SOME ANALYTICAL PROPERTIES OF BIVECTORS

by

CHARLES POLING LUEHR

A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1956

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Head of Department of Mathematics

Redacted for Privacy

Chairman of School Graduate Committee

Redacted for Privacy

Dean of Graduate School

Date thesis is presented Nov. 28, 1955

Typed by Edith S. Mimopoulos

ACKNOWLEDGMENT

I gratefully acknowledge the interest and assistance of Professor Hostetter under whose guidance this work was done.

TABLE OF CONTENTS

SOME ANALYTICAL PROPERTIES OF BIVECTORS

INTRODUCTION

In this thesis is presented some elementary theory concerning bivector spaces, which are defined as being finite dimensional vector spaces over the field of complex numbers in which there is an inner product obeying certain postulates.

Derivatives for polyadic functions are defined, a generalization of the Cauchy-Riemann conditions is derived, and necessary and sufficient conditions for the existence of the derivative are proved. Included are two theorems about analytic functions known on the boundary of a closed hypersurface. Finally a result analogous to the Cauchy-Goursat theorem but involving surface integrals is derived.

VECTOR SPACES

Definition 2A. A vector space $\mathcal{V}(\mathcal{F})$ (1, p. 162) over a field $\mathcal{F}$ is a set of elements (called vectors) with the operations of addition and multiplication defined and obeying the following postulates:

(1) $\mathcal{V}$ is an Abelian group under addition.

(2) For every vector A and every element a of $\mathcal{F}$, the product aA determines a unique vector in $\mathcal{V}$.

(3) $aA = Aa$

(4) $a(A+B) = aA + aB$

(5) $(a+b)A = aA + bA$

(6) $(ab)A = a(bA)$

(7) $1A = A$.

Definition 2B. A Euclidean vector space (1, p. 189) is a vector space $\mathcal{V}(\mathcal{R})$ over the field of real numbers $\mathcal{R}$ such that to every two vectors A and B in $\mathcal{V}(\mathcal{R})$ there corresponds a unique real number which we designate by $A \cdot B$ (called the inner product) satisfying the following properties:

(1) $A \cdot B = B \cdot A$

(2) $(aA) \cdot B = a(A \cdot B)$ a real

(3) $A \cdot (B+C) = A \cdot B + A \cdot C$

(4) $A \cdot A > o$ unless $A = 0$ $0 \cdot 0 = o$.

Definition 2C. A bivector space $\mathcal{W}_n$ is the vector space $\mathcal{V}_n(\mathcal{C})$ of dimension n (1, pp. 168-9) over the field of complex numbers $\mathcal{C}$ such that to every two elements A and B of $\mathcal{V}_n(\mathcal{C})$ there corresponds a unique complex number which we designate by $A \cdot B$ (called the inner product of A and B) satisfying the following properties:

(1) $A \cdot B = B \cdot A$

(2) $(aA) \cdot B = a(A \cdot B)$ a complex

(3) $A \cdot (B+C) = A \cdot B + A \cdot C$.

(4) There exists a linear independent set (1, p. 167) of elements $\{A_i\}$, $i = 1,2,\ldots, n$, such that $A_i \cdot A_i \neq 0$ (i not summed) for all $i = 1,2,\ldots, n$.

We shall call the elements of $\mathcal{W}_n$ bivectors.

EUCLIDEAN SUBSETS

Hereafter we shall use the summation convention, repeated indices indicating summation on those indices unless otherwise stated. Thus

$$a_i A_i = \sum_{i=1}^{n} a_i A_i.$$

Definition 3A. Given $\{A_\alpha\}$, $\alpha = 1,2,\ldots, k \leq n$, a linear independent set (1, p. 167) of vectors in $\mathcal{V}_n(\mathcal{F})$, and given a field $\mathcal{S}$ which is either $\mathcal{F}$ or a subfield (1, p. 36) of $\mathcal{F}$, we form the set P of all linear combinations $B = a_\alpha A_\alpha$, a_α in $\mathcal{S}$. We say that P is the set spanned over $\mathcal{S}$ by $\{A_\alpha\}$, $\alpha = 1,2,\ldots, k$.

Definition 3B. If a subset $\mathcal{J}$ of a bivector space $\mathcal{W}_n$ is a Euclidean vector space of dimension k, $1 \leq k \leq n$, under the addition, multiplication, and inner product operations in $\mathcal{W}_n$, we say that $\mathcal{J}$ is a Euclidean subset of $\mathcal{W}_n$.

Definition 3C. If $\mathcal{J}$ has the dimension n, $\mathcal{J}$ is a complete Euclidean subset of $\mathcal{W}_n$.

Theorem 3A. In any $\mathcal{W}_n$ there exists a complete Euclidean subset.

We shall show that there exists an independent set $\{B_i\}$, $i = 1,2,\ldots, n$, in $\mathcal{W}_n$ such that

(3.1) $$B_i \cdot B_k = \delta_{ik} ,$$

which is sufficient to prove the theorem since the subset spanned over $\mathcal{R}$ by $\{B_i\}$ is a Euclidean vector space.

