## AN ABSTRACT OF THE DISSERTATION OF

Evan Rajbhandari for the degree of Doctor of Philosophy in Mathematics presented on February 25, 2022.

Title: Optimization under Uncertainty of a Magnetohydrodynamics Generator

Abstract approved: $\qquad$
Nathan L. Gibson

Within this dissertation, we develop tools and techniques to demonstrate the feasibility of real-time optimization of a magnetohydrodynamics generator. To ease computational complexity, we work on the kinematic magnetohydrodynamic system, prescribing the fluid-flow and model the material response of the system through an updated Generalized Ohm's law. We focus on two optimization difficulties specific for this application: model accuracy and feasibility. These both are crucial in the determination of optimal operating conditions, and thus optimal power.

To address these concerns, several concepts are introduced to the model. First, we introduce the ion-slip parameter, a term which characterizes the material interactions between the fluid and electromagnetic fields. It is shown that this mechanism does not disrupt the well-posedness of the system. We then develop a function space parameter estimation convergent deterministic parameter estimation scheme, implying that recovery of a more realistic functional parameter set is possible.

We also discuss the inclusion of uncertainty within the theoretical MHD framework. This uncertainty is introduced through the parameters, viewing them as random processes rather than deterministic functions. We extend the well-posedness of the deterministic system to the stochastic system, demonstrating that the uncertain forward problem is well-posed, and that the finite-dimensional approximation to the inverse problem method stable.

We validate the theoretical results using simulations for both the deterministic and stochastic magnetohydrodynamic systems. For the numerical implementation of the deterministic system, we make use of COMSOL, a finite-element based differential equation solver. We investigate two distinct Faraday geometries, the continuous and the segmented. To verify the numerical model, we develop new ideal power equations for each geometry, again introducing the concept of the ion-slip mechanism into previous theory. Under the deterministic scheme, we also implement a numerical method for recovering parameters from fabricated 'true' data. Furthermore, the results from these numerical tests again confirmed the need for uncertainty to be included, as recovery was not only sensitive to noise, but also asymmetric with respect to expected error.

To verify the stochastic theory developed for the forward and inverse problem, we utilize the cross-platform compatibility of Matlab and COMSOL. We use the native optimization techniques and data manipulation capabilities of Matlab, paired with the deterministic forward solver in COMSOL. We apply an existing numerical method, stochastic collocation, under the new context of the kinematic MHD system, for the numerical treatment of the propagation of uncertainty within the forward problem. This method effectively capitalizes on assumed orthogonality of a finite number of random variables describing the system. We perform an error analysis of stochastic collocation, and then demonstrate that the inclusion of uncertainty does not propagate linearly through the magnetohydrodynamics system, i.e. the expected value of the solutions is not the deterministic solution of the expected value of the parameter set. We then confirm the numerical theory as well, discussing the necessary assumptions and implementation steps to apply the stochastic collocation in the uncertain parameter estimation problem. Finally, we turn to a numerical demonstration of the feasibility of a two stage optimization of an MHD generator. We use fabricated data to recover the parameters' distributions on the domain. Using these recovered distributions, we then optimize the performance of the generator, using a single optimization variable.
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Optimization under Uncertainty of a Magnetohydrodynamics Generator
by
Evan Rajbhandari

## A DISSERTATION

submitted to Oregon State University
in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Presented February 25, 2022
Commencement June 2022

Doctor of Philosophy dissertation of Evan Rajbhandari presented on February 25, 2022.

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Evan Rajbhandari, Author

## ACKNOWLEDGEMENTS

Undoubtedly, without the help of many others, this document would not exist. Without Dr. Nathan Gibson's support, and encouragement, this work would not have been possible. In particular, he gave much needed direction, guiding this project and yet allowing me to explore and grow as a mathematician. Similarly, the feedback and unique engineering perspective provided by Dr. C.R. Woodside was essential in this project's development, and whose funding allowed this work to continue. The work was also supported by the United States Department of Energy's Crosscutting Research Program, an appointment to the Science Education Programs at the National Energy Technology Laboratory, administered by ORAU through the United States Department of Energy Oak Ridge Institute for Science and Education, and by the National Science Foundation, grant number DMS-2012882.

I also would like to thank my entire family, as without their dedication to my success, I would not have had this opportunity. In particular, I thank both my parents, Amsu and Sheryl, for always stressing the importance of education, but also teaching me of balance in life, an invaluable skill for graduate school. I would also like to thank both my brother, Myles, and sister, Chloe for always providing a unique perspective on my work, and helping to bring laughter into my life.

I also would like to thank all of the faculty from the math department of Oregon State University, whose motivation and drive inspired my own, and who never ceased to lend a helpful ear, be it for class or research. As well, I must thank the office staff, whose dedication to the department allows it to thrive. Finally, I would thank Lisa Bigler, without whom, the work would have consumed me long ago.

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# Optimization under Uncertainty of a Magnetohydrodynamics Generator 

## 1 Introduction

Magnetohydrodynamics (MHD) generators harvest power from an electric field and current density which spontaneously arise from plasma flowing through an intense applied magnetic field. In this thesis, we demonstrate the feasibility of optimization under uncertainty of an MHD generator, of a given geometry. To do so, we develop both theoretical and numerical tools, derive analytic results, and implement several different numerical approaches with a focus on MHD generators.

MHD is the study of an electrically-conductive medium flowing through a magnetic field [42]. It is a multi-physics problem, governing the behavior of fluid flow, electric fields and currents, magnetic fields, and their interactions. It is complex, being a non-linear coupled system of equations of many different components and scales. Thus, to simplify the numerical implementation, we prescribe the fluid-flow, leading to the kinematic MHD model. This model consists of Maxwell's equations, which describe the coupling of the electric field, magnetic field, and electric currents within the system, as well as constitutive laws to complete the system.

In the MHD setting, the constitutive law which completes the system is the Generalized Ohm's law. This law couples the current density with the electric field and magnetic field, describing the response of the material to electric and magnetic fields, caused by the Lorenz force. This force, and thus the complete law, describes the impact of these fields on any point-charge. However, these effects are inversely related to the mass of the point-charge, and thus their effects on ions is often neglected [36]. In the law itself, this constitutes a neglect of the ion-slip parameter. For instance,
[56] analyzes the performance of a particular MHD generator, but neglects the ionslip parameter entirely. This can lead to an overestimation of the performance of the generator. Some other models go further, neglecting the Hall parameter, an analogous parameter to the ion-slip term, but for electrons; one such model is the MACH2 MHD solver [27]. Both of these terms are scalar, and take into account the mass of the associated point-charge, as well as the strength of the magnetic field, by taking the product of the two. The formal parameter definitions can be seen in Section 3.2. It will later be shown that neglecting both parameters in the MHD generator setting can lead to non-optimal operating conditions, and sub-par performance. We will also explore the qualitative impacts these parameters have on the system, such as causing a tilt in the electric field.

To address the feasibility of optimization within an MHD generator setting, we now explore some of the model-specific difficulties we must overcome. The first of these challenges is model accuracy. In order to properly assess the optimization problem, we implement and verify a numerical model of the MHD generator. This problem encompasses difficult geometrical properties, due to the physical shape of the MHD generator, and also includes solving complex multi-physics partial differential equations (PDEs). Thus, we choose to utilize the finite-element numerical software COMSOL [32] as our deterministic solver, which has a native library for solving electrostatic systems. To do so, we develop theory regarding the inclusion of the ion-slip parameter in the kinematic MHD model, which will allow the utilization of this software, as well as its LiveLink compatibility with Matlab [28]. Also included in model accuracy is the uncertainty in the prescription of system parameters. Although a reduction in the uncertainty of some parameters can be expected, such as fluid-flow, due to more accurate modeling [21], the parameters of the MHD system are not directly observable, and therefore must be inferred from exterior measurements. Thus, uncertainty must be considered in the numerical model, as well as in the optimization problem.

The second challenge we must overcome is the feasibility of real-time optimization. As mentioned previously, the state parameters of the MHD system are not directly measurable, and must be estimated by observations of the electric potential and electric current. Therefore, accurate parameter estimation methods must be available, in order to determine the optimal set of operating conditions. As well, as noted above, the prescription of these parameters is uncertain, and thus, so is the parameter estimation from data. It follows that the parameter estimation problem must also include uncertainty, much as the forward problem does.

To address these obstacles, the thesis is divided into three parts. The first is an examination of the kinematic MHD theory. We investigate the well-posedness of both the forward problem and subsequent inverse problem. We then turn to the inclusion of uncertainty in these problems, again showing existence and uniqueness of solutions. We also demonstrate that the approximations made in the parameter estimation problems result in a method-stable numerical estimation problem. The next portion of the thesis showcases numerical results that demonstrate the theory discussed in the first portion. We validate the numerical model with newly developed ideal power equations, and then demonstrate the feasibility of the parameter estimation problem. Furthermore, we examine how the shape and variance of the parameters' distributions impacts the shape and variance of the solutions, and demonstrate the continuous dependence of the solutions' distributions upon the parameters'. The final portion of the thesis is a numerical investigation into the optimization of the numerical model, representing the possibility of real-time optimization of an MHD generator. The work has resulted in the papers [38, 39, 40].

## 2 Background

### 2.1 Mangetohydrodynamics Generators

MHD generators utilize the physics of MHD to generate power within a channel. They have been investigated previously as a source of steady power [42], but recently have been shown to have increased potential [26, 54], due to improved production methods of large-scale and high-Tesla magnets, among other technological improvements. The overarching layout of all MHD generators are the same; there is some electrically conductive fluid forced through a magnetic field. The fluid and magnetic field interact to generate a Lorenz force, which lies in the direction of the curl of the two. This Lorenz force is a measure of the fields' collective impact on the point-charges within the fluid, and is what causes the generation of the electric field and current. This force is described by the generalized Ohm's law, which characterizes the material response of the fluid to the electromagnetic fields. A much more robust investigation into the Lorenz force, the generalized Ohm's law, and the parameters of the kinematic MHD system can be seen in Section 3.2 and in [50]. Further analyses on the physics behind MHD generators can also be found in Rosa [42].

In general, every direct-fired MHD generator is a complex system of sonic accelerators, combustion chambers, among other technical components. One simple schematic can be seen in Figure 2.1. For our purposes, we consider only the portion of an MHD generator in which the kinematic MHD equations apply. Thus, for the remainder of this paper, any mention of an MHD generator refers to only this specific portion, and within Figure 2.1, this is the section referred to be MHD Channel. This section of the generator contains a channel, which houses the fluid, electrodes, to harvest the power, and an array of connections in order to complete the circuit and attach the generator to some load. There are many different types of such generators, each with unique applications and benefits [50]. However, the governing equations that are pertinent in each section remain the same, and thus the theory we develop in the


FIGURE 2.1: Example schematic of an open-cycle MHD generator [54].
theoretical sections apply to a wide range of MHD generators. We choose a specific type of MHD generator, the Faraday generator, which we will use for the remainder of this thesis, for numerical implementation ease, and demonstrated performance [22].

### 2.1.1 MHD Generator Components

An MHD generator is composed of many components. However, our simple representation of an MHD generator is a domain containing a channel, electrode(s), and a load. Each of these components are governed by the MHD equations to some capacity. However, as each component is composed of very different materials, the generalized Ohm's law applied to each changes to reflect this. We now expand upon these components individually. Note that for computational simplicity, all walls are assumed to be infinitely thin.

First, we introduce the channel. The channel is the portion of the generator which the plasma flows through under an applied magnetic field. This is where the full MHD system is in effect, and is where the electric field and current spontaneously arise. Thus, there is no reduction from the full governing equations. Furthermore, the operating conditions of the MHD system render the channel unobservable, necessitating the need for the parameter estimation scheme in the real-time optimization problem.

The next component included in the model are the electrodes. By definition, these are electrically conductive materials that allow current to flow, if there is a closed circuit or path available for the electrons to move through. Thus, they have some assigned conductivity based on the material type. In our case, we use the conductivity of copper, $61 \times 10^{6} \mathrm{~S} / \mathrm{m}$ [16]. Being a solid material, electrodes have negligible ion and electron mobilities, by definition. Thus, the governing Ohm's law includes only the relation between the conductivity of the material, the current density, and the electric field.

The last component left to discuss is the load. We replicate a load being placed on a channel by placing a resistor on one electrode, and numerically implementing a periodic boundary condition between the open side of the resistor, and on an opposing electrode. Thus, we now view the load on the channel instead as a resistor on the electric current circuit, with varied resistance to vary the load. The material properties of the resistor are the same as the electrodes, save for the conductivity. This implies that there is negligible ion and electron mobilities, and a simplified Ohm's law governs their behavior. We measure the variance of the load through the load-factor (see more in 5.1.3.

By modeling the resistor, electrode, and channel connections simply as continuity or periodic boundary conditions, we now have described each physical component in our simple MHD generator model. The boundary conditions we choose represent both generator configurations and ideal conditions. Firstly, the resistor, in order to replicate a load, must represent the connection between paired electrodes, as noted previously. Thus, a periodic condition is placed on the outer boundaries of the resistor and opposing electrode. For model verification, the channel is assumed to be infinitely long, and thus periodic boundary conditions are placed on the inlet and outlet to the computational domain. For other studies of the model, the inlet and outlet are simply perfectly electrically insulating, i.e. the normal vector of the electric field is 0 on the boundaries. All other boundaries of the computational domain also satisfy the per-
fectly electrically insulating boundary condition. Any boundary between components is assumed to be a continuity condition placed on each of the solutions. We now turn to the specific geometries of two different types of Faraday generators.

### 2.1.2 MHD Generator Geometry: Continuous Faraday

There are many different electrode configurations for an MHD generator. Some that have been explored previously include the Faraday, Hall, and disc geometries [50, 25]. We focus on the Faraday geometry. Within a Faraday MHD generator framework, the channel and flow are linear, with electrodes placed to allow current to flow with the ideal Lorenz force, on opposing sides of the channel. Many other MHD numerical models use this configuration, but neglect at least the ion-slip parameter in the Generalized Ohm's law, and potentially the Hall parameter. In Section 5.1.3. we explore how this neglect can lead to exaggerated estimated power of the numerical model. To aid in later discussions, we now outline the schematics of different Faraday generator models.

There are two different Faraday geometries we consider: the continuous and segmented. We first examine the continuous Faraday channel geometry. A continuous Faraday MHD generator is simpler than a segmented one, at least qualitatively. It consists of only four components, the channel, resistor, and two block electrodes, all as described above. We choose the channel to have length 1.5 times the total electrode length to satisfy the ideal geometry restrictions presented in Rosa [42]. The electrode configuration is chosen to lie in parallel with the direction of current, i.e. in the direction of the Lorenz force. For all geometries, we arbitrarily set the coordinate system such that the flow is dominated by the $x$-direction, and that the applied magnetic field lies solely in the $z$-direction. As the curl of these two fields determines the magnitude and direction of the Lorenz force, the current density is dominated by the $y$-direction.

A simple schematic of a continuous Faraday geometry is presented in Figure 2.2.

Each component described in Section 2.1.1 is labeled. Periodic boundary conditions are noted with the coordinated dashed lines, with red lines representing the electroderesistor periodic boundary conditions, and blue lines denoting the channel inlet and outlet locations.

The continuous Faraday generator serves as an ideal model. With no Hall or ion-slip terms, this simple electrode configuration will result in an efficient generation of power. However, these terms arise naturally within an MHD generator, and may be non-negligible. We will later see that these terms alter the ideal operating conditions of an MHD generator, and inspires other MHD generator geometries. We therefore investigate a second Faraday geometry, segmented, which also is a more reasonable generator to expect to build.

### 2.1.3 MHD Generator Geometry: Segmented Faraday

Much like the continuous Faraday geometry, the segmented Faraday geometry consists of a channel, resistors, and electrodes. However, unlike the continuous geometry, segmented refers to, as one can imagine, the electrodes. These electrodes run the entire length (again, in the $y$-direction, the same as the Lorenz force) of the channel, but are separated by some inter-electrode space. The ratio of the electrode length (in $x$ ) and the inter-electrode space is critical [42, and the ideal distance is chosen such that the inter-electrode space is half the length of an electrode. We then assign the total channel length of a segmented Faraday generator, unless otherwise specified, to be $(k+1.5) 0.1$, for a segmented Faraday channel with $k$ electrodes, and with electrode width of 0.1 m . Each electrode is paired with another somewhere across the channel, again with a resistor acting as the load-interface between the two. For the segmented geometry, we will make the assumption that the load is distributed evenly across the channel. Matching the periodic boundary condition notation from the continuous geometry section, a simple schematic of a segmented Faraday MHD generator is presented in Figure 2.3 .

This schematic displays the ideal electrode configuration for an MHD system with a negligible Hall parameter. Under ideal conditions ${ }^{\text {D }}$ the electrodes connect portions of the channel which have the same electric potential, maximizing the output of the generator [42]. However, we will see in Section 5.1 .4 that the Hall parameter tilts this electric potential! Thus, for non-negligible Hall parameter, the paired electrodes must be staggered by some angle, which clearly will depend upon the Hall parameter. This further emphasizes the need for parameter estimation in the optimization problem, as the Hall parameter is not directly observable.

### 2.2 Notation

In this section, we will introduce some standard notation and concepts that will be used throughout this dissertation. Each mathematical object, function, parameter, etc., is a vector in $\mathbb{R}^{3}$ or a mapping to $\mathbb{R}^{3}$ if it is bolded. Thus, in the above schematics, both $\mathbf{u}, \mathbf{B}$ are considered fields, as they map to $\mathbb{R}^{3}$. We may refer to their scalar individual components by referencing the direction we consider, e.g. for the typical Cartesian coordinate system (which is used for the duration of this paper), we have

$$
\mathbf{u}=\left(\mathbf{u}_{x}, \mathbf{u}_{y}, \mathbf{u}_{z}\right) .
$$

Next, we define the computational domain. Let $D \subset \mathbb{R}^{3}$ denote the spatial domain. We always assume $D$ is open with compact closure, and denote the boundary as $\partial D$. In the above geometry schematics, the entire domain is $D$, as the numerical model is implemented on the channel, electrodes, and resistors.

With these basic ideas down, we now define the standard function spaces which will be used in the theoretical sections. We use the standard definitions of $L^{2}(D)$, as

[^0]

FIGURE 2.2: A simple schematic of the spatial domain for our continuous Faraday MHD generator.


FIGURE 2.3: A simple schematic of the spatial domain for our segmented Faraday MHD generator.


FIGURE 2.4: A 3D schematic of (top) continuous and (bottom) segmented Faraday geometries, as implemented in COMSOL [32].
can be seen in [17], i.e.

$$
L^{2}(D):=\left\{f: D \rightarrow \mathbb{R} \mid \int_{D} f^{2} d \mathbf{x}<\infty\right\}
$$

with norm

$$
\|f\|_{L^{2}(D)}:=\left(\int_{D} f^{2} d \mathbf{x}\right)^{1 / 2}
$$

Similarly, we define

$$
L^{\infty}(D):=\left\{g: D \rightarrow \mathbb{R} \mid \max _{\mathbf{x} \in D} g(\mathbf{x})<\infty\right\},
$$

with norm

$$
\|g\|_{L^{\infty}(D)}:=\max _{\mathbf{x} \in D} g(\mathbf{x}) .
$$

One last notational remark for this paper; when we wish to denote that some vector, $f$, of length $k \in \mathbb{N}$, has components in some space, e.g. $L^{2}(D)$, we will use the notation $f \in\left(L^{2}(D)\right)^{k}$. This implies that each component of $f$, denoted $f_{i}$, for $i \in\{1, \ldots, k\}$, is in $L^{2}(D)$, and the norm used on this tensor product space is the traditional tensor product norm [44], e.g. for $\mathbf{h} \in\left(L^{2}(D)\right)^{3}$,

$$
\|h\|_{\left(L^{2}(D)\right)^{3}}:=\left(\sum_{k=1}^{3}\left\|h_{k}\right\|_{L^{2}(D)}^{2}\right)^{1 / 2}
$$

We also use the standard Sobolev spaces [44] within the weak-form solution spaces of the MHD equations, namely

$$
W^{1,2}(D)=H^{1}(D):=\left\{f \in L^{2}(D) \mid\|\nabla f\|_{\left(L^{2}(D)\right)^{3}}<\infty\right\} .
$$

Finally, for any vector space $V$ over $\mathbb{R}$, we denote its algebraic dual, i.e. the set of all linear functions mapping from $V$ to $\mathbb{R}$, by $V^{\prime}$. Note that this also applies to function spaces, i.e. the dual of the Sobolev space $W^{1,2}=W$ is denoted $W^{\prime}$. With these function spaces defined, we now move onto an investigation into some basic finite element theory.

### 2.3 Finite Elements

Finite elements is a branch of numerical methods used to discretely approximate the solutions to differential equations. It is an extremely wide field of mathematics, and there are many different textbooks delving into a small subset of finite elements, both theoretical and nature, and those more dedicated to the implementation. It is most notably compared to finite difference, in that both were created to solve differential equations numerically. However, its most distinct difference with this other branch of numerical methods is that finite elements attempts to solve the weakform of the differential equation, and approximate integrals, whereas finite difference approximates the derivative. For a small selection of such references, see [14, 7, 1].

To examine a finite element method in general, we consider any PDE on the domain space $D$ and function space $H$, given in the weak form by

$$
\begin{equation*}
A(u, v)=f(v) \forall v \in H \tag{2.1}
\end{equation*}
$$

where $A$ is some bilinear form on $H$, and $f$ is an operator on $H$ mapping to $\mathbb{R}$. We would like to find some numerical approximation to this problem that provides a numerical approximation to $u$. We let $H^{N}$ be $\mathbb{P}^{N} \cap H$, i.e. the intersection of the space of polynomials up to order $N$, and the solution space. For the numerical analogue
to (2.1), we search for our approximate solution $u^{N} \in V^{N}$. We then choose a basis for $V^{N}$, specifically the basis where each basis element takes a value of 1 at a unique vertex, and has support in only one spatial element. We denote this basis $\left\{v_{k}\right\}_{k=1}^{K}$. There is not a convenient analytic function for generating the number of bases, as it's incredibly dependent upon the sampling distribution across the domain, and thus we simply let $K$ denote the number of basis functions. The main purpose for choosing this basis is that

$$
\int_{D} v_{j} v_{k}=\delta_{j, k}
$$

the Kronecker-delta function. Further justification of this basis can be seen in [1].
With this unique property of the basis, the bilinear form simplifies, and there is some matrix-equivalent form, $A^{N}$. Thus, the numerical implementation results in a matrix-form of

$$
A^{N} u^{N}=f^{N}
$$

where $f^{N}=\left[f\left(v_{1}\right) \ldots f\left(v_{K}\right)\right]$. Although not explicitly laid out for the kinematic MHD equations, a similar discretization method is applied within COMSOL, with piece wise linear functions as our basis for each solution space, i.e. $V$ and each component for a function in $W$. Further analysis finite element method with relation to COMSOL can be seen in Section 5.1.2.

### 2.4 Probability Theory

We now go onto describe some necessary background information for the inclusion of uncertainty within the MHD system. We let the random domain be denoted by $\Omega$, with the associated Borel sigma-algebra $\mathfrak{B}(\Omega)$, and continuous probability measure $P$. On this space, we now introduce the concept of a random-process.

Definition: 2.4.1 (Random process). Let $O$ be some function space on $D$, e.g. $L^{2}(D)$, complete under some metric $\eta$. Then $Y$ is a random process on $O$, if it
is an $(O, \eta)$-valued random variable, e.g. for any $\omega \in \Omega, Y(\omega) \in O$, and $Y$ is a measurable function, i.e. for any set $\mathcal{O} \subset O, Y^{-1}(\mathcal{O})$ is measurable by $P$.

First, note that for the purposes of this dissertation, we will only consider random processes on complete metric spaces. We now commit an abuse of notation, as for any random process $Y$ on $O$ and every $\omega, \in \Omega, Y(\omega): D \rightarrow \mathbb{R}$ is a real-valued function on $D$. Thus, we may only emphasize either the stochastic or deterministic domain by explicitly writing the arguments into the random process, e.g. $Y(\omega)$ and $Y(\mathrm{x})$. With this in mind, a random process is also a real-valued random variable for fixed $\mathbf{x} \in D$, e.g. $Y(\mathbf{x}): \Omega \rightarrow \mathbb{R}$ is a measurable function.

With a random process, much like a random variable, there is some associated probability distribution.

Definition: 2.4.2 (Probability Distribution of a Random Process). Let $Y$ be a random process on $(O, \eta)$, a complete metric space. Then the Probability Distribution of $Y$ is given by $\bar{P}: O \rightarrow \mathbb{R}$, defined by

$$
\bar{P}(o):=\rho_{Y} P\left(Y^{-1}(o)\right),
$$

where $\rho_{Y} \in \mathbb{R}$ is a normalization constant, i.e. $\rho_{Y}=P\left(Y^{-1}(Y(\Omega))\right)$. With an appropriate extension to the Borel subsets of $O$, this satisfies the definition of a probability distribution for a random variable.

These distributions will play a critical role in the exploration of including uncertainty in the kinematic MHD equations. As is traditional with probability theory, we define the expected value of an random process to be

$$
\mathbb{E}[Y]:=\int_{\Omega} Y(\omega) d P(\omega) .
$$

Equivalently, to explicitly include the distribution $\bar{P}$, one can consider the probability space $\{Y(\Omega), \mathfrak{B}(Y(\Omega)), \bar{P}\}$, and define the expected value of any random function $f: \Omega \times O \rightarrow \mathbb{R}$, which inherently depends on $Y$ as a parameter, to be

$$
\mathbb{E}[Y \mid \bar{P}]:=\int_{Y(\Omega)} Y(\omega) d P(\omega)
$$

We now define two examples of probability distributions that will be used throughout the dissertation, the beta and uniform distributions. These two definitions come from [52].

Definition: 2.4.3 (Beta Distribution). A family of continuous distributions which are characterized by two shape parameters, denoted $\alpha, \beta>0$. This distribution assumes a probability space of the interval $[0,1]$. Formally, the probability distribution function is given by

$$
p(x):=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}},
$$

where $\Gamma(x)=\int_{0}^{\infty} x^{z-1} e^{z} d z$, which simplifies to $\Gamma(x)=(x-1)$ ! if $x \in \mathbb{N}$.
Definition: 2.4.4 (Uniform Distribution). A continuous distribution which operates on a space of $[a, b]$ and whose probability distribution function is defined by

$$
p(x)= \begin{cases}\frac{1}{b-a} & \text { if } x \in[a, b] \\ 0 & \text { else }\end{cases}
$$

Note that the uniform distribution on $[0,1]$ is a beta distribution with shape parameters $\alpha=\beta=1$.

Note that the expected value of a random variable with a given beta distribution is $\alpha /(\alpha+\beta)$, with variance given by $\alpha \beta /(\alpha+\beta)^{2}(\alpha+\beta+1)$. For the uniform distribution, the expected value is $(a+b) / 2$, with a variance of $\frac{1}{12}(b-a)^{2}$. Both of these distributions will play a large role in the numerical demonstrations of the forward problem, inverse problem, and optimization problem.

We also state a critical algebraic property of random processes, how to 'combine' them. If $Y_{1}$ and $Y_{2}$ are two different random processes on $O$, then one can consider the random process $Y: \Omega \rightarrow O \times O, Y(\omega)=\left(Y_{1}(\omega), Y_{2}(\omega)\right)$. Under the continuous probability measure $P$, the associated distribution for $Y$ is the convolution of the individual processes, i.e.

$$
\bar{P}_{Y}=\bar{P}_{Y_{1}} \cdot \bar{P}_{Y_{2}} .
$$

As the kinematic MHD equations have several random processes within the model, we will have to utilize this property to generate the distributions of the solutions. In the uncertain parameter estimation scheme, we will also need to consider finitedimensional (FD) approximations to these distributions. To show convergence, we must have some way to measure the difference between distributions. To this end, we introduce the Prokhorov metric.

### 2.4.1 Prokhorov Metric

For this section, we use the sources [4, 3]. The Prokhorov metric is a metric developed to characterize the differences between different probability distributions quantitatively. We present this metric in the most general sense, and will later adapt it to the kinematic MHD framework. Let $(O, \eta)$ and $(\Omega, \mathfrak{B}(\Omega), P)$ be as before. We let $\mathcal{P}(O)$ represent the space of all probability measures on $O$, i.e. for any countable collection $\left\{E_{i}\right\}_{i \in \mathcal{I}} \subset \mathfrak{B}(O)$,

$$
\mathcal{P}(O):=\left\{P: \mathfrak{B}(O) \rightarrow[0,1] \mid P(\varnothing)=0, P(O)=1, P\left(\bigcup_{i \in \mathcal{I}} E_{i}\right)=\sum_{i \in \mathcal{I}} P\left(E_{i}\right)\right\} .
$$

Then for any closed subset $F \subset O$, and $\varepsilon>0$, define an $\varepsilon$ neighborhood of $F$ by

$$
F^{\varepsilon}:=\{o \in O: d(\tilde{o}, o)<\varepsilon, \tilde{o} \in F\} .
$$

Finally, define the metric $\rho: \mathcal{P}\left(\widetilde{Q}_{N}\right) \times \mathcal{P}\left(\widetilde{Q}_{N}\right) \rightarrow \mathbb{R}^{+}$by

$$
\rho\left(P_{1}, P_{2}\right):=\inf \left\{\varepsilon>0: P_{1}[F] \leq P_{2}\left[F^{\varepsilon}\right]+\varepsilon, F \text { closed, } F \subset \widetilde{Q}_{N}\right\}
$$

It follows from [3] that

- $\rho$ is a metric (called the Prohorov metric) on $\mathcal{P}\left(\widetilde{Q}_{N}\right)$.
- $\left(P\left(\widetilde{Q}_{N}\right), \rho\right)$ is a complete metric space.
- If $\widetilde{Q}_{N}$ is compact, then $P\left(\widetilde{Q}_{N}\right)$ is compact.

Our goal is to utilize this metric to show convergence of the uncertain parameter estimation problem. Within that framework, we require a dimension-reduction of the probability space to have a numerically implementable algorithm. To this end, we would like to further understand the convergence of the approximation distributions to the full infinite-dimensional problem, particularly under this metric.

To this end, let $P_{k} \in \mathcal{P}(O)$ be some sequence of distributions that converge to some distribution $P \in \mathcal{P}(O)$ under the Prokhorov metric. It is well known that if $\mathcal{P}(O)$ is complete under this metric, given that $(O, \eta)$ is a complete metric space. Then the following are equivalent:

1. $\rho\left(P_{k}, P\right) \rightarrow 0$;
2. $\int_{O} f d P_{k}(o) \rightarrow \int_{O} f d P(o)$ for all bounded and uniformly continuous functions $f: O \rightarrow \mathbb{R} ;$
3. $P_{k}(A) \rightarrow P(A)$ for all Borel sets $A \subset O$ with $P(\partial A)=0$. Here $\partial A$ denotes the boundary of $A$.

Thus, it immediately follows that convergence in the $\rho$ metric and convergence of the distributions $P$ are equivalent, by the equivalence of 1 and 3. For the purposes of the parameter estimation problem, we utilize the equivalence of 1 and 2, as this is the same as

$$
\mathbb{E}\left[f(o) \mid P_{k}\right] \rightarrow \mathbb{E}[f(o) \mid P]
$$

We will see in Section 4.2 how this convergence will be used to guarantee continuity of the uncertain parameter estimation problem.

To aid in the numerical implementation further, we must consider the problem of generating a countable dense subset of $\mathcal{P}(O)$. Using the same topology as seen in [3], that is we define a $\mathcal{W}$ neighborhood of $P \in \mathcal{P}(O)$ by

$$
N_{\delta}(P):=\left\{P_{1} \in \mathcal{P}(O) \mid P_{1}\left(F_{i}\right)<P\left(F_{i}\right)+\delta, i=1, \ldots, k, F_{i} \in \mathfrak{B}(O)\right\}
$$

for a given $\delta>0$ and $\left\{F_{i}\right\}_{i=1}^{k}$ is some finite set. This $\mathcal{W}$ topology is equivalent to the topology of weak convergence on $\mathcal{P}(O)$, which is equivalent to the topology of the $\rho$ metric, if $O$ is separable [5] p. 236-239. We now state a result from [3] that will be critical in showing the well-posedness of the uncertain inverse problem.

