# TWO FINITE TCHEBICHEF TRANSFORMATIONS

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## TWO FINITE TCHEBICHEF TRANSFORMATIONS

### Chapter 1

### INTRODUCTION

It is the intention of this paper to define and develop two integral transforms and their operational calculus using the Tchebichef polynomials as the kernel function. In this we shall adopt a different approach from that of Ta Li who, in a recent paper (12. vol. 11, p. 290-298), defined a new transform which used as the kernel function the Tchebichef polynomial of the first kind divided by the weight function $(1-X^2)^{\frac{1}{2}}$. He was then able to establish a closed form solution to the integral equation thus formed.

We begin by defining the Tchebichef polynomials in accordance with Lanczos (11, p. 179), Erdélyi (8, vol. 2, p. 184) and Courant and Hilbert (7, vol. 1, p. 88).

$$T_n(X) = \cos(n \cos^{-1}X),$$

$$U_n(X) = \frac{\sin\left[(n+1)\cos^{-1}X\right]}{(1-X^2)^{\frac{1}{2}}},$$

where $T_n(X)$ and $U_n(X)$ are polynomials of the first and second kinds respectively. By letting $X = \cos\theta$ we arrive at the standard form of the polynomials as given in Erdélyi (8, vol. 2, p. 184) namely:

$$T_n(\cos\theta) = \cos n\theta, \tag{1.1}$$

$$U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{1.2}$$

Though less useful for our immediate objective, the Tchebichef polynomials may also be defined as the polynomial solutions of the differential equations

$$(1-X^2)Y_1''(X) - XY_1'(X) + n^2Y_1(X) = 0, \tag{1.3}$$

$$(1-X^2)Y_2''(X) - 3XY_2'(X) + n(n+2)Y_2(X) = 0. \tag{1.4}$$

Here we have as solutions (8, vol. 2, p. 184)

$$Y_1(X) = T_n(X) = \frac{n}{2}\sum_{m=0}^{[n/2]} \frac{(-1)^m(n-m-1)!}{m!\,(n-2m)!}(2X)^{n-2m},$$

$$Y_2(X) = U_n(X) = \sum_{m=0}^{[n/2]} \frac{(-1)^m\,(n-m)!}{m!\,(n-2m)!}(2X)^{n-2m},$$

where $n = 1, 2, 3, \ldots$ .

It is readily seen that the polynomials satisfy their respective orthogonality relationships

$$\int_{-1}^{1} T_n(X)T_m(X)(1-X^2)^{½}dX = \begin{cases} 0 & m \neq n, \\ \frac{\pi}{\mathcal{E}_n} & m = n, \end{cases}$$

$$\int_{-1}^{1} U_n(X)U_m(X)(1-X^2)^{½}dX = \begin{cases} 0 & m \neq n, \\ \frac{\pi}{2} & m = n, \end{cases}$$

where $\mathcal{E}_n = \begin{cases} 1 & n = 0, \\ 2 & n = 1, 2, 3, \ldots \end{cases}$ .

The following set of relations will be useful:

$$\frac{T'_{n+1}(X)}{n+1} - \frac{T'_{n-1}(X)}{n-1} = 2T_n(X) \ , \qquad 1.5$$

$$U'_{n+1}(X) - U'_{n-1}(X) = 2(n+1)U_n(X) \ , \qquad 1.6$$

$$T'_n(X) = nU_{n-1}(X) \ , \qquad 1.7$$

$$Z_{n+1}(X) + Z_{n-1}(X) = 2XZ_n(X) \ , \qquad 1.8$$

$$Z_n(-X) = (-1)^n Z_n(X) \ , \qquad 1.9$$

where $Z_n(X)$ represents either $T_n(X)$ or $U_n(X)$.

The above formulas are taken or may be derived from material contained in Erdélyi as follows: From formulas 16, 28, and 37 (8, vol. 2, p. 185-187) we have 1.5, 1.7, and 1.8. Formula 1.6 may be derived from formulas 24 and 25 (8, vol. 2, p. 176) by recalling that

$$U_n(X) = C_n^1(X).$$

When $Z_n(X) = U_n(X)$, 1.9 follows from formula 16 (8, vol. 2, p. 175) by letting $\lambda = 1$. When $Z_n(X) = T_n(X)$, 1.9 may be established by recalling that

$$\cos^{-1}X = - \cos^{-1}(-X)$$

and making use of well known trigonometric identities.

# Chapter 2

## DEFINITION OF THE TRANSFORMS

Suppose we desire to represent a function $F(X)$ which is at least sectionally continuous by one of the following infinite series:

$$F(X) = \sum_{n=0}^{\infty} A_n \frac{T_n(X)}{(1-X^2)^{1/2}} , \qquad 2.1$$

$$F(X) = \sum_{n=0}^{\infty} B_n U_n(X)(1-X^2)^{1/2} \qquad 2.2$$

From the orthogonality conditions we may determine the constants $A_n$ and $B_n$ obtaining

$$A_n = \frac{\varepsilon_n}{\pi} \int_{-1}^{1} F(X) T_n(X) dX , \qquad 2.3$$

$$B_n = \frac{2}{\pi} \int_{-1}^{1} F(X) U_n(X) dX . \qquad 2.4$$

Substitution then shows that $F(X)$ may be represented respectively by the series

$$F(X) = \frac{1}{\pi} \sum_{n=0}^{\infty} \varepsilon_n \frac{T_n(X)}{(1-X^2)^{1/2}} \int_{-1}^{1} F(X) T_n(X) dX$$

and

$$F(X) = \frac{2}{\pi}\sum_{n=0}^{\infty} (1-X^2)^{1/2} U_n(X) \int_{-1}^{1} F(X)U_n(X)dX .$$

We now make the following definitions:

<u>Definition</u> 1. If $f_T(n) = T_n\{F(X)\}$ denotes the finite Tchebichef transform of the first kind, then

$$f_T(n) = \int_{-1}^{1} F(X)T_n(X)dX .$$

<u>Definition</u> 2. If $f_U(n) = U_n\{F(X)\}$ denotes the finite Tchebichef transform of the second kind, then

$$f_U(n) = \int_{-1}^{1} F(X)U_n(X)dX .$$

A set of inversion formulas follow immediately from equations 2.1 through 2.4 giving respectively

$$F(X) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\pi} f_T(n) \frac{T_n(X)}{(1-X^2)^{1/2}} \qquad 2.5$$

and

$$F(X) = \frac{2}{\pi}\sum_{n=0}^{\infty} f_U(n)(1-X^2)^{1/2} U_n(X) \qquad 2.6$$

provided F(X) may be represented by either 2.1 or 2.2.

