsets of* Ω*, called events and*
- P *is a measure on* $(\Omega, \mathcal{A})$ *with*

    1. $0 \leqslant P(A) \leqslant 1 \;\; \forall A \in \mathcal{A}$

2. $P(\emptyset) = 0$, $P(\Omega) = 1$ *and*

3. if $\{A_i : i \geqslant 1\} \subset \mathcal{A}$ *is a pairwise disjoint collection of events then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{2.1}$$

We are often concerned with knowing whether events are independent of each other or affected by each other.

Definition 2.2 *A collection of events* $\{A_i : i \geqslant 1\}$ *on a common probability space* $(\Omega, \mathcal{A}, P)$ *is said to be independent if for any* $m \neq n$

$$P(A_m \cap A_n) = P(A_m)P(A_n). \tag{2.2}$$

Random variables are commonly used as a tool to assign numerical values to outcomes in the sample space.

Definition 2.3 (Random Variable) *[3] A random variable on a probability space* $(\Omega, \mathcal{A}, P)$ *is a real valued function that is* $\mathcal{A}-$*measurable.*

For any event $A \in \mathcal{A}$ we denote the probability of A by

$$P(X \in A) = P\left(\{\omega : X(\omega) \in A\}\right). \tag{2.3}$$

When there is no possibility of confusion we will just write $P(A)$ for $P(X \in A)$. For any random variable, X, $\sigma(X)$ is defined to be the smallest σ-field with respect to which X is measurable. Two random variables X and Y on a common probability space $(\Omega, \mathcal{A}, P)$ are said to be independent if $\sigma(X)$ and $\sigma(Y)$ are independent. For any random variable we can compute the mean or expectation, which is a weighted average of the values of the random variable, and the variance, which is a measure of how much the random variable deviates from its mean.

Definition 2.4 (Mean and Variance of a Random Variable) *[3] Let X be a random variable on the probability space $(\Omega, \mathcal{A}, P)$. Then the mean or expectation of X is*

$$E(X) = \int_{\Omega} X(\omega)\, dP(\omega) \tag{2.4}$$

and the variance is

$$Var(X) = E\left[(X - E(X))^2\right]. \tag{2.5}$$

Given a collection of random variables on a common probability space the covariance can be used to determine how the random variables interact with each other.

Definition 2.5 *Let X, Y be random variables on a common probability space. Then the covariance of X and Y is*

$$Cov(X, Y) = E(XY) - E(X)E(Y). \tag{2.6}$$

If two random variables are independent then their covariance is zero, however the converse is not true in general.

A commonly used random variable is the Gaussian or normal random variable. The Central Limit Theorem tells us that the standardized average of a sequence of random variables converges a Gaussian random variable. Because of this, Gaussian random variables are often used for modeling situations in physics, finance and other areas.

Definition 2.6 (Gaussian Random Variable) *[3] A Gaussian random variable with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is a random variable on the probability space $(\mathbb{R}, \mathcal{B}, P)$ where $\mathcal{B}$ is the Borel $\sigma-$field and for any $B \in \mathcal{B}$*

$$P(B) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_B e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx. \tag{2.7}$$

The function

$$f_{\sigma,\mu}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2.8}$$

is called the *density* of X. In the case $\mu = 0$ and $\sigma^2 = 1$, X is called a *standard* Gaussian random variable. Gaussian random variables have several properties that make them desirable to work with. One of these properties is that two Gaussian random variables, X and Y are independent if and only if $Cov(X, Y) = 0$. Several other useful properties will be introduced in this section. First we define one notion of convergence of random variables, weak convergence or convergence in distribution.

Definition 2.7 (Weak Convergence) *[3] Let $\{X_n : n \geqslant 1\}$ and X be random variables. Then $\{X_n : n \geqslant 1\}$ converges weakly to X if for each $x \in \mathbb{R}$*

$$\lim_{n\to\infty} P(X_n \leqslant x) = P(X \leqslant x). \tag{2.9}$$

Lemma 2.1 *Let* $\{X_i : i \geqslant 1\}$ *be a sequence of Gaussian random variables with* $E(X_i) = \mu_i$ *and* $Var(X_i) = \sigma_i^2$ *for each* i *and suppose* $\lim_{n\to\infty} \mu_n = \mu$ *and* $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$. *Then*

1. *For any* n, $Y = \sum_{i=1}^{n} a_i X_i$ *is a Gaussian random variable with mean*

$$E(Y) = \sum_{i=1}^{n} a_i \mu_i \tag{2.10}$$

   and variance

$$Var(Y) = \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 2 \sum_{i<j} Cov(X_i, X_j). \tag{2.11}$$

2. X_n *converges weakly to a Gaussian random variable with mean* μ *and variance* σ^2.

The idea of a Gaussian random variable can be extended to consider a random vector $\mathcal{X} = [X_i : 1 \leqslant i \leqslant n]$ where for each i, X_i is a random variable.

Definition 2.8 (Multivariate Gaussian Random Vector) *[13] A random vector* $\mathcal{X} = [X_i : 1 \leqslant i \leqslant n]$ *is called multivariate Gaussian if for any set of real numbers* $\{a_i : 1 \leqslant i \leqslant n\}$ *the random variable*

$$Y = \sum_{i=1}^{n} a_i X_i \tag{2.12}$$

is a Gaussian random variable.

Every multivariate Gaussian random vector is characterized by a mean vector

$$\vec{\mu} = [E(X_i) : 1 \leqslant i \leqslant n] \tag{2.13}$$

and a covariance matrix

$$\Sigma = [E(X_i X_j) - \mu_i \mu_j : 1 \leqslant i, j \leqslant n]\,. \tag{2.14}$$

The density function of such a random vector is given by

$$f(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma^{-1}|}} exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu})\right\}. \tag{2.15}$$

If $\vec{\mu} = \vec{0}$ and Σ is the identity matrix then $\mathcal{X}$ is called a *standard* Gaussian random vector.

2.2 Stochastic Processes

Stochastic processes are used to model changes in random systems, such as the random motion of a particle or fluctuations in financial systems.

Definition 2.9 *[3] A stochastic process is a collection of random variables,* $\{X_t : t \in T\}$*, defined on a common probability space.*

This thesis will focus on Gaussian processes.

Definition 2.10 (Gaussian Process) *[3] A stochastic process* $\{X_t : t \in T\}$ *is said to be a Gaussian process if any finite collection* $[X_{t_1}, X_{t_2}, ..., X_{t_k}]$ *has a multivariate Gaussian distribution.*

A Gaussian process is completely characterized by its mean and a covariance function

$$Cov(X_s, X_t) = E\left[X_s X_t - E(X_s)E(X_t)\right]. \tag{2.16}$$

A stochastic process with $T \subset \mathbb{R}^d$ is often called a *random field.*

Standard Brownian motion in one dimension is a mean zero Gaussian process whose covariance function is

$$C(s,t) = s \wedge t = \frac{1}{2}(|t| + |s| - |t - s|). \tag{2.17}$$

Two common ways to generalize Brownian motion to a random field are the Brownian sheet and Lévy's Brownian motion. These are obtained by generalizing the covariance function to operate on vectors in $\mathbb{R}^d$.

Definition 2.11 (Brownian Sheet) *[10]A Brownian sheet is a Gaussian random field that is defined on $(\mathbb{R}^d_+, \mathcal{B}(\mathbb{R}^d))$ with mean 0 and covariance function*

$$C(s,t) = \prod_{i=1}^{d} s_i \wedge t_i. \tag{2.18}$$

Note that the covariance function is equal to the volume of the intersection of the d-dimensional rectangles with one vertex at the origin s and t as their diagonals.

Definition 2.12 (Lévy's Brownian Motion) *[16]Lévy's Brownian motion in d dimensions is a mean zero Gaussian random field with covariance function*

$$K(s,t) = \frac{1}{2}(|t| + |s| - |t - s|) \tag{2.19}$$

for $s, t \in \mathbb{R}^d$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$.

For the majority of this paper we will restrict our study of Levy's Brownian motion to the unit ball in $\mathbb{R}^d$, denoted B^d. In order to work more easily with this covariance function, we introduce an integral representation.

Lemma 2.2 *Define A_s to be the ball with the vector $s \in \mathbb{R}^d$ as the diameter, that is*

$$A_s = \left\{ u \in \mathbb{R}^d : |u| < \frac{u}{|u|} \cdot s \right\} \tag{2.20}$$

and let μ be the measure on $\mathbb{R}^d$ with density function

$$f_d(u) = \frac{\eta_d + 1}{\sigma_{d-1}} |u|^{-(d-1)} \tag{2.21}$$

where σ_{d-1} is the surface area of S^{d-1} and $\eta_d = \frac{d-3}{2}$. Then we can write

$$K(s,t) = \mu(A_s \cap A_t). \tag{2.22}$$

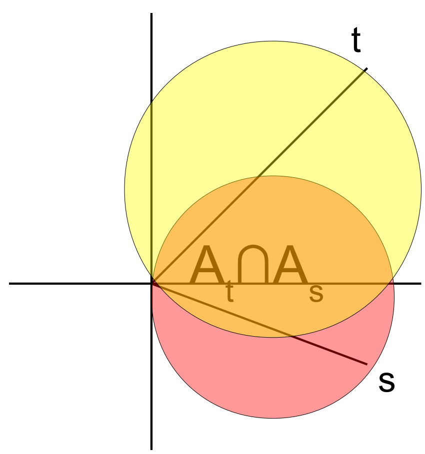

Figure 2.1: Example of $A_s \cap A_t$ in $\mathbb{R}^2$

The following will prove useful in the proof of Lemma 2.2.

Lemma 2.3 *[8] Let Φ be a bounded, integrable function on $[-1,1]$ and u be a fixed point on S^{d-1}. Then*

$$\int_{S^{d-1}} \Phi(u \cdot v)\, d\sigma(v) = \sigma_{d-1} \int_{-1}^{1} \Phi(x)(1-x^2)^{\eta_d}\, dx. \tag{2.23}$$

Proof of Lemma 2.2:

Consider the function

$$G_{t,s}(r,\theta) = sgn(\theta \cdot t - r) - sgn(\theta \cdot s - r). \tag{2.24}$$

So

$$G_{t,s}(r,\theta) = \begin{cases} 2 & \text{if } \theta \cdot t > r \text{ and } \theta \cdot s < r \\ -2 & \text{if } \theta \cdot t < r \text{ and } \theta \cdot s > r \\ 0 & \text{else} \end{cases} . \tag{2.25}$$

Therefore

$$\begin{aligned} \int\limits_{S^{d-1}} \int\limits_{\mathbb{R}} G^2_{t,s}(r,\theta)\, dr\, d\sigma(\theta) &= 4 \int\limits_{S^{d-1}} \left[\int_{\theta\cdot s}^{\theta\cdot t} \mathbf{1}_{\theta\cdot s\cdot<\theta\cdot t}\, dr + \int_{\theta\cdot t}^{\theta\cdot s} \mathbf{1}_{\theta\cdot t<\theta\cdot s}\, dr \right] d\sigma(\theta) \\ &= 4 \int_{S^{d-1}} |\theta\cdot t - \theta \cdot s|\, d\sigma(\theta). \end{aligned}$$

To evaluate this integral apply Lemma 2.3 with $v = \theta$, $u = \frac{t-s}{|t-s|}$ and $\Phi(x) = |x|$ to get

$$4\sigma_{d-1}|t-s| \int_{-1}^{1} |x|(1-x^2)^{\eta_d}\, dx \quad = \quad \frac{4\sigma_{d-1}}{\eta_d + 1}|t-s|.$$

Alternately, we could rewrite $A_s(u)$ with u converted to polar coordinates gives

$$A_s(r,\theta) = \{(r,\theta) : 0 < r < \theta \cdot s\}. \tag{2.26}$$

So we can see that $G_{t,s}$ can be written as

$$2\left[\mathbf{1}_{\theta\cdot s<r<\theta\cdot t}(r\theta) - \mathbf{1}_{\theta\cdot t<r<\theta\cdot s}(r\theta)\right] \;=\; 2\left[\mathbf{1}_{A_t\cap A_s^c}(u) - \mathbf{1}_{A_s\cap A_t^c}(u)\right].$$

Therefore

$$G_{t,s}^2(u) = 4\left(\mathbf{1}_{A_s\triangle A_t}(u)\right) \tag{2.27}$$

giving

$$\begin{aligned}\int_{\mathbb{R}^d} G_{t,s}^2(u)\, d\mu(u) &= 4\int 1_{A_s\triangle A_t}\, d\mu(u)\\ &= 4\mu(A_s\triangle A_t)\end{aligned}$$

and

$$|s-t| = \mu(A_s\triangle A_t). \tag{2.28}$$

Letting $s=0$ in (2.28) also gives

$$\mu(A_t) = |t|. \tag{2.29}$$

Therefore

$$K(s,t) = \frac{1}{2}\left[\mu(A_s) + \mu(A_t) - \mu(A_s\triangle A_t)\right] \tag{2.30}$$

$$= \mu(A_s\cap A_t). \tag{2.31}$$

■

2.3 Orthogonal Polynomials

In the following section we will construct an orthonormal basis for $L^2(B^d)$ that will later be useful in finding expansions of Gaussian random fields. This basis will be constructed using bases for $L^2((0,1))$ and $L^2(S^{d-1})$.

Definition 2.13 (Jacobi Polynomial) *[7] For parameters $\alpha, \beta > -1$ define the Jacobi polynomials on $[-1,1]$ $\{P_n^{(\alpha,\beta)} : n \geqslant 0\}$ by*

$$P_n(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}(1-x)^{\alpha+n}(1-x)^{\beta+n}. \tag{2.32}$$

The Jacobi polynomials are orthogonal with respect to the weight $(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$. i.e.

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx = 0 \text{ for } m \neq n. \tag{2.33}$$

In addition they are a complete set in $L^2([-1,1])$. Through translation they can be used to form a complete orthogonal set in $L^2([0,1])$.

Lemma 2.4 *Let $P_n^{(0,\beta)}$ denote the Jacobi polynomial with parameter $\alpha = 0$. Then $\{P_n^{(0,\beta)}(2r-1) : n \geqslant 0\}$ is orthogonal with respect to the weight r^β on $[0,1]$.*

Proof: For $n \neq m$

$$\begin{aligned} 0 &= \int_{-1}^{1} P_n^{(0,\beta)}(x) P_m^{(0,\beta)}(x)(1+x)^\beta \, dx \\ &= 2\int_0^1 P_n^{(0,\beta)}(2r-1) P_m^{(0,\beta)}(2r-1)(2r)^\beta \, dr \\ &= 2^{\beta+1} \int_0^1 P_n^{(0,\beta)}(2r-1) P_m^{(0,\beta)}(2r-1) r^\beta \, dr. \end{aligned}$$

■

Definition 2.14 (Spherical Harmonics) *[8]A polynomial p is said to be harmonic if it is homogeneous with $\triangle p = 0$. The spherical harmonics in d dimensions are the restriction of such polynomials to the unit sphere S^{d-1}.*

We will denote the space of normalized spherical harmonics of degree n and dimension d by $\mathcal{H}_n^d$ and the space of homogeneous polynomials of degree n and dimension d by $\mathcal{P}_n^d$. The dimension of $\mathcal{H}_n^d$ is

$$h_d(n) = dim\mathcal{P}_n^d - dim\mathcal{P}_{n-2}^d. \tag{2.34}$$

For every d, there is an orthonormal basis for $L^2(S^{d-1})$ consisting of spherical harmonics.

