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In 1964, Zarantonello published a constructive method for the solution of certain nonlinear problems in a Hilbert space. We extend the method in various directions including a generalization to a Banach space setting. A revealing geometric interpretation of the method yields guidelines for the analysis.

# An Iterative Procedure for the Solution of Nonlinear Equations in a Banach Space

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TABLE OF CONTENTS

AN ITERATIVE PROCEDURE FOR THE SOLUTION OF NONLINEAR EQUATIONS IN A BANACH SPACE

# CHAPTER I

# INTRODUCTION

## §1. Historical Introduction

In a paper by Birkhoff, Young and Zarantonello [5], a procedure for solving a particular equation of the form

(1) $$Hx = x$$

was proposed. In this paper, $H$ was a nonlinear mapping of a real Hilbert space into itself satisfying certain conditions. This method of solving (1) was called "contractive averaging" by Zarantonello [25]. It is an iterative procedure based on the recursion relation

(2) $$x_{k+1} = (1 - \alpha)x_k + \alpha Hx_k, \qquad k \in \omega_0$$

where $\alpha$ is a scalar and $\omega_0$ is the set of nonnegative integers.

In [5], contractive averaging is used to solve a nonlinear equation arising in conformal mapping problems. Later, the procedure was used by Birkhoff and Zarantonello [6, p. 216] to solve equations associated with free boundary problems. In 1960, Zarantonello [25] gave a theoretical discussion of contractive averaging. In his paper he dealt with mappings having domains and ranges in a real Hilbert space. To prove that the iterates converge, he showed that an associated mapping was contractive.

In 1964, Zarantonello [23; 24] generalized the method of contractive averaging working in a complex Hilbert space. He dealt with the nonlinear equation $Gx - \lambda x = y$. In the framework of solving (1), the more general method depends on recursion relations of the form

$$(3) \qquad \begin{cases} x_{k+1} = (1-a_k)x_k + a_k H x_k, & k \in \omega_0, \\ a_k = \zeta(x_k), & k \in \omega_0 \end{cases}$$

where $\zeta$ is a scalar valued function. This new procedure could be called the method of "averaged iterations". Convergence of the iterates has not yet been established by means of the contractive mapping principle. Thus, there appears to be some distinction between the theory of the two methods.

In the sequel, a revealing geometric interpretation of the method of averaged iterations is given. Also, we show that this procedure can be carried out in a Banach space setting. Instead of considering a nonlinear equation of the form (1), we shall consider the equation

$$(4) \qquad Fx = 0$$

where $F$ is a nonlinear mapping of a Banach space $\mathfrak{X}$ into itself and $F$ satisfies conditions below. Another case where $F: \mathfrak{X} \to \mathfrak{Y}$, $\mathfrak{X}$ and $\mathfrak{Y}$ Banach spaces, is mentioned.

Equation (1) can be transformed to equation (4) by defining

$$F = I - H$$

where $I$ is the identity operator on $\mathfrak{X}$. The respective recursion relations then become

$$x_{k+1} = x_k + a F x_k, \qquad k \in \omega_0, \tag{5}$$

and

$$\left\{ \begin{aligned} x_{k+1} &= x_k + a_k F x_k, && k \in \omega_0, \\ a_k &= \zeta(x_k), && k \in \omega_0 \end{aligned} \right. \tag{6}$$

where $\zeta$ is as above. In this form the recursion relations no longer appear as averages.

## §2. Banach Space Geometry

In this section and the next, basic ideas and notation are introduced. Let $F, G, H, \cdots$ be mappings of $\mathfrak{X}$ into $\mathfrak{Y}$ where $\mathfrak{X}$ and $\mathfrak{Y}$ are real or complex Banach spaces. The elements of the Banach spaces are denoted by $x, y, x', y'$, etc. The scalar field is denoted by $\mathfrak{S}$ and the elements of $\mathfrak{S}$ by $\alpha, \beta, \gamma, \cdots$. An open ball of radius $\rho > 0$ and center $x$ is denoted by $\mathfrak{B}(x, \rho)$. For brevity, $\mathfrak{B}(0, \rho) = \mathfrak{B}(\rho)$.

For each $x \in \mathfrak{X}$, let $[x]$ denote the linear subspace spanned by $x$. If $x \neq 0$, the Hahn-Banach Theorem implies the existence of at least one continuous linear projection $P_x$ of $\mathfrak{X}$ onto $[x]$ with $\|P_x\| = 1$. By the axiom of choice or other means, we associate a unique $P_x$ with each $[x]$, $x \neq 0$. The axiom of choice need not be invoked if the Banach space has a smooth unit ball [14, p. 111]; e.g., in $\ell_p$ with $1 < p < \infty$ there is only one linear projection of norm one corresponding to each $[x]$, $x \neq 0$.

If the Banach space is strictly convex (i.e., $\|x+y\| < \|x\| + \|y\|$ if $\|x\| = \|y\| \neq 0$ and $x \neq y$), then [3] for each $[x], x \neq 0$, there is a unique metric projection $Q_x$ mapping $\mathfrak{X}$ onto $[x]$ such that

(i) $\|Q_x y - y\| = \inf \{\|y' - y\| : y' \in [x]\}$, $\qquad y \in \mathfrak{X}$,

(ii) $Q_x^2 = Q_x$,

(iii) $Q_x(\alpha y) = \alpha Q_x(y)$, $\qquad y \in \mathfrak{X}, \quad \alpha \in \mathfrak{S}$,

(iv) $Q_x(y + y') = y + Q_x(y')$, $\qquad y \in [x], \; y' \in \mathfrak{X}$,

(v) $\|Q_x y - y\| \leqq \|y\|$ and $\|Q_x y\| \leqq 2\|y\|$, $\qquad y \in \mathfrak{X}$.

In general $Q_x$ is nonlinear, and continuous [ 18, p. 40]. If $\mathfrak{X}$ is a Hilbert space and $x \neq 0$, then $Q_x = P_x = O_x$, the orthogonal projection onto $[x]$:

$$O_x y = \frac{(y, x)}{\|x\|^2} x, \qquad y \in \mathfrak{X}.$$

In general,

$$\|y\| \leqq \|P_x y\| + \|(I-P_x)y\|, \qquad y \in \mathfrak{X}; \tag{7}$$

$$\|y\| \leqq \|Q_x y\| + \|(I-Q_x)y\|, \qquad y \in \mathfrak{X} \tag{8}$$

if $\mathfrak{X}$ is strictly convex; and

$$\|y\|^2 = \|O_x y\|^2 + \|(I-O_x)y\|^2, \qquad y \in \mathfrak{X}$$

if $\mathfrak{X}$ is a Hilbert space. Inequalities (7) and (8) are strict if $\mathfrak{X}$ is strictly convex and $y \notin [x]$ [ 15, p. 458].

## §3. Continuity Properties

In this section assumptions on the operator $F$ are introduced. The domain and range of $F$ are denoted by $\mathfrak{D}(F)$ and $\mathfrak{R}(F)$ respectively. Let $\Delta, \Delta', \Delta'', \cdots$ be subsets of the set $\blacktriangle$ of continuous nonnegative increasing functions which are defined on the nonnegative real numbers and vanish at zero. Elements of $\blacktriangle$

are denoted $\delta, \delta', \delta'', \cdots$.

A map on $\mathfrak{X}$ into $\mathfrak{Y}$ is continuous iff it is continuous with respect to the strong topologies on $\mathfrak{X}$ and $\mathfrak{Y}$. It is s-w (strong-weak) continuous iff it is continuous with respect to the strong topology on $\mathfrak{X}$ and the weak topology on $\mathfrak{Y}$.

Definition 1. Let $\delta \in \blacktriangle$. Then a map $F$ on $\mathfrak{X}$ into $\mathfrak{Y}$ is $\delta$-continuous if

$$\|Fx-Fy\| \leqq \delta(\|x-y\|), \qquad x, y \in \mathfrak{D}(F).$$

Special cases are Lipschitz and Hölder continuous mappings.

Definition 2. Let $F: \mathfrak{X} \to \mathfrak{X}$, $\Delta \subset \blacktriangle$ and $R_x = P_x$ or $Q_x$. Then $F$ is $\Delta$-cross continuous relative to $\{R_x : x \neq 0\}$ iff for each $[x]$, $x \neq 0$, there exists $\delta_x \in \Delta$ such that

$$\|(I-R_x)(Fy-Fy')\| \leqq \delta_x(\|y-y'\|)$$

for $y, y' \in \mathfrak{D}(F)$ and $y-y' \in [x]$.

Definition 3. Let $F: \mathfrak{X} \to \mathfrak{X}$, $\Delta \subset \blacktriangle$ and $R_x = P_x$ or $Q_x$. Then $F$ is $\Delta$-monotone relative to $\{R_x : x \neq 0\}$ iff for each $[x]$, $x \neq 0$, there exists $\delta_x \in \Delta$ such that

$$\delta_x(\|R_x(Fy-Fy')\|) \geqq \|y-y'\|$$

for $y, y' \in \mathfrak{D}(F)$ and $y-y' \in [x]$.

In [16] the modulus of continuity, modulus of cross-continuity and the parallel modulus of continuity were defined for operators having domains and ranges in a Hilbert space. Zarantonello, in [23; 24], uses particular forms of $\Delta$-cross continuity and $\Delta$-monotonicity. Browder also mentions types of $\Delta$-monotonicity in [7; 8; 9]. Both Browder and Zarantonello assume that the associated $\delta_x$ are independent of the direction $[x]$.

An example of an operator which is $\Delta$-cross continuous with $\Delta = \{\delta\}$, $\delta(s) = \kappa s$, but not Lipschitz continuous has been given in [16].

The following condition is derived from $\Delta'$-cross continuity and $\Delta''$-monotonicity.

<u>Definition 4.</u> Let $F: \mathfrak{X} \to \mathfrak{X}$, $\Delta \subset \mathbf{\Delta}$ and $R_x = P_x$ or $Q_x$. Then $F$ satisfies a <u>$\Delta$-condition</u> relative to $\{R_x : x \neq 0\}$ iff for each $[x]$, $x \neq 0$, there exists $\delta_x \in \Delta$ such that

$$\|(I-R_x)(Fy-Fy')\| \leqq \delta_x(\|R_x(Fy-Fy')\|)$$

for $y, y' \in \mathfrak{D}(F)$ and $y-y' \in [x]$.

If $F$ is $\Delta'$-cross continuous and $\Delta''$-monotone, then $F$ satisfies a $\Delta$-condition with

$$\delta_x = \delta'_x \circ \delta''_x$$

for each $[x]$, $x \neq 0$. Thus the $\Delta$-condition and $\Delta''$-monotonicity are weaker than $\Delta'$-cross continuity and $\Delta''$-monotonicity.

These are the principal concepts studied in this thesis.

