Permutation Resolutions for Specht Modules of Hecke Algebras[*]

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Abstract

In [BH], a chain complex was constructed in a combinatorial way which conjecturally is a resolution of the (dual of the) integral Specht module for the symmetric group in terms of permutation modules. In this paper we extend the definition of the chain complex to the integral Iwahori Hecke algebra and prove the same partial exactness results that were proved in the symmetric group case.

Introduction

In [BH], Hartmann and the first author constructed, for any composition λ of a positive integer r, a finite chain complex of modules for the group algebra RW of the symmetric group W on r letters over an arbitrary commutative ring R. The last module in this complex is the dual of the Specht module

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S^λ and the other modules are permutation modules with point stabilizers given by Young subgroups of W. The construction of the chain complex was completely combinatorial and characteristic-free. It was conjectured that this chain complex is exact whenever λ is a partition. In other words the chain complex is conjectured to be a resolution of the (dual) of the Specht module by permutation modules. Partial exactness results were already established in [BH] and a full proof of the exactness was given recently by Yudin and Santana (see [SY]) by translating the construction via the Schur functor. Other permutation resolutions of Specht modules have been considered by Donkin in [Dn], Akin in [A], Zelevinskii in [Z], Akin-Buchsbaum in [AB1] and [AB2], Santana in [S], Woodcock in [W1] and [W2], Doty in [Dy] and Yudin in [Y]. See [BH, Section 6] for a more detailed comparison of these constructions with the construction in [BH].

The goal of this paper is to (a) lift the construction of the chain complex in [BH] for the group algebra RW to a chain complex for modules of the Iwahori Hecke algebra $\mathcal{H}_{r,q}^R$ for any integral domain R such that the specialization $q = 1$ reproduces the original chain complex; and (b) lift the partial exactness proofs from [BH] to the new construction.

Both goals are achieved, with part (a) being relatively straightforward, but part (b) requiring much more subtle arguments. The paper is arranged as follows. In Section 1 we establish the necessary notation and recall results about the Hecke algebra $\mathcal{H}_{r,q}^R$ from [DJ]. In Section 2 we introduce a group of homomorphisms between 'permutation modules' of $\mathcal{H}_{r,q}^R$ and show that such homomorphisms are closed under composition. This is used in Section 3 to construct the chain complex which generalizes the chain complex in [BH]. In Section 4 we prove that the chain complex in question is exact in degrees -1 and 0, for arbitrary partitions λ, and that it is exact everywhere, for all partitions of the form $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_3 \leqslant 1$, cf. Theorems 4.2, 4.4, 4.5. The very technical proof of a key lemma, Lemma 4.3, is postponed to Section 5.

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1 Notation and Quoted Results

Throughout this paper we denote by R an integral domain. Unadorned tensor products and homomorphism sets will be understood to be taken over R. Moreover, we fix a positive integer r and denote by W the symmetric group on the set $\{1, \ldots, r\}$. Elements of W are composed like functions applied on the right in order to be consistent with [DJ]. Thus, W acts from the right on $\{1, \ldots, r\}$ and we write $(i)w$ or iw for $i \in \{1, \ldots, r\}$ and $w \in W$. We set $S := \{(i, i+1) \mid i = 1 \ldots, r-1\}$, the standard choice of simple transpositions. The set of positive (resp. non-negative) integers will be denoted by $\mathbb{N}$ (resp. $\mathbb{N}_0$).

1.1 *The Hecke algebra $\mathcal{H}$.* For a unit q of R we denote by $\mathcal{H} = \mathcal{H}_{r,q}^R$ the Hecke algebra as introduced in [DJ]. It has an R-basis consisting of the elements T_w, $w \in W$. The multiplication in $\mathcal{H}$ is uniquely determined by the following formulas for $w \in W$ and $s \in S$:

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } l(ws) = l(w) + 1, \text{ i.e., if } iw^{-1} < (i+1)w^{-1}, \\ qT_{ws} + (q-1)T_w, & \text{if } l(ws) = l(w) - 1, \text{ i.e., if } iw^{-1} > (i+1)w^{-1}. \end{cases}$$

The element T_1 is the identity element of $\mathcal{H}$.

By $\trianglelefteq$ we denote the *strong Bruhat order* on W: for $u, v \in W$, one has $u \trianglelefteq v$ if there exists a reduced expression $u = s_1 \cdots s_n$ of u and integers $1 \leqslant i_1 < \cdots < i_k \leqslant n$ with $v = s_{i_1} s_{i_2} \cdots s_{i_k}$. While, for given $v, w \in W$, it is in general difficult to express $T_v T_w$ in terms of the basis elements T_x, $x \in W$, Lemma 2.1(ii) in [DJ] and a quick induction argument on $l(w)$ imply that

$$T_v T_w \in \sum_{w \trianglelefteq u \in W} R T_{vu} \tag{1}$$

and that

$$T_v T_w = T_{vw} \quad \text{if } l(vw) = l(v) + l(w). \tag{2}$$

1.2 *The $\mathcal{H}$-modules M^λ.* We denote by Γ the set of *compositions* $\lambda = (\lambda_1, \lambda_2, \ldots)$ of r and by Λ the set of partitions of r. We define the *dominance order* $\trianglelefteq$ on Γ by

$$\lambda \trianglelefteq \mu : \iff \sum_{i=1}^{e} \lambda_i \leqslant \sum_{i=1}^{e} \mu_i \quad \text{for all } e \geqslant 1.$$

This defines a partial order on Γ. Note that it differs from the dominance relation defined in [DJ] which is not a partial order and is defined by

$$\lambda \leqslant \mu: \iff \mu' \trianglelefteq \lambda', \tag{3}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denotes the *dual* of λ, i.e., $\lambda'_i := |\{k \in \mathbb{N} \mid \lambda_k \geqslant i\}|$, for $i \in \mathbb{N}$. Note that λ' is always a partition.

Let λ be a composition of r. One associates to λ the set partition

$$\mathcal{P}_\lambda := \{\{1, \ldots, \lambda_1\}, \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots\}$$

of $\{1, \ldots, r\}$ and the *standard parabolic subgroup* W_λ consisting of all $w \in W$ which fix $\mathcal{P}_\lambda$ element-wise. Note that W_λ is generated by the elements $s = (i, i+1) \in S$ with the property that i and $i+1$ belong to the same element of $\mathcal{P}_\lambda$. More generally, a subgroup of W is called a *parabolic subgroup* if it arises as the stabilizer of an arbitrary set partition of $\{1, \ldots, r\}$, i.e., if it is conjugate to a subgroup of the form W_λ. Intersections of parabolic subgroups are again parabolic subgroups. The right $\mathcal{H}$-submodule M^λ of the regular module $\mathcal{H}$ is defined as

$$M^\lambda := x_\lambda \mathcal{H} \quad \text{with } x_\lambda := \sum_{w \in W_\lambda} T_w \,.$$

For $v \in W_\lambda$ one has

$$x_\lambda T_v = q^{l(v)} x_\lambda \,, \tag{4}$$

cf. [DJ, Lemma 3.2]. Every coset $W_\lambda w \in W_\lambda \backslash W$ has a unique element of smallest length. We denote the set of these distinguished coset representatives by $\mathcal{D}_\lambda$. For $w \in W_\lambda$ and $d \in \mathcal{D}_\lambda$ one has

$$l(wd) = l(w) + l(d) \quad \text{and} \quad T_{wd} = T_w T_d \,. \tag{5}$$

The elements $x_\lambda T_d$, $d \in \mathcal{D}_\lambda$, form an R-basis of M^λ, cf. [DJ, Lemma 3.2].

As usual one identifies compositions λ with Young diagrams and we say that λ is the *shape* of the corresponding Young diagram. A λ-tableau is a filling of the r boxes of the Young diagram of λ with the numbers $1, 2, \ldots, r$. We denote the set of λ-tableaux by $\mathcal{T}(\lambda)$ and usually denote elements of $\mathcal{T}(\lambda)$ by t. The λ-tableau which contains the entries $1, 2, \ldots, r$ in ascending order is denoted by t^λ. Thus, for $\lambda = (1, 0, 2)$ one has

$$t^{(1,0,2)} = \begin{array}{|c|c|} \hline 1 \\ \cline{1-1} \\ \hline 2 & 3 \\ \hline \end{array} \,.$$

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This provides a standard numbering of the boxes of the Young diagram of λ which we will use later. The group W acts from the right on $\mathcal{T}(\lambda)$ by simply applying an element $w \in W$ to the entries of the tableau $t \in \mathcal{T}(\lambda)$. This action is free and transitive, and it yields a bijection

$$W \xrightarrow{\sim} \mathcal{T}(\lambda), \quad w \mapsto t^\lambda w.$$

A λ-tableau t is called *row-standard* if its entries are increasing in each row from left to right. The row-standard λ-tableaux form a subset $\mathcal{T}^{\mathrm{rs}}(\lambda)$ of $\mathcal{T}(\lambda)$. Two λ-tableaux t_1 and t_2 are called *row-equivalent* if they arise from each other by rearranging elements within each row. We denote the row-equivalence class of t by $\{t\}$ and the set of row-equivalence classes by $\overline{\mathcal{T}}(\lambda)$. One has canonical bijections

$$\mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{T}^{\mathrm{rs}}(\lambda) \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda) \tag{6}$$

given by $d \mapsto t^\lambda d$ and $t \mapsto \{t\}$. Thus, the canonical basis $x_\lambda T_d$, $d \in \mathcal{D}_\lambda$, could also be parametrized by $\mathcal{T}^{\mathrm{rs}}(\lambda)$ or $\overline{\mathcal{T}}(\lambda)$. For $d \in \mathcal{D}_\lambda$ we denote by

$$\varepsilon_d \in \mathrm{Hom}(M^\lambda, R)$$

the R-module homomorphism with the property that $\varepsilon_d(x_\lambda T_e) = \delta_{d,e}$ for all $e \in \mathcal{D}_\lambda$. In other words, the elements ε_d, $d \in \mathcal{D}_\lambda$, form the dual basis of the R-basis $x_\lambda T_d$, $d \in \mathcal{D}_\lambda$. If $t \in \mathcal{T}^{\mathrm{rs}}(\lambda)$ corresponds to $d \in \mathcal{D}_\lambda$ under the canonical bijection in (6) we will also write ε_t instead of ε_d.