2.3.1 An orthonormal basis for $L^2(B^d)$

Lemma 2.5 *Let H_n^d be an orthonormal basis for $\mathcal{H}_n^d$ and $\varphi_{n,j} \in H_n^d$. Define $J_k^d(|u|)$ to be the Jacobi polynomial of degree k with parameter $\alpha = 0$ and $\beta = d-1$ evaluated* *at* $2|u|-1$*. Define the set*

$$P^d = \{\gamma_k J_k^d(|u|)\varphi_{n,j}(u/|u|) : k \geqslant 0, n \geqslant 0, 1 \leqslant j \leqslant h_d(n)\} \tag{2.35}$$

where γ_k *is a constant so that*

$$\gamma_k^2 \int_0^1 (J_k^d(r))^2 r^{d-1}\, dr = 1. \tag{2.36}$$

Then P^d *is an orthonormal basis for* $L^2(B^d)$*.*

Proof: To check orthogonality, observe that

$$\int\limits_{B^d} J_k^d(|u|)\varphi_{n,j}(u/|u|)J_l^d(|u|)\varphi_{m,i}(u/|u|)\, du =$$

$$\int\limits_0^1 J_k^d(r)J_l^d(r)r^{d-1}\, dr \int\limits_{S^{d-1}} \varphi_{n,j}(\theta)\varphi_{m,i}(\theta)\, d\sigma(\theta) \tag{2.37}$$

which is clearly 0 if $(k,n,j) \neq (l,m,i)$ due to the orthogonality of the Jacobi polynomials on $[0,1]$ and the spherical harmonics on S^{d-1}.

Now, assume $f \in L^2(B^d)$ with

$$\int_{B^d} f(u)J_k^d(|u|)\varphi_{n,j}(u/|u|)\, du = 0 \tag{2.38}$$

for all k, n, j. Converting to polar coordinates then gives

$$\int_{S^{d-1}} \varphi_{n,j}(\theta) \int_0^1 f(r\theta)J_k^d(r)r^{d-1}\, dr\, d\sigma(\theta) = 0. \tag{2.39}$$

Since the spherical harmonics form an orthonormal basis for $L^2(S^{d-1})$, this implies

$$\int_0^1 f(r\theta)J_k^d(r)r^{d-1}\, dr = 0 \tag{2.40}$$

for each k and each θ. However, this is only possible if $f(r\theta) = 0$ for all r, since the Jacobi polynomials form a basis for $L^2([0,1])$. Therefore, $f \equiv 0$ and P^d is an orthonormal basis.

■

Lemma 2.6 *The normalizing constant γ_k is equal to $\sqrt{2k+d}$.*

To prove this we begin by adapting two known formulas for the Jacobi polynomials for use with the polynomials $P_n^{(\alpha,\beta)}(2r-1)$.

Lemma 2.7 *[7] For any polynomial q*

$$\int_{-1}^{1} q(x) P_n^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta \, dx = \frac{1}{2^n n!} \int_{-1}^{1} (1-x)^{\alpha+n}(1+x)^{\beta+n} \frac{d^n}{dx^n} q(x) \, dx. \tag{2.41}$$

Then the corollary follows immediately.

Corollary 2.1 *For any polynomial q*

$$\int_0^1 q(r) r^{d-1} P_n^{(0,d-1)}(2r-1) \, dr = \frac{1}{n!} \int_0^1 (1-r)^n r^{n+d-1} \frac{d^n}{dr^n} q(r) \, dr. \tag{2.42}$$

Lemma 2.8 *[7] For any $n \geqslant 1$*

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.43}$$

And the following corollary is immediate.

Corollary 2.2 *For any $n \geqslant 1$*

$$\frac{d}{dr} P_n^{(\alpha,\beta)}(2r-1) = (n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(2r-1). \tag{2.44}$$

Repeated application of Corollary 2.2 gives the formula

$$\frac{d^k}{dr^k} P_k^{(0,d-1)}(2r-1) = \frac{(2k+d-1)!}{(k+d-1)!} P_0^{(k,k+d-1)}(2r-1) = \frac{(2k+d-1)!}{(k+d-1)!}. \tag{2.45}$$

Proof of Lemma 2.6: First, apply Corollary 2.1 with $q(r) = P_k^{(0,d-1)}(2r-1)$:

$$\begin{aligned} \gamma_k^{-2} &= \int_0^1 \left(P_n^{(0,d-1)}(2r-1)\right)^2 r^{d-1}\, dr && (2.46) \\ &= \frac{1}{k!} \int_0^1 (1-r)^k r^{k+d-1} \frac{d^k}{dr^k} P_k^{(0,d-1)}(2r-1)\, dr. && (2.47) \end{aligned}$$

Now use (2.45) to get

$$\begin{aligned} \binom{2k+d-1}{k} \int_0^1 (1-r)^k r^{k+d-1}\, dr &= \binom{2k+d-1}{k} \int_0^1 (1-r)^k r^{k+d-1}\, dr \\ &= \binom{2k+d-1}{k} \beta(k+1, k+d) \\ &= \binom{2k+d-1}{k} \frac{k!(k+d-1)!}{(2k+d)!} \\ &= \frac{1}{2k+d}. \end{aligned}$$

So $\gamma_k = \sqrt{2k+d}$.

■

2.3.2 Legendre Polynomials

The spherical harmonics are closely related to a family of polynomials called the *Legendre polynomials*, which are orthonormal on $[-1,1]$ with respect to the weight $(1-x^2)^{\eta_d}$.

Definition 2.15 (Legendre Polynomials) *[8] For $d \geqslant 2, n \geqslant 0$ the Legendre polynomial of degree n in d dimensions is*

$$P_n^d(t) = q_n^d \left(1-t^2\right)^{-\eta_d} \frac{d^n}{dt^n}(1-t^2)^{\eta_d+n} \tag{2.48}$$

where $\eta_d = \frac{d-3}{2}$ and

$$q_n^d = (-1)^n 2^{-n} \prod_{i=1}^{n} (\eta_d + i)^{-1}. \tag{2.49}$$

For all n and d the Legendre polynomials $\{P_n^d : n \geqslant 0\}$ have the properties that

1. $|P_n^d(t)| \leqslant 1$ for all $t \in [-1,1]$ and

2. $P_n^d(1) = 1$

The following relationships between spherical harmonics and Legendre polynomials will be useful in later calculations.

Lemma 2.9 *[8] Fix $d \geqslant 2$, $n \geqslant 0$. Let $\{\varphi_{n,i} : 1 \leqslant i \leqslant h_d(n)\}$ be an orthonormal basis for $\mathcal{H}_n^d$, the space of spherical harmonics in dimension d of degree n. Then for*

$s, t \in S^{d-1}$

$$\sum_{i=1}^{h_d(n)} \varphi_{n,i}(s)\varphi_{n,i}(t) = \frac{h_d(n)}{\sigma_{d-1}} P_n^d(s \cdot t) \tag{2.50}$$

where P_n^d is the Legendre polynomial in dimension d of degree n.

Note that in the case of $s = t$ this yields

$$\sum_{i=1}^{h_d(n)} \varphi_{n,i}^2(t) = P_m(1) = \frac{h_d(m)}{\sigma_{d-1}}. \tag{2.51}$$

Theorem 2.1 (The Funk-Hecke Theorem) *[8] If Φ is a bounded integrable function on [-1,1] and φ_n is a spherical harmonic of degree n in d dimensions then $\Phi(u \cdot v)$ is (for any fixed $u \in S^{d-1}$) an integrable function on S^{d-1} and*

$$\int_{S^{d-1}} \Phi(u \cdot v)\varphi_n(v)\, d\sigma(v) = \alpha_{d,n}(\Phi)\varphi_n(u)$$

where

$$\alpha_{d,n}(\Phi) = \sigma_{d-1} \int_{-1}^{1} \Phi(t) P_n^d(t)(1 - t^2)^{\eta_d}\, dt \tag{2.52}$$

and $\eta_d = (d-3)/2$, P_n^d is the Legendre polynomial of dimension d and degree n, σ_{d-1} is the surface area of S^{d-1} and σ is the uniform measure on S^{d-1}.

Chapter 3 The White Noise Integral

In this section we will use the Brownian sheet on $\mathbb{R}^d$ to construct a random variable called the *white noise integral* and use it to find an alternate representation for Lévy's Brownian motion. Recall that the covariance function, $K(s, t)$ can be written in integral form as

$$K(s, t) = \mu(A_s \cap A_t) \tag{3.1}$$

where μ is the measure with density

$$d\mu(u) = \frac{\eta_d + 1}{\sigma_{d-1}} |u|^{-(d-1)} \, du \tag{3.2}$$

and

$$A_s = \left\{ u \in B^d : |u - s/2| < |s/2| \right\} \tag{3.3}$$

This leads to a stochastic integral representation for Lévy's Brownian motion.

3.1 The Brownian Sheet

In order to write Lévy's Brownian motion as a stochastic integral we use a white noise integral. Recall that a Brownian sheet is a random field that is typically defined on

$\mathbb{R}_+^d$ as a Gaussian random field with mean 0 and covariance function

$$C(s,t) = \prod_{i=1}^{d} s_i \wedge t_i. \tag{3.4}$$

To extend this definition to $\mathbb{R}^d$ the covariance function must be modified so that it is positive definite. To accomplish this we introduce the following definition for the Brownian sheet in $\mathbb{R}^d$.

Definition 3.1 (Brownian Sheet) *The Brownian sheet in $\mathbb{R}^d$ is a mean zero Gaussian random field with covariance function*

$$C(s,t) = \begin{cases} \prod_{i=1}^{d} |s_i| \wedge |t_i| & \textit{if } s \textit{ and } t \textit{ are in the same orthant} \\ 0 & \textit{otherwise} \end{cases} \tag{3.5}$$

Before we can use this definition we need to verify that such a random field exists. The following theorems give criteria for the existence of Gaussian random field based on the structure of the covariance function and the finite-dimensional distributions.

Definition 3.2 (Finite-dimensional Distributions) *[3] Let $\{X_t : t \in T\}$ be a random field on $(\Omega, \mathcal{A}, P)$, $\{A_i : 1 \leqslant i \leqslant n\} \subset \mathcal{B}$ and $[X_{t_i} : 1 \leqslant i \leqslant n]$ be any finite random vector from the random field. Then define the measure $\mu_{t_1,\ldots,t_n}$ to be the joint distribution function of $[X_{t_1}, \ldots, X_{t_n}]$. That is*

$$\mu_{t_1,\ldots,t_n}(A_1 \times \ldots \times A_n) = P\left(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right). \tag{3.6}$$

These measures are called the finite-dimensional distributions of $\{X_t : t \in T\}$.

Theorem 3.1 (Kolmogorov's Consistency Conditions) *[3] Let $\{\mu_{t_{k_1},\dots t_{k_n}} : n \geqslant 1\}$ be a collection of measures on $\mathbb{R}^d$. Suppose each measure satisfies the following two conditions.*

1. *For any permutation, π, of $\{1, 2, ..., n\}$,*

$$\mu_{t_{\pi 1},\dots,t_{\pi n}}(A_{\pi 1} \times \dots \times A_{\pi n}) = \mu_{t_1,\dots,t_n}(A_1 \times \dots \times A_n) \tag{3.7}$$

   and

2. $$\mu_{t_1,\dots,t_{n-1}}(A_1 \times \dots \times A_{n-1}) = \mu_{t_1,\dots,t_n}(A_1 \times \dots \times A_{n-1} \times \mathbb{R}). \tag{3.8}$$

Then there exists a random field with these measures as the finite-dimensional distributions.

Due to the structure of the multivariate Gaussian density function, any valid covariance function defines a Gaussian random field.

Lemma 3.1 *[3] Let $K : T \times T \to \mathbb{R}$ be a symmetric function such that the matrix*

$$\Sigma = [K(t_i, t_j) : 1 \leqslant i, j \leqslant n] \tag{3.9}$$

is positive definite for any finite n. Then there exists a mean 0 Gaussian random vector $[X_{t_1}, ..., X_{t_n}]$ with

$$K(t_i, t_j) = Cov(X_{t_i}, X_{t_j}) \tag{3.10}$$

for each i, j.

Lemma 3.2 *The function $C(s,t)$ defined in (3.5) is the covariance function of a Gaussian random field.*

Proof: For $t \in \mathbb{R}^{\star d} = [\mathbb{R} \cup \{\pm\infty\}]^d$ let $\mathcal{I}_t$ denote the rectangle

$$\mathcal{I}_t = I_{t_1} \times I_{t_2} \times ... \times I_{t_d} \tag{3.11}$$

where

$$I_t = \begin{cases} (t,0] & if \quad t < 0 \\ (0,t] & if \quad 0 \leqslant t < \infty \\ (0,\infty) & if \quad t = \infty \end{cases} \tag{3.12}$$

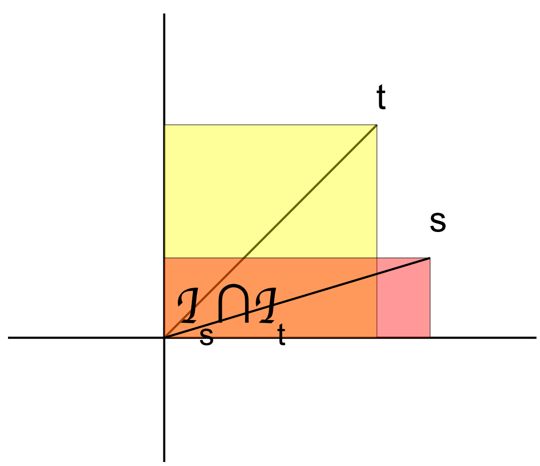

Figure 3.1: Example of $\mathcal{I}_s \cap \mathcal{I}_t$ in $\mathbb{R}^d$

Then for any $s, t \in \mathbb{R}^{\star d}$ we can write $C(s,t)$ as the integral

$$\int_{\mathbb{R}^d} 1_{\mathcal{I}_t \cap \mathcal{I}_s}(u)\, du. \tag{3.13}$$

Let $F = \{t_i : 1 \leqslant i \leqslant n\}$ be some finite subset of $\mathbb{R}^{\star d}$ and A be the $n \times n$ matrix $[C(t_i, t_j) : 1 \leqslant i, j \leqslant n]$. Suppose that for $1 \leqslant i, j \leqslant k$, t_i and t_j are in the same orthant. Then for any $x \in \mathbb{R}^d$

$$\begin{aligned}
x^T A x &= \sum_{i=1}^{k} \sum_{j=1}^{k} x_i \int_{\mathbb{R}^d} 1_{\mathcal{I}_{t_i} \cap \mathcal{I}_{t_j}}(u)\, du\, x_j && (3.14)\\
&= \int_{\mathbb{R}^d} \sum_{i=1}^{k} x_i 1_{\mathcal{I}_{t_i}}(u) \sum_{j=1}^{k} x_j 1_{\mathcal{I}_{t_j}}(u)\, du && (3.15)\\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^{k} x_i 1_{\mathcal{I}_{t_i}}(u) \right)^2 du && (3.16)\\
&\geqslant 0. && (3.17)
\end{aligned}$$

Therefore a mean 0 Gaussian random field exists with the covariance function $C(s,t)$.

