ON SYSTEMS OF BOOLEAN EQUATIONS

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CHAPTER I

INTRODUCTION

In a solvable Boolean equation or system of Boolean equations each unknown has in general more than one value. Whitehead (11) gives necessary and sufficient conditions that the solution be unique. However, Whitehead's proofs were limited to Boolean equations in two variables. Bernstein (1) in a later paper presented a general proof of Whitehead's theorem. But unfortunately the proof relies on an intuitional approach. In this paper we present a proof of Whitehead's theorem by mathematical induction. In another paper by Bernstein and Parker (2) the conditions are re-stated in a slightly different form. We generalize Whitehead's theorem to yield all of the solutions to a system of Boolean equations.

One major difficulty with Whitehead's or Bernstein's conditions is that they are based on the disjunctive canonical form of a Boolean representation. This canonical form has 2^n terms in its representation whenever n independent variables are present. And since these conditions depend upon the coefficients of the terms appearing in the canonical form all of the 2^n terms must be examined. Of course for large n this procedure would be most undesirable.

The main purpose of this paper is to develop an algorithm by which the unique solution to a system of Boolean equations can be found without appealing to the disjunctive canonical form.

In Chapter VI we discuss some applications of the results of this investigation to the solution and construction of inferential problems.

CHAPTER II

FUNDAMENTAL PROPERTIES

Before proceeding with the main part of this discussion it is necessary that we define some fundamental terminology. It will be assumed that the reader is familiar with the basic properties of a Boolean algebra. (5) Some of the basic properties are listed below for reference.

One of the simplest Boolean algebras is that defined on the set $\{0, 1\}$ with addition, $+$, and multiplication, $\cdot$, defined by the tables,

(2.0)

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

In practice the $\cdot$ may be omitted and thus $a \cdot b$ becomes ab. This particular Boolean algebra has been quite useful in applications to switching network theory and to the solution of certain logical problems. The remainder of this discussion will be devoted exclusively to it.

(2.1) We list some of the properties of the Boolean algebra defined on the set $\{0, 1\}$. If a, b and c are arbitrary elements of the set then,

P1. There is an element 0 in the set such that $a+0 = a$ for every a in the set.

P2. There is an element 1 in the set such that $a \cdot 1 = a$ for every a in the set.

P3. $a+b = b+a$

P4. $ab = ba$

P5. $a+bc = (a+b)(a+c)$

P6. $a(b+c) = ab+ac$

P7. For every a in the set there is an element a' in the set such that,

$$aa' = 0 \quad \text{and} \quad a+a' = 1$$

P8. $a+a = a$ and $a \cdot a = a$

P9. $a+1 = 1$ and $a \cdot 0 = 0$

P10. $a+ab = a$ and $a(a+b) = a$

P11. $a+(b+c) = (a+b)+c$ and $a(bc) = (ab)c$

P12. $(a')' = a$

P13. $a+b = (a'b')'$ and $ab = (a'+b')'$

P14. $0' = 1$ and $1' = 0$

(2.2) A variable a_i is a Boolean variable iff its range is the set $\{0, 1\}$.

(2.3) Let S be a set of Boolean variables and let a be an element of S. We say that a is an <u>independent</u> Boolean variable, relative to S, iff a may assume the values 0 and 1 independently of the values assumed by any other element of S. A symbol representing an independent Boolean variable will be referred to as a <u>letter</u>.

(2.4) The element a' of property P7 is called the <u>dual</u> of a.

(2.5) The set of Boolean functions of n independent Boolean variables $\{x_1, x_2, \cdots, x_n\}$ will be called the finite Boolean algebra B_n. Combining the independent Boolean variables and their duals by the properties (2.1) yields various representations of the elements of B_n.

(2.6) Two representations of elements from B_n are said to be equal iff they have the same value for every set of values of the independent variables.

(2.7) A difficulty arises here in the use of the symbol "=". We may write $x=1$ in which case we usually mean that the Boolean variable x has the value 1. Or we may write $f(x_1,x_2,\cdots,x_n) = 0$ in which case the question, "for what values of the independent variables $\{x_1,x_2,\cdots,x_n\}$ does $f(x_1,x_2,\cdots,x_n)$ have the value 0?" is implied. This is a "conditional" equality. Another example of a conditional equality might be $f(x_1,x_2,\cdots,x_n) = g(x_1,x_2,\cdots,x_n)$. However, we could mean by this exactly the definition (2.6). Finally when the definition (2.6) is implied we may sometimes write $f(x_1,x_2,\cdots,x_n) \equiv g(x_1,x_2,\cdots,x_n)$ to emphasize that (2.6) is being used.

We have this same difficulty in the algebra of real numbers and it is resolved by letting the context of the statement involved imply the use of the symbol. Throughout the rest of this discussion we shall "resolve" the difficulty in precisely the same way.

(2.8) Let $f(x_1,x_2,\cdots,x_n)$ be an arbitrary member of B_n. The Boolean function,

$$x_n\, f(x_1,x_2,\cdots,x_{n-1},1) + x_n'\, f(x_1,x_2,\cdots,x_{n-1},0)$$

is called the <u>expansion</u> of $f(x_1,x_2,\cdots,x_n)$ about the independent variable x_n.

(2.9) $$f(x_1,x_2,\cdots,x_n) \equiv x_n f(x_1,x_2,\cdots,x_{n-1},1) + x_n' f(x_1,x_2,\cdots,x_{n-1},0)$$

Proof:

Evaluate both functions for $x_n = 0$ and then $x_n = 1$. For $x_n = 0$, $f(x_1,x_2,\cdots,x_n)$ is $f(x_1,x_2,\cdots,x_{n-1},0)$ and $x_n f(x_1,x_2,\cdots,x_{n-1},1) + x_n' f(x_1,x_2,\cdots,x_{n-1},0)$ becomes $f(x_1,x_2,\cdots,x_{n-1},0)$.

If $x_n = 1$, then we see that,

$$f(x_1,x_2,\cdots,x_{n-1},1) \equiv 1\cdot f(x_1,x_2,\cdots,x_{n-1},1) + 0\cdot f(x_1,x_2,\cdots,x_{n-1},0)$$

(2.91) The elements of the set of Boolean variables $\{x_1,x_2,\cdots,x_n,x_1',x_2',\cdots,x_n'\}$ are called the literals of B_n.

