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Title: INITIAL AND BOUNDARY VALUE PROBLEM FOR A THIRD ORDER DIFFERENTIAL EQUATION OF PARABOLIC TYPE

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Ronald B. Guenther

The third order parabolic differential operator

$$Lu = \sum_{i=0}^{3} a_i(x,t) \frac{\partial^i u}{\partial x^i} - \frac{\partial u}{\partial t}, \quad a_3(x,t) \equiv 1,$$

is considered. The case where a_i, $i = 0, 1, 2$ depend only on t is also treated. Under suitable assumptions on the coefficients, the existence of a fundamental solution $\Gamma(x, t; \xi, \tau)$, for L is first proven and various properties for Γ are derived. The initial value problem

$$Lu = f(x,t) \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \phi(x)$$

where $f(x,t)$, $\phi(x)$ satisfy certain continuity and growth requirements, is then discussed. If u is restricted to a certain class of functions, this problem is shown to have a unique solution. Furthermore, a representation for the solution u in terms of the fundamental solution and the known functions f and ϕ is obtained.

Finally, the first initial boundary value problem is treated. A uniqueness theorem is proven, certain jump relations are obtained whence by standard potential theoretic techniques, one arrives at an existence theorem for this problem.

Initial and Boundary Value Problem for a Third Order Differential Equation of Parabolic Type

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Rokaya Abdel-Ghani Al-Ayat

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Dean of Graduate School

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INITIAL AND BOUNDARY-VALUE PROBLEMS FOR A THIRD ORDER DIFFERENTIAL EQUATION OF PARABOLIC TYPE

I. INTRODUCTION

Mathematical studies on third order partial differential equations of the parabolic type seem to have first appeared in 1911. However, it was known earlier that such equations play an important role in the theory of long waves. It was found, recently, that they also describe some problems in solid state and plasma physics. In the following two sections, some of the relevant physical aspects are reviewed and an account of mathematical studies on the subject is presented.

1.1 Physical Motivation

The study of the wave motion in liquids with a free surface under gravity is an old discipline. Interest in such waves were motivated both by their physical applications and the mathematical ideas and methods needed to attack them. An important part of these studies deals with long waves, i.e., those whose wave lengths are much greater than the depth of the fluid in which they are propagating. Formation of breakers in shallow water, bores in rivers and tides in the ocean are some of the important physical phenomena that belong to the theory of long waves.

Studies of long waves seem to have originated in a paper of Airy [1] in 1845. From the experimental work of Russell [16] and theoretical studies of Airy [1], Lord Rayleigh [15], and others during the last century, there appeared a contradiction which is now known as the long wave paradox. In his essay on "Tides and Waves," Airy considered the behavior of long waves in a canal with rectangular cross-section. Arguing that for long waves the vertical velocity of water-particles can be neglected and the horizontal velocity can be considered uniform over each cross-section of the canal, he found that the wave-velocity increases with the elevation from the undisturbed level. This result led him to the conclusion that progressive long waves cannot propagate without change of form. On the other hand, in the British Association Report of 1844, Russell [16] reported his experimental observations on what he called solitary waves. In his studies on waves whose wave length is about six to eight times the depth of the canal, and therefore can be properly classified as long waves, he discovered the existence of waves of small amplitude which can propagate for large distances without change of form. Russell's interpretations of his data were disputed by Airy. Later experimental work of Earnshaw [6], however, confirmed Russell's observations. In 1876 Lord Rayleigh [15] took up this problem and gave what he considered to be a perfectly satisfactory approximate theory of the solitary waves. He argued that by impressing on the water of the canal a velocity equal and opposite to that of the progressive wave, the problem would be reduced to one of a steady motion. If the x-axis is taken at the bottom of the canal and the y-axis in the vertical direction, and if $u(x, y)$ and $v(x, y)$ denote the horizontal and vertical velocities, respectively, then the Bernoulli condition for two dimensional irrotational incompressible flow

$$-2\,\frac{p-c}{\rho} = 2gy + u^2 + v^2$$

has to be satisfied on the surface, where p is the pressure, c is a constant and ρ, the density, is also constant. His idea was to examine the possibility of making $\frac{p-c}{\rho}$ constant by varying the form of y as a function of x. Such a $y(x)$ will describe a wave progressing under constant pressure, and consequently will not change its form as it propagates. Using the method of successive approximations, he succeeded in finding such a $y(x)$ and thus obtained a solitary wave which had the velocity of propagation $u = \sqrt{g\ell'}$, where, ℓ' is the distance between the summit of the wave and the bottom of the canal. This value for u is the same as that found experimentally by Russell.

The inapplicability of Airy's theory to the solitary waves gives one aspect of the long wave paradox. In an effort to settle this contradiction, Kortweg and deVries [11] in 1895, derived an equation, which describes the deformation of a system of waves of arbitrary shape moving in one direction in a rectangular canal. They supposed that by adding to the water of the canal a velocity equal and opposite to that of the wave, the surface will be reduced to approximate but not perfect rest. Assuming that the vertical and horizontal velocities $u(x, y, t)$ and $v(x, y, t)$ can be expressed by rapidly convergent series in y (an assumption made also by Rayleigh), and using the conditions of incompressibility and irrotationality of the fluid, i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$, they found that

$$u(x, y, t) = f(x, t) - \frac{1}{2} y^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{24} y^4 \frac{\partial^4 f}{\partial x^4} + \dots$$

$$v(x, y, t) = -y \frac{\partial f}{\partial x} + \frac{1}{6} y^3 \frac{\partial^3 f}{\partial x^3} - \frac{1}{120} y^5 \frac{\partial^5 f}{\partial x^5} + \dots$$

where y is the height of a particle above the bottom of the canal. Moreover, on the surface, the conditions

$$p' = p - T \frac{\partial^2 y}{\partial x^2} \qquad (1.1)$$

$$\frac{p'}{\rho} = \chi(t) - \frac{\partial \phi}{\partial t} - \frac{1}{2}(u^2+v^2) - gy \qquad (1.2)$$

and the kinematic condition

$$u \frac{\partial y}{\partial x} + v - \frac{\partial y}{\partial t} = 0 \qquad (1.3)$$

have to be satisfied, where p is the atmospheric pressure, p' is the pressure at a point just below the surface where the capillary forces cease to act, T is the surface tension and ϕ is the velocity potential, i.e., $u = \frac{\partial \phi}{\partial x}$. They used the method of successive approximations, and took $y = \ell + \eta$ and $f = q_0 + \beta$ as a first approximation, where ℓ and q_0 are constants, and η and β are functions of x and t. From the above Equations (1.1), (1.2), and (1.3), they found that

$$q_0 \frac{\partial \beta}{\partial x} + \frac{\partial \beta}{\partial t} + g \frac{\partial \eta}{\partial x} = 0$$

$$q_0 \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} + \ell \frac{\partial \beta}{\partial t} = 0$$

These equations are satisfied by taking

$$\frac{d\eta}{dt} = \frac{d\beta}{dt} = 0; \qquad \beta = -\frac{q_0}{\ell}(\eta+\alpha); \qquad q_0 = \sqrt{g\ell}$$

where α is an arbitrary small constant. This result is in agreement with the one given by Lord Rayleigh for long waves made stationary by imposing on the fluid a velocity which is equal and opposite to that of the waves. Thus, they concluded that the first approximation of their equation is a solitary wave having a velocity of propagation equal to the velocity given by Rayleigh. As a second approximation, taking $f = q_0 - \frac{q_0}{\ell}(\eta+\alpha+\gamma)$, where γ is small compared to

η and α, they found that $\eta(x,t)$ satisfies the equation

$$(1.4) \qquad \frac{\partial\eta}{\partial t} = \frac{3q_0}{2\ell} \frac{\partial(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2})}{\partial x}$$

where

$$\sigma = \frac{1}{3}\ell^3 - \frac{T\ell}{\rho g}$$

If the Equation (1.4) is linearized, i.e., the term $\frac{3q_0}{4\ell}\frac{\partial\eta^2}{\partial x}$ is neglected, we obtain a special case of the equation we shall be investigating in the sequel.

It is of interest to note that tidal waves can be treated as long waves with small amplitudes, since even in the deepest oceans, the wave lengths, due to the large periods of disturbances caused by the moon and the sun, are so great that the ratio of the wave length to the depth of the ocean is large. Moreover, since the amplitudes are small, tidal waves, can be described by linearizing (1.4).

As a final note on the long wave paradox we mention that in 1957, following a different approach, Stoker [17] in his paper "On the Formation of Breakers and Bores in Shallow Water and Open Channels" reached a conclusion which is in agreement with that of Airy. He showed that the equations describing long waves are identical in form with those arising in the study of one-dimensional unsteady flow of a compressible gas. From a known result in gas dynamics, that for a simple wave traveling in one direction crests travel faster than troughs, the front of each wave becomes steep and eventually curls over and breaks. Thus, he was able to explain the formation of breakers and support Airy's conclusion.

Recently, the Equation (1.4) in both its non-linear and linearized form, was found to be of interest in other physical problems. In 1960, Gardner and Morikawa [8] found a similarity between the behavior of certain waves in a plasma in a magnetic field and gravity waves on the surface of water of finite depth. They compared the motion of singly-charged, collision-free particles in a magnetic field and that of surface waves of water where both motions were caused by a piston started impulsively from rest and moving with a small uniform velocity. They found that an analogy between the behavior of the two phenomena exists, after a certain period of time has elapsed, in the linearized description as well as in a time-dependent non-linear approximation. In 1967, Zabusky [20] in his paper "A Synergetic Approach to Problems of Non-linear Dispersive Wave Propagation and Interaction," showed that the Equation (1.4) describes propagation of waves of finite amplitude on a collisionless, non-linear lattice which is initially excited by a progressive wave.

1.2 Mathematical Studies

The first detailed mathematical treatment for third order partial differential equations was given by Block [3] during the years 1911-1913. Motivated by the desire to find a function which for higher order parabolic equations, would play a role similar to that of the fundamental solution for the heat equation,

$$\frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}},$$

Block considered the equation $\frac{\partial^p u}{\partial x^p} + a\frac{\partial^q u}{\partial t^q} = 0$, where a is a constant, $p > q$ and p and q are relatively prime. He gave a detailed discussion of the fundamental solution for the equation

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial u}{\partial t}, \tag{1.5}$$

where he showed that it is given by

$$U(x-\xi, t-\tau) = \frac{1}{(t-\tau)^{1/3}} f(y), \quad y = \frac{\xi - x}{(t-\tau)^{1/3}}, \quad t > \tau \tag{1.6}$$

and $f(y)$ is given by

$$f(y) = \frac{1}{\pi}\int_0^\infty \cos(\lambda^3 - \lambda y)d\lambda \tag{1.7}$$

which is an integral representation for the Airy function with argument, $-y/(3)^{1/3}$ (see Appendix).

Having obtained a fundamental solution, one is able to give a representation for solutions to the relevant Cauchy problem. However, as in the case of the Cauchy problem for the heat equation, the domain of dependence is the whole x-axis, so some restriction has to be made on the behavior of the initial data.

In the case of the heat equation, it was shown by Täcklind [18] and by Tychonoff [19] that the natural region for existence and uniqueness for the solutions to the Cauchy problem is in the class of functions bounded by $Ae^{a|x|^{2-\epsilon}}$ where A, a are positive constants, and $\epsilon > 0$ is arbitrary.

In 1953, Gel'fand and Šilov [9], considered the question of existence and uniqueness of solutions to the Cauchy problem for the system of equations

$$P(\frac{\partial}{\partial x}, t)u(x,t) = \frac{\partial u(x,t)}{\partial t} \tag{1.8}$$

where $u = (u_1, u_2, \ldots, u_n)$ and $P(s,t)$ is a polynomial in s of order p, say, with coefficients depending only on t. They showed that the system (1.8) has (in a suitable generalized sense) a unique solution in the class of functions satisfying the growth condition

$$|u(x,t)| \leq c_0 e^{c_1 |x|^{p'-\epsilon}},$$

where $\epsilon > 0$ is arbitrary, c_0, c_1 are positive constants and $p' = \frac{p}{p-1}$. Consequently, the class of functions which increase more slowly than $\exp(|x|^{(3/2)-\epsilon})$, $\epsilon > 0$ is the natural class of functions in which to obtain existence (in a suitable generalized sense) and uniqueness of solutions to the Cauchy problem for third order parabolic equations.

We note also that, Gerber [10] in 1968, in his study on non-convex conservation laws, considered the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{1}{3}\frac{\partial^3 u}{\partial x^3} \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = u_0(x) \qquad x > 0$$

$$= 0 \qquad x \leq 0.$$

He showed by using Fourier transform techniques that if $u_0(x)$ is of bounded variation and if for some $c > 0$, $u_0(x) = O(e^{cx^{3/2}})$ as $x \to \infty$, then $u(x,t)$ exists in the strip $\{-\infty < x < \infty,\ 0 \leq t < \frac{4}{9c^2}\}$ and is given by

$$u(x,t) = \int_0^\infty U(x-y,t)u_0(y)dy,$$

where

$$U(x,t) = \frac{1}{t^{1/3}}\,\mathrm{Ai}\left(\frac{-x}{t^{1/3}}\right). \tag{1.9}$$

Furthermore, there exist positive constants A, a such that

(1.10) $$|u(x,t)| \le A \exp(a|x|^{3/2})$$

One should mention here, that the uniqueness class given by (1.10), cannot be extended in any essential way. For that we construct a function $u(x,t) \not\equiv 0$ satisfying

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial u}{\partial t} \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = 0 \qquad -\infty < x < \infty$$

$$|u(x,t)| \le c_0 \exp(c_1 |x|^{(3/2)+\delta})$$

where c_0 and c_1 are positive constant, $\delta > 0$ is fixed but arbitrary. Let $f(t)$ be an infinitely differentiable function satisfying $f^n(0) = 0$, $n = 0, 1, \ldots, f(t) \not\equiv 0$ and for $\epsilon > 0$

$$|f^n(t)| \le n^{(1+\epsilon)n} \qquad 0 < \epsilon < 2,$$

such function is known to exist [13]. Set

$$u(x,t) = \sum_{n=0}^{\infty} \frac{f^n(t)}{3n!} x^{3n}$$

the series converges uniformly for bounded x, $u(x,0) = 0$, $u(x,t) \not\equiv 0$ and moreover $\frac{\partial^3 u}{\partial x^3} = \frac{\partial u}{\partial t}$,

$$|u(x,t)| \leq a_0 + \sum_{n=1}^{\infty} \frac{a_1^n |x|^{3n}}{n^{3n-(1+\epsilon)n}} \leq c_0 e^{c_1 |x|^{(3/2)+\delta}} ,$$

where a_0, a_1 are constants and $\delta = \frac{3\epsilon}{2(2-\epsilon)}$.

Some work had also been done on boundary value problems for third order equations. Using the results of Block [3] on the fundamental solution of (1.5), Del Vecchio [5] in 1916, considered the boundary-value problem for the non-homogenous equation

(1.11) $$\frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} + \phi(x,t) = 0$$

in a bounded domain D. He showed, if $\phi(x,t)$, $\frac{\partial \phi(x,t)}{\partial x}$, $\frac{\partial \phi(x,t)}{\partial t}$ are continuous and uniformly bounded in D, then in the interior of D, the function

$$u(x,t) = \frac{1}{\pi} \iint_D U(x-\xi, t-\tau)\phi(\xi,\tau)d\xi d\tau$$

is a solution of (1.11), which is continuous together with its first three derivatives with respect to x, and the first derivative with respect to t. The next result on this problem was given by Cattabriga [4], in 1959. He considered the boundary-value problem for the third order equation

$$(1.12) \qquad \sum_{i=0}^{3} a_i(x) \frac{\partial^i u}{\partial x^i} - \frac{\partial u}{\partial t} = f(x,t)$$

in a bounded domain D, $D = \{(x,t) : 0 \leq t \leq 1, \chi_1(t) \leq x \leq \chi_2(t)\}$ with sufficiently regular $\chi_1(t) < \chi_2(t)$. By reducing the problem to a system of Volterra type, singular integral equations, he demonstrated the existence of solutions for (1.12). Moreover, he proved that if a_i, $i = 0, 1, 2, 3$ are functions of both x, t then the problem has a unique solution.

1.3 Outline of the Present Study

From the above survey the need for further studies on third order parabolic differential equations is obvious. Thus, in the following a study for the initial and boundary-value problems for parabolic third order equations will be considered. For the initial-value problem, we aim to prove the existence of a fundamental solution for the equation

$$(1.13) \qquad Lu = 0$$

$$L = \frac{\partial^3}{\partial x^3} + a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + c(x,t) - \frac{\partial}{\partial t}$$

under appropriate conditions on the coefficients a, b and c. Using the fundamental solution we then construct the solution for the associated Cauchy problem. However, before doing that it is necessary to obtain some results on the Cauchy problem for the equation where the coefficients depend only on t. Therefore, Chapter II is devoted to a study of this case. It will be shown that the fundamental solution for such equations can be expressed in terms of the Airy function and that in a certain class of initial data, a representation for the solution of the Cauchy problem for the homogenous and non-homogenous equations can be given.

