A mathematical model is developed for the analysis of circular cell bulkheads subjected to gravity loads of cell fill, backfill, lateral pressure, and surcharge load using the finite element method. The finite element circular cell model consists of axisymmetric triangular and/or quadrilateral ring elements for soil and cylindrical shell elements for the steel cell. To formulate and derive the governing equations for such elements, the Ritz displacement functions and the theorem of minimum potential energy are used. The loadings of the bulkhead system and corresponding displacements are expanded in Fourier series. The soil fill and foundation are assumed to be elastic, isotropic and non-homogeneous materials. It is assumed that there is no slippage in the soil-steel shell interface.

The element matrices and the Fourier harmonic analysis are verified by comparing results for several structures for which classical solutions are available.
The circular cell bulkhead is analyzed for stresses in the soil elements and shell elements and deformations for the isolated filled cell and backfilled cell cases. Results of the study are presented, discussed and compared with field measurement data obtained by other investigators.
Finite Element Analysis of Circular Cell Bulkheads

by

Likhit Kittisatra

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1976
APPROVED:

Redacted for Privacy

Professor of Civil Engineering in charge of major

Redacted for Privacy

Associate Professor of Civil Engineering

Redacted for Privacy

Head of Department of Civil Engineering

Redacted for Privacy

Dean of Graduate School

Date thesis is presented November 13, 1975

Typed by Clover Redfern for Likhit Kittisatra
ACKNOWLEDGMENT

The author wishes to express sincere appreciation and gratitude to his major professor, Dr. H.I. Laursen, for his guidance, understanding and encouragement during the preparation of this thesis.

Thanks are due to Dr. W.L. Schroeder for providing the author with invaluable guidance and philosophical influence throughout the study.

A special thanks is expressed to his wife, Supaporn, for her patience, encouragement and understanding during the year of this study.

Throughout the author's entire academic career, he has received the most enthusiastic support from his parents and his brother. This thesis is dedicated to them.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Statement and Scope of the Problem</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Method of Solution</td>
<td>4</td>
</tr>
<tr>
<td>II. FINITE ELEMENT FORMULATION</td>
<td>10</td>
</tr>
<tr>
<td>2.1 Triangular Axisymmetric Ring Element Matrix Equations</td>
<td>12</td>
</tr>
<tr>
<td>2.2 Quadrilateral Element Matrix Equations</td>
<td>27</td>
</tr>
<tr>
<td>2.3 Shell Element Matrix Equations</td>
<td>29</td>
</tr>
<tr>
<td>2.4 System Equations and Solution Process</td>
<td>40</td>
</tr>
<tr>
<td>III. TESTING OF COMPUTER PROGRAM</td>
<td>42</td>
</tr>
<tr>
<td>IV. ANALYSIS OF CIRCULAR CELL BULKHEADS</td>
<td>56</td>
</tr>
<tr>
<td>4.1 Isolated Circular Cell</td>
<td>58</td>
</tr>
<tr>
<td>4.2 Circular Cell Bulkhead</td>
<td>58</td>
</tr>
<tr>
<td>V. DISCUSSION OF RESULTS</td>
<td>105</td>
</tr>
<tr>
<td>5.1 Discussion of the Isolated Circular Cell</td>
<td>105</td>
</tr>
<tr>
<td>5.2 Discussion of Circular Cell Bulkhead</td>
<td>108</td>
</tr>
<tr>
<td>VI. SUMMARY AND CONCLUSIONS</td>
<td>118</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>121</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>124</td>
</tr>
<tr>
<td>Appendix A: Element Matrices</td>
<td>124</td>
</tr>
<tr>
<td>Appendix B: Fourier Harmonic Coefficients</td>
<td>133</td>
</tr>
<tr>
<td>Appendix C: Gaussian Quadrature Numerical Integration Procedure</td>
<td>135</td>
</tr>
<tr>
<td>Appendix D: Modulus of Elasticity of Soil</td>
<td>138</td>
</tr>
<tr>
<td>Appendix E: User's Manual for Circular Cell Bulkhead Program</td>
<td>141</td>
</tr>
<tr>
<td>Appendix F: Description of Computer Program</td>
<td>158</td>
</tr>
<tr>
<td>Appendix G: Program Listing</td>
<td>167</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Typical circular cell bulkhead structure.</td>
<td>5</td>
</tr>
<tr>
<td>1.2</td>
<td>Typical triangular soil element and shell element.</td>
<td>7</td>
</tr>
<tr>
<td>1.3</td>
<td>Finite element simulation of a circular cell bulkhead system.</td>
<td>8</td>
</tr>
<tr>
<td>2.1</td>
<td>Triangular axisymmetric element.</td>
<td>13</td>
</tr>
<tr>
<td>2.2</td>
<td>Quadrilateral element.</td>
<td>28</td>
</tr>
<tr>
<td>2.3</td>
<td>Shell element and coordinates.</td>
<td>30</td>
</tr>
<tr>
<td>3.1</td>
<td>Finite element model of rigid base soil system.</td>
<td>44</td>
</tr>
<tr>
<td>3.2</td>
<td>Vertical stress in soil due to surface load.</td>
<td>45</td>
</tr>
<tr>
<td>3.3</td>
<td>Radial stress in soil due to surface load.</td>
<td>46</td>
</tr>
<tr>
<td>3.4</td>
<td>Tangential stress in soil due to surface load.</td>
<td>47</td>
</tr>
<tr>
<td>3.5</td>
<td>Shear stress in soil due to surface load.</td>
<td>48</td>
</tr>
<tr>
<td>3.6</td>
<td>Finite element model of circular tank.</td>
<td>49</td>
</tr>
<tr>
<td>3.7</td>
<td>Radial displacement of circular tank.</td>
<td>49</td>
</tr>
<tr>
<td>3.8</td>
<td>Hoop force and longitudinal moment of circular tank.</td>
<td>50</td>
</tr>
<tr>
<td>3.9</td>
<td>Edge load applied to the cooling tower shell.</td>
<td>51</td>
</tr>
<tr>
<td>3.10</td>
<td>Approximate loading diagram for cooling tower.</td>
<td>52</td>
</tr>
<tr>
<td>3.11</td>
<td>Cooling tower stress resultants.</td>
<td>53</td>
</tr>
<tr>
<td>3.12</td>
<td>Radial and axial displacements for cooling tower at $\theta = 0^\circ$.</td>
<td>54</td>
</tr>
<tr>
<td>3.13</td>
<td>Radial and axial displacements for cooling tower at $\theta = 22.5^\circ$.</td>
<td>55</td>
</tr>
<tr>
<td>4.1</td>
<td>An isolated circular cell.</td>
<td>59</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.2.</td>
<td>A circular cell bulkhead.</td>
<td>60</td>
</tr>
<tr>
<td>4.3.</td>
<td>Circular cell finite element model.</td>
<td>61</td>
</tr>
<tr>
<td>4.4.</td>
<td>Radial displacement, hoop force and sheet stresses due to gravity load of cell fill (axisymmetric loading case).</td>
<td>62</td>
</tr>
<tr>
<td>4.5.</td>
<td>Vertical displacement of circular cell due to gravity load of cell fill.</td>
<td>63</td>
</tr>
<tr>
<td>4.6.</td>
<td>Contours of vertical stress in soil due to gravity load of cell fill.</td>
<td>64</td>
</tr>
<tr>
<td>4.7.</td>
<td>Contours of radial stress in soil due to gravity load of cell fill.</td>
<td>65</td>
</tr>
<tr>
<td>4.8.</td>
<td>Contours of circumferential stress in soil due to gravity load of cell fill.</td>
<td>66</td>
</tr>
<tr>
<td>4.9.</td>
<td>Contours of shear stress in soil due to gravity load of cell fill.</td>
<td>67</td>
</tr>
<tr>
<td>4.10.</td>
<td>Principal stresses in soil due to gravity load of cell fill.</td>
<td>68</td>
</tr>
<tr>
<td>4.11.</td>
<td>Equivalent loading diagrams due to backfill.</td>
<td>69</td>
</tr>
<tr>
<td>4.12.</td>
<td>Assumed arc tension loads.</td>
<td>71</td>
</tr>
<tr>
<td>4.13.</td>
<td>Fourier harmonic loading expansions for arc tension load $T_a$ (Case I).</td>
<td>72</td>
</tr>
<tr>
<td>4.14.</td>
<td>Fourier harmonic loading expansions for uniform distributed pressures $p_a$ and $q_b$ (Case II).</td>
<td>73</td>
</tr>
<tr>
<td>4.15.</td>
<td>Reference diagram for displacements and stresses.</td>
<td>76</td>
</tr>
<tr>
<td>4.16.</td>
<td>Horizontal displacements of cell due to arc tension loads at levels H5 and top of cell (Case I).</td>
<td>77</td>
</tr>
<tr>
<td>4.17.</td>
<td>Horizontal displacements of cell due to arc tension loadings at levels H6 and H8 (Case I).</td>
<td>78</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.18.</td>
<td>Horizontal displacements of backfilled cell at levels H5 and the top of cell.</td>
<td>79</td>
</tr>
<tr>
<td>4.19.</td>
<td>Horizontal displacements of backfilled cell at levels H6 and H8.</td>
<td>80</td>
</tr>
<tr>
<td>4.20.</td>
<td>Radial displacement of backfilled cell at $\theta = 0$ and 180 degrees.</td>
<td>81</td>
</tr>
<tr>
<td>4.21.</td>
<td>Radial displacement of backfilled cell at $\theta = 30$ and 150 degrees.</td>
<td>82</td>
</tr>
<tr>
<td>4.22.</td>
<td>Radial displacements of backfilled cell at $\theta = 60$ and 120 degrees.</td>
<td>83</td>
</tr>
<tr>
<td>4.23.</td>
<td>Radial displacement of cell at $\theta = 90$ and 270 degrees.</td>
<td>84</td>
</tr>
<tr>
<td>4.24.</td>
<td>Deformed shape of circular cell bulkhead.</td>
<td>85</td>
</tr>
<tr>
<td>4.25.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 0$ degree.</td>
<td>86</td>
</tr>
<tr>
<td>4.26.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 30$ degrees.</td>
<td>87</td>
</tr>
<tr>
<td>4.27.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 60$ degrees.</td>
<td>88</td>
</tr>
<tr>
<td>4.28.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 90$ degrees.</td>
<td>89</td>
</tr>
<tr>
<td>4.29.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 120$ degrees.</td>
<td>90</td>
</tr>
<tr>
<td>4.30.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 150$ degrees.</td>
<td>91</td>
</tr>
<tr>
<td>4.31.</td>
<td>Vertical and hoop stresses of steel cell at $\theta = 180$ degrees.</td>
<td>92</td>
</tr>
<tr>
<td>4.32.</td>
<td>Hoop force in cell vs. position.</td>
<td>93</td>
</tr>
<tr>
<td>4.33.</td>
<td>Shearing stress in cell vs. position.</td>
<td>94</td>
</tr>
<tr>
<td>4.34.</td>
<td>Contours of vertical stress in soil.</td>
<td>95</td>
</tr>
<tr>
<td>4.35.</td>
<td>Contours of radial stress in soil.</td>
<td>96</td>
</tr>
<tr>
<td>4.36.</td>
<td>Contours of shear stress ($\tau_{rZ}$) in soil.</td>
<td>97</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.37.</td>
<td>Contours of circumferential stress in soil.</td>
<td>98</td>
</tr>
<tr>
<td>4.38.</td>
<td>Radial and vertical stresses in soil vs. depth of cell.</td>
<td>99</td>
</tr>
<tr>
<td>4.39.</td>
<td>Radial and vertical stresses in soil vs. depth of cell.</td>
<td>100</td>
</tr>
<tr>
<td>4.40.</td>
<td>Coefficients of lateral earth pressure inside the cell fill.</td>
<td>101</td>
</tr>
<tr>
<td>4.41.</td>
<td>Comparison of radial deformation of backfilled cell at level H5.</td>
<td>102</td>
</tr>
<tr>
<td>4.42.</td>
<td>Settlement of the top of the steel sheet piles vs. position.</td>
<td>103</td>
</tr>
<tr>
<td>4.43.</td>
<td>Comparisons of hoop force in steel cell.</td>
<td>104</td>
</tr>
</tbody>
</table>

**Appendix**

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.1.</td>
<td>Example of finite element mesh showing node and element number scheme.</td>
<td>143</td>
</tr>
<tr>
<td>E.2.</td>
<td>Boundary pressure sign convention.</td>
<td>152</td>
</tr>
<tr>
<td>F.1.</td>
<td>Flow diagram for circular cell bulkhead analysis.</td>
<td>159</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1. Mechanical properties of soil.</td>
<td>57</td>
</tr>
<tr>
<td>5.1. Comparisons of results.</td>
<td>114</td>
</tr>
</tbody>
</table>

**Appendix**

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1. Fourier force coefficients.</td>
<td>134</td>
</tr>
<tr>
<td>C.1. Gaussian weighting functions and stations.</td>
<td>137</td>
</tr>
<tr>
<td>D.1. Modulus of elasticity of soil.</td>
<td>140</td>
</tr>
<tr>
<td>Term</td>
<td>Description</td>
</tr>
<tr>
<td>-----------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>A</td>
<td>Cross sectional area of triangular element</td>
</tr>
<tr>
<td>( A )</td>
<td>Generalized displacement coordinate in soil</td>
</tr>
<tr>
<td>( a_n )</td>
<td>Fourier coefficient for harmonic number ( n )</td>
</tr>
<tr>
<td>( a_r )</td>
<td>Radial acceleration</td>
</tr>
<tr>
<td>( a_Z )</td>
<td>Axial acceleration</td>
</tr>
<tr>
<td>( B )</td>
<td>Generalized displacement coordinate in shell</td>
</tr>
<tr>
<td>C</td>
<td>Shell contour (circle)</td>
</tr>
<tr>
<td>( [C] ), ([C_{ijkl}])</td>
<td>Material constant tensor field</td>
</tr>
<tr>
<td>D</td>
<td>Diameter of cell</td>
</tr>
<tr>
<td>d</td>
<td>Distance from the mid-plane to the surface of shell</td>
</tr>
<tr>
<td>E</td>
<td>Young's modulus</td>
</tr>
<tr>
<td>e</td>
<td>Void ratio</td>
</tr>
<tr>
<td>{F}</td>
<td>System load vector</td>
</tr>
<tr>
<td>{f}</td>
<td>Element load vector</td>
</tr>
<tr>
<td>{F_b}, {f_b}</td>
<td>Body force vector field</td>
</tr>
<tr>
<td>{F_s}, {f_s}</td>
<td>Surface traction force vector field</td>
</tr>
<tr>
<td>G</td>
<td>Shear modulus of elasticity</td>
</tr>
<tr>
<td>( G_s )</td>
<td>Specific gravity</td>
</tr>
<tr>
<td>H</td>
<td>Height of cell above dredge line</td>
</tr>
<tr>
<td>( H_d )</td>
<td>Embedment depth of cell</td>
</tr>
<tr>
<td>h</td>
<td>Depth of soil above point</td>
</tr>
</tbody>
</table>
\( I_i \) Volume integrals

\( K \) Coefficient of lateral earth pressure

\( K_a \) Rankine coefficient of active earth pressure

\([K]\) System stiffness matrix

\([k]\) Element stiffness matrix

\( l \) Length of shell element

\( M_{ij} \) Shell moment tensor field

\( M_{zz} \) Longitudinal moment in shell

\( M_{\theta\theta} \) Hoop moment in shell

\( M_{z\theta} \) Twisting moment in shell

\( N_{ij} \) Shell stress resultant tensor field

\( N_{zz} \) Longitudinal force in shell

\( N_{\theta\theta} \) Hoop force in shell

\( N_{z\theta} \) Shearing force in shell in \( z-\theta \) plane

\( n \) Harmonic number

\( p_a \) Lateral earth pressure due to backfill

\( p_i \) Prescribed surface traction on surface \( S \)

\( q_b \) Vertical pressure due to backfill

\( r \) Global radial coordinate (in cylindrical coordinate system)

\( r_a \) Radius of connecting arc

\( r_c \) Radius of circular cell

\( S \) Portion of surface on which stresses are prescribed

\([S]\) Stiffness matrix of soil element in generalized coordinates
[SS] Stiffness matrix of shell element in generalized coordinates

\( T_a \) Tension in connecting arc per unit length
\( t \) Shell thickness
\( U \) Strain energy
\( \{U\} \) Nodal point displacement for entire structure
\( \{u\} \) Displacement vector field
\( u \) Radial displacement
\( V \) Volume of the soil (solid) element
\( v \) Circumferential (tangential) displacement
\( W \) Strain energy density
\( w \) Axial displacement
\( Z \) Global axial coordinate (in cylindrical coordinate system)
\( z \) Local longitudinal coordinate of shell
\( \alpha \) Rotation of the shell surface in longitudinal axis
\( \beta \) Transverse shell coordinate
\( \gamma \) Total unit weight of soil
\( \gamma' \) Buoyant unit weight of soil
\( \gamma_d \) Dry unit weight of soil
\( \gamma_w \) Unit weight of water
\( \gamma_{rZ} \) Shearing strain in \( r-Z \) plane
\( \gamma_{r\theta} \) Shearing strain in \( r-\theta \) plane
\( \gamma_{Z\theta} \) Shearing strain in \( Z-\theta \) plane
\( \delta \) Variation symbol
\( \epsilon, \{\epsilon_{ij}\} \) Strain tensor field

\( \epsilon_{rr}, \epsilon_{ZZ}, \epsilon_{\theta \theta} \) Radial, vertical and circumferential strains, respectively

\( \sigma, \{\sigma_{ij}\} \) Stress tensor field

\( \sigma_0 \) Average confining pressure

\( \sigma_{rr}, \sigma_{ZZ}, \sigma_{\theta \theta} \) Radial, vertical and circumferential stresses, respectively

\( \sigma_{Zi}, \sigma_{Zo} \) Vertical stresses in shell on inside surface and outside surface, respectively

\( \sigma_{\theta i}, \sigma_{\theta o} \) Hoop stresses in shell on inside surface and outside surface, respectively

\( \tau_{rZ} \) Shearing stress in \( r-Z \) plane

\( \tau_{r\theta} \) Shearing stress in \( r-\theta \) plane

\( \tau_{Z\theta} \) Shearing stress in \( Z-\theta \) plane

\( \chi_{ij} \) Shell curvature change tensor field

\( \chi_{zz} \) Shell curvature change in longitudinal direction

\( \chi_{\theta \theta} \) Shell curvature change in hoop direction

\( \chi_{z\theta} \) Shell curvature change due to twisting moment

\( \nu \) Poisson's ratio

\( \rho \) Density

\( \omega \) Angular velocity

\( [\lambda_0] \) Shell coordinate transformation matrix

\( [\hat{\phi}], \{\phi\} \) Displacement expansion tensor field
\[ [\Phi] \quad \text{Strain generalized coordinate tensor field} \]

\[ \phi \quad \text{Inclination angle between local and global coordinates of shell element} \]

\[ \phi \quad \text{Angle of internal friction of soil} \]

\[ [\phi_0], [\phi_0] \quad \text{Displacement transformation matrix} \]

\[ \pi_p \quad \text{Total potential energy of the entire system} \]

\[ \pi_{pe} \quad \text{Total potential energy of element } e \]

\[ [\varphi_0], [\varphi_{00}] \quad \text{Shell displacement transformation matrix} \]

\[ \theta \quad \text{Circumferential coordinate (in cylindrical coordinate system)} \]
A cellular bulkhead is a waterfront retaining structure formed from a series of interconnected straight web steel sheet pile cells and filled with soil, usually sand or sand and gravel. The combination of steel and soil fill, which individually are unstable, forms a stable unit offering resistance to its own gravity loads, lateral loads of water and earth and surcharge loads. Current cellular bulkhead design methods have adapted methods used for cellular cofferdams which are still essentially empirical. Although various theories have been suggested to derive analytical solutions for the stresses in the cell, so far most designers in this field still rely heavily on past practice and experience.

Terzaghi (23) proposed an important method for design. In using this method, a cellular cofferdam on a rock foundation is first considered a rigid gravity structure. He proposed essentially a simple bending theory in which a linear distribution of normal stress on the base of the cofferdam is assumed and lateral pressures on vertical shearing planes are obtained from an assumed coefficient of earth pressure. The shearing stress in the fill and friction in the interlocks was considered as a critical factor in cell stability.
Terzaghi also included other possibilities of failure: interlock pullout, sliding on the base, and foundation bearing capacity failures.

The Tennessee Valley Authority (22) follows the same general method of design as proposed by Terzaghi with some modifications. In the TVA method it is assumed that the maximum lateral pressure for interlock design occurs at a point one-fourth of the exposed height of the cell above the dredge line. The full value of pressure at the dredge line is not used because of the restraint provided by embedment of the sheet piles.