# Chapter 3

## OPERATIONAL PROPERTIES OF THE FINITE TCHEBICHEF TRANSFORMATIONS

The transforms will be said to be linear if, for every pair of functions $F_1(X)$ and $F_2(X)$ and each pair of constants $C_1$ and $C_2$, the following condition is satisfied:

$$Z_n\{C_1F_1(X) + C_2F_2(X)\} = C_1Z_n\{F_1(X)\} + C_2Z_n\{F_2(X)\}. \quad 3.1$$

Application of this definition to the transforms yields

Theorem 1. The finite Tchebichef transforms are linear.

We now consider the effect of the transforms on a function whose (k-1)st derivatives are continuous on the closed interval [-1,1] and whose $k^{th}$ derivative is sectionally continuous on [-1,1]. A function shall be said to be sectionally continuous on [a,b] if the function has only a finite number of discontinuities with finite limits on [a,b], and no other discontinuities.

Theorem 2. If $F(X)$ is a function satisfying the above conditions of continuity, then

$$T_n\left\{(1-X^2)F^{(m)}(X) - 3XF^{(m-1)}(X)\right\}$$
$$= (1-n^2)T_n\left\{F^{(m-2)}(X)\right\} - F^{(m-2)}(1) - (-1)^nF^{(m-2)}(-1).$$

Proof: Let $T = T_n\left\{(1-X^2)F^{(m)}(X) - 3XF^{(m-1)}(X)\right\}$

$$= \int_{-1}^{1} F^{(m)}(X)(1-X^2)T_n(X)dX - 3\int_{-1}^{1} XF^{(m-1)}(X)T_n(X)dX .$$

Integrating the first integral by parts and then combining with the second integral, we have

$$T = -\int_{-1}^{1} F^{(m-1)}(X)\left[(1-X^2)T_n'(X) + XT_n(X)\right]dX .$$

Again integrating by parts we obtain

$$T = -F^{(m-2)}(X)\left[(1-X^2)T_n'(X) + XT_n(X)\right]_{-1}^{1}$$

$$+\int_{-1}^{1} F^{(m-2)}(X)\left[(1-X^2)T_n''(X) - XT_n'(X) + T_n(X)\right]dX .$$

Substituting 1.3 in the integral and simplifying we have the desired result

$$T = -(-1)^n F^{(m-2)}(-1) - F^{(m-2)}(1) + (1-n^2)\int_{-1}^{1} F^{(m-2)}(X)T_n(X)dX$$

$$= -F^{(m-2)}(1) - (-1)^n F^{(m-2)}(-1) + (1-n^2)T_n\{F^{(m-2)}(X)\} .$$

<u>Theorem</u> 3. If $F(X)$ is a function satisfying the above conditions of continuity, then

$$U_n\left\{(1-X^2)F^{(m)}(X) - XF^{(m-1)}(X)\right\}$$

$$= (n+1)\left[F^{(m-2)}(1) + (-1)^n F^{(m-2)}(-1)\right] - (n+1)^2 U_n\{F^{(m-2)}(X)\} .$$

The proof is similar to that of theorem 2.

While the above two theorems state the fundamental operational properties of the two transforms, additional results are of interest. Let

$$G(X) = \int_a^X F(t)dt, \qquad |a| \leq 1.$$

Then upon integrating by parts we have

$$T_n\{G'(X)\} = \int_{-1}^{1} G'(X)T_n(X)dX$$

$$= G(1) - (-1)^n G(-1) - \int_{-1}^{1} G(X)T_n'(X)dX .$$

It follows from the differential recurrence formula 1.5 that

$$\frac{T_{n-1}\{G'(X)\}}{n-1} - \frac{T_{n+1}\{G'(X)\}}{n+1}$$

$$= \frac{2}{n^2-1}[G(1) + (-1)^n G(-1)] + 2T_n\{G(X)\} .$$

Solving this difference equation, first for $T_n\{G(X)\}$ and then for $T_n\{G'(X)\}$, we arrive at the following two theorems:

<u>Theorem</u> 4. If $F(t)$ is sectionally continuous, then

$$T_n\left\{\int_a^X F(t)dt\right\} = \frac{1}{2}\left[\frac{T_{n-1}\{F(X)\}}{n-1} - \frac{T_{n+1}\{F(X)\}}{n+1}\right]$$

$$+ \frac{1}{1-n^2}\left[\int_a^1 F(t)dt + (-1)^n\int_a^{-1} F(t)dt\right], \qquad |a| \leq 1 .$$

Theorem 5. If $G(X)$ is continuous and $G'(X)$ is sectionally continuous, then

$$\frac{T_{2m}\{G'(X)\}}{2m} = -2\sum_{k=2}^{m}\left[\frac{G(1) - G(-1)}{4k(k-1)} + T_{2k-1}\{G(X)\}\right]$$

$$+ 2T_1\{G(X)\} + \tfrac{1}{2}[G(1) - G(-1)] \; .$$

$$\frac{T_{2m+1}\{G'(X)\}}{2m+1} = -2\sum_{k=1}^{m}\left[\frac{G(1) + G(-1)}{(2k+1)(2k-1)} + T_{2k}\{G(X)\}\right]$$

$$+ G(1) + G(-1) - \int_{-1}^{1} G(X)dX \; .$$

A simpler theorem for $T_n\{G'(X)\}$ is the following:

Theorem 6. If $G'(X)$ is sectionally continuous, then

$$T_n\{G'(X)\} = G(1) - (-1)^n G(-1) - nU_{n-1}\{G(X)\} \; .$$

Proof: $T_n\{G'(X)\} = \int_{-1}^{1} G'(X)T_n(X)dX$

$$= G(X)T_n(X)\Big|_{-1}^{1} - \int_{-1}^{1} G(X)T_n'(X)dX \; .$$

Making use of formula 1.7 we have the desired result after simplification.