Theorem 2.4.1 (Theorem 3.1 from [3]). Let $O$ be a complete, separable metric space with metric $\eta$, and let $\mathfrak{B}(O)$ be the class of all Borel subsets of $O$, and let $\mathcal{P}(O)$ be the space of all probability measures on $O, \mathfrak{B}(O)$. Let $O_{0}=\left\{o_{j}\right\}_{j=1}^{\infty}$ be a countable dense subset of $O$. Then the set of $P \in \mathcal{P}(O)$ such that $P$ has finite support in $O_{0}$ and rational masses is dense in $\mathcal{P}(O)$, under the $\rho$ metric. That is,

$$
\mathcal{P}_{0}(O):=\left\{P \in \mathcal{P}(O) \mid P=\sum_{j=1}^{k} p_{j} \delta o_{j}, k \in \mathbb{N}, o_{j} \in O_{0}, p_{j} \in \mathbb{Q}, p_{j} \geq 0, \sum_{j=1}^{k} p_{j}=1\right\}
$$

is dense in $\mathcal{P}(O)$ relative to $\rho$.

A proof of this theorem is provided by Banks, utilizing the Prokhorov metric.

## 3 A Deterministic Approach to Kinematic MHD

### 3.1 Introduction

In this section, we explore the deterministic side of the kinematic MHD theory. This work builds upon a previous foundation established by Rosa [42], and expand the well-posedness of the kinematic MHD model to include the ion-slip parameter. We introduce the components and parameters of the kinematic MHD model, establishing the deterministic strong-form of the system we use for the remainder of the dissertation. We then show equivalence of this form to several others, most notably a mixed-Poisson system. Using this, we establish the well-posedness using the Babuska-Brezzi-Kovalevskaya theorem [14], and establishing the continuous dependence of the solutions on the parameters.

### 3.2 Model Formulation

We begin with the governing equations for our application: Maxwell's equations, coupled with the generalized Ohm's law, as given in Rosa [42] section 2.7. As noted in the background section, we use the kinematic MHD model to simplify computation complexity, and thus prescribe the fluid flow, $\mathbf{u}: D \rightarrow \mathbb{R}^{3}$, as well as assuming a steadystate system. We also assume that all other system parameters: the conductivity, $\sigma$, electron-mobility, $\mu_{e}$, and ion-mobility, $\mu_{i}$ are prescribed. Note that these system parameters are scalar, i.e. map from $D$ to $\mathbb{R}$. Furthermore, we assume that the applied magnetic field, $\mathbf{B}$, is given, and that this field dominates the induced field [6]. Then the kinematic MHD system for $\mathbf{J}$ and $\mathbf{E}$, the electric current density and electric field respectively, is described by

$$
\begin{equation*}
\nabla \times \mathbf{E}=0, \text { on } D, \tag{3.1a}
\end{equation*}
$$

$$
\begin{gather*}
\mathbf{J}=\sigma(\mathbf{E}+\mathbf{u} \times \mathbf{B})+\frac{\beta_{e}}{\|\mathbf{B}\|}(\mathbf{J} \times \mathbf{B})+\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}((\mathbf{J} \times \mathbf{B}) \times \mathbf{B}), \text { on } D,  \tag{3.1b}\\
\nabla \cdot \mathbf{J}=0, \text { on } D . \tag{3.1c}
\end{gather*}
$$

Here, $\beta_{e}$ and $\beta_{i}$, the Hall parameter and ion-slip parameter respectively, are defined as

$$
\beta_{e}(\mathbf{x})=\mu_{e}(\mathbf{x})\|\mathbf{B}(\mathbf{x})\|_{l^{2}}, \text { and } \beta_{i}(\mathbf{x})=\mu_{e}(\mathbf{x}) \mu_{i}(\mathbf{x})\|\mathbf{B}(\mathbf{x})\|_{l^{2}}^{2} .
$$

We complete the system of equations with appropriate boundary conditions for power generation, assuming perfectly electrically insulating boundary conditions. For $\mathbf{J}$ this implies

$$
\begin{equation*}
\mathbf{J} \cdot \mathbf{n}=0, \text { on } \partial D, \tag{3.2a}
\end{equation*}
$$

while $\mathbf{E}$ satisfies

$$
\begin{equation*}
\mathbf{E} \times \mathbf{n}=0, \text { on } \partial D . \tag{3.2b}
\end{equation*}
$$

Here, $\mathbf{n}$ represents a vector normal to $\partial D$.
With some algebra, the kinematic model, (3.1), can be reduced to a mixedPoisson form. This is done by first rewriting the generalized Ohm's Law, so that the dependence on $\mathbf{E}$ is explicit. A simpler form of this transformation is presented in Rosa [42], but neglects the ion-slip parameter. The following section demonstrates how to include $\beta_{i}$.

### 3.2.1 An Explicit Generalized Ohm's Law

To rewrite (3.1b) in an explicit form, first consider that the cross product can be written as a matrix applied to a vector, e.g.

$$
\mathbf{J} \times \mathbf{B}=\left[\begin{array}{ccc}
0 & \mathbf{B}_{z} & -\mathbf{B}_{y} \\
-\mathbf{B}_{z} & 0 & \mathbf{B}_{x} \\
\mathbf{B}_{y} & -\mathbf{B}_{x} & 0
\end{array}\right]\left[\begin{array}{l}
\mathbf{J}_{x} \\
\mathbf{J}_{y} \\
\mathbf{J}_{z}
\end{array}\right]=[\mathbf{B}]_{\times} \mathbf{J},
$$

where $[\mathbf{B}]_{\times}$denotes the cross-product matrix operator. Clearly, $[\mathbf{B}]_{\times}^{2} \mathbf{J}$ denotes the vector triple product between $\mathbf{J}, \mathbf{B}, \mathbf{B}$, as $[\mathbf{B}]_{\times}^{2} \mathbf{J}=[\mathbf{B}]_{\times}(\mathbf{J} \times \mathbf{B})=(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}$. Using this, Ohm's law can be rewritten as

$$
\begin{aligned}
\mathbf{J} & =\sigma(\mathbf{E}+\mathbf{u} \times \mathbf{B})+\mu_{e}(\mathbf{J} \times \mathbf{B})+\mu_{i}((\mathbf{J} \times \mathbf{B}) \\
& =\sigma(\mathbf{E}+\mathbf{u} \times \mathbf{B})+\frac{\beta_{e}}{\|\mathbf{B}\|}[\mathbf{B}]_{\times} \mathbf{J}+\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}[\mathbf{B}]_{\times}^{2} \mathbf{J} \\
\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J} & =\sigma(\mathbf{E}+\mathbf{u} \times \mathbf{B}) .
\end{aligned}
$$

Noting that the determinant of $\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}[\mathbf{B}]_{\times}^{2}\right)$ is given by $1+\beta_{i}^{2}+\beta_{e}^{2}+2 \beta_{i}$ and since $\beta_{i}>0$ by definition, the LHS is always invertible, implying that

$$
\mathbf{J}=\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}[\mathbf{B}]_{\times}^{2}\right)^{-1} \sigma(\mathbf{E}+\mathbf{u} \times \mathbf{B})
$$

Defining $\underline{\underline{\sigma}}$ as the conductivity tensor $\underline{\underline{\sigma}}=\sigma\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}[\mathbf{B}]_{\times}^{2}\right)^{-1}$ yields the following explicit generalized Ohm's law,

$$
\begin{equation*}
\mathbf{J}_{i}=\overline{\underline{\sigma}}(\mathbf{E}+\mathbf{u} \times \mathbf{B}) . \tag{3.3}
\end{equation*}
$$

We now turn to using this to manipulate (3.1) into a mixed-Poisson form.

### 3.3 Well-Posedness: Existence and Uniqueness of Solutions

In the following section, we establish the well-posedness of the kinematic MHD system by first transforming (3.1) into a form for which the Babushka-Brezzi-Kovalevskaya (BBK) theorem is applicable. This section follows the work in Mcgregor [30] and Rosa [42] closely, but furthers the kinematic MHD theory by including the ion-slip parameter. We now state the BBK theorem in it's entirety, and then develop the operator model and operator properties necessary to apply it.

Theorem 3.3.1 (BBK theorem). Let $A: V \rightarrow V^{\prime}$ and $B: V \rightarrow W^{\prime}$ be continuous operators from the Hilbert spaces $V, W$ to their duals. In addition, assume

- $A$ is $V$-coercive on $V$, i.e. $\exists \alpha>0$ such that

$$
A v(v) \geq \alpha\|v\|_{V}^{2}, \quad v \in V,
$$

- $B$ obeys the following inf sup condition: $\exists \beta>0$

$$
\inf _{q \in W} \sup _{v \in V} \frac{|B v(q)|}{\|v\|_{V}\left\|_{q}\right\|_{W}} \geq \beta .
$$

Given these conditions, then $\forall f \in V^{\prime}, \forall g \in W^{\prime}$ there exists a unique pair $v, w \in V \times W$ such that

$$
\begin{gather*}
A v+B^{\prime} w=f \in V^{\prime},  \tag{3.4a}\\
B v=g \in W^{\prime}, \tag{3.4b}
\end{gather*}
$$

which obey the following a priori estimates:

$$
\begin{gather*}
\|v\|_{V} \leq \frac{1}{\alpha}\left(\|f\|_{V^{\prime}}+\frac{1}{\beta}\left(\|A\|_{\mathcal{L}\left(V, V^{\prime}\right)}+\alpha\right)\|g\|_{W^{\prime}}\right),  \tag{3.5a}\\
\|w\|_{W} \leq \frac{1}{\beta}\left(\|f\|_{V^{\prime}}+\|A\|_{\mathcal{L}\left(V, V^{\prime}\right)}\|g\|_{W^{\prime}}\right) . \tag{3.5b}
\end{gather*}
$$

A proof of BBK can be found in Boffi, Brezzi, and Fortin [14], Corollary 4.2.1. To apply BBK directly, we need to convert the given system to a weak mixed-Poisson system, and then to an equivalent dual operator form.

### 3.3.1 Mixed-Poisson Strong Form

First, we reduce the representation of the electric field $\mathbf{E}$ within the kinematic MHD framework, with a scalar-valued electric-potential $\mathcal{V}$. From (3.1a), and the assumption that $D$ is a bounded open domain, it follows from Stoke's theorem [45] that

$$
\begin{equation*}
\mathbf{E}=\nabla \mathcal{V} \tag{3.6}
\end{equation*}
$$

Now, define $\mathbf{J}_{i}$ to be

$$
\begin{equation*}
\mathbf{J}_{i}=\underline{\bar{\sigma}} \mathbf{E}=\underline{\bar{\sigma}} \nabla \mathcal{V}, \tag{3.7}
\end{equation*}
$$

which, coupled with the divergence-free condition, (3.1c), implies

$$
0=\nabla \cdot \mathbf{J}=\nabla \cdot\left(\mathbf{J}_{i}+\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B})\right) .
$$

This of course implies

$$
\begin{equation*}
\nabla \cdot \mathbf{J}_{i}=-\nabla \cdot(\underline{\underline{\sigma}}(\mathbf{u} \times \mathbf{B})) . \tag{3.8}
\end{equation*}
$$

Substituting the definitions of $\mathbf{J}_{i}$ and $\mathcal{V}$ (3.7) and (3.6) respectively) into (3.3) and combining the resulting equation with the divergence condition (3.8) yields the mixed-Poisson system

$$
\begin{gather*}
\underline{\bar{\sigma}}^{-1} \mathbf{J}_{i}-\nabla \mathcal{V}=0,  \tag{3.9a}\\
-\nabla \cdot \mathbf{J}_{i}=\nabla \cdot(\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B})) . \tag{3.9b}
\end{gather*}
$$

Using the definitions of $\mathbf{J}_{i}, \mathcal{V}$, we have that $\mathbf{J}, \mathbf{E}$ are described by

$$
\begin{equation*}
\mathbf{J}=\mathbf{J}_{i}+\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}), \tag{3.10a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{E}=\nabla \mathcal{V} \tag{3.10b}
\end{equation*}
$$

Implying that solving (3.9) is equivalent to solving (3.1). We now convert this into a weak-form by first defining appropriate solution function spaces.

### 3.3.2 Weak and Operator Forms

The perfectly-electrically insulating boundary conditions imply that we seek our voltage $\mathcal{V}$ in the subspace of $W^{1,2}(D)$ defined by

$$
\mathcal{V} \in W(D):=W_{0}^{1,2}(D)=\left\{f \in W^{1,2}(D): T(f)=0\right\}
$$

where $T(f)$ is the trace of $f$ on $D$. These same boundary conditions imply we seek $\mathbf{J}_{i} \in V(D)$, defined as

$$
\mathbf{J}_{i} \in V(D):=\left\{\mathbf{f} \in\left(L^{2}(D)\right)^{3}: \mathbf{f} \cdot \mathbf{n}=-(\overline{\underline{\sigma}}(\mathbf{u} \times \mathbf{B})) \cdot \mathbf{n} \text { on } \partial D\right\} .
$$

Multiplying (3.9) by test functions in the appropriate spaces and integrating, we arrive at the weak form of the MHD equations, given as

$$
\begin{array}{cc}
\int_{D} \bar{\sigma}^{-1} \mathbf{J}_{i} \cdot \phi-\int_{D} \nabla \mathcal{V} \cdot \phi=0 & \forall \phi \in V(D), \\
- & \int_{D} \mathbf{J}_{i} \cdot \nabla \psi=\int_{D}(\underline{\bar{\sigma}} \mathbf{u} \times \mathbf{B}) \nabla \cdot \psi \tag{3.11b}
\end{array} \quad \forall \psi \in W(D) .
$$

Recall the definition of the dual space, as defined in 2.2, which for any vector space $V$, we denote with $V^{\prime}$. Using this, define the mapping from $V \rightarrow V^{\prime}$ by

$$
\begin{equation*}
\mathcal{A}(\mathbf{F})(\circ):=\int_{D}\left(\underline{\bar{\sigma}}^{-1} \mathbf{F}\right) \cdot \circ, \tag{3.12}
\end{equation*}
$$

and define the mapping from $V \rightarrow W^{\prime}$ by

$$
\begin{equation*}
\mathcal{B}(\mathbf{G})(\circ):=-\int_{D} \mathbf{G} \cdot \nabla \circ . \tag{3.13}
\end{equation*}
$$

Note now that the dual, in the operator sense (See [44] Section 4.4), of $\mathcal{B}$ is given by

$$
\mathcal{B}^{\prime}(H)(\circ)=-\int_{D} \nabla H \cdot \circ .
$$

Using these operators, (3.11) can be written as

$$
\begin{array}{cl}
\left(\mathcal{A}\left(\mathbf{J}_{i}\right)+\mathcal{B}^{\prime}(\mathcal{V})\right)(\phi)=0 & \forall \phi \in V(D), \\
\mathcal{B}\left(\mathbf{J}_{i}\right)(\psi)=\int_{D}(\underline{\bar{\sigma}} \mathbf{u} \times \mathbf{B}) \cdot \nabla \psi & \forall \psi \in W(D) \tag{3.14b}
\end{array}
$$

For ease of notation, denote the operator $g \in W(D) \rightarrow \mathbb{R}$ by

$$
g(\omega)=\int_{D}(\underline{\bar{\sigma}} \mathbf{u} \times \mathbf{B}) \cdot \nabla \omega
$$

Then (3.14) is equivalent to

$$
\begin{gather*}
\mathcal{A}\left(\mathbf{J}_{i}\right)+\mathcal{B}^{\prime}(\mathcal{V})=0 \in V^{\prime}(D),  \tag{3.15a}\\
\mathcal{B}\left(\mathbf{J}_{i}\right)=g \in W^{\prime}(D) . \tag{3.15b}
\end{gather*}
$$

Clearly, (3.15) is equivalent to (3.11). We denote (3.15) the operator form. We now establish properties of these operators in the kinematic MHD framework.

### 3.3.3 Properties of the dual operators $\mathcal{A}, \mathcal{B}$

As can be seen in the assumptions, to apply Theorem 3.3.1, certain properties of the mixed-Poisson operators must be established. The following lemmas show that the operators $\mathcal{A}, \mathcal{B}$ have these necessary properties, under assumptions regarding the parameters. These assumptions are all valid within a realistic MHD generator, and thus do not overly simplify the model. Note that the domain of the functional spaces is assumed to be the given $D$, unless specifically stated otherwise.

Lemma 3.3.1. If $\boldsymbol{B} \in\left(L^{2}(D)\right)^{3}, \sigma, \beta_{e}$ and $\beta_{i}$ positive, bounded, real-valued functions on $D$, then the operator $\mathcal{A}$ is coercive and continuous.

Proof. Coercivity: $\mathcal{A}$ is coercive if and only if $\exists \alpha>0$ such that

$$
A(\mathbf{F})(\mathbf{F}) \geq \alpha\|\mathbf{F}\|_{V} \forall \mathbf{F} \in V .
$$

Let $\mathbf{F} \in V$ and consider that

$$
\begin{aligned}
A(\mathbf{F})(\mathbf{F}) & =\int_{D} \underline{\bar{\sigma}}^{-1} \mathbf{F} \cdot \mathbf{F} \\
& =\int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}[\mathbf{B}]_{\times}^{2}\right) \mathbf{F} \cdot \mathbf{F} \\
& =\int_{D} \frac{1}{\sigma}\left(\mathbf{F} \cdot \mathbf{F}-\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}(\mathbf{F} \times \mathbf{B}) \cdot \mathbf{F}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{F} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{F}\right) .
\end{aligned}
$$

Given that $\sigma$ is essentially positive and bounded, the function $\frac{1}{\sigma}$ is also essentially positive and bounded, and achieves a minimum value of $\frac{1}{\|\sigma\|_{L^{\infty}(D)}}$ implying

$$
\begin{aligned}
A(\mathbf{F})(\mathbf{F}) & \geq \frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}\left(\mathbf{F} \cdot \mathbf{F}-\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}(\mathbf{F} \times \mathbf{B}) \cdot \mathbf{F}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{F} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{F}\right) \\
& =\frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}\left(\mathbf{F} \cdot \mathbf{F}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{F} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{F}\right),
\end{aligned}
$$

as $(\mathbf{F} \times \mathbf{B}) \cdot \mathbf{F}=0$ almost everywhere. Using properties of triple vector products and dot products, we have

$$
\begin{aligned}
\frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}(\mathbf{F} \cdot \mathbf{F} & \left.-\left[\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{B} \cdot \mathbf{F}) \mathbf{B}-(\mathbf{B} \cdot \mathbf{B}) \mathbf{F}) \cdot \mathbf{F}\right]\right) \\
& =\frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}\left(\mathbf{F} \cdot \mathbf{F}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{B} \cdot \mathbf{F})(\mathbf{B} \cdot \mathbf{F})-(\mathbf{B} \cdot \mathbf{B})(\mathbf{F} \cdot \mathbf{F}))\right. \\
& =\frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}(\mathbf{F} \cdot \mathbf{F})\left(1-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}(\mathbf{B} \cdot \mathbf{B})\left(\cos ^{2}(\theta)-1\right)\right) \\
& =\frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}(\mathbf{F} \cdot \mathbf{F})\left(1+\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}(\mathbf{B} \cdot \mathbf{B}) \sin ^{2}(\theta)\right) \\
& =\frac{1}{\|\sigma\|_{L^{\infty}(D)}} \int_{D}(\mathbf{F} \cdot \mathbf{F})\left(1+\beta_{i} \sin ^{2}(\theta)\right)
\end{aligned}
$$

Thus, finally

$$
A(\mathbf{F})(\mathbf{F}) \geq \frac{1}{\|\sigma\|_{L^{\infty}(D)}}\|\mathbf{F}\|_{V}>0
$$

We show continuity of $A$ by showing that it is bounded. To see boundedness of $\mathcal{A}$, it remains to be seen that

$$
|\mathcal{A}(\mathbf{f})(\mathbf{g})|<c\|\mathbf{f}\|_{V}\|\mathbf{g}\|_{V}
$$

for some $c \in \mathbb{R}$. Thus, let $\mathbf{f}, \mathbf{g} \in V$ and consider the following.

$$
\begin{aligned}
|\mathcal{A}(\mathbf{f})(\mathbf{g})| & =\left|\int_{D}{\left.\frac{\bar{\sigma}^{-1}}{} \mathbf{f} \cdot \mathbf{g} \right\rvert\,}=\left|\int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}}[\mathbf{B}]_{\times}^{2}\right) \mathbf{f} \cdot \mathbf{g}\right|\right. \\
& =\left|\int_{D} \frac{1}{\sigma}\left(\mathbf{f} \cdot \mathbf{g}-\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}(\mathbf{f} \times \mathbf{B}) \cdot \mathbf{g}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{f} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{g}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
& \leq \int_{D}\left|\frac{1}{\sigma} \|\left(\mathbf{f} \cdot \mathbf{g}-\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}(\mathbf{f} \times \mathbf{B}) \cdot \mathbf{g}-\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{f} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{g}\right)\right| \\
& \leq \int_{D}\left|\frac{1}{\sigma}\right|\left(|f \cdot \mathbf{g}|+\left|\frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}(\mathbf{f} \times \mathbf{B}) \cdot \mathbf{g}\right|+\left|\frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}((\mathbf{f} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{g}\right|\right) \\
& \leq \frac{1}{\operatorname{ess} \inf \sigma}(\underbrace{\int_{D}|f \cdot \mathbf{g}|}_{I_{1}}+\underbrace{\int_{D} \frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}|(\mathbf{f} \times \mathbf{B}) \cdot \mathbf{g}|}_{I_{2}}+\underbrace{\left.\left.\int_{D} \frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}} \right\rvert\,(\mathbf{f} \times \mathbf{B}) \times \mathbf{B}\right) \cdot \mathbf{g} \mid}_{I_{3}}) .
\end{aligned}
$$

Rewriting the integrals $I_{1}, I_{2}, I_{3}$ in terms of $\|\mathbf{f}\|_{V}\|\mathbf{g}\|_{V}$ will yield the desired result. Consider first $I_{1}$.

$$
I_{1}=\int_{D}|\mathbf{f} \cdot \mathbf{g}| \leq\|\mathbf{f}\|_{V}\|\mathbf{g}\|_{V} \quad \text { by Cauchy-Schwartz }
$$

Now, consider $I_{2}$. First, note that $\|(\mathbf{f} \times \mathbf{B})\|_{l^{2}} \leq\|\mathbf{f}\|_{l^{2}}\|\mathbf{B}\|_{l^{2}}$. As well, note that

$$
|\mathbf{u} \cdot \mathbf{v}| \leq\|\mathbf{u}\|_{l^{2}}\|\mathbf{v}\|_{l^{2}}
$$

by Cauchy-Schwartz. Applying these two ideas to $I_{2}$ yields

$$
\begin{array}{rlr}
I_{2} & =\int_{D} \frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}|(\mathbf{f} \times \mathbf{B}) \cdot \mathbf{g}| \\
& \leq \int_{D} \frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}\|(\mathbf{f} \times \mathbf{B})\|_{l^{2}}\|\mathbf{g}\|_{l^{2}} & \\
& \leq \int_{D} \frac{\beta_{e}}{\|\mathbf{B}\|_{l^{2}}}\|\mathbf{f}\|_{l^{2}}\|\mathbf{B}\|_{l^{2}}\|\mathbf{g}\|_{l^{2}} & \text { by Cauchy-Schwarz } \\
& =\left\|\beta_{e}\right\|_{L^{\infty}(D)} \int_{D}\|\mathbf{f}\|_{l^{2}}\|\mathbf{g}\|_{l^{2}} \\
& \leq\left\|\beta_{e}\right\|_{L^{\infty}(D)}\|\mathbf{f}\|_{V}\|\mathbf{g}\|_{V} \quad \text { by Cauchy-Schwartz. }
\end{array}
$$

By a similar argument, we have for $I_{3}$

$$
\begin{aligned}
I_{3} & =\int_{D} \frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}|((\mathbf{f} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{g}| \\
& \leq \int_{D} \frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}\|(\mathbf{f} \times \mathbf{B}) \times \mathbf{B}\|_{l^{2}}\|\mathbf{g}\|_{l^{2}} \\
& \leq \int_{D} \frac{\beta_{i}}{\|\mathbf{B}\|_{l^{2}}^{2}}\|\mathbf{f}\|_{l^{2}}\|\mathbf{B}\|_{l^{2}}^{2}\|\mathbf{g}\|_{l^{2}} \\
& =\left\|\beta_{i}\right\|_{L^{\infty}(D)} \int_{D}\|\mathbf{f}\|_{l^{2}}\|\mathbf{g}\|_{l^{2}}
\end{aligned}
$$

$$
\leq\left\|\beta_{i}\right\|_{L^{\infty}(D)}\|\mathbf{f}\|_{V}\|\mathbf{g}\|_{V} \quad \text { by Cauchy-Schwartz. }
$$

Combining all three inequalities yields

$$
\mathcal{A}(\mathbf{f})(\mathbf{g}) \leq\left(1+\left\|\beta_{e}\right\|_{L^{\infty}(D)}+\left\|\beta_{i}\right\|_{L^{\infty}(D)}\right)\|\mathbf{f}\|_{V}\|\mathbf{g}\|_{V}
$$

and as $\mathcal{A}$ is both linear and bounded, we have that $\mathcal{A}(f)(\circ)$ is continuous on $V$.
To see a similar result for $\mathcal{B}$, an inequality that bounds the norm of a function by the norm of its derivative must be utilized. The Poincaré inequality does so, and is thus presented next.

Lemma 3.3.2 (Poincaré Inequality). Let $1 \leq p<\infty$ and let $\Omega$ be bounded on at least one side. Then $\exists C>0$, depending only on $p, \Omega$, such that $\forall u \in W_{0}^{1, p}(\Omega)$ where $u$ is a zero-trace function, the following holds

$$
\|u\|_{L^{p}(\Omega)} \leq C\|\nabla u\|_{L^{p}(\Omega)} .
$$

This inequality is well-known, and complete proofs are presented in a variety of textbooks. One such book is Rudin [44]. Now, to give the desired properties of $\mathcal{B}$.

Lemma 3.3.3. The operator $\mathcal{B}: V \rightarrow W^{\prime}$ is continuous and obeys the following inf-sup condition

$$
\inf _{g \in W} \sup _{f \in V} \frac{|\mathcal{B}(f)(g)|}{\|f\|_{V}\|g\|_{W}} \geq \beta>0 .
$$

Proof. Fix $g \in W$. This immediately implies that $\nabla g \in\left(L^{2}(D)\right)^{3}$, by definition of $W$. Therefore,

$$
\begin{aligned}
\sup _{\mathbf{f} \in V} \frac{|\mathcal{B}(\mathbf{f})(g)|}{\|\mathbf{f}\|_{V}\|g\|_{W}} & \geq \frac{|\mathcal{B}(\nabla g)(g)|}{\|\nabla g\|_{\mathbf{L}^{2}(D)}\|g\|_{W}} \\
& =\frac{\left|\int_{D} \nabla g \cdot \nabla g\right|}{\left(\int_{D} \nabla g \cdot \nabla g\right)^{1 / 2}\left(\|g\|_{L^{2}(D)}^{2}+\|\nabla g\|_{L^{2}(D)}^{2}\right)^{1 / 2}} \\
& =\frac{\|\nabla g\|_{L^{2}(D)}}{\left(\|g\|_{L^{2}(D)}^{2}+\|\nabla g\|_{L^{2}(D)}^{2}\right)^{1 / 2}}
\end{aligned}
$$

$$
\begin{aligned}
& \geq \frac{\|\nabla g\|_{L^{2}(D)}}{\left(\left(1+C_{p . f .}\right)\|\nabla g\|_{L^{2}(D)}^{2}\right)^{1 / 2}} \\
& =\frac{\|\nabla g\|_{L^{2}(D)}}{\left(1+C_{p . f .}\right)^{1 / 2}\|\nabla g\|_{L^{2}(D)}} \\
& =\frac{1}{\left(1+C_{p . f .}\right)^{1 / 2}} .
\end{aligned}
$$

by Poincare's Inequality,

Here, $C_{p . f \text {. represents the constant from the Poincare inequality, which has no depen- }}^{\text {r }}$ dence on $g$. This immediately implies that the supremum is bounded below independent of the choice of $g$, i.e.

$$
\inf _{g \in W} \sup _{\mathbf{f} \in V} \frac{|\mathcal{B}(\mathbf{f})(g)|}{\|\mathbf{f}\|_{V}\|g\|_{W}} \geq \frac{1}{\left(1+C_{p . f .}\right)^{1 / 2}}>0,
$$

and thus the inf-sup condition holds. To see continuity, recall that $\mathcal{B}$ is linear, and thus boundedness implies continuity. Fix $\mathbf{f} \in V, g \in W$, and consider

$$
\begin{aligned}
|\mathcal{B}(\mathbf{f})(g)|^{2} & =\left|\int_{D} \mathbf{f} \cdot \nabla g\right|^{2} \\
& \leq \int_{D}|\mathbf{f} \cdot \nabla g|^{2} \\
& \leq \int_{D}\|\mathbf{f}\|_{L^{2}}^{2}\|\nabla g\|_{L^{2}}^{2} \\
& \leq\|\mathbf{f}\|_{V}^{2}\|\nabla g\|_{\mathbf{L}^{2}(D)}^{2} \\
& \leq\|\mathbf{f}\|_{V}^{2}\left(\|\nabla g\|_{L^{2}(D)}^{2}+\|g\|_{\mathbf{L}^{2}(D)}^{2}\right) \\
& =\|\mathbf{f}\|_{V}^{2}\|g\|_{W}^{2} .
\end{aligned}
$$

Taking the square-root of both sides implies that $\mathcal{B}(\mathbf{f})(g)$ is bounded by the norms of $f, g$, which in turn implies continuity.

With these properties of the operators established, BBK can now be applied.

### 3.3.4 Existence and Uniqueness of Solutions

The operators $\mathcal{A}$, defined in (3.12), and $\mathcal{B}$, defined in (3.13), have been shown to have the desired properties to apply BBK in Lemmas 3.3.1 and 3.3.3. This leads to
the first of three main theorems, the existence and uniqueness of solutions to (3.15), the operator form of the kinematic MHD model, which in turn implies existence and uniqueness of solutions to (3.11), the weak-form, and subsequently the original model (3.1).

Theorem 3.3.2. Given $\boldsymbol{u}, \boldsymbol{B} \in\left(L^{2}(D)\right)^{3}$, bounded, $\sigma, \beta_{e}$ and $\beta_{i}$ positive, bounded, and real-valued on $D$, there exist unique solutions, $\boldsymbol{J}_{i}, \mathcal{V}$, to

$$
\begin{gather*}
\mathcal{A}\left(\boldsymbol{J}_{i}\right)+\mathcal{B}^{\prime}(\mathcal{V})=f \in V^{\prime},  \tag{3.16a}\\
\mathcal{B}\left(\boldsymbol{J}_{i}\right)=g \in W^{\prime}, \tag{3.16b}
\end{gather*}
$$

which obey the following a priori estimates:

$$
\begin{gather*}
\left\|\boldsymbol{J}_{i}\right\|_{V} \leq \frac{1}{\alpha}\|f\|_{V^{\prime}}+\frac{1}{\alpha \beta}\left(\|\mathcal{A}\|_{\mathcal{L}\left(V, V^{\prime}\right)}+\alpha\right)\|g\|_{W^{\prime}},  \tag{3.17}\\
\|\mathcal{V}\|_{W} \leq \frac{1}{\beta}\left(\|f\|_{V^{\prime}}+\|A\|_{\mathcal{L}\left(V, V^{\prime}\right)}\|g\|_{V^{\prime}}\right) \tag{3.18}
\end{gather*}
$$

where $\alpha$ is the coercive constant for $\mathcal{A}$ and $\beta$ is the bounding constant for $\mathcal{B}$, i.e., $\alpha=1+\left\|\beta_{e}\right\|_{L^{\infty}(D)}+\left\|\beta_{i}\right\|_{L^{\infty}(D)}$ and $\beta=\left(1+C_{p . f .}\right)^{1 / 2}$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be defined as above. Then by Lemmas 3.3.1 and 3.3.3, $\mathcal{A}$ is coercive, $\mathcal{B}$ obeys the inf sup condition, and both are continuous. Applying BBK implies both existence and uniqueness of the solutions $\mathbf{J}_{i}, \mathcal{V}$ in their respective spaces. The estimates follow from BBK as well.