Cummings (6) has proposed a method of cellular cofferdam analysis known as the interior sliding theory where the resistance of a cell to failure by tilting is gained largely through horizontal shear in the cell fill. Cummings' conclusions are based on model tests which indicate that the plane of rupture goes from the top of the pressure side to the bottom inner corner (toe of cofferdam). The cell fill in the rupture region acts essentially as a surcharge and only the soil below the failure plane will develop shear resistance. The Cummings' method for determining interlock tension and sliding on the base is the same as the Terzaghi method.

1.1 Statement and Scope of the Problem

Analyzing the soil-steel sheet pile interaction of the cellular bulkhead problem is difficult because the structure consists of two
very different materials. In addition, the bulkhead system is subjected to poorly defined non-axisymmetrical loading and the boundary conditions of the system are complicated.

The objective of this research is to develop a rational approach for stress analysis of circular cell bulkheads and to compare analytical results with data obtained in full scale field studies by other investigations. To do so would not only provide an aid for predicting the most likely mode of failure, but would also provide an insight to the elastic stress distribution throughout the structure.

To deal with the circular cell bulkhead problem, certain assumptions have been made. Specifically, they are as follows:

1) The soil is elastic and isotropic.

2) The thickness of steel sheet piles is uniform throughout the whole structure and no slippage occurs along the interlocks.

3) The soil-structure interface is perfectly rough with no possibility for slip.

4) The actual continuous circular bulkhead system can be represented by a single circular cell which is considered as an axisymmetric structure subjected to non-axisymmetrical loading.

5) At some finite distance beneath the bottom edge of the sheet piles, the soil foundation is rigid and rough.
The scope of the study includes development of a method for analyzing circular cell bulkhead problems and the development of a computer program to make the necessary calculations. Two load conditions are studied:

1) the gravity load of the cell fill and

2) external loading due to backfill.

These loading conditions are considered as axisymmetrical and non-axisymmetrical loads, respectively.

1.2 Method of Solution

A typical circular cell bulkhead is shown in Figure 1.1 wherein the structure is exposed to water on the river side and soil fills on the shore side. The system is assumed an axisymmetric structure subjected to non-axisymmetrical loading. The loading, however, is symmetric about a vertical plane containing the axis of the cell.

Because the bulkhead system is not a continuum, an assumption concerning the characteristics of the soil-sheet pile interface is needed. It is assumed that the interface is perfectly rough, with no possibility for slip between the soil and the sheet pile. Also, the cell itself is assumed to be a continuous circular cylindrical shell. The structure is thought of as being composed of two substructures: solid soil and the cylindrical steel shell enclosing it.
Figure 1.1. Typical circular cell bulkhead structure.
The basic concept of the finite element method is the idealization of the actual continuum as an assemblage of discrete structural elements, interconnected at a finite number of joints or nodal points. For structures that are physically axisymmetric, i.e., geometrically axisymmetric and possessing material properties that are axisymmetric, the nodes are actually circles and are called nodal circles. Figure 1.2 illustrates a typical triangular element and a shell element. The generalized displacements and loadings of a nodal circle can be expressed in terms of finite Fourier-series and the problem is uncoupled in each harmonic (27).

For each harmonic the system stiffness $[K_n]$ and system load vector $\{F_n\}$ are formed by summing appropriately the stiffness $[k_n]$ and force $\{f_n\}$ for the discrete elements of the structure, where subscript $n$ denotes the $n$th harmonic. After the system stiffness and load vectors are obtained, the structure is analyzed by the standard stiffness method (14). Since the bulkhead system is assumed as linear and elastic, the principle of superposition allows a solution for the specified load by simply adding the separate solutions that are obtained from the separate Fourier harmonic terms of separate load components.

The circular cell bulkhead can be represented by shell elements and quadrilateral elements as shown in Figure 1.3. Boundary 1 is assumed to be a firm foundation and can be considered fixed. Boundary
Figure 1.2. Typical triangular soil element and shell element.
Figure 1.3. Finite element simulation of a circular cell bulkhead system.
2 is assumed to be far from the axis of symmetry OZ and beyond the zone of failure. Therefore, boundary 2 can be assumed to be on rollers.

The finite element analyses yield the distribution of stresses in the soil elements, the steel sheet stresses and nodal point displacements.
II. FINITE ELEMENT FORMULATION

The basic finite elements used in this study are quadrilateral axisymmetric ring elements for the soil and cylindrical shell-of-revolution elements for the steel sheet pile with constant cross-section. The quadrilateral element is composed of four sub-triangles. To formulate such elements, the Theorem of Minimum Potential Energy is used.

To apply the minimum potential energy theorem it is necessary to assume a displacement field in terms of a set of unknown Ritz parameters (coordinate functions) that satisfy the hypotheses of the theorem (5). The restrictions on assumed displacement functions are that they be continuous over the entire body and possess piecewise continuous first partial derivatives. The method of analysis is to subdivide the domain into an assemblage of discrete elements and assume appropriate kinematic functions within each element such that the compatibility conditions across the element interfaces are satisfied.

The contribution to the total potential energy of one element can be written as

$$\pi_{pe} = U - \iiint_V f_i u_i dV - \iint_S p_i u_i dS \quad (2.1)$$
where

\[ U = \iiint \nabla W dV \]

\( U \) is the element strain energy and \( W \) is the strain energy density which is assumed to be positive definite. The body force vector field \( \mathbf{f} \) and the prescribed surface traction \( \mathbf{p} \) on the portion of the surface \( S \) are positive if they act in the direction of positive coordinate axes. \( \mathbf{u} \) is the displacement vector field in the volume \( V \).

The strain energy density can be written as

\[ W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \quad (2.2) \]

where \( \sigma_{ij} \) is the stress tensor field and \( \varepsilon_{ij} \) is the strain tensor field.

The total potential energy of the entire system is the sum of the potentials of the individual elements. Thus, for a system of \( L \) elements

\[ \pi_p = \sum_{e=1}^{L} \pi_{pe} \quad (2.3) \]

An absolute minimum potential energy is sought by taking the variation of the potential energy function with respect to the discrete displacement variables and set it equal to zero.
\[ \delta_{\pi_k} = 0 \]  \hspace{1cm} (2.4)

Then a set of matrix equations are formed accordingly.

### 2.1 Triangular Axisymmetric Ring Element Matrix Equations

A triangular axisymmetric ring element is used in the study as shown in Figure 2.1. Matrix notation and cylindrical coordinates are used in the analysis, that is, radial distance \( r \), axial distance \( Z \), and circumferential angle \( \theta \). The right handed system is used in the coordinate system. Since the soil is assumed a linear elastic material, the stress-strain relationship may be expressed in the form of the constitutive equation (7)

\[
\sigma_{ij} = C_{ijkl} \epsilon_{kl} \]  \hspace{1cm} (2.5a)

or, rewritten in matrix form

\[
\{\sigma\} = [C]\{\epsilon\} \]  \hspace{1cm} (2.5b)

where \( C_{ijkl} \) is a fourth-order material constant tensor.

If Equations (2.2) and (2.5b) are substituted into Equation (2.1), the element potential energy becomes
Figure 2.1. Triangular axisymmetric element.
\[ \pi_{pe} = \iiint_V \frac{1}{2} \{\epsilon(r, Z, \theta)\}^T [C]^T \{\epsilon(r, Z, \theta)\} dV \]

\[ - \iiint_V \{u(r, Z, \theta)\}^T \{f(r, Z, \theta)\} dV \]

\[ - \int_S \{u(r, Z, \theta)\}^T \{p(r, Z, \theta)\} dS \quad (2.6) \]

in which

\{ \} denotes a column or a row matrix \\
[ ] denotes a rectangular or square matrix

The superscript \( T \) denotes the transpose of the matrix.

For an isotropic material with Young's modulus \( E \) and Poisson's ratio \( v \), the constant \( C \) will be

\[
[C] = [C]^T = \frac{E}{(1+v)(1-2v)} \begin{bmatrix}
1-v & v & v & 0 & 0 & 0 \\
v & 1-v & v & 0 & 0 & 0 \\
v & v & 1-v & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2}
\end{bmatrix} \quad (2.7)
\]

The displacements within each triangular element are assumed to be a linear function of the coordinates \( r \) and \( Z \) \((27)\).
\[
\begin{align*}
\bar{u}_n &= A_{1n} + A_{2n}r + A_{3n}Z \\
\bar{w}_n &= A_{4n} + A_{5n}r + A_{6n}Z \\
\bar{v}_n &= A_{7n} + A_{8n}r + A_{9n}Z
\end{align*}
\] (2.8a)

(2.8b)

(2.8c)

where \( \bar{u}, \bar{w}, \) and \( \bar{v} \) are radial, axial and circumferential displacement components, respectively. Subscript \( n \) is an integer called the harmonic number of the Fourier series. \( A's \) are constant coefficients that represent the generalized displacement coordinates of the element.

In general cases, an axisymmetric structure subjected to arbitrary loadings, displacements and the body and surface loads are represented by Fourier series as follows:

\[
\begin{align*}
\bar{u}(r, Z, \theta) &= \sum_{n=0}^{N} \bar{u}_n(r, Z) \cos n\theta + \sum_{n=1}^{N} \bar{u}_n(r, Z) \sin n\theta \\
\bar{w}(r, Z, \theta) &= \sum_{n=0}^{N} \bar{w}_n(r, Z) \cos n\theta + \sum_{n=1}^{N} \bar{w}_n(r, Z) \sin n\theta \\
\bar{v}(r, Z, \theta) &= \sum_{n=1}^{N} \bar{v}_n(r, Z) \sin n\theta + \sum_{n=0}^{N} \bar{v}_n(r, Z) \cos n\theta
\end{align*}
\] (2.9a)

(2.9b)

(2.9c)
body forces are:

\[ f_r(r, z, \theta) = \sum_{n=0}^{N} f_{rn}(r, z) \cos n\theta + \sum_{n=1}^{N} f_{rn}(r, z) \sin n\theta \]  
\[ f_z(r, z, \theta) = \sum_{n=0}^{N} f_{zn}(r, z) \cos n\theta + \sum_{n=1}^{N} f_{zn}(r, z) \sin n\theta \]  
\[ f_\theta(r, z, \theta) = \sum_{n=1}^{N} f_{\theta n}(r, z) \sin n\theta + \sum_{n=0}^{N} f_{\theta n}(r, z) \cos n\theta \]

and surface loads are:

\[ p_r(r, z, \theta) = \sum_{n=0}^{N} p_{rn}(r, z) \cos n\theta + \sum_{n=1}^{N} p_{rn}(r, z) \sin n\theta \]  
\[ p_z(r, z, \theta) = \sum_{n=0}^{N} p_{zn}(r, z) \cos n\theta + \sum_{n=1}^{N} p_{zn}(r, z) \sin n\theta \]  
\[ p_\theta(r, z, \theta) = \sum_{n=1}^{N} p_{\theta n}(r, z) \sin n\theta + \sum_{n=0}^{N} p_{\theta n}(r, z) \cos n\theta \]

where the single barred \((\overline{\cdot})\) and double barred \((\overline{\overline{\cdot}})\) quantities represent functions of \(r, z\) and \(n\) only but not \(\theta\).

Since the circular cell bulkhead system as shown in Figure 1.1 has symmetric loading with respect to the \(rZ\) plane at \(\theta = 0\), the double barred series in Equations (2.9), (2.10) and (2.11) will not be
used in this study. The axially symmetric case is represented by use of only the $n = 0$ term of the single barred series.

Substituting Equations (2.8) into Equations (2.9), the displacements become

\[
\begin{bmatrix}
  u(r, Z, \theta) \\
  w(r, Z, \theta) \\
  v(r, Z, \theta)
\end{bmatrix}
\]

Substituting Equations (2.8) into Equations (2.9), the displacements become

\[
\begin{bmatrix}
  u(r, Z, \theta) \\
  w(r, Z, \theta) \\
  v(r, Z, \theta)
\end{bmatrix}
= \sum_{n=0}^{N} \begin{bmatrix}
  \phi(r, Z)^T \cos n\theta & 0 & 0 \\
  0 & \phi(r, Z)^T \cos n\theta & 0 \\
  0 & 0 & \phi(r, Z)^T \sin n\theta
\end{bmatrix} \begin{bmatrix}
  A_{1n} \\
  A_{2n} \\
  \vdots \\
  A_{9n}
\end{bmatrix}
\]

(2.12)

where

\[
\phi(r, Z)^T = \begin{bmatrix} 1 & r & Z \end{bmatrix}
\]

Equation (2.12) can be written symbolically as

\[
\begin{bmatrix}
  u(r, Z, \theta)
\end{bmatrix} = \sum_{n=1}^{N} [\phi(r, Z, \theta)]A_{in}
\]

(2.13)

Thus the displacement vector field for each Fourier term may be written as

\[
\begin{bmatrix}
  u_n(r, Z)
\end{bmatrix} = [\phi(r, Z)]A_{in}
\]

(2.14)
Let \( u_j \), \( w_j \), and \( v_j \) refer to the \( r \), \( Z \) and \( \theta \) direction displacement components of any corner node \( j \) of the element as shown in Figure 2.1. Using the displacement boundary conditions for \( u \) at each corner, the following matrix expression is obtained:

\[
\begin{bmatrix}
u_{1n} \\
u_{2n} \\
u_{3n}
\end{bmatrix} =
\begin{bmatrix}
1 & r_1 & Z_1 \\
1 & r_2 & Z_2 \\
1 & r_3 & Z_3
\end{bmatrix}
\begin{bmatrix}
A_{1n} \\
A_{2n} \\
A_{3n}
\end{bmatrix}
\] (2.15a)

Similarly, the displacements \( w \) and \( v \) can be expressed in terms of generalized coordinates \( A_{in} \) as follows:

\[
\begin{bmatrix}
w_{1n} \\
w_{2n} \\
w_{3n}
\end{bmatrix} =
\begin{bmatrix}
1 & r_1 & Z_1 \\
1 & r_2 & Z_2 \\
1 & r_3 & Z_3
\end{bmatrix}
\begin{bmatrix}
A_{4n} \\
A_{5n} \\
A_{6n}
\end{bmatrix}
\] (2.15b)

and

\[
\begin{bmatrix}
v_{1n} \\
v_{2n} \\
v_{3n}
\end{bmatrix} =
\begin{bmatrix}
1 & r_1 & Z_1 \\
1 & r_2 & Z_2 \\
1 & r_3 & Z_3
\end{bmatrix}
\begin{bmatrix}
A_{7n} \\
A_{8n} \\
A_{9n}
\end{bmatrix}
\] (2.15c)

Equations (2.15) can be combined together and written in symbolic form as

\[
\{u_{0n}\} = [\Phi_0]\{A_{in}\}
\] (2.16)
where \( u_{0n} \) is nodal displacement and \( \Phi_0 \) is as shown in Appendix A.

The generalized coordinates \( A_{in} \) are obtained by inversion of Equation (2.16) and expressed in terms of nodal point displacements as

\[
\{A_{in}\} = [\Phi_0^{-1}]\{u_{0n}\} \tag{2.17}
\]

\( \Phi_0^{-1} \) is shown in Appendix A.

If Equation (2.17) is substituted into Equation (2.14), the displacements at any point in the triangular element can be written in terms of the nodal displacements as

\[
\begin{bmatrix}
  u \\
  w \\
  v
\end{bmatrix} = \frac{1}{2A} \begin{bmatrix}
  n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 & 0 \\
  0 & n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 \\
  0 & 0 & n_1 & 0 & 0 & n_2 & 0 & 0 & n_3
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  w_1 \\
  v_1 \\
  u_2 \\
  w_2 \\
  v_2 \\
  u_3 \\
  w_3 \\
  v_3
\end{bmatrix} \tag{2.18}
\]

where

\[
\begin{align*}
  n_1 &= a_1 + b_1 r + c_1 z \\
  n_2 &= a_2 + b_2 r + c_2 z \\
  n_3 &= a_3 + b_3 r + c_3 z \tag{2.19a}
\end{align*}
\]
\[ a_1 = r_2 Z_3 - r_3 Z_2, \quad b_1 = Z_2 - Z_3, \quad c_1 = r_3 - r_2 \]
\[ a_2 = r_3 Z_1 - r_1 Z_3, \quad b_2 = Z_3 - Z_1, \quad c_2 = r_1 - r_3 \]  
\[ a_3 = r_1 Z_2 - r_2 Z_1, \quad b_3 = Z_1 - Z_2, \quad c_3 = r_2 - r_1 \]

and \( A \) is the cross-sectional area of the triangular element.

The element strains are obtained by differentiating Equation (2.18) to obtain (15)

\[ \epsilon_{rr} = \frac{\partial u}{\partial r} \]
\[ \epsilon_{Z Z} = \frac{\partial w}{\partial Z} \]
\[ \epsilon_{\theta \theta} = \frac{1}{r} \left( \frac{\partial v}{\partial \theta} + u \right) \]
\[ \gamma_{r Z} = 2 \epsilon_{r Z} = \frac{\partial u}{\partial Z} + \frac{\partial w}{\partial r} \]
\[ \gamma_{r \theta} = 2 \epsilon_{r \theta} = \frac{1}{r} \left( \frac{\partial u}{\partial \theta} - v \right) + \frac{\partial v}{\partial r} \]
\[ \gamma_{Z \theta} = 2 \epsilon_{Z \theta} = \frac{\partial v}{\partial Z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \]

The strain tensor can be expressed in terms of generalized coordinates

\[
\{\epsilon(r, Z, \theta)\} = \sum_{n=0}^{N} \left[ \Phi_n^r (r, Z, \theta) \right] \{ A_{in} \} \quad (2.21a)
\]

where \( \Phi_n^r \) is shown in Appendix A.

If Equation (2.17) is substituted into Equation (2.21a) a relation between element strain tensors and the nodal displacements is
obtained

\[ \{ \epsilon(r, Z, \theta) \} = \sum_{n=0}^{N} \left[ \Phi_n(r, Z, \theta) \{ \Phi_0^{-1} \} \right] \{ u_{0n} \} \]  \hspace{1cm} (2.21b)

Substituting Equation (2.21b) into Equation (2.6), the total potential energy of a single triangular element can be written in terms of its nodal displacements

\[
\pi_{pe} = \sum_{n=0}^{N} \sum_{m=0}^{M} \left( \int_{V} \left\{ \frac{1}{2} \{ u_{0n} \}^T \left[ \Phi_0^{-1} \right]^T \left[ \Phi_n(r, Z, \theta) \right]^T \left[ C \right] \left[ \Phi_m(r, Z, \theta) \right] \left[ \Phi_0^{-1} \right] \{ u_{0m} \} \right. \right.

- \left. \{ u_{0n} \}^T \left[ \Phi_0^{-1} \right]^T \left[ \Phi_n(r, Z, \theta) \right]^T \{ f_m(r, Z, \theta) \} \right) dV

- \int_{S} \left\{ \{ u_{0n} \}^T \left[ \Phi_0^{-1} \right]^T \left[ \Phi_n(r, Z, \theta) \right]^T \{ p_m(r, Z, \theta) \} \right) dS \right) \]

(2.22)

where \( n \) and \( m \) are harmonic numbers.

Since the \( \theta \)-dependence in the integrals in Equation (2.22) is known explicitly from variables \( r \) and \( Z \), it can be carried out directly. At the same time by making use of the orthogonal properties of the harmonic functions the integrals are integrated from \( \theta = -\pi \) to \( \theta = \pi \). These integrals are a well-known type in Fourier analysis. The general forms are stated as follows:
\[
\int_{-\pi}^{\pi} \sin m\theta \sin n\theta d\theta = \begin{cases} 
\pi & \text{for } m = n \neq 0 \\
0 & \text{for } m \neq n \text{ and for } m = n = 0 
\end{cases}
\]

\[
\int_{-\pi}^{\pi} \cos m\theta \cos n\theta d\theta = \begin{cases} 
2\pi & \text{for } m = n = 0 \\
\pi & \text{for } m = n \neq 0 \\
0 & \text{for } m \neq n 
\end{cases} \quad (2.23)
\]

\[
\int_{-\pi}^{\pi} \sin m\theta \cos n\theta d\theta = 0 \quad \text{for all } m \text{ and } n
\]

Thus, the sum in Equation (2.22) exists only for \( m = n \) and it becomes

\[
\pi \text{pe} = \sum_{n=0}^{N} \left( \iint_{V} \left( \frac{1}{2} \{u_{0n}\}^T [\Phi^{-1}_0]^T [\Phi_n (r, Z)]^T [C] [\Phi_n (r, Z)] [\Phi^{-1}_0] \{u_{0n}\} ight. 
\right. \\
- \{u_{0n}\}^T [\Phi^{-1}_0]^T [\Phi_n (r, Z)]^T \{f_n (r, Z)\} dV \\
\left. \left. - \iint_{S} \{u_{0n}\}^T [\Phi^{-1}_0]^T [\Phi_n (r, Z)]^T \{p_n (r, Z)\} dS \right) \quad (2.24)
\]

Let

\[
\{f_{bn} (r, Z)\} = [\Phi_n (r, Z)]^T \{f_n (r, Z)\} \quad (2.25a)
\]

\[
\{f_{sn} (r, Z)\} = [\Phi_n (r, Z)]^T \{p_n (r, Z)\} \quad (2.25b)
\]

and

\[
\{s_n (r, Z)\} = [\Phi_n (r, Z)]^T [C] [\Phi_n (r, Z)] \quad (2.25c)
\]

Substituting Equation (2.25) into Equation (2.24) we obtain
The body force vector will be (21)

\[
\{f(r, Z, \theta)\}^T = \{\rho w^2 - \rho a_r \cos \theta, -\rho a_Z, -\rho a_r \sin \theta\}
\]  

(2.27a)

where \(\rho\) is the mass density of the body, \(\omega\) is the angular velocity, and \(a_r\) and \(a_Z\) are radial and axial acceleration, respectively.