For $U_n\left\{\int_a^X F(t)dt\right\}$ and $U_n\{G'(X)\}$, we again let

$$G(X) = \int_a^X F(t)dt, \qquad |a| \leq 1 .$$

Then upon integrating by parts we have

$$U_n\{G'(X)\} = \int_{-1}^{1} G'(X)U_n(X)dX$$

$$= (n+1)\left[G(1) + (-1)^n G(-1)\right] - \int_{-1}^{1} G(X)U_n'(X)dX .$$

It follows from the differential recurrence formula 1.6 that

$$U_{n-1}\{G'(X)\} - U_{n+1}\{G'(X)\}$$

$$= -2\left[G(1) - (-1)^n G(-1) - (n+1)U_n\{G(X)\}\right] .$$

Solution of this difference equation, first for $U_n\{G(X)\}$ and then for $U_n\{G'(X)\}$ gives the following two theorems:

<u>Theorem</u> 7. If $F(t)$ is sectionally continuous, then

$$U_n\left\{\int_a^X F(t)dt\right\} = \frac{1}{2(n+1)}\left[U_{n-1}\{F(X)\} - U_{n+1}\{F(X)\}\right.$$

$$\left. -2(-1)^n\int_a^{-1} F(t)dt + 2\int_a^1 F(t)dt\right], \qquad |a| \leq 1 .$$

<u>Theorem</u> 8. If $G(X)$ is continuous and $G'(X)$ is sectionally continuous, then

$$U_{2m}\{G'(X)\} = \sum_{k=1}^{m} -4kU_{2k-1}\{G(X)\} + 2m[G(1) + G(-1)]$$
$$+ G(1) - G(-1) .$$

$$U_{2m+1}\{G'(X)\} = \sum_{k=1}^{m} -2(2k+1)U_{2k}\{G(X)\} + 2m[G(1) - G(-1)]$$
$$+ 2[G(1) + G(-1)] - 2\int_{-1}^{1} G(X)dX .$$

Another relation may be developed from the recursion formula 1.8 by multiplying both sides of the equation by $F(X)$ and integrating over $[-1,1]$, thereby obtaining

$$Z_n\{XF(X)\} = \frac{1}{2}\Big[Z_{n+1}\{F(X)\} + Z_{n-1}\{F(X)\}\Big] . \qquad 3.2$$

If in deriving 3.2 we had multiplied 1.8 by $XF(X)$ instead of $F(X)$ and then used 3.2 in the simplification, we would have obtained

$$Z_n\{X^2F(X)\} = \frac{1}{4}\Big[Z_{n+2}\{F(X)\} + 2Z_n\{F(X)\} + Z_{n-2}\{F(X)\}\Big] . \qquad 3.3$$

This suggests the following theorem which may be established by mathematical induction:

<u>Theorem</u> 9. If $F(X)$ is sectionally continuous, then

$$Z_n\{X^rF(X)\} = \frac{1}{2^r}\left[\sum_{k=0}^{r}\binom{r}{k} Z_{n+r-2k}\{F(X)\}\right] .$$

We now make the following definition for the derivative of the transform:

<u>Definition</u> 3. $Z_n^{(r)}\{F(X)\} = \lim_{s\to 1} \frac{d^r}{ds^r} \int_{-1}^{1} F(X)Z_n(sX)dX$.

Using this definition the following theorem is easily obtained:

<u>Theorem</u> 10. If $F(X)$ is sectionally continuous, then

$$Z_n^{(r)}\{F(X)\} = \int_{-1}^{1} X^r F(X) Z_n^{(r)}(X)dX.$$

A number of relations may be obtained from this theorem. For example, if we begin with the differentiation formula

$$U'_{n-1}(X) = XU'_n(X) - nU_n(X)$$

obtained from formula 24 of Erdélyi (8, vol. 2, p. 176) by recalling that $U_n(X) = C_n^1(X)$, multiply by $F(X)$ and integrate over $[-1,1]$, we have

$$\int_{-1}^{1} F(X)U'_n(X)dX = \int_{-1}^{1} XF(X)U'_{n+1}(X)dX - (n+1)\int_{-1}^{1} F(X)U_{n+1}(X)dX.$$

Application of theorem 10 reduces this to

$$\int_{-1}^{1} F(X)U'_n(X)dX = U_{n+1}^{(1)}\{F(X)\} - (n+1)U_{n+1}\{F(X)\}.$$

Integrating the left side by parts, we finally arrive at

$$U_n\{F'(X)\} = -(n+1)\Big[F(1) + (-1)^n F(-1) + U_{n+1}\{F(X)\}\Big] - U^{(1)}_{n+1}\{F(X)\}. \qquad 3.4$$

A similar expression may be obtained for $T_n\{F'(X)\}$ by using the differentiation formula

$$nT'_{n-1}(X) = (n-1)XT'_n(X) - n(n-1)T_n(X)$$

obtained by recalling that $T_n(X) = \frac{1}{2}nC^0_n(X)$ and using formula 24 of Erdélyi (8, vol. 2, p. 176). Proceeding as before we finally have

$$T_n\{F'(X)\} = (n+1)\big[F(1) - (-1)^n F(-1)\big] + n(n+1)T_{n+1}\{F(X)\} - nT^{(1)}_{n+1}\{F(X)\}. \qquad 3.5$$

Comparison of formula 3.5 with theorems 5 and 6 suggests a number of interesting identities, in particular a number of expressions for the sum of the finite series. It will also be noted that other expressions and formulas may be developed using other differentiation formulas.

A convolution theorem using the Tchebichef polynomial of the second kind may be established by using the addition theorem for the Gegenbauer polynomials (8, vol. 1, p. 177) and the relation

$$U_n(X) = C^1_n(X).$$

For the Gegenbauer polynomials we have

$$\int_0^\pi C_n^p(\cos \Phi \cos \theta + \sin \Phi \sin \theta \cos \alpha)(\sin \alpha)^{2p-1} d\alpha$$

$$= \frac{2^{2p-1} n! \left[\Gamma(p)\right]^2 C_n^p(\cos \Phi) C_n^p(\cos \theta)}{\Gamma(2p + n)},$$

which yields for $p = 1$,

$$U_n(\cos \Phi)U_n(\cos \theta) = \frac{n+1}{2} \int_0^\pi U_n(\cos \lambda)\sin \alpha \, d\alpha \qquad 3.6$$

where $\cos \lambda = \cos \Phi \cos \theta + \sin \Phi \sin \theta \cos \alpha$.