(2.92) A <u>monomial</u> in B_n is the Boolean product of literals of B_n such that no <u>letter</u> occurs more than once.

(2.93) A Boolean <u>polynomial</u> is the Boolean constant 1 or the Boolean sum of a set of monomials of B_n. If the set of monomials is empty then the polynomial is the Boolean constant 0.

(2.94) By the use of the properties (2.1), particularly

P5 and P6, every member of B_n can be expressed as a polynomial.

(2.95) In what follows we shall drop the prefix "Boolean". For example we shall write "polynomial" instead of "Boolean polynomial" or "variable" for "Boolean variable".

(2.96) If a, b and c are variables such that,

$$a = bc$$

then a is less than or equal to b ($a \leq b$, $b \geq a$) and a is less than or equal to c ($a \leq c$, $c \geq a$).

(2.97) $$a+b+bc \equiv a+b$$

Proof: By property P10 $b+bc = b$.
We shall call this procedure a <u>deletion</u>. The performance of all such deletions on a given polynomial shall be called a <u>deletion iteration</u>.

(2.98) If a, b, c and d are Boolean variables and x is an independent variable such that

$$a = cx \quad \text{and} \quad b = dx'$$

then the variable cd is called the <u>consensus</u> of a and b.

(2.99) Let Y be a polynomial in B_n and aB and aC be monomials in B_n, where a is an independent variable in B_n. Then,

$$Y + aB + a'C \equiv Y + aB + a'C + BC$$

Proof: $Y + aB + a'C + BC \equiv Y + aB + a'C + (a+a')BC$

$$Y + aB + a'C + (a+a')BC \equiv Y + aB + a'C + aBC + a'BC$$

$$Y + aB + a'C + aBC + a'BC \equiv Y + aB + a'C$$

by (2.1) and (2.97). We note that BC is the consensus of aB and $a'C$.

(2.991) We shall call the process of making every possible consensus a <u>consensus iteration</u>.

CHAPTER III

CANONICAL FORMS

By (2.6) we see that we may determine when two functions in the same number of independendt variables are equal simply evaluating each of them for every set of values of the independent variables $\{x_1, x_2, \cdots, x_n\}$. This method is usually called the truth table method. Although this can be a useful method, it is sometimes more advantageous to be able to determine equality of Boolean functions by direct inspection of the representations of the two functions.

We employ here certain transformations such that when applied to a member of B_n they transform each equal representation of a function into a unique representation of the function. These unique representations are called Boolean canonical forms. In what follows we shall make use of three standard canonical forms.

(3.0) Let $f(x_1, x_2, \cdots, x_n)$ be an element of B_n. The conjunctive canonical form of $f(x_1, x_2, \cdots, x_n)$ is,

$$f(x_1,x_2,\cdots,x_n) \equiv \prod_{i_1=0}^{i} \cdots \prod_{i_n=0}^{i} (f(1^{i_1},\cdots,1^{i_n}) + x_1^{1-i_1} + \cdots + x_n^{1-i_n})$$

where $a_j^i = a_j$ if $i = 0$ and $a_j^i = a_j'$ if $i = 1$.
The proof that this form is canonical is well known, e.g., Rosenbloom (9). This canonical representation is the product of sums of literals such that in each sum no letters appear more than once.

(3.1) The terms of the conjunctive canonical form are called <u>max-terms</u>. Max-terms that contain the Boolean constant 1 are dropped from the representation.

(3.2) As an example suppose that,

$$f(x_1,x_2) = x_1 x_2'$$

then the conjunctive canonical form of $f(x_1,x_2)$ is,

$$f(x_1,x_2) \equiv (x_1' + x_2')(1+x_1'+x_2)(x_1+x_2')(x_1+x_2)$$
$$\equiv (x_1' + x_2')(x_1+x_2')(x_1+x_2).$$

Since $f(0,0) = f(0,1) = f(1,1) = 0$ and $f(1,0) = 1$.

(3.3) An <u>alterm</u> is a sum of literals such that no letter appears more than once in the sum.

(3.4) By a <u>conjunctive</u> form of a given Boolean function we shall mean a product of alterms.

(3.5) Let $f(x_1,x_2,\cdots,x_n)$ be an element of B_n. The disjunctive canonical form of $f(x_1,x_2,\cdots,x_n)$ is,

$$f(x_1,x_2,\cdots,x_n) \equiv \sum_{i_1=0}^{1} \cdots \sum_{i_n=0}^{1} (f(1^{i_1},1^{i_2},\cdots,1^{i_n})\cdot x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})$$

Where $x_j^i = x_j$ if $i = 0$ and $x_j^i = x_j'$ if $i = 1$.

The proof that this representation is canonical is also well known (9). We note that this representation is a polynomial in which each monomial contains n distinct letters. Monomials with 0 coefficients are dropped from the representation.

(3.6) The monomials of the disjunctive canonical form are called <u>min-terms</u>.

(3.7) As an example suppose that,

$$f(x_1,x_2) = x_1 + x_2'$$

then the disjunctive canonical form of $f(x_1,x_2)$ is,

$$f(x_1,x_2) \equiv x_1x_2f(1,1) + x_1x_2'f(1,0) + x_1'x_2f(0,1) + x_1'x_2'f(0,0)$$

$$f(x_1,x_2) \equiv x_1x_2 + x_1x_2' + x_1'x_2'$$

by the properties (2.1) and since $f(1,1)=f(1,0)=f(0,0)=1$ and $f(1,0)=0$.

(3.8) A transformation which will yield the disjunctive canonical form of a member of B_n is obtained by following the rules:

i. Transform the given function to a polynomial;

ii. Multiply each monomial of the polynomial by $(x_i + x_i')$ for all independent variables x_i such that neither x_i or x_i' are factors of the monomial. See Witcraft (12).