After substituting Equation (2.27a) into Equation (2.25a) and dropping the \(\theta\)-dependence, the body force components of the triangular element are

\[
\{f_{bn} (r, Z)\}^T = \{\rho r \omega^2 - \rho a_r, -\rho a_Z, -\rho a_r \sin \theta, -\rho a_r \cos \theta, 0, 0, 0\}
\]  

for \(n = 0\)  

(2.27b)

\[
\{f_{bn} (r, Z)\}^T = \{-\rho a_r, -\rho a_r \sin \theta, -\rho a_r \cos \theta, -\rho a_r \sin \theta, -\rho a_r \cos \theta, -\rho a_r \sin \theta, -\rho a_r \cos \theta, 0, 0, 0\}
\]  

for \(n = 1\)  

(2.27c)

and

\[
\{f_{bn} (r, Z)\} = 0 \quad \text{for} \quad n \geq 2
\]  

(2.27d)
The body force components in the direction of $r$ and $\theta$ are included only for generality and will not be used in the analysis of circular cell bulkheads. The surface traction vectors will be integrated explicitly for each loading because they are arbitrary. The area integration is performed over the triangular area in the $r$-$Z$ plane by denoting

$$[S_n] = \iiint_V [s_n(r, Z)]dV \quad (2.28a)$$

Matrix $[S_n]$ is shown in Appendix A.

The system body force and surface traction vectors are

$$\{F_{bn}\} = \iiint_V \{f_{bn}(r, Z)\}dV \quad (2.28b)$$

$$\{F_{sn}\} = \iint_S \{f_{sn}(r, Z)\}dS \quad (2.28c)$$

The following notation is used for the various integrals:

$$I_1 = \iiint_V dV, \quad I_6 = \iiint_V \frac{Z^2}{r^2}dV \quad (2.29)$$

$$I_2 = \iiint_V \frac{1}{r}dV, \quad I_7 = \iiint_V rdV$$
\[ I_3 = \iiint_V \frac{1}{r^2} \, dV, \quad I_8 = \iiint_V Z \, dV \]
\[ I_4 = \iiint_V \frac{Z}{r} \, dV, \quad I_9 = \iiint_V r^2 \, dV \]  \hspace{1cm} (2.29) \text{cont.}
\[ I_5 = \iiint_V \frac{Z}{2} \, dV, \quad I_{10} = \iiint_V rZ \, dV \]

After the integration in Equation (2.28b) is carried out, the body force vector may be written as

\[
\{F_{bn}\}^T = \{\rho \omega^2 I_7, \rho \omega^2 I_9, \rho \omega^2 I_{10}, \rho a Z I_1', \rho a Z I_7, \rho a Z I_8, 0, 0, 0\} \quad (2.30a)
\]
for \( n = 0 \)

\[
\{F_{bn}\}^T = \{-\rho a I_1, -\rho a I_7, -\rho a I_8, 0, 0, 0, -\rho a I_1', -\rho a I_7', -\rho a I_8'\} \quad (2.30b)
\]
for \( n = 1 \)

For a single finite element the potential energy becomes

\[
\pi_{pe} = \sum_{n=0}^{N} \left( \frac{1}{2} \{u_{0n}\}^T \phi_0^{-1} [S_n^T] \phi_0^{-1} \{u_{0n}\} \right.
\]
\[
\left. - \{u_{0n}\}^T \phi_0^{-1} \{F_{bn}\} + \{F_{sn}\} \right) \quad (2.31)
\]

Thus for the whole system of \( L \) such elements, the total potential energy in Equation (2.3) is
\[
\pi_p = \sum_{\ell=1}^{L} \sum_{n=0}^{N} \left( \frac{1}{2} \{U_n\}^T [\Phi^{(\ell)}] T [S_n^{(\ell)}] [\Phi^{(\ell)}] \{U_n\} \right. \\
\left. - \{U_n\}^T [\Phi^{(\ell)}] T \{ \{F_n^{(\ell)}\} + \{F_{sn}^{(\ell)}\} \} \right)
\]

(2.32)

where \( \{U_n\} \) is the discretized displacement vector for the entire system and \( [\Phi^{(\ell)}] \) is the generalized coordinate transformation matrix of element \( \ell \) to the displacements in the entire system.

Taking the variation of the stationary potential energy with respect to the discrete displacement variables as stated in Equation (2.4) we obtain

\[
\sum_{\ell=1}^{L} \sum_{n=0}^{N} \left( [\Phi^{-1(\ell)}] T [S_n^{(\ell)}] [\Phi^{-1(\ell)}] \{U_n\} - [\Phi^{-1(\ell)}] T \{F_n^{(\ell)}\} \right) = 0
\]

(2.33)

where

\[
\{F_n^{(\ell)}\} = \{F_{bn}^{(\ell)}\} + \{F_{sn}^{(\ell)}\}
\]

From Equation (2.33) a set of governing simultaneous equations can be written for each Fourier harmonic term as

\[
[K_n] \{U_n\} = \{F_n\} \quad \text{for} \quad n = 0, 1, 2, \ldots, N
\]

(2.34)

where \( K_n \) is the stiffness of the assembled system and \( F_n \) is the system load vector. They may be expressed as follows:
and

\[
[K_n] = \sum_{\ell=1}^{L} [\Phi_0^{-1}(\ell)]^T S_n^{(\ell)} [\Phi_0^{-1}(\ell)]
\]  \hspace{1cm} (2.35a)

and

\[
\{F_n\} = \sum_{\ell=1}^{L} [\Phi_0^{-1}(\ell)]^T \{F_n^{(\ell)}\}
\]  \hspace{1cm} (2.35b)

in which \( [\Phi_0^{-1}(\ell)] \) is shown in Equation (A.2) in Appendix A.

From Equation (2.35a), the stiffness of element \( \ell \) in any Fourier harmonic \( n \) can be expressed as

\[
[k_n^{(\ell)}] = [\Phi_0^{-1}(\ell)]^T [S_n^{(\ell)}] [\Phi_0^{-1}(\ell)]
\]  \hspace{1cm} (2.36)

### 2.2 Quadrilateral Element Matrix Equations

The use of the quadrilateral as the discrete element to idealize the system is desirable since it reduces the required input in the computer program and the resulting set of equilibrium equations has fewer unknowns for a given number of triangular elements. A typical quadrilateral element is composed of four triangles as illustrated in Figure 2.2. The coordinates of the center node are computed as the average of the four corner point coordinates.

In the case of non-axisymmetric loads, the 12 degrees of freedom quadrilateral element matrix is formed by first combining the four 9-degree-of-freedom triangular element matrices into a 15
degree of freedom element matrix. Using a process of static condensation (5, 28) the three internal displacements are eliminated, resulting in a 12 degree of freedom quadrilateral element matrix.

The four triangular element stiffness are combined by the code number technique (14, 24). If the load vectors \{F_0\} for each of the triangular elements are similarly superimposed, a partitioned matrix equation is obtained

\[
\begin{bmatrix}
  k_{aa} & k_{ab} \\
  k_{ba} & k_{bb}
\end{bmatrix}
\begin{bmatrix}
  u_a \\
  u_b
\end{bmatrix}
+ \begin{bmatrix}
  F_{0a} \\
  F_{0b}
\end{bmatrix}
= \begin{bmatrix}
  F_a \\
  F_b
\end{bmatrix}
\] (2.37)

where subscript \(a\) is associated with nodal points 1 to 4 and subscript \(b\) is associated with point 5. Equation (2.37) may be written as two matrix equations.
\[
\{F_a\} = [k_{aa}]\{u_a\} + [k_{ab}]\{u_b\} + \{F_{0a}\} \tag{2.38a}
\]
\[
\{F_b\} = [k_{ba}]\{u_a\} + [k_{bb}]\{u_b\} + \{F_{0b}\} \tag{2.38b}
\]

Equation (2.38b) can be solved for the displacements \( u_b \):
\[
\{u_b\} = -[k_{bb}]^{-1}[k_{ba}]\{u_a\} + [k_{bb}]^{-1}\{F_b\} - \{F_{0b}\} \tag{2.38c}
\]

If Equation (2.38c) is substituted into Equation (2.38a), an expression is found relating the forces at points 1 to 4 to the unknown displacements at points 1 to 4 and the known loads as
\[
\{F_a\} = [k^{*}_{aa}]\{u_a\} + \{F^{*}_{0a}\} \tag{2.39a}
\]
where
\[
[k^{*}_{aa}] = [k_{aa}] - [k_{ab}][k_{bb}]^{-1}[k_{ba}] \tag{2.39b}
\]
the quadrilateral element stiffness matrix, and
\[
\{F^{*}_{0a}\} = \{F_{0a}\} + [k_{ab}][k_{bb}]^{-1}\{F_{0b}\} \tag{2.39c}
\]
the modified load matrix.

2.3 Shell Element Matrix Equations

Finite element analysis of shells of revolution has been developed and used for axisymmetrical loadings (9, 10, 19). For non-axisymmetrical loadings, the displacements and loadings are expanded
in Fourier series (12, 18, 21, 27). A typical shell element is illustrated in Figure 2.3.

![Shell element and coordinates.

Figure 2.3. Shell element and coordinates.]

For thin shells of revolution, the potential energy of a single element may be written as (7)

$$
\pi_{\text{pe}} = \iint_S \left\{ \frac{1}{2} (N_{ij} \epsilon_{ij} + M_{ij} \chi_{ij}) \right\} dS - \int_C p_i u_i dC 
$$

(2.40a)

where

$$
N_{ij} = \int_{-t/2}^{t/2} \sigma_{ij} d\beta
$$

(2.40b)

$$
M_{ij} = \int_{-t/2}^{t/2} \sigma_{ij} \beta d\beta
$$

(2.40c)
\( N_{ij}, M_{ij}, \varepsilon_{ij}, \) and \( \chi_{ij} \) are stress resultant, moment, extensional strain and curvature tensor fields, respectively, acting on the portion of shell surface \( S. \) \( p_i \) and \( u_i \) are nodal loads and nodal displacements along the nodal circle \( C, \) and \( t \) is the shell element thickness assumed as constant throughout the whole cell. \( \beta \) is the local transverse coordinate (see Figure 2.3).

The displacement field of the conical shell may be assumed as

\[
\begin{align*}
w'(z) &= B_1 + B_2z \\
v'(z) &= B_3 + B_4z \\
u'(z) &= B_5 + B_6z + B_7z^2 + B_8z^3
\end{align*}
\]

The shell displacements of non-axisymmetrical loading case also can be expanded into Fourier series:

\[
egin{align*}
w'(z, \theta) &= \sum_{n=0}^{N} w'_n(z) \cos n\theta \quad (2.42a) \\
v'(z, \theta) &= \sum_{n=0}^{N} v'_n(z) \sin n\theta \quad (2.42b) \\
u'(z, \theta) &= \sum_{n=0}^{N} u'_n(z) \cos n\theta \quad (2.42c)
\end{align*}
\]

where \( u', v' \) and \( w' \) are the transverse, circumferential or tangential, and longitudinal shell displacements, respectively, with
respect to the load element coordinates. \( z \) is the longitudinal shell coordinate and \( n \) is the Fourier harmonic number.

The coefficients \( B_1, B_2, \ldots, B_8 \) represent the generalized displacement coordinates of the shell element. The number of constants \( B \) are assumed equal to the number of degrees of freedom of the element. Each nodal point has four degrees of freedom that are denoted by \( u', v', w' \) and \( \alpha \). Translation \( u', v', w' \) have been defined previously; \( \alpha \) is the rotation of the meridian in the shell surface in a plane which passes through the nodal point and contains the axis of revolution of the shell. The rotation \( \alpha \) is positive if it corresponds to a positive value of \( \partial u'/\partial z \) along the meridian.

By using the boundary conditions, the nodal displacements in Equation (2.41) can be written as

\[
\begin{bmatrix}
  u_n'(1) \\
  w_n'(1) \\
  v_n'(1) \\
  \frac{\partial u_n'(1)}{\partial z} \\
  u_n'(2) \\
  w_n'(2) \\
  v_n'(2) \\
  \frac{\partial u_n'(2)}{\partial z}
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  1 & \ell & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & \ell & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 2\ell & 2\ell & 0
\end{bmatrix}
\begin{bmatrix}
  B_{1n} \\
  B_{2n} \\
  B_{3n} \\
  B_{4n} \\
  B_{5n} \\
  B_{6n} \\
  B_{7n} \\
  B_{8n}
\end{bmatrix}
\]

(2.43a)
or in symbolic form

\[ \{u'_{0n}\} = [\varphi_0]\{B_{in}\} \quad i = 1, 2, \ldots, 8 \]  

where \( I \) is the length of the shell element.

The constants \( \{B_{in}\} \) are obtained by inversion as

\[ \{B_{in}\} = [\varphi_0^{-1}]\{u'_{0n}\} \]  

where \( [\varphi_0^{-1}] \) is shown in Equation (A.12) of Appendix A.

Introducing a shell displacement transformation matrix, the relationship between local and global coordinate displacements are

\[ \{u'_{0n}\} = [\lambda_0]\{u_{0n}\} \]  

where \( [\lambda_0] \) is the shell transformation matrix and is shown in Appendix A. \( \{u_{0n}\} \) is the global coordinate displacements.

If we substitute Equation (2.44a) into Equation (2.43c), the generalized coordinates become

\[ \{B_{in}\} = [\varphi_0^{-1}][\lambda_0]\{u_{0n}\} \]

or

\[ \{B_{in}\} = [\varphi_{00}^{-1}]\{u_{0n}\} \]  

where

\[ [\varphi_{00}^{-1}] = [\varphi_0^{-1}][\lambda_0] \]  

(2.44c)
[\varphi_0^{-1}] is shown in Equation (A.15) of Appendix A.

According to the Novozhilov theory (17) of thin shells, the strain-displacement relationships are

\[
\begin{align*}
\epsilon_{zz} &= \frac{\partial w'}{\partial z} \\
\epsilon_{\theta\theta} &= \frac{1}{r} \left( \frac{\partial v'}{\partial \theta} + u' \cos \phi + w' \sin \phi \right) \\
\gamma_{z\theta} &= \frac{\partial v'}{\partial z} + \frac{1}{r} \left( \frac{\partial w'}{\partial \theta} - \sin \phi v' \right) \\
\chi_{zz} &= -\frac{\partial^2 u'}{\partial z^2} \\
\chi_{\theta\theta} &= \frac{1}{r^2} \left( \cos \phi \frac{\partial v'}{\partial \theta} - \frac{\partial^2 u'}{\partial \theta^2} \right) - \frac{1}{r} \frac{\partial u'}{\partial z} \sin \phi \\
\chi_{z\theta} &= 2\left[ \frac{1}{r^2} \sin \phi \frac{\partial u'}{\partial \theta} - \frac{1}{r} \frac{\partial^2 u'}{\partial z \partial \theta} + \left( \frac{1}{r} \frac{\partial v'}{\partial z} - \frac{1}{r^2} \sin \phi v' \right) \cos \phi \right]
\end{align*}
\]

where \( \epsilon_{zz}, \epsilon_{\theta\theta}, \) and \( \epsilon_{z\theta} \) are the longitudinal, normal and shear strains; \( \chi_{zz}, \chi_{\theta\theta} \) and \( \chi_{z\theta} \) are the curvature changes of the middle surface of the shell.

The general stress-strain relationships for a linear isotropic elastic thin shell are (17)
The potential energy in Equation (2.40a) can be expanded as

\[
\{\sigma\} = [C]\{\epsilon\}
\]  \hspace{1cm} (2.46b)

where

\[
\{\sigma\} = \{S(z)\}^T [N_{nm}(\theta)] \{S(z)\}
\]  \hspace{1cm} (2.47a)

\[
\{\epsilon\} = \{S(z)\}^T [M_{nm}(\theta)] \{S(z)\}
\]  \hspace{1cm} (2.47b)

where

\[
\{S(z)\} = \left\{ \frac{\partial w'}{\partial z}, \frac{\partial v'}{\partial z}, \frac{w'}{r}, \frac{v'}{r}, \frac{u'}{r} \right\}
\]  \hspace{1cm} (2.47c)

\[
\{S(z)\} = \left\{ \frac{\partial^2 u'}{\partial z^2}, 1 \frac{\partial v'}{\partial z}, 1 \frac{\partial u'}{\partial z}, \frac{v'}{r}, \frac{u'}{r}, \frac{u'}{r} \right\}
\]  \hspace{1cm} (2.47d)

\([N_{nm}(\theta)]\) and \([M_{nm}(\theta)]\) are developed in Appendix A.
Due to the orthogonal properties of the Fourier harmonic functions in the period of \(-\pi \leq \theta \leq \pi\), the sum of the potential energy exists only for \(m = n\).

If Equation (2.41) is differentiated and substituted into Equation (2.47c), \(\{S_{1n}(z)\}\) will become

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{r} \frac{z}{r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{r} \frac{z}{r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{r} \frac{z}{r} \frac{z^2}{r} \frac{z^3}{r}
\end{bmatrix}
\]

or written in symbolical form

\[
\{S_{1}(z)\} = [X_{1}(z)]\{B_{in}\}
\]

Substituting Equation (2.44b) into Equation (2.48b), \(\{S_{1}(z)\}\) may be written in terms of nodal displacements as follows

\[
\{S_{1}(z)\} = [X_{1}(z)][\varphi_{00}^{-1}]\{u_{0n}\}
\]

Similarly, if Equation (2.41) is differentiated and substituted into
Equation (2.47d), \( \{ S_{2n}(z) \} \) can be expressed in terms of generalized coordinates and nodal displacements, respectively, as

\[
\{ S_{2}(z) \} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2 & 6z \\
0 & 0 & 0 & \frac{1}{r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{r} & \frac{2z}{r} & \frac{3z}{r} & 2 \\
0 & \frac{1}{r} & \frac{z}{r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{r} & \frac{z}{r} & \frac{2z}{r} & \frac{3z}{r} & 2 & 2 \\
0 & 0 & 0 & \frac{1}{r} & \frac{z}{r} & \frac{2z}{r} & \frac{3z}{r} & 2 \\
\end{bmatrix}
\begin{bmatrix}
B_{1n} \\
B_{2n} \\
\vdots \\
B_{8n}
\end{bmatrix}
\] (2.49a)

or in symbolical form

\[
\{ S_{2}(z) \} = [X_{2}(z)] [B_{in}] 
\] (2.49b)

and

\[
\{ S_{2}(z) \} = [X_{2}(z)] [\phi_{00}^{-1}] [u_{0n}] 
\] (2.49c)

The terms \([X_{1}(z)]\) and \([X_{2}(z)]\) in Equations (2.48b) and (2.49b) can be expanded further as

\[
X_{1}(z)_{ik} = G_{ijk} x_{i}(z) 
\] (2.50a)

\[
X_{2}(z)_{ik} = H_{ijk} x_{i}(z) 
\] (2.50b)

where

\[
\{ x_{i}(z) \}^T = \{ 1, \frac{1}{r}, \frac{z}{r}, \frac{2}{r}, \frac{3}{r}, \frac{z}{r}, \frac{2}{r}, \frac{3}{r} \} 
\] (2.50c)
and

\[ G_{112} = G_{214} = G_{321} = G_{332} = G_{423} = G_{434} = G_{525} = G_{536} = G_{547} = G_{558} = 1 \]

\[ H_{224} = H_{326} = H_{473} = H_{484} = H_{575} = H_{586} = H_{597} = H_{5108} = 1 \]

\[ H_{117} = H_{337} = 2 \]

\[ H_{348} = 3 \]

\[ H_{168} = 6 \]

All other \( G \)'s and \( H \)'s are zero.