<u>Theorem</u> 11. If $F(X)$ and $G(X)$ are continuous functions on the interval $[-1,1]$, then

$$f_U(n) g_U(n) = \frac{n+1}{2} U_n\{H(\cos \lambda)\}$$

where

$$H(\cos \lambda) = \int_0^\pi \int_0^\pi F(\cos \theta) G(\cos \lambda \cos \theta$$

$$- \sin \lambda \sin \theta \cos \beta) \sin \alpha \sin \theta \, d\beta d\theta$$

and

$$\sin \alpha = \frac{\sin \lambda \sin \beta}{\left[\sin^2\lambda \sin^2\beta + (\sin \theta \cos \lambda + \cos \theta \sin \lambda \cos \beta)^2\right]^{\frac{1}{2}}}.$$

Proof: $f_U(n) g_U(n)$

$$= \int_0^\pi F(\cos \theta) U_n(\cos \theta) \sin \theta d\theta \int_0^\pi G(\cos \Phi) U_n(\cos \Phi) \sin \Phi d\Phi$$

$$= \int_0^\pi F(\cos\theta)\sin\theta \int_0^\pi G(\cos\Phi)U_n(\cos\Phi)U_n(\cos\theta)\sin\Phi d\Phi d\theta .$$

Substitution of the addition theorem, 3.6, yields

$$f_U(n)g_U(n) = \frac{n+1}{2}\int_0^\pi F(\cos\theta)\sin\theta$$

$$\left[\int_0^\pi\int_0^\pi G(\cos\Phi)U_n(\cos\lambda)\sin\alpha\,\sin\Phi\,d\alpha d\Phi\right]d\theta \qquad 3.7$$

where $\cos\lambda = \cos\Phi\cos\theta + \sin\Phi\sin\theta\cos\alpha$.

Let us now make the following change of variable

$$\cos\Phi = \cos\theta\cos\lambda - \sin\theta\sin\lambda\cos\beta,$$

$$\sin\Phi\cos\alpha = \sin\lambda\cos\beta\cos\theta + \sin\theta\cos\lambda,$$

$$\sin\Phi\sin\alpha = \sin\lambda\sin\beta.$$

Evaluating the Jacobian of the transformation we have

$$d\alpha d\Phi = \frac{\sin\lambda}{\sin\Phi}\,d\lambda d\beta.$$

Hence the integral in the square brackets becomes

$$\int_0^\pi\int_0^\pi G(\cos\theta\cos\lambda - \sin\theta\sin\lambda\cos\beta)U_n(\cos\lambda)$$

$$\cdot\sin\alpha\,\sin\lambda\,d\lambda d\beta \qquad 3.8$$

where

$$\sin\alpha = \frac{\sin\lambda\sin\beta}{\left[\sin^2\lambda\sin^2\beta + (\sin\theta\cos\lambda + \cos\theta\sin\lambda\cos\beta)^2\right]^{1/2}}$$

Substituting we have

$$f_U(n)g_U(n) = \frac{n+1}{2}\int_0^{\pi} F(\cos\theta)\sin\theta\left[\int_0^{\pi}\int_0^{\pi} G(\cos\theta\cos\lambda - \sin\theta\sin\lambda\cos\beta)U_n(\cos\lambda)\sin\alpha\sin\lambda\, d\lambda d\beta\right]d\theta$$

$$= \frac{n+1}{2}\int_0^{\pi}\left[\int_0^{\pi}\int_0^{\pi} F(\cos\theta)G(\cos\theta\cos\lambda - \sin\theta\sin\lambda\cos\beta)\sin\theta\sin\alpha\, d\theta\, d\beta\right]U_n(\cos\lambda)\sin\lambda\, d\lambda$$

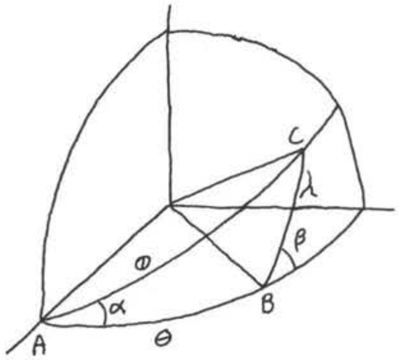
$$= \frac{n+1}{2}\int_0^{\pi} H(\cos\lambda)U_n(\cos\lambda)\sin\lambda\, d\lambda$$

which establishes the theorem.

Theorem 11 may also be established by using the convolution property for the Gegenbauer transform (6, p. 50) and the relation

$$U_n(X) = C_n^1(X)$$

It is to be noted that the iterated integral inside the square brackets in equation 3.7 can be interpreted as a surface integral over a unit hemisphere. Let $\theta$ represent arc length on the semicircle $X^2 + Y^2 = 1$, $Z = 0$, $Y > 0$ measured from the point $A(1,0,0)$ and let B denote the terminal point of the arc. If C is any point on the surface of the hemisphere, let $\Phi$ represent the arc AC of the great circle through A and C, and let $\lambda$ represent




the arc BC of the great circle through B and C. Let $\alpha$ and $\beta$ represent the angles A and B respectively. Then $\alpha$ and $\lambda$ are coordinates of the point C, and $\sin \alpha d\alpha dO$ and $\sin \lambda \, d\lambda d\beta$ represent an element of area.

Now from the cosine law of spherical trigonometry we have

$$\cos \Phi = \cos \theta \cos \lambda - \sin \theta \sin \lambda \cos \beta.$$

Also we have from spherical trigonometry

$$\sin \Phi \cos \alpha = \sin \lambda \cos \beta \cos \theta + \sin \theta \cos \lambda,$$

$$\sin \Phi \sin \alpha = \sin \lambda \sin \beta.$$

Hence the iterated integral in square brackets in equation 3.7, which is the surface integral of the function $G(\cos \Phi)U_n(\cos \lambda)\sin \alpha$, becomes that given by equation 3.8.

The Tchebichef transform of the first kind does not readily lend itself to the development of a convolution theorem at this time since an addition theorem for the Tchebichef polynomial of the first kind is unknown to this author.