Let $\{A_i\}$ be a set of n independent elements of $\mathcal{W}_n$ satisfying postulate 4, Definition 2C. Then for any set of n elements $\{B_i\}$ in $\mathcal{W}_n$ we can write

$$B_i = b_{ij}A_j \qquad\qquad i = 1,2,\ldots, n$$

since the set $\{A_i\}$ is a basis (1, pp. 168-9) of $\mathcal{W}_n$.

Then

$$B_i \cdot B_k = b_{ij}b_{km}A_j \cdot A_m = \alpha_{jm}b_{ij}b_{km},$$

where

$$\alpha_{jm} = A_j \cdot A_m \qquad\qquad (\neq o \text{ for } j = m).$$

Set

$$\alpha_{jm}b_{ij}b_{km} = \delta_{ik},$$

and we have $\frac{n}{2}(n+1)$ equations in the n^2 unknowns b_{ij}, which has an infinite number of solutions for $n > 1$ (and one solution if $n = 1$). The set $\{B_i\}$ is independent because if not, there exists a set of numbers b_i not all zero such that

$$b_i B_i = 0.$$

Suppose $b_1 \neq o$, then

$$B_1 = -\frac{1}{b_1}\, b_i B_i \qquad\qquad i = 2,\ldots, n,$$

and

$$B_1 \cdot B_1 = -\frac{1}{b_1}\, b_i B_i \cdot B_1 = o \qquad\qquad i = 2,\ldots, n,$$

which contradicts (3.1).

We note that a set of n linear independent vectors $\{C_i\}$ of a complete Euclidean subset $\mathcal{J}$ of $\mathcal{W}_n$ forms a basis of $\mathcal{W}_n$ and any bivector Z of $\mathcal{W}_n$ can be written

$$(3.2) \qquad Z = \alpha_i C_i = (a_i + ib_i)C_i$$
$$= a_i C_i + ib_i C_i = X + iY$$

where α_i are complex numbers, a_i and b_i are real numbers, and X and Y are vectors in $\mathcal{J}$.

It seems that for $\mathcal{W}_n$ there might be only one complete Euclidean subset; however this is not true.

<u>Theorem 3B</u>. For $n \geq 2$ there are more than one complete Euclidean subset in $\mathcal{W}_n$.

Consider an orthonormal basis $\{B_i\}$, $i = 1,2,\ldots, n$, of $\mathcal{J}$, a complete Euclidean subset of $\mathcal{W}_n$.

Define

$$D_1 = aB_1 + ibB_2$$
$$D_2 = ibB_1 - aB_2,$$

where a and b are any two real numbers satisfying

$$a^2 - b^2 = 1 \qquad b \neq 0.$$

Then

$$D_\alpha \cdot D_\beta = \delta_{\alpha\beta} \qquad \alpha, \beta = 1,2$$
$$D_\alpha \cdot B_i = 0 \qquad \alpha, \beta = 1,2;\ i = 3,4,\ldots, n$$

That is, the set $\{D_1, D_2, B_3, \ldots, B_n\}$ satisfies (3.1) and therefore constitutes a basis of a complete Euclidean subset.

Hereafter in our discussion of $\mathscr{W}_n$ we shall assume that a definite complete Euclidean subset $\mathscr{J}$ has been chosen and whenever we write an arbitrary bivector as a sum $Z = X + iY$, X and Y are elements of $\mathscr{J}$.

We shall call a bivector $Z = X + iY$ real or complex depending upon whether Y is or is not null. The conjugate $\bar{Z}$ of Z is $\bar{Z} = X - iY$ and the norm $|Z|$ of Z is defined as $|Z| = \sqrt{Z \cdot \bar{Z}}$.

GIBBS' RESULT

We shall generalize to $\mathcal{W}_n$ a result that Gibbs obtained for $\mathcal{W}_3$ (5, pp. 87-88). Every bivector $Z = U + iV$ can be written in the form

$$(4.1) \qquad Z = e^{ir}(P+iQ),$$

where

$$(4.2) \qquad P \cdot Q = o$$

To prove this we show that in the equation

$$(4.3) \qquad U + iV = (\cos r + i \sin r)(P + iQ)$$

r, P, and Q are determinable.

Taking the inner product of each side by itself we obtain

$$U \cdot U - V \cdot V + 2iU \cdot V = (\cos 2r + i \sin 2r)(P \cdot P - Q \cdot Q).$$

It follows that

$$U \cdot U - V \cdot V = \cos 2r(P \cdot P - Q \cdot Q)$$

$$2U \cdot V = \sin 2r(P \cdot P - Q \cdot Q).$$

Taking the quotient

$$\tan 2r = \frac{2U \cdot V}{U \cdot U - V \cdot V} \qquad U \cdot U \neq V \cdot V.$$

If $U \cdot U = V \cdot V$, make $r = \frac{\pi}{4}$.

Having found r we find P and Q from the equation

$$P + iQ = (\cos r - i \sin r)(U + iV).$$

It is readily shown that r, P, and Q thus determined do in fact satisfy equation (4.3).