After substituting stress and strain vector fields into Equation (2.40a) and using the advantage of \( \theta \)-dependence being known explicitly, the \( \theta \)-integration can be performed directly and the potential energy becomes

\[
\pi_{pe} = \sum_{n=0}^{N} \left( \int_{\ell} \left( \frac{1}{2} \{u_{0n}\}^T \varphi_{00}^{-1} \varphi_{00} \{X_1(z)\}^T \{X_1(z)\} \right. \\
+ \left. [X_2(z)]^T [M_n][X_2(z)]]\varphi_{00}^{-1}\{u_{0n}\}) d\ell \right)
- \{u_{0n}\}^T \varphi_{00}^{-1} \{F_{zn}\} \right)
\]

(2.51)

where \( \ell \) is the integration over the length of the shell element which is carried out by using numerical integration of the Gaussian Quadrature Formula (4,16).
Introducing

\[ [SS_n] = \int \left( [X_1(z)]^T [N_n] [X_1(z)] + [X_2(z)]^T [M_n] [X_2(z)] \right) \, dt \]  \hspace{1cm} (2.52a)

Equation (2.51) can be rewritten as

\[ \pi_{pe} = \sum_{n=0}^{N} \left( \frac{1}{2} \{u_{0n} \}^T [\varphi_{00}^{-1}]^T [SS_n] [\varphi_{00}^{-1}] \{u_{0n} \} - \{u_{0n} \}^T [\varphi_{00}^{-1}]^T \{F_{zn} \} \right) \]  \hspace{1cm} (2.52b)

For the assemblage of \( L \) elements, the total potential energy

\[ \pi_p = \sum_{l=1}^{L} \sum_{n=0}^{N} \left( \frac{1}{2} \{U_n \}^T [\varphi_{00}^{-1(l)}]^T [SS_n^{(l)}] [\varphi_{00}^{-1(l)}] \{U_n \} - \{U_n \}^T [\varphi_{00}^{-1(l)}]^T \{F_{zn}^{(l)} \} \right) \]  \hspace{1cm} (2.53)

The governing equations are obtained by performing a variation on the total potential energy of the system with respect to the discrete displacements and setting them equal to zero. Accordingly

\[ \delta \pi_p = 0 \]

which yields

\[ \sum_{n=0}^{N} \sum_{l=1}^{L} \left( [\varphi_{00}^{-1(l)}]^T [SS_n^{(l)}] [\varphi_{00}^{-1(l)}] \{U_n \} = [\varphi_{00}^{-1(l)}] \{F_{zn}^{(l)} \} \right) \]  \hspace{1cm} (2.54)
For each Fourier harmonic term

\[
[K_n] \{U_n\} = \{F_n\}, \quad n = 0, 1, 2, \ldots, N \tag{2.55a}
\]

where \([K_n]\) is the system stiffness matrix of the shell element and

is given by the expression

\[
[K_n] = \sum_{\ell=1}^{L} [\varphi_{00}^{-1(\ell)}]^T [S_{\ell}^S][\varphi_{00}^{-1(\ell)}] \tag{2.55b}
\]

\([F_n]\) is the system load vector of the shell element

\[
\{F_n\} = \sum_{\ell=1}^{L} [\varphi_{00}^{-1(\ell)}]^T \{F_{\ell}^{zn}\} \tag{2.55c}
\]

The corresponding shell element stiffness matrix is

\[
[k_n^{(\ell)}] = [\varphi_{00}^{-1(\ell)}][S_{\ell}^S][\varphi_{00}^{-1(\ell)}] \tag{2.55d}
\]

2.4 System Equations and Solution Process

As the element stiffness matrices \([k_n]\) for each Fourier harmonic term are generated, they are appropriately superimposed into a system matrix \([K_n]\). The superposition is accomplished by using the code number technique (14, 24).

A structural system that contains a large number of elements will involve a large amount of data preparation. This includes a code
number for each element. In order to reduce preliminary work of this nature, a subroutine was written to generate the code numbers for each element.

The load vector \( \{F_n\} \) for each harmonic is assembled at the same time as the structural stiffness matrix is formed. They are generated using the code numbers. The total system of simultaneous algebraic equations in the unknown \( \{U_n\} \) is represented as

\[
[K_n]\{U_n\} = \{F_n\}
\]  

(2.56)

The primary concern in the solution of this system is the conditioning of the system matrix. It is a banded symmetrical matrix where the band width is dependent on the direction of numbering the nodal points. The nodal points should be numbered in such a way as to minimize the difference between the largest and smallest nodal point numbers for any element. By taking advantage of symmetry, the coefficients are stored as an upper triangular matrix.

The solution of the equations is obtained using the linear equation solver BANSOL. This subroutine uses the Gaussian elimination method. The band-width is automatically computed prior to solving the system as it is required input to the solution subroutine along with the system load vector.
III. TESTING OF COMPUTER PROGRAM

The purpose of this chapter is to present verifications of the finite element formulation of this investigation and also the computer program. Because the circular cell bulkhead system consists of quadrilateral ring elements (4 triangles) and cylindrical shell elements and Fourier harmonic series are used, three different types of problems are analyzed and their results are compared with those of known classical solutions.

First, an elastic soil layer which overlies a rough rigid base is analyzed. The soil system is subjected to a constant surface loading \( q \) which is applied uniformly over a circular area of radius \( a \) as shown in Figure 3.1. The soil is assumed to be a linearly elastic, isotropic and homogeneous material. Thirty-five rectangular ring elements are used to model the system and the results are shown in Figures 3.2, 3.3, 3.4, and 3.5. It is seen that there is excellent agreement with the classical solutions proposed by Burmister (3).

Second, a circular cylindrical tank filled with water is considered in demonstrating the applicability of the shell element formulation. The shell is idealized as 12 cylindrical elements and 13 nodes as shown in Figure 3.6. The tank is subjected to hydrostatic pressure only. The displacements and stress resultants are shown in Figures 3.7 and 3.8, respectively. It is seen that there is good agreement with the exact solution (25) except that the displacements
and hoop force at the top of the cylinder are somewhat on the high side. This is due to the assumed point load at the top nodal point of the tank.

The third structure to be analyzed is a circular cylindrical thin shell cooling tower. It is included in this investigation in order to verify and demonstrate the power and versatility of using the Fourier harmonic analysis of the shell portion of the finite element method. The cooling tower rests on 8 columns and supports its own weight. The finite element model is shown in Figure 3.9(a). Five elements are shown in Figure 3.9(a). A second model, not shown, with ten elements was also considered in which each element was one-half the size of the first model. The second model was prepared because the first model proved to be very coarse. Flugge (8) has developed a classical solution for this problem with the total edge load as shown in Figure 3.9(b). The approximate loading diagram which is based on a 5 term Fourier expansion of the given stress function is shown in Figure 3.10. The results for force resultants and displacements are shown in Figures 3.11, 3.12 and 3.13, respectively.

From the results it can be seen that the hoop force of the 5-element model is on the high side but the corresponding axial force is in good agreement with the classical solution. A significant improvement is achieved with the ten element model. Therefore it is concluded that the results are quite dependent on the number of elements.
Figure 3.1. Finite element model of rigid base soil system.
Figure 3.2. Vertical stress in soil due to surface load.
Figure 3.3. Radial stress in soil due to surface load.
Tangential stress in percent of applied pressure

Numbers on curves indicate radial distances in radii

- Exact solution (3)
- Finite element

Figure 3.4. Tangential stress in soil due to surface load.
Figure 3.5. Shear stress in soil due to surface load.
Figure 3.6. Finite element model of circular tank.

D = 30 ft
H = 30 ft
t = 0.5 inch
E = 30 x 10^3 ksi
\( v = 0.3 \)

Figure 3.7. Radial displacement of circular tank.

Exact solution (25)

Finite element
Longitudinal moment, $M_{zz}$ in kips-ft/ft

Note: Positive moment causes compression inside tank

Figure 3.8. Hoop force and longitudinal moment of circular tank.
5-element model

Section C-D

(a) Cylindrical cooling tower

Section A-B

- \( E = 30 \times 10^3 \text{ ksi} \)
- \( v = 0 \)
- \( t = 0.24 \text{ inch} \)
- \( a = 3 \text{ ft} \)
- \( P = 0.4 \text{ k/ft} \)

(b) Total edge load at the bottom

Figure 3.9. Edge load applied to the cooling tower shell.
Figure 3.10. Approximate loading diagram for cooling tower.
Figure 3.11. Cooling tower stress resultants.
Figure 3.12. Radial and axial displacements for cooling tower at $\theta = 0^\circ$. 

- $w$ (axial displacement)
- $u$ (radial displacement)

Exact solution (8, p. 231)

- $\bigcirc$ 5 Element solution
- $\triangle$ 10 Element solution
Figure 3.13. Radial and axial displacements for cooling tower at $\theta = 22.5^\circ$. 

- Exact solution (8, p. 231)
- $\bigcirc$ 5 Element solution
- $\triangle$ 10 Element solution

$u$ (radial displacement) and $-w$ (axial displacement)
IV. ANALYSIS OF CIRCULAR CELL BULKHEADS

Because the mathematical model and computer program are applied to the analysis of axisymmetrical cell structures whose configurations are shown in Figures 4.1 and 4.2, no difficulty was experienced in generating the necessary input data. A user's manual for the program which describes all necessary operations for generating the data is included in Appendix E. Appendix F describes the function of the primary subroutines and Appendix G contains a listing of the program.

Two analyses of the circular cell are presented in this chapter. They are:

1. An isolated circular cell subjected to gravity load of the cell fill which simulates the behavior of the cell after the inside is filled completely as shown in Figure 4.1.

2. In addition to the weight of cell fill, the circular bulkhead is subjected to the weight and lateral pressure due to backfill.

The general configuration is shown in Figure 4.2.

The circular cell structure used in this study was of similar overall dimensions to one constructed at Port of Portland Terminal No. 4 on the Willamette River near Portland, Oregon. In that bulkhead, the front sheet piles were 18 feet longer than the back ones. The web thickness of the front sheet piling was 0.532 inch and of the
back was 0.406 inch.

Since the computer program was developed for geometrically axisymmetric structure, the average web thickness of 0.469 inch was assumed to be the uniform shell thickness. The average height of the cell (from the dredge line) was assumed to be 58.5 feet and the depth of embedment was uniformly 29.25 feet. The water level was assumed to be at the top of the cell both inside and outside.

Because of the lack of soil test data, some parameters were assumed and these properties are summarized in Table 4.1.

Poisson's ratio for the soil was assumed to have a constant value of 0.35 throughout the system. The modulus of elasticity was assumed constant only within each individual layer of elements. Values were obtained by using Richart's formula (20) as shown in Appendix D. The modulus of elasticity of the steel was assumed to be $30 \times 10^3$ ksi, and Poisson's ratio was assumed to be 0.3.

<table>
<thead>
<tr>
<th>Total Density pcf</th>
<th>$\phi$ Degree</th>
<th>Degree of Saturation %</th>
<th>Void Ratio</th>
<th>Specific Gravity</th>
<th>Poisson's Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>117.5</td>
<td>34</td>
<td>100</td>
<td>0.94</td>
<td>2.71</td>
<td>0.35</td>
</tr>
</tbody>
</table>

* Soil is classified as medium dense to dense fine sand.

A 48 soil element and 9 shell element model was generated for the structure as shown in Figure 4.3. The vertical boundary line and
the base boundary line were selected 60 feet and 29.25 feet from the cell, respectively. The selection of these boundary lines and the number of elements were based on the storage limitations of the computer and estimated zone of influence for the cell.

4.1 Isolated Circular Cell

The isolated circular cell as shown in Figure 4.1 is subjected internally to the gravity load of the cell fill and externally by water pressure. Since the water table inside the cell is at the same level as the adjacent water, the submerged unit weight of the soil is used to determine the net pressure loading. The system is obviously axisymmetrically loaded, so only one Fourier harmonic term of $n$ equal to zero is necessary for the analysis.

The computer results for the isolated circular cell are shown in Figures 4.4 to 4.10.

4.2 Circular Cell Bulkhead

The general configuration of the circular cell bulkhead is shown in Figure 4.2. The phreatic line both inside the cell fill and backfill was observed at the same elevation of the outside water (11). In this study, it was assumed at the top throughout the bulkhead system.

Figure 4.11 illustrates the equivalent loadings due to backfill. The quantity $p_a$ is the active earth pressure and can be calculated
Figure 4.1. An isolated circular cell.
Figure 4.2. A circular cell bulkhead.
Figure 4.3. Circular cell finite element model.
Figure 4.4. Radial displacement, hoop force and sheet stresses due to gravity load of cell fill (axisymmetric loading case).
Figure 4.5. Vertical displacement of circular cell due to gravity load of cell fill.
Figure 4.6. Contours of vertical stress in soil due to gravity load of cell fill.
Numbers on curves indicate stress contour in ksf.

Figure 4.7. Contours of radial stress in soil due to gravity load of cell fill.
Figure 4.8. Contours of circumferential stress in soil due to gravity load of cell fill.
Figure 4.9. Contours of shear stress in soil due to gravity load of cell fill.
Figure 4.10. Principal stresses in soil due to gravity load of cell fill.
Figure 4.11. Equivalent loading diagrams due to backfill.
by the following equations (13)

\[ p_a = \gamma' h K_a \quad (4.1) \]

\[ K_a = \tan \left( \frac{2(45^\circ - \phi)}{2} \right) \quad (4.2) \]

where

- \( K_a \) is the Rankine coefficient of active earth pressure
- \( \gamma' \) is submerged unit weight of soil
- \( h \) is depth of soil
- \( \phi \) is angle of internal friction of soil.

The quantity \( q_b \) in Figure 4.11 is assumed to be the surcharge load which is equivalent to the dead load of the backfill. \( p_a \) and \( q_b \) are uniformly distributed loads applied from one connecting arc to the other on the back side of the cell.

The connecting arc tension load was assumed to be \( T_a \) and can be obtained by

\[ T_a = p_a r_a \quad (4.3) \]

where \( r_a \) is the radius of the connecting arc.

In order to represent the arc tension load in a Fourier expansion, the arc tension load was assumed as a trigonometric loading function which subtended a small angle at the center of the cell. The area of the assumed loading must equal to \( T_a \) as shown in Figure 4.12.
Since the Fourier coefficients for the connecting arc tension load and backfill distributed loads are different, the computer analyses were done separately and the results superimposed. The loadings are divided into two cases in the following manner:

Case I - due to connecting arc tension loads. The approximate loading diagrams are represented by Fourier expansions and are shown in Figure 4.13.

Case II - includes the gravity load of the system itself and the distributed load of \( p_a \) and \( q_b \). The Fourier expansion of these loads is shown in Figure 4.14.

The corresponding Fourier coefficients used in the analyses are shown in Appendix B. Because of the very high cost of a computer
Figure 4.13. Fourier harmonic loading expansions for arc tension load $T_a$ (Case I).
Figure 4.14. Fourier harmonic loading expansions for uniform distributed pressures $p_a$ and $q_b$ (Case II).
run, the Fourier analyses were carried out using only 11 harmonics for Case I loading and 8 harmonics for Case II loading.

An extensive amount of data is obtained (see Appendix E) from the computer output for each run. In order to present it clearly, it has been divided into two separate categories. First, the data pertaining to the shell elements (sheet piles) is presented. This data includes displacements and stresses in the steel sheet piling cell. Second, the data pertaining to the soil portion of the structure is presented. A reference diagram for subsequent graphs is given in Figure 4.15.

Figures 4.16 and 4.17 show the horizontal displacement of the cell due to connecting arc tension loads (Case I). The horizontal displacement due to gravity load and uniform distributed pressures (Case II) and combined displacement are shown in Figures 4.18 to 4.23. The deformed shape of the bulkhead is shown in Figure 4.24.

Figures 4.25 to 4.31 show the vertical stress and hoop stress on both outside and inside surfaces of the cell at various angles.

Figures 4.32 and 4.33 show the variation of hoop force and shear stress in the sheet pile along the circumferential direction of the cell for each level as indicated.

The contours of stresses in soil elements are plotted in Figures 4.34 to 4.37. The radial stress and vertical stress distribution vs.
the depth of the cell in sections V1, V2 and V3 are shown in Figures 4.38 and 4.39.

Figure 4.40 is a plot of the coefficients of lateral earth pressure $K$ in the cell fill against the depth of the cell. The $K$ values were obtained directly from the ratio between the computed radial stress and vertical stress in soil elements. The radial deformation, cell settlement and hoop forces in the cell are plotted in Figures 4.41, 4.42 and 4.43, respectively, to compare with the values obtained in the field (11).
Figure 4.15. Reference diagram for displacements and stresses.
Figure 4.16. Horizontal displacements of cell due to arc tension loads at levels H-5 and top of cell (Case I).
Figure 4.17. Horizontal displacements of cell due to arc tension loadings at levels H6 and H8 (Case I).
Figure 4.18. Horizontal displacements of backfilled cell at levels H5 and the top of cell.
Figure 4.19. Horizontal displacements of backfilled cell at level H6 and H8.
Figure 4.20. Radial displacement of backfilled cell at $\theta = 0$ and 180 degrees.
Dredge line

Case II

Combined displacement
(Case I + Case II)

Displacement, inches

Depth, feet

Displacement, inches

Figure 4.21. Radial displacement of backfilled cell at $\theta = 30$ and 150 degrees.
Figure 4.22. Radial displacements of backfilled cell at $\theta = 60$ and 120 degrees.
Figure 4.23. Radial displacement of cell at $\theta = 90$ and 270 degrees.
Figure 4.24. Deformed shape of circular cell bulkhead.
Figure 4.25. Vertical and hoop stresses of steel cell at $\theta = 0$ degree.
Figure 4.26. Vertical and hoop stresses of steel cell at $\theta = 30$ degrees.
Figure 4.27. Vertical and hoop stresses of steel cell at $\theta = 60$ degrees.
Figure 4.28. Vertical and hoop stresses of steel cell at $\theta = 90$ degrees.
Figure 4.29. Vertical and hoop stresses of steel cell at $\theta = 120$ degrees.
Figure 4.30. Vertical and hoop stresses of steel cell at $\theta = 150$ degrees.
Figure 4.31. Vertical and hoop stresses of steel cell at $\theta = 180$ degrees.
Figure 4.32. Hoop force in cell vs. position.
Figure 4.33. Shearing stress in cell vs. position.
Figure 4.34. Contours of vertical stress in soil.
Figure 4.35. Contours of radial stress in soil.
Figure 4.36. Contours of shear stress ($\tau_{rz}$) in soil.
Figure 4.37. Contours of circumferential stress in soil.

Plane θ = 180°  Plane θ = 0°
Figure 4.38. Radial and vertical stresses in soil vs. depth of cell.
Figure 4.39. Radial and vertical stresses in soil vs. depth of cell.
Figure 4.40. Coefficients of lateral earth pressure inside the cell fill.
Figure 4.41. Comparison of radial deformation of backfilled cell at level H5.
Figure 4.42. Settlement of the top of the steel sheet piles vs. position.
Figure 4.43. Comparisons of hoop force in steel cell.
V. DISCUSSION OF RESULTS

5.1 Discussion of the Isolated Circular Cell

Figure 4.4 shows the radial displacement, hoop force and stresses in steel sheets for the isolated circular cell subjected to the gravity load of cell fill only. The radial displacement and hoop force of the cell gradually increase as the depth of cell increases up to the maximum bulging point and then drop off. The maximum bulging of the cell is 0.57 inch which occurs at a point one-sixth (0.17) of the exposed height above the dredge line. The maximum bulging point that was observed in the field (11) ranged from 0.18 to 0.28 of the exposed height. The value of 0.18 corresponds to field measurements on the front side of the cell where sheet pile length was the same as in the present study.

From Figure 4.4 it can be seen that all vertical stresses in the steel sheet are compressive with the maximum value of 5.3 ksi at the bottom edge of the cell. This indicates that the steel sheet transmits the downdrag due to the weight of soil adjacent to the pile, to the soil foundation. The maximum hoop stress occurs on the outside surface of the cell at a level one-sixth of the exposed height. There are also some hoop stresses in the embedment zone.

The elastic settlement of soil fill inside the circular cell is shown in Figure 4.5. It can be seen that the soil at the center of the
cell settles more than that away from the center. This may be explained by recognizing that near the sheet piles there is friction (actually no slip) between the soil fill and the wall. The stiffer steel element deforms less. The maximum settlement of the soil element at the center of the cell is 1.48 inches, while the top of the steel sheet settles only 0.66 inch.