<u>Definition</u> 4. If $\bar{f}_T(n) = \bar{T}_n\{F(X)\}$ denotes the alternate finite Tchebichef transform of the first kind and $F(X)$ is at least sectionally continuous, then

$$\bar{f}_T(n) = \int_{-1}^{1} F(X)T_n(X)(1-X^2)^{-\frac{1}{2}}dX.$$

<u>Definition</u> 5. If $\bar{f}_U(n) = \bar{U}_n\{F(X)\}$ denotes the alternate finite Tchebichef transform of the second kind and $F(X)$ is at least sectionally continuous, then

$$\bar{f}_U(n) = \int_{-1}^{1} F(X)U_n(X)(1-X^2)^{\frac{1}{2}}dX.$$

By using the orthogonality properties of the Tchebichef polynomials, inversion formulas for these two transforms may be quickly established, giving

$$F(X) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{\pi} \bar{f}_T(n)T_n(X),$$

and

$$F(X) = \frac{2}{\pi}\sum_{n=0}^{\infty} \bar{f}_U(n)U_n(X).$$

Using this approach several theorems may be derived and are listed here without proof.

Theorem 12. If $F(X)$ is a function whose $k^{th}$ derivative is at least sectionally continuous, then

$$\overline{T}_n\left\{(1-X^2)F^{(k)}(X) - XF^{(k-1)}(X)\right\} = -n^2\overline{T}_n\left\{F^{(k-2)}(X)\right\}.$$

Theorem 13. If $F(X)$ is a function whose $k^{th}$ derivative is at least sectionally continuous, then

$$\overline{U}_n\left\{(1-X^2)F^{(k)}(X) - 3XF^{(k-1)}(X)\right\} = -n(n+2)\overline{U}_n\left\{F^{(k-2)}(X)\right\}.$$

Theorem 14. If $F(X)$ and $G(X)$ are continuous functions on the interval $[-1,1]$, then

$$\overline{f}_U(n)\overline{g}_U(n) = \frac{n+1}{2}\,\overline{U}_n\{H(\cos\lambda)\}$$

where

$$H(\cos\lambda) = \int_0^{\pi}\int_0^{\pi} F(\cos\theta)G(\cos\theta\cos\lambda + \sin\theta\sin\lambda\cos\beta)\sin^2\theta\sin\beta\, d\theta d\beta\,.$$

The resemblance of theorems 12 through 14 to theorems 2, 3, and 11 respectively, should be noted, the proofs also being similar. In attempting to establish theorems similar to theorems 4, 5, 7, and 8, difficulty was encountered in evaluating the improper integral. For example

$$\bar{T}_n\{f'(X)\} = \int_{-1}^{1} f'(X)\frac{T_n(X)}{(1-X^2)^{1/2}}dX$$

$$= f(X)\frac{T_n(X)}{(1-X^2)^{1/2}}\Bigg|_{-1}^{1} - \int_{-1}^{1} f(X)\frac{(1-X^2)T_n'(X) + XT_n(X)}{(1-X^2)^{3/2}}dX,$$

the first term being discontinuous at the end points except for special cases of $f(X)$. No attempt was made to establish theorems similar to theorems 9 and 10.

Use of the transformation equations

$$\bar{T}_n\{(1-X^2)^{1/2}F(X)\} = T_n\{F(X)\}, \quad 3.9$$

$$\bar{U}_n\left\{\frac{F(X)}{(1-X^2)^{1/2}}\right\} = U_n\{F(X)\}, \quad 3.10$$

provides a connection between the two types of transforms.

In addition a convolution theorem for the alternate Tchebichef transform of the first kind may be derived using an approach similar to that for the Finite Fourier Transforms (4, p. 296-298 and 15, p. 76-79). Before establishing this theorem, we prove the following lemma:

<u>Lemma</u> 1. If $F(X)$ is a continuous function on the interval $[-1,1]$, then

$$\bar{f}_T(n)\cos n\Phi = \frac{1}{2}\int_0^{\pi} F[\cos(\mu - \Phi)]F[\cos(\mu + \Phi)]\cos n\mu\, d\mu.$$

Proof: $\bar{f}_T(n)\cos n\Phi = \int_0^{\pi} F(\cos\theta)\cos n\theta \cos n\Phi \, d\theta$

$$= \frac{1}{2}\int_0^{\pi} F(\cos\theta)[\cos n(\theta + \Phi) + \cos n(\theta - \Phi)]d\theta.$$

Since the integrand is an even function, we may write

$$\bar{f}_T(n)\cos n\Phi = \frac{1}{4}\int_{-\pi}^{\pi} F(\cos\theta)[\cos n(\theta + \Phi) + \cos n(\theta - \Phi)]d\theta$$

$$= \frac{1}{4}\int_{\pi-\pi+0}^{\pi+0} F[\cos(\mu - \Phi)]\cos n\mu \, d\mu$$

$$+ \frac{1}{4}\int_{-\pi-0}^{\pi-0} F[\cos(\mu + \Phi)]\cos n\mu \, d\mu.$$

But the integrands are periodic functions of $2\pi$, hence

$$\bar{f}_T(n)\cos n\Phi = \frac{1}{4}\int_{-\pi}^{\pi} F[\cos(\mu - \Phi)] + F[\cos(\mu + \Phi)]\cos n\mu \, d\mu$$

$$= \frac{1}{2}\int_0^{\pi} F[\cos(\mu - \Phi)] + F[\cos(\mu + \Phi)]\cos \mu \, d\mu.$$

<u>Theorem</u> 15. If $F(\cos\theta)$ and $G(\cos\theta)$ are continuous functions on $[0,\pi]$, then

$$\bar{f}_T(n)\bar{g}_T(n) = \frac{1}{2}\bar{T}_n\{F{*}G\}$$

where

$$F{*}G = \int_0^{\pi} F(\cos\theta)\{G[\cos(\mu - \theta)] + G[\cos(\mu + \theta)]\}d\theta.$$

Proof: $\overline{f}_T(n)\overline{g}_T(n) = \int_0^\pi F(\cos\theta)\overline{g}_T(n)\cos n\theta\, d\theta$

which becomes upon substitution of lemma 1

$$\overline{f}_T(n)\overline{g}_T(n) = \frac{1}{2}\int_0^\pi F(\cos\theta) \int_0^\pi \{G[\cos(\mu - \theta)]$$

$$+ G[\cos(\mu + \theta)]\}\cos n\mu\, d\mu d\theta$$

$$= \frac{1}{2}\int_0^\pi \cos n\mu \int_0^\pi \{G[\cos(\mu - \theta)]$$

$$+ G[\cos(\mu + \theta)]\}F(\cos\theta)\, d\theta d\mu$$

$$= \frac{1}{2}\overline{T}_n\{F*G\}.$$

Finally we note that if we use (8, p. 187) formulas 35 with m = 1 and n = n+1, and 37 with m = 1, we may arrive at the following formulas respectively:

$$\overline{f}_U(n) = -\frac{1}{2}\left[\overline{f}_T(n) - \overline{f}_T(n+2)\right] \qquad 3.11$$

and

$$f_T(n) = \frac{1}{2}\left[f_U(n) - f_U(n-2)\right]. \qquad 3.12$$

Use of formula 3.11 to attempt establishing a lemma for the alternate transform of the second kind, similar to lemma 1, shows

$$\overline{f}_U(n)\sin(n+1)\theta \sin\theta = \frac{1}{4}\Big[\overline{f}_T(n)\cos n\theta + \overline{f}_T(n+2)\cos(n+2)\theta$$

$$- \overline{f}_T(n)\cos(n+2)\theta - \overline{f}_T(n+2)\cos n\theta\Big].$$

Since the last two terms present difficulties, we have not established a convolution theorem similar to theorem 15.