POLYADICS

The equation

$$(A\cdot B_\alpha)C_\alpha = A\ (B_\alpha C_\alpha) \qquad \alpha = 1,\ldots, m,$$

where A is an arbitrary vector, defines the entity

$$\Phi = B_\alpha C_\alpha$$

called a dyadic by Gibbs. Likewise

$$(A\cdot B_\alpha)C_\alpha D_\alpha = A\cdot(B_\alpha C_\alpha D_\alpha)$$

defines

$$\Psi^{(3)} = B_\alpha C_\alpha D_\alpha ,$$

called a triadic. Obviously this definition can be extended by induction to define

$$\Psi^{(k)} = B^1_\alpha\ B^2_\alpha\ \ldots\ B^k_\alpha ,$$

called a polyadic or order k.

It is evident from the definition that two polyadics Φ and Ψ of the same order are equal if and only if $A\cdot\Phi = A\cdot\Psi$(or equivalently $\Phi\cdot A = \Psi\cdot A$) for an arbitrary bivector A.

The double dot product of two polydics $\Phi = M^1_i M^2_i \ldots M^k_i$ and $\Psi = N^1_j N^2_j \ldots N^k_j N^{k+1}_j \ldots N^{k+m}_j$ is defined as

$$\Phi : \Psi = \Psi : \Phi = M^1_i\cdot N^1_j\ \ M^2_i\cdot N^2_j\ \ldots\ M^k_i\cdot N^k_j\ \ N^{k+1}_j\ \ldots\ N^{k+m}_j .$$

The dot product of two polydics is defined as

$$(M^1_i M^2_i \ldots M^k_i)\cdot(N^1_j N^2_j\ \ldots\ N^m_j)$$
$$= M^k_i\cdot N^1_j\ M^1_i M^2_i \ldots M^{k-1}_i N^2_j N^3_j \ldots N^m_j\ .$$

The conjugate $\bar{\Phi}$ of a polyadic $\Phi = M_i^1 M_i^2 \ldots M_i^k$ is defined as $\bar{\Phi} = \bar{M}_i^1 \bar{M}_i^2 \ldots \bar{M}_i^k$, the norm $|\Phi|$ as

$$|\Phi| = \sqrt{\Phi : \bar{\Phi}} \ .$$

For more information about polydics we give the references 3, pp. 135-77; 5, pp. 52-84; 6, pp. 260-331; 7, pp. 93-114; 8, pp. 21-24$_5$, pp. 62-74.

DERIVATIVES OF POLYADIC FUNCTIONS

Definition 6A. For a polyadic function $\Phi(Z)$

$$\lim_{Z \to Z_o} \Phi(Z) = \Lambda$$

if and only if for an arbitrary $\epsilon > o$ there exists a $\delta > o$ such that

$$|\Phi(Z) - \Lambda| < \epsilon$$

for all Z satisfying

$$o < |Z - Z_o| < \delta .$$

We say that $\Phi(Z)$ is continuous at Z_o if and only if

$$\lim_{Z \to Z_o} \Phi(Z) = \Phi(Z_o) .$$

Consider a polyadic function $\Omega(Z)$ of a bivector variable Z. We denote by dZ any change in Z from a fixed value Z_o, that is, $dZ = Z - Z_o$. Writing $Z = X + iY$ we have $dZ = dX + idY$. Let

$$\Delta\Omega = \Delta\Omega(Z_o) = \Omega(Z) - \Omega(Z_o) .$$

Definition 6B. $\Omega(Z)$ is differentiable at Z_o if and only if we can write

(6.1) $\Delta\Omega(Z_o) = dZ \cdot D_Z\Omega(Z_o) + dX \cdot \Phi_1(dZ) + dY \cdot \Phi_2(dZ)$, where $D_Z(Z_o)$ is a polyadic independent of dZ, and $\Phi_1(dZ)$ and $\Phi_2(dZ)$ are polyadics satisfying

(6.2) $$\lim_{dZ \to 0} \Phi_m(dZ) = \Phi_m(0) = 0 \qquad m = 1,2.$$

$D_Z\Omega(Z_o)$ is called the derivative with respect to Z at Z_o .

Similarly we define a partial derivative of a function $\Omega(W,Z)$ with respect to one variable, say Z. Let

$$\Delta_Z\Omega(W_o,Z_o) = \Omega(W_o,Z) - \Omega(W_o,Z_o) .$$

<u>Definition 6C</u>. Ω is differentiable with respect to Z at $W=W_o$, $Z=Z_o$ if, and only if,

$$(6.3)\quad \Delta_Z\Omega(W_o,Z_o) = dZ\cdot\Omega_Z(W_o,Z_o) + dX\cdot\Phi_1(dZ) + dY\cdot\Phi_2(dZ),$$

where $\Omega_Z(W_o,Z_o)$ is a polyadic independent of dZ and

$$(6.4)\quad \lim_{dZ\to 0}\Phi_m(dZ) = \Phi_m(0) = 0 \qquad m = 1,2.$$

$\Omega_Z(W_o,Z_o)$ is called the partial derivative of Ω with respect to Z.