The upward displacement along the vertical boundary line outside the cell is due to the outward movement of the cell pushing against the boundary. This is unreasonable for an actual installation. In order to eliminate the upward displacement in the theoretical model it is necessary to increase the distance of the vertical boundary from the cell. By increasing this distance no significant change in the displacement and stresses along the sheet pile node is expected. The deflected shape and soil element stresses in the current boundary line, however, may be changed considerably.

The horizontal boundary line at the base of the system as used in the example is justified by observing the settlement of the nodal points directly above which indicate the uniform settlement of the soil element along the horizontal cross section. This indicates that the horizontal boundary line is deep enough from the bottom edge of the steel sheets.

Figures 4.6, 4.7 and 4.8 illustrate the contours of vertical, radial, and circumferential stresses in the soil elements for the cell
filled, gravity loading case. In the body of the circular cell, all of
the stresses are compressive with the vertical stresses $\sigma_{zz}$ greater
than the radial stresses $\sigma_{rr}$ and circumferential stresses $\sigma_{\theta\theta}$.
Outside the cell, just below the dredge line, the radial stress is con-
siderably higher than the vertical stress (Figures 4.6 and 4.7). This
distribution of normal stresses thus corresponds to development of a
passive pressure distribution where the sheet piles are being forced
into the soil.

The contours of shearing stress $\tau_{rz}$ in the soil are plotted in
Figure 4.9. It indicates that the maximum shearing stress occurs on
a plane just behind the sheet piles near mid-height above the dredge
line. Planes away from the sheet piles display a dramatic decrease
in shear stress near mid-height, while the top and bottom portions of
these planes show a slight decrease only.

Figure 4.10 shows the principal stresses in the soil for the
circular cell under the gravity load of the cell fill. The stresses
inside the cell are all compressive stresses. The directions of the
maximum principal stress above the dredge line all incline toward the
sheet piles. This shows the tendency of the soil fill inside the cell to
bulge the sheet piles. Outside the cell the principal stresses tend to
resist such lateral movement.
5.2 Discussion of Circular Cell Bulkhead

As mentioned in Chapter 4 the loadings on circular bulkheads are separated into two cases, therefore, the computed displacements are presented separately and then combined.

The deformed shape of the circular cell due to connecting arc tension loads (Case I) for various levels is shown in Figures 4.16 and 4.17. The radial displacement and tangential displacement data for 9 harmonic terms are plotted on the left side of the circular sections, while the data for 11 harmonic terms are plotted on the right side of the sections. It was found that going from 9 to 11 harmonics gave an approximately 5% increase in the maximum displacement resultant. The maximum movement occurs on the top level of the cell with 0.2 inch leaning toward the front side.

The radial displacement and tangential displacement due to Case II loading are shown in Figures 4.18 and 4.19. The displacement data for 6 and 8 harmonics are plotted on the opposite half of the circular sections in the same manner as in Case I. The discrepancy for displacement between the two harmonics was not greater than 6% in any section.

The combined displacements are plotted in Figures 4.18 to 4.23. It can be seen that the connecting arc tension loading contributes approximately 10 to 25% of the total horizontal displacements.
The deflected shape of the cell indicates that the cell was pushed toward the water side with the front portion of the embedment sheet piles forced into the front dredge line and the back portion moved into the cell.

The maximum bulging point for the circular cell bulkhead occurs on level H5 (0.17 of the exposed height above the dredge line) on the front side (θ = 180°) of the cell. This can be compared to the suggested design value of one-fourth (0.25) for TVA (22) and the field measurement value (11) of 0.18 at the front sheet position. The maximum bulging on the back portion of the cell occurs at the mid-height level (H3). This is in good agreement with the results from field measurements (11).

Figure 4.24 shows the deformed shape of the circular cell bulkhead on the front-back (180°-0°) cross-section. It indicates that the settlement of the fill in the center of the cell is consistently greater than at the edges. The shape of settlement in the cell fill has a pattern similar to that of field measurement results obtained by White (26). From Figures 4.5 and 4.24, it can be seen that placing of the backfill does not significantly change the settlement of the soil fill inside the body of the cell.

Figures 4.25 to 4.31 show the steel sheet surface stress data for various angles around the cell. The membrane stress is the average of the outside surface and inside surface stresses, whereas
the bending stress is observed by noting one-half the difference between these two stresses. It can be seen that the bending stress in the vertical direction is more significant than that in the hoop direction. It also appears that both the vertical stress and the hoop stress are maximum on the front sheet \((\theta = 180^\circ)\) of the cell.

Figures 4.32 and 4.33 show how the hoop tension force and shearing stress in the steel sheet vary around the cell at the levels indicated. The hoop forces on the level above the dredge line appear to increase from the back side \((\theta = 0^\circ)\) of the cell to a maximum at the arc connection \((\theta = 120^\circ)\). They remain almost constant from this point to the front sheet of the cell. From Figure 4.32, it can be observed that in placing the backfill, the hoop forces in the back portion decreases sharply since the back fill lateral pressures counter the internal forces from the cell fill. At the front side of the cell, the hoop forces increase to a very small degree. Moreover, it is found that the maximum hoop force does coincide with the maximum bulging point on level H5. Consequently, the TVA design rules (22) in which maximum hoop tension occurs just inside the arc connection at a point \(H/4\) above the dredge line are appropriate.

The shearing stress \(\tau_{Z\theta}\) in the steel sheet as shown in Figure 4.33 appears to increase from zero at the back side of the cell to a maximum at the arc connection. It then decreases gradually to zero at the front sheet of the cell. The shearing stress diagrams are
anti-symmetric about the front to back cross-section. The maximum shearing stress occurs on the dredge line level (H6).

The contours of vertical stress, radial stress, shearing stress and circumferential or tangential stress in the soil on vertical cross-sections of planes $\theta = 0^\circ$ and $\theta = 180^\circ$ are plotted in Figures 4.34 to 4.37, respectively. The stress contour data on other planes of the bulkhead were also computed but they are not included here because the variations from those shown here were small. The soil stresses inside the cell appear to change very slightly due to the effect of back fill (see Figures 4.6 to 4.9 and Figures 4.34 to 4.37). Outside the body of the cell, the application of backfill pressure greatly increases the vertical and radial stresses around the cell. The noticeable increase in radial stress in front of the cell results from the passive resisting force due to the outward movement of steel sheet piles in that area. The vertical stress is quite uniformly distributed across the base of the cell and approximately equal to the overburden pressure.

Backfill lateral pressure slightly affects the shear stress in the cell fill as shown in Figure 4.36. The shear stresses in the middle are much lower than near the sheet piles. It therefore seems improbable that shear failure would start in the midplane and go towards the steel sheet walls as suggested by Terzaghi (23). Instead it appears that the failure should start near the dredge line of the sheet pile and
progress from there up to the top free surface and down through the base of the sheets along a path of maximum shear stress. This type of internal failure mode was also observed in photoelastic analysis (1).

Figures 4.38 and 4.39 show the vertical and radial stresses in the soil elements inside the cell vs. the depth of the cell. Results for radial stresses show that they increase more on the back side than the front due to the lateral pressure of backfill trying to push the back sheets inward. It also can be seen that levels above the dredge line experience a small increase in stresses, while the lower portions of the cell show a considerable increase. Specifically, there is an abrupt increase in both vertical and radial stresses under the bottom edge of the sheet piles. This is due to the weight of soil fill transmitted by the sheet piles to the soil foundation. This observation may help to explain larger than anticipated settlements which have been experienced by structures of this type in Portland (11) and Long Beach (26).

The coefficient of lateral earth pressure $K$ shown in Figure 4.41 indicates that at the center plane of cell $K$ decreases slightly with depth. At the front plane, $K$ is almost constant all the way down the cell. At the back plane, it increases moderately due to backfill lateral pressure. The mean values of $K$ in the front position (unloaded side), center position and back position (loaded side) are 0.441, 0.412 and 0.64, respectively. They indicate that the
coefficient of lateral earth pressure of the fill in the cell is higher than the Rankine earth pressure coefficient. The maximum average value of $K$ in the field was 0.45 (11).

The results from the study and the field measurements obtained by Khuayjarernpanishk (11) are compared with the one obtained by using Cummings', TVA's and Terzaghi's recommended formulae as shown in Table 5.1. For the circular cell bulkhead case the maximum hoop tension force in the steel cell obtained in this study is about 0.75 of the values obtained from Cummings' and Terzaghi's formulae and 0.67 of the value obtained from TVA's formulae, for the isolated cell case. At the same time, it is only 0.61 of the field measurement result for the bulkhead case. The computed vertical pressures at the base of the cell are almost uniform and approximately equal to the overburden pressure. This indicates that the bending effect due to backfill results in a very small vertical pressure when compared to the effect of the gravity load of the cell fill. Consequently, they are not in the same trend as Terzaghi's concept which states that the bending stress is linearly distributed across the horizontal section of the cell by considering the cell as a rigid body. The calculated coefficients of lateral earth pressure in the cell fill are in good agreement with the values recommended by other investigators and those obtained from field measurements.
Table 5.1. Comparisons of results.

<table>
<thead>
<tr>
<th>Sources</th>
<th>Maximum Hoop Tension kips/inch</th>
<th>$\sigma_{ZZ}$ at front base $\sigma_{ZZ}$ overburden</th>
<th>$\sigma_{ZZ}$ at back base $\sigma_{ZZ}$ overburden</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\sigma_{ZZ}$ overburden</td>
<td>$\sigma_{ZZ}$ overburden</td>
<td>Front</td>
</tr>
<tr>
<td>Cummings (6)</td>
<td>2.53*</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>TVA (22)</td>
<td>2.85**</td>
<td>--</td>
<td>--</td>
<td>0.523***</td>
</tr>
<tr>
<td>Terzaghi (23)</td>
<td>2.53*</td>
<td>1.25</td>
<td>0.743</td>
<td>0.4</td>
</tr>
<tr>
<td>Kittisatra 1</td>
<td>1.90</td>
<td>1.04</td>
<td>1.05</td>
<td>0.441</td>
</tr>
<tr>
<td>Lacroix (13)</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>0.4</td>
</tr>
<tr>
<td>Khuayjarernpanishk (11)</td>
<td>3.10+</td>
<td>--</td>
<td>--</td>
<td>0.45+</td>
</tr>
</tbody>
</table>

* At dredge line level for isolated cell.
** At H/4 above dredge line just inside arc connection for isolated cell.
*** The Krynine factor, $K = (\cos^2 \phi)/(2 - \cos^2 \phi)$.
+ Field measurement results for bulkhead
1 Finite element results for bulkhead.
As mentioned in Chapter IV, the circular cell bulkhead used in this study is not exactly the same configuration as the one instrumented in the field (11). Nevertheless, the radial deformation at level H5, the settlement of the top of the cell and the hoop tension force in the cell have been plotted in Figures 4.41, 4.42 and 4.43, respectively, to compare and show some correlations.

Figure 4.41 shows that the computed deformations are in good agreement with the general trend of observed values in the field except for the positions near the left arc ($\theta = 120^\circ$). The exaggerated distortions at such points in the field during construction were presumed to result from wave action, compaction of fill and tightening of interlock slack (11). They are not associated with elastic deformation.

Figure 4.42 illustrates how the edge settlement varied around the cell for the top level. It appears that the settlement of the back portion of the cell is greater than the front. This is due to the back sheet piles being subjected to a higher gravity effect of the backfill than the front. The observed settlements in the field immediately after backfilling and 2-1/2 months after backfilling are approximately 5.5 and 8 times the computed elastic settlement. This seems to indicate that the total settlement in the field resulted from consolidation of the fill in response to stresses due to fill weight in addition to the immediate elastic distortion due to the same cause. The observed
settlements of the cell are in similar pattern to the computed results.

From Figure 4.43, the computed hoop tension forces in the cell have a similar pattern to the field measurement values but do illustrate again significant differences in magnitude. All hoop forces obtained from the field measurements are greater than in this study.

The main factors associated with the assumptions used in this study that influence the results are that:

1. The elastic properties of a given soil in a cell depend to a large extent on the method of filling the cell. There is no laboratory procedure that would yield reliable advance information on the state of density of the soil in the cell and on the corresponding elastic properties of the soil. The estimated moduli of elasticity $E$ used in this study seemed to be too high. They are approximately 7 times larger than that used by Brown (2). If lower values of $E$ had been used, the nodal point displacements would have been significantly greater. Since the steel sheet pile stress is a function of displacements at the nodes, a corresponding increase in sheet stress would also be expected.

2. The assumed interface condition (full friction) between the soil and the steel sheets probably resulted in higher than realistic axial stress in the steel sheets. The rougher the steel sheet surface, the better the agreement will be between
analytic and field results. It is quite likely that a boundary element could be developed to represent the actual state of friction. Such an element is not now available. Additional research and formulation would have to be undertaken to develop such an element. It is believed that modification of the analysis and the computer program can be made to account for the interface friction force, once the appropriate element is introduced and the experimental boundary interaction data are available.

3. The assumed cylindrical shell-like structure is much stiffer than the actual interlocked sheet piling cellular bulkhead. This assumption is a very essential factor, reducing the horizontal displacements in the cell because no interlock slack can exist in the shell. But on the other hand, interlock friction in the actual cell probably reduces slippage to negligible values for usual cell service conditions. The interlocks would be expected to slip if the cell were caused to distort a large amount (fail by bending).
VI. SUMMARY AND CONCLUSIONS

The finite element model and the corresponding computer program developed in this investigation can be used to compute displacements and stresses in a circular cell structure subjected to gravity loads of cell fill, backfill and surcharge loads. The structure can be founded on soil foundation or rock foundation. The loading functions are expanded in Fourier harmonic series. For different sets of Fourier force coefficients, the analyses must be carried out separately. The components of displacement and stress for each harmonic and each set of loadings are superimposed to form the total solution.

The following conclusions are drawn from this study.

1. An almost uniform vertical pressure distribution on the base of the circular cell bulkhead was obtained. The magnitude is approximately equal to the overburden pressure.

2. Horizontal stresses in the soil within the cell are mobilized as normal active and passive earth pressure.

3. The maximum elastic soil settlement occurred in the center of the cell, with smaller settlements at the edges. It also was found that the total elastic settlement at the back side of the cell is greater than at the front.

4. The vertical shear stress in the cell fill for the bulkhead is
maximum near the front and back sheet piles and not on the center plane. Therefore, the mid-plane shear failure proposed by Terzaghi (23) seems unlikely.

5. Through shear transfer to and from the soil, the sheet piles appear to transmit an appreciable amount of the load to the foundation.

6. Placement of the backfill increases the hoop tension forces in the front portion of the cell. A decrease in hoop tension in the sides and back portion was found.

7. Both maximum hoop tension force and maximum bulging of the cell occur on a level 1/6 of the exposed height above the dredge line. The maximum hoop tension occurs behind the connections of the cell and arcs, whereas maximum bulging occurs at the front sheet of the cell. This indicates that the TVA (22) suggestion to compute the maximum hoop force in the cell at the arc connecting sections is justified.

8. Both vertical and circumferential bending stresses do occur in the steel sheet piles. These bending stresses are lower than the membrane stresses. However, such stresses should be taken into account in designing.

9. The coefficients of lateral earth pressure in the cell fill are 0.441 at the front, 0.412 at the center and 0.640 at the back of the cell. These values agree closely with the values
recommended by other investigators.

10. The total eleven Fourier harmonics used in the analysis is considered adequate for the loading conditions in this study.
BIBLIOGRAPHY


APPENDICES
APPENDIX A

Element Matrices

A.1 Triangular Axisymmetric Ring Element

The $[\Phi_0]$ matrix of Equation (2.16) is (also see Figure 2.1)

$$
[\Phi_0] = \begin{bmatrix}
1 & r_1 & Z_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & r_1 & Z_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & r_1 & Z_1 \\
1 & r_2 & Z_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & r_2 & Z_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & r_2 & Z_2 \\
1 & r_3 & Z_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & r_3 & Z_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & r_3 & Z_3 
\end{bmatrix}
$$

(A.1)
The matrix \( [\Phi^{-1}_0] \) of Equation (2.17) is

\[
[\Phi^{-1}_0] = \frac{1}{2A}
\begin{bmatrix}
a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 \\
b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 \\
c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 \\
0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 \\
0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 \\
0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 \\
0 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 \\
0 & 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 \\
0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3
\end{bmatrix}
\quad \text{(A.2)}
\]

where \( A \) is the cross sectional area of the triangular element and \( a, b, c \) are nodal coordinate functions as shown in Equation (2.19b).

The matrix \( [\Phi_n'(r, Z, \theta)] \) of Equation (2.21) is shown below where \( n \) is the Fourier harmonic number.
The matrix \([ S_n ]\) in Equation (2.28a) can be found by rewriting the material constant \( C \) in Equation (2.7) as

\[
[S_n] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\cos n\theta}{r} & \frac{\cos n\theta}{r} & \frac{Z \cos n\theta}{r} & 0 & 0 & 0 \\
0 & 0 & \cos n\theta & 0 & \cos n\theta & 0 \\
-\frac{n \sin n\theta}{r} & -n \sin n\theta & -\frac{nZ \sin n\theta}{r} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{n \sin n\theta}{r} & -n \sin n\theta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\cos n\theta & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{n \cos n\theta}{r} & \frac{n \cos n\theta}{r} & \frac{nZ \cos n\theta}{r} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sin n\theta}{r} & 0 & -\frac{Z \sin n\theta}{r} & 0 & 0 \\
-\frac{nZ \sin n\theta}{r} & 0 & 0 & 0 & \sin n\theta & 0 \\
\end{bmatrix}
\] (A.3)
where

\[
\begin{align*}
C_{11} &= C_{22} = C_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\
C_{12} &= C_{13} = C_{23} = \frac{Ey}{(1+\nu)(1-2\nu)} \\
C_{44} &= C_{55} = C_{66} = \frac{E}{2(1+\nu)}
\end{align*}
\]

Substituting Equations (A. 3) and (A.4) into Equation (2.25c) and integrating Equation (2.28a), \([S_n]\) can be expressed as

\[
[S_n] = \begin{bmatrix}
[S_{UU_n}] & [S_{UW_n}] & [S_{UV_n}] \\
[S_{WU_n}] & [S_{WW_n}] & [S_{WV_n}] \\
[S_{VU_n}] & [S_{VW_n}] & [S_{VV_n}]
\end{bmatrix}
\]

(A.5)

where

\[
[S_{UU_n}] = \begin{bmatrix}
(C_{33} + n^2 C_{55})I_3 & (C_{13} + n^2 C_{55})I_1 & (C_{33} + n^2 C_{55})I_5 \\
(C_{11} + 2 C_{33} + n^2 C_{55})I_1 & (C_{13} + n^2 C_{55})I_4 & (C_{33} + n^2 C_{55})I_6 + C_{44}I_1
\end{bmatrix}
\]

(A.6)

Symmetric

\[
[S_{UW_n}] = [S_{WU_n}]^T = \begin{bmatrix}
0 & 0 & C_{23}I_2 \\
0 & 0 & (C_{12} + C_{23})I_1 \\
0 & C_{44}I_1 & C_{23}I_4
\end{bmatrix}
\]

(A.7)
\[ [SUV_n] = [SVU_n]^T \]

\[ = \begin{bmatrix}
    n(C_{33} + C_{55})_1 & nC_{33}_2 & n(C_{33} + C_{55})_5 \\
    n(C_{13} + C_{33} + C_{55})_2 & n(C_{13} + C_{33})_1 & n(C_{13} + C_{33} + C_{55})_4 \\
    n(C_{33} + C_{55})_5 & nC_{33}_4 & n(C_{33} + C_{55})_6
\end{bmatrix} \]

(A. 8)

\[ [SWW_n] = \begin{bmatrix}
    n^2C_{66}I_3 & n^2C_{66}I_2 & n^2C_{66}I_5 \\
    (C_{44} + n^2C_{66})I_1 & n^2C_{66}I_4 \\
    \text{Symmetric} & n^2C_{66} \end{bmatrix} \]

(A. 9)

\[ [SVV_n] = \begin{bmatrix}
    (n^2C_{33} + C_{55})_1 & n^2C_{33}I_2 & (n^2C_{33} + C_{55})_5 \\
    n^2C_{33}_1 & n^2C_{33}_4 \\
    \text{Symmetric} & (n^2C_{33} + C_{55})_6 + C_{66}I_1
\end{bmatrix} \]

(A. 10)

\[ [SWW_n] = [SVW_n]^T = \begin{bmatrix}
    0 & 0 & -nC_{66}I_2 \\
    0 & 0 & -nC_{66}I_1 \\
    nC_{23}I_2 & nC_{23}I_1 & n(C_{23} - C_{66})I_4
\end{bmatrix} \]

(A. 11)
A.2 Shell Element of Revolution

Matrix $[\varphi_0^{-1}]$ of Equation (2.43c) is

$$
[\varphi_0^{-1}] =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/l & 0 & 0 & 0 & 1/l & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/l & 0 & 0 & 0 & 1/l & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-3/l^2 & 0 & 0 & -2/l & 3/l^2 & 0 & 0 & -1/l \\
2/l^3 & 0 & 0 & 1/l^2 & -2/l^3 & 0 & 0 & 1/l^2 \\
\end{bmatrix}
$$

(A.12)

Matrix $[\lambda_0]$ in Equation (2.44a) is

$$
[\lambda_0] =
\begin{bmatrix}
\cos \phi & \sin \phi & 0 & 0 & 0 & 0 & 0 & 0 \\
-sin \phi & \cos \phi & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \phi & \sin \phi & 0 & 0 \\
0 & 0 & 0 & 0 & -sin \phi & \cos \phi & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

(A.13)
The matrix $[\varphi^{-1}_{00}]$ of Equation (2.44b) may be shown by denoting

$$a = Z_2 - Z_1 \quad \text{(see Figure 2.3)}$$

$$b = r_2 - r_1$$

$$\cos \phi = \frac{a}{\ell}$$

$$\sin \phi = -\frac{b}{\ell}$$

where $\ell$ is the length of the shell element.