In Chapter III we give a construction of the fundamental solution for linear, third order parabolic equations with coefficients depending on both x and t. By choosing an appropriate parametrix and using the results achieved in Chapter II, we are able to reduce the problem of finding the fundamental solution to one of solving a certain Volterra type integral equation. This integral equation is then solved by iteration. Various elementary properties of the fundamental solution are derived and the non-homogeneous Cauchy problem is solved.

Having obtained the fundamental solution for the equation with coefficients depending on both x, t we are able, in Chapter IV, to extend Cattabriga's results [4] and discuss the existence of a unique solution to the first initial-boundary value problem associated with (1.13).

II. THIRD ORDER PARABOLIC EQUATIONS WITH TIME DEPENDENT COEFFICIENTS

Let us consider the parabolic partial differential equation

$$(2.1) \qquad P(t, \frac{\partial}{\partial x})u(x,t) = \frac{\partial u(x,t)}{\partial t}$$

for

$$x \in R = \{-\infty < x < \infty\}, \qquad t \in (0, T]$$

where

$$P(t, \frac{\partial}{\partial x}) = \frac{\partial^3}{\partial x^3} + a(t)\frac{\partial^2}{\partial x^2} + b(t)\frac{\partial}{\partial x} + c(t),$$

we shall always assume that the functions $a(t)$, $b(t)$, $c(t)$ are defined and continuous for t in the finite interval $[0, T]$. Furthermore, it will be assumed that $\int_0^t a(\tau)d\tau \geq 0$ for $t \in [0, T]$.

Definition (2.1). A function $K(x, \xi; t, \tau)$ defined for all $(x, t) \in R \times [0,T]$, $(\xi, \tau) \in R \times [0,T]$ with $(x, t) \neq (\xi, \tau)$, is said to be a fundamental solution of (2.1) if:

(i) As a function of x, t, $K(x, t; \xi, \tau)$ is three times differentiable in x and once in t and satisfies (2.1) for $(x, t) \neq (\xi, \tau)$,

(ii) $K(x, t; \xi, \tau) \equiv 0$ for $t \leq \tau$

(iii) For any continuous, bounded function $\phi(x)$ which vanishes outside a finite interval,

$$\lim_{t \to \tau+} \int_{-\infty}^{\infty} K(x, t; \xi, \tau)\phi(\xi)d\xi = \phi(x)$$

2.1 Construction of the Fundamental Solution

Using standard Fourier transform techniques, one gets for the fundamental solution of (2.1), the function

$$K(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - \omega^2 A(t) - i\omega B(t) + i\omega^3 t + C(t)} \, d\omega$$

where

$$A(t) = \int_0^t a(\tau)d\tau, \quad B(t) = \int_0^t b(\tau)d\tau, \quad C(t) = \int_0^t c(\tau)d\tau$$

Obviously, the integral for $K(x, t)$ converges since $A(t) \geq 0$. We first evaluate the integral for $K(x, t)$.

$$K(x, t) = \frac{e^{C(t)}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(x+B(t)) + it(\omega^3 + \frac{iA(t)}{t}\omega^2)} \, d\omega$$

$$K(x, t) = \frac{e^{C(t) - \frac{(A(t))^3}{27t^2}}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(x - \frac{(A(t))^2}{3t} + B(t)) + it(\omega + \frac{i(A(t)}{3t})^3} \, d\omega$$

$$= \frac{e^{C(t) - \frac{(A(t))^3}{27t^2} - \frac{A(t)}{3t}y}}{2\pi} \int_{-\infty + \frac{iA(t)}{3t}}^{\infty + \frac{iA(t)}{3t}} e^{-iuy + itu^3} \, du$$

by Cuachy's theorem, where $y = x - \frac{(A(t))^2}{3t} + B(t)$.

Lemma (2. 1). For any $\mu > 0$, μ real

$$\int_{-\infty+i\mu}^{\infty+i\mu} e^{-iuy+itu^3} du = \frac{2\pi}{(3t)^{1/3}} Ai(\frac{-y}{(3t)^{1/3}}) .$$

Proof. Consider

$$I = \int_{\Gamma} e^{-izu+itz^3} dz,$$

where Γ is taken to be the contour $\overrightarrow{-\lambda, \lambda}$, $\overrightarrow{\lambda, \lambda+i\mu}$, $\overrightarrow{\lambda+i\mu, \lambda-i\mu}$, and $\overrightarrow{-\lambda+i\mu, -\lambda}$. Obviously, the integrand is an analytic function of z within Γ. Thus, by Cauchy's theorem:

$$0 = I = \int_{-\lambda}^{\lambda} e^{-iuy+itu^3} du + \int_0^{\mu} e^{-i(\lambda+iv)y+it(\lambda+iv)^3} dv$$

$$+ \int_{-\lambda+i\mu}^{\lambda+i\mu} e^{-izy+itz^3} dz + \int_0^{\mu} e^{-i(-\lambda+iv)y+it(-\lambda+iv)^3} dv$$

$$= \sum_{j=1}^{4} I_j$$

$$|I_2| \leq \int_0^{\mu} e^{vy-3\lambda^3 tv+tv^3} dv \to 0 \quad \text{as} \quad \lambda \to \infty$$

similarly for I_4. Consequently,

$$\int_{-\infty+i\mu}^{\infty+i\mu} e^{-izu+itz^3}\,dz = \int_{-\infty}^{\infty} e^{-iyu+itu^3}\,du$$

$$= 2\int_0^{\infty} \cos(uy-tu^3)\,du = \frac{2\pi}{(3t)^{1/3}}\,\mathrm{Ai}\left(\frac{-y}{(3t)^{1/3}}\right),$$

which proves the lemma.

Using the lemma, we get for the fundamental solution of (2.1)

$$\text{(2.2)}\qquad K(x,t) = \frac{e^{C(t)-\frac{(A(t))^3}{27t^2}-\frac{A(t)}{3t}y}}{(3t)^{1/3}}\,\mathrm{Ai}\left(\frac{-y}{(3t)^{1/3}}\right), \qquad t>0,$$

$$y = x - \frac{(A(t))^2}{3t} + B(t)$$

Let us observe that the function $v(x,t) = e^{-C(t)}u(x-B(t),t)$ satisfies the partial differential equation

$$\text{(2.3)}\qquad \frac{\partial^3 v}{\partial x^3} + a(t)\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

Therefore, with no loss of generality we may discuss in the following the properties of the fundamental solution of (2.3), which is given by

$$\text{(2.4)}\qquad \Gamma(x,t) = \frac{e^{-\frac{(A(t))^3}{27t^2}-\frac{A(t)}{3t}y}}{(3t)^{1/3}}\,\mathrm{Ai}\left(\frac{-y}{(3t)^{1/3}}\right),$$

$$y = x - \frac{(A(t))^2}{3t} ,$$

rather than that of (2.1) given by (2.2).

2.2 Properties of the Fundamental Solution $\Gamma(x, t)$

Property (1).

$$\int_{-\infty}^{\infty} \Gamma(x-\xi, t)d\xi = 1, \qquad t > 0. \tag{2.5}$$

The proof of (2.5) depends upon the following lemma:

Lemma (2.2). For any α, $\operatorname{Re} \alpha \geq 0$

$$I = \int_{-\infty}^{\infty} e^{\alpha\eta} Ai(\eta)d\eta$$

exists and

$$I = e^{\frac{\alpha^3}{3}} .$$

Proof.

$$I = \int_0^{\infty} e^{\alpha\eta} Ai(\eta)d\eta + \int_0^{\infty} e^{-\alpha\eta} Ai(-\eta)d\eta$$

$$= I_1 + I_2 \qquad , \text{ respectively,}$$

To evaluate I_1, we observe that,

$$|e^{\alpha\eta}Ai(\eta)| \leq e^{-\frac{2}{3}\eta^{3/2}}$$

from the bounds given in the Appendix for $Ai(x)$. Thus I_1 is uniformly convergent by the Weierstrass M-test. For I_2, using the Dirichlet's test, since $e^{-\alpha\eta}$ is monotonically decreasing and approaches zero as $\eta \to \infty$ and $Ai(-\eta)$ is an integrable function, the convergence is uniform. Therefore, the integral I converges uniformly to

$$F(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} Ai(\eta) d\eta.$$

Similarly differentiation underneath the integral sign with respect to α is justifiable and,

$$F'(\alpha) = \int_{-\infty}^{\infty} \eta e^{\alpha\eta} Ai(\eta) d\eta = \int_{-\infty}^{\infty} e^{\alpha\eta} A''i(\eta) d\eta \qquad \text{by (A.1).}$$

Thus integrating twice by parts, we get

$$F'(\alpha) = \alpha^2 \int_{-\infty}^{\infty} e^{\alpha\eta} Ai(\eta) d\eta = \alpha^2 F(\alpha)$$

i.e., $F(\alpha)$ satisfy the ordinary differential equation

$$F'(\alpha) - \alpha^2 F(\alpha) = 0$$

and for $\alpha = 0$, $F(0) = 1$, from (A. 4). Hence

(2. 6)
$$F(\alpha) = e^{\frac{\alpha^3}{3}}$$

which proves the lemma.

To prove (2. 5), using (2. 4), we have

$$\int_{-\infty}^{\infty} \Gamma(x-\xi, t)d\xi$$

$$= e^{-\frac{(A(t))^3}{27t^2}} \int_{-\infty}^{\infty} \frac{e^{\frac{A(t)}{3t}\left(-x+\xi+\frac{(A(t))^2}{3t}\right)}}{(3t)^{1/3}} Ai\left(\frac{-x+\xi+\frac{(A(t))^2}{3t}}{(3t)^{1/3}}\right) d\xi$$

Let

$$-x + \xi + \frac{(A(t))^2}{3t} = (3t)^{1/3}\eta,$$

Then

$$\int_{-\infty}^{\infty} \Gamma(x-\xi, t)d\xi = e^{-\frac{(A(t))^3}{27t^2}} \int_{-\infty}^{\infty} e^{\frac{A(t)}{(3t)^{2/3}}} Ai(\eta)d\eta = 1$$

by Lemma (2. 2) with $\alpha = \frac{A(t)}{(3t)^{2/3}} > 0$.

Property (2). $\Gamma(x, t)$ satisfies the semi-group relation

(2. 7)
$$\int_{-\infty}^{\infty} \Gamma(x-\xi, t-\tau)\Gamma(\xi, \tau)d\xi = \Gamma(x, t).$$

The proof follows directly from the following lemma.

Lemma (2. 3). For α, β real, $\beta \geq 0$,

$$I = \int_{-\infty}^{\infty} e^{\beta\eta} Ai(x-\alpha\eta)Ai(\eta)d\eta$$

exists and is given by

$$I = \frac{e^{\frac{\alpha^2\beta}{1+\alpha^2}x - \frac{\beta^3(\alpha^3-1)}{3(\alpha^3+1)^2}}}{(1+\alpha^3)^{1/3}} Ai\left(\frac{x - \frac{\alpha\beta^2}{1+\alpha^3}}{(1+\alpha^3)^{1/3}}\right).$$

Proof. The existence of I is readily verified using the properties of the Airy function. To evaluate the integral, we consider the function

$$F(x) = \int_{-\infty}^{\infty} e^{\beta\eta} Ai(x-\alpha\eta)Ai(\eta)d\eta$$

We apply the two sided laplace transform (see Apostol [2]) to $F(x)$ to get

$$G(z) = \int_{-\infty}^{\infty} e^{zx} F(x)dx = \int_{-\infty}^{\infty} e^{zx}\left[\int_{-\infty}^{\infty} e^{\beta\eta} Ai(x-\alpha\eta)Ai(\eta)d\eta\right] dx.$$

Interchanging the order of integration, we find

$$I = \int_{-\infty}^{\infty} e^{az\eta+\beta\eta} Ai(\eta)\left[\int_{-\infty}^{\infty} e^{z(x-a\eta)} Ai(x-a\eta)dx\right] d\eta$$

$$= e^{\frac{z^3}{3}} \int_{-\infty}^{\infty} e^{\eta(az+\beta)} Ai(\eta)d\eta$$

$$= e^{\frac{z^3}{3} + \frac{(az+\beta)^3}{3}}$$

by Lemma (2. 2). Taking the inverse laplace transform, we obtain

$$F(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-zx} G(z)dz$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-zx+\frac{1+a^3}{3} z^3 + a^2\beta z^2 + a\beta^2 z + \frac{\beta^3}{3}} dz$$

$$= \frac{e^{\frac{\beta^3}{3}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1+a^3}{3}\left(z+\frac{a^3\beta}{1+a^3}\right)^3 - z\left(x-\frac{a\beta^2}{1+a^3}\right) - \frac{a^6\beta^3}{3(1+a^3)^2}} dz$$

$$F(x) = \frac{e^{\frac{a^2\beta}{1+a^3}x - \frac{\beta^3(a^3-1)}{3(a^3+1)^2}}}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} e^{\frac{1+a^3}{3}\eta^3 - \eta\left(x-\frac{a\beta^2}{1+a^3}\right)} d\eta$$

where

$$c' = c + \frac{a^2\beta}{1+a^3}$$

From the definition of the Airy function (A. 2)

$$F(x) = \frac{e^{\frac{\alpha^2\beta}{1+\alpha^3} - \frac{\beta^3(\alpha^3-1)}{3(\alpha^3+1)^3}}}{(1+\alpha^3)^{1/3}} \, Ai\left(\frac{x - \frac{\alpha\beta^2}{1+\alpha^3}}{(1+\alpha^3)^{1/3}}\right)$$

From (2. 4), and using Lemma (2. 3), Property (2) follows.

Property (3). The fundamental solution, $\Gamma(x, t)$, is unique.

This property will follow from the results to be given in the next section.

Property (4). Using the bounds given for the Airy function in the Appendix one will be able to get the corresponding bounds for $\Gamma(x, t)$. More precisely for $0 < t \leq T$,

$$y = x - \frac{(A(t))^2}{3t} \leq 0$$

(2. 8)
$$\Gamma(x, t) = \frac{c_0}{(3t)^{1/3}} \, e^{-\frac{A(t)}{3t} y} \, e^{-\frac{2}{3}\frac{(-y)^{3/2}}{(3t)^{1/3}}} \left[1 + 0\left(\frac{y}{(3t)^{1/3}}\right)\right],$$

as $y \to \infty$

whence one concludes

(2. 9)
$$0 \leq \Gamma(x, t) \leq \frac{c_0}{(3t)^{1/3}} \, e^{-c\frac{(-y)^{3/2}}{(3t)^{1/3}}}$$

For

$$y = x - \frac{(A(t))^2}{3t} > 0$$

$$(2.10) \qquad \Gamma(x,t) = \frac{c_1}{(3ty)^{1/4}} e^{-\frac{A(t)}{3t} y} \sin\left(\frac{2}{3}\frac{y^{3/2}}{(3t)^{1/2}} + \theta\right) + O\left(\left(\frac{y}{(3t)^{1/3}}\right)^{-7/4}\right)$$

as $y \to \infty$, and where θ is a constant. We note that for $x < 0$, $\Gamma(x, t) > 0$ and converges to zero as $x \to \infty$, while for $x > 0$, $\Gamma(x,t)$ oscillates with decreasing magnitude converging to zero as $x \to -\infty$.

Property (5). The third derivative, Γ_{xxx}, of Γ has the bounds

$$|\Gamma_{xxx}(x,t)| \leq \frac{\text{const}}{t^{4/3}} e^{-\frac{1}{3}\frac{(-y)^{3/2}}{(3t)^{1/2}}}, \qquad y < 0$$

$$|\Gamma_{xxx}(x,t)| \leq \text{const} \frac{y^{5/4}}{t^{7/4}} e^{-\frac{a_0}{3} y} \cos\left(\frac{2}{3}\frac{y^{3/2}}{(3t)^{1/2}} + \theta\right), \qquad y > 0$$

where

$$y = x \frac{(A(t))^2}{3t}.$$

2.3 The Initial-Value Problem

The homogenous equation. From the derivation of the fundamental solution, we know that the solution of the Cauchy problem

$$(2.11) \qquad \frac{\partial^3 u}{\partial x^3} + a(t)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \qquad x \in R, \quad t \in (0, T]$$

$$u(x,0) = \phi(x), \quad x \in R \tag{2.12}$$

is formally given by

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x-\xi, t)\phi(\xi)d\xi \tag{2.13}$$

provided that the integral converges. Thus in the following we formulate conditions on $\phi(x)$ which assure the convergence of (2.13) and guarantee that $u(x,t)$ defined by (2.13) approaches the initial data $\phi(x)$ as t approaches zero.