Moreover, applying Theorem 3.3.2 when $f=0, g(\circ)=\int_{D} \nabla \cdot \underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}) \circ$ gives that there exists a pair of unique solutions to (3.15), in turn implying existence and uniqueness of solutions to (3.11). We now show that not only the solutions, $\mathbf{J}_{i}, \mathcal{V}$, to (3.11) exist uniquely, but that the solutions depend continuously upon the parameters, namely $\mathbf{u}, \mathbf{B}, \sigma, \beta_{e}$, and $\beta_{i}$.

### 3.4 Well-posedness: Continuous Dependence on Parameters

In the following section, continuous dependence of the solutions on the parameters, $\mathbf{B}, \mathbf{u}, \sigma, \beta_{e}, \beta_{i}$, is established. We show this by perturbing the parameters, and bounding the changes in the solutions by these pertubations. We do this systematically, first for the state parameters, $\sigma, \beta_{e}, \beta_{i}$, and then for the field parameters, $\mathbf{u}, \mathbf{B}$. Normally, the coupling of the full MHD system accounts for the latter dependence, but the reduction in model complexity (by the prescription of the fields and assumption of steady-state) necessitates a direct demonstration. We will make use of Kato's theorem [14], which gives a useful bound for perturbing linear operators. First, however, we have a definition of a bounding operator.

### 3.4.1 Background

In the proof of continuous dependence, we make use of Kato's theorem, which in turn requires the definition of a bounding operator.

Definition: 3.4.1. Let $V, W$ be hilbert spaces, and let $M$ be a linear mapping between them. Then $M$ is bounding if there exists a $M^{*}>0$ such that

$$
\begin{equation*}
\|M v\|_{W} \geq M^{*}\|v\|_{V} \quad \forall v \in V . \tag{3.19}
\end{equation*}
$$

Note, a bounding operator is an injective operator with a continuous inverse [14]. With this definition in mind, we state Kato's theorem.

Lemma 3.4.1 (Kato's Theorem). Let $V$, $W$ be Hilbert spaces, and let $T_{1}$ and $T_{2}$ be linear operators from $V$ to $W$. If $T_{1}$ is bounding, then there exists $\varepsilon_{0}>0$ such that for all $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon_{0}$, the perturbed operator $T_{1}+\varepsilon T_{2}$ is also bounding, and we have moreover

$$
\left\|T_{1}^{-1}-\left(T_{1}+\varepsilon T_{2}\right)^{-1}\right\|_{\mathbf{L}(V, W)} \leq C|\varepsilon|,
$$

with $C$ depending on $\varepsilon_{0}$ but independent of $\varepsilon$.
A proof is presented in Fortin, [14]. Utilizing this theorem, it is now shown that the solutions $\mathbf{J}_{i}, \mathcal{V}$ depend continuously on the parameter set.

### 3.4.2 Continuous Dependence

We now establish that a perturbation of the system parameters, whether they be the state scalar parameters or the fields used in the model, results in a change of the solutions which can be bounded by the norm of the perturbation.

Theorem 3.4.1. Solutions to (3.11) depend continuously on $\sigma, \beta_{e}$ and $\beta_{i}$.
Proof. Fix $\mathbf{u}, \mathbf{B} \in\left(L^{2}(D)\right)^{3}$, bounded, and let $\sigma, \beta_{e}$, $\beta_{i}$ be positive and bounded functions on $D$. We show that the theorem holds for $\beta_{e}$ only, as a similar proof follows for $\beta_{i}, \sigma$. Thus, consider a small pertubation function to $\beta_{e}$ given by $\delta \beta_{e}$ such that $\beta_{e}+\delta \beta_{e} \geq 0$ on $D$ and consider the following perturbed system:

$$
\begin{align*}
& \int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}^{\prime} \cdot \phi-\int_{D} \nabla \mathcal{V}^{\prime} \cdot \phi=0 \quad \forall \phi \in \mathbf{L}^{2}(D)  \tag{3.20a}\\
& \int_{D} \mathbf{J}_{i}^{\prime} \cdot \nabla \psi=-\int_{D} \nabla \cdot\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}(\mathbf{u} \times \mathbf{B}) \psi\right) \quad \forall \psi \in W \tag{3.20b}
\end{align*}
$$

Here $\mathbf{J}_{i}^{\prime}, \mathcal{V}^{\prime}$ is the unique solution as guaranteed by Theorem 3.3.2. Let $\mathbf{J}_{i}, \mathcal{V}$ be the solution to the unperturbed system, i.e the solution to (3.11) and let $\delta \mathbf{J}_{i}=\mathbf{J}_{i}^{\prime}-\mathbf{J}_{i}$ and $\delta \mathcal{V}=\mathcal{V}^{\prime}-\mathcal{V}$ denote the differences between the solutions to the perturbed and unperturbed system. Subtracting (3.11) from (3.20) yields

$$
\begin{aligned}
& 0=\left(\int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}^{\prime} \cdot \phi-\int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i} \cdot \phi\right) \\
&-\left(\int_{D} \nabla \mathcal{V}^{\prime} \cdot \phi-\int_{D} \nabla \mathcal{V} \cdot \phi\right) \\
&=\int_{D} \frac{1}{\sigma}\left[\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}^{\prime}-\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}\right] \cdot \phi \\
&-\int_{D}\left[\nabla \mathcal{V}^{\prime}-\nabla \mathcal{V}\right] \cdot \phi \\
&= \int_{D} \frac{1}{\sigma}\left[\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)\left(\mathbf{J}_{i}^{\prime}-\mathbf{J}_{i}\right)-\delta \beta_{e}[\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right] \cdot \phi-\int_{D}\left[\nabla\left(\mathcal{V}^{\prime}-\mathcal{V}\right)\right] \cdot \phi \\
&= \int_{D} \frac{1}{\sigma}\left[\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)(\delta \mathbf{J})-\delta \beta_{e}[\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right] \cdot \phi-\int_{D} \nabla \delta \mathcal{V} \cdot \phi
\end{aligned}
$$

$$
=\int_{D} \underline{\bar{\sigma}}^{-1}(\delta \mathbf{J}) \cdot \phi-\int_{D} \nabla \delta \mathcal{V} \cdot \phi-\int_{D} \frac{1}{\sigma}\left(\delta \beta_{e}[\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right) \cdot \phi
$$

Rewriting this with the operators as defined in Section 3.3.2 yields

$$
\begin{equation*}
\mathcal{A}(\delta \mathbf{J})(\phi)+B^{\prime}(\delta \mathcal{V})(\phi)=\int_{D} \frac{1}{\sigma}\left(\delta \beta_{e}[\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right) \cdot \phi \tag{3.21}
\end{equation*}
$$

Similarly, for the second equation, we have:

$$
\begin{gathered}
\int_{D} \mathbf{J}_{i}^{\prime} \cdot \nabla \psi-\int_{D} \mathbf{J}_{i} \cdot \nabla \psi=-\int_{D} \nabla \cdot\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}(\mathbf{u} \times \mathbf{B}) \psi\right) \\
+\int_{D} \nabla \cdot(\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}) \psi)
\end{gathered}
$$

Cancelling like terms yields

$$
\begin{aligned}
\int_{D} \delta \mathbf{J}_{i} \cdot \nabla \psi & =-\int_{D} \nabla \cdot\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}(\mathbf{u} \times \mathbf{B}) \psi-\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}) \psi\right) \\
\mathcal{B}\left(\delta \mathbf{J}_{i}\right)(\psi) & =-\int_{D} \nabla \cdot\left[\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\underline{\bar{\sigma}}\right)(\mathbf{u} \times \mathbf{B})\right] \psi
\end{aligned}
$$

Now, for notational simplicity, let

$$
F_{1}(\phi)=\int_{D} \frac{1}{\sigma}\left(\delta \beta_{e}[\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right) \cdot \phi
$$

and

$$
G_{1}(\psi)=-\int_{D} \nabla \cdot\left[\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\underline{\bar{\sigma}}\right)(\mathbf{u} \times \mathbf{B})\right] \psi
$$

Then combining this with 3.21 yields the following system,

$$
\begin{equation*}
\mathcal{A}(\delta \mathbf{J})(\phi)+B^{\prime}(\delta \mathcal{V})(\phi)=F_{1} \phi \tag{3.22a}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{B}\left(\delta \mathbf{J}_{i}\right)(\psi)=G_{1}(\psi) \tag{3.22b}
\end{equation*}
$$

Applying Theorem 3.3 .2 gives that $\delta \mathbf{J}_{i}, \delta \mathcal{V}$ must exist. Applying the estimates from Theorem 3.3.2 will give the desired dependence result, but first, to simplify the estimate, we apply Green's theorem [47] to the operator $G_{1} \in W^{\prime}$. For any $\psi \in W$, this gives

$$
G_{1}(\psi)=-\int_{\partial D}\left[\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\underline{\bar{\sigma}}\right)(\mathbf{u} \times \mathbf{B})\right] \psi \cdot n .
$$

Taking the absolute value of each side, we have

$$
\begin{aligned}
\left|G_{1}(\psi)\right| & =\left|\int_{\partial D}\left[\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\overline{\underline{\sigma}}\right)(\mathbf{u} \times \mathbf{B})\right] \psi \cdot n\right| \\
& \leq \int_{\partial D}\left|\left[\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\overline{\bar{\sigma}}\right)(\mathbf{u} \times \mathbf{B})\right] \psi \cdot n\right| .
\end{aligned}
$$

Applying Cauchy-Schwarz [45] and noting that $\|n\|_{l^{2}}$ on $\partial D$ yields

$$
\begin{aligned}
\left|G_{1}(\psi)\right| & \leq\left\|\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\overline{\underline{\sigma}}\right)(\mathbf{u} \times \mathbf{B}) \psi\right\|_{L^{2}(D)} \\
& \leq\left\|\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\underline{\bar{\sigma}}\right)\right\|_{V^{\prime}}\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)} .
\end{aligned}
$$

Noting now that $\underline{\bar{\sigma}}$ is bounding, define $T_{1}=\underline{\sigma}^{-1}, T_{2}=[\mathbf{B}]_{\times}, \varepsilon=\delta \beta_{e}$. Applying Kato's theorem yields $\exists C>0$, not depending $\delta \beta_{e}$ such that
$\left\|\left(\sigma\left(\mathcal{I}-\left(\beta_{e}+\delta \beta_{e}\right)[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}-\underline{\bar{\sigma}}\right)\right\|_{V^{\prime}}\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)} \leq C\left\|\delta \beta_{e}\right\|_{L^{\infty}(D)}\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)}$
Applying this to the above inequality yields

$$
\left|G_{1}(\psi)\right| \leq\left\|\delta \beta_{e}\right\|_{L^{\infty}(D)} C\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)} .
$$

For notational simplicity in this proof, let $K=\frac{1}{\alpha \beta}\left(\|\mathcal{A}\|_{\mathcal{L}\left(V, V^{\prime}\right)}+\alpha\right)$. Then, using the a-priori estimate from Theorem 3.3.2, we have

$$
\left\|\delta \mathbf{J}_{i}\right\|_{V} \leq \frac{1}{\alpha}\left\|F_{1}\right\|_{V^{\prime}}+K\left\|G_{1}\right\|_{W^{\prime}} .
$$

Now, as these operator norms are minimizations over the spaces $V, W$, where applicable, it follows that for any $\phi \in V$ and $\psi \in W$ such that $\|\phi\|_{V}=\|\psi\|_{W}=1$, it holds that

$$
\begin{aligned}
\left\|\delta \mathbf{J}_{i}\right\|_{V} & \leq \frac{1}{\alpha}\left|\int_{D} \frac{1}{\sigma}\left(\delta \beta_{e}[\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right) \cdot \phi\right|+K C\left\|\delta \beta_{e}\right\|_{L^{\infty}(D)}\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)} \\
& \leq\left\|\delta \beta_{e}\right\|_{L^{\infty}(D)}\left(\frac{1}{\alpha}\left|\int_{D} \frac{1}{\sigma}\left([\mathbf{B}]_{\times} \mathbf{J}_{i}^{\prime}\right) \cdot \phi\right|+K C\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)}\right) .
\end{aligned}
$$

As $\phi, \psi$ were arbitrary, it follows that as as $\delta \beta_{e} \rightarrow 0, \delta \mathbf{J}_{i} \rightarrow 0$. Given that $\mathcal{V}$ depends continuously on $\mathbf{J}_{i}$, as apparent from (3.11), it follows that the solutions, $\mathbf{J}_{i}, \mathcal{V}$, depends continuously upon $\beta_{e}$. Similar logic shows that the solutions also continuously depend on $\beta_{i}, \sigma$.

Theorem 3.4.2. Solutions to (3.11) depend continuously on $\boldsymbol{u}$ and $\boldsymbol{B}$.
Proof. We first show that the solutions to (3.11) depend continuously on B. Fix $\mathbf{u}, \mathbf{B} \in\left(L^{2}(D)\right)^{3}$, bounded, and let $\sigma, \beta_{e}, \beta_{i}$ be positive, bounded functions on $D$. Let $\delta \mathbf{B}$ be a small perturbation on $\mathbf{B}$ in any one direction and consider the following perturbed system

$$
\begin{align*}
& \int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}^{\prime} \cdot \phi-\int_{D} \nabla \mathcal{V}^{\prime} \cdot \phi=0, \quad \forall \phi \in V,  \tag{3.23a}\\
& \int_{D} \mathbf{J}_{i}^{\prime} \cdot \nabla \psi=-\int_{D} \nabla \cdot\left(\sigma\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right)^{-1}(\mathbf{u} \times \mathbf{B}) \psi\right), \quad \forall \psi \in W, \tag{3.23b}
\end{align*}
$$

where $\mathbf{J}_{i}^{\prime}, \mathcal{V}^{\prime}$ denote the solutions, guaranteed by Theorem 3.3.2. Let $\mathbf{J}_{i}, \mathcal{V}$ denote the solutions to the unperturbed system, (3.11), and let $\delta \mathbf{J}_{i}, \delta \mathcal{V}$ denote the differences between the perturbed and unperturbed solutions. Subtracting (3.11) from (3.23) yields

$$
\begin{aligned}
& \int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}^{\prime} \cdot \phi-\int_{D} \frac{1}{\sigma}\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i} \cdot \phi \\
&=-\left(\int_{D} \nabla \mathcal{V}^{\prime} \cdot \phi-\int_{D} \nabla \mathcal{V} \cdot \phi\right) \\
& \int_{D} \frac{1}{\sigma}\left[\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}^{\prime}-\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}\right] \cdot \phi \\
&=-\left(\int_{D}\left[\nabla \mathcal{V}^{\prime}-\nabla \mathcal{V}\right] \cdot \phi\right)
\end{aligned}
$$

By linearity it is apparent that $[\mathbf{B}+\delta \mathbf{B}]_{x}=[\mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}$and $[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}=[\mathbf{B}]_{\times}^{2}+$ $[\mathbf{B}]_{\times}[\delta \mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}[\mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}^{2}$. Applying this to the above yields

$$
\begin{array}{r}
\int_{D}\left[\frac { 1 } { \sigma } \left[\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{e}[\delta \mathbf{B}]_{\times}-\beta_{i}\left([\mathbf{B}]_{\times}^{2}+[\mathbf{B}]_{\times}[\delta \mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}[\mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}^{2}\right)\right) \mathbf{J}_{i}^{\prime}\right.\right. \\
\left.\left.-\left(\mathcal{I}-\beta_{e}[\mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}]_{\times}^{2}\right) \mathbf{J}_{i}\right] \cdot \phi\right]=-\left(\int_{D} \nabla \delta \mathcal{V} \cdot \phi\right)
\end{array}
$$

$$
\begin{aligned}
& \int_{D}\left[\frac{1}{\sigma}\left(-\beta_{e}[\delta \mathbf{B}]_{\times}-\beta_{i}\left([\mathbf{B}]_{\times}[\delta \mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}[\mathbf{B}]_{\times}+[\delta \mathbf{B}]_{\times}^{2}\right)\right) \mathbf{J}_{i}^{\prime}-\underline{\bar{\sigma}}^{-1} \delta \mathbf{J}_{i}\right] \cdot \phi \\
&=\left(\int_{D} \nabla \delta \mathcal{V} \cdot \phi\right)
\end{aligned}
$$

For simplicity, let $\chi_{i}$ denote a vector of indicator functions in the $i^{\text {th }}$ direction. Then

$$
\begin{aligned}
-\int_{D} \frac{\delta \mathbf{B}}{\sigma}\left(-\beta_{e}\left[\chi_{i}\right]_{\times}-\beta_{i}\left([\mathbf{B}]_{\times}\left[\chi_{i}\right]_{\times}+\left[\chi_{i}\right]_{\times}[\mathbf{B}]_{\times}+\delta \mathbf{B}\left[\chi_{i}\right]_{\times}^{2}\right)\right) \mathbf{J}_{i}^{\prime} \cdot \phi & \\
& =\mathcal{A}\left(\delta \mathbf{J}_{i}\right)(\phi) \mathcal{B}^{\prime}(\delta \mathcal{V})(\phi)
\end{aligned}
$$

For notational convenience, define

$$
F_{2}(\phi)=-\int_{D} \frac{\delta \mathbf{B}}{\sigma}\left(-\beta_{e}\left[\chi_{i}\right]_{\times}-\beta_{i}\left([\mathbf{B}]_{\times}\left[\chi_{i}\right]_{\times}+\left[\chi_{i}\right]_{\times}[\mathbf{B}]_{\times}+\delta \mathbf{B}\left[\chi_{i}\right]_{\times}^{2}\right)\right) \mathbf{J}_{i}^{\prime} \cdot \phi
$$

For the second equation, we have

$$
\int_{D} \delta \mathbf{J}_{i}^{\prime} \cdot \nabla \psi=-\int_{D} \nabla \cdot\left(\sigma\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right)^{-1}(\mathbf{u} \times \mathbf{B}) \psi\right)-\nabla \cdot \underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}) \psi
$$

Writing this in the notation of Section 3.3.2, we have

$$
\mathcal{B}\left(\delta \mathbf{J}_{i}\right)(\psi)=-\int_{D} \nabla \cdot\left(\left(\sigma\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right)^{-1}-\underline{\bar{\sigma}}\right)(\mathbf{u} \times \mathbf{B}) \psi\right) .
$$

Again, for notational convenience, define

$$
G_{2}(\psi)=-\int_{D} \nabla \cdot\left(\left(\sigma\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right)^{-1}-\bar{\sigma}\right)(\mathbf{u} \times \mathbf{B}) \psi\right) .
$$

Thus, we have the following system,

$$
\begin{gather*}
\mathcal{A}\left(\delta \mathbf{J}_{i}\right)(\phi)+\mathcal{B}^{\prime}(\delta \mathcal{V})(\phi)=F_{2}(\phi)  \tag{3.24a}\\
\mathcal{B}\left(\delta \mathbf{J}_{i}\right)(\psi)=G_{2}(\psi) \tag{3.24b}
\end{gather*}
$$

By applying Theorem 3.3.2, $\delta \mathbf{J}_{i}, \delta \mathcal{V}$ both exist. Similar to the proof of Theorem 3.4.1, we apply Kato's theorem to simplify the estimates given by Theorem 3.3.2. For $\psi \in W$, we have

$$
\left|G_{2}(\psi)\right| \leq\left\|\left(\left(\sigma\left(\mathcal{I}-\beta_{e}[\mathbf{B}+\delta \mathbf{B}]_{\times}-\beta_{i}[\mathbf{B}+\delta \mathbf{B}]_{\times}^{2}\right)^{-1}-\bar{\sigma}\right)(\mathbf{u} \times \mathbf{B}) \psi\right)\right\|_{L^{2}(D)}
$$

$$
\leq\|\delta \mathbf{B}\|_{L^{\infty}(D)} C\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)}
$$

for some $C \in \mathbb{R}$. Utilizing the a priori estimates given in Theorem 3.3.2, we have
$\left\|\delta \mathbf{J}_{i}\right\|_{V} \leq \frac{1}{\alpha}\left\|F_{2}\right\|_{V^{\prime}}+\frac{1}{\alpha \beta}\left(\|\mathcal{A}\|_{\mathcal{L}\left(V, V^{\prime}\right)}+\alpha\right)\left\|G_{2}\right\|_{W^{\prime}}$
Using the definition of opertator norms, for any $\phi \in V$ and $\psi \in W$ such that $\|\phi\|_{V}=$ $\|\psi\|_{W}=1$

$$
\begin{aligned}
&\left\|\delta \mathbf{J}_{i}\right\|_{V} \leq \frac{1}{\alpha}\left|\int_{D} \frac{\delta \mathbf{B}}{\sigma}\left(\beta_{e}\left[\chi_{i}\right]_{\times}+\beta_{i}\left([\mathbf{B}]_{\times}\left[\chi_{i}\right]_{\times}+\left[\chi_{i}\right]_{\times}[\mathbf{B}]_{\times}+\delta \mathbf{B}\left[\chi_{i}\right]_{\times}^{2}\right)\right) \mathbf{J}_{i}^{\prime} \cdot \phi\right|+ \\
& \frac{1}{\alpha \beta}\left(\|\mathcal{A}\|_{\mathcal{L}\left(V, V^{\prime}\right)}+\alpha\right)\|\delta \mathbf{B}\|_{L^{\infty}(D)} C\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)} \\
& \leq\|\delta \mathbf{B}\|_{L^{\infty}(D)}\left[\frac{1}{\alpha}\left|\int_{D} \frac{1}{\sigma}\left(-\beta_{e}\left[\chi_{i}\right]_{\times}-\beta_{i}\left([\mathbf{B}]_{\times}\left[\chi_{i}\right]_{\times}+\left[\chi_{i}\right]_{\times}[\mathbf{B}]_{\times}+\delta \mathbf{B}\left[\chi_{i}\right]_{\times}^{2}\right)\right) \mathbf{J}_{i}^{\prime} \cdot \phi\right|\right. \\
&\left.\quad \frac{1}{\alpha \beta}\left(\|\mathcal{A}\|_{\mathcal{L}\left(V, V^{\prime}\right)}+\alpha\right) C\|(\mathbf{u} \times \mathbf{B}) \psi\|_{L^{2}(D)}\right] .
\end{aligned}
$$

Thus, it is clear that the solutions are continuously dependent upon B. Continuous dependence on $\mathbf{u}$ follows immediately from (3.11) and the continuous dependence of integrals on their arguments.

With this last theorem, we have completed the well-posedness of the deterministic forward problem. This theory will be computationally-verified within Section 5.1, and extended to include uncertainty within Section 6.1. For now, however, we move onto establishing more of the theoretical aspects of the deterministic kinematic MHD model, the inverse problem.

### 3.5 Parameter Estimation Theory

### 3.5.1 Introduction

In this section, we introduce the first inverse problem for our real-time optimization problem. Under the framework of the real-time optimization problem of MHD
generators, there is no feasible (and in many cases, physically possible) way for direct observation of the states. However, one can expect to obtain data from the resistors ${ }^{2}$ measuring the MHD generator's electrical output through a measure of the current density, J, and electric field, E. Thus, in order to choose optimal operating conditions of an MHD generator, we must estimate the state of the MHD generator from these measurements. This problem is called the deterministic parameter-estimation problem.

In general, the well-posedness of any inverse problem equates to showing that some numerical approximation method for the inverse map of the forward problem is method-stable, in the sense of numerical stability. As the parameter estimation problem is infinitely-dimensional, we must make some reductions to the parameter space over which we search, and the norms which we will measure our residual with. We pair these reductions with likely compact function spaces that are computationally realistic, with values across $D$ lying in intervals given in [56]. We then establish the numerical method to implement the dimension-reduced problem, i.e. $\left(I D_{M}^{N}\right)$, and show that this is method-stable. This is done by showing that the method converges to any set of parameter functions, as the dimension goes to infinity, applying ideas from [4], and demonstrating that the method satisfies the necessary postulates.

### 3.5.2 Parameter Estimation Scheme

Parameter estimation is an optimization problem in which if $u$ is the solution to a system of equations, and $d$ represents provided data, the goal is to find parameter(s) $q$ to minimize the difference between the solution and data, i.e.

$$
\text { Minimize } J(q):=\|u(\cdot ; q)-d\| \text {, }
$$

where $q$ varies over some admissible parameter space, and $J$ is defined in some appropriate norm. We will denote this problem for our system of equations the ID or

[^1]identification problem. To match the notation of Banks [4], we define the following:
\[

U=\left[$$
\begin{array}{l}
\mathbf{J}_{i} \\
\mathcal{V}
\end{array}
$$\right], A(q)=\left[$$
\begin{array}{cc}
\mathcal{A} & \mathcal{B}^{\prime} \\
\mathcal{B} & 0
\end{array}
$$\right], F(q)=\left[$$
\begin{array}{l}
0 \\
G
\end{array}
$$\right],
\]

where $\mathcal{A}, \mathcal{B}, G$ are as defined in the above sections. Then the system of equations, (3.11), can be written as

$$
\begin{equation*}
A U=F . \tag{3.25}
\end{equation*}
$$

Clearly, $U \in V \times W$ for any solution of (3.11), and we denote this space $H:=V \times W$. The set of unknowns for this specific parameter estimation problem is given by

$$
q=\left(\mathbf{u}, \mathbf{B}, \sigma, \beta_{e}, \beta_{i}\right) .
$$

We now define a series of parameter spaces over which a solution to the inverse problem exists, following from the stipulations within Theorem 3.3.2. We begin with the most general, which is given by

$$
Q:=\left(L^{2}(D)\right)^{3} \times\left(L^{2}(D)\right)^{3} \times L_{+}^{\infty}(D) \times L_{+}^{\infty}(D) \times L_{+}^{\infty}(D)
$$

where

$$
L_{+}^{\infty}(D)=\left\{f \in L^{\infty}(D): f>0 \text { on } D\right\} .
$$

For any parameter set chosen within $Q$, there will be a solution to 4.7), guaranteed by Theorem 3.3.2. For simplicity, denote the solution to the system with a given parameter set $q$ as $U(\cdot ; q)$. For notational purposes, let $\mathcal{M}=L^{\infty}(D)$. We now further refine the admissible parameter set to reflect the underlying physics of the problem that is not captured by the system of equations, i.e. the realistic admissible parameter set, which will be denoted by $\widetilde{Q}$. Note that we choose this subset to be compact. First, define

$$
\begin{aligned}
\widetilde{\widetilde{Q}}:=\left\{q \in Q:\|\mathbf{u}\|_{\mathcal{M}^{3}}<u_{\max },\|\mathbf{B}\|_{\mathcal{M}^{3}}<\mathbf{B}_{\max },\right. & \sigma_{\min }<\|\sigma\|_{\mathcal{M}}<\sigma_{\max }, \\
& \left.\left\|\beta_{e}\right\|_{\mathcal{M}}<\beta_{e, \max },\left\|\beta_{i}\right\|_{\mathcal{M}}<\beta_{i, \max }\right\},
\end{aligned}
$$

and then characterize $\widetilde{Q}$ as the subset of $\widetilde{\widetilde{Q}}$ where $\mathbf{u}, \mathbf{B}$ both have uniform Lipschitz bounds, denoted $l_{1, i}, l_{2, i}$ for $i=1,2,3$. Notationally, $\forall \varepsilon>0$,

$$
\widetilde{Q}:=\left\{q \in \widetilde{\widetilde{Q}}:\left|\mathbf{u}_{i}\left(x_{i}, ;\right)-\mathbf{u}_{i}\left(x_{i}+\varepsilon, ;\right)\right| \leq \varepsilon l_{1, i},\left|\mathbf{B}_{i}\left(x_{i}, ;\right)-\mathbf{B}_{i}\left(x_{i}+\varepsilon, ;\right)\right| \leq \varepsilon l_{2, i}, \text { for } i=1,2,3\right.
$$

and $\sigma, \mu_{e}, \mu_{i}$ are equicontinuous on $\left.D\right\}$.

This implies that the identification problem that defines this parameter estimation is to minimize

$$
\begin{equation*}
J(q):=\|U(\cdot ; q)-\mathcal{U}\|_{H} \tag{ID}
\end{equation*}
$$

over $q \in \widetilde{Q}$ and for some given state $\mathcal{U}$. Now, for both numerical and theoretical considerations, we define a finite-dimensional approximation to $\widetilde{Q}$. Let $D$ be partitioned into $M$ subsets, and define
$\widetilde{Q}_{M}:=\left\{q \in \widetilde{Q}:\left.\mathbf{u}\right|_{D_{k}} \in \mathbb{P}^{n},\left.\mathbf{B}\right|_{D_{k}} \in \mathbb{P}^{n},\left.\sigma\right|_{D_{k}} \in \mathbb{P}^{0},\left.\beta_{e}\right|_{D_{k}} \in \mathbb{P}^{0},\left.\beta_{i}\right|_{D_{k}} \in \mathbb{P}^{0}\right.$ for $\left.k=1, \ldots M\right\}$, for some $n \in \mathbb{N}$. Here, $\mathbb{P}^{n}$ is the space of polynomials of degree at most $n$ on $D_{k}$, and $\mathbb{P}^{0}$ is clearly the space of constant functions. As well, note that as $\widetilde{Q}_{M} \subset \widetilde{Q}, \mathbf{u}, \mathbf{B}$ will still be continuous functions for any $q \in \widetilde{Q}_{M}$. Finally, we complete the statement of the parameter estimation problem by turning the discussion to partial observational data.

Realistically, the data provided will not be known across the domain. Thus, for incomplete or partial observational data, define $\left\{x_{k}\right\}_{k=1}^{l} \subseteq D$ to be the points at which the data, $d$, is given. Define the projection operator to be

$$
\mathcal{C}(\mathcal{F})=\left\{\mathcal{F}\left(x_{k}\right)\right\}_{k=1}^{l}
$$

where $\mathcal{F}$ denotes any function mapping $D \rightarrow \mathbb{R}^{4}$. This implies that the restricted domain ID problem is given by

$$
\begin{equation*}
J_{M}(q):=\|\mathcal{C}(U(\cdot ; q))-d\|_{\left(L^{2}\right)^{4}} \tag{M}
\end{equation*}
$$

over $q \in \widetilde{Q}_{M}$. Note now that the weak solution space, $H$, does not necessarily lie in the domain of $\mathcal{C}$. Thus, we define a continuously imbedded Banach space into $H$ by continuous functions, i.e.

$$
\widetilde{H}=\left((\mathcal{C}(D))^{3} \times \mathcal{C}(D)\right) \cap(V \times W)
$$

Then define the domain of $\mathcal{C}$ to be:

$$
\mathcal{C}: \widetilde{H} \rightarrow Z=\left(\mathbb{R}^{l}\right)^{4},
$$

where $l$ is determined by the number of observation points. This projection operator represents where the data will be provided within the domain $D$.

We also seek our solutions to (3.11) in some finite-dimensional (FD) space. The particular FD approximation to $H$ we use, denoted $H^{N}$, is defined as a space of linear functions. Define $\tau$ to be a given Delauney [11] finite element triangularization of $D$ with $N$ elements 3 Then clearly the dimension of $H^{N}$ is finite, with a basis of linear functions, $\left\{\phi_{j}\right\}$, where $\phi_{j}$ will be 1 on the $j^{\text {th }}$ node, and 0 on every other node, vanishing at the endpoints. Similar to $H^{N}$, define $\widetilde{H}^{N}=H^{N} \cap\left((\mathcal{C}(D))^{3} \times \mathcal{C}(D)\right)$. With this finite dimensional approximation space, we arrive at the final version of the identification problem, analogous to $\left(\mathrm{ID}_{M}\right)$ : find the minimum of

$$
\begin{equation*}
J_{M}^{N}(q):=\left\|\mathcal{C}\left(U^{N}(\cdot ; q)\right)-d\right\|_{l^{2}} \tag{M}
\end{equation*}
$$

over $q \in \widetilde{Q}_{M}$, with $U^{N}(\cdot ; q) \in \widetilde{H}^{N}(q)$.
With the identification problem now clear, we state the definition of a function space parameter estimation convergent or FSPEC set, as seen in [4]. This will in turn allow us to show that the inverse problem is well-posed.