## §4. Summary

In the next two chapters, further historical comments are made. For example in Chapter II, which is concerned with qualitative aspects of the theory, the existence theorems of Browder and Zarantonello are discussed. Also conditions yielding continuous dependence of the solution of $Fx = 0$ on the operator $F$ are given. In Chapter III, relations between the method of averaged iterations and the contractive mapping principle are explored. Other forms of monotonicity occur in these two chapters. For examples of these see [2; 10; 11; 20; 21; 23; 24; 25]. The term monotone operator is used to describe any of the properties given in these references.

In Chapters IV and V, the theory of the method of averaged iterations is discussed in a Hilbert space setting. The improvement of approximate solutions to $Fx = 0$ is studied in Chapter IV using the concepts introduced in §3. A geometric interpretation of these concepts is introduced there. The iterative procedure is defined in Chapter V. Also, conditions implying convergence of the iterates are given in this chapter.

In Chapter VI, the ideas necessary to carry out the theory of

Chapters IV and V in a Banach space setting are given. The geometry of the unit ball in a Banach space is examined. Further remarks about certain aspects of the Banach space theory are made.

In the sequel, a symbol such as (II, 3) refers to formula (3) in Chapter II and (3) refers to formula (3) in the current chapter.

# CHAPTER II

# QUALITATIVE THEORY

## §1. Historical Remarks

In this chapter, existence, uniqueness and continuous dependence of solutions of $Fx = 0$ are discussed. The notions of $\Delta$-monotonicity and other forms of monotonicity appear in the discussion.

Prior to 1964, there had been considerable work in the qualitative theory of nonlinear monotone operator equations by Browder and Minty. In [ 10; 11; 12; 13 ], Browder dealt with a mapping from a Hilbert space into itself. This mapping satisfied the condition

$$\text{(1)} \qquad \operatorname{Re}(Fx - Fy,\ x - y) \geqq \mathcal{K}\|x - y\|^2, \qquad x, y \in \mathcal{D}(F)$$

where $\mathcal{K} > 0$. The theory he developed was used to obtain existence and uniqueness for solutions of quasi-linear elliptic differential equations. Minty [ 20 ], using a similar condition, dealt with the existence of solutions of $Fx = 0$ where the operator mapped a reflexive Banach space into its conjugate space. In [ 23; 24 ], Zarantonello gave an existence theorem for operator equations in a Hilbert space setting. He required the operators to satisfy

$$|(Fx - Fy, x-y)| \geqq \kappa \|x-y\|^2, \qquad x, y \in \mathfrak{D}(F)$$

where $\kappa > 0$. This is a $\Delta$-monotonicity condition on $F$ with $\Delta = \{\delta\}$ where $\delta(t) = t/\kappa$. Prior to the publication of this theorem, Zarantonello's theorem and its proof were communicated to Browder by Minty (cf. [7, p. 985; 23; 24]). Then Browder generalized Zarantonello's theorem to the case of a mapping of a reflexive Banach space $\mathfrak{X}$ into its dual $\mathfrak{X}^*$ (also see [8; 9]). Precise statements of these theorems follow in the next section.

## §2. Existence and Uniqueness

In this section the existence theorems of Zarantonello and Browder are stated. A local existence theorem and its proof are given. The proof illustrates ideas developed by Zarantonello and Browder.

We remark in passing that all of the stronger forms of monotonicity used in other papers imply an inequality of the form

$$\delta(\|Fx-Fy\|) \geqq \|x-y\|, \qquad x, y \in \mathfrak{D}(F)$$

where $\delta \in \blacktriangle$. Thus with any of these types of conditions, $F$ is one-to-one and the solution to $Fx = 0$ is unique. The study of monotone operators has therefore focused on the ranges of these operators.

Zarantello's existence theorem is stated for a mapping $F$ having its domain and range in a Hilbert space $\mathcal{H}$. To state this theorem, the notions of demiclosed and locally cross-bounded mappings must be introduced. A mapping $F$ is demiclosed iff for any net $\{x_a\}$ such that $x_a$ converges strongly to $x$ and $Fx_a$ converges weakly to $y$, we have $x \in \mathcal{D}(F)$ and $Fx = y$. A mapping $F$ is locally cross-bounded if for each $x \in \mathcal{D}(F)$ there exists a neighborhood $V$ of $x$ and $k = k(V) > 0$ such that for each $[y]$, $y \neq 0$,

$$\|(I-O_y)(Fy' - Fy'')\| \leqq k$$

for $y', y'' \in V$ and $y' - y'' \in [y]$.

Theorem 1. (Zarantonello) Let $\mathcal{Y}$ be a dense linear subspace of $\mathcal{H}$. Suppose that $\mathcal{D}(F) = \overline{\mathcal{B}}(\rho) \cap \mathcal{Y}$, $F$ is continuous on each finite dimensional convex set in $\mathcal{D}(F)$, $F$ is locally cross-bounded, $F$ is $\Delta$-monotone where $\delta(t) = t/\mathcal{K}$ for each $\delta \in \Delta$, $\|F(0)\| \leqq \rho\mathcal{K}$, and $F$ is sequentially demiclosed. Then there exists a unique solution to $Fx = 0$.

Browder's extension of this theorem is stated for a reflexive Banach space $\mathfrak{X}$ with dual $\mathfrak{X}^*$. Let

$$< y, x > , \qquad x \in \mathfrak{X},\ y \in \mathfrak{X}^*$$

denote the functional relation between $x$ and $y$.

Theorem 2. (Browder) Let $F: \mathfrak{X} \to \mathfrak{X}^*$ be s-w continuous. Assume:

(i) there exists a continuous real function $c(t)$, $0 \leqq t < \infty$, with $c(t) \to \infty$ as $t \to \infty$, for which

$$|< Fx, x >| \geqq c(\|x\|)\|x\|, \qquad x \in \mathfrak{X};$$

(ii) for each $\rho > 0$ there exists $\delta_\rho \in \mathbf{A}$ such that

$$(2) \qquad |< Fx - Fy, x - y >| \geqq \delta_\rho(\|x - y\|)\|x-y\|$$

for $x, y \in \overline{\mathfrak{B}}(\rho)$. Then $F$ is a one-to-one mapping of $\mathfrak{X}$ onto $\mathfrak{X}^*$ with a continuous inverse.

These two theorems are not directly comparable since the first is a local result and the second is global. In order to compare them, a local existence theorem is stated and its proof is given. The proof is established with the aid of the following lemma.

Lemma 1. Let $\mathfrak{X}$ be a finite dimensional Banach space. Suppose that $F: \mathfrak{X} \to \mathfrak{X}$ is a homeomorphism of $\overline{\mathfrak{B}}(\rho)$ onto $F(\overline{\mathfrak{B}}(\rho))$ such that $F(0) = y_0$ and

$$\|Fx - y_0\| \geqq \delta(\|x\|), \qquad x \in \overline{\mathfrak{B}}(\rho)$$

where $\delta \in \mathbf{A}$. Then $\mathcal{R}(F) \supset \overline{\mathcal{B}}(y_0, \delta(\rho)) \equiv \mathcal{B}_0$.

<u>Proof</u> By the Brouwer Domain Invariance Theorem [1, p. 164], interior (boundary) points of $\overline{\mathcal{B}}(\rho)$ are mapped onto interior (boundary) points of the closed set $F(\overline{\mathcal{B}}(\rho))$. Assume that there exists $x_0 \in \mathcal{B}_0 \sim F(\overline{\mathcal{B}}(\rho))$ (the set theoretic difference of $\mathcal{B}_0$ and $F(\overline{\mathcal{B}}(\rho))$). Let

$$B = \{t : y_0 + t(x_0 - y_0) \in F(\overline{\mathcal{B}}(\rho)), \quad 0 \leqq t \leqq 1\} .$$

Then by using the Invariance of Domain Theorem and a standard homotopy argument, the contradiction that $B = [0, 1]$ is reached.

<u>Theorem 3.</u> Let $F : \mathcal{H} \to \mathcal{H}$ have domain $\overline{\mathcal{B}}(\rho)$, $F$ be s-w continuous and

$$|(Fx - Fy,\ x - y)| \geqq \delta(\|x - y\|)\|x-y\|, \qquad x, y \in \overline{\mathcal{B}}(\rho)$$

where $\delta \in \mathbf{A}$, and $\|F(0)\| \leqq \delta(\rho)$. Then there exists a unique solution $x_0 \in \overline{\mathcal{B}}(\rho)$ to $Fx = 0$.

<u>Proof</u> Let $M$ be a finite dimensional subspace of $\mathcal{H}$ and

$$F_M = O_M F O_M$$

where $O_M$ is the orthogonal projection of $\mathcal{H}$ onto $M$. Then $F_M$ is a continuous mapping of $M$ onto $M$. For $x, y \in M \cap \overline{\mathcal{B}}(\rho)$,

$$|(F_M x - F_M y,\ x-y)| = |(Fx - Fy, x-y)| \geqq \delta(\|x-y\|)\|x-y\|.$$

By the Schwarz inequality,

$$\|F_M x - F_M y\| \geqq \delta(\|x-y\|), \qquad x, y \in M \cap \bar{\mathcal{B}}(\rho),$$

and

$$\|F_M x - F_M(0)\| \geqq \delta(\|x\|), \qquad x \in M \cap \bar{\mathcal{B}}(\rho).$$

Thus by the lemma

$$\mathcal{R}(F_M) \supset \{y \in M : \|y - F_M(0)\| \leqq \delta(\rho)\} .$$

Since $\|F_M(0)\| \leqq \|F(0)\| \leqq \delta(\rho)$, there exists $x_M \in M \cap \bar{\mathcal{B}}(\rho)$ such that

(3) $$F_M x_M = 0 .$$

Let $\Lambda$ be the set of all finite dimensional subspaces of $\mathcal{H}$ directed by inclusion. For each $M \in \Lambda$, let $x_M \in M \cap \bar{\mathcal{B}}(\rho)$ satisfy (3). Then $\{x_M : M \in \Lambda\}$ and $\{Fx_M : M \in \Lambda\}$ are nets. By the weak compactness of $\bar{\mathcal{B}}(\rho)$, there exists a subnet $\{x_M : M \in \Lambda_0\}$, $\Lambda_0 \subset \Lambda$, and an $x_0 \in \bar{\mathcal{B}}(\rho)$ such that this subnet converges to $x_0$.

Let $\epsilon > 0$ and $\{x_i : i = 1, 2, \cdots, n\} \subset \mathcal{H}$. Then the set

$$V = \bigcap_{i=1}^{n} \{x \in \mathcal{H} : |(x, x_i)| < \epsilon\}$$

is a fundamental neighborhood of the origin in the weak topology. Let $N$ be the linear span of $\{x_i : i = 1, 2, \cdots, n\}$. Then for $M \supset N$ and $x \in N$,

$$(Fx_M, x) = (Fx_M, O_M x) = (F_M x_M, x) = 0.$$

Thus $Fx_M \in V$ and the net $\{Fx_M : M \in \Lambda\}$ converges weakly to zero.