(3.9) As an example suppose that,

$$f(x_1,x_2) = x_1 + x_2'$$

then applying (3.8),

$$f(x_1,x_2) \equiv x_1(x_2 + x_2') + x_2'(x_1 + x_1'),$$

$$f(x_1,x_2) \equiv x_1x_2 + x_1x_2' + x_1'x_2' .$$

(3.91) There is a useful relationship between the disjunctive and conjunctive representations of a Boolean function. Namely, the conjunctive (disjunctive) canonical form is the product (sum) of up to 2^n max-terms (min-terms). The conjunctive (disjunctive) canonical form of the dual of a Boolean function is the product (sum) of max-terms (min-terms) of the set complement of the conjunctive (disjunctive) canonical form of the given Boolean function relative to the set of 2^n max-terms (min-terms) of $B_n(9)$.

(3.92) As an example suppose we have,

$$f(x_1, x_2) = x_1'x_2,$$

and want the conjunctive canonical form of $f(x_1, x_2)$. By (3.9),

$$f'(x_1, x_2) \equiv x_1x_2 + x_1x_2' + x_1'x_2'.$$

By the properties P12 and P13 we have,

$$f(x_1, x_2) \equiv (f'(x_1, x_2))' = (x_1' + x_2')(x_1' + x_2)(x_1 + x_2),$$

and we have the conjunctive canonical form of $f(x_1, x_2)$.

(3.93) For our purposes we shall write the disjunctive canonical form of $f(x_1, x_2, \cdots, x_n)$ in a less concise way, i.e.,

$$f(x_1, x_2, \cdots, x_n) = a_1x_1x_2\cdots x_n + a_2x_1x_2\cdots x_{n-1}'x_n + \cdots + a_{2^n}x_1'x_2'\cdots x_n'$$

where it is understood that every combination of x_j and x_j' occurs. $(i=1,2,\cdots,2^n, j=1,2,\cdots,n)$.

(3.94) We shall call the a_i the <u>discriminants</u> of the disjunctive canonical form of a function. In what follows we shall reserve the symbols a,b and c to represent discriminants.

(3.95) We shall use the notation Q_n or $Q(x_1,x_2,\cdots,x_n)$ to represent the disjunctive canonical form of a given function in n independent variables.

(3.96) By (2.9) we may write,

$$Q_n \equiv x_n Q(x_1,x_2,\cdots,x_{n-1},1) + x_n' Q(x_1,x_2,\cdots x_{n-1},0)$$

and we note that $Q(x_1,x_2,\cdots,x_{n-1},1,)$ and $Q(x_1,x_2,\cdots,x_{n-1},0)$ are disjunctive canonical forms of functions in n-1 independent variables. We will let b_j and c_j, respectively, be the discriminants of $Q(x_1,x_2,\cdots,x_{n-1},1)$ and $Q(x_1,x_2,\cdots,x_{n-1},0)$. $j = (1,2,\cdots,2^{n-1})$. In general the b_j are not the same as the c_j but every b_j and c_j is an a_i.

(3.97) Another important canonical form is that due to Quine (7), (8) or Samson and Mills (10). We shall refer to it as Quine's canonical form.

Given an element of B_n we first reduce the element to a polynomial. Then there are three conditions that a Boolean polynomial must satisfy in order that it may be said to be in Quine's canonical form.

Condition 1.

If A and B are two monomials of a representation P such that $A \geq B$, then the monomial B is to be deleted from the expression.

Condition 2.

If A^{x} (x an independent variable) and $B^{x'}$ are two monomials of P and AB satisfies the following two conditions,

i. The monomial A does not contain as a factor the dual of any independent variable contained as a factor in B;

ii. the product AB is not less than or equal to any monomial C of D;

then the monomial AB is to be added to the representation of P.

Condition 3.

If the monomials x and x' occur in a representation (x an independent variable), the polynomial is the Boolean constant 1.

That the form is canonical is proved by Quine (7), (8) and Laxdal (6).

(3.98) The monomials of Quine's canonical form of a polynomial P are called the <u>prime implicants</u> of P.

(3.99) The following algorithm will always insure that a polynomial satisfies Quine's conditions. See Laxdal (6).

Begin with a deletion iteration and then follow with a consensus iteration which is followed by a check to see if the monomials x and x' both appear (x an independent Boolean variable). If both appear the Boolean constant 1 is the canonical representation. If not, follow with a deletion iteration.

This algorithm has been mechanized for a digital computer by both Laxdal (6) and Witcraft (12).

(3.991) The following useful theorem is due to Ghazala (4).

If, given some conjunctive form of the polynomial f,

i. we perform the indicated products by the use of properties (2.1);

ii. drop all products xx' from the resultant representation;

iii. perform a deletion iteration on the resulting polynomial;

then the new representation is Quine's canonical form of f.

CHAPTER IV

PROCEDURES BASED ON THE DISJUNCTIVE CANONICAL FORM

In what follows we shall use the symbols f_i and g_i to represent Boolean functions in the n independent variables $\{x_1,x_2,\cdots,x_n\}$. $(i = 1,2,\cdots,m)$.

Given a system of m equations in n independent variables,

$$(4.0) \qquad \begin{cases} f_1 = g_1 \\ f_2 = g_2 \\ \quad \vdots \\ f_m = g_m \end{cases}$$

if there is a set of values of the independent variables such that $f_i = g_i$ reduces to $0 = 0$ or $1 = 1$ for every i, then the system (4.0) is said to have a <u>solution</u>.

(4.1) Any Boolean equation,

$$f = g$$

can be written in the form,

$$fg' + f'g = 0 .$$

(4.2) By (4.1) the system (4.0) can be written,

$$\begin{cases} f_1 g_1' + f_1' g_1 = 0 \\ f_2 g_2' + f_2' g_2 = 0 \\ \quad \cdots \\ \quad \cdots \\ f_m g_m' + f_m' g_m = 0. \end{cases}$$

(4.3) If there is a solution to the system (4.2) then the left hand sides of each equation must have the value 0 for some set of values of the independent variables. Under this condition the system (4.2) is equivalent to,

$$H_n = \sum_{i=1}^{n} (f_i g_i' + f_i' g_i) = 0,$$

where H_n is a function of n independent variables.

(4.4) We see that any solution to the system (4.3) is a solution to the system (4.2) and any solution to the system (4.2) is a solution to the system (4.0). Therefore, to solve the system (4.0) we need only solve the single equation (4.3). We proceed by reducing H_n to the canonical form Q_n. And we shall investigate under what conditions does $H_n \equiv Q_n = 0$ have a solution.