In the sequel we will need,

Lemma (2.4). For any constant $\lambda \geq 0$, $(x+1)^{\lambda} \leq 2^{\lambda}(x^{\lambda}+1)$, for $x \geq 0$.

Proof. The inequality is obvious for $0 \leq x \leq 1$. Thus consider the function $f(x) = 2^{\lambda}(x^{\lambda}+1) - (x+1)^{\lambda}$, $f(1) = 2^{\lambda} > 0$ and $f'(x) = \lambda 2^{\lambda} x^{\lambda-1} - \lambda(x+1)^{\lambda-1} = \lambda x^{\lambda-1}[2^{\lambda} - (\frac{x+1}{x})^{\lambda-1}] > 0$ for $x \geq 0$. Therefore the inequality holds for all $x \geq 0$.

Lemma (2.5). Let $\phi(x)$ be a function of bounded variation satisfying the properties:

(i) for $x \geq 0$ there exist constants A, $\alpha \geq 0$ such that $|\phi(x)| \leq Ae^{\alpha x^{3/2}}$,

(ii) for $x < 0$, $|\phi(x)| \leq Be^{\beta(-x)}$, B, $\beta \geq 0$ constants.

Then for $\delta_1 > 0$ the integral

$$J_1(x,t) = \int_{x-\frac{(A(t))^2}{3t}+\delta_1}^{\infty} \Gamma(x-\xi,t)\phi(\xi)d\xi ,$$

converges for

$$0 < t \le T < \frac{c^2}{24a^2}$$

where c is given in (2.9). Moreover, $\lim_{t\to 0+} J_1(x,t) = 0.$

Proof. Consider first the case where $x - \frac{(A(t))^2}{3t} \ge 0$. Then from (2.9)

$$|J_1(x,t)| \le \frac{c_0}{(3t)^{1/3}} \int_{x-\frac{(A(t))^2}{3t}+\delta_1}^{\infty} e^{-c\frac{(-x+\xi+\frac{(A(t))^2}{3t})^{3/2}}{(3t)^{1/2}}} |\phi(\xi)|d\xi .$$

Taking $(3t)^{1/3}\eta = -x + \xi + \frac{(A(t))^2}{3t}$ we get

$$|J_1(x,t)| \le c_0 A \int_{\frac{\delta_1}{(3t)^{1/3}}}^{\infty} e^{-c\eta^{3/2}+a(x+(3t)^{1/3}\eta)^{3/2}} d\eta .$$

From Lemma (2.4)

$$(x+(3t)^{1/3}\eta)^{3/2} \le 2^{3/2}(x^{3/2}+(3t)^{1/2}\eta^{3/2})$$

Note that if $x - \frac{(A(t))^2}{3t} \geq 0$ then $x > 0$, and

$$|J_1(x,t)| \leq c_1 e^{2^{3/2}\alpha x^{3/2}} \int_{\frac{\delta_1}{(3t)^{1/3}}}^{\infty} e^{-c\eta^{3/2}+2\alpha(6t)^{1/2}\eta^{3/2}} d\eta .$$

Thus, for $t > 0$ such that $c - 2\alpha(6t)^{1/2} > 0$ the exponential decays and the integral converges for $0 < t < \frac{c^2}{24\alpha^2}$, moreover, as $t \to 0+$ for any $x > 0$, $\lim_{t \to 0+} J_1(x,t) = 0$. Obviously the same is true for $\delta_1 \geq x - \frac{(A(t))^2}{3t} + \delta_1 \geq 0$. For $x - \frac{(A(t))^2}{3t} + \delta_1 \leq 0$, we write

$$J_1(x,t) = \left[\int_{x-\frac{(A(t))^2}{3t}+\delta_1}^{0} + \int_0^{\infty} \right] \Gamma(x-\xi,t)\phi(\xi)d\xi$$

$$= J_{11}(x,t) + J_{12}(x,t),$$

respectively. Proceeding in an analogous manner but using condition (ii) for $\phi(x)$ in $J_{11}(x,t)$, the result of the lemma follows.

Lemma (2.6). If $\phi(x)$ satisfies the properties stated in Lemma (2.5), and if there exists a constant $a_0 > 0$ such that $a(t) \geq a_0$ for $t \geq 0$, then the integral

$$J_3(x,t) = \int_{-\infty}^{x-\frac{(A(t))^2}{3t}+\delta_2} \Gamma(x-\xi,t)\phi(\xi)d\xi, \qquad \delta_2 > 0$$

converges and

$$\lim_{t\to 0+} J_3(x,t) = 0.$$

Proof. If $x - \frac{(A(t))^2}{3t} - \delta_2 \geq 0$ the argument closely parallels that of Lemma (2.5). Thus, we consider only the case where

$$x - \frac{(A(t))^2}{3t} - \delta_2 \leq 0.$$

$$|J_3(x,t)| \leq \frac{c_1}{(3t)^{1/4}} \int_{-\infty}^{x-\frac{(A(t))^2}{3t}-\delta_2} \sin\left(\frac{2}{3}\frac{(x-\xi+\frac{(A(t))^2}{3t})^{3/2}}{3t^{1/2}}+\theta\right)$$

$$\times \frac{e^{-\frac{a_0}{3}(x-\xi+\frac{(A(t))^2}{3t})}}{(x-\xi+\frac{(A(t))^2}{3t})^{1/4}} \quad |\phi(\xi)|d\xi$$

for

$$(3t)^{1/3}\eta = x - \xi + \frac{(A(t))^2}{3t}$$

$$|J_3(x,t| \leq (3t)^{1/2} c_1 \int_{\frac{\delta_2}{(3t)^{1/2}}}^{\infty} \frac{e^{-\frac{a_0}{3}(3t)^{1/3}\eta}}{(3t)^{1/3}\eta^{1/4}} e^{\beta(-x+\frac{(A(t))^2}{3t}+(3t)^{1/3}\eta)} d\eta$$

$$\leq c_1 e^{\beta(\frac{(A(t))^2}{3t}-x)} (3t)^{1/6} \int_{\frac{\delta_2}{(3t)^{1/2}}}^{\infty} e^{-(\frac{a_0}{3}-\beta)(3t)^{1/3}\eta} \eta^{-1/4} d\eta$$

which converges for $\beta < \frac{a_0}{3}$, and $\lim_{t \to 0+} J_3(x,t) = 0$.

Remark. Since we must have $\beta < \frac{a_0}{3}$, one can see that the choice of β depends upon the fact that $a(t)$ is bounded from below by a constant $a_0 > 0$. However, if this is not the case the condition on $\phi(x)$, $x < 0$ has to be modified. A sufficient condition for the convergence in such a case is that for $x < 0$, $\phi(x)$ must satisfy a growth condition of the form $\phi(x) = O(|x|^{3/4-\epsilon})$ as $x \to -\infty$, where $\epsilon > 0$ is a constant.

Using the above two lemmas, the following theorem on the solution of (2.11) can be given.

Theorem (2.7). Suppose there exists a constant $a_0 > 0$ such that $a(t) \geq a_0$ for all $t \geq 0$. Let $\phi(x)$ be a function of bounded variation satisfying

(i) for $x \geq 0$, there exist constants A, $\alpha \geq 0$ such that

$$|\phi(x)| \le Ae^{a|x|^{3/2}},$$

(ii) for $x < 0$, $|\phi(x)| \le Be^{\beta(-x)}$, B, β are positive constants.

Then for

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x-\xi,t)\phi(\xi)d\xi, \quad 0 < t \le T < \frac{c^2}{24a^2}$$

(1) $$\frac{\partial^3 u}{\partial x^3} + a(t)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0,$$

and

(2) $$\lim_{t \to 0+} u(x,t) = \frac{1}{3}\phi(x-) + \frac{2}{3}\phi(x+)$$

Proof. The proof that differentiation under the integral sign is permissible may be carried out along the same lines as the proofs of Lemma (2.5) and (2.6). Consequently, (1) may be verified immediately. We turn now to the proof of (2). Let $\delta_2 > 0$. Consider the integral

$$\int_{x-\frac{(A(t))^2}{3t}-\delta_2}^{x-\frac{(A(t))^2}{3t}} \Gamma(x-\xi,t)d\xi$$

$$= \frac{e^{-\frac{(A(t))^3}{27t^2}}}{(3t)^{1/3}} \int_{x-\frac{(A(t))^2}{3t}-\delta_2}^{x-\frac{(A(t))^2}{3t}} Ai\left(\frac{x-\xi+\frac{(A(t))^2}{3t}}{(3t)^{1/3}}\right)e^{\frac{A(t)}{3t}\left(-x+\xi+\frac{(A(t))^2}{3t}\right)} d\xi =$$

$$= e^{-\frac{(A(t))^3}{27t^2}} \int_{\frac{-\delta_2}{(3t)^{1/3}}}^{0} e^{\frac{A(t)}{(3t)^{1/3}}\eta} Ai(\eta)d\eta .$$

We observe that $\frac{(A(t))^3}{3t^2}$ and $\frac{A(t)}{t^{2/3}}$ both approach zero as $t \to 0$.

Further, since $\int_{-\infty}^{0} Ai(\eta)d\eta = \frac{2}{3}$, we can conclude that

$$\lim_{t \to 0+} \int_{x-\frac{(A(t))^2}{3t}-\delta_2}^{x-\frac{(A(t))^2}{3t}} \Gamma(x-\xi, t)d\xi = \frac{2}{3} .$$

Similarly for $\delta_1 > 0$

$$\lim_{t \to 0+} \int_{x-\frac{(A(t))^2}{3t}}^{x-\frac{(A(t))^2}{3t}+\delta_1} \Gamma(x-\xi, t)d\xi = \frac{1}{3} .$$

Since $\phi(x)$ is of bounded variation, we can choose $\delta_1 > 0$ such that $|\phi(\xi) - \phi(x-\frac{(A(t))^2}{3t})| < \epsilon$ for $0 \leq \xi - x + \frac{(A(t))^2}{3t} \leq \delta_1$, and therefore

$$\int_{x-\frac{(A(t))^2}{3t}}^{\infty} \Gamma(x-\xi, t)\phi(\xi)d\xi = \int_{x-\frac{(A(t))^2}{3t}}^{x-\frac{(A(t))^2}{3t}+\delta_1} \Gamma(x-\xi, t)\phi(\xi)d\xi + J_1(x, t),$$

where $J_1(x, t)$ is the integral defined in Lemma (2.5). By Lemma

(2.5), $J_1(x,t)$ converges to zero as $t \to 0+$. Next,

$$\int_{x-\frac{(A(t))^2}{3t}}^{x-\frac{(A(t))^2}{3t}+\delta_1} \Gamma(x-\xi,t)\phi(\xi)d\xi$$

$$= \phi\left(x-\frac{(A(t))^2}{3t}\right)\int_{x-\frac{(A(t))^2}{3t}}^{x-\frac{(A(t))^2}{3t}+\delta_1} \Gamma(x-\xi,t)d\xi$$

$$+ \int_{x-\frac{(A(t))^2}{3t}}^{x-\frac{(A(t))^2}{3t}+\delta_1} \Gamma(x-\xi,t)\left[\phi(\xi)-\phi\left(x-\frac{(A(t))^2}{3t}\right)\right]d\xi$$

$$= \phi\left(x-\frac{(A(t))^2}{3t}\right) I_1 + I_2 \qquad \text{, respectively.}$$

$$|I_2| \le \epsilon \int_{x-\frac{(A(t))^2}{3t}}^{x-\frac{(A(t))^2}{3t}+\delta_1} \Gamma(x-\xi,t)d\xi \to \frac{1}{3}\epsilon \quad \text{as} \quad t \to 0+$$

Since ϵ was arbitrary, we get that

$$(2.14) \qquad \lim_{t\to 0+} \int_{x-\frac{(A(t))^2}{3t}}^{\infty} \Gamma(x-\xi,t)\phi(\xi)d\xi = \frac{1}{3}\phi(x+).$$

Similarly

$$(2.15)\qquad \lim_{t \to 0+} \int_{-\infty}^{x-\frac{(A(t))^2}{3t}} \Gamma(x-\xi, t)\phi(\xi)d\xi = \frac{2}{3}\phi(x-)$$

Therefore, from (2.14) and (2.15), since

$$u(x,t) = \int_{-\infty}^{x-\frac{(A(t))^2}{3t}} \Gamma(x-\xi, t)\phi(\xi)d\xi + \int_{x-\frac{(A(t))^2}{3t}}^{\infty} \Gamma(x-\xi, t)\phi(\xi)d\xi$$

$$\lim_{t \to 0+} u(x,t) = \frac{2}{3}\phi(x+) + \frac{1}{3}\phi(x-).$$

Theorem (2.8). Let $\phi(x)$ be a function of bounded variation and satisfies the properties:

(i) for $x \geq 0$ there exist constants A, $\alpha \geq 0$ such that $|\phi(x)| \leq Ae^{\alpha x^{3/2}}$

(ii) as $x \to -\infty$, $\phi(x) = 0(|x|^{-3/4-\epsilon})$ where $\epsilon > 0$ is a constantant, then for

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x-\xi, t)\phi(\xi)d\xi,$$

(1) and (2) of Theorem (2.7), hold.

The non-homogenous equation. For completeness and future reference, the solution of the Cauchy problem for the non-homogenous equation

$$(2.16) \qquad \frac{\partial^3 u}{\partial x^3} + a(t)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x,t), \qquad t > 0, \;\; x \in R$$

will be discussed in this section. With no loss of generality $u(x,t)$ will be taken to be zero for $t = 0$, i.e.,

$$(2.17) \qquad u(x,0) = 0, \qquad x \in R.$$

From Theorem (2.8) we have

Lemma (2.9). Suppose there exists a constant $a_0 > 0$ such that $a(t) \geq a_0$ for all $t \geq 0$. Suppose $f(x,t)$ is continuous in $R \times [0,T]$, and for $x \geq 0$, suppose there exist non-negative constants A, a such that $|f(x,t)| \leq Ae^{ax^{3/2}}$, and for $x < 0$, suppose there exist non-negative constants $B, b < \frac{a_0}{3}$ such that $|f(x,t)| \leq Be^{b(-x)}$. Then for $0 < t \leq T < \frac{c^2}{24a^2}$ the integral

$$J(x,t,\tau) = \int_{-\infty}^{\infty} \Gamma(x-\xi, t-\tau) f(\xi,\tau) d\xi$$

is uniformly convergent and

$$D_x^m J(x,t,\tau) = \int_{-\infty}^{\infty} D_x^m \Gamma(x-\xi, t-\tau) f(\xi,\tau) d\xi, \qquad m = 0,1,2,3.$$

Moreover,

$$(2.18) \qquad \lim_{t \to \tau} J(x,t,\tau) = f(x,t)$$

Now let

$$v(x, t) = -\int_0^t J(x, t, \tau) = -\int_0^t d\tau \int_{-\infty}^{\infty} \Gamma(x-\xi, t-\tau)f(\xi, \tau)d\xi .$$

In the following we will show that if $f(x, t)$ satisfies the conditions given in Lemma (2.9) and is uniformly Hölder continuous for $t \in (0, T]$ with respect to x, with exponent α, then $v(x, t)$ is a solution of (2.16), (2.17). To prove that, we need to show that $v(x, t)$ has three derivatives with respect to x and one with respect to t, moreover,

$$\frac{\partial^3 v}{\partial x^3} + a(t)\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = f(x, t), \quad \text{and} \quad v(x, 0) = 0.$$

Lemma (2.10). If $t > \tau$ and if $f(x, t)$ satisfies the hypothesis above, then, for the integral

$$J_{xxx}(x, t, \tau) = \int_{-\infty}^{\infty} \Gamma_{xxx}(x-\xi, t-\tau)f(\xi, \tau)d\xi$$

there exist constants $K > 0$, $0 < \beta < 1$, such that

$$|J_{xxx}(x, t, \tau)| \leq \frac{K}{(t-\tau)^{\beta}}$$

Proof. For τ, t fixed, $t > \tau$

$$J_{xxx}(x,t,\tau) = \int_{-\infty}^{\infty} \Gamma_{xxx}(x-\xi, t-\tau) f(\xi,\tau)d\xi$$

$$= \left[\int_{-\infty}^{x-\frac{(A(t,\tau))^2}{3(t-\tau)}} + \int_{x-\frac{(A(t,\tau))^2}{3(t-\tau)}}^{\infty}\right] \Gamma_{xxx}(x-\xi,t-\tau)f(\xi,\tau)d\xi$$

$$= I_1 + I_2 \qquad\qquad , \text{ respectively,}$$

where $A(t,\tau)$ denotes $\int_{\tau}^{t} a(\eta)d\eta > 0.$

From Property (5) and the Hölder continuity of $f(x,t)$ we get

$$|I_2| \leq \int_{x-\frac{(A(t,\tau))^2}{3(t-\tau)}}^{\infty} \Gamma_{xxx}(x-\xi,t-\tau)\left|f(\xi,\tau)-f(x-\frac{(A(t,\tau))^2}{3(t-\tau)})\right|d\xi$$

$$\leq \text{const} \int_{x-\frac{(A(t,\tau))^2}{3(t-\tau)}}^{\infty} \frac{(\xi-x+\frac{(A(t,\tau))^2}{3(t-\tau)})^{\alpha}}{(t-\tau)^{4/3}} e^{-c\frac{(\xi-x+\frac{(A(t,\tau))^2}{3(t-\tau)})^{3/2}}{(t-\tau)^{1/2}}} d\xi$$

$$\leq \text{const} \int_0^{\infty} \frac{e^{-\zeta}\zeta^{\frac{2\alpha}{3}-\frac{1}{3}}}{(t-\tau)^{1-\frac{\alpha}{3}}} d\xi$$

$$\leq \frac{\text{const}}{(t-\tau)^{1-\frac{\alpha}{3}}}$$

I_1 can be estimated similarly.