Substituting Equation (A.14) into Equation (2.44c), $[\varphi^{-1}_{00}]$ becomes

$$[\varphi^{-1}_{00}] = \begin{bmatrix}
\frac{b}{\ell} & \frac{a}{\ell} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-b}{\ell^2} & \frac{-a}{\ell^2} & 0 & 0 & \frac{b}{\ell^2} & \frac{a}{\ell^2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/\ell & 0 & 0 & 0 & 1/\ell & 0 \\
a/\ell & -b/\ell & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{-3a}{\ell^3} & \frac{3b}{\ell^3} & 0 & -2/\ell & \frac{3a}{\ell^3} & \frac{-3b}{\ell^3} & 0 & -1/\ell \\
\frac{2a}{\ell^4} & \frac{-2b}{\ell^4} & 0 & \frac{1}{\ell^2} & \frac{-2a}{\ell^4} & \frac{2b}{\ell^4} & 0 & \frac{1}{\ell^2} \\
\end{bmatrix}$$

(A.15)

Matrices $[N_n(\theta)]$ and $[M_n(\theta)]$ in Equations (2.47c) and (2.47d) are:
\[
\begin{bmatrix}
E_{11}\cos(\theta) & 0 & E_{12}\sin(\theta) & nE_{12}\cos(\theta) & E_{12}\sin(\theta) \\
E_{44}\sin(\theta) & -nE_{44}\sin(\theta) & -E_{44}\sin(\theta) & 0 & 0 \\
E_{22}\sin(\theta) + n^2E_{44}\sin(\theta) & n\phi(E_{22}\cos(\theta) + E_{44}\sin(\theta)) & E_{22}\sin(\theta) + n^2E_{44}\sin(\theta) & nE_{22}\cos(\theta) & E_{22}\sin(\theta) \\
E_{44}\cos(\theta) & nD_{44}\cos(\theta) & -nD_{44}\cos(\theta) & -n^2D_{44}\cos(\theta) & -nD_{44}\cos(\theta) \\
E_{22}\sin(\theta) + n^2E_{44}\sin(\theta) & n\phi(D_{22}\cos(\theta) + D_{44}\sin(\theta)) & 2\phi(n^2D_{22}\cos(\theta) + D_{44}\sin(\theta)) & 2\phi(n^2D_{22}\cos(\theta) + D_{44}\sin(\theta)) & n\phi(n^2D_{22}\cos(\theta) + D_{44}\sin(\theta)) 
\end{bmatrix}
\]

(A.16)

and

\[
\begin{bmatrix}
D_{11}\cos(\theta) & 0 & D_{12}\sin(\theta) & -nD_{12}\sin(\theta) & -n^2D_{12}\sin(\theta) \\
D_{44}\cos(\theta) & nD_{44}\cos(\theta) & -nD_{44}\cos(\theta) & -n^2D_{44}\cos(\theta) & -nD_{44}\cos(\theta) \\
D_{22}\sin(\theta) + n^2D_{44}\sin(\theta) & nD_{22}\sin(\theta) + n^2D_{44}\sin(\theta) & 2\phi(D_{22}\cos(\theta) + D_{44}\sin(\theta)) & 2\phi(D_{22}\cos(\theta) + D_{44}\sin(\theta)) & n\phi(n^2D_{22}\cos(\theta) + D_{44}\sin(\theta)) \\
D_{44}\cos(\theta) & nD_{44}\cos(\theta) & -nD_{44}\cos(\theta) & -n^2D_{44}\cos(\theta) & -nD_{44}\cos(\theta) \\
D_{22}\sin(\theta) + n^2D_{44}\sin(\theta) & nD_{22}\sin(\theta) + n^2D_{44}\sin(\theta) & 2\phi(D_{22}\cos(\theta) + D_{44}\sin(\theta)) & 2\phi(D_{22}\cos(\theta) + D_{44}\sin(\theta)) & n\phi(n^2D_{22}\cos(\theta) + D_{44}\sin(\theta)) 
\end{bmatrix}
\]

(A.17)
where

\[ \begin{align*}
    c\phi &= \cos \phi, \\
    \phi^2 &= \cos^2 \phi \\
    s\phi &= \sin \phi, \\
    \phi^2 &= \sin^2 \phi \\
    \cos \theta &= \cos \theta^2, \\
    \sin \theta &= \sin \theta^2
\end{align*} \]

\[ \begin{align*}
    E_{11} &= E_{22} = \frac{E\ell}{1-\nu^2}, \\
    E_{12} &= \frac{E\nu\ell}{1-\nu^2} \\
    E_{44} &= \frac{E\ell}{2(1+\nu)}
\end{align*} \]

\[ \begin{align*}
    D_{11} &= \frac{E\ell^3}{12(1-\nu^2)}, \\
    D_{22} &= \frac{E\ell^3}{12(1-\nu^2)} \\
    D_{12} &= \frac{E\ell^3 \nu}{12(1-\nu^2)}, \\
    D_{44} &= \frac{E\ell^3}{12(1+\nu)}
\end{align*} \]

where \( E \) is Young's modulus, \( \nu \) is Poisson's ratio.
APPENDIX B

Fourier Harmonic Coefficients

There are two sets of Fourier force coefficients in this study; the first set contains the coefficients due to the connecting arc tension load as shown in Figure 4.13, the second set contains the coefficients due to uniformly distributed lateral load and gravity load of backfill as shown in Figure 4.14.

Let

\[ p(\theta) = \sum_{n=0}^{N} a_n \cos n\theta \]  \hspace{1cm} (B.1)

then

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) d\theta \]  \hspace{1cm} (B.2)

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} p(\theta) \cos n\theta d\theta, \quad n = 1, 2, \ldots, N \]  \hspace{1cm} (B.3)

The resulting Fourier coefficients for each case of loadings are shown in Table B.1.
Table B.1. *Fourier force coefficients.*

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p(θ) = p cos θ</td>
<td>p(θ) = p, $\frac{2\pi}{3} &lt; θ &lt; \frac{2\pi}{3}$</td>
</tr>
<tr>
<td></td>
<td>p(θ) = 0, elsewhere</td>
<td>p(θ) = 0, $\frac{2\pi}{3} &lt; θ &lt; \frac{2\pi}{3}$</td>
</tr>
<tr>
<td>$a_0$</td>
<td>0.07073</td>
<td>0.66666</td>
</tr>
<tr>
<td>$a_1$</td>
<td>-0.07053</td>
<td>0.55133</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.06992</td>
<td>-0.27566</td>
</tr>
<tr>
<td>$a_3$</td>
<td>0.13783</td>
<td>0</td>
</tr>
<tr>
<td>$a_4$</td>
<td>-0.06752</td>
<td>0.13783</td>
</tr>
<tr>
<td>$a_5$</td>
<td>-0.06577</td>
<td>-0.11026</td>
</tr>
<tr>
<td>$a_6$</td>
<td>0.12732</td>
<td>0</td>
</tr>
<tr>
<td>$a_7$</td>
<td>-0.06124</td>
<td>0.07876</td>
</tr>
<tr>
<td>$a_8$</td>
<td>-0.05853</td>
<td>-0.06892</td>
</tr>
<tr>
<td>$a_9$</td>
<td>0.11111</td>
<td>0</td>
</tr>
<tr>
<td>$a_{10}$</td>
<td>-0.05236</td>
<td>0.05513</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>-0.04899</td>
<td>0</td>
</tr>
</tbody>
</table>
APPENDIX C

Gaussian Quadrature Numerical Integration Procedure

Numerical integration by the Gaussian quadrature method is chosen because high accuracy is obtained with relatively few nodal stations and singularities along the boundary can easily be handled (21).

Reference (16) gives the following Gaussian quadrature formula:

\[
\int_{a}^{b} f(X) dX = \int_{-1}^{1} g(x) \mathcal{O} dx \approx \sum_{i=1}^{N} w_i g(x_i) \mathcal{O}
\]  

(C. 1a)

where

\[
g(x) = f(h(x))
\]  

(C. 1b)

\[X = h(x)\]

and

\(w_i\) is the Gaussian weighting functions given in Table C. 1.

\(x_i, y_i\) are the Gaussian integration stations given in Table C. 1.

\(N\) is the number of Gaussian stations.

\(\mathcal{O}\) is the differential transformation containing terms to change the limits of integration to the interval [-1, 1] and terms from the differential \(dX\).

The extension to integrals involving more than one variable is straightforward. For example, in the case of two variables, we obtain
\[
\int \int_S f(X, Y) dS = \int_{-1}^{1} \int_{-1}^{1} g(x, y) \mathcal{D} dxdy
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j g(x_i, y_j) \mathcal{D}
\]

(C.2a)

where

\[g(x, y) = f(h_X(x, y), h_Y(x, y))\]

(C.2b)

\[X = h_X(x, y)\]

(C.2c)

\[Y = h_Y(x, y)\]

(C.2d)

Values of Gaussian weighting functions and stations are given in Table C.1.
Table C.1. Gaussian weighting functions and stations.

<table>
<thead>
<tr>
<th>N</th>
<th>i</th>
<th>Station, $x_i$</th>
<th>Weight, $w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>-0.57735027</td>
<td>1.00000000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.57735027</td>
<td>1.00000000</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-0.77459667</td>
<td>0.55555555</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.00000000</td>
<td>0.88888889</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.77459667</td>
<td>0.55555555</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-0.86113631</td>
<td>0.34785485</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.33998104</td>
<td>0.65214515</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.33998104</td>
<td>0.65214515</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.86113631</td>
<td>0.34785485</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-0.90617985</td>
<td>0.23692689</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.53846931</td>
<td>0.47862867</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.00000000</td>
<td>0.56888889</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.53846931</td>
<td>0.47862867</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.90617985</td>
<td>0.23692689</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-0.93246951</td>
<td>0.17132449</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.66120939</td>
<td>0.36076157</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-0.23861919</td>
<td>0.46791393</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.23861919</td>
<td>0.46791393</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.66120939</td>
<td>0.36076157</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.93246951</td>
<td>0.17132449</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>-0.97390653</td>
<td>0.06667134</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.86506337</td>
<td>0.14945135</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-0.67940957</td>
<td>0.21908636</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.43339539</td>
<td>0.26926672</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-0.14887434</td>
<td>0.29552422</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.14887434</td>
<td>0.29552422</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.43339539</td>
<td>0.26926672</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.67940957</td>
<td>0.21908636</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0.86506337</td>
<td>0.14945135</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.97390653</td>
<td>0.06667134</td>
</tr>
</tbody>
</table>
APPENDIX D

Modulus of Elasticity of Soil

The modulus of elasticity of soil used in this study was obtained by using Richert's formula (20). For an elastic soil as assumed in Chapter I, the modulus of elasticity \( E \) can be expressed in terms of shear modulus \( G \):

\[
E = 2(1 + \nu)G
\]  

(D.1)

where \( \nu \) is Poisson's ratio of soil.

The shear modulus of angular grained materials can be estimated from the empirical equation

\[
G = \frac{1230(2.97 - e)^2}{1 + e} (\sigma_o)^{0.5}
\]  

(D.2)

in which both \( G \) and \( \sigma_o \) are expressed in psi, where

\[
\sigma_o = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})
\]  

(D.3)

\( \sigma_o \) is the average confining pressure of the soil element.

\( \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \) are normal stresses in the X, Y, and Z direction, respectively.

\( e \) is void ratio which can be obtained from

\[
e = \frac{G_s \gamma_w}{\gamma_d} - 1
\]  

(D.4)
where

\[ G_s \text{ is specific gravity of soil} \]

\[ \gamma_d \text{ is dry unit weight of soil.} \]

The dry unit weight can be computed from the following equation:

\[ \gamma_d = \frac{G_s (\gamma - \gamma_w)}{G_s - 1} \]  \hfill (D.5)

where

\[ \gamma \text{ is total unit weight of soil} \]

\[ \gamma_w \text{ is unit weight of water.} \]

Once the values of \( G_s, \gamma, \gamma_w \) of 2.71, 117.5 pcf. and 62.5 pcf. respectively, are substituted into Equations (D.5) and (D.4), the void ratio will be 0.94.

The vertical stress, \( \sigma_{zz} \), in Equation (D.3) was assumed to be equal to the overburden pressure. Since the water level in this study was at the top of the cell, \( \sigma_{zz} \) can be written as

\[ \sigma_{zz} = (\gamma - \gamma_w)h \]  \hfill (D.6)

where \( h \) is depth of soil above point.

The horizontal stresses \( \sigma_{xx} \) and \( \sigma_{yy} \) may be written as

\[ \sigma_{xx} = \sigma_{yy} = \frac{\nu}{1 - \nu} \sigma_{zz} \]  \hfill (D.7)
The moduli of elasticity of soil for the circular cells (Figures 4.1 and 4.2) and the finite element model (Figure 4.3) are shown in Table D.1.

Table D.1. Modulus of elasticity of soil.

<table>
<thead>
<tr>
<th>Element Layer No.</th>
<th>Depth, ft</th>
<th>$\sigma_o$, lb/ft$^2$</th>
<th>$E$, kip/in$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.875</td>
<td>185</td>
<td>7.98</td>
</tr>
<tr>
<td>2</td>
<td>14.625</td>
<td>556</td>
<td>13.86</td>
</tr>
<tr>
<td>3</td>
<td>24.375</td>
<td>927</td>
<td>17.90</td>
</tr>
<tr>
<td>4</td>
<td>34.125</td>
<td>1298</td>
<td>21.18</td>
</tr>
<tr>
<td>5</td>
<td>43.875</td>
<td>1670</td>
<td>24.02</td>
</tr>
<tr>
<td>6</td>
<td>53.625</td>
<td>2041</td>
<td>26.55</td>
</tr>
<tr>
<td>7</td>
<td>63.375</td>
<td>2412</td>
<td>28.87</td>
</tr>
<tr>
<td>8</td>
<td>73.125</td>
<td>2783</td>
<td>31.01</td>
</tr>
<tr>
<td>9</td>
<td>82.875</td>
<td>3155</td>
<td>33.02</td>
</tr>
<tr>
<td>10</td>
<td>95.063</td>
<td>3619</td>
<td>35.36</td>
</tr>
<tr>
<td>11</td>
<td>109.687</td>
<td>4176</td>
<td>38.00</td>
</tr>
</tbody>
</table>
APPENDIX E

User's Manual for Circular Cell Bulkhead Program

The program consists of a main program and 18 subroutines. This appendix is included in order to explain the use of the computer program developed for the analysis of circular cell bulkheads as well as the capability of the computer program and restrictions concerning the preparations of the finite element model and input data.

The computer program for the non-axisymmetrically loaded axisymmetric solid and shell was written in FORTRAN IV for the CDC 3300 computer at Oregon State University.

E.1 Program Capability

The computer program is used to analyze a circular cell bulkhead which was filled with cohesionless soil. The cell can be founded on rock or a sand foundation. The program described herein does compute principal stresses on a vertical plane with $\theta = 0$, stresses in soil elements on any desired vertical plane, shell forces and deformations for a circular cell due to the gravitational effects of the fills inside and/or outside the cell. The boundary axisymmetric or non-axisymmetric surcharge loads can be added simultaneously.

The program can also analyze a shell of revolution or an axisymmetric structure subjected to external forces. Present
limitations on the finite element model's size are as follows: 66 nodal points, 57 elements, 12 different material properties, 50 Fourier terms for the boundary force condition, a maximum bandwidth of 28 and maximum degrees of freedom of 182. If the user wishes to increase these limits for other computers the following named common areas must accordingly be changed in the program: COMMON A2, COMMON A3, COMMON A6, COMMON C7, COMMON D2, COMMON D3 and COMMON F2. In addition, the indices on both DO loops in subroutine INITL, the DIMENSION and EQUIVALENCE statements in the main program and subroutines INPUT, SOISTR, SHLSTR and TSTRES must also be changed.

E.2 Preparation of the Finite Element Mesh

By making use of the axis symmetry of the circular cell system and taking advantage of orthogonal properties of the harmonic functions as mentioned in Chapter II, only one plane of revolution of the structure needs to be discretized. Figure E.1 is used as a model structure to show how to prepare the finite element mesh. Two types of elements are used to construct the model: quadrilateral elements for the soil, and shell elements for the steel sheet pile. The quadrilateral element is sub-divided by the program into four triangles. Therefore, a quadrilateral element will give the same results as four triangular elements, but in the mean time a quadrilateral element has
Figure E.1. Example of finite element mesh showing node and element number scheme.
fewer sets of equilibrium equations in the final system than four triangular elements.

Numbering of the nodal points should begin after the finite element mesh has been established. Numbering of these nodes can begin at any corner of the system and proceed either horizontally or vertically. The primary objective of the numbering scheme is to minimize the numerical difference between nodal point numbers associated with the elements. The nodal numbering in this study (Figure E.1) was done horizontally.

After the numbering of the nodes has been accomplished, all elements must be numbered sequentially.

**E. 3 Preparation of Input Data**

The following information describes the data cards which form the necessary input data for the program.

**E. 3.1 Number of Problems Card (I5)**

Columns

1-5 Number of sets of input (NPROBS)
E.3.2 Control Card (8I5, 3F10.4, 2I5)

Columns

1-5  Total number of nodal points (NUMNPT, 66 max.)
6-10  Total number of elements (NUMELT, 57 max.)
11-15 Total number of different materials (NUMMAT, 12 max.)
16-20 Total number of pressure boundary conditions (NUMPC)
21-25 Total number of Fourier terms (NUMFOU, 50 max.)
26-30 Total number of output print angles (NANGLE, 7 max.)
31-35 Total number of constrained boundary points (NCONP)
36-40 Problem type (NCUT)

Enter a 0 if no soil elements are present
Enter a 12 if both soil and shell elements are present
Enter a 13 if no shell elements are present

41-50 Radial acceleration (ACELR)
Enter a 0 or leave it blank

51-60 Axial acceleration (ACELZ)
Enter a -1.0 if own weight of soil elements is accounted

61-70 Angular velocity (ANGFQ)
Enter a 0 or leave it blank

71-73 Mesh check tag (ISTOP)
Enter a 0 on control card. The values of all essential parameters are checked against their allowable values. If
ISTOP > 0, a mesh check is obtained then the program is stopped and no further action is taken.

74-76 Total number of shell elements (NSHELT)

77-80 Total number of nodal points with axisymmetric surcharge loads (NAXISF)

E.3.3 Material Properties Cards (I5, 3F15.4)

Columns

1-5 Material identification number (MTYPE)

6-20 Material density (DENS, in kips per cu.ft.)

21-35 Modulus of elasticity (EMOD, in ksi.)

36-50 Poisson's ratio (PR)

Number of material property cards must equal to value of NUMMAT in Columns 11-15 of CONTROL CARD.

E.3.4 Nodal Point Cards (I3, 12, 6F10.4 213, F9.3)

Columns

1-3 Nodal point number (N)

4-5 Common node (NPCOM(N))

This entry indicates the number of elements being connected at this node.

6-15 Radial coordinate (COOR(N, 1) ≥ 0, in feet)

16-25 Axial coordinate (COOR(N, 2) in feet)
| 26-35 | Radial nodal force (FORCE(N, 1) in kips) |
| 36-45 | Axial nodal force (FORCE(N, 2) in kips) |
| 46-55 | Tangential nodal force (FORCE(N, 3) in kips) |
| 56-65 | Nodal moment (FORCE(N, 4) in kips-ft) |

Columns are left blank if it is a soil nodal point.

| 66-68 | Nodal point condition code (MCODE(N)) |
| 69-71 | Type of node (NPTYPE(N)) |

This entry indicates the degree of freedom the program will assign to the node. A3 is assigned all nodes except the nodes on the shell elements which are assigned a 4.

| 72-80 | Shell thickness (T in inch) |

Columns are left blank if it is a soil nodal point.