(4.5) Given the equation,

$$a_1x_1 + a_2x_1' = 0,$$

where x_1 is an independent variable and a_1, a_2 are discriminants, a necessary and sufficient condition that the equation have a solution is,

$$a_1a_2 = 0.$$

Proof: The condition is sufficient, for if $a_1 = a_2 = 0$

then, $a_1x_1 + a_2x_1' = 0x_1 + 0x_1' \equiv 0$

and any value of x_1 is a solution.

If $a_1 = 0,\ a_2 = 1$ then,

$$a_1x_1 + a_2x_1' = 1 \cdot x_1 + 0 \cdot x_1' = x_1' = 0$$

and $x_1 = 0$ is a solution. By symmetry if $a_1 = 1, a_2 = 0$ then $x_1 = 1$ is a solution.

The condition is necessary for if $a_1 = a_2 = 1$ then,

$$a_1x_1 + a_2x_1' = 1 \cdot x \ + 1 \cdot x_1' = x_1 + x_1' = 0$$

which is not satisfied for any value of x_1 since,

$$x_1 + x_1' \equiv 1 .$$

(4.6) The following theorem is due to Whitehead (11). We offer an alternative proof.

Theorem 1:

A necessary and sufficient condition that there exists a solution to $Q_n = 0$ is,

$$a_1 a_2 \cdots a_{2^n} = 0.$$

Proof: By induction on the number of independent variables n.

For $n=1$ the conjecture is valid by (4.5). Suppose that it is valid for some $n=k$. We expand Q_{k+1} about the variable x_{k+1} by (2.9.)

$$Q(x_1, x_2, \cdots, x_{k+1}) \equiv x_{k+1} Q(x_1, x_2, \cdots, x_k, 1) + x'_{k+1} Q(x_1, x_2, \cdots, x_k, 0).$$

If $a_1 a_2 \cdots a_{2^{k+1}} = 0$ then either a $b_j = 0$ and all $c_j=1$ or a $c_j=0$ and all $b_j=1$ or finally some $b_j=0$ and some $c_j=0$ at the same time.

Case 1. A b_j and a c_j are 0.

Then by the induction hypothesis $Q(x_1, x_2, \cdots, x_k, 1)$ and $Q(x_1, x_2, \cdots, x_k, 0)$ are 0 for some set of values of the variables $\{x_1, x_2, \cdots, x_n\}$ (not necessarily the same set).

In either case we may choose a value of x_{k+1} such that Q_{k+1} is 0.

Case 2. A b_j is 0 but no c_j is 0, then $Q(x_1,x_2,\cdots,x_k,1)$ has the value 0 for some set of values of the k variables $\{x_1,x_2,\cdots,x_k\}$ by the induction hypothesis. And so we take this set of k values and let $x_{k+1} = 1$ so that Q_{k+1} has the value 0.

Case 3. A c_j is 0 but no b_j is 0, then by a symmetrical argument and letting $x_{k+1} = 0$ we have a solution to $Q_{k+1} = 0$.

These three cases show the sufficiency of the condition.

Now suppose that $Q_{k+1} = 0$ has a solution then

Case 1. $Q(x_1,x_2,\cdots,x_k,1)$ is 0 and
$Q(x_1,x_2,\cdots,x_k,0)$ is 0;

Case 2. $Q(x_1,x_2,\cdots,x_k,1)$ is 0 and
$Q(x_1,x_2,\cdots,x_k,0)$ is 1;

Case 3. $Q(x_1,x_2,\cdots,x_k,1)$ is 1 and
$Q(x_1,x_2,\cdots,x_k,0)$ is 0.

If any of these three cases we have, by the induction hypothesis, that either a b_j is 0 or a c_j is 0. And since every b_j and c_j is an a_i at least one $a_i = 0$.

This shows the necessity of the condition. Hence the conjecture is valid for all n.

(4.7) The following theorem is due to Bernstein (1), but we offer an alternative proof.

Theorem 2.

A necessary and sufficient condition that $Q_n=0$ have a unique solution is,

i. $a_1a_2\cdots a_{2^n}=0$ and ii. $a_i'a_j'=0$. $i\neq j$

Proof: We first note that the conditions (i) and (ii) are equivalent to the statement that "exactly one discriminant of Q_n is 0". For by (i) at least one $a_i=0$ and if two discriminants are 0 there is an $a_i=0$ and an $a_j=0$ for $i\neq j$ so that $a_j'=1$, $a_j'=1$ and $a_i'a_j'\neq 0$.

By (4.6) the condition (i) is necessary and sufficient for the existence of a solution. Hence we need only show that the condition (ii) is necessary and sufficient for the uniqueness of the solution. The proof is by induction on the number of independent variables n.

If $n=1$ then,

$$Q_1 = a_1x_1 + a_2x_1' = 0.$$

If $a_1'a_2' = 0$ and $a_1\cdot a_2 = 0$ then there are two cases:

Case 1. $a_1=0$, $a_2=1$.

In this case $x_1'=0$, i.e., $x_1=1$ is the unique solution.

Case 2. $a_1=1$, $a_2=0$.

By symmetry $x_1=0$ is the unique solution.

Suppose now, that $Q_1=0$ has a unique solution. Then it is $x_1=0$ or $x_1=1$ since there is only one independent variable. By (4.6) at least one of a_1 and a_2 is 0 and we have three possibilities.

Case 1. $a_1=a_2=0$. Then $Q_1\equiv 0$ and $x_1=0$ or $x_1=1$ is a solution, contradicting our assumption that $Q_1=0$ has a unique solution.

Case 2. $a_1=1$, $a_2=0$. Then (ii) is satisfied.

Case 3. $a_1=0$, $a_2=1$. Then (ii) is satisfied.

This shows that the conjecture is valid for $n=1$. Suppose that it is true for $n=k$. By (3.9) we have,

$$Q_{k+1}\equiv x_{k+1}Q(x_1,x_2,\cdots,x_k,1)+x_{k+1}'\,Q(x_1,x_2,\cdots,x_k,0).$$

If the condition (ii) is satisfied then exactly one $a_j=0$. Or by (3.96) exactly one of the set $\{b_j,\ c_j\}$ is 0. $(j=1,2,\cdots,2^k)$.