Definition: 3.5.1. For notational convenience, let $x^{k}:=x_{M_{k}}^{N_{k}}$ where appropriate. Then, a set consisting of a finite-dimensional approximation to the solutions space of

[^2](3.14), $H^{N}$, a finite-dimensional approximation to the solutions of (3.14), $U^{N}(q)$, a projection operator, $\mathcal{C}$ and a finite-dimensional approximation to the admissible parameter space, $\widetilde{Q}_{M}$ is FSPEC if it satisfies the following.
i. For each $N=1,2, \ldots$ there exists a solution $\bar{q}_{M}^{N} \in \widetilde{Q}_{M}$ of $I D_{M}^{N}$.
ii. Every convergent subsequence $\left\{\bar{q}^{k}\right\}$ converges to a solution $q^{*} \in \widetilde{Q}$ of $I D$.
iii. $J^{k}\left(\bar{q}^{k}\right) \rightarrow J\left(q^{*}\right)$.
iv. $\left\|U^{N_{k}}\left(\cdot ; \bar{q}^{k}\right)-U\left(\cdot ; q^{*}\right)\right\|_{\widetilde{H}\left(\bar{q}^{k}\right)} \rightarrow 0$.
$v$. There exists at least one subsequence satisfying $i i .-i v$.

We now go on to show that the identification problem as defined above satisfies the requirements to be FSPEC.

### 3.5.3 Necessary Postulates

In this section, several propositions are presented concerning the parameter estimation problem. These are then utilized to construct a set which is function space parameter estimation convergent, implying that the parameter estimation problem is method-stable. All of the following propositions stem from similar postulates found in Banks [4], and are labeled to correspond accordingly. We begin with properties of the solution Hilbert space, and its finite dimensional approximation. Note here that we state a dependence of the solution space on the parameter set, out of notational convenience to match that in Banks.

Proposition 3.5.1 (HS). For each $q$ in the metric space $Q$, $\rho$, the space $H$ is a Hilbert space, $\widetilde{H}$ is a Banach space continuously imbedded in $H$, and $H^{N}$ is a finite dimensional (closed) linear subspace of $H$ with dimension independent of $q$.

Proof. We define $\rho$ to be the metric induced by $H$ 's inner product, i.e.

$$
\langle\cdot, \cdot\rangle_{q}=\langle\cdot, \cdot\rangle_{\left(L^{2}(D)\right)^{3}}+\langle\cdot, \cdot\rangle_{L^{2}(D)}
$$

It immediately follows that $H$ is complete, as any Cauchy sequence $f_{k}=\left(f_{1, k}, f_{2, k}\right)$ in $H$ must have the property that $f_{1, k}$ is Cauchy in $L^{2}(D)$, and by the Riesz-Fischer Theorem [43], this is a complete space, and thus $f_{1, k}$ converges. A similar result holds for $f_{2, k}$, given that $W$ is a closed subset of a complete space. Here, the closure property is immediate from $W$ being defined as the closure of the continuous test functions with compact support. Thus, $H$ is a Hilbert space.

As the continuous functions are dense in $L^{2}(D)$, it immediately follows that $\widetilde{H}$ is a Banach space continuously embedded in $H$, as the product of the dense subspace of factors is dense in the product space.

Finally, $H^{N}$ is finite-dimensional, as it has a finite basis. As well, $H^{N}$ is clearly closed under vector addition and scalar multiplication, as the compactness of $D$ guarantees the closure. Thus, $H^{N}$ is a closed linear space. The dimension depends only on the number of partitions of $\tau$, which is not a parameter of $q$, and the claim holds.

Now, to present a classic theorem for compactness in $L^{2}$ spaces. This theorem is analogous to Arzela-Ascoli [44] for $L^{p}$ spaces, and is necessary to characterize compactness in $Q$.

Theorem 3.5.1 (Frechét-Kolmogorov Theorem). Let $\mathcal{F}$ be a subset of $L^{p}(D)$ with $p \in[1, \infty)$ and let $\tau_{h} f$ denote the translation of $f \in \mathcal{F}$ by h, i.e., $\tau_{h} f=f(x-h)$. Then $\mathcal{F}$ is compact if and only if
i. $\mathcal{F}$ is closed,
ii. $\mathcal{F}$ is equicontinuous, i.e. $\lim _{|h| \rightarrow 0}\left\|\tau_{h} f-f\right\|_{L^{p}}=0$ uniformly, and
iii. $\mathcal{F}$ is equitight, or $\lim _{r \rightarrow \infty} \int_{\|x\|>r}|f|^{p}=0$ uniformly.

A proof can be found in [8]. We now apply this theorem to the first two subspaces of $\widetilde{Q}$.

Lemma 3.5.1. The space of uniformly bounded and uniformly Lipschitz continuous functions, denoted $\mathcal{F}$, is a compact subset of $L^{2}(D)$.

Proof. We clearly would like to apply the Frechét-Kolmogorov theorem. Thus, we show the three conditions stated in the theorem hold. Let $\alpha$ denote the uniform bound on $\mathcal{F}$, and let $\beta$ denote the uniform Lipschitz condition.
i. Closed. From the Lipschitz condition, it follows that $f_{k}$ must be differentiable at all but a countable number of points for each $k$. Now, as the union of countable sets is again countable, assume that these points of non-differentiability are the same for each $f_{k}$. Denote these points $\left\{x_{j}\right\}$ and consider any interval between such points. At most, $f_{k}^{\prime} \leq \beta$ on each.

Suppose that $f_{k} \rightarrow f$. It then follows that $f$ has at most the same points of non-differentiability as $\left\{f_{k}\right\}$. As well, by continuity of the derivative operator, it follows that $f_{k}^{\prime} \rightarrow f^{\prime}$. Therefore $f^{\prime} \leq \beta$ as each $f_{k}^{\prime}$ is, and thus $f$ is Lipschitz continuous with Lipschitz bound $\leq \beta$.

Now, as continuous functions with bounded derivatives are compact by ArzelaAscoli [44], it immediately follows that on each interval of differentiability,

$$
\|f\|_{L^{\infty}\left(\left[x_{k}, x_{k+1}\right]\right)}<\alpha
$$

and by continuity of $f$, it must be that $\|f\|_{\mathcal{M}}<\alpha$.
ii. Equicontinuous. Let $f \in \mathcal{F}$. The uniformly Lipschitz condition implies that

$$
\left|\tau_{h} f-f(x)\right| \leq \beta|h| .
$$

Applying this yields

$$
\left\|\tau_{h} f-f\right\|_{L^{2}(D)}^{2}=\int_{D}\left|\tau_{h} f-f\right| d V \leq \int_{D} \beta|h| \rightarrow 0
$$

and as $f$ is arbitrary, $\mathcal{F}$ must be equicontinuous.
iii. Equitight. This follows immediately from $D$ being compact and thus bounded. Therefore the support of any $f \in \mathcal{F}$ is compact, and for any ball larger than the radius of $D$, the integral of $f$ outside of that ball is zero, and therefore $\mathcal{F}$ is equitight.

We now see that our realistic admissible parameter set is compact.
Proposition 3.5.2 (HQ1). The set $\widetilde{Q}$ is a compact subset of the metric space $(Q, \rho)$.
Proof. For simplicity, label each of the product spaces of $\widetilde{Q}$ as $\widetilde{Q}_{i}$ so that

$$
\widetilde{Q}=\widetilde{Q}_{1} \times \ldots \times \widetilde{Q}_{5} .
$$

Note that $\widetilde{Q}_{1}$ and $\widetilde{Q}_{2}$ are compact by Lemma 3.5.1. Also note that $\widetilde{Q}_{3}, \widetilde{Q}_{4}, \widetilde{Q}_{5}$ are compact from Arzela-Ascolí [8]. As the product of compact spaces is compact, $\widetilde{Q}$ is compact in $Q$.

We now show that our approximations to the space pass convergent sequences to convergent sequences.

Proposition 3.5.3 (HQC). For any arbitrary sequence $q^{M} \rightarrow q^{0}$ in $Q$, we have

$$
\left\|U^{N}\left(\cdot ; q^{M}\right)-U\left(\cdot ; q^{0}\right)\right\|_{\widetilde{H}(q)} \rightarrow 0
$$

Proof. Consider the following.

$$
\begin{aligned}
\left\|U^{N}\left(\cdot ; q^{M}\right)-U\left(\cdot ; q^{0}\right)\right\|_{\widetilde{H}(q)} & =\left\|U^{N}\left(\cdot ; q^{M}\right)-U\left(\cdot ; q^{M}\right)+U\left(\cdot ; q^{M}\right)-U\left(\cdot ; q^{0}\right)\right\|_{\widetilde{H}(q)} \\
& \leq\left\|U^{N}\left(\cdot ; q^{M}\right)-U\left(\cdot ; q^{M}\right)\right\|_{\widetilde{H}(q)}+\underbrace{\left\|U\left(\cdot ; q^{M}\right)-U\left(\cdot ; q^{0}\right)\right\|_{\widetilde{H}(q)}}_{B} .
\end{aligned}
$$

As it has been shown that the solution, $U$, depends continuously on the parameters (Theorems 3.4.1 and 3.4.2, $B$ must go to 0 as $q^{M} \rightarrow q^{0}$. As well, consider that as $U^{N}$ satisfies the weak-form, i.e (3.15), it must also satisfy the minimization (Ritz) formulation, given by

$$
\left\|U^{N}\left(;, q^{M}\right)-U\left(;, q^{M}\right)\right\|_{\widetilde{H}(q)} \leq\left\|V-U\left(;, q^{M}\right)\right\|_{\widetilde{H}(q)} \forall V \in H^{N} .
$$

In particular, this is satisfied for the interpolant of $U$ in $H^{N}$, denoted $\bar{U}^{N}(q)$, and thus

$$
\left\|U^{N}\left(;, q^{M}\right)-U\left(;, q^{M}\right)\right\|_{\widetilde{H}\left(q^{M}\right)} \leq\left\|\bar{U}^{N}\left(;, q^{M}\right)-U\left(;, q^{M}\right)\right\|_{\widetilde{H}(q)} .
$$

Now, define $h_{K}=\operatorname{diam}(K)$ as the longest edge of element $K \in \tau$ and then define $h:=\max _{K \in \tau} \mathbf{h}_{k}$. Then from [7], it follows that

$$
\leq C h^{3}\left\|U\left(;, q^{M}\right)\right\|_{\widetilde{H}(q)},
$$

for some constant $C$ that does not depend on the chosen grid. Recall that $\tau$ is the Delauney finite-element triangularization of $D$. Clearly, as $N \rightarrow \infty, h \rightarrow 0$ for a Delauney finite-element mesh, and therefore the inequality holds.

We now need to show the existence of surjective maps from the admissible parameter space to the finite-dimensional representation, and that as the order of the FD representations goes to infinity, it converges to the admissible parameter space. We do so for each of the two 'types' of domains we work with, $L^{2}$ and $L^{\infty}$. We begin with $L^{2}$.

Lemma 3.5.2. Let $\mathcal{F}:=\left\{f \in L^{2}:\|f\|_{\mathcal{M}} \leq \alpha_{1},|f(x+h)-f(x)| \leq h \alpha_{2} \forall x \in D\right\}$, for some $\alpha_{1}, \alpha_{2}>0$. Now, for the $M$ parameter dimensions, $K \in \tau$, and $n$ the polynomial dimension for $\widetilde{Q}_{1}, \widetilde{Q}_{2}$, define

$$
\mathcal{F}_{M}:=\left\{f \in \mathcal{F}:\left.f\right|_{K} \in \mathbb{P}^{n}(K) k=1, \ldots, M\right\} .
$$

Then $\mathcal{F}_{M}$ is compact in $\mathcal{F}$, and there exists a surjective map $V_{M}^{1}: \mathcal{F} \rightarrow \mathcal{F}_{M}$ such that

$$
\rho\left(V_{M}^{1} q_{n}, q_{n}\right) \rightarrow 0 \text { as } M \rightarrow \infty
$$

for any convergent sequence $\left\{q_{n}\right\} \subset \mathcal{F}$.
Proof. To see that $\mathcal{F}_{M}$ is compact, we note that $\mathcal{F}$ has been shown to be compact (Lemma 3.5.1), and $\mathcal{F}_{M}$ is closed. As any closed subset of a compact set is compact, $\mathcal{F}_{M}$ must be compact.

We now define the surjective operator. For simplicity, assume that the piecewise polynomials are all of order 1 , and that $D \subset \mathbb{R}$. For higher-degree polynomials, the arguments remain similar. For $D \subset \mathbb{R}^{m}, m>1$, the arguments presented below are easily extended to each spatial dimension.

Let $\left\{x_{k}\right\}$ be the finite number of sample points. Assume that $D$ is connected (as one can extend to each disconnected subset of $D$ ), and for $f \in \mathcal{F}$, define

$$
V_{M}^{1}(f)=\frac{f\left(x_{k}\right)-f\left(x_{k-1}\right)}{x_{k}-x_{k-1}}\left(x-x_{k}\right)+f\left(x_{k}\right) \text { for } x_{k-1} \leq x<x_{k} \text {, for } k=1, \ldots, M-1 .
$$

Clearly, as $V_{M}^{1}=\operatorname{Id}$ for any $f \in \mathcal{F}_{M}$, and as $\mathcal{F}_{M} \subset \mathcal{F}$, it follows that $V_{M}^{1}$ is surjective. Now, note that the Lipschitz condition implies $V_{M}^{1}(f)<\alpha_{2}\left(x-x_{k}\right)+f\left(x_{k}\right)$ for any $f \in \mathcal{F}$ and all $x_{k-1}<x<x_{k}$. It then follows that

$$
\left\|V_{M}^{1}(f)-f\right\|_{\mathcal{M}} \leq \alpha_{2}\left(x_{k}-x_{k-1}\right)
$$

Let $\left\{q_{n}\right\} \subset \mathcal{F}$ be any convergent sequence. Then clearly, as $M \rightarrow \infty,\left(x_{k}-x_{k-1}\right) \rightarrow 0$, and thus the desired inequality holds.

We now show a similar result for $L^{\infty}$.
Lemma 3.5.3. Let $\mathcal{G}:=\left\{g \in L^{\infty}:\|g\|_{L^{\infty}} \leq \gamma,\right\}$. Let $\mathcal{G}_{M}:=\left\{g \in \mathcal{G}:\left.g\right|_{D_{k}} \in \mathbb{P}^{0}\left(D_{k}\right) k=\right.$ $1, \ldots, M\}$. Then $G_{M}$ is compact in $\mathcal{G}$, and there is a surjective map $V_{M}^{2}: \mathcal{G} \rightarrow \mathcal{G}_{M}$ such that

$$
\rho\left(V_{M}^{2} q_{n}, q_{n}\right) \rightarrow 0 \text { as } M, n \rightarrow \infty
$$

for any convergent sequence $\left\{q_{n}\right\} \subset \mathcal{G}$.
Proof. As with Lemma 3.5.2, it has already been shown that $\mathcal{G}$ is compact, and it immediately follows that $\mathcal{G}_{M}$ is closed, and thus compact as a subset of $\mathcal{G}$. Now, define the surjective map similar to Lemma 3.5.2. For any $g \in \mathcal{G}$,

$$
V_{M}^{2}(g):=g\left(x_{k}\right)-g\left(x_{k-1}\right) \text { for } x_{k-1}<x<x_{k}, \text { for } k=1, \ldots M-1 .
$$

Then $V_{M}^{2}=\operatorname{Id}$ on $\mathcal{G}_{M}$, and as a subset of $\mathcal{G}$, it is surjective. The inequality follows as

$$
\left\|V_{M}^{2}(g)-g\right\|_{\mathcal{M}} \leq \gamma \max _{K \in \mathcal{\tau}} h_{K},
$$

with $h_{k}$ defined as in the proof for Lemma 3.5.2. By the definition of Dealuney triangulation, we have $M \rightarrow \infty, h_{K} \rightarrow 0 \forall K \in \tau$ and thus the inequality holds.

Finally, we combine the ideas of the two above lemmas, applying them to the appropriate subspaces of $\widetilde{Q}$.

Proposition 3.5.4 (HQ4). There exists a sequence of finite dimensional compact sets $\widetilde{Q}_{M} \subseteq \widetilde{Q}$ and surjective maps $V_{M}: \widetilde{Q} \rightarrow \widetilde{Q}_{M}$ such that for any convergent (possible trivially convergent) sequence $\left\{q_{n}\right\}$ in $\widetilde{Q}$ we have $\rho\left(V^{M} q_{n}, q_{n}\right) \rightarrow 0$ as $n, M \rightarrow \infty$.

Proof. For simplicity, define

$$
\widetilde{Q}_{M}=\widetilde{Q}_{1}^{M} \times \ldots \times \widetilde{Q}_{5}^{M} .
$$

Then from Lemma 3.5.2 $\widetilde{Q}_{1}$ and $\widetilde{Q}_{2}$ are compact. Similarly, from Lemma 3.5.3, $\widetilde{Q}_{3}, \widetilde{Q}_{4}, \widetilde{Q}_{5}$ are compact. Thus, as the product of compact spaces is compact, $\widetilde{Q}$ is compact.

Define $V_{M}$ to be the product of the projective mappings guaranteed by the two lemmas for each of the spaces. Then it is immediate from these lemmas that the mappings are surjective, and then finally, $V_{M}$ must satisfy the desired convergence as $n, M$ go to $\infty$.

Now, with all of these propositions established, we turn to showing that the set is FSPEC.

### 3.5.4 Inverse Theorem

Following the work of Banks [4], we now define a set based on the above propositions that is FSPEC, showing that the inverse problem is well-posed. We do so by showing that the set satisfies each of piece of the definition, FSPEC.i - FSPEC.v.

Theorem 3.5.2. The set $\left\{H^{N}, U^{N}(q), C, \widetilde{Q}_{M}\right\}$ is FSPECC.
Proof. From Theorems 3.4.1 and 3.4.2, it is immediate that $U$ depends continuously on $q$, and thus so does $J_{M}^{N}(q)$. Compactness of $\widetilde{Q}_{M}$, as shown in HQ4, under the continuous map of $U$ implies the existence of solutions to $\mathrm{ID}_{M}^{N}$, as the image is compact and thus contains a minimum. Denote this solution $\bar{q}_{M}^{N}$. By definition of minimum, we have $J_{M}^{N}\left(\bar{q}_{M}^{N}\right) \leq J^{N}\left(q_{M}\right)$ for all $q_{M} \in \widetilde{Q}_{M}$. Of course, this implies that

$$
\begin{equation*}
J_{M}^{N}\left(\bar{q}_{M}^{N}\right) \leq J_{M}^{N}\left(V_{M}(q)\right) \text { for all } q \in \widetilde{Q} . \tag{3.26}
\end{equation*}
$$

As $V_{M}$ is surjective by HQ4, choose $\widetilde{q}_{M}^{N} \in \widetilde{Q}$ such that $V_{M} \widetilde{q}_{M}^{N}=\bar{q}_{M}^{N}$. As $\widetilde{Q}$ is compact (see HQ1), as $N, M \rightarrow \infty$, there exists some convergent subsequence of the above solutions, i.e. $\left\{\widetilde{q}_{M_{k}}^{N_{k}}\right\} \subset \widetilde{Q}$ such that $\widetilde{q}_{M_{k}}^{N_{k}} \rightarrow q^{*}$ for some $q^{*} \in \widetilde{Q}$. Utilizing HQ4 again, along with triangle inequality, it follows that

$$
\begin{aligned}
\rho\left(\bar{q}_{M_{k}}^{N_{k}}, q^{*}\right) & \leq \rho\left(\bar{q}_{M_{k}}^{N_{k}}, \widetilde{q}_{M_{k}}^{N_{k}}\right)+\rho\left(\widetilde{q}_{M_{k}}^{N_{k}}, q^{*}\right) \\
& =\rho\left(\mathcal{V}_{M_{k}}\left(\widetilde{q}_{M_{k}}^{N_{k}}\right), \widetilde{q}_{M_{k}}^{N_{k}}\right)+\rho\left(\widetilde{q}_{M_{k}}^{N_{k}}, q^{*}\right) \rightarrow 0 .
\end{aligned}
$$

implying that $\bar{q}_{M_{k}}^{N_{k}} \rightarrow q^{*}$ in $Q$. This implies that both FSPEC.i and FSPEC.ii are satisfied.

For notational purposes, let $\bar{q}^{k}=\bar{q}_{M_{k}}^{N_{k}}$, and similarly for $J$. Then

$$
\begin{aligned}
\left\|J^{k}\left(\bar{q}^{k}\right)-J\left(q^{*}\right)\right\|_{L^{2}} & \leq\left\|\left(\mathcal{C}\left(\bar{q}^{k}\right) U^{N_{k}}\left(\cdot ; \bar{q}^{k}\right)-d\right)-\left(\mathcal{C}\left(q^{*}\right) U\left(\cdot ; q^{*}\right)-d\right)\right\|_{L^{2}} \\
& =\left\|\mathcal{C}\left(\bar{q}^{k}\right) U^{N_{k}}\left(\cdot ; \bar{q}^{k}\right)-\mathcal{C}\left(q^{*}\right) U\left(\cdot ; q^{*}\right)\right\|_{L^{2}} .
\end{aligned}
$$

This implies the FSPEC.iii holds. Now, from HQC and the lack of dependence of $q$ on $\mathcal{C}$, it immediately follows that $\left\|J^{k}\left(\bar{q}^{k}\right)-J\left(q^{*}\right)\right\|_{L^{2}} \rightarrow 0$ by the above. This implies that

$$
\begin{equation*}
J^{k}\left(\bar{q}^{k}\right) \rightarrow J\left(q^{*}\right) \tag{3.27}
\end{equation*}
$$

and thus FSPEC.iv holds. FSPEC.v holds by all of the above. The existence of one such subsequence is guaranteed by applying Theorem 2 onto the subspaces $V_{h}, W_{h}$, with an appropriately defined parameter sequence.

We have now established the well-posedness of the deterministic parameter estimation problem. By demonstrating that the finite-dimensional approximation to $I D$ converges as the dimensions approach infinity, we have shown that the scheme is method-stable.

### 3.6 Notation

### 3.6.1 Forward Problem

| Term | Definition | Notes |
| :---: | :--- | :--- |
| $D$ | Spatial domain | Subset of $\mathbb{R}^{3}$. |
| $\partial D$ | Boundary of $D$ | Mostly in $x$-direction |
| $\mathbf{u}$ | Fluid-flow | Entirely in $z$-direction, im- <br> $\mathbf{B}$ |
| Applied Magnetic Field |  |  |
| $\sigma^{\prime}$ | Plasma Conductivity | negligible. |
| $\mu_{e}$ | Electron mobility | Ion mobiility <br> $\mu_{i}$ |
| $\beta_{e}$ | Hall parameter | This accounts for the different <br> velocity field of the electrons |
| $\beta_{i}$ | Ion-slip parameter | relative to the plasma. |
| $\mathbf{n}$ | Vector outwards-normal | This accounts for the different <br> to domain $D$. |
| $\mathcal{V}$ | Elelocity field of the ions rela- |  |
| Eive to the plasma. |  |  |


| Term | Definition | Notes |
| :---: | :--- | :--- |
| $\mathbf{J}_{i}$ | $\sigma \nabla \mathcal{V}$ | Represents the induced |
| current-density |  |  |
| $\underline{\sigma}$ | $\sigma\left(\mathcal{I}+\beta_{e}[\mathbf{B}]_{\times}+\beta_{i}[\mathbf{B}]_{\times}^{2}\right)^{-1}$ | Conductivity Tensor |
| $W(D)$ | $\left\{f \in H^{1}(D): T(f)=0\right\}$ | $T(f)$ is the trace of $f$ on $D$. |
|  |  | This is the solution space for |
|  |  | $\mathcal{V}$ |
| $V(D)$ | $\left\{\mathbf{f} \in\left(L^{2}(D)\right)^{3}: \mathbf{f} \cdot \mathbf{n}=\right.$ | This is the solution space for |
|  | $-(\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B})) \cdot \mathbf{n}$ on $\partial D\}$ | $\mathbf{J}_{i}$ |
| $\mathcal{A}(\mathbf{F})(\circ)$ | $\int_{D}\left(\bar{\sigma}^{-1} \mathbf{F}\right) \cdot \circ$ | Maps from $V \times V$ to $\mathbb{R}$. |
| $\mathcal{B}(\mathbf{G})(\circ)$ | $-\int_{D} \mathbf{G} \cdot \nabla \circ$ | Maps from $V \times W \rightarrow \mathbb{R}$. |
| $g(\omega)$ | $\int_{D}(\underline{\bar{\sigma}} \mathbf{u} \times \mathbf{B}) \cdot \nabla \omega$. |  |

### 3.6.2 Inverse Problem

| Term | Definition | Notes |
| :---: | :---: | :---: |
| $U$ | $\left[\mathbf{J}_{i}, \mathcal{V}\right]^{T}$ | Vector of solutions to the forward problem |
| H | $V \times W$ | Solution space for $U$ |
| $q$ | $\left(\mathbf{u}, \mathbf{B}, \sigma, \beta_{e}, \beta_{i}\right)$ | Parameter set for MHD system |
| $L_{+}^{\infty}(D)$ | $\left\{f \in L^{\infty}(D): f>0\right.$ on $\left.D\right\}$ |  |
| $Q$ | $\begin{aligned} & \left(L^{2}(D)\right)^{3} \times\left(L^{2}(D)\right)^{3} \times \\ & L_{+}^{\infty}(D) \times L_{+}^{\infty}(D) \times L_{+}^{\infty}(D) \end{aligned}$ | Admissible parameter space |
| $\widetilde{Q}$ | Realistic parameter space | It is both compact and continuous. |
| $\widetilde{Q}_{M}$ | Finite-dimensional approximation to $\widetilde{Q}$ |  |
| $H^{N}$ | Finite-dimensional approximation to $H$ |  |

## 4 An Introduction of Uncertainty

In the following, we establish well-posedness of the random kinematic MHD equations. We do so by developing the theory presented in the previous chapter to allow for the inclusion of dependence upon a random domain. We do the same with the inverse problem from the previous chapter. To this end, we make use of the Prokhorov metric, introduced in Section 2.4.1. We then define a finite-dimensional approximation to the uncertain identification problem, which is shown to be method stable. The following work makes great use of the sources [2, 4, (3, 46].

### 4.1 Uncertain Forward Problem

### 4.1.1 Introduction

In this section, we introduce and develop theory for the stochastic kinematic MHD equations. These equations are the stochastic equivalent to (3.11), with the major difference being that the parameters and solutions are viewed as random processes. These processes have associated distributions, and we investigate how the 'randomness' of the parameters affects the 'randomness' of the solutions, i.e. how the randomness propagates through the system to the solutions. To do so, we must first establish that the system is well-posed, extending Theorem 3.3.2 to include randomparameters.

### 4.1.2 Well-Posedness

Let the spatial domain for our system be as before, $D \subset \mathbb{R}^{3}$, open with compact closure and we refer to the boundary as $\partial D$. We refer to the stochastic probability space $(\Omega, \mathcal{H}, p)$, where $\Omega$ is the set of outcomes, $\mathcal{H}$ is a given sigma algebra of events, and $p$ is some continuous probability measure.

As with the deterministic model presented in the previous chapter, we work
with the kinematic MHD system, where the fluid-flow, $\mathbf{u}$, is given, the induced magnetic field is negligible compared to the applied magnetic field [6]. As well, all other system parameters, namely the conductivity, $\sigma$, applied magnetic field, $\mathbf{B}$, electronmobility, $\mu_{e}$, and ion-mobility, $\mu_{i}$. Recall that $\mu_{e}, \mu_{i}>0$ on their domains. However, unlike the deterministic model, these parameters are not functions, but rather random processes, as first introduced in Section 2.4. These random processes, also viewed as function-valued random variables, are some functions in the admissible parameter space associated with each. Using the notation from Section 3.5, we have that u,B are random processes on $L^{2}(D)$, while $\sigma, \mu_{e}, \mu_{i}$ act upon $L_{+}^{\infty}(D)$.

Extending the Hall parameter and ion-slip parameter, we have for $\mathbf{x} \in D, \omega \in \Omega$

$$
\beta_{e}(\mathbf{x}, \omega)=\mu_{e}(\mathbf{x}, \omega)\|\mathbf{B}(\mathbf{x})\|_{l^{2}}, \text { and } \beta_{i}(\mathbf{x}, \omega)=\mu_{e}(\mathbf{x}, \omega) \mu_{i}(\mathbf{x}, \omega)\|\mathbf{B}(\mathbf{x})\|_{l^{2}}^{2}
$$

We also utilize the conductivity tensor given in Section 3.2, but adjusted to include the stochastic domain. This is given by

$$
\underline{\bar{\sigma}}(\mathbf{x}, \omega)=\sigma(\mathbf{x}, \omega)\left(\mathcal{I}-\frac{\beta_{e}(\mathbf{x}, \omega)}{\|\mathbf{B}\|_{l^{2}}}[\mathbf{B}]_{\times}-\frac{\beta_{i}(\mathbf{x}, \omega)}{\|\mathcal{B}\|_{l^{2}}^{2}}[\mathbf{B}]_{\times}^{2}\right)^{-1}
$$

where $\mathcal{I}$ denotes the identity matrix in $\mathbb{R}^{3 \times 3}$ and $[\mathbf{B}]_{\times}$is the matrix form of the crossproduct. Invertibility of this matrix follows from the physical restriction $\mu_{e}, \mu_{i}>0$. Under these definitions, the random MHD kinematic system is given by: find the induced current density, $\mathbf{J}_{i}: D \times \Gamma \rightarrow \mathbb{R}^{3}$, and electric potential, $\mathcal{V}: D \times \Omega \rightarrow \mathbb{R}$ that satisfy

$$
\begin{gather*}
\underline{\bar{\sigma}}^{-1} \mathbf{J}_{i}(\mathbf{x}, \omega)-\nabla \mathcal{V}(\mathbf{x}, \omega)=0 \forall \mathbf{x} \in D \text {, p.a.e. } \omega \in \Omega,  \tag{4.1a}\\
-\nabla \cdot \mathbf{J}_{i}(\mathbf{x}, \omega)=\nabla \cdot(\underline{\bar{\sigma}}(\mathbf{x}, \sigma)(\mathbf{u}(\mathbf{x}, \sigma) \times \mathbf{B}(\mathbf{x}))) \forall \mathbf{x} \in D \text {, p.a.e. } \omega \in \Omega, \tag{4.1b}
\end{gather*}
$$

with boundary conditions

$$
\mathbf{J}_{i} \cdot \mathbf{n}=-(\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B})) \cdot \mathbf{n} \text { and } \operatorname{tr}(\mathcal{V})=0 \text { on } \partial D \text {, p.a.e. } \omega \in \Omega \text {, }
$$

where $n$ is a unit-vector normal to the boundary of $D$. Here, 4.1a) is the transformed Generalized Ohm's law, using the tensor conductivity $\underline{\bar{\sigma}}$ and an application of Stoke's
theorem to the Maxwell-Faraday equation. 4.1b) is the equivalent for the divergence condition for $\mathbf{J}$. It is obvious that we are not starting the uncertain problem in the same place as the deterministic equivalent. This is because we still would like to use (and have presented already) a mixed-Poisson system of equations. The algebra and other logic that converted the deterministic MHD system into its mixed-Poisson counterpart follows immediately when the parameters are viewed as random processes, and is thus not restated here.

We refer to the system (4.1) as the random strong form of the kinematic MHD governing equations. We establish well-posedness by following a similar approach to what was seen in the deterministic equivalent, converting (4.1) into a weak form, then subsequently an operator form, and finally applying the BBK theorem [14] for existence and uniqueness of solutions. Continuous dependence on the parameters follows immediately from continuity of integration, the dependence shown in Section 3.3 and the continuity of the composition of two continuous functions.