Let $M, N \in \Lambda_0$ and $M \supset N$. By an argument given above,

$$(Fx_M - Fx_N, x_M - x_N) = -(Fx_N, x_M).$$

Thus

$$|(Fx_N, x_M)| \geqq \delta(\|x_M - x_N\|) \|x_M - x_N\|.$$

By the continuity of $\delta$, this holds for $x_0$ in place of $x_M$. As the weak limit of $\{Fx_N : N \in \Lambda_0\}$ is zero, $\{x_N : N \in \Lambda_0\}$ converges strongly to $x_0$. Then s-w continuity implies that $Fx_0 = 0$.

<u>Corollary.</u> If no restrictions are made on $F(0)$, then $F$ is a one-to-one mapping of $\bar{\mathcal{B}}(\rho)$ onto $\mathcal{R}(F)$, $\mathcal{R}(F) \supset \bar{\mathcal{B}}(F(0), \delta(\rho))$ and $F^{-1}$ is continuous.

<u>Proof</u> Let $Gx = Fx - y$. Then $G$ satisfies the hypothesis of the theorem if $\|y - F(0)\| \leqq \delta(\rho)$.

By using essentially the same arguments this theorem and its corollary can be established for a mapping of a reflexive Banach space into its dual. Thus, it is a local version of Browder's theorem.

Therefore the essential difference between the two theorems is that Zarantonello only requires $F$ to be densely defined and locally cross-bounded while Browder requires $F$ to be everywhere defined. However Browder does not require $F$ to be (cross-) bounded.

These theorems depend on the reflexivity of the Banach space and thus cannot be used to yield existence theorems for Δ-monotone mappings on a non-reflexive Banach space.

## §3. Continuous Dependence

In this section the dependence of the solution of $Fx = 0$ on the operator $F$ is discussed. This discussion is given in detail for operators which are Δ-monotone relative to $\{P_x : x \neq 0\}$.

To show continuous dependence it must be established that there exists a mapping from some class of operators into the set of solutions to the equations $Fx = 0$ and that this mapping is continuous. In particular a class of operators must be defined and suitably topologized. Let $D \subset \mathfrak{X}$ and

$$\Gamma = \{F : F : D \to \mathfrak{X},\ \exists \text{ a solution of } Fx = 0 \text{ for each } F \in \Gamma\}.$$

A mapping from $\Gamma$ into the set of solutions to the equations $Fx = 0$ can be defined by the axiom of choice or in special cases, by other means. Thus for each $F \in \Gamma$, there exists a unique $x(F) \in D$

satisfying $Fx = 0$. The reason for treating the problem in this manner is indicated by an example given below.

A topology on $\Gamma$ is determined by a subbase of open sets

$$V(F_0, S, y, \epsilon) = \{F \in \Gamma : |\langle y, Fx - F_0x\rangle| < \epsilon, y \in \mathfrak{X}^*, \|y\| = 1, x \in S \subset D\}.$$

This is the topology of weak uniform convergence on $S$.

The following proposition yields the continuity of the mapping of $\Gamma$ onto $\{x(F): F \in \Gamma\}$ at $F_0 \in \Gamma$. Before stating this proposition, the following notation is introduced. For each $x \neq 0$, let $x^* \in \mathfrak{X}^*$ be defined by

$$P_x y = x^*(y)\, x/\|x\|, \qquad y \in \mathfrak{X}.$$

Then $\|x^*\| = 1$.

<u>Proposition 1</u>. Let $F_0$ be $\Delta$-monotone relative to $\{P_y : y \neq 0\}$ where $\Delta = \{\delta\}$, $F \in V(F_0, S, x^*, \epsilon)$ where $x = x(F) - x(F_0)$ and $x(F) \in S$. Then

$$\|x(F_0) - x(F)\| \leqq \delta(\epsilon).$$

<u>Proof</u> By hypothesis,

$$\|x(F) - x(F_0)\| \leqq \delta(\|P_x(F_0x(F) - F_0x(F_0))\|)$$

$$= \delta(|<x^*, F_0x(F)>|)$$

$$= \delta(|<x^*, F_0x(F) - Fx(F)>|)$$

$$\leqq \delta(\epsilon) .$$

Thus, the mapping from $\Gamma$ onto $\{x(F): F \in \Gamma\}$ is continuous at $F_0$ from the given topology to the norm topology on $\{x(F): F \in \Gamma\}$ if $S = D$.

The following proposition is a useful application of the structure which has been introduced.

Proposition 2. If $x_0 \in S$, $\|Fx_0\| \leqq \epsilon/2$, $F_0$ is $\Delta$-monotone relative to $\{P_y : y \neq 0\}$ where $\Delta = \{\delta\}$ and $F \in V(F_0, S, x^*, \epsilon/2)$ where $x = x_0 - x(F_0)$, then

$$\|x(F_0) - x_0\| \leqq \delta(\epsilon).$$

Proof Since $F$ is $\Delta$-monotone,

$$\|x(F_0) - x_0\| \leqq \delta(|<x^*, F_0x_0>|)$$

$$\leqq \delta(|<x^*, F_0x_0 - Fx_0>| + |<x^*, Fx_0>|)$$

$$\leqq \delta(\epsilon/2 + \epsilon/2) = \delta(\epsilon).$$

This proposition asserts that if $x_0$ is an approximate solution of $Fx = 0$ and $F$ is close to $F_0$, then $x_0$ is close to the solution $x(F_0)$ of $F_0x = 0$.

For operators which are $\Delta$-monotone relative to $\{P_x : x \neq 0\}$ on a non-reflexive Banach space, the open sets in the topology of $\Gamma$ need not involve all $y \in \mathfrak{X}^*$ such that $\|y\| = 1$. The set of $x^* \in \mathfrak{X}^*$ corresponding to the set of projections $P_x$ suffices to establish the propositions given above. This set of functionals is a total subset of $\mathfrak{X}^*$.

Similar propositions can be established for operators which are $\Delta$-monotone relative to $\{Q_x : x \neq 0\}$. But the topology on $\Gamma$ generated by the open sets

$$\{F \in \Gamma : \|Q_x(Fy - F_0y)\| < \epsilon, \quad x \neq 0, \quad y \in S \subset D\}$$

apparently has not yet been investigated. This topic will not be pursued further.

Next two examples are given. The first indicates a reason for defining the set $\Gamma$ as above. The second example illustrates further properties of $\Gamma$ and is of historical interest.

<u>Example 1.</u> Let $F_0 : \mathfrak{H} \to \mathfrak{H}$ be defined on $\bar{\mathfrak{B}}(\rho)$, be uniformly continuous, bounded and $\Delta$-monotone with $\Delta = \{\delta\}$. Let $\mathfrak{H}$ be separable with an orthonormal basis $\{\phi_k : k \in \omega\}$ ($\omega$ is the set of

positive integers) and $O_k$ be the orthogonal projection of $\mathcal{H}$ onto the linear span of $\{\phi_j : j = 1, 2, \cdots, k\}$. The mappings

$$F_k = O_k F_0 O_k, \qquad k \in \omega,$$

approach $F_0$ uniformly on $\overline{\mathfrak{B}}(\rho)$. By previous arguments there exist unique solutions $x_k \in O_k(\mathcal{H}) \cap \overline{\mathfrak{B}}(\rho)$ to the equation

$$\text{(4)} \qquad F_k x = 0, \qquad k \in \omega.$$

If $x_k \in O_k(\mathcal{H})$ is a solution, then any $x \in \mathcal{H}$ such that $O_k x = x_k$ is also a solution to (4). For purposes of computation, an obvious choice of $x(F_k)$ is $x_k$. In this case, let $y_k = (x(F_k) - x(F_0))/\|x(F_k) - x(F_0)\|$. Then

$$\begin{aligned} \|x(F_k) - x(F_0)\| &\leqq \delta(|(F_0 x(F_k), y_k)|) \\ &= \delta(|(F_0 x(F_k) - F_k x(F_k), y_k)|) \\ &\to 0 \end{aligned}$$

as $k \to \infty$ .

<u>Example 2.</u> Let $G : \mathcal{H} \to \mathcal{H}$ be defined on the unit ball and be s-w continuous. Let

$$\mathcal{R}_n(G) = \{ \frac{(Gx-Gy, x-y)}{\|x-y\|^2} : x \neq y, \quad x, y \in \mathcal{D}(G) \} .$$

If $d_0 = \inf \{ | \lambda_0 - \mu | : \mu \in \mathcal{R}_n(G) \} > 0$ and

$$F_0 x = Gx - \lambda_0 x - y,$$

then for each $[x]$, $x \neq 0$,

$$\| O_x(F_0 y - F_0 y') \| \geqq d_0 \| y - y' \|$$

for $y, y' \in \mathcal{D}(G)$ and $y - y' \in [x]$. Thus there exists a solution $x(F_0)$ to $F_0 x = 0$. Let $\epsilon/2 < d_0 < \epsilon$, $|\lambda - \lambda_0| < \epsilon$ and

$$F_\lambda x = Gx - \lambda x - y.$$

Since it can happen that $\lambda \in \mathcal{R}_n(G)$, there may not exist $x(F_\lambda)$ such that $F_\lambda x(F_\lambda) = 0$. Nevertheless

$$\| F_\lambda x - F_0 x \| = |\lambda - \lambda_0| \quad \| x \| < \epsilon .$$

Thus the nearness of $F_\lambda$ to $F_0 \in \Gamma$ does not imply that $F_\lambda \in \Gamma$. The set $\mathcal{R}_n(G)$ is called the numerical range by Zarantonello [23; 24]. Zarantonello asserts that the spectrum of $G$ is contained in $\mathcal{R}_n(G)$. This example also shows where a $\Delta$-monotone condition may arise.

# CHAPTER III

## THE CONTRACTIVE MAPPING PRINCIPLE AND MONOTONE OPERATORS

### §1. Introduction

As mentioned in Chapter I, Zarantonello [ 25] gave a proof that the iterative method of contractive averaging converges by showing that a related operator is contractive. In [ 24], he states that under suitable conditions on the operator $F$, the method of averaged iterations converges with a constant mapping $\zeta$. This second result can also be established by showing that a related operator is contractive. This is a generalization of the earlier result [ 25] which is stated and proved in the next section.

### §2. Contractive Mappings

In this section it is demonstrated that certain monotonicity conditions on an operator imply that a related operator is contractive. Theorem 1 below is essentially the above mentioned result which is given in [ 24]. In the discussion $F$ is an operator having its domain and range in a Hilbert space $\mathcal{H}$.

Theorem 1. (Zarantonello) Let $F : \overline{\mathcal{B}}(\rho) \to \mathcal{H}$ satisfy a $\Delta$-condition with $\delta(t) = \kappa t$ for each $\delta \in \Delta$,

$$\mathrm{Re}(Fx - Fy,\ x-y) \geqq d\|x - y\|^2, \qquad x, y \in \overline{\mathfrak{S}}(\rho),$$

$\|F(0)\| \leqq \beta \rho d, \quad \beta < 1$ and

$$|(Fx - Fy,\ x-y)| \leqq k\|x - y\|^2, \qquad x, y \in \overline{\mathfrak{S}}(\rho).$$

Let $G_\alpha = I + \alpha F, \quad \alpha \in \mathfrak{s}$.