<u>Theorem 6A</u>. If $D_ZW(Z)$ exists at Z_o and $D_W\Omega(W)$ exists at $W = U + iV = W(Z_o)$, then

$$(6.5)\quad D_Z\Omega(W(Z_o)) = D_ZW(Z_o)\cdot D_W\Omega(W_o) \qquad W_o = W(Z_o) .$$

Write

$$\Delta W = W(Z) - W(Z_o)$$

$$\Delta\Omega = \Omega(W) - \Omega(W_o)$$

$$(6.6)\quad \Delta W = dZ\cdot D_ZW(Z_o) + dX\cdot\Phi_1(dZ) + dY\cdot\Phi_2(dZ)$$

$$(6.7)\quad \Delta\Omega = \Delta W\cdot D_W\Omega(W_o) + \Delta U\cdot\Psi_1(\Delta W) + \Delta V\cdot\Psi_2(\Delta W),$$ where

$$\lim_{dZ\to 0}\Phi_m(dZ) = \Phi_m(0) = 0 \qquad m = 1,2$$

$$\lim_{\Delta W\to 0}\Psi_m(\Delta W) = \Psi_m(0) = 0 .$$

Substituting (6.6) into (6.7) we obtain

$$\Delta\Omega = dZ\cdot D_Z W(Z_o)\cdot D_W\Omega(W_o) + dX\cdot\Gamma_1(dZ) + dY\cdot\Gamma_2(dZ),$$

where

$$\Gamma_1(dZ) = \Phi_1(dZ)\cdot D_W\Omega(W_o) + [\mathcal{R}e\, D_Z W(Z_o) + \mathcal{R}e\,\Phi_1(dZ)]\cdot\Psi_1(\Delta W) + [\mathcal{I}m\, D_Z W(Z_o) + \mathcal{I}m\,\Phi_1(dZ)]\cdot\Psi_2(\Delta W)$$

$$\Gamma_2(dZ) = \Phi_2(dZ)\cdot D_W\Omega(W_o) + [-\mathcal{I}m\, D_Z W(Z_o) + \mathcal{R}e\,\Phi_2(dZ)]\cdot\Psi_1(\Delta W) + [\mathcal{R}e\, D_Z W(Z_o) + \mathcal{I}m\,\Phi_2(dZ)]\cdot\Psi_2(\Delta W).$$

The symbols Re and Im followed by a polyadic represent respectively the real and the imaginary components of the polyadic. Since every term of Γ_1 and Γ_2 contains an infinitesimal factor we have

$$\lim_{dZ\to 0}\Gamma_m(dZ) = \Gamma_m(0) = 0 \qquad\qquad m = 1,2,$$

and our result is obtained.

It is readily shown that

(6.8) $\quad D_Z(aZ) = aI,$

(6.9) $\quad D_Z\Phi = 0,$

where a is a complex number, I is the idemfactor (5, p. 58) and Φ is a constant polyadic.

GENERALIZATION OF THE CAUCHY-RIEMANN CONDITIONS

For a polyadic function $\Omega(Z)$, we may write

$$\Omega(Z) = \Gamma(X,Y) + i\Lambda(X,Y),$$

where $Z = X + iY$ and all the bivectors composing Γ and Λ are real.

Then by (6.8) and (6.9)

(7.1) $\quad Z_X = I$

(7.2) $\quad Z_Y = iI.$

By (6.5), (7.1) and (7.2)

$$\Omega_X = Z_X \cdot D_Z \Omega = I \cdot D_Z \Omega = D_Z \Omega$$

$$\Omega_Y = Z_Y \cdot D_Z \Omega = iI \cdot D_Z \Omega = iD_Z \Omega,$$

that is

$$D_Z \Omega = \Omega_X = -i\Omega_Y .$$

Upon replacing $\Omega(Z)$ by $\Gamma(X,Y) + i\Lambda(X,Y)$ we obtain

$$D_Z \Omega = \Gamma_X + i\Lambda_X = -i\Gamma_Y + \Lambda_Y .$$

Setting the real parts equal and the imaginary parts equal, we obtain

(7.3) $\quad \Gamma_X = \Lambda_Y . \qquad \Lambda_X = -\Gamma_Y$

which are the bivector equivalents of the Cauchy-Riemann conditions.

<u>Definition 7A</u>. A neighborhood of a point Z_o is a set of points satisfying

$$|Z - Z_o| < c \qquad\qquad c > o .$$

<u>Theorem 7A</u>. Consider a function

$$\Omega(Z) = \Gamma(X,Y) + i\Lambda(X,Y)$$

and let Γ, Λ, Γ_X, Γ_Y, Λ_X, and Λ_Y exist and be continuous in a neighborhood of Z_o. Then for $D_Z\Omega$ to exist at Z in the neighborhood of Z_o it is necessary and sufficient that

$$\Gamma_X = \Lambda_Y, \qquad \Gamma_Y = -\Lambda_X .$$

The necessary condition has already been proved. For Z and Z + dZ in the neighborhood of Z_o write

$$(7.4) \quad \Delta\Gamma = \Gamma(X+dX,\ Y+dY) - \Gamma(X,Y) = \Gamma(X+dX,\ Y+dY) - \Gamma(X,Y+dY) + \Gamma(X,Y+dY) - \Gamma(X,Y) .$$