Lemma (2. 11). For $f(x, t)$ satisfying the same hypothesis as in Lemma (2. 10), $v(x, t) = -\int_0^t J(x, t, \tau) d\tau$ admits a first derivative with respect to t, $t > 0$, given by

$$v_t(x, t) = -f(x, t) - \int_0^t d\tau \int_{-\infty}^{\infty} \Gamma_t(x-\xi, t-\tau) f(\xi, \tau) d\xi .$$

Proof. For $t > \tau$, we have that

$$J_t(x, t, \tau) = \int_{-\infty}^{\infty} \Gamma_t(x-\xi, t-\tau) f(\xi, \tau) d\xi$$

$$= \int_{-\infty}^{\infty} [\Gamma_{xxx}(x-\xi, t-\tau) + a(t)\Gamma_{xx}(x-\xi, t-\tau)] f(\xi, \tau) d\xi$$

since $\Gamma(x-\xi, t-\tau)$ satisfies (2. 3). Using similar steps to that used in Lemma (2. 10), we find that there exist constants $K > 0$, $0 < \beta < 1$ such that

(2. 19) $$|J_t(x, t, \tau)| \leq \frac{K}{(t-\tau)^{\beta}}$$

Thus $\int_0^t J_t(x, t, \tau) d\tau$ exists.

For each positive integer n, $t - \frac{1}{n} > 0$, let

$$v^n(x, t) = -\int_0^{t-\frac{1}{n}} J(x, t, \tau) d\tau .$$

Obviously, $\lim_{n \to \infty} v^n(x,t) = v(x,t)$ uniformly with respect to t.
Moreover, we have

$$v_t^n(x,t) = -J(x,t,t-\frac{1}{n}) - \int_0^{t-\frac{1}{n}} J_t(x,t,\tau)d\tau$$

but

$$\lim_{n \to \infty} J(x,t,t-\frac{1}{n}) = \lim_{t \to \tau} \int_{-\infty}^{\infty} \Gamma(x-\xi, t-\tau) f(\xi,\tau) d\xi$$

$$= f(x,t)$$

From (2.19), and for a fixed x, the sequence $v_t^n(x,t)$ converges uniformly on $0 < t \leq T$ to $v_t(x,t)$ and we have

$$v_t(x,t) = \lim_{n \to \infty} v_t^n(x,t) = -f(x,t) - \int_T^t d\tau \int_{-\infty}^{\infty} \Gamma_t(x-\xi, t-\tau) f(\xi,\tau) d\xi \,,$$

which was to be proved.

From Lemmas (2.10), and (2.11), we have that
$\frac{\partial^i v}{\partial x^i}$, $i = 1, 2, 3$, and $\frac{\partial v}{\partial t}$ exist and

$$\frac{\partial^3 v}{\partial x^3} + a(t)\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = f(x,t) - \int_0^t d\tau \int_{-\infty}^{\infty} [\frac{\partial^3}{\partial x^3} - a(t)\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}]\Gamma f d\xi$$

$$= f(x,t)$$

and this concludes the proof of the fact that the function

$$v(x,t) = -\int_0^t d\tau \int_{-\infty}^{\infty} \Gamma(x-\xi, t-\tau) f(\xi,\tau) d\xi$$

is a solution of (2.16), (2.17).

Remark. The Cauchy problem

$$\frac{\partial^3 v}{\partial x^3} + a(t)\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = f(x,t)$$

$$v(x,0) = \phi(x)$$

has the solution

$$v(x,t) = \int_{-\infty}^{\infty} \Gamma(x-\xi, t)\phi(\xi)d\xi - \int_0^t d\tau \int_{-\infty}^{\infty} \Gamma(x-\xi, t-\tau) f(\xi,\tau) d\xi$$

for admissible function $f(x,t)$, $\phi(x)$.

III. THIRD ORDER PARABOLIC EQUATIONS WITH COEFFICIENTS DEPENDING ON BOTH SPACE AND TIME

3.1 Preliminaries

In this section we shall give several preliminary lemmas which will be needed in the proof of the existence of a fundamental solution for the equation

$$(3.1) \qquad Mu = \frac{\partial^3 u}{\partial x^3} + a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u - \frac{\partial u}{\partial t} = 0$$

Lemma (3.1) is a well known property of the exponential function.

Lemma (3.1). If $x \geq 0$, and if $\alpha > 0$, $0 < h < 1$ are given constants, then there exists a constant $k > 0$ independent of x, such that

$$(3.2) \qquad x^{\alpha} e^{-x} \leq k e^{-hx}$$

Lemma (3.2). Let

$$I(x,\xi,t,\tau,\sigma) = \int_x^{\xi} \frac{e^{-c\left\{\frac{(\eta-x)^{3/2}}{(t-\sigma)^{1/2}} + \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}\right\}}}{(t-\sigma)^{1/3}(\sigma-\tau)^{1/3}}\, d\eta,$$

for $\xi > x$, $c > 0$ and $\tau < \sigma < t$. Then there exist positive constants c_0, c_1 such that

$$I(x, \xi, t, \tau, \sigma) \leq \frac{c_0}{(t-\tau)^{1/3}} e^{-c_1 \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}$$

Proof. Let

$$g(x, \xi, \eta, t, \tau, \sigma) = \frac{(\eta-x)^{3/2}}{(t-\sigma)^{1/2}} + \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}$$

$$\frac{\partial g(x, \xi, \eta, t, \tau, \sigma)}{\partial \sigma} = \frac{1}{2} \frac{(\eta-x)^{3/2}}{(t-\sigma)^{3/2}} - \frac{1}{2} \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{3/2}}$$

From this we see that at $\sigma_0 = \frac{(\xi-\eta)t + (\eta-x)\tau}{\xi-x}$, g as a function of σ attains a minimum. At σ_0

$$\eta = \frac{\xi(t-\sigma_0) + x(\sigma_0-\tau)}{(t-\tau)},$$

and

$$g(x, \xi, \eta, t, \tau, \sigma)\Big|_{\sigma=\sigma_0} = \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}},$$

$$I(x, \xi, t, \tau, \sigma) \leq e^{-\frac{c}{2} \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}} \int_x^\xi \frac{e^{-\frac{c}{2} g(x, \xi, \eta, t, \tau, \sigma)}}{(t-\sigma)^{1/3} (\sigma-\tau)^{1/3}} d\eta .$$

For $\tau \leq \sigma \leq \frac{t+\tau}{2}$,

$$g(x, \xi, \eta, t, \tau, \sigma) \geq \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}},$$

thus

$$I(x,\xi,t,\tau,\sigma) \le e^{-\frac{c}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}} \int_x^{\xi} \frac{e^{-\frac{c}{2}\frac{(\xi-x)^{3/2}}{(\sigma-\tau)^{1/2}}}}{(\frac{t-\tau}{2})^{1/3}(\sigma-\tau)^{1/3}} d\eta$$

$$\le \frac{2^{1/3}}{2}\left(\frac{2}{c}\right)^{2/3} \frac{e^{-\frac{c}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}}{(t-\tau)^{1/3}} \int_0^{\frac{c}{2}\frac{(\xi-x)^{3/2}}{(\sigma-\tau)^{1/2}}} e^{-\xi}\xi^{-1/3} d\xi$$

$$\le \frac{2\Gamma(\frac{2}{3})}{3c^{2/3}} \frac{e^{-\frac{c}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}}{(t-\tau)^{1/3}} .$$

Since the case $\frac{t+\tau}{2} \le \sigma < t$ can be treated similarly, the lemma follows.

Remark. If $a(x,t)$ is twice continuously differentiable with respect to x, and once with respect to t, then if $u(x,t)$ is a solution of (3.1), the function $v(x,t) = \exp\left[\frac{1}{3}\int^{x} a(\xi,t)d\xi\right]u(x,t)$ satisfies the differential equation

$$\frac{\partial^3 v}{\partial x^3} + a_0(x,t)\frac{\partial v}{\partial x} + a_1(x,t)v = \frac{\partial v}{\partial t}$$

where

$$a_0(x,t) = a_x(x,t) - (a(x,t))^2 + b(x,t)$$

and

$$a_1(x,t) = -\frac{1}{3}\, a_{xx}(x,t) - \frac{1}{3} a_x(x,t)a(x,t) - \frac{1}{27}(a(x,t))^3 + \frac{1}{9}(a(x,t))^2$$

$$- \frac{1}{3} a_x(x,t) - \frac{1}{3} a(x,t) + c(x,t) + \frac{1}{3}\int^x a_t(\xi,t)d\xi \, .$$

Therefore, we will only be concerned with a study for the parabolic differential equation

$$\text{(3.3)} \qquad Lu = \frac{\partial^3 u}{\partial x^3} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)u - \frac{\partial u}{\partial t}$$

As was mentioned earlier, the fundamental solution $U(x-\xi, t-\tau)$ of $\frac{\partial^3 u}{\partial x^3} = \frac{\partial u}{\partial t}$ will be taken as a first approximation for the fundamental solution $\Gamma(x,t,\xi,\tau)$ of the Equation (3.3). From (1.6), $U(x-\xi, t-\tau)$ is given by

$$U(x-\xi, t-\tau) = \frac{1}{(3(t-\tau))^{1/3}}\, Ai\left(\frac{\xi - x}{(3(t-\tau))^{1/3}}\right) \, .$$

Therefore, using the bounds for the Airy function given in the Appendix one gets

<u>Lemma (3.3)</u>. For $\xi \geq x$ there exist constants a_m, $m = 0,1,2,3,$ and $c > 0$ such that

$$\text{(3.4)} \qquad D_x^m U(x-\xi, t-\tau) = \frac{a_m}{(t-\tau)^{(m+1)/3}}\, e^{-c\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}\left[1 + O\left(\frac{(\xi-x)^{-3/2}}{(t-\tau)^{-1/2}}\right)\right]$$

and for $x \geq \xi$, there exist constants b_m, $m = 0, 1, 2, 3$ and θ such that

$$(3.5) \quad U(x-\xi, t-\tau) = b_0 \frac{(x-\xi)^{-1/4}}{(t-\tau)^{1/4}} \sin\left(\frac{2}{3}\frac{(x-\xi)^{3/2}}{(t-\tau)^{1/2}} + \theta\right) + O\left(\frac{(x-\xi)^{-7/4}}{(t-\tau)^{-7/12}}\right)$$

Moreover, for $x \geq \xi$,

$$D_x^m U(x-\xi, t-\tau) \leq b_m \frac{(x-\xi)^{\frac{2m-1}{4}}}{(t-\tau)^{\frac{2m+1}{4}}}$$

The following two lemmas will be needed. Their proof follows arguments similar to the corresponding lemmas for $\Gamma(x, t; \xi, \tau)$ (Lemma (2.9) and (2.11) given in Chapter II).

Lemma (3.4). Let $f(x,t)$ be continuous in $R \times [0, T]$ and once continuously differentiable with respect to x, for all $t \in [0, T]$, suppose

(i) for $x \geq 0$ there exist positive constants c_0, λ such that $|D_x^i f(x,t)| \leq c_0 e^{\lambda x^{3/2}}$, $i = 0, 1$,

(ii) for $x < 0$ there exists, $c_2 > 0$ such that for $i = 0, 1$, $|D_x^i f(x,t)| \leq c_2 |x|^{-1/4-\epsilon}$ for any $\epsilon > 0$.

Then the integral

$$J(x, t, \tau) = \int_{-\infty}^{\infty} U(x-\xi, t-\tau) f(\xi, \tau) d\xi ,$$

exists and is continuous for $0 < \tau < t < \frac{c^2}{24\lambda^2}$ and for each fixed $x \in R$

$$\lim_{t \to \tau} J(x, t, \tau) = f(x, t) .$$

Lemma (3.5). Let $f(x, t)$ be as in Lemma (3.4). Then

$$v(x, t) = \int_0^t J(x, t, \tau) d\tau ,$$

has a first derivative with respect to t given by

$$v_t(x, t) = f(x, t) + \int_0^t J_t(x, t, \tau) d\tau .$$

In the next section using the parametrix method, we will show the existence of a fundamental solution for the Equation (3.3) under the following assumption:

(*) $a(x, t)$, $b(x, t)$ are continuous and uniformly bounded functions in $R \times [0, T]$ together with their first partial derivatives with respect to x.

The fundamental solution $U(x-\xi, t-\tau)$ of (1.5) will be taken as the parametrix.

Remark. For $a(x,t)$, $b(x,t)$ satisfying (*), we have

$$LU(x-\xi, t-\tau) = a(x,t)U_x(x-\xi, t-\tau) + b(x,t)U(x-\xi, t-\tau).$$

Therefore, by Lemma (3.3) for $x - \xi \leq 0$

$$\text{(3.6)} \qquad LU \leq \left\{\frac{A}{(t-\tau)^{2/3}} + \frac{B}{(t-\tau)^{1/3}}\right\} e^{-c\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}$$

$$\leq \frac{C_0}{(t-\tau)^{2/3}} e^{-c\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}$$

while, for $x - \xi > 0$

$$\text{(3.7)} \qquad LU \leq C_1 \frac{(x-\xi)^{1/4}}{(t-\tau)^{3/4}}.$$

3.2 Existence of the Fundamental Solution

Let $\Gamma(x,t;\xi,\tau)$ denote a fundamental solution of (3.3). Taking $U(x-\xi, t-\tau)$ as a first approximation, we suppose that

$$\text{(3.8)} \quad \Gamma(x,t;\xi,\tau) = U(x-\xi, t-\tau) + \int_\tau^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta$$

where Φ is to be determined. Suppose that for

$$V(x,t;\xi,\tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta$$

as a function of x, t, the result of Lemma (3.5) holds then we will have

$$0 = L\Gamma = LU + \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} LU(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta - \Phi(x, t, \xi, \tau)$$

From which one can deduce that $\Phi(x, t; \xi, \tau)$ has to satisfy the integral equation

(3.9)

$$\Phi(x, t; \xi, \tau) = LU(x-\xi, t-\tau) + \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} LU(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

It is the objective of this section to prove the following theorem.

Theorem (3.6). For all (x, t), (ξ, τ), $t > \tau$, the Equation (3.3) $Lu = 0$ with the coefficients satisfying (*) has a fundamental solution $\Gamma(x, t; \xi, \tau)$ given by (3.8) where $\Phi(x, t; \xi, \tau)$ satisfies the integral equation (3.9).

To prove the theorem we shall show first that the Volterra integral equation (3.9) has a unique solution $\Phi(x, t; \xi, \tau)$. Using this we show that $L\Gamma(x, t; \xi, \tau) = 0$, and for continuous bounded functions $\phi(x)$ with compact support

$$\lim_{\tau \to t} \int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi = \phi(x).$$

Step 1. It is the purpose of this step to show that there exists a solution $\Phi(x, t; \xi, \tau)$ to the integral equation

$$\Phi(x, t; \xi, \tau) = U(x-\xi, t-\tau) + \int_\tau^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

of the form

(3.10) $$\Phi(x, t; \xi, \tau) = \sum_{m=0}^{\infty} G_m(x, t; \xi, \tau)$$

where

$$G_0(x, t; \xi, \tau) = LU(x-\xi, t-\tau)$$

and,

$$G_m(x, t; \xi, \tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} G_0(x, t; \eta, \sigma)G_{m-1}(\eta, \sigma; \xi, \tau)d\eta .$$

To prove the convergence of this iteration scheme, we will need the following lemma.