For $\lambda \in \Lambda$ we have an obvious bijection $\mathcal{T}(\lambda) \xrightarrow{\sim} \mathcal{T}(\lambda')$, $t \mapsto t'$, where t' is the reflection of t with respect to the diagonal axis. Note that $(tw)' = t'w$ for all $t \in \mathcal{T}(\lambda)$ and $w \in W$.

1.3 *The homomorphisms $\varphi_d^{\lambda,\mu} \colon M^\mu \to M^\lambda$.* Let $\mu, \lambda \in \Gamma$ be compositions of r. The elements of $\mathcal{D}_{\lambda,\mu} := \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ form a set of representatives of the double cosets $W_\lambda \backslash W / W_\mu$, and each element $d \in \mathcal{D}_{\lambda,\mu}$ is the unique element of shortest length in its double coset $W_\lambda d W_\mu$. By [DJ, Theorem 3.4], the set $\mathcal{D}_{\lambda,\mu}$ parametrizes an R-basis $\varphi_d^{\lambda,\mu}$, $d \in \mathcal{D}_{\lambda,\mu}$, of $\mathrm{Hom}_{\mathcal{H}}(M^\mu, M^\lambda)$ given by

$$\varphi_d^{\lambda,\mu}(x_\mu) = \sum_{w \in W_\lambda d W_\mu} T_w = x_\lambda \sum_{e \in \mathcal{D}_\nu \cap W_\mu} T_{de} = x_\lambda T_d \sum_{e \in \mathcal{D}_\nu \cap W_\mu} T_e, \tag{7}$$

where $\nu \in \Gamma$ is determined by $W_\nu = d^{-1} W_\lambda d \cap W_\mu$. The last equality follows from (5). We want to mention that the cited theorem requires that R is a

principal ideal domain. But a careful examination of the canonical isomorphisms in [DJ, Theorems 2.5, 2.6, 2.7] involved in the proof of the theorem show that this hypothesis is unnecessary. (More precisely, the R-modules $\operatorname{Hom}_{\mathcal{H}}(M_{\mathcal{H}_\nu}, N \otimes T_d)$ in [DJ, Theorem 2.8] are free of rank one in the case that M and N are trivial modules.)

We will use a combinatorial description of the set $\mathcal{D}_{\lambda,\mu}$ in terms of the set $\mathcal{T}(\lambda,\mu)$ of generalized tableaux T of shape λ and content μ. Such a generalized tableau T is a filling of the r boxes of the Young diagram of shape λ with μ_1 entries equal to 1, μ_2 entries equal to 2, etc. We denote by $T_\mu^\lambda \in \mathcal{T}(\lambda,\mu)$ the generalized tableau which has its boxes filled in the natural order. If we number the boxes of the Young diagram of λ according to the entries of $t^\lambda \in \mathcal{T}(\lambda)$, we may view $\mathcal{T}(\lambda,\mu)$ as the set of functions $T\colon \{1,\ldots,r\} \to \mathbb{N}$ with the property $|T^{-1}(i)| = \mu_i$ for all $i \in \mathbb{N}$. The group W acts transitively on $\mathcal{T}(\lambda,\mu)$ from the left by $(wT)(i) := T(iw)$, for $T \in \mathcal{T}(\lambda,\mu)$, $w \in W$ and $i \in \{1,\ldots,r\}$. The stabilizer of T_μ^λ is equal to W_μ. This defines a bijection $W/W_\mu \xrightarrow{\sim} \mathcal{T}(\lambda,\mu)$, $wW_\mu \mapsto wT_\mu^\lambda$. We call two generalized tableaux $T_1, T_2 \in \mathcal{T}(\lambda,\mu)$ *row-equivalent* if they arise from each other by rearranging the entries within the rows, i.e., if $T_2 = wT_1$ for some $w \in W_\lambda$. If we denote the row equivalence class of $T \in \mathcal{T}(\lambda,\mu)$ by $\{T\}$ and the set of such classes by $\overline{\mathcal{T}}(\lambda,\mu)$ then we have obtain a bijection $W_\lambda \backslash W / W_\mu \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda,\mu)$, $W_\lambda w W_\mu \mapsto \{wT_\mu^\lambda\}$. A generalized tableau $T \in \mathcal{T}(\lambda,\mu)$ is called *row-semistandard* if in each of its rows the entries are in their natural order from left to right. We denote the set of these tableaux by $\mathcal{T}^{\mathrm{rs}}(\lambda,\mu)$. Each row-equivalence class contains a unique row-semistandard element. Thus, $\mathcal{T}^{\mathrm{rs}}(\lambda,\mu) \to \overline{\mathcal{T}}(\lambda,\mu)$, $T \mapsto \{T\}$, is a bijection. Altogether, we now have canonical bijections

$$\mathcal{D}_{\lambda,\mu} \xrightarrow{\sim} W_\lambda \backslash W / W_\mu \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda,\mu) \xleftarrow{\sim} \mathcal{T}^{\mathrm{rs}}(\lambda,\mu) \tag{8}$$

and we may use each of these sets to parametrize the basis $\varphi_d^{\lambda,\mu}$, $d \in \mathcal{D}_{\lambda,\mu}$, of $\operatorname{Hom}_{\mathcal{H}}(M^\mu, M^\lambda)$. So, if $T \in \mathcal{T}^{\mathrm{rs}}(\lambda,\mu)$ and $d \in \mathcal{D}_{\lambda,\mu}$ correspond under the above bijection, we also write $\varphi_T^{\lambda,\mu}$ instead of $\varphi_d^{\lambda,\mu}$.

We leave it to the reader to check that $d \in \mathcal{D}_{\lambda,\mu}$ corresponds to $T \in \mathcal{T}^{\mathrm{rs}}(\lambda,\mu)$ if and only if $\varphi_T^{\lambda,\mu} = \theta_T \colon M^\mu \to M^\lambda$ in the case $q = 1$, with θ_T the homomorphism defined in [BH, Subsection 1.5].

1.4 *The Specht modules S^λ.* Let $\lambda \in \Lambda$ be a partition of r. One defines elements

$$y_\lambda := \sum_{w \in W_\lambda} (-q)^{-l(w)} T_w \quad \text{and} \quad z_\lambda := x_\lambda T_{w_\lambda} y_{\lambda'} \in M^\lambda$$

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of $\mathcal{H}$, where $w_\lambda \in W$ is defined by the property that $(t^\lambda w_\lambda)' = t^{\lambda'}$. The *Specht module* S^λ is defined by

$$S^\lambda := z_\lambda \mathcal{H}.$$

Since $z_\lambda \in M^\lambda$, S^λ is an $\mathcal{H}$-submodule of M^λ.

A λ-tableau $t \in \mathcal{T}(\lambda)$ is called a *standard* λ-tableau if $t \in \mathcal{T}^{\text{rs}}(\lambda)$ and $t' \in \mathcal{T}^{\text{rs}}(\lambda')$. We denote the set of standard λ-tableaux by $\mathcal{T}^{\text{st}}(\lambda)$. More generally, for $\mu \in \Gamma$, we say that a generalized tableau $T \in \mathcal{T}(\lambda, \mu)$ is *standard* if $T \in \mathcal{T}^{\text{rs}}(\lambda, \mu)$ and if the entries of T are strictly increasing along each column from top to bottom. We denote the set of standard λ-tableaux of content μ by $\mathcal{T}^{\text{st}}(\lambda, \mu)$.

We extend the definition of S^λ, $\mathcal{T}^{\text{st}}(\lambda)$, and $\mathcal{T}^{\text{st}}(\lambda, \mu)$ to arbitrary compositions $\lambda \in \Gamma$ by setting $S^\lambda := 0$, $\mathcal{T}^{\text{st}}(\lambda) = \emptyset$, and $\mathcal{T}^{\text{st}}(\lambda, \mu) = \emptyset$ if λ is not a partition.