But

$$(7.5) \quad \Gamma(X,Y+dY) - \Gamma(X,Y) = dY\cdot\Gamma_Y(X,Y) + dY\cdot\Phi_1(dY) ,$$

where

$$\lim_{dZ\to 0} \Phi_1(dY) = \Phi_1(0) = 0 .$$

Also

$$(7.6) \quad \Gamma(X+dX,\ Y+dY) - \Gamma(X,Y+dY) = dX\cdot\Gamma_X(X,Y+dY) + dX\cdot\Phi_2(dX) ,$$

where

$$\lim_{dZ\to 0} \Phi_2(dX) = \Phi_2(0) = 0 .$$

Since Γ_X is continuous we have

$$(7.7) \quad \Gamma_X(X,Y+dY) = \Gamma_X(X,Y) + \Phi_3(dY),$$

where

$$\lim_{dZ\to 0} \Phi_3(dY) = \Phi_3(0) = 0 .$$

Substituting (7.7) into (7.6) and then (7.5) and (7.6) into (7.4) we obtain

$$\Delta\Gamma = dX\cdot\Gamma_X(X,Y) + dY\cdot\Gamma_Y(X,Y) + dX\cdot\Psi_1(dZ) + dY\cdot\Psi_2(dZ)\ ,$$

where

$$\Psi_1(dZ) = \Phi_2(dX) + \Phi_3(dY)\ ,$$
$$\Psi_2(dZ) = \Phi_1(dY)\ .$$

It immediately follows that

$$\lim_{dZ\to 0}\Psi_m(dZ) = \Psi_m(0) = 0 \qquad m = 1,2.$$

Similarly

$$\Delta\Lambda = dX\cdot\Lambda_X(X,Y) + dY\cdot\Lambda_Y(X,Y) + dX\cdot\Psi_3(dZ) + dY\cdot\Psi_4(dZ)\ ,$$

where

$$\lim_{dZ\to 0}\Psi_m(dZ) = \Psi_m(0) = 0 \qquad m = 3,4.$$

Therefore

$$\begin{aligned}\Delta\Omega = \Omega(Z+dZ) - \Omega(Z) &= \Delta\Gamma + i\,\Delta\Lambda\\ &= dX\cdot\Gamma_X + dY\cdot\Gamma_Y + dX\cdot\Psi_1 + dY\cdot\Psi_2\\ &\quad + i(dX\cdot\Lambda_X + dY\cdot\Lambda_Y + dX\cdot\Psi_3 + dY\cdot\Psi_4)\ .\end{aligned}$$

Replace Γ_Y by $-\Lambda_X$ and Λ_Y by Γ_X and obtain

$$\begin{aligned}\Delta\Omega &= (dX+idY)\cdot\Gamma_X + i(dX+idY)\cdot\Lambda_X + dX\cdot\Psi_5 + dY\cdot\Psi_6\\ &= dZ\cdot(\Gamma_X+i\Lambda_Y) + dX\cdot\Psi_5 + dY\cdot\Psi_6\ ,\end{aligned}$$

where

$$\Psi_5 = \Psi_1 + i\Psi_3 , \qquad \Psi_6 = \Psi_2 + i\Psi_4 .$$

Obviously

$$\lim_{dZ \to 0} \Psi_m = \Psi_m(0) = 0 \qquad m = 5,6.$$

Therefore $D_Z \Omega$ exists and

$$D_Z \Omega = \Gamma_X + i\Lambda_X .$$

GEOMETRICAL CONSIDERATIONS

Definition 8A. A line in $\mathcal{W}_n$ is a set of points Z described by the equation

$$Z = A + bB$$

where A and B are constant bivectors and b is a real parameter. It can be shown easily that two points determine a line.

Definition 8B. A manifold of type 1 in $\mathcal{W}_n$ is a set of points Z described by

$$Z = A + bB + cC ,$$

where A, B, and C are constant bivectors, and b and c are real parameters.

It is evident that a line through any two points in a manifold of type 1 lies entirely in the manifold.

Definition 8C. A manifold of type 2 is a set described by

$$Z = A + aB + b(iB)$$

where A and B are constant bivectors and a and b are real parameters.

We may write $w = a + ib$ and put this equation in the form

$$Z = A + wB ,$$

where w is a complex parameter.

<u>Definition 8D</u>. A regular curve in $\mathcal{W}_n$ is a set

$$Z = F(t) \qquad -\infty \leq a \leq t \leq b \leq +\infty ,$$

where F is a continuous function with a continuous first derivative except possibly at a finite number of points in every finite interval of the variable t.

<u>Definition 8E</u>. A simple closed curve is a regular curve where a and b are finite and F gives a one to one correspondence between points on the curve and numbers t in the interval except that

$$F(a) = F(b) .$$

<u>Definition 8F</u>. A closed regular hypersurface is a set of points satisfying the condition that every intersection with a manifold of type 1 is a simple closed curve.

An example of such a surface is the one described by the equation

$$|Z - B| = 2 ,$$

where B is a constant bivector.

ANALYTIC FUNCTIONS

Definition 9A. A polyadic function $\Omega(Z)$ is analytic at a point Z_o if and only if $D_Z\Omega$ exists at every point in some neighborhood of Z_o. Ω is analytic in a region if Ω is analytic at every point in the region.

Theorem 9A. If $f(Z)$ is analytic in the interior and on the boundary of a region bounded by a closed regular hypersurface, and if f is known on the boundary, its value at every point in the interior is determined.