$\blacksquare$

3.2 The White Noise Integral

We begin by considering the family of simple functions built from indicators on rectangles of the form described in (3.11) and (3.12).

Note that it suffices to consider rectangles of this form since they can be used to construct an indicator function on a general rectangle of the form $R = (a_1, b_1] \times ... \times$

$(a_d, b_d]$. To construct R, begin by writing

$$1_R(\vec{x}) = \prod_{i=1}^{d} 1_{(a_i,b_i]}(x_i). \tag{3.18}$$

Then if a_i and b_i have the same sign $(a_i, b_i]$ can be written as $I_{b_i} \setminus I_{a_i}$. In addition, intersections commute with the Cartesian product so we can write

$$1_{(a_i,b_i]}(x_i) = 1_{I_{a_i} \setminus I_{b_i}}(x_i). \tag{3.19}$$

If $a_i < 0 < b_i$ then I_{a_i} and I_{b_i} are disjoint so we can write

$$1_{(a_i,b_i]}(x_i) = 1_{I_{a_i}} + 1_{I_{b_i}}. \tag{3.20}$$

Let $\{t_i : 1 \leqslant i \leqslant n\}$ be a subset of $\mathbb{R}^d$, $\{a_i : 1 \leqslant i \leqslant n\}$ a sequence of real numbers and let f be the simple function

$$f(u) = \sum_{i=1}^{n} a_i 1_{\mathcal{I}_{t_i}}(u). \tag{3.21}$$

Define the random variable $I(f)$ to be

$$I(f) = \sum_{i=1}^{n} a_i W(t_i) \tag{3.22}$$

where $W(\cdot)$ is the Brownian sheet.

Definition 3.3 (White noise integral of a general function) *Let $f \in L^2(\mathbb{R}^d)$. Then the white noise integral of f, denoted*

$$\int f(u)\, dW_u \tag{3.23}$$

is a mean zero Gaussian random variable with variance

$$\int f^2(t)\, dt. \tag{3.24}$$

Lemma 3.3 *For a simple function f*

$$I(f) = \int f(u)\, dW_u. \tag{3.25}$$

Proof: The fact that $I(f)$ is Gaussian and mean 0 is immediate since the sum of Gaussian random variables is also Gaussian and the Brownian sheet has mean 0.

To calculate the variance, consider

$$\begin{aligned}
E\left(I^2(f)\right) &= E\left(\sum_{i=1}^{n} a_i W(t_i)\right)^2 \\
&= \sum_{i=1}^{n} a_i^2 E(W(t_i)^2) + 2 \sum_{1\leqslant i<j\leqslant n} a_i a_j E(W(t_i)W(t_j)) \\
&= \sum_{i=1}^{n} a_i^2 \prod_{k=1}^{d} |t_{i,k}| + 2 \sum_{1\leqslant i<j\leqslant n, t_i t_j>0} a_i a_j \prod_{k=1}^{d} |t_{i,k}| \wedge |t_{j,k}| \\
&= \sum_{i=1}^{n} a_i a_j \int_{\mathbb{R}^d} 1_{\mathcal{I}_{t_i}\cap\mathcal{I}_{t_j}}\, dt \\
&= \int_{\mathbb{R}^d} f^2(t)\, dt.
\end{aligned}$$

■

Next we will show that the white noise integral of a general function can be considered as the limit of the white noise integral of simple functions. First we define another notion of convergence of random variables, convergence in probability, which is stronger than weak convergence.

Definition 3.4 (Convergence in Probability) *[3] Let $\{X_n : n \geqslant 1\}$ and X be random variables on $(\Omega, \mathcal{A}, P)$. Then X_n converges to X in probability, denoted*

$$X_n \to^P X \tag{3.26}$$

if

$$\lim_{n\to\infty} P\left(|X_n - X| \geqslant \epsilon\right) = 0 \tag{3.27}$$

for all $\epsilon > 0$.

Lemma 3.4 *Let $\{f_n : n \geqslant 1\}$ be a sequence of simple functions that converge to $f \in L^2(\mathbb{R}^d)$. Then*

$$\int f_n(u)\, dW_u \to^P \int f(u)\, dW_u. \tag{3.28}$$

Proof: Since $f_n \to f$ in $L^2(\mathbb{R}^d)$, $\{f_n : n \geqslant 1\}$ is Cauchy and therefore for any m, n

$$\begin{aligned} E\left(\int f_n(u)\, dW_u - \int f_m(u)\, dW_u\right)^2 &= \int \left(f_n(t) - f_m(t)\right)^2\, dt & (3.29)\\ &\to 0. & (3.30)\end{aligned}$$

So there exists some random variable X such that

$$\int f_n(u)\, dW_u \to^P X. \tag{3.31}$$

Since convergence in probability implies weak convergence, X must be a Gaussian random variable with mean zero and variance

$$\lim_{n\to\infty} \int f_n^2(t)\, dt = \int f^2(t)\, dt. \tag{3.32}$$

■

If the convergence of f_n to f is fast enough then we get one more type of convergence, convergence almost surely.

Definition 3.5 (Almost Sure Convergence) *[9] Let $\{X_n : n \geqslant 1\}$, X be random variables on $(\Omega, \mathcal{A}, P)$. Then X_n converges to X almost surely if*

$$P\left(\limsup_{n\to\infty} |X_n - X| \geqslant \varepsilon\right) = 0 \tag{3.33}$$

for all $\varepsilon > 0$.

Almost sure convergence implies convergence in probability and weak convergence.

Lemma 3.5 *If $\{f_n : n \geqslant 1\}$ is a sequence of simple functions converging to f such that*

$$\sum_{n=1}^{\infty} E\left(\int f_n(u)\, dW_u\right)^2 < \infty \tag{3.34}$$

then

$$\int f_n(u)\, dW_u \to \int f(u)\, dW_u \quad \textit{almost surely.} \tag{3.35}$$

To prove this we will make use of two common results.

Lemma 3.6 (The Borel-Cantelli Lemma) *[3] Let $\{A_n : n \geqslant 1\}$ be a sequence of events. If*

$$\sum_{n=1}^{\infty} P(A_n) < \infty \tag{3.36}$$

then

$$P\left(\limsup_{n\to\infty} A_n\right) = 0. \tag{3.37}$$

Lemma 3.7 (Markov's Inequality) *[3] Let X be a mean zero random variable on $(\Omega, \mathcal{A}, P)$ and $\epsilon > 0$. Then*

$$P(|X| \geqslant \varepsilon) \leqslant \frac{Var(X)}{\varepsilon^2}. \tag{3.38}$$

Proof of Lemma 3.5

By Markov's Inequality

$$P\left(\left|\int f_n(u)\, dW_u - \int f(u)\, dW_u\right| \geqslant \varepsilon\right) \leqslant \frac{1}{\varepsilon^2}\int (f_n(t) - f(t)\, dt)^2\ dt \tag{3.39}$$

for any $\varepsilon > 0$. So by the Borel-Cantelli Lemma, for f_n converging to f in L^2

$$P\left(\limsup_{n\to\infty}\left|\int f_n(u)\, dW_u - \int f(u) dW_u\right| \geqslant \varepsilon\right) = 0 \tag{3.40}$$

if

$$\sum_{n=1}^{\infty} ||f_n - f||_{L^2}^2 < \infty \tag{3.41}$$

which proves almost sure convergence of the white noise integral.

■

The covariance of two white noise integrals will allow us to find the white noise integral representation of Lévy's Brownian motion.

Lemma 3.8 *Let $f, g \in L^2(\mathbb{R}^d)$. Then*

$$E\left(\int_{\mathbb{R}^d} f(u)\, dW_u \int_{\mathbb{R}^d} g(u)\, dW_u\right) = \int_{\mathbb{R}^d} f(t)g(t)\, dt. \tag{3.42}$$

Proof: We can write

$$\begin{aligned} E\left(\int_{\mathbb{R}^d} f(u) + g(u)\, dW_u\right)^2 &= \int_{\mathbb{R}^d} (f(t) + g(t))^2\, dt & (3.43) \\ &= \int_{\mathbb{R}^d} f^2(t) + g^2(t) + 2f(t)g(t)\, dt. & (3.44) \end{aligned}$$

On the other hand,

$$E\left(\int_{\mathbb{R}^d} f(u)\, dW_u + \int_{\mathbb{R}^d} g(u)\, dW_u\right)^2 =$$
$$E\left(\int f(u)\, dW_u\right)^2 + E\left(\int g(u)\, dW_u\right)^2 + 2E\left(\int f(u)g(u) dW_u\right). \tag{3.45}$$

Since the stochastic integral is a linear function, these two are equal and the result follows. ■

These properties immediately yield the white noise integral representation for Lévy's Brownian motion.

Lemma 3.9 *Lévy's Brownian motion has white noise integral representation*

$$X_t = \sqrt{\frac{\eta_d + 1}{\sigma_{d-1}}} \int_{B^d} |u|^{-(d-1)/2} 1_{A_t}(u)\, dW_u. \tag{3.46}$$

Chapter 4 Expansions for Lévy's Brownian Motion in Terms of Spherical Harmonics

In this section we will find an expansion for Lévy's Brownian Motion and its covariance function in d dimensions as a linear combination of standard Gaussian random variables. Such expansions can be extremely useful in simulating the processes. The expansions will be based on the *Karhunen-Loeve Theorem* and *Mercer's Theorem.*

4.1 The Karhunen-Loeve Expansion

Theorem 4.1 (Mercer's Theorem) *[6] Let T be a compact subset of $\mathbb{R}^d$ and K be a continuous positive definite function on $T \times T$. Then there exists an orthonormal basis $\{e_i : i \geqslant 0\}$ of L^2 consisting of eigenfunctions of K with corresponding non-negative eigenvalues $\{\lambda_i^2 : i \geqslant 0\}$ and*

$$K(s,t) = \sum_{j=0}^{\infty} \lambda_j^2 e_j(s) e_j(t) \tag{4.1}$$

where convergence is absolute and uniform.

Theorem 4.2 (The Karhunen-Loeve Theorem) *[6] Let T be a compact subset of $\mathbb{R}^d$ and $\{X_t : t \in T\}$ be a mean zero stochastic process with continuous covariance* *function $K(s,t)$. Then there is an orthonormal basis $\{e_i(t) : i \geqslant 0\}$ for $L^2(T)$ consisting of eigenfunctions of $K(s,t)$ and*

$$X_t = \sum_{n=0}^{\infty} \lambda_n Z_n e_n(t) \tag{4.2}$$

where λ_n^2 is the eigenvalue corresponding to e_n and

$$Z_n = \frac{1}{\lambda_n} \int_T X_t e_n(t)\, dt. \tag{4.3}$$

The series in (4.2) is called the **Karhunen-Loeve expansion** *for X_t and converges uniformly to X_t in $L^2(T)$.*

The Karhunen-Loéve Expansion for standard Brownian motion in 1 dimension can easily be found by finding the eigenvalues of the covariance function.

Lemma 4.1 *Let $\{Z_n : n \geqslant 0\}$ be a sequence of independent standard Gaussian random variables and $t \in [0,1]$. Then the Karhunen-Loéve expansion for standard Brownian motion in 1 dimension is*

$$X_t = \sqrt{2} \sum_{n=0}^{\infty} \frac{\sin[(n+\frac{1}{2})\pi t]}{(n+\frac{1}{2})\pi} Z_n. \tag{4.4}$$

Proof: We seek to find an orthonormal basis $\{\varphi_n : n \geqslant 0\}$ of $L^2((0,1))$ such that for each n

$$\int_0^1 (s \wedge t)\varphi_n(s)\, ds = \lambda_n \varphi_n(t). \tag{4.5}$$

Let

$$I = \int_0^1 (s \wedge t) f(s)\, ds \tag{4.6}$$

$$= \int_0^t s f(s)\, ds + t \int_t^1 f(s)\, ds. \tag{4.7}$$

Then

$$\frac{\partial^2 I}{\partial t^2} = -f(t) \tag{4.8}$$

which yields the differential equation

$$f''(t) = -\lambda f(t) \tag{4.9}$$

with the initial condition $f(0) = 0$. This has solutions of the form

$$f(t) = c \sin(t\lambda^{-1/2}) \tag{4.10}$$

where c is some normalization constant. To solve for λ solve the equation:

$$\int_0^1 (s \wedge t) \sin(s\lambda^{-1/2})\, ds = \lambda \sin(t\lambda^{-1/2}) \tag{4.11}$$

which gives

$$\lambda = \frac{1}{\left((n + \frac{1}{2})\pi\right)^2} \tag{4.12}$$

for any $n \in \mathbb{Z}$. Normalize the solution by setting

$$\int_0^1 c^2 \sin^2\left((n + \frac{1}{2})\pi s\right)\, ds = 1 \tag{4.13}$$

which has the solution $c = \sqrt{2}$. So the eigenfunctions are

$$\varphi_n(t) = \sqrt{2}\sin\left((n+\frac{1}{2})\pi t\right) \tag{4.14}$$

with corresponding eigenvalues

$$\lambda_n = \frac{1}{\big((n+\frac{1}{2})\pi\big)^2}. \tag{4.15}$$

Now, recall

$$Z_n = \sqrt{2}\int_0^1 X(t)\sin\left((n+\frac{1}{2})\pi t\right)\, dt \tag{4.16}$$

Since $\sqrt{2}X(t)\sin\big((n+\frac{1}{2})\pi t\big)$ is a standard Gaussian random variable for each n and t, the integral will also be a standard Gaussian random variable. Apply the Karhunen-Loéve Theorem to obtain the expansion.

$\blacksquare$.

These expansions can be extremely useful in simulation, however solving the eigenvalue problem in Mercer's Theorem can be difficult or even impossible and often yields complicated expansions. In the next section we will use the integral representation of the covariance function given in Lemma 2.2 to derive an expansion for Lévy's Brownian motion.

4.2 Lévy's Brownian Motion

Theorem 4.3 *Let $\{X_t : t \in B^d\}$ be Lévy's Brownian motion and $K(s,t) = E(X_s X_t)$ be its covariance function. Then*

$$K(s,t) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \lambda_{m,k}(|t|)\lambda_{m,k}(|s|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|)\varphi_{m,j}(s/|s|) \tag{4.17}$$

where

$$\lambda_{m,k}(|t|) = \frac{\sigma_{d-1}^2}{\eta_d + 1}\gamma_k \int_0^1 P_m^d(x)(1-x^2)^{\eta_d} \int_0^{x|t|} J_k(r) r^{(d-1)/2}\, dr\, dx \tag{4.18}$$

P_m^d is the Legendre polynomial of degree m in d dimensions and $\{Z_{m,k,j} : m \geqslant 0, k \geqslant 0, 1 \leqslant j \leqslant h_d(m)\}$ is an array of independent standard Gaussian random variables and $\{\varphi_{m,j} : m \geqslant 0, 1 \leqslant j \leqslant h_d(m)\}$ is an orthonormal basis of spherical harmonics.