We now define a random function space. We only seek solutions that are squareintegrable with respect to the random domain. Notationally, this space is given by

$$
L_{2, p}(\Omega):=\left\{f: \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} f^{2}(\omega) p(\omega) d \omega<\infty\right\}
$$

with norm

$$
\|f\|_{L_{2, p}}=\mathbb{E}[f]=\left(\int_{\Omega}|f|^{2}(\omega) p(\omega) d \omega\right)^{1 / 2}
$$

Recall the deterministic solution spaces for $\mathbf{J}_{i}$ and $\mathcal{V}$ defined in Section 3.3.2. Taking the tensor product between these two, we define the random solution space for $\mathbf{J}_{i}$ as

$$
\bar{V}:=V(D) \times L_{2, p}(\Omega)=\left\{f: D \times \Omega \rightarrow \mathbb{R}^{3} \mid f(\cdot, y) \in\left(L_{2, p}(\Omega)\right)^{3}, f(\mathbf{x}, \cdot) \in V\right\} .
$$

Similarly, define $\bar{W}:=W(D) \times L_{2, p}(\Omega)$ as the random solution space for $\mathcal{V}$. We define the norm on $\bar{V}$ as the averaging norm, i.e.

$$
\|\mathbf{F}\|_{\bar{V}}=\left(\mathbb{E}\left[\|\mathbf{F}\|_{V}^{2}\right]\right)^{1 / 2}
$$

and similarly define the norm on $\bar{W}$ to be the averaging norm using $\|\cdot\|_{W}$. Given that $V, W$ are both Hilbert spaces, as shown in 3.3, it follows that $\bar{V}, \bar{W}$ are also Hilbert spaces. Multiplying (4.1) by appropriate test functions and integrating, the spatially-weak form of the system is given by: find $\mathbf{J}_{i} \in \bar{V}, \mathcal{V} \in \bar{W}$ that satisfy

$$
\begin{gather*}
\mathbb{E}\left[\int_{D} \underline{\bar{\sigma}}^{-1} \mathbf{J} \cdot \phi\right]-\mathbb{E}\left[\int_{D} \nabla \mathcal{V} \cdot \phi\right]=0, \forall \phi \in \widetilde{V},  \tag{4.2a}\\
-\mathbb{E}\left[\int_{D} \mathbf{J}_{i} \cdot \nabla \psi\right]=\mathbb{E}\left[\int_{D}(\underline{\underline{\sigma}}(\mathbf{u} \times \mathbf{B})) \cdot \nabla \psi\right], \forall \psi \in \widetilde{W} . \tag{4.2b}
\end{gather*}
$$

We refer to (4.2) as the random weak form. We define the bilinear operator $\overline{\mathcal{A}}: \bar{V} \times \bar{V} \rightarrow$ $\mathbb{R}$ as

$$
\overline{\mathcal{A}}(\mathbf{F})(\mathbf{G})=\mathbb{E}\left[\int_{D} \underline{\underline{\sigma}}^{-1} \mathbf{F} \cdot \mathbf{G}\right]
$$

Similarly, we define $\overline{\mathcal{B}}: \bar{V} \times \bar{W} \rightarrow \mathbb{R}$ as

$$
\overline{\mathcal{B}}(\mathbf{F})(g)=-\mathbb{E}\left[\int_{D} \mathbf{F} \cdot \nabla g\right] .
$$

Finally, define

$$
\bar{G}(\psi)=\mathbb{E}\left[\int_{D}(\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B})) \cdot \nabla \psi\right] .
$$

Using these, the random operator form is given: find $\mathbf{J}_{i} \in \bar{V}, \mathcal{V} \in \bar{W}$ such that

$$
\begin{equation*}
\overline{\mathcal{A}}\left(\mathbf{J}_{i}\right)+\overline{\mathcal{B}}^{\prime}(\mathcal{V})=0 \in \bar{V}^{\prime} \tag{4.3a}
\end{equation*}
$$

$$
\begin{equation*}
\overline{\mathcal{B}}\left(\mathbf{J}_{i}\right)=\bar{G} \in \bar{W}^{\prime} . \tag{4.3b}
\end{equation*}
$$

With this operator form defined, we now present a theorem stating that the system (4.3) is well-posed.

Theorem 4.1.1. Given $\boldsymbol{u}, \boldsymbol{B} \in L_{2, p}(\Omega) \times\left(L^{2}(D)\right)^{3}$, bounded, and $\sigma, \beta_{e}, \beta_{i} \in L_{2, p}(\Omega) \times$ $L_{+}^{\infty}(D)$, i.e. for each $\omega \in \Omega$, assume $\sigma, \beta_{e}$, and $\beta_{i}$ are positive and bounded. Then there exist unique solutions, $\boldsymbol{J}_{i}, \mathcal{V}$, to

$$
\begin{equation*}
\overline{\mathcal{A}}\left(\boldsymbol{J}_{i}\right)+\overline{\mathcal{B}}^{\prime}(\mathcal{V})=F_{1} \in \bar{V}^{\prime}, \tag{4.4a}
\end{equation*}
$$

$$
\begin{equation*}
\overline{\mathcal{B}}\left(\boldsymbol{J}_{i}\right)=F_{2} \in \bar{W}^{\prime}, \tag{4.4b}
\end{equation*}
$$

which, for some $b \in \mathbb{R}^{+}$, independent of the choice of parameters, obey the following $a$ priori estimates:

$$
\begin{gather*}
\left\|\boldsymbol{J}_{i}\right\|_{\bar{V}} \leq\left\|F_{1}\right\|_{\bar{V}^{\prime}}+\frac{1}{b}\left(\|\overline{\mathcal{A}}\|_{\mathcal{L}\left(\bar{V}, \bar{V}^{\prime}\right)}+1\right)\left\|F_{2}\right\|_{\bar{W}^{\prime}},  \tag{4.5}\\
\|\mathcal{V}\|_{\bar{W}} \leq \frac{1}{b}\left(\left\|F_{1}\right\|_{\bar{V}^{\prime}}+\|\overline{\mathcal{A}}\|_{\mathcal{L}\left(\overline{\bar{V}}, \bar{V}^{\prime}\right)}\left\|F_{2}\right\|_{\bar{V}^{\prime}}\right), \tag{4.6}
\end{gather*}
$$

The proof of this theorem follows immediately from Theorem 3.3.2, where $b$ is the bounding constant for $\overline{\mathcal{B}}$, with appropriate extensions to the arguments regarding the coercivity and bounding constants of the operators, and the fact that, as $V, W$ are Hilbert spaces 3.3, $\bar{V}, \bar{W}$ must be as well. Note now that letting $F_{1}=0, F_{2}=G$ implies well-posedness of our system. The bounds on the solutions will prove vital in the error analysis of our numerical method.

### 4.2 Inverse Problem

We now introduce the concept of uncertainty to the parameter estimation problem, first presented in Section 3.5, and investigate the well-posedness of the inverse problem. To do this, we combine the uncertain kinematic MHD model developed above, (4.3), with the probability theory presented in Section 2.4 and the parameter estimation problem presented in Section 3.5. We demonstrate that an approximation to the full uncertain parameter estimation problem is method-stable, and that we can expect our results to converge to the true parameter distributions under ideal conditions. This provides a numerically implementable algorithm, whose implementation can be seen in Section 6.2. This section builds upon the work in [3].

### 4.2.1 Identification Problem

We turn the focus on to the inverse problem for the above system, parameter estimation. In Section 4.2, we assigned both the random and deterministic parameters, $\left\{\mathbf{u}, \sigma, \beta_{e}, \beta_{i}\right\}$ and $\{\mathbf{B}\}$ respectively. We only attempt to recover the random parameters, as we have already demonstrated in Section 5.2 that the deterministic B can be recovered. As well, the Hall and ion-slip parameters, $\beta_{e}, \beta_{i}$ respectively, are both assigned from the norm of $\mathbf{B}$ and their respective mobilities, $\mu_{e}, \mu_{i}$. Thus, when attempting the parameter recovery, it is sufficient to recover $\mu_{e}$ and $\mu_{i}$. Therefore, our random parameter set, $q$, is given by the vector of random processes, $q=\left\{\mathbf{u}, \sigma, \mu_{e}, \mu_{i}\right\}$.

To update the parameter estimation problem to include uncertainty, three distinct components must be stochastic in nature: the norm which defines the identification problem, the data and what it is understood to represent, and the space over which we search. The norm changes as the solution space changes. To simplify the notation, we use the operator notation as defined in the previous section, but updated to include the dependence on the random domain. With these, define
$U(\mathbf{x} ; q(\omega))=\left[\begin{array}{l}\mathbf{J}_{i}(\mathbf{x} ; q(\omega)) \\ \mathcal{V}(\mathbf{x} ; q(\omega))\end{array}\right], A(q(\omega))=\left[\begin{array}{cc}\overline{\mathcal{A}}(q(\omega)) & \overline{\mathcal{B}}^{\prime}(q(\omega)) \\ \overline{\mathcal{B}}(q(\omega)) & 0\end{array}\right], F(q(\omega))=\left[\begin{array}{c}0 \\ G(q(\omega))\end{array}\right]$.
It follows that the (4.2), can be written as

$$
\begin{equation*}
A U=F \tag{4.7}
\end{equation*}
$$

It also follows that $U \in \bar{V} \times \bar{W}$ from Theorem 4.1.1. Thus, as with the deterministic parameter estimation problem, let $\bar{H}=\bar{V} \times \bar{W}$, with norm $\|\cdot\|_{\bar{H}}^{2}=\|\cdot\|\left\|_{\bar{V}}^{2}+\right\| \cdot \|_{\bar{W}}^{2}$. Given that $\bar{H}$ is simply the tensor product of $H$ from Section 3.5 and $L^{2}(\Omega)$, it follows that $\bar{H}$ is a Hilbert space, and that we can use the norm or an approximation of it for the uncertain parameter estimation scheme.

Similarly, the definition of 'data,' must change. To include uncertainty, first assume we are given some data $\mathcal{U}$ corresponding to $U$ on $D$, as with the deterministic problem. However, we view the data as a realization of the expectation of the system,
and thus there will be no dependence on $\omega$. Thus, the way we fabricate data will not change between this and the deterministic problem, only how we view it.

Finally, the definition of the identification problem itself must change to reflect that we search for a distribution rather than a parameter set. However, recall that in the deterministic theory, it was vital that we searched over a compact parameter space. To address these concerns in the more complex space of distributions, we utilize the Prokhorov metric. For the definition of this metric, we refer the reader to Section 2.4 Using this metric, we adapt the probability distribution space to reflect the compactness of the underlying parameter spaces. We also use this metric to demonstrate the convergence of the approximations to the true distributions.

Theorem 4.1.1 defines the admissible parameter space for our parameter set q. However, unlike the deterministic case, rather than $\mathbf{u}$ simply being a function in $L^{2}(D), \mathbf{u}$ is a random process which acts on $\left(L^{2}(D)\right)^{3}$, i.e. $\mathbf{u}(\omega) \in L^{2}(D) \forall \omega \in \Omega$. For a formal definition of random process, see Section 2.4. Similarly, we have that $\sigma, \mu_{e}, \mu_{i}$ are random processes which all act upon the space of bounded functions on $D, L^{\infty}(D)$. Thus, we have the uncertain admissible parameter space given by $\bar{Q}:=L^{2}(D) \times L^{\infty}(D) \times L^{\infty}(D) \times L^{\infty}(D)$. We now go into more depth on how the we convert the identification problem into one in which we recover distributions.

In this problem, we seek to identify a distribution on the parameter space $\bar{Q}$, rather than a specific element $q \in \bar{Q}$. Thus, we must review the concept of a probability distribution on $\bar{Q}$. Recall the probability space of events, $\Omega$, the Borel sigma-algebra of outcomes, $B(\Omega)$, and the continuous measure $P$, from the previous section. As well, recall that we stated a random process $Y$ is a random variable for fixed $\mathbf{x} \in D$, and a function for fixed $\omega \in \Omega$. We now restate a definition from the introduction, the definition of a probability distribution for a random process. For convenience, let $Y$ denote any of the random parameters, acting on $Q_{Y}$, the associated function space. We define the probability distribution of $Y$ on $Q_{Y}$ as a measure $\bar{P}_{Y}: Q_{Y} \rightarrow \mathbb{R}$, which
for $y \in Y(\Omega)$ is given by

$$
\bar{P}_{Y}(y):=p_{Y} P\left(Y^{-1}(y)\right)
$$

Note that we must also include a normalization factor, $p_{Y}$, to ensure that $\bar{P}(Y(\Omega))=$ 1 , and thus satisfy the definition of a probability measure. If $Y$ is onto, then this normalization factor is 1 .

With this new measure, we have actually generated a new probability space, $\left(Y(\Omega), B(Y(\Omega)), \bar{P}_{Y}\right)$, where $B(Y(\Omega))$ denotes the Borel sigma-algebra of events, as with the original probability space. With this, we can now treat every realization of each parameter simply as an element in the event space. We extend this same idea to the entirety of the parameter space by taking the convolution of these distributions, which gives the probability distribution on $\bar{Q}$. If these are discrete distributions, as they will be for the numerical case, we have for $\omega \in \Omega$

$$
\begin{equation*}
\bar{P}(q(\omega)):=\bar{P}_{\mathbf{u}}(\mathbf{u}(\omega)) \cdot \bar{P}_{\sigma}(\sigma(\omega)) \cdot \bar{P}_{\mu_{e}}\left(\mu_{e}(\omega)\right) \cdot \bar{P}_{\mu_{i}}\left(\mu_{i}(\omega)\right) \tag{4.8}
\end{equation*}
$$

This definition easily extends to also include subsets rather than elements of $\bar{Q}$. Note that here, the dependence on $\omega \in \Omega$ is made explicit, but only to show that the inputs to each probability distribution are elements of their respective functional fields, e.g. $L^{2}(D)$. This distribution is equivalent to viewing: for $q \in \bar{Q}$,

$$
\bar{P}(q)=\bar{P}_{\mathbf{u}}(\mathbf{u}) \cdot \bar{P}_{\sigma}(\sigma) \cdot \bar{P}_{\mu_{e}}\left(\mu_{e}\right) \cdot \bar{P}_{\mu_{i}}\left(\mu_{i}\right)
$$

where we view the probability distributions as under the new probability spaces, i.e. (if $\mathbf{u}$ is onto) $\left(L^{2}(D), B\left(L^{2}(D)\right), \bar{P}_{\mathbf{u}}\right)$. This of course leads to the final equivalent probability space, $(\bar{Q}, B(\bar{Q}), \bar{P})$, under the assumption that the random variables are all onto. Thus, for the remainder of this section, the dependence of the parameters of $q$ on $\omega$ is dropped, for simplicity of notation, and we view the random parameters each as elements of their respective probability spaces.

We would also like to define an expected value on the probability space of $\bar{Q}$. To this end, we let $\mathcal{P}(\bar{Q})$ be the set of all probability distributions on $\bar{Q}$. Then for
$f \in \bar{Q}^{\prime}$, the expected value of $f$ under some distribution $P^{*} \in \mathcal{P}(\bar{Q})$ is given by:

$$
\mathbb{E}\left[f \mid P^{*}\right]=\int_{\bar{Q}} f(q) d P^{*}(q)
$$

Using this notation, the uncertain identification problem is: find the probability distribution $P^{*}$ in $\mathcal{P}(\bar{Q})$ that minimizes

$$
\begin{equation*}
J(P):=\|\mathbb{E}[U(q) \mid P]-\mathcal{U}\|_{H}^{2} \tag{UID}
\end{equation*}
$$

Note that here, the data $\mathcal{U}$ represents the expected value of the solution $U$ and thus, (UID) is analogous to finding the distribution which minimizes the difference between the data and the expected value of the solution. However, to guarantee the stability of (UID), further discussion of the continuity of the recovered distributions on the data must be done.

Similar to the deterministic parameter estimation in Section 3.5, well-posedness of the uncertain identification problem (UID) is equivalent to showing that some finite-dimensional approximation is method-stable, i.e. that the distributions of the parameters depend continuously on the data $\mathcal{U}$. We make several approximations to (UID) to generate implementable algorithms for the uncertain parameter estimation, namely reducing the dimension. To guarantee the convergence of the minimization problem, we must only search over some finite-dimensional and compact approximation of this function space. To this end, we use the compact metric space defined in Section 3.5. $\widetilde{Q}$. For the deterministic parameter estimation scheme, this is a sufficient space to search for minimizers to (ID). We now go into detail into the dimension reduction of (UID).

### 4.2.2 Dimension Reduced Uncertain Identification Problem

There are three components that are infinite-dimensional in (UID): the compact parameter search space, $\widetilde{Q}$, the distribution space, $\mathcal{P}(\widetilde{Q})$, and the spatial domain, $D$. Each dimension reduction is done separately, and then combined to form a complete FD approximation to the uncertain identification problem.

First, we note that every sample $q \in \widetilde{Q}$, is a vector of continuous functions, and that the polynomials are dense in continuous functions [43]. For $N \in \mathbb{N}$, let the space of polynomials up to degree $N$ on $D$ be denoted $\mathbb{P}^{N}$. Then for any index $a$, let

$$
\widetilde{\mathcal{Q}}_{a}^{N}:=\widetilde{\mathcal{Q}}_{a} \cap \mathbb{P}^{N} .
$$

Similarly, define

$$
\dot{\mathcal{Q}}_{a}^{N}:=\dot{\mathcal{Q}}_{a} \cap \mathbb{P}^{N} .
$$

Note that this $N$ is arbitrary, and may differ on each subspace. However, for the sake of notation, we assume this $N$ is uniform across all subspaces of $\widetilde{Q}$. Then we define our finite-dimensional approximation to $\widetilde{Q}$ as

$$
\widetilde{Q}_{N}:=\widetilde{\mathcal{Q}}_{\mathbf{u}}^{N} \times \dot{\mathcal{Q}}_{\sigma}^{N} \times \dot{\mathcal{Q}}_{\mu_{e}}^{N} \times \dot{\mathcal{Q}}_{\mu_{i}}^{N}
$$

Note that we now have a countable dense subset of $\widetilde{Q}$, given by

$$
\widetilde{Q}_{d}=\bigcup_{N \in \mathbb{N}} \widetilde{Q}_{N} .
$$

$\widetilde{Q}_{d}$ being dense follows from the polynomials being dense in the continuous functions. Analogously, we define a FD approximation to the distributions.

We first do so in general. Let $(O, \rho)$ be any complete metric space with dimension $M_{1}$, and let $\mathcal{P}(O)$ be the space of distributions for the random processes acting upon $O$. For $M \in \mathbb{N}, M_{1} \leq M$, let $\left\{o_{j}\right\}_{j=1}^{M}$ be any basis for $O$. Then we define the $M$-pole approximation of $\mathcal{P}(O)$ as

$$
\mathcal{P}_{M}(O):=\left\{P \in \mathcal{P}(O): P(o)=\sum_{j=1}^{M} p_{j} \delta_{o_{j}}(o), p_{j} \in \mathbb{R} \text { for } j=1, \ldots, m, \text { and } \sum_{j=1}^{M} p_{j}=1\right\},
$$

with $o_{j}$ defined as above. Now, one could consider this an $M$-dimensional approximation to the space $\mathcal{P}(O)$, with uniqueness determined by the choice of basis $\left\{o_{j}\right\}_{j=1}^{M}$. We now extend the approximation space to a dense subspace of distributions.

We define a countable dense subset of $\mathcal{P}\left(\widetilde{Q}_{N}\right)$ by

$$
\mathcal{P}_{d}(\widetilde{Q})=\bigcup_{M \geq N} \mathcal{P}_{M}\left(\widetilde{Q}_{N}\right) .
$$

The density of $\mathcal{P}_{d}\left(\widetilde{Q}_{N}\right)$ follows from Theorem 3.1 in [3]. With these now defined, note that for $P \in \mathcal{P}_{M}\left(\mathcal{Q}_{N}\right), M \geq N$, the expected value of a random process $f: \widetilde{Q}_{N} \rightarrow H$ is given by

$$
\mathbb{E}[f \mid P]=\int_{\widetilde{Q}} f(q) d P(q)=\sum_{j=1}^{M} f\left(q_{j}^{N}\right) p_{j}
$$

with $\left\{q_{j}^{N}\right\}_{j=1}^{M}$ a basis for $\widetilde{Q}_{N}$. Here, we must make clear that $\mathbb{E}[f \mid P]$ is a function in $H$, much like the discussion in Section 4.1.2, in that the expected value of a random process is a function on $D$.

This leads to an intermediate approximation of (UID), denoted (IUID): for given data $\mathcal{U}$ and fixed integers $M, N \in \mathbb{N}, M \leq N$, find $P^{*} \in \mathcal{P}_{M}\left(\mathcal{Q}_{N}\right)$ that satisfies

$$
\begin{equation*}
\min _{P \in \mathcal{P}_{M}\left(\mathcal{Q}_{N}\right)} J_{M, N}(P)=\left\|\sum_{j=1}^{M} U\left(q_{j}^{N}\right) p_{j}-\mathcal{U}\right\|_{H} . \tag{IUID}
\end{equation*}
$$

Here, $\mathcal{U}$ represents data in the expected-value of the deterministic solution, i.e. for (IUID), $\mathcal{U} \in H$. Computationally, we cannot expect to obtain a function in $H$, but rather data at some collection of sample points. To this end, for $K \in \mathbb{N}$, let $\left\{x_{j}\right\}_{j=1}^{K}$ be a collection sample points of the spatial domain $D$ such that $\left\{x_{j}\right\}_{j=1}^{K} \rightarrow D$ as $K \rightarrow \infty$. Now, define $\mathcal{U}_{K}=\left[\mathcal{U}_{1}^{K}, \ldots, \mathcal{U}_{K}^{K}\right]^{T}$, where $\mathcal{U}_{j}^{K}$ corresponds to the data at the point $x_{j}$. Clearly, as $K \rightarrow \infty, \mathcal{U}_{K} \rightarrow \mathcal{U}$ on every point of the domain, $D$, which is sufficient.

To relate the solutions $U$ with this new interpretation of the data, we must project $U$ onto the densely embedded subset of $H, H \cap C(D)$, i.e. solutions which are continuous on $D$. This is to ensure the evaluation of $U\left(\mathbf{x}_{j}, \omega\right)$ exists for every $\omega \in \Omega$. For any vector of functions $F \in H$, let $F^{c}$ be the projection of $F$ onto $H \cap(C(D))^{4}$, i.e. the space of continuous function on $D$ mapping to $\mathbb{R}^{4}$. We then denote the projection mapping $\mathcal{C}_{K}: H \rightarrow \mathbb{R}^{K}$ as

$$
\mathcal{C}_{K}(U)=\left[U^{c}\left(x_{j}\right)\right]_{j=1}^{K}
$$

Using this projection, the norm on $H$ can be approximated by a grid-norm, i.e.

$$
\|f\|_{H} \approx \frac{1}{K} \sum_{j=1}^{K}\left\|f^{c}\left(x_{j}\right)\right\|_{l^{2}} .
$$

With this final approximation, the dimension-reduced uncertain identification problem is given by: for fixed $K, M, N \in \mathbb{N}, M \leq N$ and data $\left\{\mathcal{U}_{k}\right\}_{k=1}^{K}$, find $P^{*} \in \mathcal{P}_{M}\left(\mathcal{Q}_{N}\right)$ that satisfies:

$$
\begin{equation*}
\min _{P \in \mathcal{P}_{M}\left(\mathcal{Q}_{N}\right)} J_{M, N}^{K}(P)=\frac{1}{K} \sum_{k=1}^{K}\left|\sum_{j=1}^{M} U\left(x_{k}, q_{j}^{N}\right) p_{j}-\mathcal{U}_{k}\right| \tag{DRUID}
\end{equation*}
$$

Note that here, the interpretation of the data is the expected value of the solutions at a given point $\mathbf{x}_{k} \in D$. Where the data is given, $\left\{\mathbf{x}_{k}\right\}_{k=1}^{K}$, represents the points at which we approximate our $H$-norm. We now present a major theorem regarding the stability of the method.

Theorem 4.2.1. Let $\widetilde{Q}, \widetilde{Q}_{N}, \widetilde{Q}_{d}, \mathcal{P}(\widetilde{Q}), \mathcal{P}_{M}\left(\widetilde{Q}^{N}\right)$ be as above. We consider the reduceddimension identification problem, (DRUID), and compare it to the original distribution identification problem (UID). Let $P_{M, N}^{*}\left(\mathcal{U}^{K}\right)$ be the set of minimizers for (DRUID), for some given data $\mathcal{U}^{K}$. Let $P^{*}(\mathcal{U})$ be the set of minimizers for (UID), corresponding to data $\mathcal{U}$, where $\mathcal{U}^{K} \rightarrow \mathcal{U}$ as $K \rightarrow \infty$. These minimizers satisfy $\operatorname{dist}\left({ }^{*} P_{M, N}\left(\mathcal{U}^{K}\right), P^{*}(\mathcal{U})\right)$ as $K, M, N \rightarrow \infty$. Thus, the distributions depend continuously on the data, and the approximate problem is method stable.

The proof of this theorem follows from applying Theorem 4.1 from [3]. With this theorem, we have shown that our uncertain parameter estimation problem is well-posed in an inverse-problem sense.

### 4.3 Notation

### 4.3.1 Stochastic Forward Problem

| Term | Definition | Notes |
| :---: | :---: | :---: |
| D | Spatial domain |  |
| $(\Omega, \mathcal{H}, p)$ | Original probability space |  |
| u | Fluid-flow |  |
| B | Applied magnetic field |  |
| $\sigma$ | Conductivity |  |
| $\mu_{e}$ | Electron Mobility |  |
| $\mu_{i}$ | Ion Mobility |  |
| $\underline{\underline{\sigma}}$ | Conductivity Tensor |  |
| $[B]_{\times}$ | Cross-product matrix operator on $\mathbf{B}$ | $[\mathrm{B}]_{\times} \mathbf{J}_{i}=\mathbf{J}_{i} \times \mathbf{B}$ |
| n | unit normal vector to surface. $V(D) / W(D)$ | Spatial solution space for $\mathbf{J}_{i} / V$ |
| $\bar{V}(D) / \bar{W}(D)$ | Random solution spaces | $V / W \times L_{\rho}^{2}(\Gamma)$, norm is the averaging norm |
| $\overline{\mathcal{A}}$ | Bilinear operator | Maps $\bar{V} \times \bar{V} \rightarrow \mathbb{R}$ |
| $\overline{\mathcal{B}}$ | Bilinear operator | Maps $\bar{V} \times \bar{W} \rightarrow \mathbb{R}$ |
| $\bar{G}$ | RHS of operator form | Used for bounds only. Not to be confused with $g$. |

### 4.3.2 Stochastic Inverse Problem

| Term | Definition | Notes |
| :---: | :---: | :---: |
| $b$ | Bounding constant for $\overline{\mathcal{B}}$ |  |
| $U$ | $\left[\mathbf{J}_{i} \mathcal{V}\right]^{T}$ |  |
| $A U=F$ | Overall PDE |  |
| $\bar{H}$ | $\bar{V} \times \bar{W}$ | Total (Stochastic + Deterministic) solution space for U |
| $\widetilde{Q}$ | Compact realistic parameter space | This is one of many things we are trying to approximate. |
| $\widetilde{Q}_{N}$ | FD approximation of $\widetilde{Q}$ | $\widetilde{Q} \cap \mathbb{P}^{N}$ |
| $(O, \rho)$ | Some general complete metric space |  |
| $\mathcal{P}(O)$ | Set of all probability distributions on $(O, \rho)$ |  |
| $\mathcal{P}_{M}(O)$ | Set of all $M$-pole distributions on $(O, \rho)$ |  |
| $\left\{x_{j}\right\}$ | Sample points of $D$ | Goes from $j=1$ to $K$ |
| U | Represents data for the problem | Meaning changes throughout |
| $\mathcal{U}_{K}$ | Data evaluated at all $\left\{x_{j}\right\}$, a long vector |  |


| Term | Definition | Notes |
| :---: | :---: | :---: |
| $\mathcal{C}_{K}$ | Projection! Turns a function into a vector of length $K$, evaluated at each $x_{j}$ |  |
| ID | Deterministic identification problem |  |
| UID | Uncertain identification problem |  |
| IUID | FD in the parameter/random sense, but still too broad |  |
| DRUID | FD ID in both stochastic and deterministic sense |  |
| $\left\{q_{j}^{N}\right\}_{j=1}^{M}$ | M-component basis for $\widetilde{Q}_{N}$ | $M \geq N$ obviously |
| M | Number of samples of $\tilde{Q}$, correlates to number of poles for discrete probability space. | Really the resolution of the probability space |
| $N$ | Polynomial degree for approximation of $\widetilde{Q}$ | Can be thought of as the resolution of $\widetilde{Q}$ |
| K | Number of samples of $D$ | Can be thought of as the resolution of $D$ |

## 5 Deterministic Numerical Demonstrations and Model Verification

With well-posedness of both the forward and inverse problem established, we now turn to the numerical implementation of the governing equations, specifically in relation to an MHD generator. The past two chapters, although theoretical, establish the framework for which we approach the investigation of the feasibility of real-time optimization for an MHD generator. We now take the finite-dimensional approximations and implement them numerically, utilizing a finite-element multi-physics software, COMSOL [32], along with the cross-platform compatibility of LiveLink for Matlab [28]. Using these and with both the continuous and segmented Faraday geometry, we validate the deterministic numerical model using a set of ideal ' 0 -D' equations, testing the sensitivity of the power within the channel to the expected power, in turn numerically confirming the existence of a solution to the deterministic kinematic MHD model. Following this, we will provide a numerical demonstration of the realistic parameter estimation problem, finding the state of an MHD generator from some fabricated data of external measurements.

### 5.1 Numerical Forward Problem

### 5.1.1 Introduction

There are many different electrode configurations for an MHD channel [42, 50, [25] . However, for our purposes, we focus on two different configurations of a Faraday generator. Faraday generators have electrodes configured to allow current to flow in the direction of the Lorenz-force [13]. To see a more robust description of the physics and definition of individual components to the MHD generator, we refer the reader to the Section 2.1 .

As mentioned previously, for the deterministic forward problem, we make use of an finite-element multi-physics software, COMSOL [32]. This software has appli-
cations in a variety of fields, and comes equipped with a multitude of solver-packages and meshing techniques, while also allowing for complex spatial domains. With the reworked system of equations and prescription of fluid-flow, among other parameters, the deterministic model is easily adapted from the AC/DC electrostatics module within COMSOL [34]. In order to use this numerical model to infer the state of an MHD generator, we must first validate the numerical implementation and COMSOL software. In lieu of true solutions to this model, we use theoretical ideal-power equations, derived from [42], to validate the numerical implementation. We will then explore some of the qualitative effects of the various parameters on the system.

### 5.1.2 COMSOL

### 5.1.2.1 Governing Equations

We first delve deeper into the software we use for the deterministic forward problem, COMSOL [32]. As noted in the introduction, COMSOL is known for coupling complex multi-physics on complex geometries. For our purposes, the application is straight-forward. The AC/DC module utilizes a mixed-Poisson form of the electrostatic equations. As we have rewritten our system in such a form, i.e. (3.11), we can adapt the COMSOL governing equations to match the kinematic MHD model with the addition of the Lorenz force, and an alteration to the conductivity tensor based on the assumed material properties. Recall that each component in an MHD generator ${ }^{1 / 2}$ is governed by some form of the kinematic MHD equations, with varied values for the parameters. With the model equations established, we can now discuss how exactly the material properties of the different components alter the governing equations, and are then implemented in COMSOl.

We begin with the channel. It should be obvious that the plasma is governed by the full kinematic MHD system. With the walls assumed to be infinitely-thin, for implementation simplicity, we assume the entirety of the channel has no physics-

[^3]based reductions to the governing equations. This is not the case for the other two components, the electrodes and resistor (which represents the load on the channel). There is no plasma in the electrodes or load, and thus there is no Lorenz force in either. Similarly, as they are composed of a solid material, there is no electron or ion mobility. This simplifies the governing equations to just Maxwell's equations and a reduced Ohm's law, given by
\[

$$
\begin{gather*}
\mathbf{J}=\sigma \mathbf{E}  \tag{5.1a}\\
\nabla \times \mathbf{E}=0  \tag{5.1b}\\
\nabla \cdot \mathbf{J}=0 \tag{5.1c}
\end{gather*}
$$
\]

Coincidentally, this system is the set of governing equations implemented in the $\mathrm{AC} / \mathrm{DC}$ module. Thus, to set-up the governing equations within these components, we simply assign the material parameter of conductivity to accurately represent the materials we wish to model. For the electrodes, we assume a conductivity of $58 \mathrm{MS} / \mathrm{m}$, due to the assumed copper material they are composed of. As the resistors represent a load being placed on the channel, these values are swept across, to validate the model. The range of appropriate values we test are on a large interval, anywhere from $10^{-7}$ to $10^{4}$. We discuss how we measure the load placed on the channel through the use of a load-factor in the numerical validation of the model.