If exactly one b_j is 0 then every c_j is 1 and there is a unique set of values of the variables $\{x_1, x_2, \cdots, x_k\}$ such that $Q(x_1, x_2, \cdots, x_k, 1)$ is 0 by the induction hypothesis. If we add to that unique set of values $x_{k+1} = 0$ then Q_{k+1} does not have the value 0. But if we add $x_{k+1} = 1$ we have a unique set of values of the variable such that Q_{k+1} is 0. By a symmetrical argument we see that when exactly one c_j is 0 we also have a unique solution to $Q_{k+1} = 0$. This shows the sufficiency of the condition (ii) for $n=k+1$.

Now suppose that $Q_{k+1} = 0$ has a unique solution. Then $Q(x_1, x_2, \cdots, x_k, 1)$ and $Q(x_1, x_2, \cdots, x_k, 0)$ cannot have the value 0 for the same set of values of the variables. For this would imply that x_{k+1} is arbitrary. In which case there would be two or more solutions to $Q_{k+1}=0$ contrary to our assumption.

Now $Q(x_1, x_2, \cdots, x_k, 1) = 0$ has a unique solution for otherwise we could take $x_{k+1} = 1$ and Q_{k+1} is 0 for more than one set of values of the variables contrary to our assumption. By symmetry the set of values for which $Q(x_1, x_2, \cdots, x_k, 0)$ is 0 is also unique. By (4.6) at least one a_i is 0. Suppose two or more a_i is 0, then there are three cases.

Case 1. Two or more $b_j = 0$ and all $c_j = 1$, in which case $Q(x_1, x_2, \cdots, x_k, 1) = 0$ does not have a unique solution by the induction hypothesis.

Case 2. Two or more $c_j = 0$ and all $b_j = 1$. Again by a symmetrical argument $Q(x_1, x_2, \cdots x_k, 0) = 0$ does not have a unique solution.

Case 3. One or more $c_j = 0$ and one or more $b_j = 0$. But if more than one b_j or c_j is 0 then as before our assumption that there is a unique solution is contradicted. And if exactly one $b_j = 0$ and exactly one $c_j = 0$ then there are two distinct solutions to $Q_{k+1} = 0$. Since $Q(x_1, x_2, \cdots, x_k, 1)$ and $Q(x_1, x_2, \cdots, x_k, 0)$ cannot be 0 for the same set of values of the variables, we may take the set of values for which the first is 0 and $x_{k+1} = 1$ and the set of values for which the second is 0 and $x_{k+1} = 0$.

By these three cases we see that exactly one of the following occurs:

1. exactly one $b_j = 0$ and all $c_j = 1$,

2. exactly one $c_j = 0$ and all $b_j = 1$.

In either case exactly one a_i is 0. This proves the necessity of condition (ii). Therefore by mathematical induction the conjecture is valid for all n.

(4.8) For any min-term A of Q_n there is one and only one set of values of the variables such that A has the value 1.

Proof: We take $x_i = 1$ if x_i is a factor of A and take $x_i = 0$ if x_i' is a factor of A.

(4.9) If A_h and A_k are min-terms of Q_n and A_h has the value 1 for a set of values of the variables then $A_k = 0$ for that set of values.
$(k=1, 2,\cdots, h-1, h+1,\cdots,2^n)$.

(4.91) Theorem 3.

Given $Q_n = 0$, T is the set of all a_j such that $a_i = 1$, t is the cardinality of T and S is the set of all solutions to $Q_n = 0$ then the cardinality of S is 2^n-t. $(t=1,2,\cdots,2^n)$.

Proof: By induction on t and for a fixed n.

If $t=0$ then every $a_i=0$, $Q_n \equiv 0$ and any one of the 2^n possible solutions satisfy the equation. And so the cardinality of S is 2^n.

If $t=1$ then exactly one a_i is 1 and the disjunctive canonical form Q_n is a single min-term. By (4.8) there is exactly one set of values of the variables for which Q_n has the value 1. This implies that of the 2^n possible solutions to $Q_n = 0$, for exactly one of the possibilities Q_n is 1. Therefore there are 2^n-1 solutions to $Q_n = 0$ when t is 1.

We suppose that the conjucture is valid for $t = k$. Let $Q_n^{(k)}$ and $Q_n^{(k+1)}$ be polynomials in disjunctive canonical form for which k and $k+1$ discriminants, respectively, are 1. Now if $k+1 \leq 2^n$,

$$Q_n^{(k+1)} \equiv Q_n^{(k)} + A$$

for some $Q_n^{(k)}$ and A is a min-term belonging to $Q_n^{(k+1)}$ but not to $Q_n^{(k)}$. If $Q_n^{(k)}$ and A have the value 0 for a set of values of the variables then $Q_n^{(k+1)}$ has the value 0 for that same set of values. By the induction hypothesis there are 2^n-k solutions to $Q_n^{(k)} = 0$. By

(4.8) and (4.9) there is one and only one set of values of the variables such that $A=1$ and $Q_n^{(k)} = 0$ and hence for which $Q_n^{(k+1)} = 1$. Therefore for the other 2^n-k-1 sets of values of the variables for which $Q_n^{(k)}$ is 0, A is also 0. That is, $Q_n^{(k+1)}$ is 0 for $2^n-(k+1)$ sets of values of the variables and the cardinality of S is $2^n-(k+1)$.

If $k<2^n$ the above argument is valid for any $Q_n^{(k+1)}$. If $k=2^n$ then $Q_n \equiv 0$ and we have 2^n solutions. Therefore by mathematical induction the conjecture is valid for $t \leq 2^n$.

(4.92) It is instructive to examine several examples using (4.91).

Example 1.

$$Q(x_1,x_2) = x_1'x_2 + {}_1x_1x_2' = 0 .$$

By (4.91) and since two $a_i = 1$ there are $2^2-2 = 2$ solutions to this equation. We verify this with a truth table.

x_1	x_2	$Q(x_1,x_2)$
0	0	0
0	1	1
1	0	1
1	1	0

Example 2.

$$Q(x_1,x_2,x_3) = x_1'x_2x_3 + x_1x_2'x_3 + x_1x_2x_3' = 0$$

By (4.91) there are $2^3-3 = 5$ solutions to the equation. We verify this with a truth table.

x_1	x_2	x_3	$Q(x_1,x_2,x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

(4.93) Theorem 4.