We will begin by finding an expansion for a function $y_t(\cdot)$ which has the property that

$$K(s,t) = c_d \int_{B^d} y_t(u)\, y_s(u)\, du = < y_t, y_s >_{L^2(B^d)} \tag{4.19}$$

for a constant c_d.

Lemma 4.2 *For each $u, t \in B^d$ define the function*

$$y_t(u) = |u|^{-(d-1)/2} \mathbb{1}_{A_t}(u) \tag{4.20}$$

and let

$$p_{m,k,j}(u) = \gamma_k J_k^d(|u|)\varphi_{m,j}(u/|u|) \tag{4.21}$$

be an element of the orthonormal basis P^d as defined in (2.35). Then

$$\int_{B^d} y_t(u) p_{m,k,j}(u)\, du = \alpha_{m,k}(|t|)\varphi_{m,j}(t/|t|) \tag{4.22}$$

where

$$\alpha_{m,k}(|t|) = \sigma_{d-1}\gamma_k \int_0^1 P_m(x)(1-x^2)^{\eta_d} \int_0^{x|t|} J_k(r) r^{(d-1)/2}\, dr\, dx \tag{4.23}$$

where P_m the Legendre polynomial of degree m in d dimensions and σ_{d-1} is the surface area of S^{d-1}.

Proof: We begin by writing the integral in polar coordinates.

$$\begin{aligned} \int\limits_{B^d} y_t(u) p_{m,k,j}(u)\, du &= \gamma_k \int\limits_{\theta\cdot t>0} \int\limits_0^{\theta\cdot t} J_k(r)\varphi_{m,j}(\theta) r^{(d-1)/2}\, dr\, d\sigma(\theta) & (4.24) \\ &= \int\limits_{\theta\cdot t/|t|>0} \Phi_{t,k}(\theta\cdot t/|t|)\varphi_{m,j}(\theta)\, d\sigma(\theta) & (4.25) \end{aligned}$$

where

$$\Phi_{t,k}(x) = \gamma_k \int\limits_0^{x|t|} J_k^d(r) r^{(d-1)/2}\, dr \tag{4.26}$$

We can now apply the Funk-Hecke Theorem to get

$$\int\limits_{B^d} y_t(u) p_{m,k,j}(u)\, du = \alpha_{m,k}(|t|)\varphi_{m,j}(t/|t|) \tag{4.27}$$

where

$$\alpha_{m,k}(|t|) = \sigma_{d-1} \int\limits_0^1 P_m(x)(1-x^2)^{\eta_d}\Phi_{t,k}(x)\, dx. \tag{4.28}$$

$\blacksquare$

This allows us to expand $y_t(u)$ in terms of the spherical harmonics as

$$
\begin{aligned}
y_t(u) &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{h_d(m)} < y_t, p_{m,k,j} > p_{m,k,j}(u) && (4.29)\\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \alpha_{m,k}(|t|) \sum_{j=1}^{h_d(m)} p_{m,k,j}(u)\varphi_{m,j}(t/|t|). && (4.30)
\end{aligned}
$$

Now, note that

$$\frac{\eta_d + 1}{\sigma_{d-1}} \int_{B^d} y_t(u) y_s(u)\, du = K(s,t). \tag{4.31}$$

Using the expansion for y_t given in equation 4.30 gives the following expansion for K in terms of spherical harmonics.

$$
\begin{aligned}
K(s,t) &= \frac{\eta_d + 1}{\sigma_{d-1}} < y_t, y_s > && (4.32)\\
&= \frac{\eta_d + 1}{\sigma_{d-1}} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{h_d(m)} < y_t, p_{m,k,j} >< y_s, p_{m,k,j} > && (4.33)\\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{m,k}(|t|)\lambda_{m,k}(|s|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|)\varphi_{m,j}(s/|s|) && (4.34)
\end{aligned}
$$

where

$$\lambda_{m,k}(|t|) = \sqrt{\frac{\eta_d + 1}{\sigma_{d-1}}}\alpha_{m,k}(|t|). \tag{4.35}$$

■

This also leads to an expansion for Lévy's Brownian motion. Note that since the basis vectors used are not eigenfunctions of the covariance function this will not be the Karhunen-Loéve expansion, but can still be shown to converge with probability 1.

Theorem 4.4 *Let* $\{Z_{m,k,j} : m \geqslant 0, k \geqslant 0, 1 \leqslant j \leqslant h_d(m)\}$ *be an array of independent standard Gaussian random variables and define*

$$\tilde{X}_t = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \lambda_{m,k}(|t|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|) Z_{m,k,j}. \tag{4.36}$$

Then $\{\tilde{X}_t : t \in T\}$ *is equal in distribution to Lévy's Brownian motion and the sum converges with probability 1.*

To prove that $\{X_t : t \in B^d\}$ is Lévy's Brownian motion we need to show that it is a mean 0 Gaussian random field with covariance function $K(s,t)$. Fubini's Theorem will be used to justify exchanging the sums with the expectation.

Theorem 4.5 (Fubini's Theorem) *[11] Suppose* $(\Omega, \mathcal{A}, P)$ *and* $(\Lambda, \mathcal{F}, \nu)$ *are* σ-*finite measure spaces. Let* f *be a real-valued* $\mathcal{A} \times \mathcal{F}$ *measurable function on* $\Omega \times \Lambda$ *such that at least one of the quantities*

1. $$\int_{\Omega\times\Lambda} |f(x,y)| \, d(P \times \nu)(x,y)$$

2. $$\int_{\Omega} \left[\int_{\Lambda} |f(x,y)| d\nu(y) \right] dP(x)$$

3. $$\int_{\Lambda} \left[\int_{\Omega} |f(x,y)| \, dP(x) \right] d\nu(y)$$

is finite. Then

$$\begin{aligned}\int_{\Omega\times\Lambda} f(x,y)\, d(P\times\nu)(x,y) &= \int_{\Omega}\left[\int_{\Lambda} f(x,y)d\nu(y)\right] dP(x)\\ &= \int_{\Lambda}\left[\int_{\Omega} f(x,y)\, dP(x)\right] d\nu(y).\end{aligned}$$

Proof of Theorem 4.4: The fact that X_t is mean 0 and Gaussian follows immediately from the fact that it is a sum of mean 0 Gaussian random variables.

To simplify notation for the following, let

$$\sum_{m,k,j} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{h_d(m)}. \tag{4.37}$$

Then

$$X_tX_s = \sum_{m,k,j}\sum_{n,l,i}\lambda_{m,k}(|t|)\lambda_{n,l}(|s|)\varphi_{m,j}(t/|t|)\varphi_{n,i}(s/|s|)Z_{m,k,j}Z_{n,l,i}. \tag{4.38}$$

So $E(X_tX_s)$ can be thought of as the iterated integral of integration with respect to a counting measure, ν, and the density of the normal random variable, P. Recall that $\{Z_{m,k,j}\}$ forms an array of i.i.d. standard normal random variables so

$$E(Z_{m,k,j}Z_{n,l,i}) = \delta_{(m,k,j),(n,l,i)}. \tag{4.39}$$

Also,

$$\sum_{m,k,j}|\lambda_{m,k}(|t|)\lambda_{m,k}(|s|)\varphi_{m,j}(t/|t|)\varphi_{m,j}(s/|s|)| =$$

$$c_d\sum_{m,k,j}|< y_t, p_{m,k,j} >< y_s, p_{m,k,j} >| \tag{4.40}$$

where $< \cdot, \cdot >$ is the standard L^2 inner product and $y_t, p_{m,k,j}$ are defined as in (4.20) and (4.21).

By Holder's inequality this is bounded above by

$$c_d \sqrt{\sum_{m,k,j} | < y_t, p_{m,k,j} > |^2 \sum_{m,k,j} | < y_s, p_{m,k,j} > |^2} = c_d \|y_t\|_{L^2(B^d)} \|y_s\|_{L^2(B^d)}. \quad (4.41)$$

This is finite so Fubini's Theorem implies

$$E(X_s X_t) = \sum_{m,k,j} \lambda_{m,k}(|t|)\lambda_{m,k}(|s|)\varphi_{m,j}(t/|t|\varphi_{m,j}(s/|s|)E(Z^2_{m,k,j}) = K(s,t). \quad (4.42)$$

Therefore X_t converges to Lévy's Brownian motion. All that remains is to show it converges with probability 1, which will use the following.

Lemma 4.3 *[3] Let $\{X_n : n \geqslant 1\}$ be a sequence of independent random variables on a common probability space. If $\sum_{n=1}^{\infty} Var(X_n) < \infty$ then $\sum_{n=1}^{\infty} X_n$ converges with probability 1.*

Define

$$X_{m,k,j} = \lambda_{m,k}(|t|)\varphi_{m,j}(t/|t|)Z_{m,k,j}. \quad (4.43)$$

Then $\{X_{m,k,j}\}$ is a sequence of independent random variables with variance

$$Var(X_{m,k,j}) = \lambda^2_{m,k}(|t|)\varphi^2_{m,k,j}(t/|t|). \quad (4.44)$$

So

$$\sum_{m,k,j} Var(X_{m,k,j}) = K(t,t) < \infty. \quad (4.45)$$

Therefore X_t converges to Lévy's Brownian motion with probability 1. ■

The expression for $K(s,t)$ given in (4.17) can be simplified significantly by exchanging sums and integration and applying Lemma 2.9. First, by Lemma 2.9

$$\sum_{j=1}^{h_d(m)} \varphi_{m,j}(s/|s|)\varphi_{m,j}(t/|t|) = \frac{h_d(m)}{\sigma_{d-1}} P_m(s \cdot t/|s||t|). \tag{4.46}$$

Next we will work on switching the double integral with one of the infinite sums. To simplify notation, let

$$Q_m(x,y) = P_m(x)(1-x^2)^{\eta_d} P_m(y)(1-y^2)^{\eta_d} \tag{4.47}$$

where P_m is the Legendre polynomial of degree m. Note that $|Q_m(x,y)| \leqslant 1$ for all $x \in [0,1]$.

Let

$$f_n(s,t) = \sum_{k=0}^{n} \gamma_k^2 \int_0^1 \int_0^1 Q_m(x,y) \int_0^{x|t|} J_k^d(r) r^{(d-1)/2}\, dr \int_0^{y|s|} J_k^d(u) u^{(d-1)/2}\, du\, dx\, dy. \tag{4.48}$$

And recall Lebesgue's Dominated Convergence Theorem.

Theorem 4.6 *[11] Suppose $\{f_n : n \geqslant 1\}$ is a sequence of measurable functions that converges point wise to a real-valued function. Further suppose that there is a nonnegative integrable function, g, such that $|f_n| \leqslant g$ for all n. Then*

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n. \tag{4.49}$$

If we can show that there is some function, g, integrable on $B^d \times B^d$ such that for every $(s,t) \in B^d \times B^d$ and every n

$$|f_n(s,t)| \leqslant g(s,t) \tag{4.50}$$

then Lebesgue's Dominated Convergence Theorem can be applied to (4.17).

First, using the fact that $|Q_m(x,y)| \leqslant 1$, we obtain the bound

$$|f_n(s,t)| \leqslant \int_0^1 \int_0^1 \sum_{k=0}^{n} \left| \gamma_k^2 \int_0^{x|t|} J_k^d(r) r^{(d-1)/2}\, dr \int_0^{y|s|} J_k^d(u) u^{(d-1)/2}\, du \right| \, dx\, dy. \tag{4.51}$$

By the Cauchy-Schwarz inequality, this is bounded by

$$\int_0^1 \int_0^1 \sqrt{\sum_{k=0}^{n} \left| \int_0^{x|t|} \gamma_k J_k^d(r) r^{(d-1)/2}\, dr \right|^2 \sum_{k=0}^{n} \left| \int_0^{y|s|} \gamma_k J_k^d(u) u^{(d-1)/2}\, du \right|^2} \, dx\, dy. \tag{4.52}$$

However, since all the terms in the summation are positive, this is bounded by

$$\int_0^1 \int_0^1 \sqrt{\sum_{k=0}^{\infty} \left| \int_0^{x|t|} \gamma_k J_k^d(r) r^{(d-1)/2}\, dr \right|^2 \sum_{k=0}^{\infty} \left| \int_0^{y|s|} \gamma_k J_k^d(u) u^{(d-1)/2}\, du \right|^2} \, dx\, dy. \tag{4.53}$$

Now rewrite this as

$$\int_0^1 \int_0^1 \sqrt{\sum_{k=0}^{\infty} \left| \left\langle \gamma_k J_k^d(r) r^{(d-1)/2}, 1_{(0,x|t|)}(r) \right\rangle \right|^2 \sum_{k=0}^{\infty} \left| \left\langle \gamma_k J_k^d(u) u^{(d-1)/2}, 1_{(0,y|s|)}(u) \right\rangle \right|^2} \, dx\, dy \tag{4.54}$$

where $< \cdot, \cdot >$ is the $L^2((0,1))$ inner product. Since $\{\gamma_k J_k^d(r) r^{(d-1)/2} : k \geqslant 0\}$ forms an orthonormal basis for $L^2((0,1))$ this is equal to

$$\int_0^1 \int_0^1 \left\| 1_{(0,x|t|)} \right\|_{L^2((0,1))} \left\| 1_{(0,y|s|)} \right\|_{L^2((0,1))} dx\, dy \leqslant 1 \tag{4.55}$$

Since $g(s,t) = 1$ is integrable on $B^d \times B^d$, the Dominated Convergence Theorem implies

$$\sum_{k=0}^{\infty} \gamma_k^2 \int_0^1 \int_0^1 Q_m(x,y) \int_0^{x|t|} J_k^d(r) r^{(d-1)/2}\, dr \int_0^{y|s|} J_k^d(u) u^{(d-1)/2}\, du\, dx\, dy =$$
$$\int_0^1 \int_0^1 Q_m(x,y) \sum_{k=0}^{\infty} \int_0^{x|t|} \gamma_k J_k^d(r) r^{(d-1)/2}\, dr \int_0^{y|s|} \gamma_k J_k^d(u) u^{(d-1)/2}\, du\, dx\, dy. \tag{4.56}$$

Again, we can rewrite the inner integrals as inner products and this is equal to

$$\int_0^1 \int_0^1 Q_m(x,y) \left\langle 1_{(0,x|t|)}, 1_{(0,y|s|)} \right\rangle dx\, dy = \int_0^1 \int_0^1 Q_m(x,y)(x|t| \wedge y|s|)\, dx\, dy. \tag{4.57}$$

We have just proven the following:

Lemma 4.4 *The covariance function for Lévy's Brownian motion can be written as*

$$K(s,t) = c_d \sum_{m=0}^{\infty} h_d(m) P_m(s \cdot t/|s||t|) \int_0^1 \int_0^1 Q_m(x,y)(x|t| \wedge y|s|)\, dx\, dy. \tag{4.58}$$

This expression will be very useful in simulating Lévy's Brownian motion.