In general every nodal point must be defined but since the program has an automatic mesh generation feature, a minimum of two nodal points per row need be input and the intervening points will be assigned coordinates based on a linear interpolation procedure. For example, if nodal point 1 is the first point in a row with coordinates (0, 0) and nodal point 4 is the next point defined with coordinates (33, 0), then the nodal point 2 will be located (11, 0) and etc.

The nodal point condition code (MCODE) will be set 0 unless points 1 and 4 have the same MCODE, in which case all intervening points will be assigned the same MCODE as the two end points.
The common node (NPCOM) and type of nodal point (NPTYPE) of the omitted intervening points will be assigned the same NPCOM and NPTYPE as the preceding point on that row.

The radial, axial, tangential forces and moments of all omitted intervening points will be assigned 0 in all cases.

The shell thickness at each nodal point should be non-negative and it is assumed constant over the shell length in each element.

All loads are total forces acting on a one-radian segment.

The nodal point condition code (MCODE in columns 66-68) is interpreted in the following manner:

<table>
<thead>
<tr>
<th>MCODE</th>
<th>Radial (r)</th>
<th>Axial (Z)</th>
<th>Tangential ((\theta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>X</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>X</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>X</td>
</tr>
<tr>
<td>5</td>
<td>X</td>
<td>X</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>X</td>
<td>-</td>
<td>X</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

The nodal point cards must be in order, starting with nodal point number 1. The last nodal point card must be supplied.

**E. 3. 5 Connectivity Cards or Element Cards (615)**

This data connects the element number with the nodal point numbers on the element parameters. The nodal points for a
quadrilateral and triangular elements are listed counter-clockwise sequentially around the element. The nodal points for a shell element are arbitrary but all shell elements' nodal points must be in the same pattern.

**Columns**

<table>
<thead>
<tr>
<th>Columns</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-5</td>
<td>Element number (M)</td>
</tr>
<tr>
<td>6-10</td>
<td>Nodal point I (ICONN(M, 1))</td>
</tr>
<tr>
<td>11-15</td>
<td>Nodal point J (ICONN(M, 2))</td>
</tr>
<tr>
<td>16-20</td>
<td>Nodal point K (ICONN(M, 3)). Columns are left blank if it is a shell element</td>
</tr>
<tr>
<td>21-25</td>
<td>Nodal point L (ICONN(M, 4)). Columns are left blank if it is a shell element or a triangular element</td>
</tr>
<tr>
<td>26-30</td>
<td>Material identification (ICONN(M, 5))</td>
</tr>
</tbody>
</table>

In general, every element must be defined but with the automatic mesh generation feature a minimum of 1 element per row need be input. The program generates the omitted elements by incrementing by 1 the preceding I, J, K and L. The material identification code for the generated elements is the value specified on the first generated card. For example if element 1 is read with values I = 1, J = 2, K = 13, L = 12, material type = 11 and the next element card read is element 11 with values I = 12, J = 13, K = 24, L = 23, material type = 10 then element 2 would be assigned value 2, 3, 14,
13, 11, element 3 values 3, 4, 15, 14, 11 and etc. The last element card must be supplied.

The nodal point array on the element cards must of course correspond to nodal points on the nodal point cards and the material identification must correspond to the materials in the material cards.

The element cards must be in order, starting with element number 1.

E. 3. 6 Constrained Boundary Cards (S15)

This data is applied primarily to nodal points on the boundaries of the finite element model in the non-axisymmetric loading case. A 1 is used to indicate the constraint of a nodal point.

Columns

1-5 Nodal point number of constrained node (KNOVA(N, 1))

6-10 Radial direction constraint (KNOVA(N, 2), 1 or blank)

11-15 Axial or vertical direction constraint (KNOVA(N, 3), 1 or blank)

16-20 Tangential direction constraint (KNOVA(N, 4), 1 or blank)

21-25 Rotational constraint (KNOVA(N, 5), 1 or blank)

Enter 1 if the nodal point is not on the shell element

The number of cards must equal to NCONP, columns 31-35 of control card. The program has an automatic constrained nodal point
generation i.e., it will assign a 1 to tangential displacement (KNOVA(N, 4)) of all nodal points of the system and radial displacement (KNOVA(N, 2)) of all nodal points which are on Z-axis when Fourier harmonic number equal to zero. The program will also assign a 1 to rotation displacement (KNOVA(N, 5)) of all soil nodal points.

E.3.7 Boundary Pressure and/or Shear Cards (3I5, 2F10.4)

If NUMPC in columns 16-20 of control card is zero or blank, then omits these cards.

Columns

1-5 Element number (LBC)
6-10 Boundary nodal point I (IBC)
11-15 Boundary nodal point J (JBC)
16-25 Normal traction (PN in kips per sq. ft.)
26-35 Shear traction (PT in kips per sq. ft.)

Nodes I and J must be chosen such that the sequence of nodal points on the traction cards are in the same manner as on the element cards. The tractions are assumed constant over the length of the element boundary. The positive senses of normal pressure and shear are shown in Figure E.2.
The number of cards must correspond to the value of NUMPC input in columns 16-20 of CONTROL CARD.

**E. 3.8 Axisymmetric Nodal Surcharge Loading Cards (I5, 4F15. 5)**

If NAXISF in columns 77-80 of CONTROL CARD is zero, then omits these cards.

**Columns**

1-5  
Nodal point number (NPAX(N))

6-20  
Radial axisymmetric nodal force (FORC(N, 1) in kips)

21-35  
Axial axisymmetric nodal force (FORC(N, 2) in kips)

36-50  
Tangential axisymmetric nodal force (FORC(N, 3) in kips)

51-65  
Axisymmetric nodal moment (FORC(N, 4) in kips-ft)
If the surcharge loads are distributed over the boundary surface of the elements, then the surface integral must be carried out explicitly for the desired loading. The value of the nodal point forces are total forces acting on a one-radian segment.

The number of these cards must correspond to the value of NAXISF.

E.3.9 Fourier Force Coefficient Cards (8F10.5)

The Fourier force coefficients of the surface tractions and nodal point forces of the non-zero displacement boundary conditions are as follow:

Card 1  Columns  1-10  Fourier coefficient $a_0$ (FORCOF(1))
           11-20  "       "   $a_1$ (FORCOF(2))
           :      :        :  
           71-80  "       "   $a_7$ (FORCOF(8))
Card 2 (if need)  1-10  "       "   $a_8$ (FORCOF(9))
           :      :        :  
           71-80  "       "   $a_{15}$ (FORCOF(16))
Card 3 (if need)  1-10  "       "   $a_{16}$ (FORCOF(17))
           ;      ;        ;  
           1-10  "       "   $a_{N-1}$ (FORCOF(N))

The value of N must correspond to the value of NUMFOU input in columns 21-25 of CONTROL CARD.
E.3.10 Angles Station of Displacements and Stresses Print Out Card (7F10.5)

If value of NANGLE in columns 26-30 of CONTROL CARD less than or equal zero, no angle station card is needed.

**Columns**

<table>
<thead>
<tr>
<th>Column Range</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-10</td>
<td>Angle station 1 (XANG(1) in radians)</td>
</tr>
<tr>
<td>11-20</td>
<td>Angle station 2 (XANG(2) in radians)</td>
</tr>
<tr>
<td>21-30</td>
<td>Angle station 3 (XANG(3) in radians)</td>
</tr>
<tr>
<td>31-40</td>
<td>Angle station 4 (XANG(4) in radians)</td>
</tr>
<tr>
<td>41-50</td>
<td>Angle station 5 (XANG(5) in radians)</td>
</tr>
<tr>
<td>51-60</td>
<td>Angle station 6 (XANG(6) in radians)</td>
</tr>
<tr>
<td>61-70</td>
<td>Angle station 7 (XANG(7) in radians)</td>
</tr>
</tbody>
</table>

E.3.11 Output Control Card (2I5)

**Columns**

<table>
<thead>
<tr>
<th>Column Range</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-5</td>
<td>IPRIN1. Enter a 1 if the output summation of total nodal displacements are needed to be punched on cards, otherwise, these columns are left blank.</td>
</tr>
<tr>
<td>6-10</td>
<td>IPRIN2. Enter a 1 if the output summation of total soil element and shell nodal stresses are needed to be punched on cards, otherwise, these columns are left blank.</td>
</tr>
</tbody>
</table>
E. 4 Printed Information and Data

The input data is printed first. This includes the control numbers, the material properties, the nodal point coordinates, forces and properties, the element connectivity, the constraint data, the Fourier coefficients, and the angles of the results to be printed out.

For each non-zero Fourier coefficient harmonic, some computed information is printed out next. First, the generated displacement or rotation numbers for each nodal point in the model are printed. For each soil nodal point there are two possible displacements for Fourier numbers equal to zero and three for the succeeding Fourier terms. For each shell nodal point there is one more rotational displacement in addition to the translational displacements. Each number represents one equation in the system. Secondly, the code numbers for each element are printed. This information is generated from the input connectivity and the displacement numbers and is used to assemble the system of simultaneous equations. Finally, the maximum half-bandwidth for the system of simultaneous equations for each Fourier harmonic is printed.

The output data is printed next. First, for every nodal point the global displacement components and rotations (for shell node) are printed. The units are feet for displacements and radians for rotations.
The soil element stress coefficient on plane $0$ equal zero are printed next. A quadrilateral element will have four sets of components, one for the center of each sub-triangle. A triangular element will have one set of components. Listed for each element are the radial stress $\sigma_{rr}$, the vertical stress $\sigma_{zz}$, the tangential or circumferential stress $\sigma_{\theta\theta}$, the shearing stresses: $\tau_{rz}$, $\tau_{r\theta}$ and $\tau_{z\theta}$, the maximum stress, the minimum stress, the maximum shearing stress, and the direction of maximum stress from $r$-axis. The unit of stresses is kips per square foot, the unit of angle is degrees.

The next data printed is the shell nodal resultant coefficients on plane $0$ equal zero. Listed for each shell nodal point are the vertical membrane force $N_{zz}$, the hoop membrane force $N_{\theta\theta}$, the shearing force $N_{z\theta}$, the vertical moment $M_{zz}$, the circular moment $M_{\theta\theta}$, and the cross moment $M_{z\theta}$.

Finally, the summation of nodal displacements, soil element stresses, shell nodal stresses which were contributed from each non-zero Fourier term for different desired angles are printed.

The shell stresses can be computed from the following equations:

\[
\sigma_{zz} = \frac{N_{zz}}{12t} + \frac{12M_{zz}d}{t^3} \tag{E. 1}
\]
where

\[ \sigma_{\theta \theta} = \frac{N_{\theta \theta}}{12t} + \frac{12M_{\theta \theta}d}{t^3} \quad (E. 2) \]

\[ \sigma_{z \theta} = \frac{N_{z \theta}}{12t} + \frac{12M_{z \theta}d}{t^3} \quad (E. 3) \]

\[ \sigma_{zz}, \sigma_{\theta \theta}, \sigma_{z \theta} \] are vertical, hoop and shearing stresses, respectively, in ksi.

\[ N_{zz}, N_{\theta \theta}, N_{z \theta} \] are shell resultant forces, in kips/ft.

\[ M_{zz}, M_{\theta \theta}, M_{z \theta} \] are shell resultant moments, in kips-ft/ft.

\[ d \] is the distance along the normal from the mid-plane to the surface of shell, in inch.

\[ t \] is the thickness of shell, in inch.
Eighteen subroutines comprise the body of the computer program. They are controlled by call statements. The main program modifies the initial input boundary constrained nodal point data and input nodal forces for each non-zero Fourier term. The calling sequence of the subroutines is as follow:

1 INPUT
2 SUBCOD
3 GENCOD
4 INITL
5 STIFF
6 BIGK
7 FVECTR
8 BANWID
9 BANSOL
10 SOISTR
11 SHLSTR

Recycles subroutines 3-11 for each Fourier term

12 TSTRES

Recycles subroutines 1-12 for each problem.

The flow diagram for circular cell bulkhead analysis is illustrated in Figure F.1. The functions of the subroutines are as follows:
Figure F.1. Flow diagram for circular cell bulkhead analysis.
Initialize the structure to the initial condition
CALL INITL

Read boundary nodal forces from LUN 15 and modify for each non-zero Fourier term

Is there any surcharge load? NAXISF > 0
Yes
Add surcharge loads to nodal forces

No
Start DO-Loop (NN=1, NUMELT) to calculate and assemble element stiffness and load vector
CALL STIFF

Is this a soil element?
Yes

Is this a quadrilateral element?
Yes

Compute triangular element stiffness and body forces
CALL TRISTF, CALL SOILIN

No

Compute four sub-triangular element stiffness and body forces and assemble into one quadrilateral element
CALL QUAD

No

Compute shell element stiffness
CALL SHLSTF
CALL SHLING
CALL SHLTRN

Figure F.1. Continued.
Assemble element stiffness matrices into system stiffness matrix
CALL BIGK

Assemble system load vectors
CALL FVECTR

NN=NN+1
Is DO-Loop satisfied?

Is this the first Fourier term or first succeeding Fourier term?
NF1=0

Calculate band width
CALL BANWID

Solve the system governing equations
CALL BANSOL

Compute and output the magnitude of nodal displacement and soil element stress coefficients
CALL SOISTR

Compute and output the magnitude of shell nodal force coefficients
CALL SHLSTR

Figure F.1. Continued.
Is there any next Fourier term? 
NFOUR < NUMFOU

Is this the axisymmetric loading problem? 
NANGLE = 0

Is there any next problem? 
NPORB < NPROBS

Compute and output the total displacements and total stresses for the required θ -stations 
CALL TSTRES

STOP

Figure F.1. Continued.
1) Subroutine INPUT. The input data are read and written. This subroutine generates a mesh of quadrilateral or shell elements in the r-Z plane, checks the magnitude of all essential variables against their maximum permissible sizes. This subroutine also computes geometric properties of shell elements and stores on file LUN19. It calculates the nodal point force from the boundary tractions.

2) Subroutine SUBCOD. This subroutine generates the 4 x 9 matrix of code numbers used subsequently to combine four triangular elements into one quadrilateral element.

3) Subroutine GENCOD. This subroutine generates a code number for each element in the finite element model. These are subsequently used to assemble the system matrix of algebraic equations.

4) Subroutine INITL. This subroutine initializes a given problem to the initial conditions.

5) Subroutine SOILIN. This subroutine calculates the 10 required volume integrals for a triangular element and stores these data on file LUN16.

6) Subroutine TRISTF. This subroutine generates the 9 x 9 triangular element stiffness matrix and computes the element body force vectors for the non-zero Fourier term. Subroutine SOILIN is called for first non-zero Fourier term,
or read the corresponding volume integral data of triangular element from LUN16 for succeeding Fourier term.

7) Subroutine QUAD. This subroutine computes the center nodal point of each quadrilateral element using the mean value of the 4 nodes of that element and then uses code numbers to assemble four triangular elements into a quadrilateral element. The 15 x 15 quadrilateral stiffness matrix and associated load vector are obtained. Subroutine TRISTF is called.

8) Subroutine SHLING. This subroutine computes shell volume integrals using a 10 point Gaussian Quadrature Formula and stores the results on file LUN17.

9) Subroutine SHLTRN. This subroutine generates 8 x 8 displacement transformation matrix for shell element.

10) Subroutine SHLSTF. This subroutine computes 8 x 8 shell element stiffness matrix referred to the local coordinates system, then transforms element stiffness matrix to global coordinates by calling subroutine SHLTRN.

11) Subroutine STIFF. This subroutine performs the static condensation process which reduces the 15 x 15 quadrilateral stiffness matrix to a 12 x 12 element matrix. Subroutine QUAD is called.
12) Subroutine BIGK. This subroutine uses the code number technique to assemble the element matrices into a system matrix.

13) Subroutine FVECTR. This subroutine uses the code number technique to assemble the system load vector.

14) Subroutine BANWID. This subroutine computes the band width of the system stiffness matrix for each Fourier term.

15) Subroutine BANSOL. This is a standard subroutine used to solve the system of simultaneous equations by Gaussian elimination.

16) Subroutine SOISTR. This subroutine outputs the magnitude of the Fourier coefficients of the nodal point displacements, stores the nodal point displacement coefficients of each Fourier term on file LUN8. It computes and outputs the magnitude of the Fourier coefficients of the element stresses at the center node of each element, then stores the element stress coefficients on file LUN10. This subroutine also computes the principal stresses and their directions at the plane \( \theta = 0 \) (plane of symmetric loading).

17) Subroutine SHLSTR. This subroutine computes and outputs the magnitude of the Fourier coefficients of the generalized shell stresses at the shell nodal points. The shell element stress coefficients for each Fourier term are stored on
18) Subroutine TSTRES. This subroutine computes and outputs the total displacement for each of the 3 components at each soil nodal point and the 4 components at each shell nodal point for the required 0-stations by summing the contribution of each Fourier term at each nodal point. It also computes and outputs the total stress components for each element.
APPENDIX G

Program Listing
PROGRAM BULKHEAD

COMMON /41/ WINPT, NUMELT, NCONP, NSMEL'
COMMON /12/ COOR(57,2), IS0NNI57(5), NPTYPE(66), NPCOM(66)

IF BOTH SOIL ELEMENT AND SHELL ELEMENT ARE PRESENT
NCUT=12

COMMON /a3/ KNOVAC66, 30, NUM00m(56,4)

GO TO 13

GoMmuN /91/ ACELR, ACELZ, ANGFQ, DENS(12), 0, 0, NP, 0.
COMMON /C1/ FORCE(66,4)

IF(NP.NE.4.NTEM(NC,1)) GO TO 3

COMMON /F1/ NFO11R, NFOF2, NFTIPMF1, NFOUNTNF.