# Chapter 4

## METHODS OF COMPUTATION AND APPLICATIONS

Some methods of computation of transforms may best be presented by a few illustrative examples. The results obtained are also presented in the two tables in the Appendix. We first remark, however, that considerable use is made of formula 1.9 in the reduction of the expressions.

By letting $X = \cos\theta$ the transform of a constant is immediately obtained, for

$$\int_{-1}^{1} CT_n(X)dX = C\int_0^{\pi} \cos n\theta \, \sin\theta \, d\theta = \frac{C}{1-n^2}\left[1+(-1)^n\right], \quad n \neq 1.$$

$$\int_{-1}^{1} CT_1(X)dX = 0$$

$$\int_{-1}^{1} CU_n(X)dX = C\int_0^{\pi} \sin\,(n+1)\theta \, d\theta = \frac{C}{n+1}\left[1+(-1)^n\right].$$

A similar procedure enables us to obtain the transforms of $F(X) = X$, the integration being accomplished either by tables or by the use of well known trigonometric formulas.

To obtain the transform of higher powers of X, use may be made of theorems 2 and 3 with $m = 2$. For example, consider the transform of $F(X) = X^2$. We have

$$T_n\left\{(1-X^2)F^{(2)}(X) - 3XF^{(1)}(X)\right\}$$

$$= - F(1) - (-1)^n F(-1) + (1-n^2)T_n\{F(X)\}.$$

Letting $F(X) = X^2$, we have on substitution

$$T_n\left\{2(1-X^2) - 6X^2\right\} = -\left[1+(-1)^n\right] + (1-n^2)T_n\{X^2\},$$

or transposing and collecting terms

$$T_n\{X^2\} = \frac{T_n\{2\} + \left[1+(-1)^n\right]}{9-n^2} .$$

But $T_n\{2\} = 2\left[\frac{1+(-1)^n}{1-n^2}\right]$. Substituting and simplifying we obtain finally

$$T_n\{X^2\} = \frac{(3-n^2)\left[1+(-1)^n\right]}{(1-n^2)(9-n^2)}, \qquad n \neq 1,\ 3.$$

Similarly, since

$$U_n\left\{(1-X^2)F^{(2)}(X) - XF^{(1)}(X)\right\}$$

$$= (n+1)\left[F(1) + (-1)^n F(-1)\right] - (n+1)^2 U_n\{F(X)\},$$

we have

$$U_n\{X^2\} = \frac{(n+1)\left[1+(-1)^n\right] - U_n\{2\}}{(n+1)^2 - 4}.$$

But $U_n\{2\} = \frac{2}{n+1}\left[1+(-1)^n\right]$. Hence

$$U_n\{X^2\} = \frac{n^2 + 2n - 1}{(n^2-1)(n+3)}\left[1+(-1)^n\right], \qquad n \neq 1.$$

The special cases were all found to be zero, the computation being made by means of the definition and the substitution $X = \cos\theta$.

The computation of the transforms $T_n\{T_m(X)\}$, $U_n\{T_m(X)\}$, and $T_n\{U_m(X)\}$ follow immediately upon making the transformation $X = \cos\theta$. However, two cases must be considered for $T_n\{U_m(X)\}$, or, since $U_m\{T_n(X)\} = T_n\{U_m(X)\}$, $U_m\{T_n(X)\}$, namely, $n \geq m$ and $m > n$. We consider the case when $n \geq m$, the other being similar. We have

$$T_n\{U_m(X)\} = U_m\{T_n(X)\} = \int_0^{\pi} \sin\,(m+1)\theta\,\cos n\theta\,d\theta$$

$$= -\frac{1}{2}\left[\frac{\cos\,(n+m+1)\theta}{n+m+1} + \frac{\cos\,(n+1-m)\theta}{n+1-m}\right]_0^{\pi}$$

$$= \left[1+(-1)^{n+m}\right]\left[\frac{n+1}{(n+1)^2-m^2}\right], \qquad n \geq m.$$

For the transform $U_n\{U_m(X)\}$ we have

$$U_n\{U_m(X)\} = \int_0^{\pi}\frac{\sin\,(m+1)\theta\,\sin\,(n+1)\theta}{\sin\theta}\,d\theta$$

$$= \int_0^{\pi}\left[\frac{\cos\,(m-n)\theta}{2\sin\theta} - \frac{\cos\,(m+n+2)\theta}{2\sin\theta}\right]d\theta, \qquad m \geq n.$$

To evaluate this integral we make use of the following formula (10, p. 126, formula 6e):

$$\int \frac{\cos nx}{\sin x}\,dx = 2\sum_{k=0}^{r-1} \frac{\cos\,(n-2k-1)x}{n-2k-1} + s \ln \sin x$$

$$+ (1-s) \ln \tan \frac{x}{2} + C$$

where $n = 2r+s$ and $s = \begin{cases} 0 \text{ if } n = 2m, \\ 1 \text{ if } n = 2m+1. \end{cases}$

We have then

$$U_n\{U_m(X)\} = \left[\sum_{k=0}^{r-1} \frac{\cos\,(m-n-2k-1)\theta}{m-n-2k-1} - \sum_{k=0}^{q-1} \frac{\cos\,(m+n-2k+1)\theta}{m+n-2k+1}\right.$$

$$\left. + \frac{s}{2} \ln \sin \theta + \frac{1-s}{2} \ln \tan \frac{\theta}{2} - \frac{t}{2} \ln \sin \theta - \frac{1-t}{2} \ln \tan \frac{\theta}{2}\right]_{0^0}^{\pi}$$