Recall that the *weak Bruhat order* $\leqslant$ on W is defined by $u \geqslant v$ if and only if there exists a reduced expression $v = s_1 \cdots s_n$ for v with $s_1, \ldots, s_n \in S$ and $k \in \{0, \ldots, n\}$ such that $u = s_1 \cdots s_k$. Given $\lambda \in \Lambda$, [DJ, Lemma 1.5] states that

$$\mathcal{D}_\lambda^{\text{st}} := \{d \in W \mid d \geqslant w_\lambda\} \xrightarrow{\sim} \mathcal{T}^{\text{st}}(\lambda), \quad d \mapsto t^\lambda d. \tag{9}$$

is a bijection. Thus, $\mathcal{D}_\lambda^{\text{st}} \subseteq \mathcal{D}_\lambda$. Note that for $d \in W$ one has $t^{\lambda'} d = (t^\lambda w_\lambda)' d = (t^\lambda w_\lambda d)'$ and therefore,

$$d \geqslant w_{\lambda'} \iff t^{\lambda'} d \in \mathcal{T}^{\text{st}}(\lambda') \iff t^\lambda w_\lambda d \in \mathcal{T}^{\text{st}}(\lambda) \iff w_\lambda d \in \mathcal{D}_\lambda^{\text{st}}. \tag{10}$$

By [DJ, Theorem 5.6], for given $\lambda \in \Lambda$, the elements $z_\lambda T_d$ with $w_{\lambda'} \leqslant d \in W$ form an R-basis of S^λ. By the above, this set has cardinality $|\mathcal{T}^{\text{st}}(\lambda')| = |\mathcal{T}^{\text{st}}(\lambda)|$, since $t \in \mathcal{T}(\lambda)$ is standard if and only if $t' \in \mathcal{T}(\lambda')$ is standard. Moreover, [DJ, Lemma 5.1] states that if $d \geqslant w_{\lambda'}$ and if one expands the basis element $z_\lambda T_d$ of S^λ in terms of the standard basis of M^λ, i.e., $z_\lambda T_d = \sum_{e \in \mathcal{D}_\lambda} \alpha_{d,e} x_\lambda T_e$, with $\alpha_{d,e} \in R$, then

$$\alpha_{d, w_\lambda d} = q^{l(d)} \tag{11}$$

and

$$\alpha_{d,e} \neq 0 \Rightarrow \left(e = w_\lambda d \text{ or } l(e) > l(w_\lambda d) \right). \tag{12}$$

Thus, if we set $\mathcal{D}_\lambda^{\text{ns}} := \mathcal{D}_\lambda \setminus \mathcal{D}_\lambda^{\text{st}}$ then the R-span of the elements $x_\lambda T_e$, $e \in \mathcal{D}_\lambda^{\text{ns}}$, is a direct complement of S^λ in M^λ, and we state for later reference:

> *For every $\lambda \in \Gamma$, the Specht module S^λ is R-free of rank $|\mathcal{T}^{\text{st}}(\lambda)|$ and S^λ has an R-complement in M^λ.* $\tag{13}$

2 Ascending Generalized Tableaux and Homomorphisms

In this section we introduce, for any compositions $\lambda, \mu \in \Gamma$, an R-submodule $\mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^{\mu}, M^{\lambda})$ of $\mathrm{Hom}_{\mathcal{H}}(M^{\mu}, M^{\lambda})$. In Proposition 2.2 we show that for $\varphi \in \mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^{\mu}, M^{\lambda})$ and $\psi \in \mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^{\nu}, M^{\mu})$ one has $\varphi \circ \psi \in \mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^{\nu}, M^{\lambda})$.

2.1 Let $\lambda, \mu \in \Gamma$ be compositions of r. We say that $T \in \mathcal{T}(\lambda, \mu)$ is *ascending* if, for every $i \in \mathbb{N}$, the i-th row of T contains only entries which are greater than or equal to i. As in [BH], we denote the set of ascending and row semistandard elements of $\mathcal{T}(\lambda, \mu)$ by $\mathcal{T}^{\wedge}(\lambda, \mu)$. One has

$$\mathcal{T}^{\wedge}(\lambda, \mu) \neq \emptyset \iff \mu \trianglelefteq \lambda \iff T_{\mu}^{\lambda} \in \mathcal{T}^{\wedge}(\lambda, \mu) \tag{14}$$

and $\mathcal{T}^{\wedge}(\lambda, \lambda) = \{T_{\lambda}^{\lambda}\}$.

We define $\mathcal{D}_{\lambda, \mu}^{\wedge}$ as the image of $\mathcal{T}^{\wedge}(\lambda, \mu)$ under the canonical bijection in (8). Moreover, we define

$$\mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^{\mu}, M^{\lambda}) := \bigoplus_{d \in \mathcal{D}_{\lambda, \mu}^{\wedge}} R\varphi_d^{\lambda, \mu} \subseteq \mathrm{Hom}_{\mathcal{H}}(M^{\mu}, M^{\lambda}). \tag{15}$$

By (14) we have

$$\mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^{\mu}, M^{\lambda}) \neq \{0\} \Rightarrow \mu \trianglelefteq \lambda.$$

For $i \in \mathbb{N}$, we define the intervals $P_i := \{\lambda_1 + \cdots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_i\}$ and $Q_i := \{\mu_1 + \cdots + \mu_{i-1} + 1, \ldots, \mu_1 + \cdots + \mu_i\}$. One has for any $w \in W$:

$$wT_{\mu}^{\lambda} \text{ is ascending}$$
$$\iff P_i w \subseteq Q_i \cup Q_{i+1} \cup \cdots \text{ for all } i \geqslant 1$$
$$\iff (P_i \cup P_{i+1} \cup \cdots)w \subseteq Q_i \cup Q_{i+1} \cup \cdots \text{ for all } i \geqslant 1.$$

We define $W_{\lambda, \mu}$ to be the set of elements $w \in W$ satisfying the above equivalent properties. Clearly, $W_{\lambda, \mu}$ is a union of double cosets in $W_{\lambda} \backslash W / W_{\mu}$. The following properties are also immediate from the definition:

$$\begin{aligned} W_{\lambda, \mu} \neq \emptyset \quad &\text{if and only if } \mu \trianglelefteq \lambda, \\ W_{\lambda, \lambda} &= W_{\lambda}, \\ W_{\lambda, \mu} W_{\mu, \nu} &\subseteq W_{\lambda, \nu} \quad \text{if } \nu \trianglelefteq \mu \trianglelefteq \lambda, \\ W_{\lambda} 1 W_{\mu} &\subseteq W_{\lambda, \mu} \quad \text{if } \mu \trianglelefteq \lambda, \\ W_{\lambda, \mu} \cup W_{\mu, \nu} &\subseteq W_{\lambda, \nu} \quad \text{if } \nu \trianglelefteq \mu \trianglelefteq \lambda. \end{aligned} \tag{16}$$

The goal of this section is the proof of the following proposition.

2.2 Proposition *Let* $\lambda, \mu, \nu \in \Gamma$ *be compositions of* r *and let* $\alpha \in \mathrm{Hom}^{\wedge}_{\mathcal{H}}(M^{\nu}, M^{\mu})$ *and* $\beta \in \mathrm{Hom}^{\wedge}_{\mathcal{H}}(M^{\mu}, M^{\lambda})$. *Then* $\beta \circ \alpha \in \mathrm{Hom}^{\wedge}_{\mathcal{H}}(M^{\nu}, M^{\lambda})$.

Before we can prove the proposition, we need three lemmas.

2.3 Lemma *Let* $\lambda, \mu \in \Gamma$ *be compositions of* r *and let* $w \in W_{\lambda,\mu}$ *(in particular,* $\mu \trianglelefteq \lambda$*). If* $w = sv$ *with* $s \in S$, $v \in W$ *and* $l(w) = l(v) + 1$ *then there exists* $\rho \in \Gamma$ *such that* $\mu \trianglelefteq \rho \trianglelefteq \lambda$, $s \in W_{\lambda,\rho}$ *and* $v \in W_{\rho,\mu}$.

Proof Write $s = (i, i+1)$ with $i \in \{1, \ldots, r-1\}$ and note that $iw > (i+1)w$, since $l(w) > l(sw)$. Let $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ be the subsets of $\{1, \ldots, r\}$ associated to λ and μ, respectively, as above. If there exists $j \in \mathbb{N}$ with $\{i, i+1\} \in P_j$ then $s \in W_{\lambda}$ and $\rho := \lambda$ satisfies the required conditions, since $W_{\lambda} = W_{\lambda,\lambda}$ and $v = sw \in W_{\lambda}W_{\lambda,\mu} = W_{\lambda,\mu}$. Hence, we may assume that there exist positive integers $j < k$ such that $i \in P_j$ and $i+1 \in P_k$ (we need to allow the possibility that $P_{j+1} = \cdots = P_{k-1} = \emptyset$). Set $P_l' := P_l$ for $l \notin \{j, k\}$, and set $P_j' := P_j \smallsetminus \{i\}$ and $P_k' := P_k \cup \{i\}$. Moreover, let $\rho \in \Gamma$ be the composition defined by $P_1', P_2', \ldots$. Then clearly, $\rho \trianglelefteq \lambda$ and $s \in W_{\lambda,\rho}$. We still need to show that $v \in W_{\rho,\mu}$. This also implies $\mu \trianglelefteq \rho$ by (16). For $l \notin \{j+1, \ldots, k\}$ we have $P_l' \cup P_{l+1}' \cup \cdots = P_l \cup P_{l+1} \cup \cdots$ and $(P_l' \cup P_{l+1}' \cup \cdots)v = (P_l \cup P_{l+1} \cup \cdots)v = (P_l \cup P_{l+1} \cup \cdots)sv = (P_l \cup P_{l+1} \cup \cdots)w \subseteq Q_l \cup Q_{l+1} \cup \cdots$. For $l \in \{j+1, \ldots, k\}$ we have $i+1 \in P_l' \cup P_{l+1}' \cup \cdots = (P_l \cup P_{l+1} \cup \cdots) \cup \{i\}$ and $(P_l' \cup P_{l+1}' \cup \cdots)v = (\{i\} \cup P_l \cup P_{l+1} \cup \cdots)v = (\{i\} \cup P_l \cup P_{l+1} \cup \cdots)sv = (\{i\} \cup P_l \cup P_{l+1} \cup \cdots)w \subseteq \{iw\} \cup Q_l \cup Q_{l+1} \cup \cdots$, since $w \in W_{\lambda,\mu}$. It suffices to show that $iw \in Q_l \cup Q_{l+1} \cup \cdots$. But since $w \in W_{\lambda,\mu}$, we have $(i+1)w \in (P_k \cup P_{k+1} \cup \cdots)w \subseteq Q_k \cup Q_{k+1} \cup \cdots \subseteq Q_l \cup Q_{l+1} \cup \cdots$, and $iw > (i+1)w$ now implies also $iw \in Q_l \cup Q_{l+1} \cup \cdots$. This completes the proof of the lemma. $\square$