For an arbitrary point $Z = A$ in the interior consider any manifold of type 2 passing through $Z = A$:

$$Z = A + wB \ .$$

The manifold intersects the closed regular hypersurface in a simple closed curve

$$Z = A + w(t)B \qquad t_1 \leq t \leq t_2$$
$$w(t_1)=w(t_2)$$

where $w(t)$ is a simple closed curve about the origin in the complex number plane. We have

$$f(Z) = f(A+wB) = g(w) \ .$$

Then

$$\frac{dg}{dw} = B \cdot D_Z f$$

in some neighborhood of every point in the complex plane inside and on the curve $w(t)$. Therefore $g(w)$ is analytic inside and on the curve, and by the Cauchy integral formula

$g(o)$ is determined. Since

$$f(A) = g(o)$$

$f(A)$ is determined.

Theorem 9B. Under the hypotheses of Theorem 9A $f(Z)$ is continuous along every line in the interior of the region.

An arbitrary line

$$Z = A + bB \qquad\qquad b \text{ real}$$

lies in some manifold of type 2,

$$Z = A + wB \; .$$

Since $g(w)$ is continuous along the line $w=b$ (the line of reals) in the complex plane, $f(Z)$ is continuous along the line $Z = A + bB$.

SURFACE INTEGRALS

Consider the bivector space $\mathcal{W}_n$. In a complete Euclidean subset $\mathcal{J}$ of $\mathcal{W}_n$ there exists an orthonormal basis $E_1, E_2, \dots, E_n$ (1, pp. 192-93). We may regard $\mathcal{J}$ as being a subspace (1, p. 164) of a Euclidean vector space $\mathcal{E}_{2n}$. Then the set $E_1, E_2, \dots, E_n$ can form part of an orthonormal basis of $\mathcal{E}_{2n}$ (1, p. 193). Let $F_1, F_2, \dots, F_n$ be a set of vectors in $\mathcal{E}_{2n}$ which together with the set $E_1, E_2, \dots, E_n$ form an orthonormal basis of $\mathcal{E}_{2n}$. Then

$$(10.1) \qquad E_i \cdot E_j = \delta_{ij} \qquad F_i \cdot F_j = \delta_{ij}$$

$$E_i \cdot F_j = 0 \qquad i, j = 1, 2, \dots, n.$$

Let

$$(10.2) \qquad I_1 = E_i E_i \qquad I_2 = F_i F_i$$

$$\Phi = F_i E_i \qquad I = I_1 + I_2$$

We observe that

$$I_1 \cdot \Phi = 0 \qquad \Phi \cdot I_2 = 0$$

$$I_1 \cdot I_2 = I_2 \cdot I_1 = 0 \; .$$

If R is an element of $\mathcal{E}_{2n}$ and $Z = X + iY$ is an element of $\mathcal{W}_n$, a one to one correspondence between the elements of $\mathcal{W}_n$ and $\mathcal{E}_{2n}$ is established by the relation

$$(10.3) \qquad R = X + \Phi \cdot Y \; .$$

This makes E_k in $\mathcal{W}_n$ correspond to E_k in $\mathcal{E}_{2n}$ and iE_k in $\mathcal{W}_n$

correspond to F_k in $\mathcal{E}_{2n}$, $k = 1,2,\ldots, n$. The inverse transformation is

$$X = R \cdot I_1 , \tag{10.4}$$
$$Y = \Phi_T \cdot R = R \cdot \Phi ,$$

where Φ_T is the transpose of Φ; or we may write

$$Z = R \cdot (I_1 + i\Phi) . \tag{10.5}$$

We shall call $\mathcal{W}_n$ and $\mathcal{E}_{2n}$ associated spaces and the elements Z and $R = X + \Phi \cdot Y$ corresponding elements.

Consider $\Psi(R)$, a continuous real polyadic function of R in $\mathcal{E}_{2n}$; and $\Omega(Z) = \Gamma(X,Y) + i\Lambda(X,Y)$, an analytic polyadic function of $Z = X + iY$ in $\mathcal{W}_n$. We designate by

σ, a closed regular hypersurface in $\mathcal{W}_n$

σ', surface in $\mathcal{E}_{2n}$ corresponding to σ

σ'_i, subdivisions of σ'

σ_i, subdivisions of σ corresponding to σ'_i

R_i, any point in σ'_i

Z_i, any point in $\sigma_i (Z_i = X_i + iY_i)$

Δa_i, the hyperarea of σ'_i

N, the unit normal to σ'

$L = N \cdot (I_1 + i\Phi)$, the element of $\mathcal{W}_n$ corresponding to N. We shall call L the unit normal to σ.