DO 5 L=1,5

COMMON /F2/ FCRCOF(50), XAMG(7), NANGLE

GO TO 6

DIMENSION KNTEM(66,5), 3 CONTINUE

REAL NF, NF2

KNOVA(NP,3)=0

READ(60,100) NPROBS

KN04A(NP,4)=1

100 FORMAT(I5)

IF(NPTYPE(NP).E0.3) KNOVA(NP,5)=1

NPROBS=1

IF (COOR(NP,1).NE.0.3) GO TO 2

10 CALL INPUT KNOVA(NP,2)=1

CALL CORR(COORIN,3,0.00001

2 CONTINUE

STORE NODAL BOUNDARY DATA ON FILE UNIT NO. 15

NCONP=NUMNPT

GO TO 45

REMIND 19

7 00 B NP,1, NMNPT WRITE115,1, NCONP, I(KNOVATAII), I21.5), N.1, NCONP)

40 4 NC=1, NCONP

IF(NP.NE.KNTEM(NC,1)) GO TO 9

STORE NODAL FORCING ON FILE UNIT 15

DO 11 1=1,9

KNOVA(NP,1)=NPWILL STORED ON FILES, LJM 6,7,8,10,12,20,21,22,23

C 00 4 NC=1, NCONP

C BEGIN FOURIER LOOP

NFOUQ=1

DO 20 M=1,4

IF(MF0UQ).GT.0.01 GO TO 300

DO 21 M=1,4

DO 20 M=1,4

DO 16 NP=1, NUMNPT

DO 17 NC=1, NCONP

IF(TNp.NE.KNTEM(NC,1)) GO TO 17

NCN=NGN+1

IF(X.FS.0.0) GO TO 310

00 15 L=1,9

NF.NF.N=1

NF2=NF2+1

KNOVA(NP,L)=KNTEM(NG,L)

GO TO 16

16 CONTINUE

IF(NPTYPE(NP).NE.3) GO TO 16

KNOVA(NP,1)=NP

KN04A(NP,2)=

KNOVA(NP,3)=1

KNOVA(NP,4)=

KNOVA(NP,5)=1

CONTINUE

NCW4PI(KNOVAINP,I/0.1,05), N31, NCONP)

16 CONTINUE

NDONP=NCN

C C017.1. E3', JNOARY CONDITIONS FOR EACH NON-ZERO

FOURIER TERM OF DIFFERENT SYSTEMS

DO 26 N=1, NCONP

26 KTEM(NC,L)=KNOVAINP(NC,K)

IF(NFOUQ.EQ.1.1) GO TO 1

IF(NF1.EQ.1) GO TO 31

*** IF NO SHELL ELEMENTS ARE PRESENT, NCUT=0

IF(NF1.EQ.13) GO TO 45

*** IF NO SHELL ELEMENTS ARE PRESENT, NCUT=13

IF(NF1.EQ.13) GO TO 7

*** IF OTHERS ARE PRESENT AND SHELL ELEMENT ARE PRESENT

NCUT=12

GO TO 13

100 FORMAT(15)

DO 3 0=1, NUMNPT

IF(NF1.EQ.1) GO TO 3

5 0=1, NUMNPT

KDAM(NP,1)=KNTEM(NC,1)

CONTINUE

KNOVAINP(NP,1)=NP

KN04A(NP,2)=

KNOVA(NP,3)=1

KNOVA(NP,4)=

KNOVA(NP,5)=1

CONTINUE

NCW4PI(KNOVAINP,I/0.1,05), N31, NCONP)

16 CONTINUE

*** GENERATE ELEMENT CODE NUMBERS AND INITIALIZE THE STRUCTURES

45 CALL DENCOD

41 CALL INITL

*** READ NODAL POINT BOUNDARY LOADS FROM FILE LUN 15

READ(15) (FORCE(NP,1),I=1,4), NP=1, NUMNPT

*** MODIFY BOUNDARY NODAL FORCES FOR FOURIER TERM

DO 109 N=1, NUMNPT

109 FORCE(NP,2)=FORCE(NP,1)*X

IF(NF1.EQ.14) GO TO 108
ADD AXI-SYMMETRIC SURCHARGE LOADS

DATA (MAXNP=66), (MAXEL=57), (MAXMAT=12), (MAXFOU=5)

REAL NP, NF

*** READ AND PRINT OF CONTROL DATA

WRITE(61,1)
1 FORMAT(11,NUMMAT, NUMEL, NUMPC, NUMFOU, NANGLE, REAL NP, NF)

READ INPUT DATA

DO 1,5 K=1,4

105 FORCE(N,K)=FORCE(N,K)+FORCE(J,K)

GO TO 10

END

*** ADD AXI SYMMETRIC SURCHARGE LOADS

DATA (MAXNP=66), (MAXEL=57), (MAXMAT=12), (MAXFOU=5)

REAL NP, NF

IF(NF.NE.N) GO TO 1E1

READ AND PRINT OF CONTROL DATA

DO 1,5 K=1,4

105 FORCE(N,K)=FORCE(N,K)+FORCE(J,K)

GO TO 10

END

*** GENERATE SYSTEM STIFFNESS MATRIX AND SYSTEM LOAD VECTOR

DO 110 NPT=1,MAXNP

110 CONTINUE

END

*** ADD AXI SYMMETRIC SURCHARGE LOADS

DATA (MAXNP=66), (MAXEL=57), (MAXMAT=12), (MAXFOU=5)

REAL NP, NF

IF(NF.NE.N) GO TO 1E1

READ AND PRINT OF CONTROL DATA

DO 1,5 K=1,4

105 FORCE(N,K)=FORCE(N,K)+FORCE(J,K)

GO TO 10

END

*** GENERATE SYSTEM STIFFNESS MATRIX AND SYSTEM LOAD VECTOR

DO 110 NPT=1,MAXNP

110 CONTINUE

END

*** ADD AXI SYMMETRIC SURCHARGE LOADS

DATA (MAXNP=66), (MAXEL=57), (MAXMAT=12), (MAXFOU=5)

REAL NP, NF

IF(NF.NE.N) GO TO 1E1

READ AND PRINT OF CONTROL DATA

DO 1,5 K=1,4

105 FORCE(N,K)=FORCE(N,K)+FORCE(J,K)

GO TO 10

END

*** GENERATE SYSTEM STIFFNESS MATRIX AND SYSTEM LOAD VECTOR

DO 110 NPT=1,MAXNP

110 CONTINUE

END

*** ADD AXI SYMMETRIC SURCHARGE LOADS

DATA (MAXNP=66), (MAXEL=57), (MAXMAT=12), (MAXFOU=5)

REAL NP, NF

IF(NF.NE.N) GO TO 1E1

READ AND PRINT OF CONTROL DATA

DO 1,5 K=1,4

105 FORCE(N,K)=FORCE(N,K)+FORCE(J,K)

GO TO 10

END
**READ AND PRINT ELEMENT PROPERTIES**

**COMPUTE SHELL ELEMENT PROPERTIES**

**READ AND PRINT CONstrained BOUNDARY NODAL POINTS**

**READ PRESSURE AND/OR SHEAR BOUNDARY STRESSES**

**COMPUTE SURFACE INTEGRALS**

**READ AXISYMMETRIC SURCHARGE LOADS**
COMMON /AI/ NPT, NELT, NCONP, NCUT, NSHEL
COMMON /05/ KNOVAT, N330M
COMMON /11/ NFOUR, NFOF2, NDNFOU, XNE, NFT, NF1
DO 412 N, NAXISF
REAL NF, NF2
FORMAT(15.5)
NF = NFOUR - 1
RHET(61, 4141, RAx(N), FORC(N, L), L=1, 4)
FORMAT(15.5)
ENDDO
C *** TO GENERATE CODE NUMBERS IN TERM OF NODAL POINT NUMBER
DO 12 I=1, NELT
C TO
C 2 CONTINUE
IF (ICONN(L).NE.0) GO TO 18
DO 111 O=1, 4
IF (ICONN(L).NE.0) GO TO 19
DO 199 J=1, 9
C IF
IF (ICODE(I, J).EQ.0) CONTINUE
STRO(15, 20)
CONTINUE
STOP
12 CONTINUE
17 SUBROUTINE GENCOD
COMMON /A5/ ICODE(40)
C
DO 199 I=1, 9
DO 199 I=1, 9
IF (ICODE(I).EQ.0) RETURN
STRO(15, 20)
CONTINUE
STOP
17 SUBROUTINE GENCOD
COMMON /A5/ ICODE(40)
**SUBROUTINE INIT**

**COMMON** /Al/ NUMNPT, NUMELT, NCONP, NCUT

**COMMON** /A5/ ICOOE(NQUAD,N)

**COMMON** /C1/ 2F(9), 0(15), 5TF(1)

**COMMON** /01/ S(9,0), SYSK(15,15), SA(9,9)

**COMMON** /El/ XI(10), A(9,9)

**COMMON** /F1/ NFOUROF, NFO2, NUMFOU, ANF, NFT, NFA

**REAL** NE, NF2

**DIMENSION** NP(5)

**SUBROUTINE QUAO(NN)**

**SUBROUTINE QUAD(NN)**

**SUBROUTINE TRISTF(NN,II,JJ, KK)**

**SUBROUTINE QUAD(NN)**

***** THIS SUBROUTINE USES THE CODE NUMBER TECHNIQUE TO ASSEMBLE QUADRILATERAL ELEMENTS INTO A QUADRILATERAL ELEMENT ***

**COMMON** //26// C002(67,2), ICONN(57,5), NPTYPE(66), NPCON(66)

**COMMON** //27// MDL, AC3, ACELL, GFG, DENSI(12)

**COMMON** //28// C11, C12, C13, C23, C33, C44, C55, C66

**COMMON** //29// NFOUR, NFO2, NUMFOU, NFX, NFT, NFI

**EXTERNAL** MDL, ACELL, GFG, DENSI(12)

**EXTERNAL** C11, C12, C13, C23, C33, C44, C55, C66

**CALL** TRISTF(NN)

**CALL** QUAO(NN)

**DIMENSION** NE, NF2

**SUBROUTINE TRISTF(NN,II,JJ, KK)**

**SUBROUTINE TRISTF(NN,II,JJ, KK)**

***** THIS SUBROUTINE FORMS THE TRIANGULAR ELEMENT STIFFNESS MATRIX AND BODY FORCE VECTOR FOR THE NON-ZERO FOURIER TERM ***

**COMMON** /27// C002(67,2), ICONN(57,5), NPTYPE(66), NPCON(66)

**COMMON** /28// MDL, AC3, ACELL, GFG, DENSI(12)

**COMMON** /29// C11, C12, C13, C23, C33, C44, C55, C66

**COMMON** /30// NFOUR, NFO2, NUMFOU, NFX, NFT, NFI

**EXTERNAL** MDL, ACELL, GFG, DENSI(12)

**EXTERNAL** C11, C12, C13, C23, C33, C44, C55, C66

**CALL** TRISTF(NN)

**CALL** QUAO(NN)

**ITERATE** MDL, ACELL, GFG, DENSI(12)

**ITERATE** C11, C12, C13, C23, C33, C44, C55, C66
**FORM STIFFNESS MATRICES IN GENERALIZED COORDINATES**

CALL SOILIN(NN,II,JJ,KK)

GO TO 17

**15** RE32(IJ) = XI(NN,II,JJ,10), IF(NFOUR.GT.2) GO TO 70

**17** X = M15*NF2

**S(1,1) = 1**

**S(1,2) = XI(1)**

**S(1,3) = XI(5)**

**S(2,1) = XI(7)**

**S(2,2) = XI(1)**

**S(2,3) = XI(4)**

**S(3,1) = XI(6)**

**S(3,2) = XI(2)**

**S(3,3) = XI(3)**

**S(4,1) = XI(8)**

**S(4,2) = XI(2)**

**S(4,3) = XI(4)**

**S(5,1) = XI(9)**

**S(5,2) = XI(4)**

**S(5,3) = XI(1)**

**S(6,1) = XI(10)**

**S(6,2) = XI(1)**

**S(6,3) = XI(7)**

**S(7,1) = XI(1)**

**S(7,2) = XI(2)**

**S(7,3) = XI(5)**

**S(8,1) = XI(4)**

**S(8,2) = XI(4)**

**S(8,3) = XI(1)**

**S(9,1) = XI(6)**

**S(9,2) = XI(1)**

**S(9,3) = XI(5)**

**S(10,1) = XI(3)**

**S(10,2) = XI(5)**

**S(10,3) = XI(7)**

**END**

**SUBROUTINE SOILIN(NN,II,JJ,KK)**

THIS SUBROUTINE CALCULATES THE TRIANGULAR VOLUME INTEGRALS AND FORMS THE DISPLACEMENT TRANSFORMATION MATRIX FOR THE STIFFNESS MATRIX AND BODY FORCE VECTOR

**COMMON /A2/ CONN(67,21),NPTY(166),NPCOM(66)**

**COMMON /E1/ XNFOUR,NF,NF2**

**DIMENSION XM(6),R(6),Z(6)**

**REAL**

**NF,NF2**

**DO 20 I=1,3**

**XMI(I) = R(I)**

**DO 30 I=1,6**

**A = XM(I)**

**B = R(I)**

**C = Z(I)**

**D = AB**

**XIB = XT(I)**

**END**

**DO 20 J=1,3**

**XIB = XI(J)**

**CONTINUE**

**DO 20 K=1,9**

**BF(K) = BF(I)**

**DO 20 L=1,9**

**SA(K,L) = 0.0**

**SA(K,L) = SA(K,L) + (M,K) * SA(M,L)**

**RETURN**

**END**
**SUBROUTINE FVECTR (N)**

**This subroutine uses code numbers to assemble the system load vector.**

**COMMON /A4/ LADOM**
**COMMON /A2/ CooP(67,2), ICONN(57,5), NPTYPE(166), NPCON(66)**
**COMMON /AL/ NNOV(60,5)**
**COMMON /A4/ NNOO(57)12)**
**COMMON /AL/ F(182)**

**SUBROUTINE BANSOL**

**REAL NF, NF2**

'**Calculate displacements**'

**GO TO 5**

**IF(KK .EQ. 3) GO TO 9**

**IF(K - L) 11, 12, 11**

**12 END**

**SUBROUTINE SDCSTR**

**COMMON /A1/ NNUMPT, NUMELT, NCONP, NNCUT, NSHELT**
**COMMON /A2/ COD(467,2,1), ICONN(57,1), NPTYPE(66), NPCOM(66)**
**COMMON /A3/ COOR(467,2), ICONN(57,5), NPCOM(66)**
**COMMON /A1/ NCONP, NCUT, NGMELT**

**SUBROUTINE SDCSTR**

**RETURN**

**C**

**COMMON /A1/ NNUMPT, NUMELT, NCONP, NNCUT, NSHELT**
**COMMON /A2/ COD(467,2,1), ICONN(57,1), NPTYPE(66), NPCOM(66)**
**COMMON /A3/ COOR(467,2), ICONN(57,5), NPCOM(66)**
**COMMON /A1/ NCONP, NCUT, NGMELT**

**RETURN**
DIMENSION TOISP(15), ST(12), DISP(66,4)
EQUIVALENCE (FORC(1,1),DISP(1,11))

*** PRINT OF NODAL POINT VARIABLES ***
DO 41 IN=1,NUMNPT
41 DISP(IN,4)=0.0
NFR = FORC(1,1)
DO 55 IN=1,NUMNPT
IF(CONN(IN,1).LT.0) GO TO 50
J=IN
55 DISP(IN,J)=DISP(IN,J)+F(K)
41 J=1,4
GO TO 12

*** IF LOADINGS ARE AXISYMMETRIC, MANGLE LESS THAN ZERO ***
IF(NEQ2).LE.-LJ) GO TO 51

*** STORE NODAL DISPLACEMENT COEFFICIENTS ON FILE, LUN 67,6 ***
IF(FORC(1,1).LT.0) GO TO 1
NFR= FORC(1,1)
GO TO 31
41 J=1,4
24 WRITE(6,50) (DISP(IN,J),J=1,4)
GO TO 50

*** IF NO SOIL ELEMENT IS PRESENT, NCUT=0 ***
IF(ICONN(NN,4).EQ.0) GO TO 50

*** CALCULATION OF CENTER NODE VARIABLES ***
DO 12 IN=1,NUMNPT
ST(IN)=0.0
IF(ICONN(IN,4).EQ.0) GO TO 104

*** INITIALIZE STRAINS ***
EP2=0.0
EP7=0.0
EP8=0.0
EP9=0.0
EP10=0.0
EP11=0.0
GO TO 15
15 IF(CONN(NN,4).EQ.0) GO TO 104

*** THIS IS A QUADRILATERAL ELEMENT ***
CALL QUAD4(NN)

C *** CALCULATE STRESSES ***
ST(1)=EP2
ST(2)=EP7
ST(3)=EP8
ST(4)=EP9
ST(5) = C55*EPRT
ST(6) = C66*EPRT

IF(NFOUR > 1) GO TO 26
EPZZ = EP17 + E77
EPTT = ITOISR(10) + NFTOISR(12/RK + EPTT
ST(5) = 0.0
LPRZ = FPR7 + ERZ
ST(6) = 6.0
EPRT = EPRT + CERT - NFTOISP(10) / RK

26 IF(NAIiSLE.LT.0) GO TO 24
EPZT = EPZI + EZT - NFTOISP(11/RK
IF(INFO2...T.5) GO TO 5
IF(NFOUR.LT.9) GO TO 6

CIR(4FOUR.LT.17) GO TO 8
ST(1) = C11*EPRR + C12*EPZZ + C13*EPTT
WRITE(23) NN,II,JJ, (ST(N).04 = 1.6
ST(2) = C12*EPRR + C22*EPZZ + C23*EPTT
GO TO 24
ST(3) = C13 + EPRP + C23*EPZZ + C33*EPTT
WRITE(101 NN,II,JJ, (ST(N).N = 1.6)
ST(4) = C44*EPRZ
GO TO 24
ST(5) = G55 + ERRT
WRITE(23) NN,II,JJ, (ST(N).N = 1.6
ST(6) = 0.6 + EPZT
GO TO 24
IF(N0!GLE.LT.0) GO TO 20
ST(7) = OM0IRP + OR0HR
WRITE(23) NN,II,JJ, (ST(N).N = 1.6
ST(8) = CM0HR - OM0HR
WRITE(23) NN,II,JJ, (ST(N).N = 1.6
ST(9) = RMOH2
WRITE(23) NN,II,JJ, (ST(N).N = 1.6

CONTINUE
GO TO 23

22 IF(ST(1).LT.ST(2)) GO TO 22
CONTINUE
GO TO 33

10 IF(ST(1).LT.ST(2)) GO TO 10
IF(ST(1).LT. ST(2)) GO TO 33

WRITE(211 NN,II,JJ, (ST(N).N = 1.6
IF(NFOUR.LT.5) GO TO 42
GO TO 24
IF(NFOUR.LT.9) GO TO 43

WRITE(22) NN,II,JJ, (ST(N).N = 1.6
IF(NFOUR.LT.13) GO TO 44

WRITE(23) NN,II,JJ, (ST(N).N = 1.6

WRITE(24) NN,II,JJ, (ST(N).N = 1.6
WRITE(25) NN,II,JJ, (ST(N).N = 1.6
WRITE(26) NN,II,JJ, (ST(N).N = 1.6

WRITE(27) NN,II,JJ, (ST(N).N = 1.6
WRITE(28) NN,II,JJ, (ST(N).N = 1.6
WRITE(29) NN,II,JJ, (ST(N).N = 1.6
WRITE(30) NN,II,JJ, (ST(N).N = 1.6
WRITE(31) NN,II,JJ, (ST(N).N = 1.6
WRITE(32) NN,II,JJ, (ST(N).N = 1.6
WRITE(33) NN,II,JJ, (ST(N).N = 1.6
WRITE(34) NN,II,JJ, (ST(N).N = 1.6
WRITE(35) NN,II,JJ, (ST(N).N = 1.6
WRITE(36) NN,II,JJ, (ST(N).N = 1.6
WRITE(37) NN,II,JJ, (ST(N).N = 1.6
WRITE(38) NN,II,JJ, (ST(N).N = 1.6
WRITE(39) NN,II,JJ, (ST(N).N = 1.6
WRITE(40) NN,II,JJ, (ST(N).N = 1.6
WRITE(41) NN,II,JJ, (ST(N).N = 1.6
WRITE(42) NN,II,JJ, (ST(N).N = 1.6
WRITE(43) NN,II,JJ, (ST(N).N = 1.6
WRITE(44) NN,II,JJ, (ST(N).N = 1.6
WRITE(45) NN,II,JJ, (ST(N).N = 1.6
WRITE(46) NN,II,JJ, (ST(N).N = 1.6
WRITE(47) NN,II,JJ, (ST(N).N = 1.6
WRITE(48) NN,II,JJ, (ST(N).N = 1.6
WRITE(49) NN,II,JJ, (ST(N).N = 1.6
WRITE(50) NN,II,JJ, (ST(N).N = 1.6
WRITE(51) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(52) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(53) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(54) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(55) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(56) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(57) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(58) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(59) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(60) NN,II,JJ, (ST(L).L = 1.9, ANGLE
WRITE(61) NN,II,JJ, (ST(L).L = 1.9, ANGLE
RETURN
END
**COMPUTE GENERALIZED DISPLACEMENT COORDINATES, PN**

\[ \text{RETURN} \]

---

**SUBROUTINE TSTRES**

\[ \text{END} \]

---

**COMPUTE DISPLACEMENTS**

\[ \text{RETURN} \]

---

**PRINT SHELL STRESSES FOR EACH FOURIER COEFFICIENT**

\[ \text{WRITE}((1X,200)) \]

---
DO J=1,4  
K=J;  
DO 26 M=5,6  
IF(K.EQ.0) GO TO 70  
STRES(N,M)=STRES(N,M)+ST(4)  
260 CONTINUE  
DO 25 L=1,4  
K=41';  
DO 26 M=5,6  
IF(K.LT.3) GO TO 60  
IF(IPRINT2.EQ.0) GO TO 411  
DISL)=DIS(N,L)*ISP(N,L)  
GO TO 50  
411 IF(NFOUR.LE.NF3) GO TO 270  
WRITE(61,410) N,II,JJ,(STRES(N01)021,6)  
70 OIG(N,L)=0.0  
CONTINUE  
READ(23) II,JJ,ISET(J),J=1,6)  
CONTINUE  
READ(21) M,II,JJ,(ST(J),J=1.6)  
WRITE(51.25) N,(0IS(N,L).L=1.4)  
58 IF(NFOUR.LE.NF3) GO TO 270  
CONTINUE  
READ(22) M,II,JJ,(ST(J),J=1.6)  
DO 27 N=1,4  
WRT15,410) N,(STRES(N.K).K=1,6)  
35 FORMAT(A1 TOTAL FOURIER SOIL ELEMENT STRESSES AT :.F7.3,* DEGREES,)  
GO TO 36  
36 NCUT= NCUT -1  
CONTINUE  
READ(SOIL ELEMENT STRESS COEFFICIENTS FROM LUN 12)  
100 CONTINUE  
READ(SHELL STRESS COEFFICIENTS FROM LUN 12)  
250 STRES(N,M)=STRES(N,M)+ST(M)*CO  
CONTINUE  
READ SHELL STRESS COEFFICIENTS FROM LUN 12