Given $Q_n = 0$, U is the set of all a_j such that a_j is 0 and u is the cardinality of U then the solutions to $Q_n = 0$ are the same as the solutions to $A_j = 1$.

Where A_j is a min-term missing from the expansion Q_n. $(j = 1,2,\cdots,2^n)$.

Proof: By induction on u and for a fixed n.

For $u = 0$ there are no solutions since the cardinality of T by (4.91) is 2^n. For $u = 1$ we examine the min-term A_j whose discriminant a_j is 0. By (4.91) there is only one solution. By (4.8) and (4.9) there is a set of values of the variables such that A_j is 1 and all other min-terms are 0. But this implies that this set of values is a solution to $Q_n = 0$. And since the solution is unique we have found it by finding the set of values for which A_j is 1.

We now suppose that the conjecture is valid for $u=k$. Let $Q_n^{(k)}$ and $Q_n^{(k+1)}$ be disjunctive canonical form associated with $u=k$ and $u = k+1$ respectively. We note that $Q_n^{(k)}$ has one more min-term in its representation than $Q_n^{(k+1)}$. Thus for some $Q_n^{(k+1)}$,

$$Q_n^{(k)} = Q_n^{(k+1)} + A,$$

where A is a min-term of $Q_n^{(k)}$ but not of $Q_n^{(k+1)}$. $(k>0)$.

By (4.91) $Q_n^{(k)}$ is 0 for k sets of values of the variables. By the induction hypothesis and (4.9) the k solutions to $Q_n^{(k)} = 0$ are determined. Every solution to $Q_n^{(k)} = 0$ is a solution to $Q_n^{(k+1)} = 0$ since if $Q_n^{(k)}$ has the value 0 then both $Q_n^{(k+1)}$ and A have the value 0. By (4.8) and (4.9) there is one and only one set of values of the variables such that A is 1 and $Q_n^{(k+1)}$ is 0. This set of values is a solution to $Q_n^{(k+1)} = 0$ but not to $Q_n^{(k)} = 0$ and since $Q_n^{(k+1)} = 0$ has exactly one more solution than $Q_n^{(k)} = 0$ this set must be the extra solution. And furthermore it is exactly the set of values of the variables for which A is 1. The above argument is valid for any $Q_n^{(k)}$ if $0 < k < 2^n$. If $k=0$ then $Q_n^{(k)} \equiv 1$ and there are no solutions. If $k = 2^n$ then $Q_n^{(k)} \equiv 0$ and there are 2^n solutions and we merely take all of the logical possibilities. But this is equivalent to setting each min-term equal to 1.

Therefore by mathematical induction the conjecture is valid for all u and hence for all n.

(4.94) Examples using (4.92)

Example 1:

$$Q(x_1,x_2,x_3) = x_1x_2x_3' + x_1x_2'x_3 + x_1'x_2x_3 + x_1x_2'x_3'=0.$$

Let M be the set of all min-terms of three variables. Then,

$$M = \{x_1x_2x_3, x_1x_2x_3', x_1x_2'x_3, x_1x_2'x_3', x_1'x_2x_3, x_1'x_2x_3', x_1'x_2'x_3, x_1'x_2'x_3'\}$$

and we see that the min-terms whose discriminants are 0 are $x_1x_2x_3$, $x_1'x_2x_3'$, $x_1'x_2'x_3$ and $x_1'x_2'x_3'$. By (4.92) the solutions are,

$$x_1 = 0, x_2 = 0, x_3 = 0, \qquad x_1=0,\ x_2=0,\ x_3=1,$$

$$x_1=0,\ x_2=1,\ x_3=0, \qquad x_1=1,\ x_2=1,\ x_3=1.$$

We verify this with a truth table.

x_1	x_2	x_3	Q_3
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Example 2:

$$x_1x_2x_3x_4 + x_1x_2x_3x_4' + x_1x_2x_3'x_4' + x_1x_2x_3'x_4 = 0.$$

The missing min-terms are:

$x_1x_2'x_3x_4$ $x_1x_2'x_3x_4'$ $x_1x_2'x_3'x_4$ $x_1x_2'x_3'x_4'$ $x_1'x_2x_3x_4$ $x_1'x_2x_3x_4'$

$x_1'x_2x_3'x_4$ $x_1'x_2x_3'x_4'$ $x_1'x_2'x_3x_4$ $x_1'x_2'x_3x_4'$ $x_1'x_2'x_3'x_4$ $x_1'x_2'x_3'x_4'$

By (4.92) the 12 solutions are,

$x_1=1,\ x_2=0,\ x_3=1,\ x_4=1$ $\qquad$ $x_1=1,\ x_2=0,\ x_3=1,\ x_4=0$

$x_1=1,\ x_2=0,\ x_3=0,\ x_4=1$ $\qquad$ $x_1=1,\ x_2=0,\ x_3=0,\ x_4=0$

$x_1=0,\ x_2=1,\ x_3=1,\ x_4=1$ $\qquad$ $x_1=0,\ x_2=1,\ x_3=1,\ x_4=0$

$x_1=0,\ x_2=1,\ x_3=0,\ x_4=1$ $\qquad$ $x_1=0,\ x_2=1,\ x_3=0,\ x_4=0$

$x_1=0,\ x_2=0,\ x_3=1,\ x_4=1$ $\qquad$ $x_1=0,\ x_2=0,\ x_3=1,\ x_4=0$

$x_1=0,\ x_2=0,\ x_3=0,\ x_4=1$ $\qquad$ $x_1=0,\ x_2=0,\ x_3=0,\ x_4=0.$

We verify this with a truth table.

x_1	x_2	x_3	x_4	Q_4
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

CHAPTER V

PROCEDURES BASED ON QUINE'S CANONICAL FORM

(5.0) Theorem 5.