Lemma (3.7). If $G_0(x, t; \xi, \tau)$ satisfies the estimates

$$|G_0(x, t; \xi, \tau)| \leq \frac{a_0}{(t-\tau)^{2/3}} e^{-\mu_0 \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}} \qquad \xi - x > 0$$

$$|G_0(x, t; \xi, \tau)| \leq b_0 \frac{(x-\xi)^{1/4}}{(t-\tau)^{3/4}} \qquad x - \xi \geq 0$$

and if $G_m(x, t; \xi, \tau)$ satisfies the estimates

$$|G_m(x,t;\xi,\tau)| \le \frac{a_m}{(t-\tau)^{\frac{2}{3}-\frac{2m}{3}}} e^{-\mu_m \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}} \qquad \xi - x > 0$$

$$|G_m(x,t;\xi,\tau)| \le b_m \frac{(x-\xi)^{\frac{1}{4}+\frac{5m}{4}}}{(t-\tau)^{\frac{3}{4}-\frac{m}{4}}} \qquad x - \xi \ge 0,$$

then

$$G_{m+1}(x,t;\xi,\tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} G_0(x,t;\eta,\sigma) G_m(\eta,\sigma;\xi,\tau)d\eta$$

(3.11) $|G_{m+1}(x,t;\xi,\tau)|$

$$\le c[b_0 a_m \beta(\tfrac{1}{4},\tfrac{3}{4}+\tfrac{2m}{3}) + a_0 a_m \beta(\tfrac{2}{3},\tfrac{2}{3}+\tfrac{2m}{3}) + a_0 b_m \beta(\tfrac{m+1}{4},\tfrac{3}{4}+\tfrac{5m}{12})]$$

$$\times \frac{e^{-\mu_{m+1}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}}{(t-\tau)^{-\frac{2m}{3}}}$$

where $\mu_{m+1} \le \mu_m$ and $c > 0$ is a numerical constant. For $x - \xi \ge 0$

(3.12)

$$|G_{m+1}(x,t;\xi,\tau)| \le c b_0 b_m \beta(\tfrac{1}{4},\tfrac{1}{4}+\tfrac{m}{4})\beta(\tfrac{5}{4},\tfrac{5}{4}+\tfrac{5m}{4}) \frac{(x-\xi)^{\frac{1}{4}+\frac{5}{4}(m+1)}}{(t-\tau)^{\frac{3}{4}-\frac{1}{4}(m+1)}}$$

where $\beta(x,y)$ is the Beta function and c is a numerical constant.

Proof. (i) For $x - \xi \le 0$

$$G_{m+1} = \int_\tau^t d\sigma \left[\int_{-\infty}^{x} + \int_x^{\xi} + \int_{\xi}^{\infty} \right] G_0(x, t; \eta, \sigma) G_m(\eta, \sigma; \xi, \tau) d\eta$$

$$= I_1 + I_2 + I_3, \qquad \text{respectively.}$$

Therefore,

$$|I_1| \le \int_0^t d\sigma \left[b_0 a_m \int_{-\infty}^{x} \frac{(x-\eta)^{1/4}}{(t-\sigma)^{3/4}} \frac{e^{-\mu_m \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}}}{(\sigma-\tau)^{\frac{2}{3}-\frac{2m}{3}}} d\eta \right].$$

Using Lemma (3.1) and noticing that $x - \eta \le \xi - \eta$, we have

$$|I_1| \le \int_\tau^t d\sigma \left[b_0 a_m \int_{-\infty}^{x} \frac{e^{-\frac{\mu_m}{3} \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}}}{(t-\sigma)^{\frac{3}{4}} (\sigma-\tau)^{\frac{7}{12}-\frac{2m}{3}}} d\eta \right].$$

But, since

$$\int_{-\infty}^{x} e^{-\frac{\mu_m}{3} \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}} d\eta \le \frac{2}{3} e^{-\frac{\mu_m}{6} \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}} \int_{\frac{(\xi-x)^{3/2}}{(\sigma-\tau)^{1/2}}}^{\infty} e^{-\frac{\mu_m}{6}\zeta} (\sigma-\tau)^{1/3} \zeta^{-1/3} d\zeta$$

$$= \frac{2}{3} (\sigma-\tau)^{1/3} e^{-\frac{\mu_m}{6} \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}},$$

$$|I_1| \leq \frac{2}{3} a_m b_0 \int_\tau^t \frac{e^{-\frac{\mu_m}{6}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}}{(t-\sigma)^{\frac{3}{4}}(\sigma-\tau)^{\frac{1}{4}-\frac{2m}{3}}} d\sigma$$

(3.13) $$|I_1| \leq \frac{2}{3} a_m b_0 \beta\left(\frac{1}{4}, \frac{3}{4}+\frac{2m}{3}\right) e^{-\frac{\mu_m}{6}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}$$

Similarly the third integral is estimated, and

(3.14) $$|I_3| \leq \frac{2}{3} a_0 b_m \beta\left(\frac{1}{4}+\frac{m}{4}, \frac{3}{4}+\frac{5m}{12}\right) \frac{e^{-\frac{\mu_0}{6}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}}{(t-\tau)^{-\frac{2m}{3}}}.$$

For the second integral, we use Lemma (3.2) to get

$$|I_2| \leq a_0 a_m \int_\tau^t d\sigma \int_x^\xi \frac{e^{-\mu_m\left\{\frac{(\eta-x)^{3/2}}{(t-\sigma)^{1/2}} + \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}\right\}}}{(t-\sigma)^{\frac{2}{3}}(\sigma-\tau)^{\frac{2}{3}-\frac{2m}{3}}} d\eta$$

since $\mu_m \leq \mu_0$,

$$|I_2| \leq \frac{2\Gamma(\frac{2}{3})}{3} a_0 a_m \int_\tau^t \frac{e^{-\frac{\mu_m}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}}{(t-\tau)^{\frac{1}{3}}(t-\sigma)^{\frac{1}{3}}(\sigma-\tau)^{\frac{1}{3}-\frac{2m}{3}}} d\sigma$$

$$(3.15) \qquad |I_2| \le \frac{2\Gamma(\frac{2}{3})}{3} a_0 a_m e^{-\frac{\mu_m}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}$$

Thus, from (3.13), (3, 14) and (3.15), taking μ_{m+1} such that

$$\mu_{m+1} = \min(\frac{\mu_0}{6}, \frac{\mu_m}{2}, \frac{\mu_m}{6}) = \frac{\mu_m}{6} ,$$

(3.11) follows.

(ii) For $x - \xi \ge 0$

$$G_{m+1} = \int_\tau^t d\sigma \left[\int_{-\infty}^{\xi} + \int_{\xi}^{x} + \int_{x}^{\infty} \right] G_0(x, t; \eta, \sigma) G_m(\eta, \sigma; \xi, \tau) d\eta$$

$$= J_1 + J_2 + J_3 , \qquad \text{respectively.}$$

$$|J_1| \le a_m b_0 \int_\tau^t d\sigma \int_{-\infty}^{\xi} \frac{(x-\eta)^{1/4}}{(t-\sigma)^{3/4}} \frac{e^{-\mu_m \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}}}{(\sigma-\tau)^{\frac{2}{3}-\frac{2m}{3}}} d\eta$$

$$\le \frac{2}{3} a_m b_0 \int_\tau^t d\sigma \int_0^\infty \frac{(x-\xi+(\sigma-\tau)^{1/3}\zeta^{2/3})^{1/4}}{(t-\sigma)^{\frac{3}{4}}} \frac{e^{-\mu\zeta}(\zeta-\tau)^{1/3}\zeta^{-1/3}}{(\sigma-\tau)^{\frac{2}{3}-\frac{2m}{3}}} d\zeta$$

$$= J_{11} + J_{12} .$$

Using Lemma (2.4),

$$|J_{11}| \leq \frac{2^{5/4}}{3} a_m b_0 \int_\tau^t d\sigma \int_0^\infty \frac{(x-\xi)^{1/4} \zeta^{-1/3} e^{-\mu_m \zeta}}{(t-\sigma)^{\frac{3}{4}} (\sigma-\tau)^{\frac{1}{3}-\frac{2m}{3}}} d\zeta$$

(3.16) $$|J_{11}| \leq \frac{2^{5/4}\Gamma(\frac{2}{3})}{3} a_m b_0 \beta(\frac{1}{4}, \frac{2}{3}+\frac{2m}{3}) \frac{(x-\xi)^{1/4}}{(t-\tau)^{\frac{1}{12}-\frac{2m}{3}}},$$

and,

$$|J_{12}| \leq \frac{2^{5/4}}{3} a_m b_0 \int_\tau^t d\sigma \int_0^\infty \frac{\zeta^{-1/2} e^{-\mu_m \zeta}}{(t-\sigma)^{\frac{3}{4}} (\sigma-\tau)^{\frac{1}{4}-\frac{2m}{3}}} d\zeta .$$

(3.17) $$|J_{12}| \leq \frac{2^{5/4}}{3} \Gamma(\frac{1}{2}) a_m b_0 \frac{\beta(\frac{1}{4}, \frac{3}{4}+\frac{2m}{3})}{(t-\tau)^{-\frac{2m}{3}}} .$$

Similarly J_3 is estimated.

$$|J_2| \leq b_0 b_m \int_\tau^t d\sigma \int_\xi^x \frac{(x-\eta)^{1/4}}{(t-\sigma)^{\frac{3}{4}}} \frac{(\eta-\xi)^{\frac{1}{4}+\frac{5m}{4}}}{(\sigma-\tau)^{\frac{3}{4}-\frac{m}{4}}} d\eta$$

(3.18) $$|J_2| \leq b_0 b_m \beta(\frac{1}{4}, \frac{1}{4}+\frac{m}{4}) \beta(\frac{5}{4}, \frac{5}{4}+\frac{5m}{4}) \frac{(x-\xi)^{\frac{3}{4}+\frac{5m}{4}}}{(t-\tau)^{\frac{1}{2}-\frac{m}{4}}} .$$

From (3.16), (3.17), and (3.18) one can see that the bound satisfied by J_2 will dominate the others and thus,

$$|G_{m+1}| \leq cb_0 b_m \beta(\frac{1}{4}, \frac{1}{4}+\frac{m}{4})\ \beta(\frac{5}{4}, \frac{5}{4}+\frac{5m}{4})\ \frac{(x-\xi)^{\frac{3}{4}+\frac{5m}{4}}}{(t-\sigma)^{\frac{1}{2}-\frac{m}{4}}}\ .$$

This proves (3.12) and concludes the proof of the lemma.

Using this lemma we find that

$$|\Phi(x, t; \xi, \tau)| \leq \frac{1}{(t-\tau)^{2/3}} \sum_{m=0}^{\infty} a_m (t-\tau)^{2m/3} e^{-\mu_m \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \quad \xi - x > 0$$

and

$$|\Phi(x, t; \xi, \tau)| \leq \frac{(x-\xi)^{1/4}}{(t-\tau)^{3/4}} \sum_{m=0}^{\infty} b_m (x-\xi)^{5m/4} (t-\tau)^{m/4}, \qquad x - \xi \geq 0$$

where $\frac{a_{m+1}}{a_m}$, $\frac{b_{m+1}}{b_m}$ are given by

$$\frac{a_{m+1}}{a_m} = b_0 \beta(\frac{1}{4}, \frac{3}{4}+\frac{2m}{3}) + a_0 \beta(\frac{2}{3}, \frac{2}{3}+\frac{2m}{3}) + a_0 \frac{b_m}{a_m} \beta(\frac{1}{4}+\frac{m}{4}, \frac{3}{4}+\frac{5m}{12})$$

$$\frac{b_{m+1}}{b_m} = b_0 \beta(\frac{1}{4}, \frac{1}{4}+\frac{m}{4})\ \beta(\frac{5}{4}, \frac{5}{4}+\frac{5m}{4})$$

but

$$\frac{b_m}{a_m} = \frac{b_m}{a_{m-1}\dfrac{a_m}{a_{m-1}}} < \frac{b_m}{a_0 a_{m-1}\dfrac{b_{m-1}}{a_{m-1}}\beta(\frac{m}{4}, \frac{1}{4}+\frac{5m}{12})}$$

$$= \frac{b_0}{a_0}\,\frac{\beta(\frac{1}{4},\frac{m}{4})\beta(\frac{5}{4},\frac{5m}{4})}{\beta(\frac{m}{4},\frac{1}{4}+\frac{5m}{12})}$$

Therefore,

$$\frac{a_{m+1}}{a_m} = b_0\beta(\frac{1}{4},\frac{3}{4}+\frac{2m}{3}) + a_0\beta(\frac{2}{3},\frac{2}{3}+\frac{2m}{3})$$

$$+ b_0\frac{\beta(\frac{1}{4},\frac{m}{4})\beta(\frac{5}{4},\frac{5m}{4})}{\beta(\frac{m}{4},\frac{1}{4}+\frac{5m}{12})}\ \beta(\frac{1}{4}+\frac{m}{4},\frac{3}{4}+\frac{5m}{12})$$

$$\frac{\beta(\frac{1}{4},\frac{m}{4})\beta(\frac{5}{4},\frac{5m}{4})\beta(\frac{1}{4}+\frac{m}{4},\frac{3}{4}+\frac{5m}{12})}{\beta(\frac{m}{4},\frac{1}{4}+\frac{5m}{12})}$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{m}{4})\Gamma(\frac{5}{4})\Gamma(\frac{5m}{4})\Gamma(\frac{1}{4}+\frac{m}{6})\Gamma(\frac{3}{4}+\frac{5m}{12})\,\Gamma(\frac{1}{4}+\frac{2m}{3})}{\Gamma(\frac{m+1}{4})\,\Gamma(\frac{5m}{4}+\frac{5}{4})\Gamma(1+\frac{2m}{3})\Gamma(\frac{m}{4})\Gamma(\frac{1}{4}+\frac{5m}{12})}$$

$$= 0[\{m^{-5/4}(1+\frac{1}{8m})\}\,\{m^{-3/4}(1+\frac{9}{64m})\}m^{1/4}] \quad \to 0 \quad \text{as} \quad m \to \infty$$

using Stirling's formula. Thus one can conclude that $\dfrac{a_{m+1}}{a_m} \to 0$ as $m \to \infty$, and $\dfrac{b_{m+1}}{b_m} \to 0$ as $m \to \infty$. Therefore, the series (3.10) for Φ converges, and

$$(3.19) \qquad |\Phi(x,t;\xi,\tau)| \le \frac{K_1}{(t-\tau)^{2/3}} e^{-\mu_0 \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \quad \xi - x > 0$$

$$(3.20) \qquad |\Phi(x,t;\xi,\tau)| \le K_2 \frac{(x-\xi)^{1/4}}{(t-\tau)^{3/4}}, \qquad x - \xi \ge 0$$

where K_1 and K_2 are positive constants.

From (3.4), (3.11) and Lemma (3.4), we can see that

$$G_m(x,t;\xi,\tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} G_0(x,t;\eta,\sigma) G_{m-1}(\eta,\sigma;\xi,\tau) d\eta$$

for each $m \ge 1$ is a continuous function of (x,t), (ξ,τ) for $t > \tau$. Since Φ is given by

$$\Phi(x,t;\xi,\tau) = \sum_{m=0}^{\infty} G_m(x,t;\xi,\tau),$$

we conclude that $\Phi(x,t;\xi,\tau)$ is a continuous function for all (x,t), (ξ,τ), $t > \tau$.

Step 2. We prove that $L\Gamma = 0$. For that, we consider the integral

$$V(x,t;\xi,\tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma) \Phi(\eta,\sigma;\xi,\tau) d\eta,$$

and show that it has three derivatives with respect to x, which can be found by differentiating under the integral sign, and once with respect to t. We then show that V_t exists and is given by the formula

$$V_t(x, t; \xi, \tau) = \Phi(x, t; \xi, \tau) + \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} U_t(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta,$$

and since $\Phi(x, t; \xi, \tau)$ satisfies the integral equation (3.9), we can conclude that $L\Gamma = 0$. We turn therefore to prove the fact that V can be differentiated three times with respect to x and once with respect to t.

Lemma (3.8). If $\Phi(x, t; \xi, \tau)$ satisfies (3.9), then

$$V(x, t; \xi, \tau) = \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta,\sigma ; \xi, \tau)d\eta$$

exists for $t > \tau$, and $V(x, t; \xi, \tau) = O((t-\tau)^{1/4})$.