Then there exists $\alpha \neq 0$ such that $G_\alpha$ is a contractive mapping of $\overline{\mathfrak{S}}(\rho)$ into $\overline{\mathfrak{S}}(\rho)$.

Proof For each $[x], \quad x \neq 0$,

$$\begin{aligned}\|Fy - Fy'\|^2 &= \|O_x(Fy - Fy')\|^2 + \|(I-O_x)(Fy-Fy')\|^2 \\ &\leqq k^2(\mathcal{K}^2 + 1)\|y - y'\|^2,\end{aligned}$$

$y, y' \in \mathfrak{D}(F)$ and $y - y' \in [x]$. Further

$$\|G_\alpha y - G_\alpha y'\|^2 = \|y - y'\|^2 + 2\mathrm{Re}\,\alpha(Fy-Fy', y-y') + |\alpha|^2\|Fy-Fy'\|^2.$$

Thus if $\alpha < 0$,

(1) $$\|G_\alpha y - G_\alpha y'\| \leqq \|y-y'\|^2(1+2\alpha d + \alpha^2 k^2(\mathcal{K}^2+1)).$$

For $$-\frac{2d}{k^2(\mathcal{K}^2+1)} < \alpha < 0,$$

$$0 \leqq \theta(\alpha) \equiv 1 + 2\alpha d + \alpha^2 k^2(\mathcal{K}^2+1) < 1.$$

Further

$$\|G_\alpha x\| \leqq \|G_\alpha x - G_\alpha(0)\| + \|G_\alpha(0)\| \leqq \theta^{\frac{1}{2}}(\alpha)\|x\| + |\alpha| \ \|F(0)\|$$

$$\leqq (\theta^{\frac{1}{2}}(\alpha) + |\alpha|\beta d)\rho.$$

For $\alpha \leqq 0$,

$$\phi(\alpha) \equiv \theta^{\frac{1}{2}}(\alpha) + |\alpha|\beta d = \theta^{\frac{1}{2}}(\alpha) - \alpha\beta d$$

and for $\alpha < 0$, $\phi(\alpha) \leqq 1$ iff

$$\theta^{\frac{1}{2}}(\alpha) \leqq 1 + \alpha\beta d.$$

The latter inequality holds for $-\dfrac{2d(1-\beta)}{k^2(\kappa^2+1)-\beta^2 d} \leqq \alpha \leqq 0$. Thus there exists $\alpha \neq 0$ such that $G_\alpha$ is contractive and $G_\alpha : \bar{\mathcal{B}}(\rho) \to \bar{\mathcal{B}}(\rho)$.

The fixed point of $G_\alpha$ is also the solution of $Fx = 0$. Thus this theorem provides an iterative procedure for finding the solution of $Fx = 0$.

The contractive mapping principle can be used in another manner to investigate the existence of solutions to $Fx = 0$. It follows from the consideration of continuous mappings of the real line into itself. If such a mapping satisfies either

$$|F(x) - F(y)| \leqq \theta|x - y|, \qquad \theta < 1,$$

or

$$|F(x) - F(y)| \geqq \theta|x - y|, \qquad \theta > 1,$$

then $F$ has a fixed point. In the first case $F$ is contractive while in the second, $F^{-1}$ is contractive. A similar result can be established for monotone operators.

Theorem 2. Let $F: \overline{\mathcal{B}}(\rho) \to \mathcal{H}$ be s-w continuous,

$$\mathrm{Re}(Fx - Fy,\ x - y) \geqq d\ \|x - y\|^2, \qquad x, y \in \overline{\mathcal{B}}(\rho),$$

and $\|F(0)\| \leqq \rho d$. Let $G_\alpha = I + \alpha F$, $\alpha \in \mathfrak{S}$. Then there exists $\alpha \neq 0$ such that $\mathcal{R}(G_\alpha) \supset \overline{\mathcal{B}}(\rho)$ and $G_\alpha^{-1}$ is contractive.

Proof If $\alpha > 0$,

$$\mathrm{Re}(G_\alpha x - G_\alpha y,\ x - y) = \|x - y\|^2 + \mathrm{Re}\,\alpha(Fx - Fy,\ x - y)$$
$$\geqq (1 + \alpha d)\ \|x - y\|^2 .$$

This implies that

(2) $$\|G_\alpha x - G_\alpha y\| \geqq (1 + \alpha d)\ \|x - y\| .$$

The corollary to Theorem (II, 3) yields

$$\mathcal{R}(G_\alpha) \supset \overline{\mathcal{B}}(G_\alpha(0),\ (1 + \alpha d)\rho) = \overline{\mathcal{B}}(\alpha F(0),\ (1 + \alpha d)\rho).$$

Then for $\alpha > 0$ and $x \in \overline{\mathcal{B}}(\rho)$,

$$\|x - \alpha F(0)\| \leqq \|x\| + \alpha\|F(0)\| \leqq (1 + \alpha d)\rho .$$

Further (2) implies

$$\|G_\alpha^{-1}x - G_\alpha^{-1}y\| \leqq \frac{1}{1+\alpha d} \|x-y\|$$

for $x, y \in \overline{\mathfrak{B}}(G_\alpha(0), (1+\alpha d)\rho)$.

The problem of inverting $G_\alpha$ or $F$ is equivalently difficult. Thus this theorem and others similar to it are not particularly useful in finding a constructive means of obtaining the solution to $Fx = 0$. In later chapters, the approximate inversion of $F$ can be carried out under less restrictive monotonicity conditions than used in this theorem. Thus the parallels between the method of averaged iterations and the contractive mapping principle are not pursued further. An example given by Wong [22] also discourages further investigation. His example shows that there exist mappings such that the iterates converge while the operator is noncontractive.

# CHAPTER IV

# IMPROVING APPROXIMATIONS

## §1. Preliminary Remarks

In this chapter sufficient conditions for improving an arbitrary approximate solution are given. These conditions are of interest relative to iterative procedures. If an arbitrary approximate solution can be sufficiently improved, then it can be established that an iterative procedure converges.

In this chapter and the next, $F$ is a mapping of a Hilbert space $\mathcal{H}$ into itself. The global case where $\mathfrak{D}(F) = \mathcal{H}$ and the local case where $\mathfrak{D}(F) = \overline{\mathfrak{S}}(\rho)$ are considered. It is assumed that $F$ is s-w continuous, satisfies a $\Delta$-condition and is $\Delta'$-monotone. In the following sections, the roles of the $\Delta$-condition and $\Delta'$-monotonicity are considered. A revealing geometric interpretation of the $\Delta$-condition is given.

Using the results of this chapter, the convergence of an iterative procedure will be established in a Hilbert space setting and the iterative procedure will be generalized to a Banach space setting.

The problem is as follows. Given an approximate solution $x_0$ of

(1) $$Fx = 0 ,$$

find a better approximation of the form

(2) $$x_\alpha = x_0 + \alpha F x_0, \qquad \alpha \in \mathfrak{S}.$$

Also a measure of the improvement is sought.

A necessary and sufficient condition for the existence of better approximations of the form (2) is that

$$F(x_0 + [Fx_0]) \cap \mathfrak{B}(\|Fx_0\|) \neq \emptyset$$

(cf. Figure 1). Since this condition is not very useful, the intersection of $\mathfrak{B}(\|Fx_0\|)$ with a larger set determined by the $\Delta$-condition is studied.

## §2. The $\Delta$-condition

For each $\delta \in \mathfrak{A}$ and $x \neq 0$, let

$$K(\delta, x) = \{y : \|(I-O_x)y\| \leqq \delta(\|O_x y - x\|)\} .$$

We call this set the $\delta$-cone with vertex $x$ and axis $[x]$ (cf. Figure 2). If $y \in K(\delta, x)$ and