2.4 Lemma *Let* $\lambda, \mu \in \Gamma$ *and let* $w \in W_{\lambda,\mu}$. *If* $u \in W$ *satisfies* $u \trianglerighteq w$ *then also* $u \in W_{\lambda,\mu}$.

Proof We proceed by induction on $l(w)$. Note that $\mu \trianglelefteq \lambda$, since $W_{\lambda,\mu} \neq \emptyset$. If $w = 1$ we have $u = 1$ and $u = w \in W_{\lambda,\mu}$. Now assume that $l(w) \geqslant 1$ and write $w = sv$ with $l(w) = l(v) + 1$ and $s \in S$. By Lemma 2.3 there exists $\rho \in \Gamma$ such that $\mu \trianglelefteq \rho \trianglelefteq \lambda$, $s \in W_{\lambda,\rho}$ and $v \in W_{\rho,\mu}$. Since $u \trianglerighteq w = sv$ and $l(w) > l(v)$, we have $u \trianglerighteq v$ or $su \trianglerighteq v$. If $u \trianglerighteq v$ then $l(u) \leqslant l(v) < l(w)$

and, by induction, $v \in W_{\rho,\mu}$ implies $u \in W_{\rho,\mu}$. But $W_{\rho,\mu} \subseteq W_{\lambda,\mu}$ by (16). If $su \trianglerighteq v$ then, again by induction, $v \in W_{\rho,\mu}$ implies $su \in W_{\rho,\mu}$, and further, $u = s(su) \in W_{\lambda,\rho}W_{\rho,\mu} \subseteq W_{\lambda,\mu}$. Now the proof is complete. $\square$

2.5 Lemma Let $\lambda, \mu, \nu \in \Gamma$, $v \in W_{\lambda,\mu}$, and $w \in W_{\mu,\nu}$. Then $T_v T_w \in \sum_{u \in W_{\lambda,\nu}} RT_u$.

Proof This follows immediately from Equation (1), Lemma 2.4 and $W_{\lambda,\mu}W_{\mu,\nu} \subseteq W_{\lambda,\nu}$. $\square$

Proof of Proposition 2.2 By the definition in (15) we may assume that $\alpha = \varphi_d^{\mu,\nu}$ and $\beta = \varphi_e^{\lambda,\mu}$ for some $d \in \mathcal{D}_{\mu,\nu}^\wedge$ and $e \in \mathcal{D}_{\lambda,\mu}^\wedge$. By Equation (7) and since $d \in W_{\mu,\nu}$ and $e \in W_{\lambda,\mu}$, we have $\varphi_d^{\mu,\nu}(x_\nu) \subseteq x_\mu \cdot \sum_{w \in W_{\mu,\nu}} RT_w$ and $\varphi_e^{\lambda,\mu}(x_\mu) \subseteq \sum_{v \in W_{\lambda,\mu}} RT_v$. Together with Lemma 2.5 this implies

$$(\varphi_e^{\lambda,\mu} \circ \varphi_d^{\mu,\nu})(x_\nu) \in \varphi_e^{\lambda,\mu}(x_\mu) \cdot \sum_{w \in W_{\mu,\nu}} RT_w \subseteq \sum_{\substack{v \in W_{\lambda,\mu} \\ w \in W_{\mu,\nu}}} RT_v T_w$$

$$\subseteq \sum_{u \in W_{\lambda,\nu}} RT_u = \sum_{f \in \mathcal{D}_{\lambda,\nu}^\wedge} \sum_{u \in W_\lambda f W_\nu} RT_u \,.$$

Since the elements $\varphi_f^{\lambda,\nu}$, $f \in \mathcal{D}_{\lambda,\nu}$, form an R-basis of $\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\lambda)$ we can write $\varphi_e^{\lambda,\mu} \circ \varphi_d^{\mu,\nu} = \sum_{f \in \mathcal{D}_{\lambda,\nu}} a_f \varphi_f^{\lambda,\nu}$ with $a_f \in R$, for $f \in \mathcal{D}_{\lambda,\nu}$. Thus,

$$(\varphi_e^{\lambda,\mu} \circ \varphi_d^{\mu,\nu})(x_\nu) = \sum_{f \in \mathcal{D}_{\lambda,\nu}} a_f \varphi_f^{\lambda,\nu}(x_\nu) = \sum_{f \in \mathcal{D}_{\lambda,\nu}} a_f \sum_{u \in W_\lambda f W_\nu} T_u \,.$$

Comparing this with the above yields $a_f = 0$ for all $f \in \mathcal{D}_{\lambda,\nu} \smallsetminus \mathcal{D}_{\lambda,\nu}^\wedge$, and the proof is complete. $\square$

3 The Chain Complex $\widetilde{C}_*^\lambda$

Throughout this section we fix a composition $\lambda \in \Gamma$. We will use the R-modules $\mathrm{Hom}_{\mathcal{H}}^\wedge(M^\mu, M^\lambda)$ from Section 2 and the result from Proposition 2.2 to construct a chain complex $\widetilde{C}_*^\lambda$ of left $\mathcal{H}$-modules. In the case $q = 1$, this chain complex coincides with the chain complex constructed in [BH] for the symmetric group algebra.

3.1 *The definition of $\tilde{C}_*^\lambda$.* For every strictly ascending chain

$$\gamma = (\lambda^{(0)} \lhd \cdots \lhd \lambda^{(n)}) \tag{17}$$

of *length n* in Γ we set

$$M_\gamma := \operatorname{Hom}_{\mathcal{H}}^\wedge(M^{\lambda^{(0)}}, M^{\lambda^{(1)}}) \otimes \operatorname{Hom}_{\mathcal{H}}^\wedge(M^{\lambda^{(1)}}, M^{\lambda^{(2)}}) \otimes \cdots$$
$$\cdots \otimes \operatorname{Hom}_{\mathcal{H}}^\wedge(M^{\lambda^{(n-1)}}, M^{\lambda^{(n)}}) \otimes \operatorname{Hom}(M^{\lambda^{(n)}}, R).$$

We view M_γ as left $\mathcal{H}$-module via

$$h(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \varepsilon) := \varphi_1 \otimes \cdots \otimes \varphi_n \otimes h\varepsilon,$$

for $h \in \mathcal{H}$, $\varphi_i \in \operatorname{Hom}_{\mathcal{H}}^\wedge(M^{\lambda^{(i-1)}}, M^{\lambda^{(i)}})$, $i = 1, \ldots, n$, and $\varepsilon \in \operatorname{Hom}(M^{\lambda^{(n)}}, R)$, where $(h\varepsilon)(m) := \varepsilon(mh)$ for $m \in M^{\lambda^{(n)}}$.

For every integer $n \geqslant 0$, we write Δ_n^λ for the set of chains γ as in (17) with $\lambda^{(0)} = \lambda$. Further, we define the left $\mathcal{H}$-module

$$C_n^\lambda := \bigoplus_{\gamma \in \Delta_n^\lambda} M_\gamma.$$

For $n \geqslant 1$, $\gamma \in \Delta_n^\lambda$ and $i \in \{1, \ldots, n\}$, we denote by $\gamma_i \in \Delta_{n-1}^\lambda$ the chain obtained from γ by omitting $\lambda^{(i)}$. We define an $\mathcal{H}$-module homomorphism

$$d_{n,i}^\lambda \colon M_\gamma \to M_{\gamma_i}$$

by

$$\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \varepsilon \mapsto \varphi_1 \otimes \cdots \otimes \varphi_{i+1} \circ \varphi_i \otimes \cdots \otimes \varphi_n \otimes \varepsilon,$$

where φ_{i+1} is interpreted as ε if $i = n$. The direct sum of these homomorphisms yields an $\mathcal{H}$-module homomorphism $d_{n,i}^\lambda \colon C_n^\lambda \to C_{n-1}^\lambda$ and we set

$$d_n^\lambda := \sum_{i=1}^n (-1)^{i-1} d_{n,i}^\lambda \colon C_n^\lambda \to C_{n-1}^\lambda.$$

Since the maps $d_{n,i}^\lambda \colon C_n^\lambda \to C_{n-1}^\lambda$ satisfy the usual simplicial relations, we obtain $d_n^\lambda \circ d_{n+1}^\lambda = 0$ for $n \geqslant 1$. Thus, we have constructed a chain complex

$$C_*^\lambda \colon \quad 0 \longrightarrow C_{a(\lambda)}^\lambda \xrightarrow{d_{a(\lambda)}^\lambda} C_{a(\lambda)-1}^\lambda \xrightarrow{d_{a(\lambda)-1}^\lambda} \cdots \xrightarrow{d_1^\lambda} C_0^\lambda \longrightarrow 0$$

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of finitely generated left $\mathcal{H}$-modules and $\mathcal{H}$-module homomorphisms. Here, $a(\lambda)$ is defined as the length of the longest possible strictly ascending chain γ as in (17) with $\lambda^{(0)} = \lambda$, in other words, $a(\lambda)$ is the largest integer n with $\Delta_n^\lambda \neq \emptyset$.