Proof. The existence of $V(x, t; \xi, \tau)$ is obvious. Using the bounds for $\Phi(\eta, \sigma; \xi, \tau)$ given by (3.19), (3.20) and those for $U(x-\eta, t-\sigma)$ given by (3.4) and (3.5) we will show that $V(x, t; \xi, \tau) = O((t-\tau)^{1/4})$. We consider for $x - \xi \leq 0$

$$V(x,t;\xi,\tau) = \int_\tau^t d\sigma \left[\int_{-\infty}^{x} + \int_x^{\xi} + \int_{\xi}^{\infty}\right] U(x-\eta, t-\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta$$

$$= I_1 + I_2 + I_3, \qquad \text{respectively,}$$

$$|I_1| \leq \int_\tau^t d\sigma \int_{-\infty}^{x} b_0 K_1 \frac{(x-\eta)^{-1/4}}{(t-\sigma)^{1/4}} \frac{e^{-\gamma \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}}}{(\sigma-\tau)^{2/3}} d\eta$$

$$\leq b_0 K_1 \int_\tau^t e^{-\frac{\gamma}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}} \left[\int_0^\infty \frac{e^{-\frac{\gamma}{2}\zeta}(\sigma-\tau)^{1/3}\zeta^{-1/2}}{(t-\sigma)^{\frac{1}{4}}(\sigma-\tau)^{\frac{2}{3}+\frac{1}{12}}} d\zeta\right] d\sigma$$

$$\leq b_0 K_1 \sqrt{\frac{\pi}{2\gamma}}\,(t-\tau)^{1/3} e^{-\frac{\gamma}{2}\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}.$$

Similarly for I_3. For I_2

$$|I_2| \leq a_0 K_1 \int_\tau^t d\sigma \int_x^{\xi} \frac{e^{-\gamma\left[\frac{(\eta-x)^{3/2}}{(t-\sigma)^{1/2}} + \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}\right]}}{(t-\sigma)^{1/3}(\sigma-\tau)^{1/3}} d\eta$$

$$\leq a_0' K_1 (t-\tau)^{1/3} e^{-\gamma' \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}.$$

However, for $x - \xi > 0$

$$V(x,t;\xi,\tau) = J_1 + J_2 + J_3$$

where

$$|J_2| \leq \int_\tau^t d\sigma \int_\xi^x \frac{(x-\eta)^{-1/4}}{(t-\sigma)^{1/4}} \sin\left(\frac{(x-\eta)^{3/2}}{(t-\sigma)^{1/2}} + \theta\right) \frac{(\eta-\xi)^{1/4}}{(\sigma-\tau)^{3/4}} \, d\eta$$

$$= O((t-\tau)^{1/4}).$$

J_1 and J_3 can be treated in a similar fashion and the conclusion of the lemma is shown to hold.

Consider for $\tau < \sigma < t$, the integral

$$I(x, t; \xi, \tau, \sigma) = \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta .$$

Obviously the derivatives $D_x^m I$, $m = 1, 2, 3,$ exist and are given by

$$D_x^m I(x, t; \xi, \tau, \sigma) = \int_{-\infty}^{\infty} D_x^m U(x-\eta, t-\sigma)\Phi(\eta, \sigma, \xi, \tau)d\eta .$$

Moreover, following similar steps as those used above we can conclude that for $m = 1, 2$

$$(3.21) \qquad D_x^m V(x, t; \xi, \tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} D_x^m U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

holds for $t > \tau$, and for c_m, b_m, $\mu > 0$, $m = 1, 2$

$$|D_x^m V(x,t;\xi,\tau)| \leq \frac{c_m}{(t-\tau)^{\frac{m-1}{3}}} e^{-\mu \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \quad \xi - x \geq 0$$

$$|D_x^m V(x,t;\xi,\tau)| \leq b_m \frac{(x-\xi)^{\frac{2m-3}{4}}}{(t-\tau)^{\frac{m+1}{4}}}, \quad x - \xi > 0.$$

For $m = 3$, however, a sharper bound for $D_x^3 I$ is needed. To achieve such a bound we consider the following two cases (1) $\tau < \sigma < \frac{t+\tau}{2}$, and (2) $\frac{t+\tau}{2} < \sigma < t$ separately. For $\tau < \sigma < \frac{t+\tau}{2}$, one can easily show that for all $x, \xi \in R$, there exists an $M_0 > 0$ such that

$$(3.22) \qquad |I_{xxx}| \leq \frac{M_0}{(t-\tau)^{5/4}(\sigma-\tau)^{5/6}} .$$

However, for case (2), we will need to establish the following lemma on Φ.

Lemma (3.9). $\Phi(x,t;\xi,\tau)$ given by the integral equation (3.9) has a continuous first derivative with respect to x, for all $x \in R$, and $t > \tau$.

Proof. From above we have seen that for $m = 1, 2$, $D_x^m V$ exist and can be obtained by differentiating under the integral sign. Since

$$\Phi(x,t;\xi,\tau) = LU(x-\xi,t-\tau) + \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} LU(x-\eta,t-\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta,$$

where

$$LU(x-\xi,t-\tau) = a(x,t)U_x(x-\xi,t-\tau) + b(x,t)U(x-\xi,t-\tau),$$

we can conclude that $\frac{\partial\Phi}{\partial x}$ exists. Moreover, the continuity of $\frac{\partial\Phi}{\partial x}$ is a direct consequence of the continuity of $D_x^m U$, $m = 1, 2$ and the fact that $a(x,t), b(x,t)$ satisfies (*).

From the above lemma we can write

$$I_{xxx}(x,t;\xi,\tau,\sigma) = \int_{-\infty}^{\infty} U_{xxx}(x-\eta,t-\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta$$

$$= -\int_{-\infty}^{\infty} U_{xx\eta}(x-\eta,t-\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta$$

$$= \int_{-\infty}^{\infty} U_{xx}(x-\eta,t-\sigma)\Phi_{\eta}(\eta,\sigma;\xi,\tau)d\eta$$

since $U_{xx}\Phi\Big|_{-\infty}^{\infty} = 0$. Therefore, having that $\frac{\partial\Phi}{\partial x}$, satisfies:

$$\left|\frac{\partial\Phi}{\partial x}(x,t;\xi,\tau)\right| \le \frac{C_0}{(t-\tau)} e^{-\gamma\frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \qquad \xi - x \ge 0$$

$$\left|\frac{\partial\Phi}{\partial x}(x,t;\xi,\tau)\right| \le b_0 \frac{(x-\xi)^{3/4}}{(t-\tau)^{5/4}}, \qquad x - \xi > 0$$

enables us to get a sharper bound on I_{xxx} and show,

Lemma (3.10). For $t > \tau$, $D_x^3 V(x, t; \xi, \tau)$ exists and is given by

$$D_x^3 V(x, t; \xi, \tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} D_x^3 U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

Proof. Consider

$$V(x, t; \xi, \tau)$$

$$= \int_\tau^{\frac{t+\tau}{2}} d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta + \int_{\frac{t+\tau}{2}}^{t} d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

$$= V_1 + V_2, \qquad \text{respectively.}$$

For V_1 the result of the lemma can be seen to hold using (3.22).
For V_2, we consider for $\frac{t+\tau}{2} < \sigma < t$

$$D_x^3 I = \int_{-\infty}^{\infty} U_{xx}(x-\eta), t-\sigma)\Phi_\eta(\eta, \sigma; \xi, \tau)d\eta .$$

Since $t - \tau < 2(\sigma - \tau)$, we can estimate I_{xxx}, for $x - \xi \leq 0$ as follows

$$|I_{xxx}| \leq \left[\int_{-\infty}^{x} + \int_{x}^{\xi} + \int_{\xi}^{\infty}\right] |U_{xxx}| \, |\Phi_\eta| d\eta \leq$$

$$\leq \text{const}\left[\int_{-\infty}^{x} \frac{(x-\eta)^{3/4}}{(t-\sigma)^{5/4}} \frac{e^{-\gamma \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}}}{(\sigma-\tau)} \, d\eta\right.$$

$$+ \int_{x}^{\xi} \frac{e^{-\gamma\left[\frac{(\eta-x)^{3/2}}{(t-\sigma)^{1/2}} + \frac{(\xi-\eta)^{3/2}}{(\sigma-\tau)^{1/2}}\right]}}{(t-\sigma)(\sigma-\tau)} \, d\eta$$

$$\left. + \int_{\xi}^{\infty} \frac{e^{-\gamma \frac{(\eta-x)^{3/2}}{(t-\sigma)^{1/2}}}}{(t-\sigma)} \frac{(\eta-\xi)^{3/4}}{(\sigma-\tau)^{5/4}} \, d\eta\right]$$

$$\leq \frac{M_1}{(t-\tau)(t-\sigma)^{5/6}} .$$

Similarly for $x - \xi > 0$, we can find an $M_2 > 0$ such that

$$|I_{xxx}| \leq \frac{M_2}{(t-\tau)^{5/4}(t-\sigma)^{2/3}} \tag{3.23}$$

Therefore, the lemma holds for V_2, and

$$D_x^3 V(x, t; \xi, \tau) = \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} D_x^3 U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau) d\eta \tag{3.24}$$

exists and therefore

$$\int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} U_t(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta, \quad t > \tau$$

also exists. Moreover, using Lemma (3.5), we have that V_t is given by

$$(3.25) \quad V_t(x, t; \xi, \tau) = \Phi(x, t; \xi, \tau) + \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} U_t(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

and from (3.8), (3.21), (3.25) we have that

$$L\Gamma = LU + \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} LU(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta - \Phi(x, t; \xi, \tau)$$

$$= 0, \qquad \text{by (3.9).}$$

And thus concludes Step 2.

<u>Step 3.</u> We finally show that

$$\lim_{t \to \tau} \int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi = \phi(x)$$

for every continuous ϕ with compact support. This follows directly from the lemmas proved for Step 2. In Lemma (3.8), it was shown that

$$\int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta = O((t-\tau)^{1/4}),$$

therefore, for $\phi(x)$, continuous and uniformly bounded with compact support

$$\lim_{t \to \tau} \int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi$$

$$= \lim_{t \to \tau} \Big[\int_{-\infty}^{\infty} U(x, t; \xi, t)\phi(\xi)d\xi$$

$$+ \int_{-\infty}^{\infty} \Big\{ \int_{\tau}^{t} d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta \Big\} \phi(\xi)d\xi \Big]$$

$$= \phi(x).$$

This completes the proof of Theorem (3.6).

Before we conclude this section, we note that from the representation

$$\Gamma(x, t; \xi, t) = U(x-\xi, t-\tau) + V(x, t; \xi, t), \quad t > \tau$$

and from Lemma (3.8), $\Gamma(x, t; \xi, \tau)$ can be easily shown to satisfy the following bounds,

$$(3.26) \qquad |\Gamma(x, t; \xi, \tau)| \leq \frac{M_0}{(t-\tau)^{1/3}} e^{-\delta_1 \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \quad \xi - x \geq 0$$

$$(3.27) \qquad |\Gamma(x, t; \xi, \tau)| \leq M_1 \frac{(x-\xi)^{-1/4}}{(t-\tau)^{1/4}}, \quad \xi - x < 0,$$

where M_0, M_1 and δ_1 are positive constants. Moreover, since

$$D_x^m V(x,t;\xi,\tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} D_x^m U(x,t;\eta,\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta, \qquad m = 0, 1, 2$$

one can find the bounds for Γ_x, Γ_{xx} which are given for later use:

$$|\Gamma_x(x,t;\xi,\tau)| \leq \frac{M_2}{(t-\tau)^{2/3}} e^{-\delta_2 \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \qquad \xi - x \geq 0$$

$$|\Gamma_x(x,t;\xi,\tau)| \leq M_3 \frac{(x-\xi)^{1/4}}{(t-\tau)^{3/4}}, \qquad \xi - x < 0$$

and

$$|\Gamma_{xx}(x,t;\xi,\tau)| \leq \frac{M_4}{(t-\tau)} e^{-\delta_3 \frac{(\xi-x)^{3/2}}{(t-\tau)^{1/2}}}, \qquad \xi - x \geq 0$$

$$|\Gamma_{xx}(x,t;\xi,\tau)| \leq M_5 \frac{(x-\xi)^{3/4}}{(t-\tau)^{5/4}}, \qquad \xi - x < 0$$

where M_2, M_3, M_4, M_5, δ_2 and δ_3 are positive constants.

3.3 The Adjoint Equation

Consider the operator L^*,

(3.28) $$L^* v = -\frac{\partial^3 v}{\partial x^3} - \frac{\partial}{\partial x}(a(x,t)v) + b(x,t)v + \frac{\partial v}{\partial t},$$

provided that $\frac{\partial a}{\partial x}$ exists. The differential operator L^* is obviously the adjoint operator for L, and the form

$$(3.29) \quad vLu - uL^*v = \frac{\partial}{\partial x}\left(v\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + auv\right) - \frac{\partial}{\partial t}(uv).$$

is the Green's identity for L.

The existence of a fundamental solution for $L^*v = 0$, for all $(x, t) \in R \times [0, T]$, $(\xi, \tau) \in R \times [0, T]$, $t < \tau$, can be demonstrated in a similar fashion as that for L in the preceeding section. However, to be able to carry out the procedure the following assumption on $\frac{\partial a}{\partial x}$ has to be added

(**) $\frac{\partial a}{\partial x}$ is continuous, uniformly bounded for all $(x, t) \in R \times [0, T]$, together with $\frac{\partial^2 a}{\partial x^2}$.

Taking as a parametrix the function

$$U(\xi - x, \tau - t) = \frac{1}{(3(\tau - t))^{1/3}} Ai\left(\frac{x - \xi}{(3(\tau - t))^{1/3}}\right),$$

then the fundamental solution of $L^*v = 0$ is $\Gamma^*(x, t; \xi, \tau)$ given by

$$(3.30) \quad \Gamma^*(x, t; \xi, \tau) = U(\xi - x, \tau - t) + \int_\tau^t d\sigma \int_{-\infty}^{\infty} U(\eta - x, \sigma - \tau)\Phi^*(\eta, \sigma; \xi, \tau) d\eta$$

where Φ^* is the solution of the integral equation

(3.31)

$$\Phi^*(x, t; \xi, \tau) = L^*U(\xi - x, \tau - t) + \int_\tau^t d\sigma \int_{-\infty}^{\infty} L^*U(\eta - x, \sigma - \tau)\Phi^*(\eta, \sigma; \xi, \tau) d\eta$$

By repeating the steps of Section 3.2, we can conclude that $\Gamma^*(x,t;\xi,\tau)$ exists and satisfies $\Gamma^*(x,t;\xi,\tau) = 0$ moreover, for all $\tau > t$

(3.32) $$|\Gamma^*(x,t;\xi,\tau)| \le \frac{m_1}{(\tau-t)^{1/3}} e^{-m_0 \frac{(x-\xi)^{3/2}}{(\tau-t)^{1/2}}} \qquad x-\xi \ge 0$$

(3.33) $$|\Gamma^*(x,t;\xi,\tau)| \le m_2 \frac{(\xi-x)^{-1/4}}{(\tau-t)^{1/4}} \qquad \xi - x > 0$$

with m_0, m_1 and m_2 are positive constants.

3.4 Properties of the Fundamental Solution

Property 1. The fundamental solution of (3.3), satisfies for $(x,t)\in R\times[0,T]$, $(\xi,\tau)\in R\times[0,T]$, $(x,t) \ne (\xi,\tau)$, $t > \tau$ the bounds given by (3.26) and (3.27).

Property 2. Suppose that $\Gamma^*(x,t;\xi,\tau)$ of $L^*v = 0$ exists. Then $\Gamma(x,t;\xi,\tau) = \Gamma^*(\xi,\tau;x,t)$, $t > \tau$.

Proof. In (3.29) let us take

$$u(\eta,\sigma) = \Gamma(\eta,\sigma;\xi,\tau),$$

and

$$v(\eta,\sigma) = \Gamma^*(\eta,\sigma;x,t)$$

and integrate Green's identity over the domain $|\eta| < N$,

$\tau + \epsilon < \sigma < t - \epsilon$, and noticing that $Lu = 0$, $L^* v = 0$, we get

(3. 34)

$$0 = \int_{\tau+\epsilon}^{t-\epsilon} d\sigma \int_{-N}^{N} (vLu - uL^* v) d\eta$$

$$= \int_{\tau+\epsilon}^{t-\epsilon} [v\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - auv]_{-N}^{N} d\sigma - \int_{-N}^{N} uv\Big|_{\tau+\epsilon}^{t-\epsilon} d\eta$$

From the bounds on Γ, Γ_x and Γ^*, Γ_x^* we can see that as $N \rightarrow \infty$, the first integral on the right hand side of (3. 34) will vanish and

$$\int_{-\infty}^{\infty} u(\eta, \tau-\epsilon)v(\eta, \tau-\epsilon)d\eta = \int_{-\infty}^{\infty} u(\eta, t+\epsilon)v(\eta, t+\epsilon)d\eta$$

Now, taking the limit as $\epsilon \rightarrow 0$ we get

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} v(\eta, \tau-\epsilon)\Gamma(\eta, \tau-\epsilon; \xi, \tau)d\eta = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} u(\eta, t+\epsilon)\Gamma^*(\eta, t+\epsilon; x, t)d\eta$$

i.e.,

$$u(x, t) = \Gamma(x, t; \xi, \tau) = v(\xi, \tau) = \Gamma^*(\xi, \tau; x, t)$$

Thus establishing Property 2.