Before evaluating the integral at its end points, we observe that if $m-n = 2h$, then $m+n+2 = 2n+2h+2 = 2p$; and if $m-n = 2h+1$, then $m+n+2 = 2n+2h+3 = 2p+1$. Consequently $s = t$ and we have

$$U_n\{U_m(X)\} = \left[\sum_{k=0}^{r-1} \frac{\cos\,(m-n-2k-1)\theta}{m-n-2k-1} - \sum_{k=0}^{q-1} \frac{\cos\,(m+n-2k+1)\theta}{m+n-2k+1}\right]_0^{\pi},$$

which finally becomes

$$U_n\{U_m(X)\} = \sum_{k=0}^{\frac{m-n-s-2}{2}} \frac{(-1)^{m-n-2k-1}-1}{m-n-2k-1} - \sum_{k=0}^{\frac{m+n-s}{2}} \frac{(-1)^{m+n-2k+1}-1}{m+n-2k+1}$$

where $s = \begin{cases} 0 \text{ if } m-n = 2h, \\ 1 \text{ if } m-n = 2h+1, \end{cases}$ and $m \geq n$. If $n < m$ interchange m and n in the final formula.

From the uniformly convergent expansions for the generating functions, namely

$$\frac{1 - Z^2}{1 - 2XZ + Z^2} = 1 + 2\sum_{m=1}^{\infty} T_m(X)Z^m, \qquad 4.1$$

$$\ln(1 - 2XZ + Z^2)^{-1} = -1 - 2\sum_{m=1}^{\infty} \frac{T_m(X)}{m} Z^m, \qquad 4.2$$

$$(1 - 2XZ + Z^2)^{-1} = \sum_{m=0}^{\infty} U_m(X)Z^m, \qquad 4.3$$

we obtain another set of formulas.

Differentiation of both sides of 4.3 with respect to Z followed by multiplication by Z leads to the additional result

$$T_n\left\{\frac{2XZ - 4Z^2}{(1-2XZ+Z^2)^2}\right\} = A(n) + B(m,n)$$

where

$$A(n) = \frac{1}{1-n^2}\left[1+(-1)^n\right] \quad \text{if } n \neq 1, \text{ 0 if } n = 1,$$

$$B(m,n) = \begin{cases} 2\sum_{m=1}^{\infty} (-1)^{n+m+1} \dfrac{2m^2 n}{\left[(n+m)^2-1\right]\left[(n-m)^2-1\right]} Z^m & \text{if } n \pm m \neq \pm 1, \\ 0 \quad \text{if } n \pm m = \pm 1. \end{cases}$$

If this process is continued, namely differentiation of both sides with respect to Z followed by multiplication by Z, additional transforms of more complicated functions may be found.

Similarly, use of the other generating functions will yield additional transforms although in the case of 4.2, we note that differentiation with respect to Z followed by multiplication by -Z and the addition of 1 to both sides yields 4.1.

A few additional formulas which may be considered as finite Tchebichef transforms may be found in Erdélyi (9, p. 271-275).

As an example of the application of the theory, consider the problem of solving the following differential equation

$$(1-X^2)f''(X) - 3Xf'(X) + af(X) = g(X)$$

subject to the boundary conditions

$$f(1) = A,$$
$$f(-1) = B.$$

Taking the transform of the differential equation, we have

$$T_n\{(1-X^2)f''(X) - 3Xf'(X) + af(X)\} = T_n\{g(X)\}.$$

Applying theorems 1 and 2, this reduces to

$$(1-n^2+a)T_n\{f(X)\} - f(1) - (-1)^n f(-1) = T_n\{g(X)\}$$

or

$$T_n\{f(X)\} = \frac{A + (-1)^n B + T_n\{g(X)\}}{1-n^2+a}$$

Solving this equation by means of inversion formula 2.5, we have

$$f(X) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\pi} \frac{A + (-1)^n B + T_n\{g(X)\}}{1-n^2+a} (1-X^2)^{-\frac{1}{2}} T_n(X). \qquad 4.5$$

Another example is to find the solution to the differential equation

$$(1-X^2)f''(X) - Xf'(X) + af(X) = g(X)$$

subject to the boundary conditions

$$f(1) = A,$$

$$f(-1) = B.$$

Taking the transform of the differential equation, applying theorems 1 and 3, and simplifying, we have

$$U_n\{f(X)\} = \frac{(n+1)A + (-1)^n B - U_n\{g(X)\}}{(n+1)^2-a}. \qquad 4.6$$

Using inversion formula 2.6, we arrive at

$$f(X) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(n+1)\left[A + (-1)^{n}B\right] - U_n\{g(x)\}}{(n+1)^2 - a}(1-X^2)^{\frac{1}{2}}U_n(X). \quad 4.7$$

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# APPENDIX

Table 1

| $F(X)$ | $T_n\{F(X)\}$ |
|---|---|
| $C$ | $\frac{C}{1-n^2}\left[1+(-1)^n\right]$ if $n \neq 1$; $0$ if $n = 1$ |
| $X$ | $\frac{1}{4-n^2}\left[1-(-1)^n\right]$ if $n \neq 2$; $0$ if $n = 2$ |
| $X^2$ | $\frac{3-n^2}{(1-n^2)(9-n^2)}\left[1+(-1)^n\right]$ if $n \neq 1, 3$; $0$ if $n = 1, 3$ |
| $X^3$ | $\frac{10-n^2}{(4-n^2)(16-n^2)}\left[1-(-1)^n\right]$ if $n \neq 2, 4$; $0$ if $n = 2, 4$ |
| $T_m(X)$ | $\frac{2mn}{\left[(n+m)^2-1\right]\left[(n-m)^2-1\right]}\left[1+(-1)^{n+m}\right]$ if $n \pm m \neq \pm 1$; <br> $0$ if $n \pm m = \pm 1$ |

| Table 1 (cont.) | |
|---|---|
| $F(X)$ | $T_n\{F(X)\}$ |
| $U_m(X)$ | $\frac{m+1}{(m+1)^2-n^2}\left[1+(-1)^{m+n}\right]$ if $m \geq n$; $0$ if $m+1 = n$ <br> $\frac{n+1}{(n+1)^2-n^2}\left[1+(-1)^{m+n}\right]$ if $n \geq m$; $0$ if $n+1 = m$ |
| $\ln(1 - 2XZ + Z^2)$ | $A(n) + B(m,n)$ where <br> $A(n) = \frac{-1}{1-n^2}\left[1+(-1)^{n}\right]$ if $n \neq 1$; $0$ if $n = 1$ <br> $B(m,n) = -2 \sum_{m=1}^{\infty} 1+(-1)^{m+n} \frac{2mn}{\left[(n+m)^2-1\right]\left[(n-m)^2-1\right]} Z^m$ <br> if $n \pm m \neq \pm 1$; $0$ if $n \pm m = \pm 1$ |