Let Z_1^i and Z_2^i be points in σ_i, and R_1^i and R_2^i be points in σ'_i. We define

$$\Delta = \max_i \left(\sup |Z_1^i - Z_2^i| \ ; \ Z_1^i, Z_2^i \text{ in } \sigma_i\right)$$

$$\Delta' = \max_i \left(\sup |R_1^i - R_2^i| \ ; \ R_1^i, R_2^i \text{ in } \sigma_i'\right) .$$

Then a surface integral of Ω over σ is

$$(10.6) \qquad \int_\sigma L\,\Omega\, da = \lim_{\substack{k\to\infty \\ \Delta\to 0}} \sum_{i=1}^{k} L(Z_i)\,\Omega(Z_i)\,\Delta a_i$$

Also a surface integral of Ψ over σ' is

$$(10.7) \qquad \int_{\sigma'} N\,\Psi\, da = \lim_{\substack{k\to\infty \\ \Delta'\to 0}} \sum_{i=1}^{k} N(R_i)\,\Psi(R_i)\,\Delta a_i \ .$$

SOME DIFFERENTIAL IDENTITIES

Differentiating (10.4) we obtain

$$\text{(11.1)} \qquad DX = I_1$$

$$DY = \Phi,$$

where the derivatives are understood to be taken with respect to R. Also

$$D\Gamma = DX\cdot\Gamma_X + DY\cdot\Gamma_Y = I_1\cdot\Gamma_X + \Phi\cdot\Gamma_Y$$

$$= \Gamma_X + \Phi\cdot\Gamma_Y \,.$$

Therefore

$$D\cdot\Gamma = (\Gamma_X)_s + (\Phi\cdot\Gamma_Y)_s$$

where by the subscript s we indicate contraction on the first two vector files of the polyadic. Thus if

$$\Gamma_X = A^1_\alpha A^2_\alpha A^3_\alpha \;\dots\; A^k_\alpha$$

then

$$(\Gamma_X)_s = A^1_\alpha\cdot A^2_\alpha A^3_\alpha \;\dots\; A^k_\alpha \,.$$

When we write

$$\Gamma_Y = \gamma'_{i_1 i_2 \dots i_k} E_{i_1} E_{i_2} \dots E_{i_k}$$

we find that

$$\Phi\cdot\Gamma_Y = \delta_{i_1 j}\, \gamma'_{i_1 i_2 \dots i_k}\, F_j E_{i_2} E_{i_3} \dots E_{i_k}$$

$$= \gamma'_{i_1 i_2 \dots i_k}\, F_{i_1} E_{i_2} E_{i_3} \dots E_{i_k} \,,$$

and

$$(\bar{\Phi}\cdot\Gamma_Y)_s = \gamma'_{i_1 i_2\ldots i_k}\ F_{i_1}\cdot E_{i_2}E_{i_3}\ldots E_{i_k} = 0\ .$$

Therefore

$$D\cdot\Gamma = (\Gamma_X)_s\ . \tag{11.2}$$

By Hostetter's method (8, pp. 46-48)

$$D(\bar{\Phi}\Gamma) = D[\bar{\Phi}]\Gamma$$

$$D(\bar{\Phi}\cdot\Gamma) = D[\bar{\Phi}\cdot]\Gamma = (I)(\bar{\Phi})\ :\ D\Gamma$$

$$= (I)(\bar{\Phi})\ :\ (\Gamma_X+\bar{\Phi}\cdot\Gamma_Y)\ .$$

If we set

$$\Gamma_X = \gamma_{i_1 i_2\ldots i_k}\ E_{i_1}E_{i_2}\ldots\ E_{i_k}\ ,$$

$$\Gamma_Y = \gamma'_{i_1 i_2\ldots i_k}\ E_{i_1}E_{i_2}\ldots\ E_{i_k}\ ,$$

then

$$(I)(\bar{\Phi})\ :\ \Gamma_X = \gamma_{i_1 i_2\ldots i_k}E_{i_1}F_{i_2}E_{i_3}E_{i_4}\ldots E_{i_k}\ ,$$

and

$$[(I)(\bar{\Phi})\ :\ \Gamma_X]_s = \gamma_{i_1 i_2\ldots i_k}\ E_{i_1}\cdot F_{i_2}E_{i_3}E_{i_4}\ldots E_{i_k}$$

$$= 0\ .$$

Also

$$\bar{\Phi}\cdot\Gamma_Y = \gamma'_{i_1 i_2\ldots i_k}\ F_{i_1}E_{i_2}E_{i_3}\ldots\ E_{i_k}$$

$$(I)(\bar{\Phi})\ :\ (\bar{\Phi}\cdot\Gamma_Y) = \gamma'_{i_1 i_2\ldots i_k}\ F_{i_1}F_{i_2}E_{i_3}E_{i_4}\ldots E_{i_k}$$

$$[(I)(\bar{\Phi}) : (\bar{\Phi}\cdot\Gamma_Y)]_s = \gamma'_{i_1 i_2\ldots i_k}\ F_{i_1}\cdot F_{i_2}E_{i_3}E_{i_4}\ldots E_{i_k}\ .$$

But

$$F_{i_1} \cdot F_{i_2} = E_{i_1} \cdot E_{i_2} ,$$

and

$$[(I)(\Phi) : (\Phi \cdot \Gamma_Y)]_s = \gamma'_{i_1 i_2 \ldots i_k} E_{i_1} \cdot E_{i_2} E_{i_3} E_{i_4} \cdots E_{i_k}$$

$$= (\Gamma_Y)_s .$$

Therefore

(11.3) $$D \cdot (\Phi \cdot \Gamma) = (\Gamma_Y)_s .$$

AN INTEGRAL THEOREM

We now prove a theorem that is an analogue of the Cauchy-Goursat Theorem (4, pp. 81-82).