The proof of the following two properties depends on the results to be given in the next section.

Property 3. The fundamental solution $\Gamma(x, t; \xi, \tau)$ is unique.

Property 4. $\Gamma(x, t; \xi, \tau)$ given by (3.8) satisfies the semi-group relation

$$\int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau)\Gamma(\xi, \tau)d\xi = \Gamma(x, t). \quad (3.35)$$

3.5 The Cauchy Problem

In this section we consider the question of existence and uniqueness of solutions to the Cauchy problem

$$Lu(x, t) = f(x, t), \quad x \in R, \quad t \in (0, T] \quad (3.36)$$

$$u(x, 0) = \phi(x), \quad x \in R \quad (3.37)$$

for sufficiently small T.

We aim to show that for suitably chosen functions $f(x, t)$, $\phi(x)$ the function

$$u(x, t) = \int_{-\infty}^{\infty} \Gamma(x, t; \xi, 0)\phi(\xi)d\xi - \int_0^t d\tau \int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau)f(\xi, \tau)d\xi \quad (3.38)$$

$$= u_1(x, t) + u_2(x, t), \qquad \text{respectively,}$$

is a solution of the Cauchy problem (3.36) and (3.37).

Lemma (3.11). Let $\phi(x)$ be a continuous function which satisfies

(1) for $x \geq 0$, $|\phi(x)| \leq A_1 e^{a_1 x^{3/2}}$, for A_1, a_1 appropriate positive constants.

(2) for $x < 0$, $|\phi(x)| = O(|x|^{-3/4-\epsilon})$ for some $\epsilon > 0$.

Then

$$u_1(x, t) = \int_{-\infty}^{\infty} \Gamma(x, t; \xi, 0)\phi(\xi)d\xi ,$$

is a continuous function for which $Lu_1(x, t) = 0$ and

$$\lim_{t \to 0} u_1(x, t) = \phi(x).$$

<u>Proof.</u> From (3.8), we have

$$u_1(x, t) = \int_{-\infty}^{\infty} U(x-\xi, t)\phi(\xi)d\xi + \int_{-\infty}^{\infty} \phi(\xi)\left[\int_0^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta\right]d\xi$$

$$= u_{11}(x, t) + u_{12}(x, t)$$

Let

$$\Psi(\eta, \sigma) = \int_{-\infty}^{\infty} \Phi(\eta, \sigma; \xi, 0)\phi(\xi)d\xi ,$$

then by changing the order of integration in $u_{12}(x, t)$ we have

$$u_{12}(x, t) = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Psi(\eta, \sigma)d\eta .$$

Since $\Phi(\eta, \sigma; \xi, \tau)$ is once continuously differentiable with respect to η, we can show that the same is true for $\psi(\eta, \sigma)$. Therefore, using a similar argument as that in Section 2, we can conclude that

$$Lu_{12}(x,t) = -\psi(x,t) + \int_0^t d\sigma \int_{-\infty}^{\infty} LU(x-\eta, t-\sigma)\psi(\eta,\sigma)d\eta$$

and thus

$$Lu_1 = \int_{-\infty}^{\infty} \phi(\xi) \Big[LU(x-\xi, t) - \Phi(x,\sigma;\xi,0) + \int_0^t d\sigma \int_{-\infty}^{\infty} LU(x-\eta, t-\sigma)\Phi(\eta,\sigma;\xi,0)d\eta \Big] d\xi$$

$$= 0, \qquad \text{by (3.9)}.$$

Moreover,

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} \int_{-\infty}^{\infty} U(x-\xi,t)\phi(\xi)d\xi + \lim_{t\to 0} u_{12}(x,t)$$

$$= \phi(x) + \lim_{t\to 0} \int_0^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\psi(\eta,\sigma)d\eta .$$

The second term can be shown to vanish, by using an argument similar to that of Lemma (3.8).

<u>Lemma (3.12)</u>. Let $f(x,t)$ be a continuous function in $R \times [0,T]$, and once continuously differentiable with respect to x, and for $x \geq 0$, suppose that there exist positive constants a_i, A_i, $i = 0, 1$ such that $|D_x^i f(x,t)| \leq A_i e^{a_i x^{3/2}}$ for all $t \in [0,T]$ while for $x < 0$, $|D_x^i f(x,t)| \leq B_i |x|^{-1/4-\epsilon}$, $i = 0, 1$, for some $\epsilon > 0$, and B_i are a positive constant. Then

$$u_2(x, t) = -\int_0^t d\sigma \int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau$$

satisfies

$$Lu_2 = f(x, t), \qquad u_2(x, 0) = 0.$$

The proof is similar to that given for Lemma (3.11). Summing up we get:

Theorem (3.13). (Existence). Let $\phi(x)$ satisfy the conditions of Lemma (3.12). Then, for sufficiently small t, $u(x, t)$ given by (3.38) is a solution to the Cauchy problem (3.36) and (3.37). Moreover, there exist positive constants c_0, c_1, c_2 such that

(3.39) $$|u(x, t)| \leq c_1 e^{c_0 x^{3/2}} \qquad \text{for} \quad x \geq 0$$

(3.40) $$|u(x, t)| \leq c_2 |x|^{-1/4-\epsilon} \qquad x < 0$$

where $\epsilon > 0$.

We only need to estimate $u(x, t)$. This can be easily achieved by using the appropriate bounds for Γ and the properties of $\phi(x)$ and $f(x, t)$.

Theorem (3.14). (Uniqueness). Suppose that the function $u(x, t)$ satisfies the conditions

(i) $u(x,t)$ continuous in $0 \leq t \leq T$.

(ii) for $0 < t < T$, $u(x,t)$ satisfies $Lu = 0$, where the derivatives appearing in the equation are continuous.

(iii) there exist positive constants c_0, c_1, c_2 such that

$$|u(x,t)| \leq c_1 e^{c_0 x^{3/2}} \qquad \text{for} \quad x \geq 0$$

$$|u(x,t)| \leq c_2 |x|^{-1/4-\epsilon} \qquad x < 0$$

(iv) $u(x,0) \equiv 0$.

Then $u(x,t)$ vanishes identically in $0 \leq t \leq T$.

Proof. Let $h(x)$ be a three times continuously differentiable function such that

$$h(x) = \begin{cases} 1 & \text{for} \quad |x| \leq N \\ 0 & \text{for} \quad |x| \geq N+1 \end{cases}$$

$$0 \leq h(x) \leq 1, \quad \text{and} \quad \left|\frac{\partial h}{\partial x}\right| + \left|\frac{\partial^2 h}{\partial x^2}\right| + \left|\frac{\partial^3 h}{\partial x^3}\right| \leq M$$

where M is a constant independent of N, such function can be constructed [7]. Apply the Green's identity to $u = u(\xi, \tau)$, $v = h(\xi)\Gamma(x,t;\xi,\tau)$ for (x,t) fixed but arbitrary, in the region $\{|\xi| \leq N+2,\ 0 < \tau < t-\epsilon\}$ where $0 < \epsilon < t$ is arbitrary and N is chosen so large that $|x| \leq N$. We find

$$\int_0^{t-\epsilon} d\tau \int_{-(N+2)}^{N+2} u(\xi,\tau) L^*(\Gamma(x,t;\xi,\tau)h(\xi))d\xi$$

$$= \int_0^{t-\epsilon} u\frac{\partial^2(h\Gamma)}{\partial\xi^2} + h\Gamma\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial u}{\partial\xi}\frac{\partial(h\Gamma)}{\partial\xi} + ah\Gamma u\Big|_{-(N+2)}^{N+2} d\tau - \int_{-(N+2)}^{N+2} uh\Gamma\Big|_0^{t-\epsilon} d\xi .$$

We note that as $\epsilon \to 0$, the second term on the right hand side will reduce to $h(x)u(x,t) = u(x,t)$ for $|x| \leq N$. Since $L^*\Gamma(x,t;\xi,\tau) = L^*\Gamma^*(\xi,\tau;x,t) = 0$ by Property 2, we have

$$(3.41) \qquad u(x,t) = \int_0^t d\tau \left[\int_{-(N+1)}^{-N} + \int_N^{N+1}\right] u(\xi,\tau)$$

$$\times \left[3\frac{\partial h}{\partial\xi}\frac{\partial^2\Gamma}{\partial\xi^2} - 3\frac{\partial^2 h}{\partial\xi^2}\frac{\partial\Gamma}{\partial\xi} - \Gamma\frac{\partial^3 h}{\partial\xi^3} - a\Gamma\frac{\partial h}{\partial\xi}\right]d\xi .$$

Now, using the fact that $u(x,t)$ satisfies condition (iii) and using the bounds on Γ, Γ_x, Γ_{xx}, we find that for t restricted to a sufficiently small interval, say $0 \leq t \leq \delta$, that the right hand side of (3.41) tends to zero as $N \to \infty$. The left side was independent of N, whence we conclude that $u(x,t) \equiv 0$, for $0 \leq t \leq \delta$. This argument is repeated on parallel strips of width δ, which then yields the statement of the theorem.

In the following we will give the proof of Properties (3) and (4) of Section 4.

<u>Proof of Property (3).</u> Let $K(x, t; \xi, \tau)$ be a fundamental solution of (3.3). Consider, for all (x, t), $x \in R$, $\tau < t \leq T$, where τ is arbitrary but fixed, and T is sufficiently small, the Cauchy problem

$$Lu(x, t) = 0$$

$$u(x, \tau) = \phi(x, \tau)$$

for all $\phi(x, \tau)$, which as a function of x, are continuous with compact support. Then for $u(x, t)$ satisfying the conditions of Theorem (3.14), $u(x, t)$ is given by

$$u(x, t) = \int_{-\infty}^{\infty} \Gamma(x, t; \xi, \tau)\phi(\xi, \tau)d\xi$$

and

$$u(x, t) = \int_{-\infty}^{\infty} K(x, t; \xi, \tau)\phi(\xi, \tau)d\xi .$$

Thus

$$0 = \int_{-\infty}^{\infty} [\Gamma(x, t; \xi, \tau) - K(x, t; \xi, \tau)]\phi(\xi, \tau)d\xi$$

holds true for all such ϕ. Since τ was arbitrary, one can conclude that for all $(x, t) \in R \times (0, T]$, $(\xi, \tau) \in R \times (0, T]$, $(x, t) \neq (\xi, \tau)$

$$\Gamma(x, t; \xi, \tau) = K(x, t; \xi, \tau) ,$$

hence proves the uniqueness of Γ.

As for Property (4), we consider the Cauchy problem

$$(\text{I}) \quad \begin{cases} Lu(x, t) = 0 & (x, t) \in R \times (0, T] \\ u(x, 0) = \phi(x) & x \in R \end{cases}$$

From Lemma (3.11) we have that

$$u(x, t) = \int_{-\infty}^{\infty} \Gamma(x, t; \xi, 0)\phi(\xi)d\xi ,$$

for all continuous $\phi(x)$ with compact support.

Let t_0 be fixed, $0 < t_0 < T$ and consider

$$(\text{II}) \quad \begin{cases} Lv(x, t) = 0 & x \in R, \quad t_0 < t \leq T \\ v(x, t_0) = u(x, t_0) & x \in R \end{cases}$$

for all (x, t), $t_0 < t \leq T$. By Theorem (3.13) the solution for (II) is given by

$$v(x, t) = \int_{-\infty}^{\infty} \Gamma(x, t; \xi, t_0)u(\xi, t_0)d\xi$$

$$= \int_{-\infty}^{\infty} \Gamma(x, t; \xi, t_0)\left[\int_{-\infty}^{\infty} \Gamma(\xi, t_0; \eta, 0)\phi(\eta)d\eta\right]d\xi$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \Gamma(x, t; \xi, t_0)\Gamma(\eta, t_0; \xi, 0)d\eta\right]\phi(\xi)d\xi$$

On the other hand, Problem II is equivalent to

$$Lw(x, t) = 0$$

$$w(x, 0) = \phi(x)$$

which has the solution, for $t_0 < t < T$

$$w(x, t) = \int_{-\infty}^{\infty} \Gamma(x, t; \xi, 0)\phi(\xi)d\xi \ .$$

However, by Theorem (3. 14) we have

$$\int_{-\infty}^{\infty}\left[\Gamma(x, t; \xi, 0) - \int_{-\infty}^{\infty}\Gamma(x, t; \eta, t_0)\Gamma(\eta, t_0; \xi, 0)d\eta\right]\phi(\xi)d\xi = 0$$

holds true for all admissible $\phi(\xi)$. Thus

$$\Gamma(x, t; \xi, 0) = \int_{-\infty}^{\infty}\Gamma(x, t; \eta, t_0)\Gamma(\eta, t_0; \xi, 0)d\eta$$

which proves Property (4).

IV. SOME REMARKS ABOUT THE FIRST INITIAL BOUNDARY VALUE PROBLEM

Let D be the region, $D = \{(x,t) : 0 < t \leq 1, \chi_1(t) < x < \chi_2(t)\}$ where χ_1, χ_2 are given smooth functions such that $\chi_1(t) < \chi_2(t)$ for all $0 \leq t \leq 1$. Consider the problem

$$(4.1)\qquad \begin{cases} Lu(x,t) = 0 & \text{in } D \\ u(x,0) = \phi(x), & \chi_1(0) \leq x \leq \chi_2(0) \\ u(\chi_1(t),t) = \psi_0(t), & u(\chi_2(t),t) = \psi_1(t) \\ u_x(\chi_1(t),t) = \psi_2(t) & \end{cases}$$

where $\phi(x)$, $\psi_i(t)$, $i = 0, 1, 2$ are arbitrary functions to be specified, and L is given by (3.3). In the following we will give a brief discussion on the existence and uniqueness of solutions of (4.1).

For the uniqueness question one can easily state and prove the following theorem.

<u>Theorem (4.1).</u> Let $a(x,t)$, $b(x,t)$ be continuous in D together with the first derivative of $a(x,t)$ with respect to x. Then in D there is at most one solution to the problem (4.1).

<u>Proof.</u> Let u_1, u_2 be two solutions of (4.1). The difference $u = u_1 - u_2$ satisfies $Lu = 0$ with identically vanishing initial and boundary conditions. Let

$$u(x,t) = v(x,t)e^{\alpha x+\beta t}$$

where α and β are constants to be chosen. Accordingly, $v(x,t)$ satisfies

$$Mv = \frac{\partial^3 v}{\partial x^3} + 3\alpha\frac{\partial^2 v}{\partial x^2} + (3\alpha^2 + a(x,t))\frac{\partial v}{\partial x}$$

$$+ (\alpha^3 + \alpha a(x,t) + b(x,t) - \beta)v - \frac{\partial v}{\partial t} = 0$$

Consider

$$\iint_{\overline{D}} vMvdxdt = -3\iint_{\overline{D}} \alpha\left(\frac{\partial v}{\partial x}\right)^2 dxdt$$

$$+\iint_{\overline{D}}\left[\alpha^3 + \alpha a(x,t) + b(x,t) - \beta - \frac{1}{2}\frac{\partial a(x,t)}{\partial x}\right]v^2 dxdt$$

$$-\frac{1}{2}\int_{\chi_1(1)}^{\chi_2(1)} v^2(x,1)dx - \frac{1}{2}\int_0^t \left(\frac{\partial v}{\partial x}(\chi_2,(\tau),\tau)\right)^2 d\tau$$

Therefore, choosing $\alpha = \alpha_0 > 0$ and β such that $\beta > \alpha_0^3 + \alpha_0 a(x,t) + b(x,t) - \frac{1}{2}\frac{\partial a(x,t)}{\partial x}$ for all $(x,t) \in \overline{D}$, we get that $\frac{\partial v}{\partial x}$, v must vanish identically in D. Thus proves the uniqueness.

<u>Remark.</u> If $\chi_i(t)$, $i = 1, 2$ are continuous together with their first derivatives in $0 \leq t \leq 1$, then the transformation

$$\xi = \frac{x-\chi_1(t)}{\chi_2(t)-\chi_1(t)}, \qquad \eta = \int_0^t \frac{d\tau}{[\chi_2(t)-\chi_1(t)]^3},$$

transfer the domain D to a rectangular one say, $\mathbb{R}$, while the form of the differential equation is preserved. Therefore, we will restrict the discussion on the existence question to the case of a rectangular domain.