(3) $$\delta^2(\|O_x y-x\|) + \|O_x y\|^2 < \|x\|^2,$$

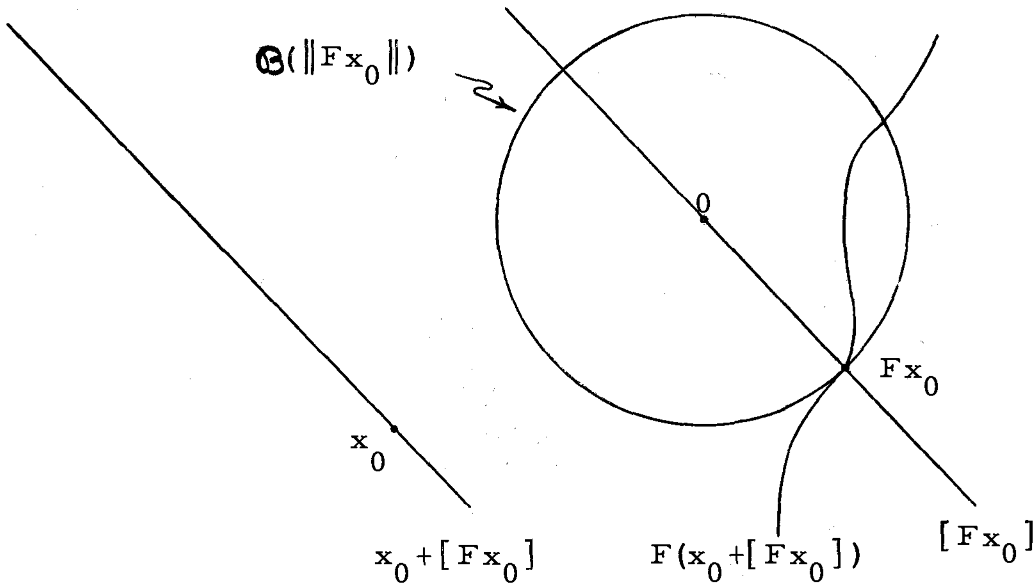



Figure 1

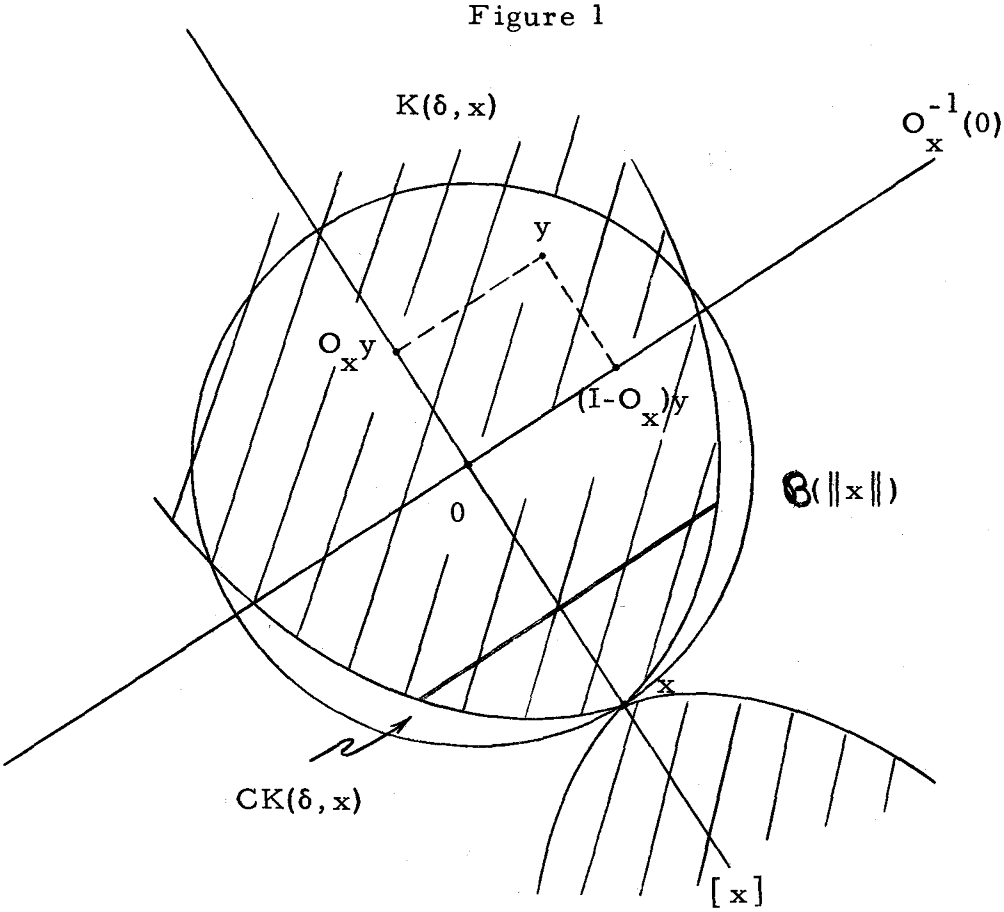



Figure 2

then $y \in \mathcal{B}(\|x\|)$.

Inequality (3) implies more than $y \in \mathcal{B}(\|x\|)$. For each $C \subset \mathcal{S}$, let

$$CK(\delta, x) = \{y : O_x y = \beta x, \quad \beta \in C\} \cap K(\delta, x).$$

We call this set a C-cross section of the $\delta$-cone with vertex $x$. Thus (3) implies that

$$CK(\delta, x) \subset \mathcal{B}(\|x\|)$$

where $C = \{\beta\}$ and $O_x y = \beta x$ (cf. Figure 2).

The following considerations indicate the significance of the $\delta$-cones. Let $x_a \in x_0 + [Fx_0]$. Then

$$Fx_a = (I-O_0)Fx_a + O_0 Fx_a$$

where $O_0 = O_{Fx_0}$. If $F$ satisfies a $\Delta$-condition with $\delta \in \Delta$ corresponding to $[Fx_0]$, then

$$\begin{aligned} \|(I-O_0)Fx_a\| &= \|(I-O_0)(Fx_a - Fx_0)\| \\ &\leqq \delta(\|O_0(Fx_a - Fx_0)\|) = \delta(\|O_0 Fx_a - Fx_0\|). \end{aligned}$$

Thus $F(x_0 + [Fx_0]) \subset K(\delta, Fx_0)$. If

$$CK(\delta, Fx_0) \subset \mathcal{B}(\|Fx_0\|)$$

for some $C$, the next problem is to insure that there exists $\alpha$ such that

$$(4) \qquad Fx_\alpha \in \{y : O_0 y = \beta F x_0, \ \beta \in C\} .$$

This leads to the study of $\Delta'$-monotonicity.

## §3. The $\Delta'$-monotonicity Condition

To insure that there exists $\alpha$ such that (4) holds, the following conditions are assumed:

(i) $\mathfrak{D}(F) = \mathcal{H}$;

(ii) $F$ is s-w continuous;

(iii) $F$ is $\Delta'$-monotone where the $\delta' \in \Delta'$ corresponding to $[Fx_0]$ is strictly increasing and $(\delta')^{-1}(t) \to \infty$ as $t \to \infty$.

The analysis involves a function which is used to study the behavior of $O_0 F x_\alpha$ as a function of $\alpha$.

Definition 1. Given $F$ and $x_0$, define $\gamma : \mathfrak{R} \to \mathfrak{R}$ by

$$\gamma(\alpha) F x_0 = O_0 F x_\alpha$$

where $x_\alpha = x_0 + \alpha F x_0$.

Note that

$$\gamma(0) = 1,$$

$$\gamma(\alpha) = \frac{(Fx_\alpha, Fx_0)}{\|Fx_0\|^2},$$

and $\gamma$ is continuous whenever $F$ is s-w continuous.

To show that suitable $\alpha$ exists, $\mathcal{R}(\gamma)$ must be studied. The following proposition gives information about the range of $\gamma$.

<u>Proposition 1.</u> Let $F$ satisfy (i), (ii) and (iii) above and suppose that $Fx_0 \neq 0$. Then for every $\rho > 0$,

$$\mathcal{R}(\gamma) \supset \{\alpha : |\alpha - 1| \leqq \frac{(\delta')^{-1}(\rho \|Fx_0\|)}{\|Fx_0\|}\}.$$

<u>Proof</u> Let $x_i = x_0 + \alpha_i Fx_0$, $i = 1, 2$. Then

$$(\gamma(\alpha_1) - \gamma(\alpha_2))Fx_0 = O_0(Fx_1 - Fx_2).$$

This implies

$$\delta'(|\gamma(\alpha_1) - \gamma(\alpha_2)| \, \|Fx_0\|) \geqq \|x_1 - x_2\| = |\alpha_1 - \alpha_2| \, \|Fx_0\|.$$

Thus $\gamma$ is one-to-one and the inverse is $\delta'$-continuous. Also

$$|\gamma(\alpha)-1| \geqq \frac{(\delta')^{-1}(|\alpha| \, \|Fx_0\|)}{\|Fx_0\|} .$$

Since $\{\alpha : |\alpha| \leqq \rho\} \subset \mathfrak{D}(\gamma) = \mathfrak{S}$ for every $\rho > 0$, the result follows by Lemma (II, 1).

Corollary. $\gamma$ is one-to-one, $\gamma^{-1}$ is $\delta'$-continuous and $\mathfrak{R}(\gamma) = \mathfrak{S}$.

Thus there exists $\alpha$ such that (4) holds for any $C$.

The case where $\mathfrak{D}(F) = \overline{\mathfrak{B}}(\rho)$ is discussed in §5.

## §4. Improving Approximations

The following proposition summarizes the results obtained above.

Proposition 2. Let $F$ satisfy (i), (ii) and (iii) of §3 and a $\Delta$-condition with $\delta \in \Delta$ corresponding to $[Fx_0]$. Suppose there exists $C \subset \mathfrak{S}$ such that

$$CK(\delta, Fx_0) \subset \mathfrak{B}(\|Fx_0\|). \tag{5}$$

Then where $\gamma(\alpha) = \beta \in C$,

$$\|Fx_\alpha\|^2 \leqq \delta^2(|1-\beta| \, \|Fx_0\|) + (|\beta| \, \|Fx_0\|)^2 < \|Fx_0\|^2. \tag{6}$$

Thus $Fx_\alpha$ is a better approximation.

It is desirable to know for which $\delta$ there exist $C \neq \emptyset$

such that (5) holds. Before providing an answer, a lemma is needed.

Lemma 1. Let $\delta \in \mathbf{\Delta}$. There exist $\mu, \xi$ such that $0 \leq \mu < \infty$, $0 < \xi \leq 1$ and

$$\delta^2(\theta t^{\mu+1}) \leq \theta t^{\mu+2}, \qquad 0 \leq t \leq 1, \quad 0 \leq \theta \leq \xi$$

iff there exist $\mathcal{K}, \nu, \eta$ such that $0 < \mathcal{K} < \infty$, $\frac{1}{2} < \nu < \infty$, $0 < \eta$ and

$$\delta(s) \leq \mathcal{K} s^\nu, \qquad 0 \leq s \leq \eta.$$

Proof Necessity: Let $\mathcal{K}^2 = (\frac{1}{\xi})^{\frac{1}{\mu+1}}$, $\nu = \frac{\mu+2}{2(\mu+1)}$, $\eta = \xi$ and $s = \xi t^{\mu+1}$. Then $\mathcal{K}, \nu, \xi$ satisfy the appropriate inequalities and

$$\delta^2(s) \leq st = s(\frac{s}{\xi})^{\frac{1}{\mu+1}} = \mathcal{K}^2 s^{\frac{\mu+2}{\mu+1}}, \qquad 0 \leq s \leq \eta.$$

Equivalently,

$$\delta(s) \leq \mathcal{K} s^\nu, \qquad 0 \leq s \leq \eta.$$

Sufficiency: Suppose $\frac{1}{2} < \nu < 1$. Let $\mu = \frac{2(1-\nu)}{2\nu - 1}$ and $\xi = \min\{1, \mathcal{K}^{-2(\mu+1)}, \eta\}$. Then $\mu$ and $\xi$ satisfy the appropriate inequalities and for $0 \leq \theta \leq \xi$, $0 \leq t \leq 1$,

$$\delta^2(\theta t^{\mu+1}) \leq \mathcal{K}^2(\theta t^{\mu+1})^{2\nu} = \mathcal{K}^2 \theta^{2\nu-1} \cdot \theta t^{2\nu(\mu+1)}$$

$$= \mathcal{K}^2 \theta^{\frac{1}{\mu+1}} \theta t^{\mu+2} \leq \theta t^{\mu+2}.$$

Suppose $1 \leqq \nu$. Let $\mu = 0$ and $\xi = \min\{1, \mathcal{K}^{-2}, \eta\}$. Then for $0 \leqq \theta \leqq \xi$, $0 \leqq t \leqq 1$,

$$\delta^2(\theta t) \leqq \mathcal{K}^2(\theta t)^{2\nu} .$$

Since $t \leqq 1$, $\theta \leqq 1$ and $2\nu \geqq 2$,

$$\delta^2(\theta t) \leqq \mathcal{K}^2\theta\ \theta t^2 \leqq \theta t^2 .$$

<u>Proposition 3.</u> Let $\delta \in \mathbf{A}$.

(i) If $\delta(s) \leqq \mathcal{K} s^{\nu}$, $\mathcal{K} > 0$, $\nu > \frac{1}{2}$ and $0 \leqq s \leqq \eta$, $\eta > 0$, then there exists $C \subset \mathfrak{S}$ such that

$$CK(\delta, x) \subset \mathfrak{B}(\|x\|)$$

for each $x \neq 0$, $\|x\| \leqq 1$.

(ii) If $\delta(s) \geqq \mathcal{K} s^{\frac{1}{2}}$, $0 \leqq s$ and some $\mathcal{K}$, $0 < \mathcal{K} \leqq 1$, then there exists $x \neq 0$ such that

$$CK(\delta, x) \cap \mathfrak{B}(\|x\|) = \emptyset$$

for every $C$.