Finally, to implement the governing equations within the channel, we still use the AC/DC COMSOL module, with some slight modifications. First, within the channel, we no longer use the material conductivity, but rather a user-defined conductivity tensor, i.e. $\underline{\bar{\sigma}}$ as defined in Section 3.2.1 Secondly, we introduce an artificial current, which is the effect of the Lorenz force on the plasma. Within COMSOL, this is given by

$$
\mathbf{J}_{e}=\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}),
$$

which is just a vector in $\mathbb{R}^{3}$. With these two slight adjustments to the governing physics, the full kinematic MHD is in place, with appropriately defined parameters, whose function spaces can be seen in Section [3.5, with the scalar values described in Table 5.1.

We also apply the periodic boundary conditions described in 2.1, namely that the electrodes and resistors are connected with periodic boundary conditions, and the perfectly-electrically-insulating conditions are in place elsewhere, except the channel inlet and outlet. For model verification purposes, we assume that the channel is infinitely long. We replicate this numerically by setting a periodic boundary condition at the channel inlet and outlet. However, note that we assume the perfectly electrically insulating at the channel inlet and outlet for the other numerical examples we run.

### 5.1.2.2 Meshing Techniques

With these governing equations established, we now discuss the meshing techniques used in our numerical model. We choose to use the native Delauney meshing technique, of the two that COMSOL provides. A mesh, sometimes also referred to as a triangulation, is a discretization of a domain into nodes, and then joining specific nodes to form elements. In the 3-D case, every element has a triangular side. However, withing the Delauney meshing algorithm, the elements are chosen to maximize the minimum angle of all the angles within the triangulation. This helps prevent 'sliver' elements within the mesh, aiding in reducing numerical error [11. Formally, a Delauney mesh is any mesh such that no nodes within the mesh lie within the circumference of a circle generated by three other connected nodes. As example of two meshes, one Delauney and one not, is presented in Figure 5.1.

The generation of such a mesh does take some thought. COMSOL uses a variety of techniques, such as a flipping algorithm, to generate this mesh. A flipping algorithm is a technique in which if a set of four nodes is connected in a non-Delauney way, then simply flip the interior edge. An example of this is also presented in Figure 5.1.


FIGURE 5.1: Two sample triangulations of four nodes. (left) Non-Delauney triangulation. (Right) Deulauney triangulation. Demonstrates the flipping algorithm that transforms a non-Delauney triangulation into a Delauney triangulation.

Finally, the coarseness of the nodes across the domain is important question. As always, a finer mesh will require greater computational requirements. COMSOL allows for a choice of user-defined grids, or the selection of a 'coarseness level,' from 'very coarse' to 'extremely fine.' Although these is no analytic function for the difference in nodes between these levels, Figure 5.2(a) and Figure 5.2(b) shows the qualitative difference between a 'normal' mesh and a 'fine' mesh for a segmented Faraday MHD generator. The final method COMSOL allows to generate a mesh is a hybrid method, in which one can select a coarseness level on each subsection of the domain, and choose to further refine specific edges or boundaries. This is in fact the method we choose, refining the measure where previous iterations of our numerical model show steep gradients for the solutions. We allow for a coarsening where this gradient is small. The result for a segmented Faraday MHD generator can be seen in Figure 5.2(c), with a refinement of the mesh near the channel-electrode boundary, and the electrode-resistor boundaries. It is on this mesh we solve for our solutions.

### 5.1.2.3 Solver

Numerically, COMSOL is a finite element based solver. Recall from Section 2.3 the general finite element method. It is easy to adapt (3.15) to fit the general

(a)

(b)

(c)

FIGURE 5.2: Three different example COMSOL generated meshes. (a): physicsbased, normal distribution, $\sim 40,000$ nodes, (b): physics-based, fine distribution, ~ 4,000 nodes, (c): user-defined, ~ 28, 000 nodes.
differential equation, by using a matrix and vector representation of the operators and solutions respectively. Thus, discretization of the PDE follows immediately. COMSOL utilizes linear polynomials as the approximation of the continuous solutions, as well as the standard finite element basis of piece-wise polynomials with a single element of support, as well as a unique node taking a value of 1 . Thus, solving the system is equivalent to finding the weights of these basis functions [1]. In order to find the weights for our approximate solution, COMSOL uses an iterative solver to converge to the solution of the resulting linear system of equations. The algorithm utilized for this purpose is functional generalized residual method, or FGMRES. This is a preconditioned method of the generalized minimal residual method (GMRES), which searches through the Krylov subspace with the Arnoldi iteration algorithm. A further discussion of this algorithm can be seen in (31].

### 5.1.3 Model Validation

We now demonstrate that the numerical approximations made within COMSOL produce results with satisfactorily small relative error. Typically, in lieu of an analytic function, this would occur by demonstrating that the approximation to the solution converges to some fine-grid solution, or the solution generated by another numerical method. However, we validate our models alternatively, by comparing the powerdensity within the channel to some set of ideal equations. The following work builds upon the foundations established in [42].

The electrical power contained within any generator can be determined by the dot product of the current density with the electric field. Thus, the power within any MHD channel is given by

$$
P=-\mathbf{J} \cdot \mathbf{E} .
$$

Under varying geometries and ideal conditions, this simplifies. We first explore the ideal conditions, which are synonymous across all Faraday geometries.

In order to compare the numerical power within the model to some 'idealized
power,' we consider all of the parameters to be constant on $D$. This is in order to neglect the size of the generator, e.g. we would like to compare the average power within the channel, as opposed to the total power within the channel. As well, given that the magnetic field neglects the induced component, we assume that $\mathbf{B}$ is unidirectional, i.e., $\mathbf{B}=\left(0,0, \mathbf{B}_{z}\right)$. Under ideal conditions, we also assume that the fluid flow is uni-directional, i.e. $\mathbf{u}=\left(\mathbf{u}_{x}, 0,0\right)$. Of course, this implies that the Lorenz force, $\mathbf{u} \times \mathbf{B}$, lies only in the $y$-direction. Using the definition of $\mathbf{J}_{i}$, we have that

$$
\underline{\bar{\sigma}} \mathbf{E}=-\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}) .
$$

It follows from the definition of the conductivity tensor that this implies $\mathbf{E}_{z}=0$, and thus that $\mathbf{J}_{z}$ has no effect on the power within the channel. We will see that other geometries imply further simplifications to the electric field and current density.

Conventionally, one also defines a load-factor, representing the ratio of the average load placed on the channel over the maximum load allowed, as

$$
K=\frac{\mathbf{E}_{y}}{\mathbf{u}_{x} \mathbf{B}_{z}} .
$$

Moving forward, we now the generation of ideal power equations based on the given geometry of the generator. For the simpler continuous Faraday geometry as well as the more complex segmented Faraday geometry, we explore the implications the geometries have on the solutions and thus the ideal-power equations, followed by sweeping across a variety of parameter values and comparison of COMSOL's measured average power to the ideal power.

### 5.1.3.1 Conductivity Tensor: An Explicit Examination

To do so, we must first give a brief examination into the conductivity tensor. Recall from Section 3.2 .1 that this was defined as

$$
\underline{\bar{\sigma}}:=\sigma\left(\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|_{L^{2}}}[\mathbf{B}]_{\times}-\frac{\beta_{i}}{\left\|\mathbf{B}^{2}\right\|_{L^{2}}}[\mathbf{B}]_{\times}^{2}\right)^{-1},
$$

where the inverse of the matrix on the right-hand side of the equation is guaranteed by the physical restriction $\mu_{e}, \mu_{i}>0$ on $D$. Under the assumptions for the numerical problem, namely that $\mathbf{B}=\left(0,0, \mathbf{B}_{z}\right)$, it immediately follows that

$$
\begin{aligned}
\mathcal{I}-\frac{\beta_{e}}{\|\mathbf{B}\|}[\mathbf{B}]_{\times} & -\frac{\beta_{i}}{\|\mathbf{B}\|^{2}}[\mathbf{B}]_{x}^{2} \\
& =\left[\begin{array}{ccc}
\left(\mathbf{B}_{y}^{2}+\mathbf{B}_{z}^{2}\right) \frac{\beta_{i}}{\|\mathbf{B}\|^{2}}+1 & -\mathbf{B}_{z} \frac{\beta_{e}}{\|\mathbf{B}\|}-\mathbf{B}_{x} \mathbf{B}_{y} \frac{\beta_{i}}{\| \| \|^{2}} & \mathbf{B}_{y} \frac{\beta_{e}}{\|\mathbf{B}\|}-\mathbf{B}_{x} \mathbf{B}_{z} \frac{\beta_{i}}{\|\mathbf{B}\|^{2}} \\
\mathbf{B}_{z} \frac{\beta_{e}}{\|\mathbf{B}\|}-\mathbf{B}_{x} \mathbf{B}_{y} \frac{\beta_{i}}{\|\mathbf{B}\|^{2}} & \left(\mathbf{B}_{x}^{2}+\mathbf{B}_{z}^{2}\right) \frac{\beta_{i}}{\|\mathbf{B}\|^{2}}+1 & -\mathbf{B}_{x} \frac{\beta_{e}}{\|\mathbf{B}\|}-\mathbf{B}_{y} \mathbf{B}_{z} \frac{\beta_{i}}{\|\mathbf{B}\|^{2}} \\
-\mathbf{B}_{y} \frac{\beta_{e}}{\|\mathbf{B}\|}-\mathbf{B}_{x} \mathbf{B}_{z} \frac{\beta_{i}}{\|\mathbf{B}\|^{2}} & \mathbf{B}_{x} \frac{\beta_{e}}{\|\mathbf{B}\|}-\mathbf{B}_{y} \mathbf{B}_{z} \frac{\beta_{i}}{\|\mathbf{B}\|^{2}} & \left(\mathbf{B}_{x}^{2}+\mathbf{B}_{y}^{2}\right) \frac{\beta_{i}}{\|\mathbf{B}\|^{2}}+1
\end{array}\right] \\
& =\left[\begin{array}{ccc}
1+\beta_{i} & -\beta_{e} & 0 \\
\beta_{e} & 1+\beta_{i} & 0 \\
0 & 0 & 1
\end{array}\right]
\end{aligned}
$$

Computing the inverse yields:

$$
\underline{\bar{\sigma}}=\frac{\sigma}{1+2 \beta_{i}+\beta_{i}^{2}+\beta_{e}^{2}}\left[\begin{array}{ccc}
1+\beta_{i} & \beta_{e} & 0 \\
-\beta_{e} & 1+\beta_{i} & 0 \\
0 & 0 & 1+2 \beta_{i}+\beta_{i}^{2}+\beta_{e}^{2}
\end{array}\right]
$$

For simplicity in the code and notation, we let $\gamma=1+2 \beta_{i}+\beta_{i}^{2}+\beta_{e}^{2}$. Thus, the conductivity tensor implemented is:

$$
\underline{\bar{\sigma}}=\frac{\sigma}{\gamma}\left[\begin{array}{ccc}
\left(1+\beta_{i}\right) & \left(\beta_{e}\right) & 0  \tag{5.2}\\
-\left(\beta_{e}\right) & \left(1+\beta_{i}\right) & 0 \\
0 & 0 & \gamma
\end{array}\right]
$$

### 5.1.3.2 Continuous Faraday: Ideal Power Equation

Under the continuous Faraday geometry, the electrodes connect the upstream and downstream portions of the channel. This implies that $\mathbf{E}_{x} \approx 0$ [42]. Using a manipulation of the explicit generalized Ohm's law derived in Section 3.2.1, i.e. J = $\underline{\underline{\sigma}} \mathbf{E}+\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B})$, and then applying the matrix-form of the conductivity tensor as defined above, it follows that

$$
\mathbf{J}_{y}=\frac{\sigma\left(1+\beta_{i}\right)}{\left(1+\beta_{i}\right)^{2}+\beta_{e}^{2}}\left(\mathbf{E}_{y}-\mathbf{u}_{x} \mathbf{B}_{z}\right) .
$$

Of course, if $\mathbf{E}_{x}, \mathbf{E}_{z} \approx 0$, the power within the ideal continuous Faraday generator must be given by the scalar product of $\mathbf{E}_{y}$ and $\mathbf{J}_{y}$. Using the traditional load factor as defined above, the ideal continuous Faraday power equation is given by

$$
\begin{equation*}
P_{c f}=-\mathbf{J} \cdot \mathbf{E} \approx-\mathbf{J}_{y} \mathbf{E}_{y}=-\frac{\sigma\left(1+\beta_{i}\right)}{\left(1+\beta_{i}\right)^{2}+\beta_{e}^{2}} K(1-K) \mathbf{u}_{x}^{2} \mathbf{B}_{z}^{2} \tag{5.3}
\end{equation*}
$$

This ideal-power equation shows the detrimental effect that the hall parameter can have on the power output of a continuous Faraday MHD generator. It also shows that a neglect in the ion-slip parameter would also result in unexplained power losses, although, the magnitude would be less than if the hall parameter was neglected. This is not the case for the segmented geometry.

### 5.1.3.3 Segmented Faraday: Ideal Power Equation

Unlike the continuous Faraday geometry, the segmented Faraday geometry no longer guarantees that $\mathbf{E}_{x} \approx 0$. However, the segmented electrodes short-circuit the electric current in the $x$-direction, implying that $\mathbf{J}_{x} \approx 0$ ! Thus, to generate the ideal power equation for the segmented Faraday, we make use of the explicit form of the Generalized Ohms law, i.e. 3.9a Solving this for $\mathbf{E}_{x}$ and using the traditional load factor $K$, yields that the ideal segmented Faraday power equation, is given by

$$
\begin{equation*}
P_{s f}=\frac{\sigma}{\left(1+\beta_{i}\right)} K(1-K) \mathbf{u}_{x}^{2} \mathbf{B}_{z}^{2} \tag{5.4}
\end{equation*}
$$

Immediately, this equation showcases some important differences between the segmented and continuous geometries. First and foremost, there is no dependence on the hall parameter. This is to be expected, as the segmenting of the electrodes is meant to short-circuit the naturally arising Hall current, i.e. $\mathbf{J}_{x}$. However, there is a much simpler scaling factor in front of the segmented ideal power equation. Thus, by neglecting ion-slip, MHD generators could have power-losses on the order of $1 /\left(1+\beta_{i}\right)$, which is significant for non-negligible $\beta_{i}$.

TABLE 5.1: Distributions and fixed values for parameters used in the model validation.

| Parameter | Lower-bound | Upper-bound | Fixed value |
| :---: | :---: | :---: | :---: |
| $\mathbf{B}_{z}$ | 0 | 20 | 6 |
| $\sigma$ | 10 | 100 | 60 |
| $K^{*}$ | 0 | 1 | N/A |
| $\mathbf{u}$ | 0 | 2800 | 1600 |
| $\mu_{e}$ | 0 | $10 / 6$ | 0 |
| $\mu_{i}$ | 0 | 1 | 0 |

*: The load-factor is altered by changing the resistance of the resistors. The appropriate distributions for the resistance are: $[1 \mathrm{E}-7,1 \mathrm{E} 4]$ for bounds, $7 \mathrm{E}-1$ for fixed-value.

### 5.1.3.4 Validation

Using these ideal power equations, we now validate the numerical model, using a distribution of parameter values. We fix all but one parameter, and then sweep across the given distribution, computing an ensemble of solutions. We then use a numerical approximation to

$$
\int_{\text {channel }}-\mathbf{J} \cdot \mathbf{E} d V,
$$

and compare this to the ideal power equations of the associated geometry. We choose an appropriate distribution of parameter values to represent a variety of operating conditions for the generator. When fixed, the values can be seen in Table 5.1. Also, note that the load factor is varied by changing the resistivity of the resistor, as discussed in the components section.

For the continuous geometry, Figures 5.3 and 5.4 compare the ideal and measured power in the top figure, with the lower figure giving the relative difference between the two. Clearly, for all parameters varied, good agreement is seen, with difference less than $1 \%$.

The differences between the (5.4) and the measured COMSOL power within the segmented Faraday model are seen in Figures 5.5 and 5.6. Again, as with the continuous Faraday, good agreement between the ideal power equation and measured power is noted. However, poorer agreement than the continuous Faraday geometry can be attributed to geometrical errors from the more complex geometry, as well as numerical difficulties from this complex geometry for higher electron mobilities.

### 5.1.4 Deterministic Solutions

With the numerical model validated, we now provide some examples of the solutions to (3.1). These showcase the complexity of the solutions, and the robustness of the implementation within COMSOL. Furthermore, these will demonstrate the qualitative effects of some system parameters.

We examine the qualitative effects of non-negligible Hall and ion-slip parameter. To do so, we recall the matrix-form of the conductivity tensor as presented in Section 5.1.3.1 We restate it here for posterity,

$$
\underline{\bar{\sigma}}=\frac{\sigma}{\gamma}\left[\begin{array}{ccc}
\left(1+\beta_{i}\right) & \left(\beta_{e}\right) & 0 \\
-\left(\beta_{e}\right) & \left(1+\beta_{i}\right) & 0 \\
0 & 0 & \gamma
\end{array}\right],
$$

where $\gamma=\left(1+\beta_{i}\right)^{2}+\beta_{e}^{2}$. As a representation of the generalized Ohm's law, this conductivity tensor reflects the material response due to the magnetic field and resulting Lorenz force. To view this idea simply, consider (3.3). With some slight algebra, this becomes

$$
\mathbf{J}=\underline{\bar{\sigma}} \mathbf{E}+\underline{\bar{\sigma}}(\mathbf{u} \times \mathbf{B}) .
$$

Now, consider the Hall parameter and it's effect on the Lorenz force, $(\mathbf{u} \times \mathbf{B})$. Clearly, if $\mathbf{u} \times \mathbf{B}$ is dominated in the $y$ direction, as guaranteed by the ideal assumptions, the Hall parameter, being in the off diagonal, will tilt the electric field and current density. The ion-slip parameter, being on the diagonals, will result in a dampening. Quantitatively, these effects are noted in the ideal power equations; however, the qualitative effects are more apparent, and can be seen in Figure 5.7.


FIGURE 5.3: Continuous Faraday geometry numerical validation. For all paired plots (top) displays the ideal power output (dashed) for the given system compared to the COMSOL measured power (solid) of the channel, (bottom) displays the relative error. Run for $\mathbf{B}, \sigma, K$, u.


FIGURE 5.4: Continuous Faraday geometry numerical validation for varied $\mu_{i}, \mu_{e}=$ $1 / 6,5 / 6,10 / 6$. (Top) displays (dashed) ideal power and (solid) comsol measured power within channel under the given conditions. (Bottom) displays the relative error.

### 5.2 Numerical Implementation of Parameter Estimation

With the numerical model validated, we now turn to a numerical demonstration of the parameter estimation scheme presented above, using the segmented Faraday geometry. We choose to use only the segmented Faraday geometry as it is sufficiently complex to investigate the reliability of the estimation scheme. Similarly, for simplicity, we only attempt to recover two parameters, $\beta_{e}, \beta_{i}$, by recovering their respective mobilities, $\mu_{e}$ and $\mu_{i}$. To perform the parameter estimation, we utilize the livelink cross software compatibility between Matlab and Comsol [28] for the scheme, using the optimization solver lsqnonlin [29, 20] to recover the parameters.

As is standard when other data is unavailable, we synthesize the data, using our model with a fine-mesh. We then perturb the data to represent instrument noise,. We investigate the reliability of the recovery of the parameters under additive Gaussian noise. Under the previous notation from Section 3.5, let $U(\cdot ; q)$ represent the solution


FIGURE 5.5: Segmented Faraday geometry numerical validation. For all paired plots (top) displays the ideal power output (dashed) for the given system compared to the COMSOL measured power (solid) of the channel, (bottom) displays the relative error. Run for $\mathbf{B}, \sigma, K, \mathbf{u}$.


FIGURE 5.6: Segmented Faraday geometry numerical validation for varied $\mu_{i}, \mu_{e}=$ $1 / 6,5 / 6,10 / 6$. (Top) displays (dashed) ideal power and (solid) comsol measured power within channel under the given conditions. (Bottom) displays the relative error.
to (3.11) for some $q$. Then define

$$
D_{A}\left(n_{L}\right):=U(\cdot ; q)+N \cdot n_{L},
$$

as the additive-noise data. Here, $n_{L}$ represents the noise-level and $N \sim(\mathcal{N}(0,\|U\|))^{m}$, i.e. it is a random-vector of dimension equal to $U$, with standard deviation given by the norm of $U$. After adding the noise as described above, we compare the recovered parameter to the true parameter as a function of noise-level, within the framework of the segmented Faraday geometry.

Of particular interest is the shape of the residual function. It is important to investigate whether the estimation scheme could recover the true parameter, despite noise, but the initial guess of the parameter values play a crucial role in determining the success of the numerical parameter estimation scheme. This is due to the fact that we implement a local minimization scheme only, rather then searching through the entire parameter space described in 5.1. Therefore, we run the parameter estimation scheme


FIGURE 5.7: Examples of electric potential $\left(\mathcal{V}\right.$, colored) and currenty density ( $\mathbf{J}_{i}$, streamlines) for (top) $\beta_{e}=0, \beta_{i}=0$, (center) $\beta_{e}=10, \beta_{i}=0$, (botton) $\beta_{e}=10, \beta_{i}=$ $0.1 / 6$. Note the tilt from the hall parameter, and the dampening effect of the ion-slip parameter.


FIGURE 5.8: Mean relative error (solid) with asymmetrical error bars for recovery of $\mu_{e}$ (left) and $\mu_{i}$ (right). Asymmetrical error bars were generated by determining the standard deviation of errors for over-estimates and under-estimates separately.
multiple (3) times for every given noise-level, and randomly assign 3 different initial guesses to give to our optimization solver. This will allow a better understanding on if the minimization scheme has a tendency to under or over estimate the true parameter values, aiding in the ultimate goal of real-time optimization of an MHD generator.

Under this framework, we choose our initial guesses for the true parameters with an initial error of between 9 and $11 \%$. We then average the recovered parameter values to better understand how the estimation scheme can be expected to recover at a given noise-level. To qualitatively investigate the tendency of the scheme to over or under estimate the true parameters, asymmetrical error bars are added to the relative error plot, which represent the standard deviation of the errors for the 'over-valued' recovered parameters and 'under-valued' recovered parameters, respectively. These results are seen in Figure 5.8 for the electron and ion mobilities scalar recoveries.

Comparing these two figures, we see that $\mu_{e}$ is more easily and accurately recovered that $\mu_{i}$. For instance, at a noise level of 0.04 , we have that the average relative error over 10 runs for $\mu_{e}$ is roughly 0.02 , while the average relative error over 10 runs for $\mu_{i}$ is greater than 0.1 . The variance on the estimates of $\mu_{i}$ also grows much more quickly with respect to the noise level. However, neither is reliably recovered. Note here the lack of monotinicty and large variances for different noise levels. As well,
note the asymmetric error bars on both $\mu_{e}, \mu_{i}$, implying the need for distributional recovery to better describe the likely recovered parameter.

## 6 Stochastic Collocation and Uncertain Parameter Estimation

In the introduction, two major challenges were presented that hinder the feasibility of the real-time optimization of an MHD generator. The first was model accuracy, or how the numerical model could better reflect the world. The first approach in dealing with this was theoretical in nature; including oft-neglected terms within the governing equations. The next approach for improving model accuracy is the introduction of uncertainty to the MHD numerical framework.

Uncertainty must be included in the parameter estimation scheme in order to optimize for the expected power out, rather than a single realization of the power out. This leads to a more realistic and less sensitive expected power, over the deterministic equivalent [?], as we are optimizing with a distribution of parameters. Inclusion of uncertainty in the parameter estimation scheme starts with the propagation of uncertainty within the forward problem. To include uncertainty within the system, we view the parameters as random processes (See Section 2.4), which implies that the solutions are random processes as well. Thus, the parameter space has some associated distribution, as do the solutions. In Chapter 4, it was shown that the solutions' distributions depend upon the parameters' distributions. In the follow work, we demonstrate this dependence in several ways. To do so, we implement stochastic collocation, a numerical method used to approximate solutions to random differential equations. We then investigate an error analysis of the method, and conclude with several numerical demonstrations. These include a demonstration of convergence in the random-sense, a comparison between the mean of the uncertain solutions, and a deterministic set of solutions corresponding to the mean of the distributions, as well as an investigation of the sensitivity of the shape of the solutions' distributions to the shape of the parameters' distributions.

Following the numerical work for the uncertain forward problem, we turn to the equivalent for the uncertain inverse problem. Analogous to the work in the determin-
istic parameter estimation problem, we verify the theoretical framework presented in Chapter 4, and examine the effect of noise on the recovery. We demonstrate how to apply a Karhouneun-Loève expansion in the inverse problem framework, and, under a stochastic collocation forward problem, generate an implementable algorithm to recover the distribution of the parameters within the system. We then investigate the sensitivity of the distributional recovery to the number and location of spatial nodes, restricting the fabricated data to solely within the resistors.

### 6.1 Stochastic Collocation

Many different uncertainty quantification techniques exist, and are used in a variety of applications. One simple approach is the Monte Carlo (MC) method [48]. This involves sampling the distribution, and averaging the deterministic solutions to give an expected value and standard deviation [46]. Although robust, with a convergence rate on the order of $N^{-1 / 2}$, where $N$ is the number of samples, it is quite slow [35]. For an example of using an MC approach to uncertainty within MHD, see [37.

The method we implement to quantify the uncertainty propagation dramatically increases the convergence rate through the choice of stochastic-grid, as well as the method of computing the expected solutions. In the following, we implement an approximation method called stochastic collocation (SC) [55]. Utilizing a sparse grid, we show that the 'curse of dimensionality' which often afflcits uncertainty quantification, can be reduced while maintaining a high degree of accuracy [9, 53]. This will allow for the model to be used in both the parameter estimation and optimization problems under uncertainty.

As a method, stochastic collocation is similar to MC. Both are non-intrusive and involve sampling the random space. Each uses these sampled values of random parameters to solve the deterministic form of (4.2), i.e. (3.11), and approximate the
desired moments by averaging the sampled solutions appropriately [46]. However, the choice of sampling of the random inputs is not done arbitrarily, as with MC. Rather, we specifically choose to sample at the extremas of orthogonal polynomials of the random-space, and thus greatly reduce the number of samples for the same accuracy comparatively to Monte Carlo [19]. As well, stochastic collocation uses a weighted average over a numerical average, with weights corresponding from the choice of approximation to the random space, and subsequently the basis chosen to represent it. Under appropriate assumptions on the regularity of the system, SC will result in sub-exponential convergence in the random direction, as can be seen in the error analysis of the method. With this in mind, we now describe in detail the collocation method, but first make a necessary assumption to apply the method.

### 6.1.1 Finite-Dimensional Noise

To use the stochastic collocation method, one must assume that there are a finite number of random variables describing the noise [2]. One such way to satisfy this is to truncate a KL expansion of each random parameter. Thus, in the framework of uncertain kinematic MHD, we assume that the only random processes in the system are real-valued random parameters describing the electron mobility, $\mu_{e}$, ion-mobility, $\mu_{i}$, conductivity, $\sigma$, and fluid-flow, $\mathbf{u}$, and furthermore, that each are described by a finite number of independent random variables. For notational simplicity, let the set of random variables be denoted $\left\{m_{k}, k=1, \ldots, M\right\}$ for some $M \in \mathbb{N}$. Finally, for simplicity of exposition, we make the final assumption that $\forall k, m_{k} \sim U(0,1)$, but other distributions may be used as well.

Now, let $\Gamma_{k}:=m_{k}(\Omega)$, or the image of the events under the real-valued random variable, and define $\Gamma:=\Pi_{k=1}^{M} \Gamma_{k}$, the tensor product of each $\Gamma_{k}$. Let $\rho$ be the joint independent probability distribution for the random variables $\left[m_{1}, \ldots, m_{M}\right.$ ], $\rho: \Gamma \rightarrow$ $\mathbb{R}^{+}, \rho \in L^{\infty}(\Gamma)$. Thus, by the Doob-Dynkin's Lemma, [35], we have that the solutions
$\mathbf{J}_{i}, \mathcal{V}$ can be described by a finite number of random variables as well, e.g.

$$
\mathbf{J}_{i}(\mathbf{x}, \omega)=\mathbf{J}_{i}\left(\mathbf{x}, m_{1}(\omega), \ldots, m_{M}(\omega)\right)
$$

and similarly for $\mathcal{V}$. Applying this idea to the above problem, we attempt to find $\mathbf{J}_{i} \in V \times L_{2, \rho}(\Gamma), \mathcal{V} \in W \times L_{2, \rho}(\Gamma)$ such that for $\rho . a . e . y \in \Gamma$,

$$
\begin{align*}
& \int_{D} \bar{\sigma}^{-1} \mathbf{J}_{i}(\mathbf{x}, y) \cdot \phi(\mathbf{x}, y) d \mathbf{x}-\int_{D} \nabla \mathcal{V}(\mathbf{x}, y) \cdot \phi(\mathbf{x}, y) d \mathbf{x}=0 \quad \forall \phi \in V,  \tag{6.1a}\\
& -\int_{D} \mathbf{J}_{i}(\mathbf{x}, y) \cdot \nabla \psi(\mathbf{x}, y) d \mathbf{x}=\int_{D} \underline{\bar{\sigma}}(\mathbf{x}, y)(\mathbf{u}(\mathbf{x}, y) \times \mathbf{B}(\mathbf{x})) \cdot \nabla \psi(\mathbf{x}, y) d \mathbf{x} \quad \forall \psi \in W . \tag{6.1b}
\end{align*}
$$

For notational convenience, we now define $\widetilde{V}:=V \times L_{2, \rho}(\Gamma)$ and $\widetilde{W}:=W \times L_{2, \rho}(\Gamma)$ as the new random solution spaces for which we seek a numerical approximation. Note now that this is equivalent to 4.2 , only with the alternative probability space $(\Gamma, H, \rho)$.

We can now define the finite-dimensional (FD) random solution subspaces in which we search for our approximate solutions. We begin with the spatial dimension. Define $V_{h} \subset V$ to be the standard finite element approximation to $V$, with quadratic polynomials, on some Delauney triangular prism mesh $t_{h}$ with max side length $h$. Similarly define $W_{h} \subset W$ on the same mesh $t_{h}$. The meshes used for this problem have been described extensively in Section 5.1.2.2, and thus are not discussed further here. We now turn to the FD approximation to the random function space, $L_{\rho}^{2}(\Gamma)$. We do so iteratively, for each subspace $\Gamma_{k}$. For $k=1, \ldots, M$, define $P_{N_{k}}\left(\Gamma_{k}\right) \subset L_{2, \rho}\left(\Gamma_{k}\right)$ as the span of all polynomials on $\Gamma_{k}$ of degree up to $N_{k}$, for $N_{k} \in \mathbb{N}$. In each direction $\Gamma_{k}$, we choose a basis of orthogonal Chebyshev polynomials $\left\{r_{k}^{j}\right\}_{j=0}^{N_{k}-1}$ [46] that satisfy

$$
\int_{\Gamma} r_{k}^{j} r_{k}^{l} \rho(y) d y=\delta_{j l}
$$

where $\delta_{j l}$ is the Dirac delta function. Note that Chebyshev polynomials have welldocumented extremas and allows for the nesting of nodes at subsequent levels of the

Clenshaw-Curtis or Fejér grid, as defined in [9], and with plotted examples of the mesh seen in Figure 6.1 Define $N=\left[N_{1}, \ldots, N_{M}\right]$, as a set of orders for each dimension of $\Gamma$. Then we approximate our random $L_{2}(\Gamma)$ space with a tensor product of these polynomial spaces, i.e.

$$
P_{N}(\Gamma)=\prod_{k=1}^{M} P_{N_{k}}\left(\Gamma_{k}\right)
$$

Note that the dimension of $P_{N}$ is $\prod_{k=1}^{M} N_{k}<\infty$. The FD approximation to the random solution spaces are thus given by:

$$
\widetilde{V}_{h, N}:=V_{h} \otimes P_{N}(\Gamma) \text { and } \widetilde{W}_{h, N}:=W_{h} \otimes P_{N}(\Gamma)
$$

With these spaces defined, we can apply the stochastic collocation method. It is worth noting that the FD noise assumption is a crucial step in turning the stochastic system of equations into a set of deterministic equations, and subsequently allowing the use of finite-element techniques in approximating the solutions [2]. It follows that this system is well-posed, by the continuity of the measure $\rho$, and the well-posedness of the equivalent form, as seen in Section 4.2 .