A necessary and sufficient condition that $f(x_1,x_2,\cdots,x_n) = 0$ have a unique solution is that Quine's canonical form of $f(x_1,x_2,\cdots,x_n)$ is a sum of literals such that no letter appears more than once and every letter appearing in the representation $f(x_1,x_2,\cdots,x_n)$ appears in Quine's canonical form of $f(x_1,x_2,\cdots,x_n)$.

Proof: Let $QF{:}f_n$ stand for the phrase, "Quine's canonical form of $f(x_1,x_2,\cdots,x_n)$." The condition is sufficient for if,

$$QF{:}f_n = \sum_{i=1}^{n} \alpha_i = 0,$$

where α_i is a literal then every $\alpha_i=0$. And a unique solution is determined.

If there is a unique solution then by (4.7) or (4.91) there is exactly one discriminant which has the value 0 in the disjunctive canonical form Q_n, of f_n.

By (3.91)

$$f_n' = Q_n' = M,$$

where M is the min-term whose discriminant is 0. There are n literals which are factors of M and no letter appears more than once. Furthermore

$$f_n = (f_n')' = M' \sum_{i=1}^{n} \alpha_i$$

But

$\sum_{i=1}^{n} \alpha_i$ is an alterm and hence is also a conjunctive form of f_n. (a conjunctive of one alterm).

If we apply (3.991), since there are no deletions to be made and there is no consensus to take,

$\sum_{i=1}^{n} \alpha_i$ is in Quine's canonical form. This shows the necessity of the condition.

(5.1) We note that (5.0) also determines the solution $f_n = 0$.

(5.2) Theorem 6.

If $QF{:}f_n$ has in its representation prime implicants that are literals then in any solution to $f_n = 0$ the values of the variables associated with these literals are uniquely determined.

Proof:

Suppose,

$$QF{:}f_n = \psi + \Sigma\ \alpha_i = 0$$

where ψ is a polynomial no monomial of which is a literal and $\Sigma\alpha_i$ is an alterm. Then if f_n has the value 0 for a set of values of the independent variables every $\alpha_i = 0$ in which case the variable associated with a particular α_i is uniquely determined.

(5.3) Theorem 7.

If $QF{:}f_n$ is not the Boolean constant 1 and if a letter appearing in the representation f_n does not appear in $QF{:}f_n$ then the variable associated with the missing letter is arbitrary in any solution to $f_n = 0$.

Proof:

From (3.99) the only way a letter can be deleted from a representation which is not the Boolean constant 1 is for a deletion iteration to be made. Hence if x is a letter appearing in f_n but not in $QF{:}f_n$ then the form $Ax + A$ must appear either in f_n or after the consensus iteration. But by (2.97),

$$Ax + A \equiv A$$

and the value of x is arbitrary.

(5.8) Theorem 8.

If f_n is a Boolean polynomial then a necessary and sufficient condition that $f_n = 0$ have a unique solution is that $QF{:}f_n'$ is a product of n literals so that no letter appears more than once.

Proof:

By (4.2) a necessary and sufficient condition that $f_n = 0$ have a unique solution is that exactly one min-term is missing from the disjunctive canonical form of f_n. But by (3.91),

$$Q_n' = M,$$

where M is the missing min-term. But M is in Quine's canonical form since no deletions can be made and since there is no consensus to take.

(5.9) Theorem 9.

If $QF{:}f_n'$ is a product of n literals such that no letter appears more than once then the unique solution to $f_n = 0$ is the solution to $QF{:}f_n' = 1$.

Proof: By the proof of (5.8) $QF{:}f_n'$ is the min-term missing from the disjunctive canonical form of f_n. Therefore by (4.93) the unique solution to $f_n = 0$ is determined by finding the unique set of values of the variables for which $QF{:}f_n' = 1$.

(5.91) We now have an algorithm for finding the unique solution to the equation $f_n = 0$, if it exists. And for finding the arbitrary variables.

Step 1. transform f_n to a polynomial P_n

Step 2. find P_n'

Step 3. perform the indicated multiplications

Step 4. perform a deletion iteration

Step 5. if the result is a monomial M_n in n literals $f_n = 0$ has a unique solution and the solution is found by solving $M_n = 1$.

We illustrate the procedure with some examples.

Example 1.

$$f(x_1,x_2) = x_2(x_1+x_1'x_2) + x_1x_2'$$

Step 1.

$$f(x_1,x_2)=x_2(x_1+x_1'x_2)+x_1x_2' = x_1x_2+x_1'x_2+x_1x_2'$$

Step 2.

$$f'(x_1,x_2) = (x_1'+x_2')(x_1+x_2')(x_1'+x_2)$$

Step 3.

$$(x_1'+x_2')(x_1+x_2')(x_1'+x_2)=(x_1x_2'+x_1'x_2'+x_2')(x_1'+x_2)=x_1'x_2'$$

Step 4. in this case there are no deletions to be made.

Step 5. if $x_1'x_2' = 1$ then $x_1' = 1$ and $x_2' = 1$

and the solution to $f(x_1,x_2) = 0$ is $x_1=0, x_2=0$.

Example 2.

$$f(x_1,x_2,x_3) = x_1+x_2x_1' + x_1x_2x_3$$

Step 1. we already have a polynomial

Step 2. $f'(x_1,x_2,x_3) = (x_1')(x_2'+x_1)(x_1'+x_2'+x_3')$

Step 3. $(x_1')(x_2'+x_1)(x_1+x_2+x_3) = (x_1'x_2')(x_1'+x_2'+x_3')$

$= x_1'x_2' + x_1' \cdot x_2' \cdot x_3'$

Step 4. $x_1'x_2' + x_1' \cdot x_2' \cdot x_3' = x_1'x_2'$

Step 5. if $x_1'x_2' = 1$ then $x_1 = 0$, $x_2 = 0$ and x_3 is arbitrary.

CHAPTER VI

ON INFERENTIAL PROBLEMS

In this chapter we investigate the applicability of the preceding results to the solution and construction of certain inferential problems. It is shown in most standard textbooks on symbolic logic, e.g., Rosenbloom (9), that the Boolean algebra defined in Chapter II is equivalent to the propositional calculus. For instance, if the proposition A is false we say that A has the truth value 0 or $A=0$. Similarly if the proposition A is true we write $A=1$. The symbol "+" and "·" of the Boolean algebra correspond respectively to the "or" and "and" of the propositional calculus.