<u>Proof</u> Note that $CK(\delta, x) \subset \mathfrak{B}(\|x\|)$ iff

$$\delta^2(\|\beta x - x\|) + (|\beta|\ \|x\|)^2 < \|x\|^2$$

for each $\beta \in C$.

(i) Let $x \neq 0$ and $\|x\| \leqq 1$. Let $\mu, \xi$ be as in the lemma and

$$\beta = 1 - \theta \|x\|^{\mu}, \qquad 0 < \theta < \xi .$$

Then

$$\delta^2(\|\beta x - x\|) + (|\beta| \|x\|)^2 = \delta^2(\theta \|x\|^{\mu+1}) + (1-\theta\|x\|^{\mu})^2 \|x\|^2$$

$$\leqq (1-\theta\|x\|^{\mu}(1-\theta\|x\|^{\mu}))\|x\|^2 < \|x\|^2 .$$

Thus the assertion holds for

$$C = \{\beta : \beta = 1-\theta\|x\|^{\mu}, \quad 0 < \theta < \xi \} .$$

(ii) Let $K(x)$ be the $\delta'$-cone determined by $x$ and $\delta'(s) = \kappa s^{\frac{1}{2}}$. Then

$$K(x) \subset K(\delta, x).$$

Thus if the assertion is established for $K(x)$, it holds for $K(\delta, x)$. Let $x$ be such that $\|x\| = \lambda \kappa^2$, $0 < \lambda \leqq \frac{1}{2}$. Then

$$[\delta'(\|\beta x - x\|)]^2 + (|\beta| \|x\|)^2 = \|x\|^2 \left(\frac{|1-\beta|}{\lambda} + |\beta|^2\right)$$

$$\geqq \|x\|^2 (2|1-\beta| + |\beta|^2) .$$

If $|\beta| \geqq 1$, then

$$[\delta'(\|\beta x - x\|)]^2 + (|\beta| \; \|x\|)^2 \geqq \|x\|^2 .$$

If $|\beta| \leqq 1$, then

$$\begin{aligned} [\delta'(\|\beta x - x\|)]^2 + (|\beta| \; \|x\|)^2 &\geqq \|x\|^2(2(1-|\beta|) + |\beta|^2) \\ &= \|x\|^2(1+(1-|\beta|)^2) \\ &\geqq \|x\|^2 . \end{aligned}$$

This implies (ii) of the proposition.

In (i) of Proposition 3, it is required that $\delta(s) \leqq \mathcal{K} s^{\nu}$ for $0 \leqq s \leqq \eta$. This indicates that the ability to improve any approximation depends on the shape of the $\delta$-cone near the vertex. In (ii) of the proposition it is required that $\delta(s) \geqq \mathcal{K} s^{\frac{1}{2}}$ for $0 \leqq s$. This condition can be relaxed. To establish (ii) it is sufficient that $\delta(s) \geqq \mathcal{K} s^{\frac{1}{2}}$ for $0 \leqq s \leqq \eta$, $\eta > 0$. This fact also reflects the importance of the shape of the cone near the vertex.

In [24], Zarantonello examines a mapping $F$ which is $\Delta'$-monotone and $\Delta''$-cross continuous. He assumes that $\Delta' = \{\delta'\}$ and $\Delta'' = \{\delta''\}$ where $\delta'(s) = ds$ and $\delta''(s) = \mathcal{K} s^{\nu}$, $\nu \geqq 0$. He asserts [24, p. 15] that he could obtain results only for $\nu > \frac{1}{2}$. Part (ii) of Proposition 3 explains this fact as $F$ satisfies a $\Delta$-condition with $\Delta = \{\delta\}$ where $\delta(s) = \mathcal{K} d^{\nu} s^{\nu}$.

Propositions 2 and 3 answer the question concerning the ability to improve any approximation $x_0$ in the case that $\mathfrak{D}(F) = \mathfrak{H}$. Inequality (6) gives a measure of the improvement. In the next section, the effect on the theory of restricting the domain of $F$ is considered.

## §5. $\mathfrak{D}(F) = \overline{\mathfrak{S}}(\rho)$

If $\mathfrak{D}(F) = \overline{\mathfrak{S}}(\rho)$, then the domain of $\gamma$ and, hence, the range of $\gamma$ are restricted. In particular $\mathfrak{D}(\gamma)$ is determined by the condition that

$$x_\alpha = x_0 + \alpha F x_0 \in \overline{\mathfrak{S}}(\rho).$$

Equivalently,

$$\mathfrak{D}(\gamma) = \left\{ \alpha : \left| \alpha + \frac{(Fx_0, x_0)}{\|Fx_0\|^2} \right|^2 \leqq \frac{\rho^2 - \|x_0\|^2}{\|Fx_0\|^2} + \left| \frac{(Fx_0, x_0)}{\|Fx_0\|^2} \right|^2 \right\}.$$

If $\|x_0\| = \rho$ and $(Fx_0, x_0) = 0$, then $\mathfrak{D}(\gamma) = \{0\}$ which implies that $\mathfrak{R}(\gamma) = \{1\}$. This difficulty can be avoided by requiring that $\|x_0\| < \rho$.

It is desirable to find conditions such that $C \subset \mathfrak{R}(\gamma)$ $(C \subset \mathfrak{S})$ and

$$CK(\delta, Fx_0) \subset \mathfrak{S}(\|Fx_0\|)$$

where $\delta(s) \leqq \mathcal{K} s^{\nu}$, $0 \leqq s \leqq \eta$, $\nu > \frac{1}{2}$.

The set

$$C(\delta, x_0) = \{\beta : \delta^2(|\beta - 1| \, \|Fx_0\|) + (|\beta| \, \|Fx_0\|)^2 < \|Fx_0\|^2\}$$

is the largest subset of $\mathfrak{S}$ such that (5) holds. The continuity of $\delta$ implies that $C(\delta, x_0)$ is open. Proposition 3 implies that $C(\delta, x_0)$ contains an interval of the form $\{\beta : \beta = 1 - \theta \|Fx_0\|^{\mu},\ 0 < \theta < \xi\}$. However it has not been established that $C(\delta, x_0)$ is connected. Let $C^*(\delta, x_0)$ denote the component of $C(\delta, x_0)$ containing $\{\beta : \beta = 1 - \theta \|Fx_0\|^{\mu},\ 0 < \theta < \xi\}$.

If the approximate solution $x_0$ is required to be in a certain set, then an interesting result can be obtained concerning

$$C^*(\delta, x_0) \cap \mathfrak{R}(\gamma).$$

Let $F$ be $\Delta'$-monotone where $\delta' \in \Delta'$ corresponds to $[Fx_0]$ and let $\delta'$ be strictly increasing. Let

$$A = \{x : \|x\| < \rho,\quad \delta'(\|Fx\| + \|F(0)\|) \leqq \rho\}.$$

Elements of $A$ are called admissable approximations by Zarantonello [24].

Proposition 4. If $x_0 \in A$, then $C^*(\delta, x_0) \subset \mathfrak{R}(\gamma)$.

Proof Since $\|x_0\| < \rho$, 0 is an interior point of $\mathfrak{D}(\gamma)$. Therefore by the Brouwer Domain Invariance Theorem, 1 is an interior point of $\mathfrak{R}(\gamma)$. Thus

$$\mathfrak{R}(\gamma) \cap C^*(\delta, x_0) \neq \emptyset .$$

Let $\beta \in \mathfrak{R}(\gamma) \cap C^*(\delta, x_0)$ and $\gamma(\alpha) = \beta$. Then

$$\|x_\alpha\| \leqq \delta'(\|Fx_\alpha - F(0)\|) \leqq \delta'(\|Fx_\alpha\| + \|F(0)\|)$$

$$< \delta'(\|Fx_0\| + \|F(0)\|) \leqq \rho.$$

Hence $\alpha$ is an interior point of $\mathfrak{D}(\gamma)$ and by the Domain Invariance Theorem, $\beta$ is an interior point of $\mathfrak{R}(\gamma)$.

Since $C^*(\delta, x_0)$ and $\mathfrak{R}(\gamma)$ are connected and the boundary of $\mathfrak{R}(\gamma)$ does not intersect $C^*(\delta, x_0)$, the conclusion follows.

Corollary 1. If $x_0 \in A$ and $\gamma(\alpha) = \beta \in C^*(\delta, x_0)$, then $x_\alpha = x_0 + \alpha F x_0 \in A$.

Corollary 2. If $\|x_0\| < \rho$, then $\mathfrak{R}(\gamma) \cap C^*(\delta, x_0) \neq \emptyset$ .

Corollary 1 asserts that membership in A is inherited by improvements of approximations. This fact is useful in the discussion of iterative procedures. Corollary 2 insures that the approximate solution $x_0$ can be improved. However if $x_0 \notin A$, a more complicated set (i.e. the intersection) must be examined to find suitable C-cross sections.

The set $A$ contains only the solution to Equation (1) if $\delta'(\|F(0)\|) = \rho$. Thus to obtain a satisfactory set $A$, stronger conditions are placed on $F$.

We remark that if $\delta^2(\theta s) \leq \theta\delta^2(s)$ for each $s$, then it can be shown that $C(\delta, x_0)$ is connected and thus

$$C(\delta, x_0) = C^*(\delta, x_0) .$$

We end this section with an example of $C(\delta, x_0)$. Let $\delta(s) = s$. Then

$$C(\delta, x_0) = \{\beta : |\beta - \tfrac{1}{2}| < \tfrac{1}{2}\} .$$

Note that in this case $C(\delta, x_0)$ is a connected open set which is independent of $x_0$. Since this set is nice, the problem of finding, for example, $\gamma^{-1}(\frac{1}{2})$ can be done approximately. This example illustrates the desirability of having

$$C(\delta, x_0) \subset \mathcal{R}(\gamma).$$

# CHAPTER V

# ITERATION

## §1. Preliminary Remarks

In this chapter, an iterative procedure of the form (I, 6) is defined. A proposition stating sufficient conditions for the convergence of the iterates is given. Also an upper bound on the number of iterations required to obtain an accuracy $\epsilon > 0$ is given.

We continue considering an operator $F$ having its domain and range in a Hilbert space $\mathcal{H}$. In order to define the iterative procedure certain conditions on $F$ are assumed. The form of the conditions depends on whether $\mathcal{D}(F) = \mathcal{H}$ or $\mathcal{D}(F) = \overline{\mathcal{B}}(\rho)$. These conditions are:

(i) $F$ is s-w continuous ;

(ii) $F(0)$ is in the interior of $\mathcal{D}(F)$;

(iii) $F$ satisfies a $\Delta$-condition where the family $\{\delta_x : x \neq 0\}$ is bounded above by $\delta(s)$ and $\delta(s) \leq \mathcal{K} s^{\nu}$, $0 \leq s \leq \eta$, $\nu > \frac{1}{2}$ ;

(iv) if $\mathcal{D}(F) = \mathcal{H}$, then $F$ is $\Delta'$-monotone where the family $\{\delta'_x : x \neq 0\}$ is bounded above by $\delta'(t)$, $\delta'$ is strictly increasing and $(\delta')^{-1}(t) \to \infty$ as $t \to \infty$;

(v) if $\mathfrak{D}(F) = \overline{\mathfrak{G}}(\rho)$, then $F$ is $\Delta'$-monotone where the family $\{\delta'_x : x \neq 0\}$ is bounded above by $\delta'(t)$ and $\delta'$ is strictly increasing.