Chapter 5 The P-Processes

The ideas in the preceding section can be generalized to find the expansions of a family of random fields, $\{\mathbb{X}^p : p+d>0\}$ with similar covariance functions.

5.1 Definition

For each $p>-d$ define the measure ν_p on B^d by

$$\nu_p(A) = \frac{\eta_d+1}{\sigma_{d-1}} \int_A |u|^p \, du. \tag{5.1}$$

Then define the function $K_p : B^d \times B^d \to \mathbb{R}$ by

$$K_p(s,t) = \nu_p(A_s \cap A_t) \tag{5.2}$$

where

$$A_s = \left\{u \in \mathbb{R}^d : |u - s/2| < |s|/2\right\}. \tag{5.3}$$

For each p the p-process is defined to be a mean zero Gaussian random field with K_p as its covariance function.

Kolmogorov's consistency conditions will allow us to conclude that such a random field exists.

Lemma 5.1 *For any finite* $F = \{t_i : 1 \leqslant i \leqslant n\}$ *with* $t_i \in B^d$ *the matrix*

$$A = [K_p(t_i, t_j) : 1 \leqslant i, j \leqslant n] \tag{5.4}$$

is positive definite.

Proof: Let

$$y_{t,p}(u) = |u|^{p/2}\ 1_{A_t}(u). \tag{5.5}$$

Then we have the integral representation

$$K_p(s,t) = \frac{\eta_d + 1}{\sigma_{d-1}} \int_{B^d} y_{t,p}(u) y_{s,p}(u)\ du. \tag{5.6}$$

So the matrix A is

$$A = \left[\int_{B^d} y_{t_i,p}(u) y_{t_j,p}(u)\ du : 1 \leqslant i, j \leqslant n \right] \tag{5.7}$$

for some $n \in \mathbb{Z}^+$.

Therefore, for any $x \in \mathbb{R}^n$

$$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \int_{B^d} y_{t_i,p}(u) y_{t_j,p}(u)\ du\ x_j \tag{5.8}$$

$$= \int_{B^d} \left(\sum_{i=1}^{n} x_i y_{t_i,p}(u) \right)^2 du \tag{5.9}$$

which is non-negative for any choice of x.

∎

Note that we can calculate the variance of the process $X_{t,p}$ by calculating the integral

$$\begin{aligned} \nu_p(A_t) &= \int_{A_t} |u|^p \, du & (5.10) \\ &= \int\limits_{\theta\cdot t>0} \int_0^{\theta\cdot t} r^{p+d-1} \, dr \, d\sigma(\theta) & (5.11) \\ &= \frac{1}{p+d} \int\limits_{\theta\cdot t>0} (\theta\cdot t)^{p+d} \, d\sigma(\theta). & (5.12) \end{aligned}$$

Now, apply Lemma 2.3 with $v = \theta$, $u = t/|t|$ and $\Phi(x) = x^{p+d} 1_{\{x>0\}}$ to get

$$\frac{\sigma_{d-1}}{p+d} |t|^{p+d} \int_0^1 x^{p+d}(1-x^2)^{\eta_d} \, dx \quad = \quad \beta_{p,d}|t|^{p+d} \tag{5.13}$$

where

$$\beta_{p,d} = \frac{\sigma_{d-1}}{2(p+d)} \beta\left((p+d+1)/2, \eta_d + 1\right) \tag{5.14}$$

So the variance will be close to zero near the origin and increasing as it moves away from the origin. Note also that since $|t|^{p+d}$ is decreasing in $p+d$ for $0 < |t| < 1$ that the variance at a fixed point decreases as $p+d$ increases.

It is also helpful to consider the process along a ray from the origin. Let $r, s > 0$ and consider

$$K_p(r\theta, s\theta) = \nu_p(A_{(r\wedge s)\theta}) = \beta_{p,d}(r \wedge s)^{p+d}. \tag{5.15}$$

Now, to examine the behavior of the process in increments along this ray, suppose the length of the increment is held constant at $\tau > 0$ and consider

$$Var(X_{(r+\tau)\theta,p} - X_{r\theta,p}) = \beta_{p,d}\left((r+\tau)^{p+d} - r^{p+d}\right). \tag{5.16}$$

This is constant if $p+d=1$, increasing in r if $p+d>1$ and decreasing in r if $0<p+d<1$.

As in the case of Lévy's Brownian motion, the integral representation for the covariance function allows a white noise integral representation of the p-processes.

Lemma 5.2 *The p-processes have the white noise integral representation*

$$X_{t,p} = \sqrt{\frac{\eta_d + 1}{\sigma_{d-1}}} \int_{B^d} |u|^{p/2} \mathbb{1}_{A_t}(u)\, dW_u. \tag{5.17}$$

5.2 Expansions of P-Processes in Terms of Orthogonal Polynomials

As with Lévy's Brownian Motion, the p-processes and their covariance function have expansions in terms of spherical harmonics and independent standard Gaussian random variables.

Theorem 5.1 *Let $\{X_{t,p} : t \in B^d\}$ be a p-process and $K_p(s,t) = E(X_{s,p}X_{t,p})$ be its covariance function. Then*

$$K_p(s,t) = c_{p,d} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \lambda_{m,k,p}(|t|)\lambda_{m,k,p}(|s|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|)\varphi_{m,j}(s/|s|) \tag{5.18}$$

where

$$\lambda_{m,k,p}(|t|) = \sqrt{\frac{\eta_d + 1}{\sigma_{d-1}}} \int_0^1 P_m(x)(1-x^2)^{\eta_d} \int_0^{x|t|} J_k(r) r^{d+p/2-1}\, dr\, dx \tag{5.19}$$

P_m is the Legendre polynomial of degree m and $\{Z_{m,k,j} : m \geqslant 0, k \geqslant 0, 1 \leqslant j \leqslant h_d(m)\}$ is an array of i.i.d standard Gaussian random variables.

We begin by finding an expansion for a function $y_{t,p}$ with the property that

$$K_p(s,t) = c_d < y_{t,p}, y_{s,p} >_{L^2(B^d)} . \tag{5.20}$$

Lemma 5.3 *For each $u, t \in B^d$ define the function*

$$y_{t,p}(u) = |u|^{p/2} \mathbb{1}_{A_t} \tag{5.21}$$

and let

$$p_{m,k,j}(u) = \gamma_k J_k^d(|u|)\varphi_{m,j}(u/|u|) \tag{5.22}$$

be an element of the orthonormal basis as defined in equation 2.35. Then

$$\int_{B^d} y_{t,p}(u) p_{m,k,j}(u)\, du = \alpha_{m,k,p}(|t|)\varphi_{m,j}(t/|t|) \tag{5.23}$$

where

$$\alpha_{m,k,p}(|t|) = \gamma_k \int_0^1 P_m(x)(1-x^2)^{\eta_d} \int_0^{x|t|} J_k(r) r^{d+p/2-1}\, dr\, dx. \tag{5.24}$$

Proof: We begin by converting the integral to polar coordinates.

$$\begin{aligned}
\int\limits_{B^d} y_{t,p}(u) p_{m,k,j}(u)\, du &= \gamma_k \int\limits_{\theta\cdot t>0} \int\limits_0^{\theta\cdot t} J_k^d(r)\varphi_{m,j}(\theta) r^{d+p/2-1}\, dr\, d\sigma(\theta) \\
&= \int\limits_{\theta\cdot t/|t|>0} \Phi_{t,k,p}(\theta\cdot t/|t|)\varphi_{m,j}(\theta)\, d\sigma(\theta)
\end{aligned}$$

where

$$\Phi_{t,k,p}(x) = \gamma_k \int\limits_0^{x|t|} J_k^d(r) r^{d+p/2-1}\, dr. \tag{5.25}$$

We can now apply the Funk-Hecke Theorem to get

$$\int\limits_{B^d} y_{t,p}(u) p_{m,k,j}(u)\, du = \alpha_{m,k,p}(|t|)\varphi_{m,j}(t/|t|) \tag{5.26}$$

where

$$\alpha_{m,k,p}(|t|) = \sigma_{d-1} \int\limits_0^1 P_m(x)(1-x^2)^{\eta_d}\Phi_{t,k,p}(x)\, dx. \tag{5.27}$$

■

This yields the expansion

$$y_{t,p}(u) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \alpha_{m,k,p}(|t|) \sum_{j=1}^{h_d(m)} p_{m,k,j}(u)\varphi_{m,j}(t/|t|). \tag{5.28}$$

Now, note that

$$\begin{aligned} K_p(s,t) &= \frac{\eta_d + 1}{\sigma_{d-1}} < y_{t,p}, y_{s,p} > & (5.29)\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{h_d(m)} < y_{t,p}, p_{m,k,j} >< p_{m,k,j}, y_{s,p} > & (5.30) \end{aligned}$$

and Theorem 5.2 follows with

$$\lambda_{m,k,p}(|t|) = \sqrt{\frac{\eta_d + 1}{\sigma_{d-1}}}\alpha_{m,k,p}(|t|). \tag{5.31}$$

This also yields an expansion for the p-process.

Theorem 5.2 *Let $p + d > 0$ and define*

$$\tilde{X}_{t,p} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{m,k,p}(|t|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|) Z_{m,k,j}. \tag{5.32}$$

Then $\tilde{X}_{t,p}$ converges to a p-process with probability 1.

Proof: To prove that $\{\tilde{X}_{t,p} : t \in T\}$ is equal to the p-process we need to show that it is a mean 0 Gaussian random field with $E(\tilde{X}_{s,p}\tilde{X}_{t,p}) = K_p(s,t)$. The fact that $\tilde{X}_{t,p}$ is mean 0 and Gaussian follows immediately from the fact that it is a sum of mean 0 Gaussian random variables. Also,

$$\tilde{X}_{t,p}\tilde{X}_{s,p} = \sum_{m,k,j}\sum_{n,l,i} \lambda_{m,k,p}(|t|)\lambda_{n,l,p}(|s|)\varphi_{m,j,}(t/|t|)\varphi_{n,i}(s/|s|)Z_{m,k,j}Z_{n,l,i}. \tag{5.33}$$

Now

$$\sum_{m,k,j} |\lambda_{m,k,p}(|t|)\lambda_{m,k,p}(|s|)\varphi_{m,j}(t/|t|)\varphi_{m,j}(s/|s|)| =$$

$$c_d \sum_{m,k,j} |< y_{t,p}, p_{m,k,j} >< y_{s,p}, p_{m,k,j} >| \tag{5.34}$$

where $< \cdot, \cdot >$ is the standard L^2 inner product and $y_{t,p}, p_{m,k,j}$ are defined as in (5.21) and (4.21).

By Holder's inequality this is bounded above by

$$c_d \sqrt{\sum_{m,k,j} | < y_{t,p}, p_{m,k,j} > |^2 \sum_{m,k,j} | < y_{s,p}, p_{m,k,j} > |^2} = c_d \|y_{t,p}\|_{L^2(B^d)} \|y_{s,p}\|_{L^2(B^d)}. \tag{5.35}$$

This is finite so the sums and expectations can be exchanged. Therefore

$$E(\tilde{X}_{s,p}\tilde{X}_{t,p}) = \sum_{m,k,j} \lambda_{m,k,p}(|t|)\lambda_{m,k,p}(|s|)\varphi_{m,j}(t/|t|\varphi_{m,j}(s/|s|)E(Z^2_{m,k,j}) = K_p(s,t). \tag{5.36}$$

Therefore $\tilde{X}_{t,p}$ converges to the p-process. It remains to show that it converges with probability 1.

Define

$$X_{m,k,j} = \lambda_{m,k,p}(|t|)\varphi_{m,j}(t/|t|)Z_{m,k,j}. \tag{5.37}$$

Then $\{X_{m,k,j}\}$ is a sequence of independent random variables with variance

$$Var(X_{m,k,j}) = \lambda^2_{m,k,p}(|t|)\varphi^2_{m,j}(t/|t|). \tag{5.38}$$

So

$$\sum_{m,k,j} Var(X_{m,k,j}) = K_p(t,t) < \infty. \tag{5.39}$$

Therefore $X_{t,p}$ converges to a p-process with probability 1. ■

The expression for $K_p(s,t)$ given in (4.17) can be simplified significantly by exchanging sums and integration.

Let

$$f_{n,p}(s,t) = \sum_{k=0}^{n} \gamma_k^2 \int_0^1 \int_0^1 Q_m(x,y) \int_0^{x|t|} J_k^d(r) r^{d+p/2-1}\, dr \int_0^{y|s|} J_k^d(u) u^{d+p/2-1}\, du\, dx\, dy \tag{5.40}$$

where Q_m is defined as in (4.47).

If we can show that there is some function, g_p, integrable on $B^d \times B^d$ such that for every $(s,t) \in B^d \times B^d$ and every n

$$|f_{n,p}(s,t)| \leqslant g_p(s,t) \tag{5.41}$$

then Lebesgue's Dominated Convergence Theorem can be applied.