Note that the definition of d_n^λ implies immediately that

$$
\begin{aligned}
d_n^{\lambda^{(0)}}(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \varepsilon) = (\varphi_2 \circ \varphi_1) \otimes \cdots \otimes \varphi_n \otimes \varepsilon \\
- \varphi_1 \otimes d_{n-1}^{\lambda^{(1)}}(\varphi_2 \otimes \cdots \otimes \varphi_n \otimes \varepsilon),
\end{aligned} \tag{18}
$$

for $n \geqslant 2$, γ as in (17), $\varphi_i \in \mathrm{Hom}_{\mathcal{H}}^\wedge(M^{\lambda^{(i-1)}}, M^{\lambda^{(i)}})$, $i = 1, \ldots, n$, and $\varepsilon \in \mathrm{Hom}(M^{\lambda^{(n)}}, R)$.

Finally, we extend the chain complex C_*^λ by the $\mathcal{H}$-module homomorphism

$$
d_0^\lambda \colon C_0^\lambda = \mathrm{Hom}(M^\lambda, R) \to \mathrm{Hom}(S^\lambda, R) =: C_{-1}^\lambda, \quad \varepsilon \mapsto \varepsilon|_{S^\lambda},
$$

and obtain a chain complex

$$
\widetilde{C}_*^\lambda \colon \quad 0 \longrightarrow C_{a(\lambda)}^\lambda \xrightarrow{d_{a(\lambda)}^\lambda} C_{a(\lambda)-1}^\lambda \xrightarrow{d_{a(\lambda)-1}^\lambda} \cdots \xrightarrow{d_1^\lambda} C_0^\lambda \xrightarrow{d_0^\lambda} C_{-1}^\lambda \longrightarrow 0
$$

in the category of finitely generated left $\mathcal{H}$-modules. In fact, in the next proposition we show that $d_0^\lambda \circ d_1^\lambda = 0$. Note that $C_{-1}^\lambda = 0$ if λ is not a partition. Also note that every $\mathcal{H}$-module in the chain complex $\widetilde{C}_*^\lambda$ is finitely generated and free as R-module so that after applying the functor $\mathrm{Hom}(-, R)$ we obtain a chain complex starting with $0 \to S^\lambda \to M^\lambda \to \cdots$ and involving direct sums of modules of the form M^μ with $\mu \triangleright \lambda$ from there on.

3.2 Proposition *With the notation from 3.1 one has $d_0^\lambda \circ d_1^\lambda = 0$.*

Proof We may assume that λ is a partition, since otherwise $C_{-1}^\lambda = 0$. Let $\lambda \triangleleft \mu$ be a chain of length 1 in Γ and let $d \in \mathcal{D}_{\mu,\lambda}$. It suffices to show that $\varphi_d^{\mu,\lambda}(S^\lambda) = 0$. Since $S^\lambda = z_\lambda \mathcal{H}$, it suffices to show that $\varphi_d^{\mu,\lambda}(z_\lambda) = 0$. But,

$$
\varphi_d^{\mu,\lambda}(z_\lambda) = \varphi_d^{\mu,\lambda}(x_\lambda T_{w_\lambda} y_{\lambda'}) = \varphi_d^{\mu,\lambda}(x_\lambda) T_{w_\lambda} y_{\lambda'} \in x_\mu \mathcal{H} y_{\lambda'}
$$

and it suffices to show that $x_\mu T_w y_{\lambda'} = 0$ for all $w \in W$. So assume that $x_\mu T_w y_{\lambda'} \neq 0$ for some $w \in W$. Then [DJ, Lemma 4.1] implies $\lambda'' \geqslant \mu$, with respect to relation $\leqslant$ defined in (3). But $\lambda'' \geqslant \mu$ is equivalent to $\lambda''' \trianglelefteq \mu'$. Now $\lambda''' = \lambda'$ and $\lambda' \trianglelefteq \mu'$ implies $\mu'' \trianglelefteq \lambda''$. Since λ is a partition, we have

$\lambda'' = \lambda$ and obtain $\mu \trianglelefteq \mu'' \trianglelefteq \lambda$, a contradiction to $\lambda \triangleleft \mu$. This shows that $x_\mu T_w y_{\lambda'} = 0$ for all $w \in W$ and the proof is complete. $\square$

3.3 For $\lambda \in \Gamma$ and an integer $n \geqslant 0$ we denote by $\mathcal{B}_n^\lambda$ the set of symbols

$$(\lambda^{(0)} \underset{T_1}{\triangleleft} \lambda^{(1)} \underset{T_2}{\triangleleft} \cdots \underset{T_n}{\triangleleft} \lambda^{(n)}, t) \tag{19}$$

with $\gamma = (\lambda^{(0)} \triangleleft \cdots \triangleleft \lambda^{(n)}) \in \Delta_n^\lambda$ (so $\lambda^{(0)} = \lambda$) and $T_i \in \mathcal{T}^\wedge(\lambda^{(i)}, \lambda^{(i-1)})$ for $i = 1, \ldots, n$, and with $t \in \mathcal{T}^{\mathrm{rs}}(\lambda^{(n)})$. By the definition of C_n^λ, the elements of $\mathcal{B}_n^\lambda$ parametrize and R-basis of C_n^λ, by associating the symbol in (19) with the element

$$\varphi_{T_1}^{\lambda^{(1)}, \lambda^{(0)}} \otimes \cdots \otimes \varphi_{T_n}^{\lambda^{(n)}, \lambda^{(n-1)}} \otimes \varepsilon_t \in M_\gamma \subseteq C_n^\lambda.$$

Note that $\mathcal{B}_0^\lambda = \mathcal{T}^{\mathrm{rs}}(\lambda)$. For completeness we set $\mathcal{B}_{-1}^\lambda := \mathcal{T}^{\mathrm{st}}(\lambda)$ and associate to a standard tableau $t \in \mathcal{T}^{\mathrm{st}}(\lambda)$ the element $\psi_t \in \mathrm{Hom}(S^\lambda, R)$ with the property $\psi_t(z_\lambda T_d) = 1$ if $d \in \mathcal{D}_\lambda^{\mathrm{st}}$ corresponds to t under the bijection (9) and $\psi_t(z_\lambda T_d) = 0$ otherwise.

4 Exactness Results

The goal of this section is to state three exactness results, Theorems 4.2, 4.4 and 4.5, on the chain complex $\widetilde{C}_*^\lambda$. The lengthy and technical proof of a key lemma, Lemma 4.3, which is needed in the proof of the two latter theorems, will be postponed to the next section. The strategy of the proofs is the same as in [BH]. The proofs in this section are adaptations of the proofs in [BH]. We will present them for the reader's convenience in the Hecke algebra setting. However, the proof of Lemma 4.3 is more difficult if $q \neq 1$.

4.1 Assume that $\lambda \in \Gamma$, that $n \geqslant -1$ is an integer and that K_n^λ is an R-submodule of C_n^λ satisfying

$$\mathrm{im}(d_{n+1}^\lambda) + K_n^\lambda = C_n^\lambda \tag{A_n^λ}$$

and

$$\ker(d_n^\lambda) \cap K_n^\lambda = \{0\}. \tag{B_n^λ}$$

Then it is easy to see that the chain complex $\widetilde{C}^\lambda_*$ is exact in degree n. In fact, the conditions (A^λ_n) and (B^λ_n) together are equivalent to

$$\operatorname{im}(d^\lambda_{n+1}) = \ker(d^\lambda_n) \quad \text{and} \quad \ker(d^\lambda_n) \oplus K^\lambda_n = C^\lambda_n.$$

We will produce, for certain choices of λ and n, submodules K^λ_n which satisfy (A^λ_n) and (B^λ_n), by taking the R-span of the basis elements of C^λ_n parametrized by a subset $\mathcal{K}^\lambda_n$ of the canonical R-basis $\mathcal{B}^\lambda_n$, cf. Subsection 3.3. By abuse of notation we then will say that $\mathcal{K}^\lambda_n$ satisfies (A^λ_n) or (B^λ_n), if its R-span does.