If $\mathfrak{D}(F) = \mathcal{H}$, we let $A = \mathcal{H}$. Condition (ii) implies that $\{x : Fx \neq 0\} \cap A \neq \emptyset$ for the cases $\mathfrak{D}(F) = \mathcal{H}$ and $\mathfrak{D}(F) = \overline{\mathfrak{G}}(\rho)$. Assumption (iii) implies that for each $x \in \{x : Fx \neq 0\} \cap A$ that

$$1 - \theta \|Fx\|^{\mu} \in C^*(\delta, x), \qquad 0 < \theta < \xi$$

where $\mu, \xi$ are given by Lemma (IV, 1). Define $\gamma : \mathfrak{S} \times [\mathfrak{D}(F) \cap \{x : Fx \neq 0\}] \to \mathfrak{S}$ by

$$\gamma(a, x)Fx = O_{Fx}(F(x + aFx)).$$

Then conditions (i) and either (iv) or (v) imply that

$$C^*(\delta, x) \subset \mathfrak{R}(\gamma(\cdot, x))$$

for each $x \in A \cap \{x : Fx \neq 0\}$.

Let

$$(1) \qquad \begin{cases} x_{k+1} = x_k + a_k Fx_k, & k \in \omega_0, \\ \\ (1 - \theta\sigma_k^{\mu})Fx_k = O_k(F(x_k + a_k Fx_k)), & k \in \omega_0 \end{cases}$$

where $\sigma_k = \|Fx_k\|$, $\theta$ is fixed, $0 < \theta < \xi$, and $O_k$ is the orthogonal projection of $\mathcal{H}$ onto $[Fx_k]$. Then if $x_0 \epsilon A \cap \{x : Fx \neq 0\}$, $\|Fx_0\| \leqq 1$, (1) defines an iterative sequence which is either finite or infinite. The sequence is finite iff there exists $k \epsilon \omega_0$ such that $Fx_k = 0$.

## §2. Convergence

From (IV, 6),

(2) $$\sigma_{k+1}^2 \leqq \delta^2(\theta\sigma_k^{\mu+1}) + (1-\theta\sigma_k^{\mu})^2\sigma_k^2 .$$

Without loss of generality, let $\sigma_k > 0$, $k \epsilon \omega_0$. Then (2) may be rewritten as

$$\sigma_{k+1}^2 \leqq \sigma_k^2 [ (\delta^2(\theta\sigma_k^{\mu+1})/\sigma_k^2) + (1-\theta\sigma_k^{\mu})^2 ] .$$

By induction,

(3) $$\sigma_{k+1}^2 \leqq \sigma_0^2 \prod_{j=0}^{k} [ (\delta^2(\theta\sigma_j^{\mu+1})/\sigma_j^2) + (1-\theta\sigma_j^{\mu})] .$$

Inequality (2) implies that $\sigma_{k+1} < \sigma_k$, $k \epsilon \omega_0$. Thus the sequence $\{\sigma_k : k \epsilon \omega_0\}$ converges. The following proposition shows that $\sigma_k \to 0$. This proposition is stated for mappings $F$ such that $\mathcal{D}(F) = \mathcal{H}$ or $\mathcal{D}(F) = \overline{\mathcal{B}}(\rho)$.

<u>Proposition 1.</u> Suppose that $F$ satisfies (i), (ii), (iii) and either

(iv) or (v). Let $x_0 \epsilon A$, $\|Fx_0\| \leqq 1$ and define the sequence $\{x_k : k \epsilon \omega_0\}$ by (1). Then $\sigma_k \to 0$ and there exists $y_0$ such that $x_k \to y_0$ and $Fy_0 = 0$.

<u>Proof</u> Assume that $\sigma_k \to \epsilon$, $\epsilon > 0$. By the monotonicity of $\{\sigma_k\}$,

$$\epsilon \leqq \sigma_k \leqq 1, \qquad k \epsilon \omega_0 .$$

By (3),

$$\sigma_{k+1}^2 \leqq \sigma_0^2 \prod_{j=0}^{k} [(\delta^2(\theta\sigma_j^{\mu+1})/\sigma_j^2) + (1-\theta\sigma_j^{\mu})^2]$$

$$\leqq \prod_{j=0}^{k} [1-\theta\sigma_j^{\mu}(1-\theta\sigma_j^{\mu})].$$

Now

$$r = \min\{|\theta\epsilon^{\mu}(1-\theta\epsilon^{\mu})|, \ |\theta(1-\theta)|\} > 0$$

and

$$1-\theta\sigma_j^{\mu}(1-\theta\sigma_j^{\mu}) \leqq 1-r, \qquad j \epsilon \omega_0 .$$

Thus $\sigma_{k+1}^2 \leqq (1-r)^k$, $k \epsilon \omega_0$ which implies that $\epsilon \leqq 0$. This contradiction implies that $\epsilon = 0$.

By either (iv) or (v),

$$\|x_k - x_j\| \leqq \delta'(\|Fx_k - Fx_j\|) \leqq \delta'(\sigma_k + \sigma_j).$$

Hence $\{x_k\}$ is Cauchy and there exists $y_0$ such that $x_k \to y_0$. Since $F$ is closed,

$$Fy_0 = 0.$$

<u>Corollary.</u> If $r = \min \{ |\theta \epsilon^{\mu}(1-\theta\epsilon^{\mu})|, \ |\theta(1-\theta)| \}$ and $k' = \min \{ k : (1-r)^k \leqq \epsilon^2 \}$, then $\sigma_{k'} \leqq \epsilon$ and $\|x_k - y_0\| \leqq \delta'(\epsilon)$.

The corollary gives an upper bound on the number of steps required to obtain an accuracy $\epsilon > 0$, i.e., $\|Fx_k\| \leqq \epsilon$. Note that if $\mu = 0$, then $r$ is independent of $\epsilon$.

# CHAPTER VI

# BANACH SPACE THEORY

## §1. Preliminary Remarks

In this chapter, results are obtained which can be used to generalize the theory given in Chapters IV and V to a Banach space setting. The geometry of the unit ball is examined more carefully in order to obtain conditions for the improvement of approximations.

Most of the concepts introduced in Chapter IV can be used in the present context. The differences between the Banach and Hilbert space settings are pointed out in the discussions below. As in Chapters IV and V, most of the analysis is devoted to geometric concepts and to improving approximations.

## §2. Improving Approximations

The problem to be discussed is the same one posed in Section (IV, 1). Let $R_x = P_x$ (or $Q_x$ if $\mathfrak{X}$ is strictly convex). Then to solve this problem when $\mathfrak{D}(F) = \mathfrak{X}$, the following conditions are assumed:

(i) $F$ is s-w continuous (continuous if $R_x = Q_x$);

(ii) $F$ satisfies a $\Delta$-condition relative to $\{R_x : x \neq 0\}$

with $\delta \in \Delta$ corresponding to $[Fx_0]$;

(iii) $F$ is $\Delta'$-monotone relative to $\{R_x : x \neq 0\}$ with $\delta' \in \Delta'$ corresponding to $[Fx_0]$ where $\delta'$ is strictly increasing and $(\delta')^{-1}(t) \to \infty$ as $t \to \infty$.

Given $\delta \in \Delta$, $x \in \mathfrak{X}$ and $C \subset \mathfrak{S}$, the $\delta$-cone and C-cross section are defined as in Section (IV, 2) (substituting $P_x$ or $Q_x$ for $O_x$). The major difference in the theory occurs in the condition used to show that a C-cross section is contained in $\mathfrak{B}(\|x\|)$. Inequality (I, 7) or (I, 8) is used in the derivation of a general condition which implies that

(1) $$C\,K(\delta, x) \subset \mathfrak{B}(\|x\|)$$

for some $C \subset \mathfrak{S}$. This general condition is

(2) $$\delta(|1-\beta| \; \|x\|) + |\beta| \; \|x\| < \|x\| .$$

There exists $\beta$ such that (2) holds if $\delta(s) \leq \kappa s$, $0 \leq s \leq \eta$, $\kappa < 1$ (if $\mathfrak{X}$ is strictly convex $\kappa \leq 1$ suffices). The use of the triangle inequality in the derivation of (2) reflects the "worst possible shape" of $\mathfrak{B}(\|x\|)$ at $x$. This shape occurs, for example, on the unit ball of $\ell_1$ at $x = (1, 0, 0, \cdots)$. If $P_x(\cdot) = x^*(\cdot)x/\|x\|$ and $x^* \in (\ell_1)^*$ corresponds to $(1, 0, 0, \cdots) \in \ell_\infty$, then $y = (y_1, y_2, \cdots) \in K(\delta, x)$ iff

$$\sum_{i>1} |y_i| \leq \delta(|y_1 - 1|).$$

Hence

(3) $$\|y\| \leq \delta(|y_1 - 1|) + |y_1|.$$

Inequality (3) implies that $\|y\| < 1$ if $\delta(s) \leq \mathcal{K} s$, $0 \leq s \leq \eta$, $\mathcal{K} < 1$ and $0 \leq |y_1 - 1| \leq \eta$.

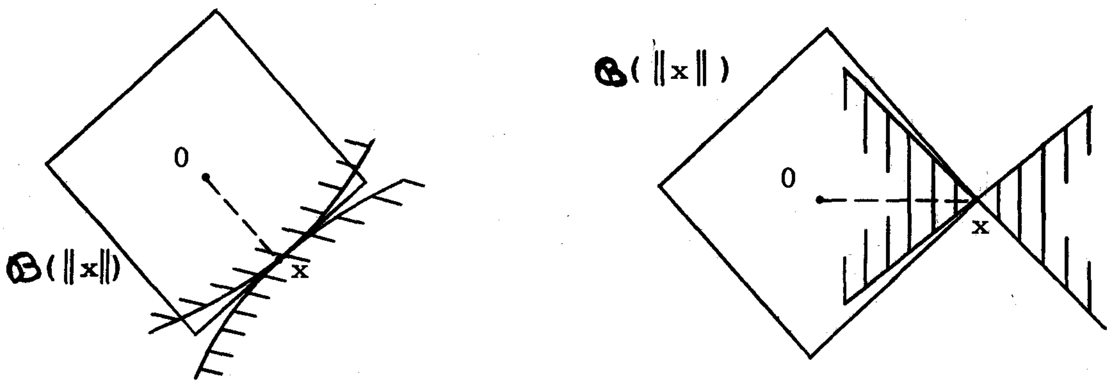
Further examples show that the worst possible shape does not, in general, occur at every point on the boundary of $\mathcal{B}(\|x\|)$. Let $\mathfrak{X} = \ell_p$, $p > 1$, and $x = (x_1, 0, 0, \cdots)$. Then $y \in K(\delta, x)$ iff

$$\|y\|^p \leq \delta^p(\|P_x y - x\|) + \|P_x y\|^p .$$

If $\delta(s) \leq \mathcal{K} s^\nu$, $0 \leq s \leq \eta$, $\nu > \frac{1}{p}$, then for each $x_1 \neq 0$, there exists $C \neq \emptyset$ such that (1) holds. For large $p$, this condition on $\delta$ begins to look like

$$\delta(s) < \mathcal{K}, \qquad 0 \leq s \leq \eta .$$

Figure 3 illustrates the "best" and the worst possible shape of $\mathcal{B}(\|x\|)$ at $x$.