First, using the fact that $|Q_m(x,y)| \leqslant 1$, we obtain the bound

$$|f_{n,p}(s,t)| \leqslant \int_0^1 \int_0^1 \sum_{k=0}^{n} \left| \gamma_k^2 \int_0^{x|t|} J_k^d(r) r^{d+p/2-1}\, dr \int_0^{y|s|} J_k^d(u) u^{d+p/2-1}\, du \right| \, dx\, dy. \tag{5.42}$$

By Holder's inequality, this is bounded by

$$\int_0^1 \int_0^1 \sqrt{\sum_{k=0}^{n} \left| \int_0^{x|t|} \gamma_k J_k^d(r) r^{d+p/2-1}\, dr \right|^2 \sum_{k=0}^{n} \left| \int_0^{y|s|} \gamma_k J_k^d(u) u^{d+p/2-1}\, du \right|^2} \, dx\, dy. \tag{5.43}$$

However, since all the terms in the summation are positive, this is bounded by

$$\int_0^1 \int_0^1 \sqrt{\sum_{k=0}^{\infty} \left| \int_0^{x|t|} \gamma_k J_k^d(r) r^{d+p/2-1}\, dr \right|^2 \sum_{k=0}^{\infty} \left| \int_0^{y|s|} \gamma_k J_k^d(u) u^{d+p/2-1}\, du \right|^2} \, dx\, dy. \tag{5.44}$$

Now rewrite this as

$$\int_0^1 \int_0^1 \sqrt{\sum_{k=0}^{\infty} \left| \left\langle \gamma_k J_k^d(r) r^{(d-1)/2}, r^{(d+p-1)/2} 1_{(0,x|t|)}(r) \right\rangle \right|^2} \times \sqrt{\sum_{k=0}^{\infty} \left| \left\langle \gamma_k J_k^d(u) r^{(d-1)/2}, u^{(d+p-1)/2} 1_{(0,y|s|)}(u) \right\rangle \right|^2} \, dx\, dy \tag{5.45}$$

where $< \cdot, \cdot >$ is the $L^2((0,1))$ inner product. Since $\{\gamma_k J_k^d(r) r^{(d-1)/2} : k \geqslant 0\}$ forms an orthonormal basis for $L^2((0,1))$ this is equal to

$$\int_0^1 \int_0^1 \left\| r^{(d+p-1)/2} 1_{(0,x|t|)}(r) \right\|_{L^2((0,1))} \left\| u^{(d+p-1)/2} 1_{(0,y|s|)}(u) \right\|_{L^2((0,1))} \, dx\, dy \leqslant 1 \tag{5.46}$$

Since $g(s,t) = 1$ is integrable on $B^d \times B^d$ with respect to the measure $d\mu(u) = |u|^{p+d-1}\,du$ for $p > -d$, the Dominated Convergence Theorem implies

$$\sum_{k=0}^{\infty} \gamma_k^2 \int_0^1 \int_0^1 Q_m(x,y) \int_0^{x|t|} J_k^d(r) r^{d+p/2-1}\,dr \int_0^{y|s|} J_k^d(u) u^{d+p/2-1}\,du\,dx\,dy =$$

$$\int_0^1 \int_0^1 Q_m(x,y) \sum_{k=0}^{\infty} \int_0^{x|t|} \gamma_k J_k^d(r) r^{d+p/2-1}\,dr \int_0^{y|s|} \gamma_k J_k^d(u) u^{d+p/2-1}\,du\,dx\,dy. \quad (5.47)$$

Again, we can rewrite the inner integrals as inner products and this is equal to

$$\int_0^1 \int_0^1 Q_m(x,y) \left\langle r^{(d+p-1)/2} 1_{(0,x|t|)}(r), r^{(d+p-1)/2} 1_{(0,y|s|)}(r) \right\rangle \,dx\,dy =$$

$$\frac{1}{p+d} \int_0^1 \int_0^1 Q_m(x,y)(x|t| \wedge y|s|)^{p+d}\,dx\,dy. \quad (5.48)$$

We have just proven the following:

Lemma 5.4 *The covariance function for the p-process can be written as*

$$K_p(s,t) = c_{d,p} \sum_{m=0}^{\infty} h_d(m) P_m(s \cdot t/|s||t|) \int_0^1 \int_0^1 Q_m(x,y)(x|t| \wedge y|s|)^{p+d}\,dx\,dy. \quad (5.49)$$

This expression will be very useful in simulating the p-processes.

5.3 Properties of P-Processes

5.3.1 Stationarity

Definition 5.1 (Strictly Stationary Random Field) *[17] A random field $\{X_t : t \in T\}$ is called stationary if all finite dimensional distributions are invariant under translations.*

Since the variance of the p-processes is a power of $|t|$ for every p, all p-processes, including Levy's Brownian Motion are non-stationary. Levy's Brownian Motion does, however, have a local stationarity property.

Definition 5.2 (Locally Stationary) *[17] Let $\mathcal{G}(T)$ be the group of all rotations and translations in $T \subset \mathbb{R}^n$. The random field $\{X_t : t \in T\}$ is locally stationary if*

$$\{X_{g(t)} - X_{g(0)} : t \in T\} =^d \{X_t - X_0 : t \in T\} \qquad \forall g \in \mathcal{G}(T). \tag{5.50}$$

Lemma 5.5 *Levy's Brownian Motion is the only p-process that is locally stationary.*

Proof It was shown in (5.15) that for $p + d \neq 1$, $\{X_{t,p} : t \in B^d\}$ is not locally stationary. If $p + d = 1$ then the p-process is Lévy's Brownian motion and

$$Var(X_{g(t)} - X_{g(0)}) = |g(t) - g(0)| = |t| \quad \forall g \in \mathcal{G}. \tag{5.51}$$

■

5.3.2 Isotropy

Definition 5.3 (Isotropic Random Field) *Let $G(T)$ be the group of all rotations of $T \subset \mathbb{R}^d$. A random field with covariance function $C(s,t)$ is called isotropic if*

$$C(s,t) = C(g(s), g(t)) \qquad \forall g \in G(T), \forall s,t \in T. \tag{5.52}$$

Lemma 5.6 *All p-processes are isotropic in all dimensions.*

Proof: The rotation of a vector $t \in \mathbb{R}^d$ can be expressed by Qt where Q is a real unitary matrix with determinant 1. Then for any p, d with $p + d > 0$

$$\begin{aligned} K_p(Qt, Qs) &= \iint_{\{0<r<\theta\cdot(Qt)\wedge\theta\cdot(Qs)\}} r^{p+d-1}\, dr\, d\sigma(\theta) \\ &= \frac{1}{p+d} \int_{\{(Q^T\theta)\cdot t\wedge(Q^T\theta)\cdot s>0\}} ((Q^T\theta)\cdot t \wedge (Q^T\theta)\cdot s)^{p+d}\, d\sigma(\theta) \\ &= \frac{1}{p+d} \int_{\{\alpha\cdot t\wedge\alpha\cdot s>0\}} (\alpha\cdot t \wedge \alpha\cdot s)^{p+d}\, d\sigma(\alpha) \\ &= K_p(s, t). \end{aligned}$$

Theorem 5.3 *Let*

$$X_{t,p}^{M,K} = \sum_{m=0}^{M}\sum_{k=0}^{K} \lambda_{m,k,p}(|t|) \sum_{j=1}^{h_d(m)} \varphi_m(t/|t|) Z_{m,k,j}. \tag{5.53}$$

Then $\{X_{t,p}^{M,K} : t \in B^d\}$ *is isotropic in all dimensions and for all* M, K*.*

The proof relies on the following property of spherical harmonics:

Lemma 5.7 *[8] If* ρ *is a rotation on* B^d *and* $H = \{h_i : 1 \leqslant i \leqslant h_d(n)\}$ *is an orthonormal basis for* $\mathcal{H}_n^d$ *then* $\rho^{-1}(H) := \{h_i \circ \rho \dot{:} 1 \leqslant i \leqslant h_d(n)\}$ *is also an orthonormal basis for* $\mathcal{H}_n^d$*.*

Proof of Lemma To show that $\rho^{-1} \circ h$ is harmonic, we take

$$\frac{\partial^2}{\partial x_i^2}(h(\rho(x)) = \frac{\partial^2 h}{\partial x_i^2}(\rho(x))\rho'(x) + \frac{\partial h}{\partial x_i}(\rho(x))\rho''(x). \tag{5.54}$$

Since h is harmonic and ρ is linear, this gives $\triangle h \circ \rho = 0$. Further, for any inner product, $< \cdot, \cdot >$ and $g, h \in \mathcal{H}_n^d$ we must have

$$< \rho^{-1}(g), \rho^{-1}(h) >=< g, h >$$

so the image $\rho^{-1}(H)$ will also be an orthonormal set. Since ρ is an injective function, $\rho^{-1}(H)$ will also be the same size as H, and therefore $\rho^{-1}(H)$ is an orthonormal basis for $\mathcal{H}_n^d$.

Proof of Theorem:

Let ρ be any rotation of B^d and

$$K_p^{M,K}(s,t) = \sum_{m=0}^{M} \sum_{k=0}^{K} \lambda_{m,k,p}(|t|)\lambda_{m,k,p}(|s|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|)\varphi_{m,j}(s/|s|) \tag{5.55}$$

be the covariance function for the random field $\{X_{t,p}^{M,K} : t \in B^d\}$.

Then

$$\begin{aligned} K_p^{M,K}(\rho(s),\rho(t)) &= \sum_{m=0}^{M} \sum_{k=0}^{K} \lambda_{m,k,p}(|t|)\lambda_{m,k,p}(|s|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(\rho(t/|t|))\varphi_{m,j}(\rho(s/|s|)) \\ &= K_p^{M,K}(s,t). \end{aligned}$$

5.4 Generalization to Balls of Radius a.

The Gaussian random fields from the previous sections can be generalized to be defined on any sized ball in $\mathbb{R}^d$.

5.4.1 An Orthonormal Basis

First, we need to define a new orthonormal basis. Let B_a^d denote the ball of radius a centered at the origin in $\mathbb{R}^d$.

Lemma 5.8 *Let $P_n^{(\alpha,\beta)}$ denote the Jacobi polynomial of degree n with parameters α and β. Then $\left\{P_n^{(0,\beta)}\left(\frac{2}{a}x-1\right) : n \leqslant 0\right\}$ is orthogonal with respect to the weight x^β on $[0,a]$. i.e.*

$$\int_0^a P_n^{(0,\beta)}\left(\frac{2}{a}x-1\right) P_m^{(0,\beta)}\left(\frac{2}{a}x-1\right) x^\beta \, dx = 0 \quad \textit{for } m \neq n. \tag{5.56}$$

Proof: Using the change of variables $y = x/a$ we can rewrite (5.56) as

$$a^{\beta+1} \int_0^1 P_n^{(0,\beta)}(2y-1) P_m^{(0,\beta)}(2y-1) y^\beta \, dy \tag{5.57}$$

which is 0 for $n \neq m$ by Lemma (2.4). ■

This leads to the orthonormal basis for B_a^d,

$$P_a^d = \{\gamma_{a,k} J_k^d(|u|/a) \varphi_{n,j}(u/|u|) : n,k \geqslant 0, 1 \leqslant j \leqslant h_d(n)\}. \tag{5.58}$$

where J_k^d is the Jacobi polynomial of degree k with parameters $\alpha = 0$ and $\beta = d-1$, $\{\varphi_n, j : 1 \leqslant j \leqslant h_d(n)\}$ is an orthonormal basis for the spherical harmonics of order n and $\gamma_{a,k}$ is a constant chosen so that

$$\gamma_{a,k}^2 \int_0^a (J_k^d)^2(x/a) \, dx = 1. \tag{5.59}$$

A simple calculation and change of variables shows that

$$\gamma_{a,k}^2 = a^{-d}\gamma_k^2 = a^{-d}(2k+d) \tag{5.60}$$

where $\gamma_k = \sqrt{2k+d}$ is the normalization constant from P^d, the orthonormal basis for $L^2(B^d)$.

5.4.2 Expansion in Terms of Orthogonal Polynomials

The expansion for the p-processes on the ball of radius a is similar to the expansion on the unit ball and can be found using a similar method.

Theorem 5.4 *Let $\{X_{t,p} : t \in B_a^d\}$ be the p-process on the ball of radius a and K_p be its covariance function. We can write*

$$K_p(s,t) = a^p \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{m,k,p}(|t|) \sum_{m=0}^{\infty} \varphi_{m,j}(t/|t|) \tag{5.61}$$

and

$$X_{t,p} = a^{p/2} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{m,k,p}(|t|) \sum_{j=1}^{h_d(m)} \varphi_{m,j}(t/|t|) \tag{5.62}$$

where $\lambda_{m,k,p}$ and $\varphi_{m,j}$ are the same as in Theorem 5.2.

Proof: We begin, as in the proof of Theorem 5.2 by using the function

$$y_{t,p}(u) = |u|^{p/2} 1_{A_t}(u). \tag{5.63}$$

Then

$$\begin{aligned}\int\limits_{B_a^d} y_{t,p}(u) J_k^d(|u|/a)\varphi_{m,j}(u/|u|)\, du &= \int\limits_{\theta\cdot t>0} \varphi_{m,j}(\theta) \int\limits_0^{\theta\cdot t} J_k^d(r/a) r^{d+p/2-1}\, dr\, d\sigma(\theta) \\ &= a^{d+p/2} \int\limits_{\theta\cdot t>0} \varphi_{m,j}(\theta)\Phi_{t/a,k,p}(\theta\cdot t/|t|)\, d\sigma(\theta)\end{aligned}$$

where $\Phi_{t,p,k}(x)$ is as defined in (5.25). Now, we apply the Funk-Hecke Theorem to obtain

$$\int\limits_{B_a^d} y_{t,p}(u) J_k^d(|u|/a)\varphi_{m,j}(u/|u|)\, du = a^{d+p/2}\alpha_{m,k,p}(|t|/a)\varphi_{m,j}(t/|t|). \tag{5.64}$$

where $\alpha_{m,k,p}$ is as defined in (5.27).

The rest of the proof is identical to that of Theorem 5.2 . ∎

Chapter 6 Simulations

In this chapter the expansions for Standard Brownian motion, Lévy's Brownian motion and the p-processes found in the preceding chapters will be used to simulate the random fields for values of t along a ray from the origin.

6.1 Standard Brownian Motion

Recall the Karhunen-Loéve expansion for standard Brownian motion on $[0, 1]$ is

$$X_t = \sqrt{2} \sum_{n=0}^{\infty} \frac{\sin[(n + \frac{1}{2})\pi t]}{(n + \frac{1}{2})\pi} Z_n \tag{6.1}$$

where $\{Z_n : n \geqslant 0\}$ is a sequence of independent standard Gaussian random variables. Finite truncations of this sum can be used to generate simulations. Figure 6.1 used the first 100 terms in the expansion with 100 time steps. Figure6.1 used 1000 terms with 100 time steps

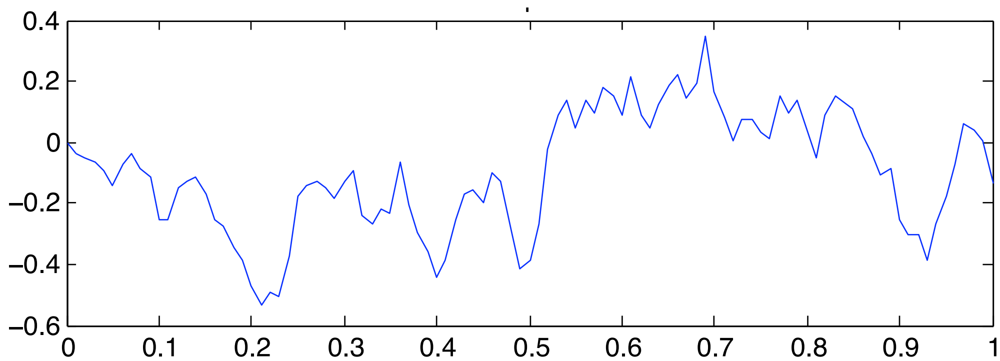

Figure 6.1: Brownian motion with 100 terms

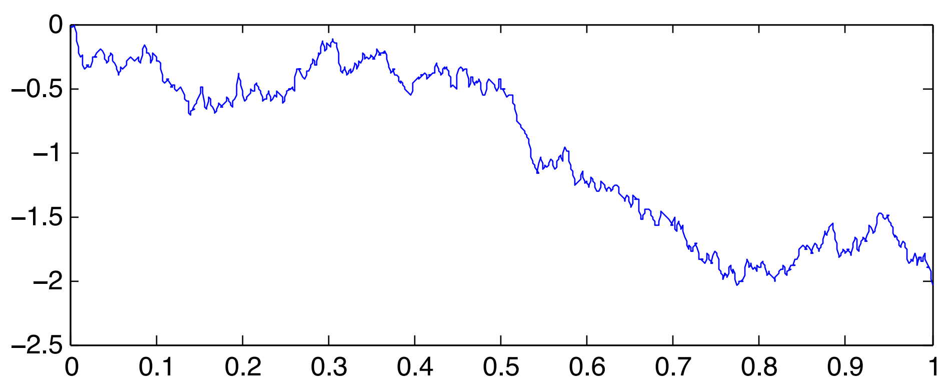

Figure 6.2: Brownian motion with 1000 terms

6.2 Lévy's Brownian Motion and the P-Processes

In the case of Lévy's Brownian motion and the p-processes the expansions for the covariance function can be used to simulate the random fields much more efficiently than the expansions for the random fields themselves.