4.2 Theorem *The chain complex $\widetilde{C}^\lambda_*$ is exact in degree -1 for every composition $\lambda \in \Gamma$.*

Proof Since S^λ has an R-complement in M^λ, cf. (13), the restriction map $\operatorname{Hom}(M^\lambda, R) \to \operatorname{Hom}(S^\lambda, R)$ is surjective, and the proof is complete. $\qquad\square$

The following lemma will be a key ingredient to the proof of Theorem 4.4, which in turn forms the base case for the inductive proof of Theorem 4.5. To state the lemma we need to introduce the following notation. For $\lambda \in \Gamma$, every tableau $t \in \mathcal{T}(\lambda)$ can be thought of as a sequence of numbers by appending the second row at the end of the first, and so on. On these sequences we have the lexicographic total order. We set $t' < t$ if and only if the first entry in the sequence of t which differs from the corresponding entry of t' is greater than the one for t'. Moreover, for $t \in \mathcal{T}^{\mathrm{rs}}(\lambda)$ we denote by $C^\lambda_{0,<t}$ the R-span of the canonical basis elements $\varepsilon_{t'}$ of $\operatorname{Hom}(M^\lambda, R)$ with $t > t' \in \mathcal{T}^{\mathrm{rs}}(\lambda)$. See the end of Subsection 1.2 for the definition of ε_t, $t \in \mathcal{T}^{\mathrm{rs}}(\lambda)$.

4.3 Lemma *Let $\lambda \in \Gamma$ be a composition and let $t \in \mathcal{T}^{\mathrm{rs}}(\lambda)$. If t is not standard then $\varepsilon_t \in \operatorname{im}(d^\lambda_1) + C^\lambda_{0,<t}$.*

The proof of Lemma 4.3 will be given in the next section.

4.4 Theorem *Let $\lambda \in \Gamma$ be a composition and set*

$$\mathcal{K}^\lambda_0 := \left\{ (\lambda, t) \in \mathcal{B}^\lambda_0 \mid t \in \mathcal{T}^{\mathrm{st}}(\lambda) \right\}.$$

Then (A^λ_0) and (B^λ_0) are satisfied. In particular, $\widetilde{C}^\lambda_*$ is exact in degree 0 for every composition $\lambda \in \Gamma$.

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Proof In order to show that (A_0^λ) holds for $\mathcal{K}_0^\lambda$, we order the elements in $\mathcal{T}^{rs}(\lambda)$ in ascending lexicographic order: $t_1 < t_2 < \cdots < t_n$. For every $i \in \{1, \ldots, n\}$, Lemma 4.3 implies that $t_i \in \mathcal{T}^{st}(\lambda)$ or $\varepsilon_{t_i} \in \operatorname{im}(d_1^\lambda) + C_{0,<t_i}^\lambda$. An easy induction argument on $i \in \{1, \ldots, n\}$ now shows that $\varepsilon_{t_i} \in \operatorname{im}(d_i^\lambda) + K_0^\lambda$, and (A_0^λ) holds.

If λ is not a partition then (B_0^λ) holds trivially. Suppose that λ is a partition. In order to show that (B_0^λ) holds for $\mathcal{K}_0^\lambda$, we number the standard λ-tableaux $t_1, \ldots, t_m$ in such a way that the following holds for $i, j \in \{1, \ldots, m\}$: If $t_i = t^\lambda w_\lambda d_i$ with unique $w_{\lambda'} \leqslant d_i \in W$, cf. (9) and (10), then $l(w_\lambda d_i) < l(w_\lambda d_j)$ implies $i > j$. Next assume that there exists $0 \neq \varepsilon \in K_0^\lambda \cap \ker(d_0^\lambda)$ and write $\varepsilon = \sum_{i=1}^m a_i \varepsilon_{t_i}$ with $a_i \in R$. Let i be minimal with $a_i \neq 0$. Then, since $0 = d_0^\lambda(\varepsilon) = \varepsilon|_{S^\lambda}$, we have

$$0 = \varepsilon(z_\lambda T_{d_i}) = \sum_{j=i}^m a_j \varepsilon_{t_j}(z_\lambda T_{d_i}).$$

By (11) we obtain $\varepsilon_{t_i}(z_\lambda T_{d_i}) = q^{l(d_i)}$. Moreover, by (12) we obtain $\varepsilon_{t_j}(z_\lambda T_{d_i}) = 0$ for $j > i$. This implies $0 = a_i q^{l(d_i)}$, and since q is a unit, we have $a_i = 0$, a contradiction. Thus, (B_0^λ) holds and the proof is complete. □

We recall from [BH] that a composition $\mu \in \Gamma$ is called a *quasi-partition* if $\mu^* = \bar{\mu}$. Here, $\bar{\mu}$ is the unique smallest partition in Λ which dominates μ (cf. [BH, Lemma 4.2]) and $\mu^* \in \Lambda$ denotes the unique partition obtained by reordering the parts $\mu_1, \mu_2, \ldots$ of μ in weakly descending order. Recall also that a composition $\lambda \in \Gamma$ is called *tame* if every composition $\mu \in \Gamma$ with $\lambda \trianglelefteq \mu$ is a *quasi-partition*. Tame compositions are classified in [BH, Proposition 4.6]. They include all partitions of the form $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_3 \in \{0, 1\}$.

The technical definition of a tame composition is dictated by the inductive proof of the following theorem. The proof is a straight forward adaptation of the proof of Theorem 5.1 in [BH] and will not be repeated here.

4.5 Theorem *Let $\lambda \in \Gamma$ be a tame composition. For $n = -1$ set $\mathcal{K}_{-1}^\lambda := \emptyset$, for $n = 0$ set $\mathcal{K}_0^\lambda := \mathcal{T}^{st}(\lambda)$ and for $n \geqslant 1$ set*

$$\mathcal{K}_n^\lambda := \left\{ (\lambda = \lambda^{(0)} \underset{T_1}{\triangleleft} \lambda^{(1)} \underset{T_2}{\triangleleft} \cdots \underset{T_n}{\triangleleft} \lambda^{(n)}, t) \in \mathcal{B}_n^\lambda \mid t \in \mathcal{T}^{st}(\lambda^{(n)} \right\}.$$

Then (A_n^λ) and (B_n^λ) hold for all $n \geqslant -1$. In particular, the chain complex $\widetilde{C}_*^\lambda$ is exact.

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5   Proof of Lemma 4.3

The exclusive goal of this section is to prove Lemma 4.3. The proof is a substantial refinement of the proof of [BH, Lemma 3.4]. Throughout we fix a composition $\lambda \in \Gamma$ and a λ-tableau $t \in \mathcal{T}^{\mathrm{rs}}(\lambda) \smallsetminus \mathcal{T}^{\mathrm{st}}(\lambda)$. We first need to establish some notation that will be used throughout this section.

5.1 Since $t \in \mathcal{T}^{\mathrm{rs}}(\lambda)$ is not standard, there exist consecutive rows of t, say rows k and $k+1$, which we write as

$$a_1 \ a_2 \ \cdots \ a_m \quad (m = \lambda_k)$$

and

$$b_1 \ b_2 \ \cdots \ b_n \quad (n = \lambda_{k+1}),$$

such that there exist $i \in \{0, \ldots, \min\{m, n-1\}\}$ satisfying

$$a_1 < b_1, \ a_2 < b_2, \ \cdots, a_i < b_i, \quad \text{and} \quad (a_{i+1} > b_{i+1} \text{ or } m = i).$$

The case $m = i$ can arise when $\lambda_{k+1} > \lambda_k$. We also set

$$Z := \{a_1, \ldots, a_m, b_1, \ldots, b_n\}.$$

For every subset $X \subseteq Z$ with $|X| \leqslant n$ we set

$$\mu_X := (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + n - |X|, |X|, \lambda_{k+2}, \ldots) \in \Gamma$$

and denote by $t_X \in \mathcal{T}^{\mathrm{rs}}(\mu_X)$ the tableau which coincides with t in all rows except row k and row $k+1$, and which has the elements of X in row $k+1$ and those of $Z \smallsetminus X$ in row k. Moreover, we denote by $d_X \in \mathcal{D}_{\mu_X}$ the element which satisfies $t_X = t^{\mu_X} d_X$, and by $\varepsilon_X \in \mathrm{Hom}(M^{\mu_X}, R)$ the element $\varepsilon_{t_X} = \varepsilon_{d_X}$. We will also use the composition

$$\nu_X := (\lambda_1, \ldots, \lambda_k, n - |X|, |X|, \lambda_{k+2}, \ldots) \in \Gamma.$$

We then have

$$\nu_X \trianglelefteq \lambda \trianglelefteq \mu_X \quad \text{and} \quad W_{\nu_X} = W_{\mu_X} \cap W_\lambda.$$

Finally, we define the adjacent, pairwise disjoint intervals of integers,

$$A_X := \{\lambda_1 + \cdots + \lambda_{k-1} + 1, \lambda_1 + \cdots + \lambda_{k-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_{k-1} + \lambda_k\},$$
$$B_X := \{\lambda_1 + \cdots + \lambda_k + 1, \ldots, \lambda_1 + \cdots + \lambda_k + (n - |X|)\},$$
$$C_X := \{\lambda_1 + \cdots + \lambda_k + (n - |X| + 1), \ldots, \lambda_1 + \cdots + \lambda_k + n\},$$

(with $n = \lambda_{k+1}$) and set

$$D := \{1, \ldots, r\} \smallsetminus (A_X \cup B_X \cup C_X)$$

Thus, (A_X, B_X, C_X) are the k-th, $(k+1)$-th, $(k+2)$-th respective subsets of the set partition associated with ν_X, $(A_X, B_X \cup C_X)$ are the k-th and $(k+1)$-th respective subsets of the set partition associated with λ, and $(A_X \cup B_X, C_X)$ are the k-th and $(k+1)$-th subsets associated with μ_X.