In the following we will suppose that the coefficients of Lu satisfies (*). Thus, from the preceding chapter the fundamental solution of $Lu = 0$ exists and is given by

$$\Gamma(x, t; \xi, \tau) = U(x-\xi, t-\tau) + V(x, t; \xi, \tau),$$

$$V(x, t; \xi, \tau) = \int_\tau^t d\sigma \int_{-\infty}^{\infty} U(x-\eta, t-\sigma)\Phi(\eta, \sigma; \xi, \tau)d\eta$$

where Φ is given by (3.9).

In obtaining existence theorem by potential theoretic methods, one derives a system of Volterra type integral equations via certain jump relations, which we now establish for our problem.

Lemma (4.2). Let $\alpha(t)$ be a function of bounded variation for $0 < t \leq 1$, and $u(x, t) = \int_0^t U_x(x, t-\tau)\alpha(\tau)d\tau$. Then the derivative $u_x(x, t)$ exists,

$$u_x(x,t) = \int_0^t U_{xx}(x, t-\tau)a(\tau)d\tau$$

and

$$\lim_{x \to 0} \frac{\partial u(x,t)}{\partial x} = \frac{2}{3}a(t).$$

Proof. We need only to prove that $\lim_{x \to 0} \frac{\partial u(x,t)}{\partial x} = \frac{2}{3}a(t)$. For that, we write

$$\frac{\partial u(x,t)}{\partial x} = a(t)\int_0^t U_{xx}(x, t-\tau)d\tau - \int_0^t [a(t) - a(\tau)]U_{xx}(x, t-\tau)d\tau$$

$$= a(t)I_1 + I_2.$$

For $\delta > 0$,

$$\int_0^{t-\delta} U_{xx}(x, t-\tau)d\tau = \int_0^{t-\delta} \frac{x}{(3(t-\tau))^{1/3}} Ai\left(\frac{-x}{(3(t-\tau))^{1/3}}\right)d\tau$$

$$= \int_{\frac{-x}{(3\delta)^{1/3}}}^{\frac{-x}{(3t)^{1/3}}} Ai(\zeta)d\zeta \to \int_{-\infty}^{\frac{-x}{(3t)^{1/3}}} Ai(\zeta)d\zeta, \qquad \delta \to 0$$

therefore, for $t > 0$

(4.2) $$\lim_{x \to 0} I_1 = \lim_{x \to 0} \int_{-\infty}^{\frac{-x}{(3t)^{1/3}}} Ai(\zeta)d\zeta = \frac{2}{3}$$

On the other hand, for $x > 0$, $\delta > 0$

$$I_2 = \left[\int_0^{t-\delta} + \int_{t-\delta}^{t}\right][\alpha(\tau)-\alpha(t)]U_{xx}(x, t-\tau)d\tau$$

$$= I_{21} + I_{22}, \qquad \text{respectively,}$$

$$\lim_{x \to 0} I_{21} = 0.$$

For I_{22}, we have by Bonnett's theorem [2], that there exists δ_0, $0 < \delta_0 < \delta$, such that

$$I_{22} = [\alpha(t-\delta)-\alpha(t)] \int_{t-\delta}^{t-\delta_0} U_{xx}(x, t-\tau)d\tau$$

$$= [\alpha(t-\delta)-\alpha(t)] \int_{\frac{-x}{(3\delta)^{1/3}}}^{-\frac{x}{(3\delta_0)^{1/3}}} Ai(\zeta)d\zeta .$$

Since α is of bounded variation, we can find an $\epsilon > 0$ such that

(4.3) $$|I_{22}| \leq \epsilon M, \quad \text{where} \quad M = \max_x \left| \int_{\frac{-x}{(3\delta)^{1/3}}}^{-\frac{x}{(3\delta_0)^{1/3}}} Ai(\zeta)d\zeta \right| .$$

From (4.2), (4.3) and since ϵ was arbitrary the lemma is proved.

Lemma (4.3). Let $v(x,t) = \int_0^t V_x(x, t; 0, \tau)\alpha(\tau)d\tau$, where α is a function of bounded variation. Then $v_x(x, t)$ exists and is continuous in a neighborhood of $x = 0$, i.e.,

$$\lim_{x \to 0} \frac{\partial v(x,t)}{\partial x} = \int_0^t V_{xx}(0,t;0,\tau)\alpha(\tau)d\tau \ .$$

The proof can easily be demonstrated by using the same ideas used, in Chapter III, to find the bounds on V and its derivatives.

Summing the result of the above two lemmas we get,

Theorem (4.4). Suppose $\alpha(t)$ is a function of bounded variation and let $w(x,t) = \int_0^t \Gamma_x(x,t;0,\tau)\alpha(\tau)d\tau$ then $w_x(x,t)$ exists and

$$\lim_{x \to 0} w_x(x,t) = \frac{2}{3}\alpha(t) + \int_0^t V_{xx}(0,t;0,\tau)\alpha(\tau)d\tau \ .$$

Having established the jump relation, we seek a solution $u(x,t)$ for (4.1) of the form

$$(4.4) \quad u(x,t) = \int_0^1 \Gamma(x,t;\xi,0)\phi(\xi)d\xi + \int_0^t \Gamma(x,t;0,\tau)\alpha(\tau)d\tau$$

$$+ \int_0^t \Gamma(x,t;1,\tau)\beta(\tau)d\tau + \int_0^t \Gamma_x(x,t;0,\tau)\gamma(\tau)d\tau$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ belong to certain classes of functions which we will specify in the following lemmas.

Lemma (4.5). Let $\alpha(t)$ be a function of bounded variation, $\alpha'(t)$

exists and $\alpha' \in L_p(0,T)$. $p > \frac{4}{3}$. Then for $x > 0$, the function $u(x,t) = \int_0^t \Gamma(x,t;0,\tau)\alpha(\tau)d\tau$ satisfies the differential equation $Lu = 0$.

Proof. Let

$$w(x,t-\eta) = \int_\eta^t U(x,t-\tau)d\tau = \int_\eta^t \frac{1}{(3(t-\tau))^{1/3}} Ai\left(\frac{-x}{(3(t-\tau))^{1/3}}\right) d\tau$$

$$= \int_{-\infty}^{-\frac{x}{(3(t-\eta))^{1/3}}} \frac{x^2}{\zeta^3} Ai(\zeta)d\zeta .$$

Then

$$w_t(x,t-\eta) = \frac{1}{(3(t-\eta))^{1/3}} Ai\left(\frac{-x}{(3(t-\eta))^{1/3}}\right)$$

On the other hand, from the bounds on $U(x,t-\eta)$ it is possible to differentiate w twice with respect to x and obtain that

$$w_{xx}(x,t-\eta) = -\int_\eta^t \frac{x}{(3(t-\eta))^{1/3}} Ai\left(\frac{-x}{(3(t-\eta))^{1/3}}\right) d\tau$$

$$= -\int_{-\infty}^{-\frac{x}{(3(t-\eta))^{1/3}}} Ai(\zeta)d\zeta .$$

Therefore,

$$w_{xxx}(x,t-\eta) = \frac{1}{(3(t-\eta))^{1/3}} Ai\left(\frac{-x}{(3(t-\eta))^{1/3}}\right) = w_t(x,t-\eta). \tag{4.5}$$

We note here that

$$\bar{u}(x, t) = \int_0^t U(x, t-\tau)\alpha(\tau)d\tau = \alpha(0)w(x, t) + \int_0^t w(x, t-\eta)\alpha'(\eta)d\eta .$$

Since $\alpha' \in L_p(0, T)$, $p > \frac{4}{3}$, we can find a constant $M < \infty$ such that for $x > 0$,

$$\left| \int_{t_0}^t w_{xxx}(x, t-\eta)\alpha'(\eta)d\eta \right| \leq M, \quad t_0 < t$$

and thus

$$\bar{u}_{xxx} = \alpha(0)w_{xxx}(x, t) + \int_0^t w_{xxx}(x, t-\eta)\alpha'(\eta)d\eta$$

holds, moreover from (4.5)

$$\bar{u}_{xxx} = \bar{u}_t . \tag{4.6}$$

To complete the proof of the lemma we note that

$$u(x, t) = \int_0^t U(x, t-\tau)\alpha(\tau)d\tau + \int_0^t V(x, t; 0, \tau)\alpha(\tau)d\tau$$

$$Lu(x, t) = \int_0^t LU(x, t-\tau)\alpha(\tau)d\tau + \int_0^t LV(x, t; 0, \tau)\alpha(\tau)d\tau$$

$$- \alpha(t) \lim_{\tau \to t} V(x, t; 0, \tau)$$

by (4.6). By Lemma (3.8), we can conclude that

$$Lu(x, t) = \int_0^t L\Gamma(x, t; 0, \tau)\alpha(\tau)d\tau = 0$$

which proves the lemma.

Remark. For $\beta(t)$ satisfying the conditions of the above lemma, we can show similarly that

$$v(x, t) = \int_0^t \Gamma(x, t; 1, \tau)\beta(\tau)d\tau$$

satisfies the differential equation $Lv = 0$.

Lemma (4.6). Let $\gamma(t)$ be a function of bounded variation for which $\gamma' \in L_p(0, T)$, $p > 4$. Then the function

$$w(x, t) = \int_0^t \Gamma_x(x, t; 0, \tau)\gamma(\tau)d\tau$$

satisfies $Lw = 0$.

The proof is similar to that of Lemma (4.4).

From Lemmas (4.5), (4.6) and the Remark we see that (4.4) is a solution for Lu.

At this point, the derivation of the system of the Volterra type, singular integral equations for α, β, γ is standard. The solution of such a system can be achieved by an iteration procedure, however, the computations are complicated.

BIBLIOGRAPHY

1. Airy, G. B. Tides and waves, Encyclopedia Metropolitana, Article 192, 1845.

2. Apostol, T. M. Mathematical analysis, Addison-Wesley, Redding, Mass., 1957.

3. Block, H. Sur les equations lineaires aux derivees partielles a caracteristiques multiples, I e II, Arkiv For Mat. Astro. Och Fysik Bd 7, 1912, e III ibidem Bd. 8, 1913.

4. Cattabriga, L. Un problem al contorno per una equation parabolica di ordine dispari, Annah della Scoula Norm. Sup. Pisa. 13, 1959, 163-203.

5. Del Vecchio, E. Sulle equazioni: $\frac{\partial^3 z}{\partial x^3} - \frac{\partial z}{\partial y} + \phi_1(x, y) = 0$, $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^2 z}{\partial y^2} + \phi_2(x, y) = 0$, Memorie della Reale Academia della Scienze di Torino, Serie II, 66, 1955, 1-41.

6. Earnshow. Cambridge Translations, Volume VIII.

7. Friedman, A. Partial differential equation of parabolic type, Prentice-Hall, Englewood Cliffs, N. J., 1964.

8. Gardner, C. S. and G. K. Morikawa. Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves, New York Univ., Courant Inst. Math. Sci., Res. Rep. NYO-9082, 1960.

9. Gel'fand, I. M. and G. E. Šilov. Fourier transforms of rapidly increasing functions and questions of the uniqueness of the solution of Cauchy's problem, Uspehi Mat. Nauk., N.S., 8, No. 6, 58, 1953, 3-54. (Transla. Amer. Math. Soc., 2, 5, 1957, 221-274.)

10. Gerber, P.D. Reimann's problem for the equation $v_t - \frac{1}{3}(v^3)_x = \frac{1}{3}\epsilon^2 v_{xx} - \frac{1}{2}\epsilon(v^2)_{xxx}$, Thesis, New York Univ., 1968.

11. Kortweg, D. J. and G. DeVries. On the change of form of long waves advancing in a rectangular canal and a new type of long stationary waves, Phil. Mag., Vol. 39, 1895, 422-443.

12. Magnus, W., F. Oberhettinger and R. Soni. Formulas and theorems for special functions of mathematical physics, Springer-Verlag, New York, 1966.

13. Mandelbrojt, S. Series de Fourier et classes quasi-analytique de fonction, Gauthier Villars, Paris, 1935.

14. National Bureau of Standards. Handbook of mathematical functions, Washington, D.C., 1962, 446-452.

15. Rayleigh, Lord. On waves, Phil. Mag., Vol. 5, 1, 1876, 257-279. (Scientific Papers, Vol. 1, 251-271.)

16. Russell, S.J. Report on waves, Report to the Fourteenth Meeting (1844) of British Asscn. for Advancement of Science, London, 1845, 311-390.

17. Stoker, J.J. The formation of breakers and bores in shallow water and open channels, Comm. Appl. Math., 1, 1948, 1-87.

18. Täcklind, S. Sur les classes quasianalytiques des solutions des equations aux derivees partielles du type parabolique. Thesis, Uppsala, 1936. Also Nova Acta Regiae Societatis Scientiarum Upsaliensis, Vol. 10, 1936, 1-57.

19. Tychonoff, A. Theoremes d'uniciti pour l'equation de la Chaleur, Math. Sbr., 42, 1935, 199-216.

20. Zabusky, M.J. A synergetic approach to problems of nonlinear dispersive wave propagation and interaction, Academic Press, N.Y., 1967.

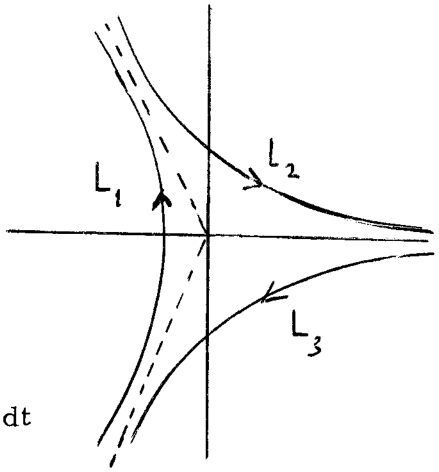
APPENDIX

APPENDIX

The Airy Function

Consider the ordinary differential equation

(A. 1) $$\frac{d^2w}{dz^2} - zw = 0.$$



It can be easily verified that the integral

$$f(z) = \frac{1}{2\pi i}\int e^{tz-\frac{1}{3}t^3}\,dt$$

taken along any of the three paths L_1, L_2, or L_3 provided that the real part of t tends to ∞ at the termini, satisfy the differential Equation (A. 1). We define theAiry function by

(A. 2) $$Ai(z) = \frac{1}{2\pi i}\int_{L_1} e^{tz-\frac{1}{3}t^3}\,dt$$

which for real z, takes the form

(A. 3) $$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(sx+\frac{1}{3}s^2)}\,ds = \frac{1}{\pi}\int_0^{\infty}\cos(sx+\frac{1}{3}s^3)\,ds$$

or

$$\frac{1}{(3a)^{1/3}}\,Ai(\pm\frac{x}{(3a)^{1/3}}) = \frac{1}{\pi}\int_0^{\infty}\cos(sx \pm as^3)\,ds$$

Thus the Airy function $Ai(z)$ is an entire function which satisfies the differential Equation (A.1) and has the integral representation (A.2) and (A.3) for real z. Moreover, it can be shown that

$$\text{(A.4)} \qquad \int_{-\infty}^{0} Ai(t)dt = \frac{2}{3}, \qquad \int_{0}^{\infty} Ai(t)dt = \frac{1}{3}.$$

We also note that, if we let, for $x < 0$

$$\eta = (-x)^{1/4} w(x), \qquad \xi = \frac{2}{3}(-x)^{3/2}$$

then η satisfies

$$\frac{d^2\eta}{d\xi^2} + \left(\frac{5}{12\xi^2} + 1\right)\eta = 0$$

which has the two independent solutions η_1, η_2 where

$$\eta_1 = (c_0 + w_0(\xi))e^{i\xi}, \qquad \eta_2 = (c_1 + w_1(\xi))e^{-i\xi}$$

where

$$w_i(\xi) = O\left(\frac{1}{\xi}\right), \quad i = 0, 1, \quad c_0, c_1 \text{ are positive constants}.$$

Therefore

$$Ai(x) = \frac{1}{(-x)^{1/4}}\left(c_2 \sin\left(\frac{2}{3}(-x)^{3/2}\right) + c_3 \cos\left(\frac{2}{3}(-x)^{3/2}\right)\right) + O((-x)^{-7/4})$$

or,

$$\text{(A.5)} \qquad Ai(x) = \frac{c_4}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} + \theta\right) + O((-x)^{-7/4})$$

for suitably chosen θ. On the other hand, as $x \to \infty$, we can show similarly that

$$\text{(A.6)} \qquad Ai(x) = c_5 e^{-\frac{2}{3}x^{3/2}} + O(x^{-3/2}) \,.$$

Moreover,

$$\text{(A.7)} \qquad A'(x) = -\frac{x^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}} (1+O(x^{-3/2})) \quad \text{as} \quad x \to \infty$$

and

$$\text{(A.8)} \qquad A'(x) = -\frac{x^{1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2}+\theta\right) + O(x^{-5/4}) \quad \text{as} \quad x \to \infty$$