Lemma 6.1 *[18] Suppose K is a positive-definite symmetric matrix. Then there exists an upper triangular matrix, R, such that $RR^T = K$.*

The product RR^T is called the *Cholesky decomposition* of K and is very useful for simulations of multivariate Gaussian random variables.

Lemma 6.2 *[18] Let K be an $n \times n$ positive-definite symmetric matrix with Cholesky decomposition RR^T and let $\mathcal{Z}$ be an $n \times 1$ vector of standard Gaussian random variables. Then $R\mathcal{Z}$ is a mean 0 Gaussian random vector with covariance matrix K.*

Recall that the expansion for the covariance function K was obtained by taking

$$K(s,t) = c_d \sum_{e \in E} < y_t, e >< y_s, e > \tag{6.2}$$

where $y_t, y_s \in L^2(B^d)$ and E is an orthonormal basis for $L^2(B^d)$. For a fixed integer $N \geqslant 0$ define K_N to be a finite truncation of the sum in (6.2),

$$K_N(s,t) = c_d \sum_{i=1}^{N} < y_t, e_i >< y_s, e_i > \tag{6.3}$$

where $\{e_i : 1 \leqslant i \leqslant N\} \subset E$. Then for any finite matrix n, define the matrix

$$M = [K_N(t_i, t_j) : 1 \leqslant i, j \leqslant n]. \tag{6.4}$$

For any vector $x \in \mathbb{R}^n$

$$\begin{aligned} x^T M x &= c_d \sum_{1 \leqslant i,j \leqslant n} \sum_{k=1}^{N} x_i < y_{t_i}, e_k >< y_{t_j}, e_k > x_j & (6.5) \\ &= c_d \sum_{k=1}^{N} \left(\sum_{i=1}^{n} x_i < y_{t_i}, e_k > \right)^2 & (6.6) \\ &\geqslant 0. & (6.7) \end{aligned}$$

So M is positive definite and therefore any finite truncation of the sum in (6.2) can be used to simulate a random field.

6.3 Lévy's Brownian Motion

Recall that the covariance function for Lévy's Brownian motion can be written as

$$K(s,t) = c_d \sum_{m=0}^{\infty} h_d(m) P_m(s \cdot t/|s||t|) \int_0^1 \int_0^1 Q_m(x,y)(x|t| \wedge y|s|)\, dx\, dy \tag{6.8}$$

where $Q_m(x,y) = P_m(x)P_m(y)(1-x^2)^{\eta_d}(1-y^2)^{\eta_d}$ and P_m is the Legendre polynomial of degree m in d dimensions.

To reduce computation time, we will simplify this expression further. Recall that the Legendre polynomial is equal to

$$P_m(x) = (-1)^m 2^{-m} \prod_{i=1}^{m} (\eta_d + i)^{-1} (1-x^2)^{-\eta_d} \frac{d^m}{dx^m} (1-x^2)^{\eta_d+m}. \tag{6.9}$$

So

$$Q_m(x,y) = p_m^2 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \tag{6.10}$$

where

$$p_m = (-1)^m 2^{-m} \prod_{i=1}^{m} (\eta_d + i)^{-1}. \tag{6.11}$$

So

$$K(s,t) = c_d \sum_{m=0}^{\infty} P_m(s \cdot t/|s||t|) q_m^2 \int_0^1 \int_0^1 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} (x|t| \wedge y|s|)\, dx\, dy \tag{6.12}$$

where $q_m = \sqrt{h_d(m)} p_m$. Consider the double integral

$$\int_0^1 \int_0^1 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} (x|t| \wedge y|s|)\, dx\, dy \tag{6.13}$$

for $m \geqslant 2$. The cases $m = 0$ and $m = 1$ can be computed directly. If we assume that $|t| \geqslant |s| > 0$ then this can be rewritten as

$$|t| \int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \int_0^{y|s|/|t|} x \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m}\, dx\, dy \quad + \\ |s| \int_0^1 y \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \int_{y|s|/|t|}^1 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m}\, dx\, dy. \tag{6.14}$$

Using integration by parts

$$\int_0^{y|s|/|t|} x \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m}\, dx = \\ y|s|/|t| \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \Big|_{x=y|s|/|t|} - \frac{d^{m-2}}{dx^{m-2}}(1-x^2)^{\eta_d+m} \Big|_{x=0}^{y|s|/|t|}. \tag{6.15}$$

Also, since $\frac{d^k}{dx^k}(1-x^2)^{\eta_d+m} \big|_{x=1} = 0$ for all $k < \eta_d + m$,

$$\int_{y|s|/|t|}^1 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m}\, dx = -\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \Big|_{x=y|s|/|t|}. \tag{6.16}$$

Incorporating these into (6.14) gives

$$-|t| \int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \frac{d^{m-2}}{dx^{m-2}}(1-x^2)^{\eta_d+m} \Big|_{x=0}^{y|s|/|t|} \, dy \, . \tag{6.17}$$

However, for odd $k < \eta_d + m$,

$$\frac{d^k}{dx^k}(1-x^2)^{\eta_d+m} \,|_{x=0} = 0 \tag{6.18}$$

so

$$\frac{d^{m-2}}{dx^{m-2}}(1-x^2)^{\eta_d+m} \,|_{x=0} \int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \, dy = 0 \tag{6.19}$$

and therefore

$$K(s,t) = f_0(s,t) + f_1(s,t) -$$

$$c_d|t| \sum_{m=2}^{\infty} h_d(m) q_m^2 P_m(s \cdot t/|s||t|) \int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \frac{d^{m-2}}{dx^{m-2}}(1-x^2)^{\eta_d+m} \,\Big|_{x=y|s|/|t|} \, dy \, . \tag{6.20}$$

Now, if we rewrite

$$\frac{d^{m-2}}{dx^{m-2}}(1-x^2)^{\eta_d+m} \Bigg|_{x=y|s|/|t|} = (|t|/|s|)^{m-2} \frac{d^{m-2}}{dy^{m-2}} \left(1-(|s|/|t|y)^2\right)^{\eta_d+m} \tag{6.21}$$

and perform repeated integration by parts starting with

$$u = \frac{d^{m-2}}{dy^{m-2}} \left(1-(|s|/|t|y)^2\right)^{\eta_d+m} \text{ and } dv = \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \, dy \tag{6.22}$$

we get

$$\int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \frac{d^{m-2}}{dy^{m-2}} \left(1-(|s|/|t|y)^2\right)^{\eta_d+m} \, dy =$$

$$(-1)^m \int_0^1 (1-y^2)^{\eta_d+m} \frac{d^{2m-2}}{dy^{2m-2}} \left(1-(|s|/|t|y)^2\right)^{\eta_d+m} \, dy. \tag{6.23}$$

So

$$K(s,t) = f_0(s,t) + f_1(s,t) -$$
$$c_d|t| \sum_{m=2}^{\infty} (-|s|/|t|)^m q_m^2 P_m(s \cdot t/|s||t|) \int_0^1 (1-y^2)^{\eta_d+m} \frac{d^{2m-2}}{dx^{2m-2}} \left(1-x^2\right)^{\eta_d+m} |_{x=y|s|/|t|}\, dy. \tag{6.24}$$

Now we will simulate Lévy's Brownian on an interval by approximating the covariance function at discrete points along a vector. Since Lévy's Brownian motion is isotropic it suffices to consider

$$\{X_t : t = ke_n\} \tag{6.25}$$

where e_n is a standard basis vector in $\mathbb{R}^d$ and k is any real number with $|k| \leqslant 1$. Additionally, if $t_i = k_1 e_n$ and $t_j = k_2 e_n$ then for any m,

$$P_m(t_i \cdot t_j/|t_i||t_j|) = P_m(1) = 1. \tag{6.26}$$

So in this case we can think of the covariance function K as a function of two real numbers. For $0 < i \leqslant j \leqslant 1$ define

$$K'_N(i,j) = f_1(i,j) + f_2(i,j) -$$
$$c_d j \sum_{m=2}^{N} (-i/j)^m q_m^2 \int_0^1 (1-y^2)^{\eta_d+m} \frac{d^{2m-2}}{dx^{2m-2}} (1-x^2)^{\eta_d+m} |_{x=yi/j}\,. \tag{6.27}$$

This is a finite truncation of the expansion of covariance function K evaluated at s and t where $|s| = i$ and $|t| = j$. So in order to simulate Lévy's Brownian motion on an interval we evaluate

$$M = [K'_N(i,j) : 0 < i \leqslant j \leqslant 1] \tag{6.28}$$

then find the Cholesky factorization of M and multiply it by a matrix of standard Gaussian random variables. The following simulations were created using this process.

Figure 6.3: Lévy's Brownian motion with 500 terms, d=3

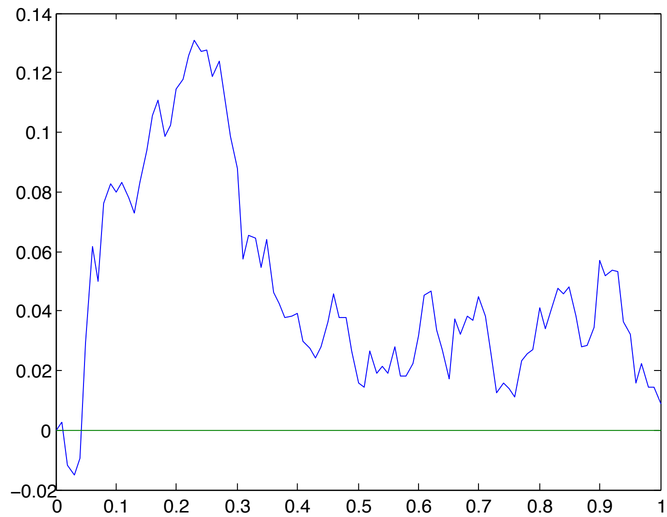

Figure 6.4: Lévy's Brownian motion with 1000 terms, d=3

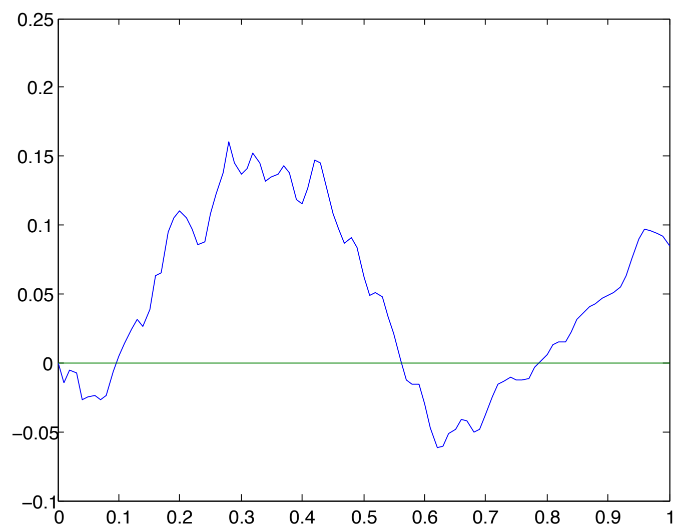

Figure 6.5: Lévy's Brownian motion with 100 terms, d=5

6.4 p-processes

In this section we will discuss simulation of the p-processes along a ray from the origin. Recall that the covariance function for the a p-process can be written as

$$K_p(s,t) = \\ c_{d,p} \sum_{m=0}^{\infty} h_d(m) P_m(s \cdot t/|s||t|) \int_0^1 \int_0^1 Q_m(x,y)(x|t| \wedge y|s|)^{p+d} \, dx \, dy. \quad (6.29)$$

As with Lévy's Brownian motion, further simplification of the double integral will reduce computation time. If we assume that $|t| \geqslant |s| > 0$ and $m \geqslant 2$ then we can rewrite the double integral

$$\int_0^1 \int_0^1 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m}(x|t| \wedge y|s|)^{p+d} \quad (6.30)$$

as

$$|t|^{p+d} \int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \int_0^{y|s|/|t|} x^{p+d} \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \, dx \, dy \quad + \\ |s|^{p+d} \int_0^1 y^{p+d} \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \int_{y|s|/|t|}^1 \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \, dx \, dy. \quad (6.31)$$

Using integration by parts

$$\int_0^{y|s|/|t|} x^{p+d} \frac{d^m}{dx^m}(1-x^2)^{\eta_d+m} \, dx = (y|s|/|t|)^{p+d} \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \Big|_{x=y|s|/|t|} \\ - (p+d) \int_0^{y|s|/|t|} x^{p+d-1} \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \Big|_{x=0}^{y|s|/|t|} dy \, . \quad (6.32)$$

Incorporating this into (6.31) gives

$$-(p+d)|t|^{p+d}\int_0^1 \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m}\int_0^{y|s|/|t|} x^{p+d-1}\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m}\,dx\,dy. \quad (6.33)$$

Now this can be simplified even further using integration by parts by taking

$$u=\int_0^{y|s|/|t|} x^{p+d-1}\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m}\,dx \text{ and } dv=\frac{d^m}{dy^m}(1-y^2)^{\eta_d+m}\,dy \quad (6.34)$$

Since $u(0)=0$ and $v(1)=0$, (6.31) is equal to

$$(p+d)|s|^{p+d}\int_0^1 y^{p+d-1}\frac{d^{m-1}}{dy^{m-1}}(1-y^2)^{\eta_d+m}\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m}\Big|_{x=y|s|/|t|}\,dy\;. \quad (6.35)$$

Further, recall that the constant, $c_{p,d}$ in the expansion of K_p given in is equal to $c_d/(p+d)$, where c_d is the constant in the expansion of K. So the form of the covariance function that will be used to simulate the p-processes is

$$K_p(s,t)=f_{p,0}(s,t)+f_{p,1}(s,t)+$$
$$c_d|s|^{p+d}\sum_{m=2}^{\infty} q_m^2 P_m(s\cdot t/|s||t|)\int_0^1 y^{p+d-1}\frac{d^m}{dy^m}(1-y^2)^{\eta_d+m}\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m}\Big|_{x=y|s|/|t|}\,dy\;. \quad (6.36)$$

Now we will do a more explicit calculation for the case $p+d=2$. The method described below can be generalized to work for any integer value of p. Consider the integral

$$\int_0^1 y\frac{d^m}{dy^m}(1-y^2)^{\eta_d+m}\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m}\Big|_{x=y|s|/|t|}\,dy\;. \quad (6.37)$$