Recall from Subsection 1.3 that, for X as above, one has an element $\varphi_1^{\mu_X, \lambda} \in \mathrm{Hom}_{\mathcal{H}}^{\wedge}(M^\lambda, M^{\mu_X})$.

5.2 Lemma *Let $X \subseteq Z$ be a subset with $|X| \leqslant n$.*
  (a) *For $d \in \mathcal{D}_\lambda$, the set*

$$W_{X,d} := W_{\mu_X} \cap (\mathcal{D}_{\nu_X} \cap W_\lambda) dd_X^{-1}$$

has at most one element.
  (b) *In $\mathrm{Hom}(M^\lambda, R)$, one has the equation*

$$\varepsilon_X \circ \varphi_1^{\mu_X, \lambda} = \sum_{\substack{d \in \mathcal{D}_\lambda \\ W_{X,d} \neq \emptyset}} q^{l(w_{X,d})} \varepsilon_d \,,$$

where we write $W_{X,d} = \{w_{X,d}\}$ when $W_{X,d} \neq \emptyset$.

Proof (a) Let $d \in \mathcal{D}_\lambda$ and assume that one has elements $w_1, w_2 \in W_{\mu_X}$ and $e_1, e_2 \in \mathcal{D}_{\nu_X} \cap W_\lambda$ with $w_i = e_i dd_X^{-1}$ for $i = 1, 2$. Then $e_1 e_2^{-1} = w_1 w_2^{-1} \in W_{\mu_X} \cap W_\lambda = W_{\nu_X}$. Since $e_1, e_2 \in \mathcal{D}_{\nu_X}$, we have $e_1 = e_2$ and $w_1 = w_2$.

(b) Recall that the elements $x_\lambda T_d$, $d \in \mathcal{D}_\lambda$, form an R-basis of M^λ. For $d \in \mathcal{D}_\lambda$ we have

$$\varepsilon_X \left(\varphi_1^{\mu_X, \lambda}(x_\lambda T_d) \right) = \varepsilon_X \left(\varphi_1^{\mu_X, \lambda}(x_\lambda) T_d \right) = \sum_{e \in \mathcal{D}_{\nu_X} \cap W_\lambda} \varepsilon_X(x_{\mu_X} T_{ed})$$

by (7). For every $e \in \mathcal{D}_{\nu_X} \cap W_\lambda$ there exists a unique $f_{d,e} \in \mathcal{D}_{\mu_X}$ with $ed \in W_{\mu_X} f_{e,d}$ and a unique element $w_{e,d} \in W_{\mu_X}$ with $ed = w_{e,d} f_{e,d}$. We continue the above computation:

$$\varepsilon_X \left(\varphi_1^{\mu_X, \lambda}(x_\lambda T_d) \right) = \sum_{e \in \mathcal{D}_{\nu_X} \cap W_\lambda} \varepsilon_X(x_{\mu_X} T_{w_{e,d} f_{e,d}})$$

$$= \sum_{e \in \mathcal{D}_{\nu_X} \cap W_\lambda} q^{l(w_{e,d})} \varepsilon_X(x_{\mu_X} T_{f_{e,d}}) = \sum_{\substack{e \in \mathcal{D}_{\nu_X} \cap W_\lambda \\ f_{e,d} = d_X}} q^{l(w_{e,d})} \,.$$

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But, if $e \in \mathcal{D}_{\nu_X} \cap W_\lambda$ satisfies $f_{e,d} = d_X$ then $w_{e,d} = edd_X^{-1} \in W_{X,d} = \{w_{X,d}\}$ and e is the only element in $\mathcal{D}_{\nu_X} \cap W_\lambda$ with $f_{e,d} = d_X$. Conversely, if $W_{X,d} \neq \emptyset$ and we write $w_{X,d} = edd_X^{-1}$ with $e \in \mathcal{D}_{\nu_X} \cap W_\lambda$ then e satisfies $f_{e,d} = d_X$. Thus, we have

$$
\varepsilon_X\left(\varphi_1^{\mu_X,\lambda}(x_\lambda T_d)\right) = \begin{cases} 0 & \text{if } W_{X,d} = \emptyset, \\ q^{l(w_{X,d})} & \text{if } W_{X,d} \neq \emptyset. \end{cases}
$$

Now the statement in (b) is immediate and the proof is complete. $\qquad\square$

5.3 Lemma *Let $X \subseteq Z$ be a subset with $|X| \leqslant n$.*
  (a) *The function*

$$
\{X \subseteq Y \subseteq Z \mid |Y| = n\} \to \mathcal{D}_\lambda, \quad Y \mapsto d_Y,
$$

is injective with image $\{d \in \mathcal{D}_\lambda \mid W_{X,d} \neq \emptyset\}$.
  (b) *In $\mathrm{Hom}(M^\lambda, R)$, one has*

$$
\varepsilon_X \circ \varphi_1^{\mu_X,\lambda} = \sum_{\substack{X \subseteq Y \subseteq Z \\ |Y| = n}} q^{l(w_{X,Y})} \varepsilon_Y,
$$

where $w_{X,Y} := w_{X,d_Y}$.
  (c) *For $X \subseteq Y \subseteq Z$ with $|Y| = n$ one has*

$$
l(w_{X,Y}) = \sum_{y \in Y \smallsetminus X} |\{z \in Z \smallsetminus Y \mid y < z\}|.
$$

Proof (a) For $X \subseteq Y \subseteq Z$ with $|Y| = n$ we have $d_Y \in \mathcal{D}_\lambda$, since $\mu_Y = \lambda$. Moreover, the map $Y \mapsto d_Y$ is clearly injective. Our next goal is to show that $W_{X,d_Y} \neq \emptyset$. Note that $X \subseteq Y = (B_X \cup C_X)d_Y$ implies $(X)d_Y^{-1} \subseteq B_X \cup C_X$. We define an element $e \in W$ by

$$
\begin{aligned}
&e|_{A_X \cup D} \text{ is the identity}, \\
&e|_{C_X} : C_X \xrightarrow{\sim} (X)d_Y^{-1} \text{ is monotonous}, \\
&e|_{B_X} : B_X \xrightarrow{\sim} (B_X \cup C_X) \smallsetminus (X)d_Y^{-1} \text{ is monotonous.}
\end{aligned} \qquad (20)
$$

Then, $e \in \mathcal{D}_{\nu_X}$, since e is monotonous on $A_X \cup B_X$, on C_X and on D. Moreover, $e \in W_\lambda$, since $(B_X \cup C_X)e = B_X \cup C_X$ and e is the identity on $A_X \cup D$. We set

$$
w := e d_Y d_X^{-1}. \qquad (21)
$$

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We claim that

$$w|_{C_X \cup D} \text{ is the identity map.} \tag{22}$$

In fact, for $x \in D$ we have $(x)ed_Y d_X^{-1} = (x)d_Y d_X^{-1} = x$, since d_X and d_Y coincide on D. Also, $w|_{C_X}$ is the composition

$$C_X \xrightarrow{\;e\;} (X)d_Y^{-1} \xrightarrow{\;d_Y\;} X \xrightarrow{\;d_X^{-1}\;} C_X \;.$$

This composition is monotonous, since each of the three factors is monotonous (because $e \in \mathcal{D}_{\nu_X}$, $d_Y \in \mathcal{D}_\lambda$, $d_X \in \mathcal{D}_{\mu_X}$ and $(C_X)e \subseteq B_X \cup C_X$). Thus, $w|_{C_X}$ is the identity map and Claim (22) is proven. Since w is the identity on $D \cup C_X$, we obtain $(A_X \cup B_X)w = A_X \cup B_X$. This implies $w \in W_{\mu_X}$ and $w \in W_{X,d_Y}$.

Conversely, let $d \in \mathcal{D}_\lambda$ with $W_{X,d} \neq \emptyset$. Then there exits $w \in W_{\mu_X}$ and $e \in \mathcal{D}_{\nu_X} \cap W_\lambda$ with $w = edd_X^{-1}$. Since $w \in W_{\mu_X}$, we have $(C_X)w = C_X$, and since $e \in W_\lambda$, we have $(C_X)e \subseteq (B_X \cup C_X)e = B_X \cup C_X$. Therefore,

$$Y := (B_X \cup C_X)d \supseteq (C_X)ed = (C_X)wd_X = (C_X)d_X = X$$

and $|Y| = |B_X \cup C_X| = n$. We claim that $d = d_Y$. For $j \notin \{k, k+1\}$ and $D_j := \{\lambda_1 + \cdots + \lambda_{j-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_{j-1} + \lambda_j\}$ we have

$$(D_j)d = (D_j)ed = (D_j)wd_X = (D_j)d_X = (D_j)d_Y \;,$$

since $e \in W_\lambda$ and $w \in W_{\mu_X}$. Moreover, d and d_Y are monotonous on D_j, since $d, d_Y \in \mathcal{D}_\lambda$. Therefore, d and d_Y coincide on D_j. Further, $(B_X \cup C_X)d_Y = Y = (B_X \cup C_X)d$ and d and d_Y are monotonous on $B_X \cup C_X$. Therefore, d and d_Y coincide on $B_X \cup C_X$. The above implies that $(A_X)d = (A_X)d_Y$ and since d and d_Y are monotonous on A_X, they also coincide on A_X. Thus, $d = d_Y$.