Table 1 (cont.)

| $F(X)$ | $T_n\{F(X)\}$ |
|---|---|
| $\dfrac{1 - Z^2}{1 - 2XZ + Z^2}$ | $-A(n) - B(m,n)$ where<br>$A(n) = \dfrac{1}{1-n^2}\left[1+(-1)^n\right]$ if $n \neq 1$; $0$ if $n = 1$<br>$B(m,n) = -2\sum_{m=1}^{\infty}\left[1+(-1)^{n+m}\right]\dfrac{2mn}{[(n+m)^2-1][(n-m)^2-1]}Z^m$<br>if $n \pm m \neq \pm 1$; $0$ if $n \pm m = \pm 1$ |
| $\mathrm{Sin}^{-1}X$ | $\dfrac{1}{2(1-n^2)}\left[1-(-1)^n\right]$ if $n \neq 1$; $-\frac{1}{4}$ if $n = 1$ |
| $\mathrm{Cos}^{-1}X$ | $\dfrac{(-1)^n}{1-n^2}$ if $n \neq 1$; $-\frac{1}{4}$ if $n = 1$ |

Table 1 (cont.)

| $F(X)$ | $T_n\{F(X)\}$ |
|---|---|
| $\dfrac{1}{1 - 2XZ + Z^2}$ | $\sum_{m=1}^{\infty}\left[(-1)^{n+m+1}-1\right]\dfrac{n+1}{(n+1)^2-m^2}Z^m$ if $n \geq m$;<br>$\sum_{m=1}^{\infty}\left[(-1)^{n+m+1}-1\right]\dfrac{m+1}{(m+1)^2-n^2}Z^m$ if $m \geq n$;<br>$0$ if $n \pm 1 = m$ |

Table 2

| $F(X)$ | $U_n\{F(X)\}$ |
|---|---|
| $C$ | $\frac{C}{n+1}\left[1+(-1)^n\right]$ |
| $X$ | $\frac{n+1}{n(n+2)}\left[1-(-1)^n\right]$ if $n \neq 0$; $0$ if $n = 0$ |
| $X^2$ | $\frac{n^2 + 2n - 1}{(n^2-1)(n+3)}\left[1+(-1)^n\right]$ if $n \neq 1$; $0$ if $n = 1$ |
| $X^3$ | $\frac{(n+1)(n^2+2n-6)}{n(n^2-4)(n+4)}\left[1-(-1)^n\right]$ if $n \neq 0, 2$; $0$ if $n = 0, 2$ |
| $\sin^{-1}X$ | $\frac{\pi}{2(n+1)}\left[1+(-1)^n\right] + \frac{(-1)^n}{n+1}$ |

**Table 2 (cont.)**

| $F(X)$ | $U_n\{F(X)\}$ |
|---|---|
| $T_m(X)$ | $\frac{n+1}{(n+m+1)(n-m+1)}\left[1+(-1)^{n+m}\right]$ if $n \geq m$; 0 if $n \pm m = -1$<br>$\frac{m+1}{(n+m+1)(m-n+1)}\left[1+(-1)^{n+m}\right]$ if $m \geq n$; 0 if $m \pm n = -1$ |
| $U_m(X)$ | $\sum_{k=0}^{\frac{m-n-s-2}{2}} \frac{(-1)^{m-n-2k-1}-1}{m-n-2k-1} - \sum_{k=0}^{\frac{m+n-s}{2}} \frac{(-1)^{m+n-2k+1}-1}{m+n-2k+1}$ where $m \geq n$;<br>$\sum_{k=0}^{\frac{n-m-s-2}{2}} \frac{(-1)^{n-m-2k-1}-1}{n-m-2k-1} - \sum_{k=0}^{\frac{m+n-s}{2}} \frac{(-1)^{m+n-2k+1}-1}{m+n-2k+1}$ where $n \geq m$<br>$s = 0$ if $m-n = 2h$; 1 if $m-n = 2h+1$ |

| Table 2 (cont.) | |
|---|---|
| $F(X)$ | $U_n\{F(X)\}$ |
| $\dfrac{1 - Z^2}{1 - 2XZ + Z^2}$ | $\frac{1}{n+1}\left[1+(-1)^n\right] + 2\sum_{m=1}^{\infty}\left[1+(-1)^{m+n}\right]\frac{m+1}{(m+1)^2-n^2}Z^m$<br>if $m \geq n$ and $m+1 \neq n$;<br>$\frac{1}{n+1}\left[1+(-1)^n\right] + 2\sum_{m=1}^{\infty}\left[1+(-1)^{m+n}\right]\frac{n+1}{(n+1)^2-m^2}Z^m$<br>if $n \geq m$ and $n+1 \neq m$;<br>$\frac{1}{n+1}\left[1+(-1)^n\right]$ if $n-m = \pm 1$ |
| $\mathrm{Cos}^{-1}X$ | $\frac{(-1)^n}{n+1}$ |

Table 2 (cont.)

| $F(X)$ | $U_n\{F(X)\}$ |
|---|---|
| $\ln(1 - 2XZ + Z^2)$ | $\frac{1}{n+1}\left[1+(-1)^n\right] + 2\sum_{m=1}^{\infty}\left[1+(-1)^{n+m}\right]\frac{m+1}{(m+1)^2-n^2}\,\frac{Z^m}{m}$<br>if $m \geq n$ and $m+1 \neq n$;<br>$\frac{-1}{n+1}\left[1+(-1)^n\right] + 2\sum_{m=1}^{\infty}\left[1+(-1)^{n+m}\right]\frac{n+1}{(m+1)^2-n^2}\,\frac{Z^m}{m}$<br>if $n \geq m$ and $n+1 \neq m$;<br>$\frac{-1}{n+1}\left[1+(-1)^n\right]$ if $n-m = \pm 1$ |