### 6.1.2 Method

The SC method solves the deterministic system numerous times, whose solutions are then used to build an interpolate approximation to the solutions of 6.1). The deterministic system is given by: for fixed $y^{\prime} \in \Gamma$, find $\mathbf{J}_{i}^{h} \in V_{h}, \mathcal{V}^{h} \in W_{h}$ such that

$$
\begin{align*}
& \int_{D} \underline{\bar{\sigma}}^{-1}\left(\mathbf{x}, y^{\prime}\right) \mathbf{J}_{i}^{h}\left(\mathbf{x}, y^{\prime}\right) \cdot \phi\left(\mathbf{x}, y^{*}\right) d \mathbf{x}-\int_{D} \nabla \mathcal{V}^{h}\left(\mathbf{x}, y^{\prime}\right) \cdot \phi\left(\mathbf{x}, y^{\prime}\right) d \mathbf{x}=0, \quad \forall \phi \in V_{h},  \tag{6.2a}\\
& -\int_{D} \mathbf{J}_{i}^{h}\left(\mathbf{x}, y^{\prime}\right) \cdot \nabla \psi\left(\mathbf{x}, y^{\prime}\right) d \mathbf{x}=\int_{D} \underline{\bar{\sigma}}\left(\mathbf{x}, y^{\prime}\right)\left(\mathbf{u}\left(\mathbf{x}, y^{\prime}\right) \times \mathbf{B}(\mathbf{x})\right) \cdot \nabla \psi\left(\mathbf{x}, y^{\prime}\right) d \mathbf{x}, \quad \forall \psi \in W_{h} . \tag{6.2b}
\end{align*}
$$

We now perform the collocation, i.e. the collecting of solutions sampled at the zeros $\left\{y_{k, l}^{m}\right\}, l=1, \ldots, m$ of each polynomial $r_{k}^{m}$ in each direction $\Gamma_{k}$ and building a
polynomial chaos interpolant. By using the Chebyshev polynomials, we are able to use the Clenshaw-Curtis (CC) or Fejér method of numerical quadrature, which guarantees nesting of the nodes in each random direction [46]. Furthermore, we reduce the number of points for multi-dimensional quadrature by constructing a Smolyak sparse grid [9]. Although there is not a closed-form method of giving the number of nodes required in each domain [19], we let $\widetilde{N}_{k}$ denote the total number of points in the $\Gamma_{k}$ direction. A more detailed discussion of the construction of such grids can be seen in [9] or [19]. A representation of the difference between full and sparse CC or Fejér grids can be seen in Figure 6.1.

With this in mind, we let $y_{k}^{m_{k}}$ for $m_{k}=1, \ldots, \tilde{N}_{k}, k=1, \ldots, M$ be the $m^{\text {th }}$ unique zero in the direction $\Gamma_{k}$. To ease the notation, define $m=\left[m_{1}, \ldots, m_{M}\right]$ as an array of indices, and define $y_{m}=\left[y_{1}^{m_{1}}, \ldots, y_{M}^{m_{M}}\right]$ as a collection of zeroes in each random direction. Lastly, define the product of the polynomials of a given order in each direction as

$$
r_{m}(y)=\prod_{j=1}^{M} r_{j}^{m_{j}}\left(y_{j}\right) .
$$

Thus, the polynomial chaos expansion of $\mathbf{J}_{i}$ is given by

$$
\begin{equation*}
\mathbf{J}_{i}^{h, N}(\mathbf{x}, y)=\sum_{m_{1}=1}^{\widetilde{N}_{1}} \ldots \sum_{m_{M}=1}^{\widetilde{N}_{M}} \mathbf{J}_{i}\left(\mathbf{x}, y_{m}\right) r_{m}(y) \tag{6.3}
\end{equation*}
$$

and for $\mathcal{V}$,

$$
\begin{equation*}
\mathcal{V}^{h, N}(\mathbf{x}, y)=\sum_{m_{1}=1}^{\widetilde{N}_{1}} \ldots \sum_{m_{M}=1}^{\widetilde{N}_{M}} \mathcal{V}\left(\mathbf{x}, y_{m}\right) r_{m}(y) \tag{6.4}
\end{equation*}
$$

Let $C^{0}(\Gamma)$ denote the set of continuous functions on $\Gamma$ and we can define an interpolation operator, $\mathcal{I}_{V}^{N}: C^{0}(\Gamma) \times V(D) \rightarrow P_{N}(\Gamma) \times V(D)$ as, for $\mathbf{f} \in C^{0}(\Gamma) \times V(D)$

$$
\begin{equation*}
\mathcal{I}_{V}^{N}(\mathbf{f}):=\sum_{m_{1}=1}^{\widetilde{N}_{1}} \ldots \sum_{m_{M}=1}^{\widetilde{N}_{M}} \mathbf{f}\left(\mathbf{x}, y_{m}\right) r_{m}(y) . \tag{6.5}
\end{equation*}
$$

This implies immediately that $\mathbf{J}_{i}^{h, N}=\mathcal{I}_{V}^{N}\left(\mathbf{J}_{i}\right)$, under appropriate assumptions. We similarly define


FIGURE 6.1: Quadrature grids with level = 5. (a) CC Full: 289 nodes, CC, (b) Sparse: 65 nodes, (c) Fejér Sparse: 55 nodes. Constructed using J. Burkhardt's repository [10].

$$
\begin{align*}
\mathcal{I}_{W}^{N}(g): C^{0}(\Gamma) \times W(D) & \rightarrow P_{N}(\Gamma) \times W(D) \text { as, for } g \in C^{0}(\Gamma) \times W(D) \\
& \mathcal{I}_{W}^{N}(g):=\sum_{m_{1}=1}^{\widetilde{N}_{1}} \cdots \sum_{m_{M}=1}^{\widetilde{N}_{M}} g\left(\mathbf{x}, y_{m}\right) r_{m}(y) \tag{6.6}
\end{align*}
$$

and we have that $\mathcal{V}^{h, N}=\mathcal{I}_{W}^{N}(\mathcal{V})$, under appropriate assumptions. For both $\mathbf{J}_{i}$ and $\mathcal{V}$, these assumptions are discussed in the Section 6.1.3. Using the interpolation, we arrive at a deterministic form of estimating the expected values of the true solutions. Using Gaussian quadrature, the approximate integral is given by

$$
\mathbb{E}\left[\mathbf{J}_{i}^{h, N}\right]=\sum_{m_{1}=1}^{\widetilde{N}_{1}} \ldots \sum_{m_{M}=1}^{\widetilde{N}_{M}} w_{m} \mathbf{J}_{i}\left(\mathbf{x}, y_{m}\right), \text { and } \mathbb{E}\left[\mathcal{V}^{h, N}\right]=\sum_{m_{1}=1}^{\widetilde{N}_{1}} \ldots \sum_{m_{M}=1}^{\widetilde{N}_{M}} w_{m} \mathcal{V}\left(\mathbf{x}, y_{m}\right)
$$

where $w_{m}:=\prod_{j=1}^{M} w_{j}^{m_{j}}$ and $w_{j}^{m_{j}}:=\int_{\Gamma_{j}}\left(r_{j}^{m_{j}}\left(y_{j}\right)\right)^{2} \rho\left(y_{j}^{m_{j}}\right) d y_{j}$, e.g. the weights of the polynomial in each direction.

### 6.1.3 Regularity Assumptions

Before going through the error analysis of the stochastic collocation method, we must first establish some regularity properties of the solutions. We do so through assumptions about the random parameters, and then show the implications on $\mathbf{J}_{i}, \mathcal{V}$. These results will prove necessary to guarantee convergence of the collocation method. For simplicity of discussing these assumptions, we define

$$
\mathbf{g}(\mathbf{x}, y):=\overline{\underline{\sigma}}(\mathbf{x}, y)(\mathbf{u}(\mathbf{x}, y) \times \mathbf{B}(\mathbf{x})) .
$$

We begin with a lemma that will be used to show that the continuity of the random parameters with respect to $y$ can be directly transferred to $\mathbf{J}_{i}$, $\mathcal{V}$, under suitable conditions.

Lemma 6.1.1. Under the assumption that $\mathbf{g} \in C^{0}(\Gamma) \times W, \mathbf{J}_{i} \in C^{0}(\Gamma) \times V$ and $\mathcal{V} \in C^{0}(\Gamma) \times W$.

The proof of this lemma follows immediately from the bounds given in Theorem 4.1.1. We thus turn to the more complex task of bounding the derivatives of $\mathbf{J}_{i}, \mathcal{V}$ in
each random direction. For notational simplicity, let

$$
\partial_{n}^{k}:=\frac{\partial^{k}}{\partial y_{n}^{k}}
$$

We now make another major assumption regarding the implementation of the SC method. We require that for the remainder of this paper, $\Gamma_{k}$ is bounded for each $k=1, \ldots, M$. For the power-generation application, this assumption is rooted in the physical limitations of what the expected parameters values can take, and how regular their distributions are.

To bound the derivatives of the solutions, we will make use of the bounds in Theorem 4.1.1, and begin by taking the derivative of (6.2) with respect to one of the random directions. We also consider the deterministic form of 4.2), i.e. for fixed $y$, and let $\mathcal{A}, \mathcal{B}$ represent the deterministic equivalent operators of $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}$ respectively. Through an iterative application of the product rule and solving for the desired operators on the left-hand side of the system, the differentiated solutions to (6.1) satisfy

$$
\begin{gather*}
\mathcal{A}\left(\partial_{n}^{k} \mathbf{J}_{i}\right)+\mathcal{B}^{\prime}\left(\partial_{n}^{k} \mathcal{V}\right)=F^{k} \in V^{\prime},  \tag{6.7a}\\
\mathcal{B}\left(\partial_{n}^{k} \mathbf{J}_{i}\right)=G^{k} \in W^{\prime} . \tag{6.7b}
\end{gather*}
$$

Here,

$$
\begin{equation*}
F^{k}(\phi):=-\int_{D}\left(\sum_{j=0}^{k-1}\binom{k}{j} \partial_{n}^{k-j}\left(\underline{\bar{\sigma}}^{-1}\right) \partial_{n}^{j}\left(\mathbf{J}_{i}\right)\right) \cdot \phi, \text { and } G^{k}(\psi):=\int_{D} \partial_{n}^{k} \mathbf{g} \cdot \nabla \psi . \tag{6.8}
\end{equation*}
$$

Note that it is obvious by their definition that $F^{k} \in V^{\prime}$ and $G^{k} \in W^{\prime}$ for all $k \in \mathbb{N}$, by construction. Through an application of Theorem 4.1.1, we have that the solutions $\partial_{n}^{k} \mathbf{J}_{i}, \partial_{n}^{k} \mathcal{V}$ exist, are unique, and obey the following bounds:

$$
\begin{gather*}
\left\|\partial_{n}^{k} \mathbf{J}_{i}\right\|_{V} \leq\left\|F^{k}\right\|_{V^{\prime}}+C\left\|G^{k}\right\|_{W^{\prime}},  \tag{6.9}\\
\left\|\partial_{n}^{k} \mathcal{V}\right\|_{W} \leq \frac{1}{b}\left(\left\|F^{k}\right\|_{V^{\prime}}+\|A\|_{\mathcal{L}\left(V, V^{\prime}\right)}\left\|G^{k}\right\|_{V^{\prime}}\right), \tag{6.10}
\end{gather*}
$$

where $C=\frac{1}{a b}\left(\|\mathcal{A}\|_{\mathcal{L}\left(V, V^{\prime}\right)}+a\right)$. Using this system, we now place bounds on the derivatives of $\mathbf{J}_{i}, \mathcal{V}$ in the random direction that will guarantee the convergence of the SC method. We will need restrictive assumptions regarding the decay of the derivatives of $\underline{\sigma}^{-1}, \mathbf{g}$. These restrictions may differ in each random direction by the values of the constants only. Thus, for the purposes of the rest of this section, fix $n \in\{1, \ldots, M\}$, and thus fix our direction $\Gamma_{n}$. Dependence of a parameter on the direction is denoted through the subscript $n$. We now state the following regularity assumptions and the resulting regularity properties of $\mathbf{J}_{i}, \mathcal{V}$.

Lemma 6.1.2. Assume that

$$
\left\|\partial_{n}^{k} \overline{\bar{\sigma}}^{-1}\right\|_{V^{\prime}} \leq \alpha_{1, k} \frac{k!}{\gamma_{n}^{k}},\left\|\partial_{n}^{k} \mathbf{g}\right\|_{W} \leq \alpha_{2, k} \frac{k!}{\gamma_{n}^{k}} \quad \forall k \in \mathbb{N} .
$$

where $\gamma_{n}>0$, and the other constants satisfy the inequality

$$
\frac{\alpha_{2, k}}{\|g\|_{W}}+\left(\alpha_{1, k}+\sum_{j=1}^{k-1} \alpha_{1, k-j}\right) \leq 1
$$

Then

$$
\left\|\partial_{n}^{k} \mathbf{J}_{i}\right\|_{V} \leq C_{1, k} \frac{k!}{\gamma_{n}^{k}}, \text { and }\left\|\partial_{n}^{k} \mathcal{V}\right\|_{V} \leq C_{2, k} \frac{k!}{\gamma_{n}^{k}},
$$

where

$$
C_{1, k} \leq C\|\mathbf{g}\|_{W}:=C_{0}, C_{2, k} \leq \frac{1}{b}\|\mathbf{g}\|_{W} \quad \forall k \in \mathbb{N} .
$$

Proof. (By induction for $\mathbf{J}_{i}$ ) First, let $k=1$. Then by (6.9), we have

$$
\left\|\partial \mathbf{J}_{i}\right\|_{V} \leq\left\|F^{1}\right\|_{V^{\prime}}+C\left\|G^{1}\right\|_{V^{\prime}}
$$

By definition of $F^{1}, G^{1}$, we have

$$
\left\|\partial_{n} \mathbf{J}_{i}\right\|_{V} \leq\left\|\partial_{n} \underline{\bar{\sigma}}^{-1}\right\|_{V^{\prime}}\left\|\mathbf{J}_{i}\right\|_{V}+C\left\|\partial_{n} \mathbf{g}\right\|_{V}
$$

By the bounds given in Theorem 4.1.1, we have

$$
\left\|\partial_{n} \mathbf{J}_{i}\right\|_{V} \leq\left\|\partial_{n} \underline{\bar{\sigma}}^{-1}\right\|_{V^{\prime}} C\|\mathbf{g}\|_{W}+C\left\|\partial_{n} \mathbf{g}\right\|_{V}
$$

$$
\leq \alpha_{1,1} C\|\mathbf{g}\|_{W}+C \alpha_{2,1} .
$$

Letting $C_{1, k}=\alpha_{1,1} C\|g\|_{W}+C \alpha_{2,1}$ yields the first desired inequality. To see that $C_{1, k} \leq C_{0}$, consider that as $j=1$, the sum in (6.9) is 0 . Then we have

$$
1 \geq \alpha_{1,1}+\frac{\alpha_{2,1}}{\|\mathbf{g}\|_{W}} \quad \quad \text { by assumption }
$$

Multiplying each side by $C_{0}$ gives

$$
C_{0}=C\|g\|_{W} \geq C \alpha_{1,1}\|\mathbf{g}\|_{W}+C \alpha_{2,1}=C_{1, k},
$$

and the second desired inequality, the bounding constant inequality, is satisfied.
Now, let $k \in \mathbb{N}$, and assume that

$$
\left\|\partial^{j} \mathbf{J}_{i}\right\|_{V} \leq C_{1, j} \frac{j!}{\alpha^{j}}
$$

holds for some $C_{1, j} \leq C_{0}, j=1, \ldots, k-1$. Again, by (6.9), we have

$$
\begin{aligned}
\left\|\partial_{n}^{k} \mathbf{J}_{i}\right\|_{V} & \leq C\left\|G^{k}\right\|_{W^{\prime}}+\left\|F^{k}\right\|_{V^{\prime}} \\
& \leq C\left\|\partial_{n}^{k} \mathbf{g}\right\|_{W}+\sum_{j=0}^{k-1}\binom{k}{j}\left\|\partial_{n}^{k-j} \underline{\underline{\sigma}}^{-1}\right\|_{V^{\prime}}\left\|\partial^{j} \mathbf{J}_{i}\right\|_{V}
\end{aligned}
$$

By the induction assumption, we have

$$
\left\|\partial_{n}^{k} \mathbf{J}_{i}\right\|_{V} \leq C \alpha_{2, k} \frac{k!}{\gamma_{n}^{k}}+\left(\alpha_{1, k} \frac{k!}{\gamma_{n}^{k}} C_{0}+\sum_{j=1}^{k-1}\binom{k}{j} \alpha_{1, k-j} \frac{(k-j)!}{\gamma_{n}^{k-j}} C_{1, j} \frac{j!}{\gamma_{n}^{j}}\right) .
$$

Here, we separated the $j=0$ case from the rest, as these bounds stem from the deterministic inequality, and not the lemma assumptions. As well, note that $\binom{a}{b}=$ $\frac{a!}{b!(a-b)!}$. Thus, we have

$$
\begin{aligned}
\left\|\partial_{n}^{k} \mathbf{J}_{i}\right\|_{V} & \leq C \alpha_{2, k} \frac{k!}{\gamma_{n}^{k}}+\left(\alpha_{1, k} \frac{k!}{\gamma_{n}^{k}} C_{0}+\sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} \alpha_{1, k-j} \frac{(k-j)!}{\gamma_{n}^{k-j}} C_{1, j} \frac{j!}{\gamma_{n}^{j}}\right) \\
& =\frac{k!}{\gamma_{n}^{k}} \underbrace{\left(C \alpha_{2, k}+\left(\alpha_{1, k} C_{0}+\sum_{j=0}^{k-1} \alpha_{1, k-j} C_{j}\right)\right)}_{C_{1, k}} .
\end{aligned}
$$

which yields the first inequality. To see that $C_{1, k}<C_{0}$, consider that by assumption

$$
1 \geq \frac{\alpha_{2, k}}{\|\mathbf{g}\|_{W}}+\left(\alpha_{1, k}+\sum_{j=1}^{k-1} \alpha_{1, k-j}\right) .
$$

Multiplying each side by $C_{0}$ yields

$$
C_{0} \geq C \alpha_{2, k}+\left(C_{0} \alpha_{1, k}+\sum_{j=1}^{k-1} C_{0} \alpha_{1, k-j}\right)
$$

Using the induction assumption that $C_{1, j} \leq C_{0}$ yields

$$
C_{0} \geq C \alpha_{2, k}+\left(\alpha_{1, k} C_{0}+\sum_{j=0}^{k-1} \alpha_{1, k-j} C_{1, j}\right)=C_{1, k} .
$$

and the second inequality follows. A similar proof will achieve the same result for $\mathcal{V}$ and the constants $C_{2, k}$.

Note that the constants $\alpha_{1, k}$ and $\alpha_{2, k}$ may also depend on $n$, but that dependence is neglected for notational sake. With these regularity assumptions, we can define an appropriate analytic extension to each function of $\mathbf{J}_{i}$ and $\mathcal{V}$.

Lemma 6.1.3. Under the assumptions of Lemma 6.1.2, $\mathbf{J}_{i}$ and $\mathcal{V}$ admit analytic extensions in the region of the complex plane $\Sigma\left(\Gamma_{n}, \gamma_{n}\right):=\left\{z=y_{n}+i y_{2} \in \Gamma_{n} \times \mathbb{C}: y_{n} \in \Gamma_{n}\right.$ and $\left.\left|y_{2}\right| \leq a\right\}$.

Proof. We show this for $\mathcal{V}$ only, as a similar approach will show the same in any spatial direction $x_{j}$ for $\mathbf{J}_{i, x_{j}} \cdot{ }^{5}$ Here, we define $y_{n}^{*}=\left\{y_{k}\right\}_{k \neq n}$ as a set of values in $\Gamma_{k}$ for $k \neq N$. Define the extension of $\mathcal{V}$ on $\Sigma\left(\Gamma_{n}, \gamma_{n}\right)$ with an analytic power-series,

$$
\begin{equation*}
\mathcal{V}\left(z, y_{n}^{*}, x\right)=\sum_{k=0}^{\infty} \frac{\left(z-y_{n}\right)^{k}}{k!}\left\|\partial_{n}^{k} \mathcal{V}\left(y_{n}, y_{n}^{*}, \mathbf{x}\right)\right\|_{W} \tag{6.11}
\end{equation*}
$$

We now seek to show that this series converges. To this end, consider

$$
\mathcal{V}\left(z, y_{n}^{*}, x\right)=\sum_{k=0}^{\infty} \frac{\left(z-y_{n}\right)^{k}}{k!}\left\|\partial_{n}^{k} \mathcal{V}\left(y_{n}, y_{n}^{*}, \mathbf{x}\right)\right\|_{W}
$$

[^4]\[

$$
\begin{array}{lr}
\leq \sum_{k=0}^{\infty} \frac{\left(z-y_{n}\right)^{k}}{k!} C_{2, k} \frac{k!}{\gamma_{n}^{k}} & \text { by Lemma 6.1.2, } \\
\leq \frac{1}{b}\|\mathbf{g}\|_{W} \sum_{k=0}^{\infty} \frac{\left(z-y_{n}\right)^{k}}{\gamma_{n}^{k}} & \text { as } C_{2, k}<\frac{1}{b}\|\mathbf{g}\|_{W} \forall k .
\end{array}
$$
\]

This converges by a geometric series argument for $\left(z-y_{n}\right)<\gamma_{n}$, implying that the region of convergence for our series is given by the subspace $\Sigma\left(\Gamma_{n}, \gamma_{n}\right)$, yielding the desired result.

Note that this implies for both $\mathbf{J}_{i}, \mathcal{V}$, the domain in which there exists an analytic extension is directly related to the decay of the derivatives of $\mathbf{g}, \underline{\sigma}^{-1}$ in the direction $\Gamma_{n}$, as this domain is given by, for each $\Gamma_{n}, \Sigma\left(\Gamma_{n}, \gamma_{n}\right)$. A faster decay of the derivatives results in a larger area in which the analytic extension exists. We now move onto the error analysis section.

### 6.1.4 Error Analysis

Prior to proving the convergence of the SC method, we present two crucial lemmas necessary for bounding the random approximation error. These are as given in Babuska [2], and pertain to bounding 1-D random polynomial interpolation. They will be then adapted to bound the interpolation error on the entire random-space $\Gamma$. To this end, similar to the notation seen in Section 6.1.3, fix $n \in\{1, \ldots, M\}$, which in turn fixes our direction, $\Gamma_{n}$, and all other constants dependent upon the direction $\Gamma_{n}$, such as the $\Gamma_{n}$ interpolation order, $N_{n}$. Then we have a general interpolation lemma on a 1-D random space.

Lemma 6.1.4. For any Banach function space, $H$, define the tensor product norm of $H$ with $L_{\rho}^{2}\left(\Gamma_{n}\right)$ as

$$
\|f\|_{L_{\rho}^{2}\left(\Gamma_{n}\right) \times H}:=\int_{\Gamma_{n}}\|f\|_{H}^{2} \rho_{n}\left(y_{n}\right) d y_{n} .
$$

Then the 1-D random interpolation $\mathcal{I}_{H}^{N_{n}}: C^{0}\left(\Gamma_{n}\right) \times H \rightarrow L_{\rho}^{2}\left(\Gamma_{n}\right) \times H$, defined for
$f \in C^{0}\left(\Gamma_{n}\right) \times H$,

$$
\mathcal{I}_{H}^{N_{n}}(f):=\sum_{m_{n}=1}^{\widetilde{N}_{n}} f\left(y_{n}, \boldsymbol{x}\right) r_{n}^{m_{k}}\left(y_{n}\right),
$$

is continuous under this norm. We define the tensor product norm of $H$ with $C^{0}\left(\Gamma_{n}\right)$, as

$$
\|f\|_{C^{0}\left(\Gamma_{n}\right) \times H}:=\max _{y_{n} \in \Gamma_{n}}\left\|f\left(y_{n}, \boldsymbol{x}\right)\right\|_{H} .
$$

Then the interpolation error satisfies

$$
\begin{equation*}
\left\|f-\mathcal{I}_{H}^{N_{n}}(f)\right\|_{L_{\rho}^{2}\left(\Gamma_{n}\right) \times H} \leq K_{H} \inf _{w \in P_{N_{n}}\left(\Gamma_{n}\right) \times H}\left\|f\left(y_{n}\right)-w\left(y_{k}\right)\right\|_{C^{0}\left(\Gamma_{n}\right) \times H}, \tag{6.12}
\end{equation*}
$$

with constant $K_{H}$ independent of the choice of $N_{n}$.
We also wish to bound the best-approximation error. Thus, we consider this next lemma.

Lemma 6.1.5. For some Banach function space, $H$, let $v \in C^{0}\left(\Gamma_{n}\right) \times H$. Assume $v$ also admits an analytic extension in the region of the complex plane $\Sigma\left(\Gamma_{n} ; \tau\right)$ for some $\tau>0$. Then it holds:

$$
\min _{w \in P_{N_{n}}\left(\Gamma_{n}\right) \times H}\left\|v\left(y_{n}\right)-w\left(y_{n}\right)\right\|_{C^{0}(\Gamma) \times H} \leq \frac{2}{\Psi_{n}-1} \exp \left(-N_{n} \log \left(\Psi_{n}\right)\right) \max _{z \in \Sigma\left(\Gamma_{n} ; \tau\right)}\|v(z)\|_{H}
$$

where $1<\Psi_{n}=\frac{2 \tau}{\left|\Gamma_{n}\right|}+\sqrt{1+\frac{4 \tau^{2}}{\left|\Gamma_{n}\right|^{2}}}$.
As stated previously, proofs of both Lemma 6.1.4 and Lemma 6.1.5 can be found in [2]. With these established, we now turn to bounding our complete $\Gamma$ interpolation error. We do so iteratively. We first break apart our function space, separating as

$$
L_{\rho}^{2}(\Gamma) \times V \equiv L_{\rho}^{2}\left(\Gamma_{k}\right) \times\left(L_{\rho}^{2}\left(\Gamma_{k^{*}}\right) \times V\right)
$$

where $\Gamma_{k^{*}}$ are all random directions except $\Gamma_{k}$. Similarly define the product of the values of any parameter in every direction but the $k^{t h}$ as $\circ_{k^{*}}:=\prod_{j \neq k} \circ_{j}$, with this being a real product, tensor product, or set of values where appropriate. We define the norm on the latter space as the averaging norm, i.e. for $f \in L_{\rho}^{2}\left(\Gamma_{k^{*}}\right) \times V$,

$$
\|f\|_{L_{\rho}^{2}\left(\Gamma_{k^{*}}\right) \times V}:=\int_{\Gamma_{k^{*}}}\left\|f\left(y_{k^{*}}, \mathbf{x}\right)\right\|_{V}^{2} \rho_{k^{*}}\left(y_{k^{*}}\right) d y_{k^{*}}
$$

which corresponds to the Hilbert tensor product construction. Of course, this also implies that it is a Banach space, and thus Lemmas 6.1.4 and 6.1.5 both apply. Denote the product space of the analytic extension region as

$$
\Sigma(\Gamma, \gamma):=\prod_{j=1}^{M} \Sigma\left(\Gamma_{j}, \gamma_{j}\right)
$$

where $\gamma_{j}$ is as defined in Lemma 6.1.3. Then we have the following interpolation bounds for our specific operators.

Lemma 6.1.6. The interpolation operators, $\mathcal{I}_{V}^{N}, \mathcal{I}_{W}^{N}$ are continuous, and obey the following interpolation bounds. For $\boldsymbol{f} \in C^{0}(\Gamma) \times V$ and $g \in C^{0}(\Gamma) \times W$, with analytic extensions in the region $\Sigma(\Gamma, \gamma)$, we have

$$
\begin{align*}
& \left\|\boldsymbol{f}-\mathcal{I}_{V}^{N}(f)\right\|_{L_{\rho}^{2}(\Gamma) \times V} \leq K_{1} \max _{z \in \Sigma(\Gamma, \tau)}\|\boldsymbol{f}(z)\|_{V} \sum_{j=1}^{M} \frac{2}{\Psi_{j}-1} \exp \left(-N_{j} \log \left(\Psi_{j}\right)\right),  \tag{6.13a}\\
& \left\|g-\mathcal{I}_{W}^{N}(g)\right\|_{L_{\rho}^{2}(\Gamma) \times W} \leq K_{2} \max _{z \in \Sigma(\Gamma, \tau)}\|g(z)\|_{W} \sum_{j=1}^{M} \frac{2}{\Psi_{j}-1} \exp \left(-N_{j} \log \left(\Psi_{j}\right)\right), \tag{6.13b}
\end{align*}
$$

where $\Psi_{j}:=\frac{2 \gamma_{j}}{\left|\Gamma_{j}\right|}+\sqrt{1+\frac{4 \gamma_{j}^{2}}{\left|\Gamma_{j}\right|^{2}}}>1$, and $K_{1}, K_{2}$ are two constants independence of the choice of $N$.

Proof. We begin with continuity, and must prove this iteratively. As it will follow with similar logic, we show these properties for the interpolation operator $\mathcal{I}_{V}^{N}$ only. For notational convenience, define

$$
V_{k^{*}}:=L_{\rho}^{2}\left(\Gamma_{k^{*}}\right) \times V \text {. }
$$

We define the $k^{\text {th }}$ interpolation operator $\mathcal{I}_{V_{K^{*}}}^{N_{k}}: C^{0}\left(\Gamma_{k}\right) \times V_{k^{*}} \rightarrow L_{\rho}^{2}\left(\Gamma_{k}\right) \times V_{k^{*}}$ as

$$
\begin{equation*}
\mathcal{I}_{V_{k^{*}}}^{N_{k}}(\mathbf{f}):=\sum_{m_{k}=1}^{\widetilde{N}_{k}} \mathbf{f}\left(y_{k}^{m_{k}}, \mathbf{x}\right) r_{k}^{m_{k}}\left(y_{k}\right) . \tag{6.14}
\end{equation*}
$$

It follows that the interpolation operator satisfies

$$
\mathcal{I}_{V}^{N}=\mathcal{I}_{V_{1^{*}}}^{N_{1}} \circ \ldots \circ \mathcal{I}_{V_{M^{*}}}^{N_{M}} .
$$

Note now that $V_{k^{*}}$ is a Hilbert space for all $k=1, \ldots, M$, as it is the finite tensor product of Hilbert spaces. Therefore, Lemma 6.1.4 applies, and as the composure of continuous functions are continuous, continuity holds. To see that the bounds hold, consider that

$$
\begin{aligned}
\left\|\mathbf{f}-\mathcal{I}_{V}^{N}(\mathbf{f})\right\|_{L_{\rho}^{2}(\Gamma) \times V} & \leq\left\|\mathbf{f}-\left(\mathcal{I}_{V_{1^{*}}}^{N_{1}} \circ \ldots \circ \mathcal{I}_{V_{M^{*}}}^{N_{M}}\right)(\mathbf{f})\right\|_{L_{\rho}^{2}(\Gamma) \times V} \\
& \leq\left\|\left(\mathbf{f}-\mathcal{I}_{V_{1^{*}}}^{N_{1}} \mathbf{f}\right)+\left(\mathcal{I}_{V_{1^{*}}}^{N_{1}} \mathbf{f}-\mathcal{I}_{V_{1^{*}}}^{N_{1}} \circ \mathcal{I}_{V_{2^{*}}}^{N_{2}} \mathbf{f}\right)+\ldots\right\|_{L_{\rho}^{2}(\Gamma) \times V} \\
& \leq\left\|\mathbf{f}-\mathcal{I}_{V_{1^{*}}}^{N_{1}} \mathbf{f}\right\|_{L_{\rho}^{2}(\Gamma) \times V}+\left\|\mathcal{I}_{V_{1^{*}}}^{N_{1}} \mathbf{f}-\mathcal{I}_{V_{1^{*}}}^{N_{1}} \circ \mathcal{I}_{V_{2^{*}}}^{N_{2}} \mathbf{f}\right\|_{L_{\rho}^{2}(\Gamma) \times V}+\ldots
\end{aligned}
$$

We now note that $L_{\rho}^{2}(\Gamma) \times V=L_{\rho}^{2}\left(\Gamma_{k}\right) \times V_{k^{*}}$, and thus $\|\cdot\|_{L_{\rho}^{2}(\Gamma) \times V}=\|\cdot\|_{L_{\rho}^{2}\left(\Gamma_{k}\right) \times V_{k^{*}}}$. It is also immediate that $\mathbf{f} \in C^{0}(\Gamma)$ implies that $\mathcal{I}_{V_{k}}^{N_{k}}(\mathbf{f}) \in C^{0}\left(\Gamma_{k}\right)$. Finally, for notational convenience in the purposes of this proof, we define $P_{k}:=P_{N_{k}}\left(\Gamma_{k}\right) \times V_{k^{*}}$ WIth this in mind, we apply the bounds from Lemma 6.1.4, and have

$$
\begin{aligned}
\left\|\mathbf{f}-\mathcal{I}_{V}^{N}(\mathbf{f})\right\|_{L_{\rho}^{2}(\Gamma) \times V} \leq K_{V_{1^{*}}} \inf _{w_{1} \in P_{1}} & \left\|\mathbf{f}-w_{1}\right\|_{C^{0}\left(\Gamma_{1}\right) \times V_{1^{*}}} \\
& +K_{V_{2^{*}}} \inf _{w_{2} \in P_{2}}\left\|\mathcal{I}_{V_{1^{*}}}^{N_{1}}(\mathbf{f})-w_{2}\right\|_{C^{0}\left(\Gamma_{2}\right) \times V_{2^{*}}}+\ldots
\end{aligned}
$$

Choosing $K_{1}:=\max _{j \in\{1, \ldots, M\}} K_{V_{j^{*}}}$ yields

$$
\begin{aligned}
\left\|\mathbf{f}-\mathcal{I}_{V}^{N}(\mathbf{f})\right\|_{L_{\rho}^{2}(\Gamma) \times V} \leq K_{1}\left(\inf _{w_{1} \in P_{1}} \| \mathbf{f}\right. & -w_{1} \|_{C^{0}\left(\Gamma_{k}\right) \times V_{1^{*}}} \\
& \left.+\inf _{w_{2} \in P_{2}}\left\|\mathcal{I}_{V_{1^{*}}}^{N_{1}}(\mathbf{f})-w_{2}\right\|_{C^{0}\left(\Gamma_{2}\right) \times V_{2^{*}}}+\ldots\right)
\end{aligned}
$$

Now, given that all of $V_{k^{*}}$ are Hilbert spaces, it follows that they are Banach spaces, and we can apply Lemma 6.1.5, which yields

$$
\left\|\mathbf{f}-\mathcal{I}_{V}^{N}(\mathbf{f})\right\|_{L_{\rho}^{2}(\Gamma) \times V} \leq K_{1} \max _{z \in \Sigma(\Gamma, \tau)}\|\mathbf{f}(z)\|_{V} \sum_{j=1}^{M} \frac{2}{\Psi_{j}-1} \exp \left(-N_{j} \log \left(\Psi_{j}\right)\right)
$$

with $\Psi_{j}$ defined as in Lemma 6.1.5. The independence of $K_{1}$ on $N$ follows from the independence of each constant $K_{V_{j^{*}}}$ on $N$. This is the exact approximation bound we claimed. With similar logic, the continuity and approximation error bounds for $\mathcal{I}_{W}^{N}$ also follow.