(a) (b)

Figure 3

The shape of $\mathfrak{B}(\|x\|)$ is described further in §3 below.

The function $\gamma: \mathfrak{S} \rightarrow \mathfrak{S}$ is defined as in Section (IV, 3) (replacing $O_0$ by $P_0$ or $Q_0$). Using (i) and (iii), the results of Section (IV, 3) follow in a Banach space setting. Hence, we may state the following proposition which summarizes the above remarks.

<u>Proposition 1.</u> Let $F: \mathfrak{X} \rightarrow \mathfrak{X}$ have $\mathfrak{D}(F) = \mathfrak{X}$.

Suppose that $F$ satisfies (i), (ii) and (iii) and there exists $C \subset \mathfrak{S}$ such that

$$CK(\delta, Fx_0) \subset \mathfrak{B}(\|Fx_0\|).$$

Let $\gamma(\alpha) = \beta \in C$ and $x_\alpha = x_0 + \alpha F x_0$. Then

$$\|Fx_\alpha\| < \|Fx_0\|.$$

Thus $x_\alpha$ is a better approximation.

This proposition parallels Proposition (IV, 2). A measure of the improvement (cf. (IV, 6)) is given in §3.

## §3. Further Banach Space Geometry

In this section conditions weaker than (2) which imply (1) are given. The investigation yields a measure for the improvement.

Consider the following figure.

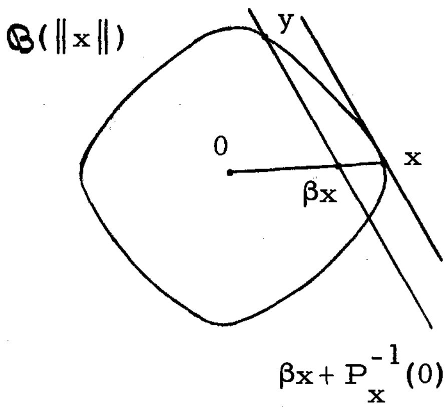



Figure 4

This figure illustrates one of the problems encountered in Banach spaces. That is, if $y \in \mathcal{B}(\|x\|)$, then it may be the case that $\beta x-(y-\beta x) \notin \mathcal{B}(\|x\|)$. This asymmetry is dealt with by means of the following function. For each $[x]$, $x \neq 0$, define $\psi_x : \{\beta : |\beta| \leqq 1\} \rightarrow \mathcal{S}$ by

$$\|x\| \; \psi_x(\beta) = \inf\{\, \|(I-P_x)y\| : P_x(y) = \beta x, \quad \|x\| = \|y\| \,\}.$$

(If $\mathfrak{X}$ is strictly convex, $P_x$ may be replaced by $Q_x$). Note that $\psi_x(\beta) \geqq 1 - |\beta|$. The following propositions indicate the use of $\psi_x$.

<u>Proposition 2.</u> Let $\delta(|1-\beta| \, \|x\|) < \|x\| \psi_x(\beta)$ for $\beta \in C \subset \{\beta : |\beta| < 1\}$. Then

$$CK(\delta, x) \subset \mathfrak{B}(\|x\|).$$

<u>Proof</u> Let $y \in K(\delta, x)$ and $P_x y = \beta x$, $\beta \in C (Q_x y = \beta x$, $\beta \in C$ if $\mathfrak{X}$ strictly convex). Suppose $\|y\| \geq \|x\|$. Then there exists $s$, $0 < s \leqq 1$ such that $\|y'\| = \|x\|$ where

$$y' = sy + (1-s)\beta x .$$

Now $P_x y' = \beta x$. (Note that (iii) and (iv) of Section (I, 2) imply $Q_x y' = \beta x$ if $\mathfrak{X}$ strictly convex.) Hence

$$\|x\| \psi_x(\beta) \leqq \|y' - \beta x\| = s \, \|y - \beta x\| < s \, \|x\| \psi_x(\beta).$$

This contradiction implies $\|y\| < \|x\|$ and

$$CK(\delta, x) \subset \mathfrak{B}(\|x\|).$$

<u>Proposition 3.</u> Let $\delta(|1-\beta| \, \|x\|) \leqq \mathcal{K} \|x\| \psi_x(\beta)$, $\mathcal{K} < 1$, for $|\beta| < 1$. Then $y \in \{\beta\} K(\delta, x)$ implies that

(4) $$\|y\| \leqq [\mathcal{K} + (1-\mathcal{K}) \, |\beta|] \, \|x\| .$$

Proof Let $y \in \{\beta\} K(\delta, x)$. Then $\|y\| < \|x\|$. Choose $s > 0$ such that $\|y'\| = \|x\|$ where

$$(5) \qquad y' = s(y - \beta x) + \beta x.$$

Since $P_x y' = \beta x$ (note that $Q_x y' = \beta x$ by (iii) and (iv) of Section (I, 2)),

$$\|x\| \psi_x(\beta) \leqq \|y' - \beta x\| \leqq s\|y - \beta x\| \leqq s\mathcal{K}\|x\|\psi_x(\beta).$$

Hence $1/s \leqq \mathcal{K}$.

By (5),

$$y = \frac{1}{s} y' + (1 - \frac{1}{s}) \beta x .$$

Thus

$$\begin{aligned} \|y\| &\leqq [\frac{1}{s} + (1 - \frac{1}{s})|\beta|] \ \|x\| \\ &\leqq [\mathcal{K} + (1 - \mathcal{K})|\beta|] \ \|x\| . \end{aligned}$$

Formula (4) provides a measure for the error.

Using the ideas introduced above, the theory could now be developed paralleling Section (IV, 5). In particular,

$$C(\delta, x_0) = \{\beta : \delta(|1-\beta| \ \|Fx_0\|) < \|Fx_0\| \psi_{Fx_0}(\beta)\} .$$

Extending the above ideas, an iterative procedure can be defined and shown to converge. We observe that to carry out this procedure, the family $\{\delta_x : x \neq 0\}$ given by the $\Delta$-condition on $F$ does not need to be uniformly bounded above by a particular $\delta \in \Delta$. Instead, each $\delta_x \in \Delta$ must be bounded above by $\psi_x$ which may vary considerably for each $[x]$, $x \neq 0$ (cf. Figure 3). It is in the Banach space setting that the definitions given in Section (I, 3) are used in their greatest generality.

It is interesting to note that if $\mathfrak{X}$ is a Hilbert space, then the above notions yield the theory of Chapter IV. For example,

$$\psi_x(\beta) = \sqrt{[1-|\beta|^2]}$$

for each $[x]$, $x \neq 0$. Thus $C(\delta, x_0)$ is as defined in Section (IV, 5). Further the requirement that

$$\delta(\|x\|t) < \|x\|\psi_x(1-t) = \|x\|\sqrt{[2t-t^2]}, \qquad 0 \leqq t \leqq \eta,$$

for each $x$, $0 < \|x\| \leqq 1$, implies that $\delta(t) \leqq \mathcal{K}\, t^{\nu}$, $0 \leqq t \leqq \eta$, $\nu > \frac{1}{2}$.

Another example which is of interest is $\mathfrak{X} = C[0,1]$, the Banach space of continuous functions on the unit interval with the supremum norm. Let $e_t \in C^*[0, 1]$ be defined by $e_t(x) = x(t)$, $x \in C[0,1]$. Let $x \neq 0$ and $t \in \{t: |x(t)| = \|x\|\}$. Then

$$P_t(\cdot) = e_t(\cdot)x/\|x\| \tag{6}$$

is a projection of norm one onto $[x]$. Simple arguments show that $\psi_x(1-t) = t$, $0 \leqq t \leqq 1$. Thus, using the set of projections given by (6), the ball $\mathcal{B}(\|x\|)$ has the worst possible shape at every point of its boundary.

## §4. Further Remarks

We note that the conditions which imply that (1) holds are not related to the study of $Fx = 0$. They yield some insight into the shape of the unit ball in two dimensional subspaces of $\mathfrak{X}$.

From these conditions, finer estimates than those obtained by using the triangle inequality can be realized. That is, if

$$y = (I-P_x)y + P_x y,$$

then the triangle inequality yields

$$\|y\| \leqq \|(I-P_x)y\| + \|P_x y\|.$$

We are not able to compare $\|y\|$ and $\|x\|$ by means of this inequality. If

$$\|(I-P_x)y\| \leqq \|x\| \psi_x(\beta) \tag{7}$$

where $P_x y = \beta x$, then $\|y\| \leqq \|x\|$. Thus (7) yields a finer estimate of $\|y\|$ than the triangle inequality gives. This may prove useful in applications, since vectors are often decomposed in terms of their projections onto subspaces.

In the development of the theory, uniform convexity was not useful since it is stated in terms of the norms of two vectors, the norm of their sum and the norm of their difference. In our discussion, these norms are $\|P_x y\|$, $\|(I-P_x)y\|$ $\|y\|$ and $\|P_x y-(I-P_x)y\|$. Figure 4 illustrates the difficulties which may be encountered.

Finally we note that the theory can be applied in special cases of mappings $F$ having $\mathfrak{D}(F) \subset \mathfrak{X}$ and $\mathfrak{R}(F) \subset \mathfrak{Y}$. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be such that there exists a continuous one-to-one mapping $J$ of $\mathfrak{Y}$ into $\mathfrak{X}$. Instead of considering recursion relations of the form (I, 6), we consider the relations,

$$(8) \qquad \begin{cases} x_{k+1} = x_k + a_k JF x_k, & k \in \omega_0, \\ a_k = \zeta(x_k), & k \in \omega_0 . \end{cases}$$

The development is the same as above if $JF$ satisfies the usual conditions. In the case that $\mathfrak{X} = \mathfrak{Y}$, the effect is to compare $F$ with $J$ instead of the identity $I$.

The geometrical ideas presented above are an accurate picture of what occurs when iterative procedures of the form (I, 2), (I, 3), (I, 5), (I, 6) or (8) are used for operators which are even Lipschitz continuous and monotone.

# EPILOGUE

Although this report is essentially theoretical, we should like to mention some examples. As indicated in Section (I, 1), the origin of the theory was in problems of conformal mapping and fluid flow [ 5; 6; 25 ]. Other applications [ 11; 12; 13 ] have been to existence and uniqueness problems for solutions of elliptic partial differential equations.

The theory developed by quite a number of investigators over the past few years is now in a reasonably definitive state. Thus, in the future, the emphasis should be on further applications.

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