In 1952 Fletcher (3) gave examples illustrating the use of Boolean algebra in solving certain types of problems, the solutions to which might be difficult to obtain by other means.

(6.0) Example 1.

Out of six boys, two were known to have been stealing apples. But who? Harry said, "Charlie and George". James said, "Donald and Tom". Donald said "Tom and Charlie". George said, "Harry and Charlie". Charlie said, "Donald and James". Tom couldn't be found.

Four of the boys interrogated named one miscreant correctly. The fifth had lied outright. Who stole the apples?

Fletcher solves this problem in the following manner. Let H, J, D, G, C, T denote the propositions "Harry, James, Donald, George, Charlie, Tom did it" respectively. Each person who makes a statement names at least one miscreant incorrectly. That is,

(1) $CG = DT = TC = HC = DJ = 0.$

Four of the five statements are true taken in disjunction, but one is false because one of the boys lied outright. Thus,

(2) $(C+G)(D+T)(T+C)(H+C)(D+J) = 0.$

Performing the indicated multiplication and using the relations (1) we have,

$$(3) \qquad CD = 0$$

But one set of four out of the five statements in disjunction are true. Hence,

$$(4) \quad (C+G)(D+T)(T+C)(H+C)+(C+G)(D+T)(T+C)(D+J) \\ +(C+G)(D+T)(H+C)(D+J)+(C+G)(T+C)(H+C)(D+J) \\ +(D+T)(H+C)(D+J)(T+C) = 1.$$

Using (1) and (3) this reduces to,

(5) $$CJ = 1.$$

And this means that Charlie and James stole the apples.

The question might now arise, "is this the only solution?" If we take the results of (1), (3) and (5) we have the system,

$$CG+DT+TC+HC+DJ+CD = 0$$
$$CJ = 1.$$

Applying P13 to the second equation we have

$$CG+DT+TC+HC+DJ+CD+C'+J' = 0.$$

Putting this equation in Quine's canonical form gives,

$$C' + J' + G + T + H + D = 0.$$

Therefore by (5.0) Theorem 5, the given system has a unique solution. Hence, providing the analysis of problem is correct, the equations result in a unique solution.

(6.1) Example 2.

Alice, Brenda, Cissie and Doreen competed for a scholarship. "What luck have you had?" someone asked them. Said Alice, "Cissie was top, Brenda was second." Said Brenda, "Cissie was second and Doreen was third." Said Cissie, "Doreen was bottom, Alice was second." Each of the three girls had made two assertions, of which only one was true. Who won the scholarship?

If we let A_1 denote the proposition, "Alice was first" and similarly for the other statements we have,

from Alice, $C_1B_2 = 0$ and $C_1 + B_2 = 1$

from Brenda, $C_2D_3 = 0$ and $C_2 + D_3 = 1$

from Cissie, $D_4A_2 = 0$ and $D_4 + A_2 = 1.$

These equations yield the system

$$C_1B_2 + C_1'B_2' = 0$$

$$C_2D_3 + C_2'D_3' = 0$$

$$A_2D_4 + A_2'D_4' = 0.$$

From this we have the system,

$$C_1B_2' + C_1'B_2 = 1$$

$$C_2D_3' + C_2'D_3 = 1$$

$$A_2D_4' + A_2'D_4 = 1.$$

Finally we have,

$$(C_1B_2' + C_1'B_2)(C_2D_3'+C_2'D_3)(A_2D_4'+A_2'D_4) = 1.$$

This yields,

$$C_1B_2'C_2D_3'D_4A_2' + C_1B_2'C_2'D_3D_4'A_2 + C_1B_2'C_2'D_3D_4A_2' + C_1B_2'C_2D_3'D_4'A_2$$
$$+ C_1'B_2C_2D_3'D_4A_2' + C_1'B_2C_2'D_3D_4'A_2 + C_1'B_2C_2'D_3D_4A_2'$$
$$+ D_1'B_2C_2D_3'D_4'A_2 = 1.$$

Certainly $C_1C_2 = 0$, $D_3D_4 = 0$, $B_2C_2 = 0$, $A_2B_2 = 0$,

$D_3D_4 = 0$,

(Fletcher calls these exclusion relations) and this yields

$$C_1B_2'C_2'D_3C_4'A_2 = 1.$$

From this we conclude that,

Cissie was first, Alice second, Doreen third and therefore Brenda was fourth.

(6.2) The two examples and the results of the preceding chapters indicate a procedure by which this type of problem can be constructed so as to yield a unique solution.

Suppose we start with the following equation,

(1) $A_r+A_b'+A_w+B_r+B_b'+B_w+C_r+C_b+C_w' = 0,$

which we know has a unique solution.

We will let A, B and C represent colored beads. We will use the obvious suffix notation A_r with the statement A is red. Similarly, B_b stands for B is blue, C_w' stands for C is not white and so on.

Using (1) as a guide we construct the following example.

Example 3.

Out of a box of red, blue and white beads three are drawn at random. Two of those drawn are blue and only one of the following statements is true;

(i) A is red (ii) B is blue (iii) C is not white.

Can you tell what color the beads that were drawn are?

Solution: Since only one of the given statements (i), (ii), and (iii) is true, we know one of three alternatives to be true. The symbolic expression for this is,

(2) $A_r B_b' C_w + A_r' B_b C_w + A_r' B_b' C_w' = 1.$

Since two beads are blue we have,

(3) $A_b B_b + A_b C_b + B_b C_b = 1.$

Again use is made of the exclusion relations which can be constructed from the nature of the problem. Some of them are,

(4)
$$A_b\ B_b\ C_b = 0$$
$$C_b\ C_w = 0$$

Multiplying (2) and (3), using (4) and the rule $xx' = 0$ we have,

$$A_r' B_b C_w A_b = 1.$$

This means that A is blue, B is blue and C is white and this is exactly what we started with.

(6.3) In general to construct this type of problem we can start with a linear combination of variables set equal to 0. We can use any of the properties of (2.1) to transform the original equation into a new, but identical representation. A verbal cloak is then given to the symbolic representation.

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