Use integration by parts with $u=y$ and

$$dv=\frac{d^m}{dy^m}(1-y^2)^{\eta_d+m}\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m}\Big|_{x=y|s|/|t|}\,dy \quad (6.38)$$

to get

$$\int_0^1 y \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \big|_{x=y|s|/|t|} \, dy = \int_0^1 v(y)\, dy \tag{6.39}$$

where

$$v(y) = \int_y^1 \frac{d^m}{du^m}(1-u^2)^{\eta_d+m} \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \big|_{x=u|s|/|t|} \, du \; . \tag{6.40}$$

Using repeated integration by parts on v, starting with

$$u = \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \Big|_{x=u|s|/|t|} \quad \text{and} \quad dw = \frac{d^m}{du^m}(1-u^2)^{\eta_d+m}\, du \tag{6.41}$$

v simplifies to

$$\sum_{k=1}^{m} (-1)^k \frac{d^{m+k-2}}{dx^{m+k-2}}(1-x^2)^{\eta_d+m} \big|_{x=y|s|/|t|} \, \frac{d^{m-k}}{du^{m-k}}(1-y^2)^{\eta_d+m} + \\ (-1)^m \int_y^1 (1-y^2)^{\eta_d+m} \frac{d^{2m-1}}{du^{2m-1}}(1-u^2)^{\eta_d+m}\, du. \tag{6.42}$$

However, note that for each $k \geqslant 1$,

$$\int_0^1 \frac{d^{m+k-2}}{dx^{m+k-2}}(1-x^2)^{\eta_d+m} \big|_{x=y|s|/|t|} \, \frac{d^{m-k}}{dy^{m-k}}(1-y^2)^{\eta_d+m}\, dy = \\ -\int_0^1 \frac{d^{m+k-1}}{dx^{m+k-1}}(1-x^2)^{\eta_d+m} \big|_{x=u|s|/|t|} \, \frac{d^{m-k-1}}{dy^{m-k-1}}(1-y^2)^{\eta_d+m}\, dy. \tag{6.43}$$

Therefore

$$\int_0^1 \sum_{k=1}^{m} (-1)^k \frac{d^{m+k-2}}{dx^{m+k-2}}(1-x^2)^{\eta_d+m} \big|_{x=y|s|/|t|} \, \frac{d^{m-k}}{du^{m-k}}(1-y^2)^{\eta_d+m}\, dy = \\ (-1)^m m \int_0^1 (1-y^2)^{\eta_d+m} \frac{d^{2m-2}}{dy^{2m-2}}(1-y^2)^{\eta_d+m}\, dy. \tag{6.44}$$

So

$$\int_0^1 v(y)\ dy = (-1)^m m \int_0^1 (1-y^2)^{\eta_d+m} \frac{d^{2m-2}}{dy^{2m-2}}(1-y^2)^{\eta_d+m}\ dy +$$
$$(-1)^m \int_0^1 \int_y^1 (1-u^2)^{\eta_d+m} \frac{d^{2m-1}}{du^{2m-1}}(1-u^2)^{\eta_d+m}\ du\ dy. \quad (6.45)$$

Replacing the integral

$$\int_0^1 y \frac{d^m}{dy^m}(1-y^2)^{\eta_d+m} \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\eta_d+m} \Big|_{x=y|s|/|t|}\ dy \quad (6.46)$$

given in (6.36) gives an expansion for the covariance function $K_{2-d}(s,t)$. The following simulations were obtained using this representation for the covariance function in the manner described in the previous section.

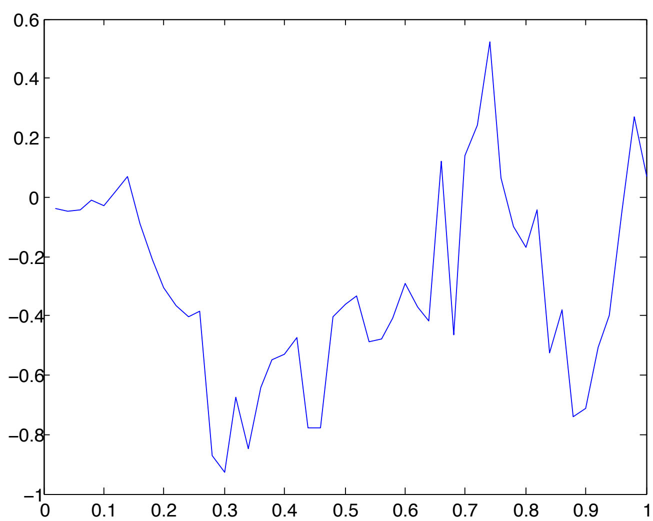

Figure 6.6: d=3, p=-1, 100 terms, 50 time steps

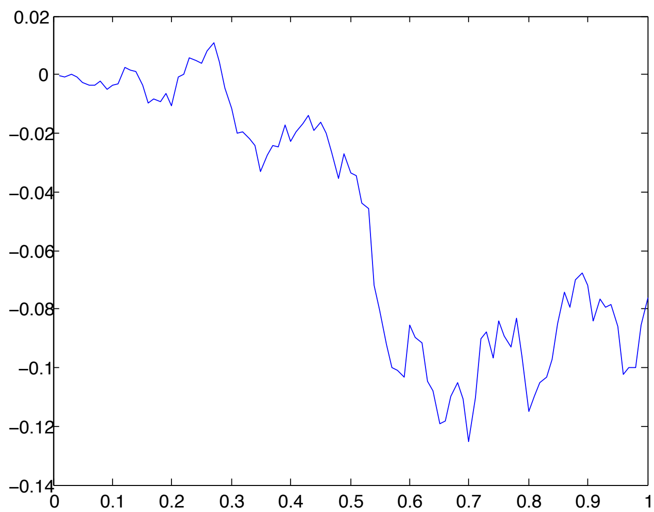

Figure 6.7: d=3, p=-1, 30 terms, 100 time steps

Chapter 7 Conclusion

This thesis introduces a new family of isotropic Gaussian random fields that are not locally stationary and presents two types of representations for isotropic Gaussian random fields. The first is a stochastic integral representation. The second is a new expansions as an infinite linear combination of independent standard Gaussian random variables that converges with probability 1. Finite truncations of this representation are used to simulate the random fields along a ray through the origin in $\mathbb{R}^d$.

Bibliography

[1] Axler, S., Bourdon, P. & Ramey, W. (2001). *Harmonic Function Theory*. New York:Springer-Verlag.

[2] Balan, R.M. &Ivanoff B.G. (2002)*A Markov Property for Set-Indexed Processes*. Journal of Theoretical Probability, Vol. 15, No. 3.

[3] Billingsley, P. (1995). *Probability and Measure*, 3rd ed. Wiley Series in Probability and Mathematical Statistics. New York: John Wiley & Sons.

[4] Bosq,D. (2000).*Linear Processes in Function Spaces*. New York: Springer-Verlag.

[5] Brezis, H.(2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York: Springer Business+Science Media.

[6] Bricio Hernandez, D. (1995). *Probability and Second Order Random Fields*. Series on advances in mathematics for applied sciences, Vol 30. Singapore: World Scientific.

[7] Dunkl, C.F & Xu, Y. (2001).*Orthogonal Polynomials of Several Variables*. Encyclopedia of Mathematics and it's Applications, Vol. 81. Cambridge: Cambridge University Press.

[8] Groemer, H. (1996). *Geometric Applications of Fourier Series and Spherical Harmonics*. Encyclopedia of Mathematics and it's Applications, Vol. 61. NewYork: Cambridge University Press.

[9] Khoshnevisan, D. (2002). *Multiparameter Processes*. New York: Springer-Verlag.

[10] Kuo, H. (2006). *Introduction to Stochastic Integration*. New York: Springer Science+Business Media.

[11] McDonald, J. & Weiss, N. (1999) *A Course in Real Analysis*. San Diego:Academic Press.

[12] McKean, H.P. (1963). *Brownian Motion with a Several Dimensional Time*. Journal of Applied Probability, Vol. 8, pp.335-354.

[13] Mukhopadhyay, N. (2000).*Probability and Statistical Inference.* New York: Marcel Dekker.

[14] Noda, A. (1987) *Generalized Radon Transform and Lévy's Brownian Motion, I.* Nagoya Math Journal, Vol. 105, pp71-87.

[15] Ossiander, M & Pyke, R. (1985). *Lévy's Brownian Motion as a Set-Indexed Process and a Related Central Limit Theorem.* Stochastic Processes and their Applications, vol. 21, pp133-145.

[16] Rozanov, Y.(1982).*Markov Random Fields.* NewYork: Springer-Verlag.

[17] Samorodnitsky, G. & Taqqu, M. (1994). *Stable Non-Gaussian Random Processes.* New York: Chapman & Hall.

[18] Williams, D. (2001). *Weighing the Odds: A Course in Probability and Statistics.* Cambridge: Cambridge University Press.

[19] Yadrenko, M.I. (1983). *Spectral Theory of Random Fields.* New York: Springer-Verlag.

APPENDIX

.1 Code for Simulating Standard Brownian Motion

```
step=.01;
m=1000;
U=0:step:1;

%Create a vector of standard Gaussian random variables
W=normrnd(0,1,m+1);

lambda=zeros(1,m+1);
E=zeros(length(U),length(U));
Eprime=zeros(length(U),length(U));
for k=1:m+1

%%Define the eigenvalues
   lambda(k)=1/((k-.5)*pi);

   %kth column of E is the kth eigenfunction
   %% evaluated at each point of U
  E(:,k)=sqrt(2)*sin((k-.5)*pi*U);
   %kth column of Eprime is kth column of E times
  %%the kth eigenvalue and a standard Gaussian  random variables
  Eprime(:,k)=E(:,k)*lambda(k)*W(k);

end

figure;plot(U,Y);
```

.2 Code for simulating Lévy's Brownian Motion Along a Ray

```
d=5;etad=(d-3)/2;
M=100;
syms x  s t y
%%Create the constant sigma_d-1
sigd=2*pi^((d+1)/2)/gamma((d+1)/2);


 c(1)=1;
```

```
 c(2)=d;
for m=3:M
 u=2*m-4;
 %%Create the constants q_m and h_d(m)
       qv=(d-1):2:d-3+2*(m-1);
   c(m)=nchoosek(d+m-2,m-1)-nchoosek(m+d-4,m-3);

%%Create the vector of order 2m-2 derivatives
    P(m)=c(m)*prod(qv)^(-2)*diff((1-x^2)^(etad+m-1),x,u);
    Q(m)=int((1-y^2)^(etad+m-1)*subs(P(m),x,y*s/t),y,0,1);

end
%%The case m=0 and m=1 treated separately
Q(1)=c(1)*(s*int(y*(1-y^2)^etad*int((1-x^2)^etad,x,y*s/t,1),y,0,1)...
   +t*int((1-y^2)^etad*int((1-x^2)^etad*x,x,0,y*s/t),y,0,1));
Q(2)=c(2)*4*(etad+1)^2*(t*int(y*(1-y^2)^etad*.. .
int(x^2*(1-x^2)^etad,x,0,y*s/t),y,0,1)...
   +s*int(y^2*(1-y^2)^etad*int(x*(1-x^2)^etad,x,y*s/t,1),y,0,1));

step=.01 ;
T=step:step:1;
K=zeros(length(T),length(T));
lam1=zeros(1,M);

for i=1:length(T)
    for j=i:length(T)
    %%Create the terms in the expansion of the covariance function
    %%evaluated along (0,1]
        lam1(1)=subs(subs(Q(1),s,T(i)),t,T(j));
        lam1(2)=subs(subs(Q(2),s,T(i)),t,T(j));
        for m=3:M
      lam1(m)=-T(j)*(-T(i)/T(j))^((m-1))*...
           subs(subs(Q(m),s,T(i)),t,T(j));
        end

       %%Create the matrix, K, of the covariance function evaluated
       %%at all points in the square (0,1]X(0,1]
        K(i,j)=sum(lam1);
```

```
    end
   end

Tz=0:step:1;
%%Compute the Cholesky factorization of K
R=chol(sigd^-1*(etad+1)*K);
%%Multiply the Cholesky factorization by a
%%standard Gaussian vector
Z=normrnd(0,1,size(transpose(T)));
S=transpose(R)*Z;

Sc=zeros(size(S)+1);
for k=1:size(S)
   Sc(k+1)=S(k);
end

   figure; plot(Tz,Sc)
```

.3 Code for simulating the P-Processes Along a Ray

```
d=2;etad=(d-3)/2;
p=2-d;
M=100;
sigd=2*pi^((d+1)/2)/gamma((d+1)/2);
for m=1:M
   u=2*m-3;
   w=2*m-4;
   %%Create the constants q_m and h_d(m)
       qv=(d-1)/2:1:etad+(m-1);
    if m<3
      c(m)=nchoosek(d+m-2,m-1);
  else
   c(m)=nchoosek(d+m-2,m-1)-nchoosek(m+d-4,m-3);
   pc(m)=c(m)/prod(qv)^2;
    end

    %%Create the vectors of order 2m-1 and 2m-2 derivatives
```

```
    P1(m)=pc(m)*diff((1-x^2)^(etad+m-1),x,u);
    P2(m)=pc(m)*diff((1-x^2)^(etad+m-1),x,w);
m
end
Pp(1)=1;
step=.01;
T=step:step:1;
lamp=zeros(1,M);
%%Create the terms in the expansion of the covariance function
%%evaluated along (0,1]
Kp=zeros(length(T));
for i=1:length(T)
   for j=i:length(T)
  %%Compute terms for m=0 and m=1 separately
       lamp(1)=T(j)^2*int((1-y^2)^etad*...
           int((1-x^2)^etad*x^2,x,0,y*T(i)/T(j)),y,0,1)...
           +T(i)^2*int(y^2*(1-y^2)^etad*...
           int((1-x^2)^etad,x,y*T(i)/T(j),1),y,0,1);
       lamp(2)=-pc(2)*(etad+1)^2*(T(j)^2*int(y*(1-y^2)^etad*...
           int(x^3*(1-x^2)^etad,x,0,y*T(i)/T(j)),y,0,1)...
           +T(i)^2*int(y^3*(1-y^2)^etad...
           *int(x*(1-x^2)^etad,x,y*T(i)/T(j),1),y,0,1));

       for m=3:M
    lamp(m)=T(i)^2*(m-1)*(-1/4)^(m-1)*...
        int((1-x^2)^(etad+m-1)*P2(m),x,0,1)+...
        T(i)^2*(-1/4)^(m-1)*int(int((1-x^2)^(etad+m-1)*...
        P1(m),x,y,1),y,0,1);
       end
%%Create the matrix K_p of the covariance function
%%evaluated at all points in (0,1]X(0,1]
       Kp(i,j)=sum(-lamp);
       Kp(j,i)=Kp(i,j);
   end
end

%%Find the Cholesky factorization of K_p
Rp=chol(sigd^-1*(etad+1)*Kp);
```

```
%%Multiply K_p by a standard Gaussian random vector
Zp=normrnd(0,1,size(transpose(T)));
S=transpose(Rp)*Zp;
Tz=0:step:1;
Sz=zeros(1,length(S));
for i=1:length(S)
   Sz(i+1)=S(i);
end

figure; plot(Tz,Sz);
```