(b) Using Lemma 5.2(b) and Part (a), we have

$$\varepsilon_X \circ \varphi_1^{\mu_X, \lambda} = \sum_{\substack{d \in \mathcal{D}_\lambda \\ W_{X,d} \neq \emptyset}} q^{l(w_{X,d})} \varepsilon_d = \sum_{\substack{X \subseteq Y \subseteq Z \\ |Y| = n}} q^{l(w_{X,d_Y})} \varepsilon_{d_Y}$$

and (b) is proven.

(c) By (22), $w_{X,Y}$ is the identity on $D \cup C_X$, and we can view $w_{X,Y}$ as a permutation of $A_X \cup B_X$. We claim that $w := w_{X,Y}$ is the permutation of $A_X \cup B_X$ with maps B_X monotonously onto the positions of $Y \smallsetminus X$ in the

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k-th row of t_X. In fact, we need to show that $(B_X)wd_X = Y \setminus X$ and that w is monotonous on B_X. But by (20) we have

$$(B_X)wd_X = (B_X)ed_Y = \big((B_X \cup C_X) \setminus (X)d_Y^{-1}\big)d_Y = Y \setminus X\,,$$

where e is defined as in Part (a). Moreover, since $(B_X)w \subseteq A_X \cup B_X$ and since d_X is monotonous on $A_X \cup B_X$, w is monotonous on B_X if and only if $wd_X = ed_Y$ is. Again, since $(B_X)e \subseteq B_X \cup C_X$ and d_Y is monotonous on $B_X \cup C_X$, ed_Y is monotonous on B_X if and only if e is. But, $e \in \mathcal{D}_{\mu_X}$ is monotonous on $A_X \cup B_X$ and so also on B_X. This shows the claim.

Since $l(w)$ is equal to the number of inversions of w, where w is viewed as a permutation of $A_X \cup B_X$, our claim immediately implies the desired equation if we can also show that w is monotonous on A_X. But, w is monotonous on A_X if and only if $wd_X = ed_Y$ is, by the argument above. Further, e is the identity on A_X, cf. (20), so that with d_Y also ed_Y is monotonous on A_X, and the proof of the lemma is complete. $\qquad\square$

Next we consider the element

$$c_1 := \sum_{X \subseteq \{a_1,\dots,a_i\}} (-1)^{|X|} q^{f(X)} \Big(\varphi_1^{\mu_{\widetilde{X}},\lambda} \otimes \varepsilon_{\widetilde{X}}\Big) \in C_1^\lambda\,,$$

where

$$\widetilde{X} := X \cup \{b_{i+2}, \cdots, b_n\}$$

and

$$f(X) := \sum_{\substack{j \in \{1,\dots,i\} \\ a_j \in X}} (m+1-j)\,.$$

5.4 Lemma *With the above notation, one has*

$$d_1^\lambda(c_1) \in q^l \varepsilon_t + C_{0,<t}\,,$$

for some integer l.

Proof By Lemma 5.3(b) we have

$$
\begin{aligned}
d_1^\lambda(c_1) &= \sum_{X \subseteq \{a_1,\dots,a_i\}} (-1)^{|X|} q^{f(X)} \Big(\varepsilon_{\widetilde{X}} \circ \varphi_1^{\mu_{\widetilde{X}},\lambda}\Big) \\
&= \sum_{X \subseteq \{a_1,\dots,a_i\}} (-1)^{|X|} q^{f(X)} \sum_{\substack{\widetilde{X} \subseteq Y \subseteq Z \\ |Y|=n}} q^{l(w_{\widetilde{X},Y})} \varepsilon_Y \\
&= \sum_{\substack{\{b_{i+2},\dots,b_n\} \subseteq Y \subseteq Z \\ |Y|=n}} \Big(\sum_{X \subseteq \{a_1,\dots,a_i\} \cap Y} (-1)^{|X|} q^{f(X)+l(w_{\widetilde{X},Y})}\Big) \varepsilon_Y\,.
\end{aligned}
$$

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For every $Y \subseteq Z$ with $\{b_{i+2}, \ldots, b_n\} \subseteq Y$ and $|Y| = n$ we study the contribution of the corresponding summand at the end of the last equation. We distinguish three cases.

If $Y = \{b_1, \ldots, b_n\}$ then $\varepsilon_Y = \varepsilon_t$ and the only possible set X in the inner sum of the last equation is the empty set. Therefore, the summand corresponding to Y is equal to $q^l \varepsilon_t$ with $l = f(\emptyset) + l(w_{\widetilde{\emptyset}, Y})$.

If $Y \cap \{a_1, \ldots, a_i\} = \emptyset$ and $Y \neq \{b_1, \ldots, b_n\}$ we claim that $t_Y < t$. Note that in this case we have $m \geqslant 1$. Let $j \in \{1, \ldots, i+1\}$ be minimal with $b_j \notin Y$ (note that $a_1 < b_j$) and let $p \in \{j-1, \ldots, i\}$ be maximal with $a_p < b_j$ (note that $a_{j-1} < b_{j-1} < b_j$). Then the k-th row of t_Y begins with $a_1, a_2, \ldots, a_p, b_j$ and $b_j < a_{p+1}$. Note that $p < m$, since if $p = m$ then $a_1, \ldots, a_m \in Z \smallsetminus Y$ and therefore $Y \subseteq \{b_1, \ldots, b_n\}$, a contradiction. Now our claim is proved and the contribution of the summand corresponding to Y is an element in $C_{0, <t}^\lambda$.

We are left with the case that $Y \cap \{a_1, \ldots, a_i\} \neq \emptyset$. If $t_Y < t$ we are done. So assume that $t_Y > t$. Our goal is to show that

$$\sum_{X \subseteq \{a_1, \ldots, a_i\} \cap Y} (-1)^{|X|} q^{f(X) + l(w_{\widetilde{X}, Y})} = 0 \,.$$

First let $j \in \{1, \ldots, i\}$ be minimal with $a_j \in Y$. We claim that

$$\{z \in Z \smallsetminus Y \mid z < a_j\} = \{a_1, \ldots, a_{j-1}\} \,. \tag{23}$$

Clearly, the right hand side is contained in the left hand side. Now let $z \in Z \smallsetminus Y$ with $z < a_j$ and assume that z is not contained in the right hand side. Then $z = b_p$ for some $p \in \{1, \ldots, i+1\}$, since $b_{i+2}, \ldots, b_n \in Y$. Let $r \in \{1, \ldots, i+1\}$ be minimal with $b_r \in Z \smallsetminus Y$ and $b_r < a_j$, and let $s \in \{1, \ldots, j-1\}$ be maximal with $a_s < b_r$ (note that $j > 1$, since $b_r < a_j$ and that $a_1 < b_r$). Then the k-th row of t_Y starts with $a_1, a_2, \ldots, a_s, b_r$ and one has $b_r < a_{s+1}$. This implies $t_Y < t$, a contradiction. Therefore, the claim (23) is proven. For our given set Y, the subsets X of $\{a_1, \ldots, a_i\} \cap Y = \{a_j, \ldots, a_i\} \cap Y$ fall into two classes, the ones that contain a_j and the ones that don't. So let X_1 be a subset of $\{a_1, \ldots, a_i\} \cap Y$ with $a_j \in X_1$ and let $X_2 := X_1 \smallsetminus \{a_j\}$. Since $X_1 \mapsto X_2$ defines a bijection between these two classes, it suffices to show that

$$(-1)^{|X_1|} q^{f(X_1) + l(w_{\widetilde{X}_1, Y})} + (-1)^{|X_2|} q^{f(X_2) + l(w_{\widetilde{X}_2, Y})} = 0 \,,$$

or, equivalently, that

$$l(w_{\widetilde{X}_1, Y}) + m + 1 - j = l(w_{\widetilde{X}_2, Y}) \,. \tag{24}$$

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But by Lemma 5.3(c) and Equation (23), and since $Y \smallsetminus \widetilde{X}_2 = Y \smallsetminus \widetilde{X}_1 \cup \{a_j\}$ and $|Z \smallsetminus Y| = m$, we have

$$
\begin{aligned}
& l(w_{\widetilde{X}_2,Y}) - l(w_{\widetilde{X}_1,Y}) \\
= \; & \sum_{y \in Y \smallsetminus \widetilde{X}_2} |\{z \in Z \smallsetminus Y \mid y < z\}| - \sum_{y \in Y \smallsetminus \widetilde{X}_1} |\{z \in Z \smallsetminus Y \mid y < z\}| \\
= \; & |\{z \in Z \smallsetminus Y \mid a_j < z\}| = m - |\{z \in Z \smallsetminus Y \mid z < a_j\}| = m - (j-1) \,.
\end{aligned}
$$

This shows Equation (24) and the proof of the lemma is complete. $\qquad\square$

Now Lemma 4.3 is an immediate consequence of Lemma 5.4.

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