Theorem 12A. If a polyadic function $\Omega(Z)$ is analytic inside and on a closed regular hypersurface σ in $\mathcal{W}_n$ then

$$(12.1) \qquad \Upsilon_1 = \int_\sigma L \cdot \Omega \, da = 0$$

Proof:

$$\Upsilon_1 = \lim_{\substack{k \to \infty \\ \Delta \to o}} \sum_{i=1}^{k} L(Z_i) \cdot \Omega(Z_i) \Delta a_i$$

$$= \lim_{\substack{k \to \infty \\ \Delta' \to o}} \sum_{i=1}^{k} N(R_i) \cdot (I + i\Phi) \cdot [\Gamma(X_i, Y_i) + i\Lambda(X_i, Y_i)] \Delta a_i$$

$$= \Psi_1 + i\Psi_2$$

where

$$\Psi_1 = \lim_{\substack{k \to \infty \\ \Delta' \to o}} \sum_{i=1}^{k} N(R_i) \cdot [I_1 \cdot \Gamma(X_i, Y_i) - \Phi \cdot \Lambda(X_i, Y_i)] \Delta a_i \ ,$$

$$\Psi_2 = \lim_{\substack{k \to \infty \\ \Delta' \to o}} \sum_{i=1}^{k} N(R_i) \cdot [I_1 \cdot \Lambda(X_i, Y_i) + \Phi \cdot \Gamma(X_i, Y_i)] \Delta a_i \ .$$

Therefore

$$\Psi_1 = \int_{\sigma'} N \cdot (\Gamma - \Phi \cdot \Lambda) da \ ,$$

$$\Psi_2 = \int_{\sigma'} N \cdot (\Lambda + \Phi \cdot \Gamma) da \ .$$

By the divergence theorem for polyadics in $\mathcal{E}_{2n}$, the proof of which is an immediate generalization of the proof given by Hostetter for $\mathcal{E}_3$ (8, pp. 96-97$_1$),

$$\Psi_1 = \int_\tau D\cdot(\Gamma - \Phi\cdot\Lambda)d\tau$$

$$\Psi_2 = \int_\tau D\cdot(\Lambda + \Phi\cdot\Gamma)d\tau,$$

where the integral is taken over the region τ inclosed by σ' and $d\tau$ is an element of hypervolume.

By (11.2) and (11.3)

$$\Psi_1 = \int_\tau [(\Gamma_X)_s - (\Lambda_Y)_s]d\tau$$

$$\Psi_2 = \int_\tau [(\Lambda_X)_s + (\Gamma_Y)_s]d\tau \,.$$

By the generalization of the Cauchy-Riemann conditions (7.3),

$$\Gamma_X - \Lambda_Y = 0 \qquad \Lambda_X + \Gamma_Y = 0 \,.$$

Therefore

$$(\Gamma_X)_s - (\Lambda_Y)_s = 0 \qquad (\Lambda_X)_s + (\Gamma_Y)_s = 0 \,,$$

and

$$\Psi_1 = 0 \,, \quad \Psi_2 = 0 \,.$$

Therefore

$$\Upsilon_1 = 0 \,,$$

which completes the proof.

Since we can write

$$(12.2) \qquad \Upsilon_2 = \int_\sigma L\Omega\, da = \int_\sigma L\cdot I_1\Omega\, da$$

$$= \int_\sigma L\cdot\Theta\, da \,,$$

where

$$\Theta = I_1 \Omega ,$$

and since from (14.1)

$$\int_\sigma L \cdot \Theta \, da = 0 ,$$

we have

$$\Upsilon_2 = 0 , \tag{12.3}$$

which is a more general result than $\Upsilon_1 = 0$.

BIBLIOGRAPHY

1. Birkhoff, Garrett and Saunders MacLane. A survey of modern algebra. Rev. ed. New York, Macmillan, 1953. 472p.

2. Bochner, Salmon and William Ted Martin. Several complex variables. Princeton, Princeton university, 1948. 216p. (Princeton mathematical series, no. 10.)

3. Brand, Louis. Vector and tensor analysis. New York, John Wiley, 1947. 439p.

4. Churchill, Ruell V. Introduction to complex variables and applications. New York, McGraw-Hill, 1948. 216p.

5. Gibbs, Josiah Willard. The collected works of Josiah Willard Gibbs. Vol. 2. Part 2. New Haven, Yale university, 1928. 284p.

6. Gibbs, J. Willard and Edwin Bidwell Wilson. Vector analysis. New Haven, Yale university, 1901. 436p.

7. Hostetter, I. M. Polyadic products. Journal of mathematics and physics 22: 93-114. June, 1943.

8. Hostetter, I. M. Notes on vector analysis. Corvallis, Oregon state college, 1952. 140p. (Mimeographed.)

9. Knopp, Konrad. Theory of functions. Tr. by Frederick Bagemihl. Vol. 1. New York, Dover, 1945. 146p.

10. Lohr, Erwin. Vektor-und Dyadenrechnung für Physiker und Techniker. Berlin, Walter De Gruyter, 1939. 411p.

11. Sommerville, Duncan McLaren Young. An introduction to the geometry of n dimensions. New York, E. P. Dutton, 1929. 196p.