We have now established that the random polynomial chaos interpolation has error bounded by the max value of the true solutions. However, we used the complete deterministic solution space, $V$ and $W$, only out of notational convenience. Both $V$ and its finite-dimensional approximation, $V_{h}$, and subsequently $W, W_{h}$, are assumed to have the same random-space regularity, and thus Lemma 6.1.6 also applies when considering the stochastic interpolation operators acting on $V_{h}, W_{h}$. Combining this lemma with the regularity assumptions in Section 6.1 .3 and the finite-dimensionality of $\Gamma$ (see Section 6.1.1), we now bound the approximation error in the SC method.

Theorem 6.1.1. Under the assumptions of Lemma 6.1.1, Lemma 6.1.2, the finitedimensional noise assumption, and that the $\Gamma$ is bounded, we have the following error bounds on the approximate solutions resulting from the SC method. They satisfy

$$
\begin{align*}
& \left\|\boldsymbol{J}_{i}-\boldsymbol{J}_{i}^{h, N}\right\|_{\widetilde{V}} \leq C\left\|\mathbf{g}-\mathbf{g}^{h}\right\|_{\widetilde{W}}+K_{1} \max _{z \in \Sigma(\Gamma, \tau)}\left\|\boldsymbol{J}_{i}(z)\right\|_{V} \sum_{j=1}^{M} \frac{2}{\Psi_{j}-1} \exp \left(-N_{j} \log \left(\Psi_{j}\right)\right)  \tag{6.15a}\\
& \left\|\mathcal{V}-\mathcal{V}^{h, N}\right\|_{\widetilde{W}} \leq \frac{1}{b}\left\|\mathbf{g}-\mathbf{g}^{h}\right\|_{\widetilde{W}}+K_{2} \max _{z \in \Sigma(\Gamma, \tau)}\|\mathcal{V}(z)\|_{W} \sum_{j=1}^{M} \frac{2}{\Psi_{j}-1} \exp \left(-N_{j} \log \left(\Psi_{j}\right)\right) \tag{6.15b}
\end{align*}
$$

with $\Psi_{j}<1$ defined as in Lemma 6.1.5. and $\mathbf{g}^{h}$ is the polynomial interpolation of $\mathbf{g}$ on $t_{h}$, of the same degree as $\boldsymbol{J}_{i}^{h}$ and $\mathcal{V}^{h}$.

Proof. We will show that the bounds hold for $\mathbf{J}_{i}$ only, as similar logic will follow for $\mathcal{V}$. We first expand the approximations into the spatial discretization and random polynomial interpolations, e.g.

$$
\left\|\mathbf{J}_{i}-\mathbf{J}_{i}^{h, N}\right\|_{\widetilde{V}} \leq\left\|\mathbf{J}_{i}-\mathbf{J}_{i}^{h}\right\|_{\widetilde{V}}+\left\|\mathbf{J}_{i}^{h}-\mathbf{J}_{i}^{h, N}\right\|_{\widetilde{v}}
$$

The first of the two terms stem from standard finite-element theory and polynomial interpolation of $\mathbf{J}_{i}$, and corresponds to the first term in the bound above. It is clear that the spatial approximation of $\mathbf{J}_{i}^{h}$ satisfies the system

$$
\widetilde{\mathcal{A}}\left(\mathbf{J}_{i}^{h}\right)+\widetilde{\mathcal{B}}^{\prime}\left(\mathcal{V}^{h}\right)=0 \in \widetilde{V}^{\prime},
$$

$$
\widetilde{\mathcal{B}}\left(\mathbf{J}_{i}^{h}\right)=G_{h} \in \widetilde{W}^{\prime} .
$$

where $G_{h}(\psi):=\mathbb{E}\left[\int_{D} \mathbf{g}^{h} \cdot \nabla \psi\right]$, and $\mathbf{g}^{h}$ is as stated in the theorem. Subtracting this system from (4.2) and applying the bounds from Theorem 4.1.1, we have

$$
\left\|\mathbf{J}_{i}-\mathbf{J}_{i}^{h}\right\|_{\widetilde{V}} \leq C\left\|\mathbf{g}-\mathbf{g}^{h}\right\|_{\widetilde{W}}
$$

We now turn to the second of the two terms, which stems from the random space polynomial chaos expansion and subsequent approximation. It is clear from Lemma 6.1.6 that the interpolation operator corresponding to $V$, when applied to $\mathbf{J}_{i}^{h}$ satisfies

$$
\begin{equation*}
\left\|\mathbf{J}_{i}^{h}-\mathcal{I}_{V}^{N}\left(\mathbf{J}_{i}^{h}\right)\right\|_{L_{\rho}^{2}(\Gamma) \times V} \leq K_{1} \max _{z \in \Sigma(\Gamma, \tau)}\left\|\mathbf{J}_{i}^{h}(z)\right\|_{V} \sum_{j=1}^{M} \frac{2}{\Psi_{j}-1} \exp \left(-N_{j} \log \left(\Psi_{j}\right)\right) \tag{6.17}
\end{equation*}
$$

with the previously defined $\psi_{j}$, and $K_{1}$. This yields the desired result for $\left\|\mathbf{J}_{i}-\mathbf{J}_{i}^{h, N}\right\|_{\widetilde{V}}$ and a similar argument argument holds for the bounds on the error of approximating $\mathcal{V}$.

Note now that it is sufficient to bound the spatial approximation error by $\| \mathbf{g}$ $\mathbf{g}^{h} \|_{\widetilde{W}}$, as this converges as $h \rightarrow \infty$ by standard polynomial interpolation arguments [14], and is thus not discussed further here. We also can see that the random error, as desired, achieves the sub-exponential convergence, as $\psi_{j}>1$. Thus, the SC method will converge, as desired. We now move onto numerical demonstrations of the forward problem.

### 6.1.5 Numerical Experiments

We now turn to the numerical investigation into the propagation of uncertainty within the kinematic MHD framework. There are three distinct demonstrations that are showcased below. First, we perform a numerical demonstration of the convergence of the method. In lieu of analytic solutions to (6.1), we show a convergence to a higherorder approximation to (6.2). We also demonstrate that the uncertain problem obtains useful information that the deterministic equivalent does not provide. This is shown
by comparing the expected value of the solutions to the deterministic solutions of the system coupled with the expected value of the parameters. Finally, we investigate the impact the shape of the parameters' distributions has on the solutions' distributions, by comparing two distributions with equivalent expected values and variances, but different shapes.

### 6.1.5.1 Convergence Demonstration

Within the error analysis and Theorem6.1.1, we showed that the approximation and the true solution differ by the sum of two terms, one which stems from standard finite-element theory and polynomial interpolation, the latter stemming from the polynomial chaos expansion and subsequent approximations. For demonstration purposes, we neglect showcasing the finite-element convergence, as this has been done extensively before, [1, 7, 30]. Thus, we focus on showing convergence of the approximations solely in the random sense. As mentioned previously, we showcase the convergence to a 'high-order' approximation.

To aid the computational complexity, we make several assumptions regarding the randomness of the system. First and foremost, we assume that the electron mobility, $\mu_{e}$, is the only random parameter within our model. Furthermore, we assume that $\mu_{e}$ is spatially-constant, implying that it is described by a real-valued random variable, with

$$
\mathbb{E}\left[\mu_{e}\right]=10 / 6, \text { and } \operatorname{var}\left(\mu_{e}\right)=0.5
$$

This random variable is assumed to have a beta distribution, with shape parameters of $a=4, b=4$. Recall that we make use of the Fejér Smolyak sparse-grid, which can be seen in Figure 6.1. To make the 'high-order' approximation, we set an arbitrary level of 6 , and then compare the results to the samples corresponding to the lower levels. The error between the lower-order approximations and the level 5 node assignment can be seen in the Table 6.1.

TABLE 6.1: Demonstration of convergence of solutions for different levels of the Fejér grid, compared to the SC solutions for level 6 grid. Note that the use of the $l^{2}$ grid-norm is an approximation to the $L^{2}$ average norm.

| Level | $l^{2}$ Grid-norm Difference |
| :---: | :---: |
| 2 | 2.18 |
| 3 | 1.38 |
| 4 | 0.24 |
| 5 | 0.02 |

### 6.1.5.2 Uncertain versus Deterministic Solutions

We now investigate how the uncertain forward problem and the deterministic forward problem differ. We investigate the differences between the two functions $\mathbb{E}[U(q)]$ and $U(\mathbb{E}[q])$, qualitatively and numerically. For the sake of succinctness, we will refer to the former as the expected solution, and the latter as the deterministic solution, although these don't quite describe the functions in their entirety. To examine the qualitative differences between the solutions, we examine a single plane of solutions. For these results, we run the stochastic collocation method for a constant electron mobility, with a distribution as described in Table 6.2. We also compute the deterministic solution for the solution fixed at the mean of $\mu_{e}, 10 / 6$. All other parameters are considered deterministic, and are fixed as described in Section 5.1. For demonstration, we examine only the electric potential, $\mathcal{V}$, within the channel, across a plane for $z=0.05[m]$, i.e. half the channel height. We then compare their difference across this same plane, with results seen in Figure 6.2.

Although perhaps not immediately obvious, there is clearly a quantitative difference between the deterministic and expected solutions. Numerically, we see that the approximate difference between these two values is on the order of 10 , when using the grid-norm approximation to the $H$ norm. Qualitatively, however, the difference plot within Figure 6.2 demonstrates this result. Interestingly, the difference between
the two remains positive throughout the domain. Regardless, it is obvious that

$$
\mathbb{E}[[U(q)] \neq U(\mathbb{E}[q])
$$

We now compare the effects of different distributional shapes on the distribution of the solutions.

### 6.1.5.3 Comparing Distributional Shapes

To extend the demonstration of robustness for the model, we include the fluidflow to be a random parameter for which we investigate the propagation of uncertainty. However, we further simply the electron mobility to be spatially constant, to diminish the computational complexity. Furthermore, the fluid flow is assumed to be in one direction, $\mathbf{u}=\left(\mathbf{u}_{x}, 0,0\right)$, described by a single random variable and is spatially constant. For demonstration purposes, we assume both random variables are described by either a uniform or beta distribution, with a given mean and variance. The distributional choices are described in Table 6.2. To directly compare the difference of the solutions' distributions for varying parameter distributions, we make use of the Kullback Leibler (KL) divergence, also known as relative entropy [12, [23]. This is a statistical difference, in the sense that it gives a quantitative value of the difference between two distributions. Formally, given two discrete distributions $P_{1}, P_{2}$ that act on the event and outcome spaces $\Omega, \mathfrak{B}(\Omega)^{6}$ respectively, the KL divergence between them is defined as

$$
D_{\mathrm{KL}}\left(P_{1} \| P_{2}\right):=\sum_{\omega \in \Omega} P_{1}(\omega) \log \left(\frac{P_{1}(\omega)}{P_{2}(\omega)}\right)
$$

If they are assumed to be continuous, then the analogous definition of the KL divergence is given by

$$
D_{\mathrm{KL}}\left(P_{1} \| P_{2}\right):=\int_{\Omega} p(\omega) \log \left(\frac{p_{1}(\omega)}{p_{2}(\omega)}\right)
$$

where $p_{1}, p_{2}$ are the probability density functions of $P_{1}, P_{2}$ respectively.

[^5]TABLE 6.2: Distribution parameters for the random parameters. Beta distribution shape parameters recovered from mean and standard deviation, and then translation maps from $[0,1]$ to the bounded space. Uniform distribution determines the bounds from the given mean and standard deviation.

| Parameter | Distribution | Lower Bound | Upper Bound | Mean | Stdev. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mu_{e}$ | beta | $8 / 6$ | $12 / 6$ | $10 / 6$ | $0.5 / 6$ |
| $\mathbf{u}_{x}$ | beta | 1200 | 1800 | 1700 | 100 |
| $\mu_{e}$ | uniform | $9.13 / 6$ | $10.86 / 6$ | $10 / 6$ | $0.5 / 6$ |
| $\mathbf{u}_{x}$ | uniform | 1527 | 1873 | 1700 | 100 |

The results from these numerical experiments can be seen in Figure 6.3. In these, we examine a 1-D center line of the full 3-D model, running from channel inlet to outlet, of the electric potential $\mathcal{V}$. The random-grid and weights are implemented using Burkhardt's repository [10], while the deterministic solutions are computed with the COMSOL [32] model described in Section 5.1. In Figure 6.3, we see a spatial dependence for the variation of the solution, with higher variances seen at the inlet of the channel. However, despite having the same mean and variance, the shape of the parameter's distribution clearly also had an impact on the distribution shape of the solutions, as seen in the differences plot of Figure 6.3. This is further verified by plotting the distribution of the random process $\mathcal{V}$ for fixed $\mathbf{x} \in D$, i.e. generating a random variable. Assuming that the 'shape' of the solutions is the same shape as the parameters, i.e. a uniform parameter distribution implies a uniform solution distribution, we also calculate the relative entropy between the uniform and beta distributed $\mathcal{V}$. The random variable's distributions can be see in Figure 6.4, while the KL divergence of the distribution at each point are seen in Table 6.3.

Thus, the shape and variance of the random variables have a significant effect on the variance of the solutions, and inclusion of uncertain parameters within the model is necessary for reliable simulations.

TABLE 6.3: Relative entropies of solutions distributions, at fixed $\mathbf{x} \in D$, for assumed parameter distribuions of beta or uniform. Distributions themselves can be seen in Figure 6.4

| $\mathbf{x}$ | $\mathrm{D}_{\mathrm{KL}}$ (beta\\| \|uniform) | $\mathrm{D}_{\mathrm{KL}}$ (uniform\\|beta) |
| :---: | :---: | :---: |
| $(0.011,0.5,0.05)$ | 0.0053 | 0.0152 |
| $(0.035,0.5,0.05)$ | 0.148 | 0.7229 |
| $(0.055,0.5,0.05)$ | 0.0042 | 0.019 |

### 6.2 Uncertain Parameter Estimation

In Section 4.2.2, we stated that the dimension-reduced uncertain parameter estimation problem (DRUID) was a method-stable FD approximation to (UID), the distribution parameter estimation problem. Rather than recovering the complete probability distribution for each random process $Y, P_{Y}: \widetilde{Q}_{Y} \rightarrow \mathbb{R}$, where $Q_{Y}$ is the subspace of $\widetilde{Q}$ associated with $Y$, (DRUID) attempts to recover an M-pole approximation to $P_{Y}$. To do so, we reduced the dimension of the parameter function space, by approximating the continuous functions with polynomials.

We desire to apply the stochastic collocation (SC) method described in the previous section as the uncertain forward solver. To this end, we must satisfy the requirements, namely the finite-dimensional noise assumption. This assumption states that to utilize the SC method, the problem must be described by a finite-number of random variables. By reducing the dimension of (UID), the approximate distribution used in (DRUID) satisfies this requirement. The exact implementation of (DRUID), however, remains unclear.

There are two main obstacles to overcome in this problem: how to choose the sampling $\left\{q_{j}\right\}_{j=1}^{M} q_{j} \in \widetilde{Q}^{N}$, and how to determine the appropriate weights $p_{j}, j=$ $1, \ldots, M$ associated with each. The choice of $p_{j}$ can be done in many ways, using general optimization schemes, such as a local-optimization gradient based search, or a more robust global optimization scheme. Thus, we focus on how to choose the
sampling of $\widetilde{Q}^{N}$, by focusing again on a single random process, and then expanding to include every random parameter. This will guide us in how to choose the weights $\left\{p_{j}\right\}_{j=1}^{M}$.

### 6.2.1 Karhunen-Loève Expansion

In the deterministic parameter estimation setting, the sampling of $\widetilde{Q}$ was done naturally, by using an orthogonal basis of the approximation space, $\widetilde{Q}^{N}$. For the UPE, we again use polynomials to represent the spatial component of the random parameters. However, as these random parameters are viewed as realizations of a random process, we also use a series of independent random variables with mean zero, via a Karhunen-Loève expansion (KLE) 46].

Let $Y$ represent any of the random parameters, mapping from $D \times \Omega \rightarrow \mathbb{R}$ or $\mathbb{R}^{3}$, and let $Q_{Y}$ denote the function space on which $Y$ acts upon, i.e. for fixed $\omega \in \Omega, Y(\omega) \in Q_{Y}$. Recall that $Q_{Y}^{N}$ represents the approximation of the random parameter function space, with $N \in \mathbb{N}$ being the dimension of the approximation. We use the definition of the probability distribution of a random process first introduced in Section 2.4. We now layout some necessary arguments for the KLE of any random parameter. As before, let $\mathbf{x} \in D, \omega \in \Omega$.

First, recall that by definition of random-process, the expected value will be some function in $Q_{Y}$. We let $Y_{\mathcal{M}}(\mathbf{x})$ represent the mean of the random parameter $Y$. Furthermore, any random process inherently has some associated covariance function. The covariance function is a measure of how two random variables change together. In the case of random processes, we view the covariance as a measure of how the random processes change between two different spatial sample points, as for fixed $\mathbf{x} \in D, Y(\mathbf{x})$ is a random variable. Let $\mathcal{C}$ denote the covariance function for the random parameter, i.e. $\mathcal{C}: D \times D \rightarrow \mathbb{R}, \mathcal{C}(\mathbf{t}, \mathbf{s})=\operatorname{Cov}(Y(\mathbf{t}, \omega), Y(\mathbf{s}, \omega))$. Although not necessarily directly observable, the structure of the covariance function for an MHD generator can be estimated, using an ensemble of observations [15]. By structure of
the covariance function, we mean that the eigenfunctions, $\left\{\phi_{j}\right\}_{j=1}^{\infty}$, which satisfy the Frendholm integral equation of the second kind,

$$
\begin{equation*}
\lambda_{n} \phi_{n}(s)=\int_{D} \mathcal{C}(\mathbf{x}, s) \phi_{n}(\mathbf{x}) d \mathbf{x} \tag{6.18}
\end{equation*}
$$

are known. Simply put, assuming that $\mathcal{C}$ is of a known structure is equivalent to assuming that the eigenfunctions are known. Note that unlike with $Y_{\mathcal{M}}, \phi_{j}$ may not be in the deterministic function space for $Y$. We also define $\left\{\psi_{j}\right\}_{j=1}^{\infty}$ to be a set of random variables on $\Omega$ with mean zero and unit-variance, that are uncorrelated with respect to $\mathcal{C}$, i.e. $C\left(\psi_{j}, \psi_{k}\right)=\delta_{j, k}$, where $\delta(j, k)$ denotes the Kronecker-delta function. With these functions defined for the random process $Y$, the KLE is expressed by 55]

$$
\begin{equation*}
Y(\mathbf{x}, \omega)=Y_{M}(\mathbf{x})+\sum_{j=1}^{\infty} \sqrt{\lambda_{j}} \phi_{j}(\mathbf{x}) \psi_{j}(\omega) \tag{6.19}
\end{equation*}
$$

To satisfy the finite-dimensional noise assumption, we truncate the KLE to a finite number of terms. In fact, we truncate the KLE to exactly match the dimension of the approximation space of $\widetilde{Q}_{Y}$, i.e. $N$, as defined in Section 4.2.2. The truncated KLE for $Y$ is given by

$$
\begin{equation*}
Y(\mathbf{x}, \omega) \approx Y_{M}(\mathbf{x})+\sum_{j=1}^{N} \sqrt{\lambda_{j}} \phi_{j}(\mathbf{x}) \psi_{j}(\omega) \tag{6.20}
\end{equation*}
$$

Given that we are approximating our parameters with polynomials in the deterministic setting, the mean can also be expressed as a linear combination of polynomial basis functions. Letting $\left\{r_{k}\right\}_{k=1}^{N}$ be an orthogonal basis for $\mathbb{P}^{N}(D)$, we have that

$$
Y_{M}(\mathbf{x}) \approx \sum_{k=1}^{N} a_{k} r_{k}(\mathbf{x})
$$

for some set $\left\{a_{k}\right.$ : for $\left.k=1, \ldots, N, a_{k} \in \mathbb{R}\right\}$. Thus, we define the FD approximation to $Y$ as

$$
\begin{equation*}
Y^{N}(\mathbf{x}, \omega):=\sum_{k=1}^{N} a_{k} r_{k}(\mathbf{x})+\sum_{j=1}^{N} \sqrt{\lambda_{j}} \phi_{j}(\mathbf{x}) \psi_{j}(\omega) . \tag{6.21}
\end{equation*}
$$

It follows from Theorem 3.1 of Banks [3] that $Y^{N} \rightarrow Y$ as $N \rightarrow \infty$ in the Prokhorov
metrid 7 in the sense that the associated probability distributions converge, i.e. that

$$
P_{Y^{N}} \rightarrow P_{Y} \text { as } N \rightarrow \infty .
$$

Thus, we have defined a series of random processes which approximates the true random process. Now, let $Y_{k}^{N}$ be the sequence whose sum is the series $Y^{N}$. As the sum of random variables equates to the convolution of their probabilities, we let

$$
P_{Y^{N}}=\prod_{k=1}^{N} P_{Y_{k}^{N}}
$$

denote their convolution. Note that in the discrete case, this equates to a product. Furthermore, if each covariance structure is further assumed to generate symmetric distributions, the convolution of discrete distributions simplifies to simply the product of the discrete probability distributions, evaluated at the same $\omega$. For convenience, we will assume symmetry of these independent random variables. Now, further consider the sequence which makes up the series $Y^{N}$, i.e.

$$
Y_{k}^{N}(\mathbf{x}, \omega)=a_{k} r_{k}(\mathbf{x})+\sqrt{\lambda_{k}} \phi_{k}(\mathbf{x}) \psi_{k}(\omega)
$$

Despite this in-depth analysis of the KLE, we have yet to provide direction in choosing the sampling, nor the weights associated. However, the KLE aids is us in separating these two questions, and gives the probability association through the independent random variables $\psi_{k}$. Thus the probability associated with $Y_{k}^{N}$ is uniquely determined by the random variable $\psi_{k}$. As well, it is clear (as $\psi_{k}$ is mean zero, unit-variance), that the mean of $Y_{k}^{N}$ is $a_{k} r_{k}(\mathbf{x})$. Similarly, it follows that the variance, in the sense of the random-process being a $\left(Q_{Y}, \eta_{Y}\right)$-valued random-variable (where $\eta_{Y}$ is the metric on $Q_{Y}$ ), is given by $\lambda_{k}\left(\phi_{k}(\mathbf{x})\right)^{2}$. We now use these to build the associated probability distribution for $Y^{N}$.

Which allows us to take the product of the distributions in each direction to obtain the distribution of $Y$, much the same as we did for the probability on $\widetilde{Q}$ as

[^6]the product of the individual random processes. We let $P_{Y_{k}^{N}}$ denote the $M$-pole distribution for $Y_{k}^{N}$ for $k=1, \ldots, N$. This is of course simply a shifted and scaled version of the random-variable's distribution $P_{\psi_{k}}$. As this distribution is known, under the assumption of some covariance structure, then so is $P_{Y_{k}^{N}}$.

This is an important characteristic of the KLE, and thus we summarize and reiterate for clarity's sake. The KLE separates the deterministic and stochastic domains of any random process. We truncate the KLE to exactly match the order of the approximation to the function space. By doing so, we generate $N$ random processes, whose sum approximates the random process itself, all with known distributions, under the assumption of a known covariance structure. This assumption itself is valid as one can estimate the covariance of the system with an ensemble of samples. Furthermore, as we work with some basis for the approximate function space, we can uniquely determine the sample $Y_{j}^{N}$ for any given sample $\omega_{j}$. In conclusion, finding a solution to (DRUID) is equivalent to determining the coeffecients of the mean in $Q^{N}$, i.e. $\left\{a_{k}\right\}_{k=0}^{N}$ and the eigenvalues of the covariance matrix, i.e. $\left\{\lambda_{k}\right\}_{k=0}^{N}$. Now, noting that these distributions are assumed to be discrete and symmetric, it of course follows that

$$
P_{Y^{N}}=\prod_{k=1}^{N} P_{Y_{k}^{N}} .
$$

Extending this to the every random parameter in the system, we have

$$
p_{j}=\prod_{Y \in q} P_{Y^{N}},
$$

where of course $q=\left\{\mathbf{u}, \sigma, \mu_{e}, \mu_{i}\right\}$. For a more robust discussion on the KLE, we refer the reader to [55].

In summary, as the KLE has separated the stochastic and spatial dependence of the random-process, the weight $p_{j}$ can be uniquely determined from the real-valued random variable $\psi_{j}$. Thus, to generate each weight $p_{j}$, we simply use the sample point $\omega_{j}$, be it in $[-1,1]$ or in $\mathbb{R}$, to determine the probability associated with $Y_{k}^{N}$, and then take the scalar product to build the probability of the associated with the
function $\sum_{k=1}^{N} a_{k} r_{k}(\mathbf{x})+\sqrt{\lambda_{k}} \phi_{k}(\mathbf{x})$. Taking the product as above, we have generated an implementable algorithm, concurrent with stochastic collocation, to perform the uncertain parameter estimation. We now go into detail in how this is implemented numerically.

### 6.2.2 Data Fabrication

In lieu of experimental data, we must simulate some data for the parameter estimation scheme. To do so, we assign a true q value, and simulate data using the COMSOL deterministic model previously mentioned. To avoid an inverse crime, as we are using the same model for the data and forward solver, we solve for the true solution values on a different mesh for $D$. Furthermore, to simulate measurement error, we corrupt the data with random Gaussian noise of some given level. Thus, for true fabricated data $\mathcal{U}^{\prime} \in \mathbb{R}^{K}$, where $K \in \mathbb{N}$ is the number of sample points in $D$, as above, we have

$$
\mathcal{U}=\mathcal{U}^{\prime}+\eta_{L}\left\|\mathcal{U}^{\prime}\right\|_{l^{2}} M
$$

where $M$ is a random-Gaussian vector of the same length as $\mathcal{U}$, with mean zero and variance 1 , and $\eta_{L}$ is the assigned noise-level of the random noise. Using this, we employ the lsqnonlin MATLAB function to search for a minimum, over the $\left\{a_{k}, \lambda_{k}\right\}_{k=1}^{M}$, to (DRUID). Recall from Section 4.2 that we also do not expect to have samples corresponding to every point in the domain. Thus, for the numerical sections below, we distinguish between the impractical case of knowing the full domain as 'full', while the 'partial' data is assumed to be a single line through the resistors. For an exact location of this line, we fix $y=-0.1, z=0.05$, which happens to be in the center of the resistors, as can be seen in Section 2.1. To see some examples of simulated data, corresponding to the 'partial data', and under varying noise levels, see Figure 6.5.

### 6.2.3 Numerical Scheme

In order to implement the uncertain parameter estimation scheme, we will need several tools. First and foremost, their must be some deterministic forward solver, that gives the solution to (4.2) for some given set of parameter values. For this, we use the numerical model described in Section 5.1, which is implemented in COMSOL [32]. We also must be able to store and average the solutions to some capacity. For the algorithm choice of averaging, we use the stochastic collocation method outlines in Section 6.1.2. To store the solutions, we use the livelink compatability with Matlab [28]. Finally, for the choice of minimization, we use the native Matlab software lsqnonlin [29]. We now outline the actual numerical scheme, including defining our approximation spaces.

Up to this point, we have kept the approximation space general. We now discuss a numerical implementation of this method, and it's results. To begin, we make several further simplifications to the unknown model, as this serves as a numerical demonstration of the feasibility of the method, and we will not attempt to investigate the computational requirements.

First and foremost, we reduce the amount of uncertain parameters, to only $\mu_{e}$. Secondly, to simplify the model further, we assume that the ion-slip parameter is negligible, i.e. $\mu_{i} \approx 0$. Third, we assume that mean of $\mu_{e}$ is symmetric across the channel in the $y$-direction, and uniform in all others. We also assume an exponential covariance structure with a correlation length of 10 . This gives that the eigenfunctions of the covariance are given by

$$
\phi_{k}(y):= \begin{cases}\sin \left(w_{i}(y-1 / 2)\right) / \sqrt{1 / 2-\frac{\sin \left(w_{i}\right)}{2 w_{i}}} & \text { for } i \text { even } \\ \cos \left(v_{i}(y-1 / 2)\right) / \sqrt{1 / 2-\frac{\sin \left(v_{i}\right)}{2 v_{i}}} & \text { for } i \text { odd }\end{cases}
$$

where $w_{i}, v_{i}$ are solutions to the transcendental equations

$$
\begin{cases}10 w_{i}+\tan \left(w_{i} / 2\right)=0 & \text { for } i \text { even } \\ 1-10 v_{i} \tan \left(v_{i} / 2\right)=0 & \text { for } i \text { odd }\end{cases}
$$

Based on the exponential decay of exponential covariance functions, as can be seen in Dongbin [55] Figure 4.2, we choose to use only quadratic polynomials to approximation $Q_{\mu_{e}}$. Thus, we use the polynomial basis of $\left\{1,(y+w / 2),(y+w / 2)^{2}\right\}$, where $w$ is the width of the channel in the $y$-direction. We choose the basis $(y+w / 2)$ so that the function operates on a domain of $[-w / 2, w / 2]$, as required to apply the exponential covariance structure. Note now that we shift the parameter index to correspond with the traditional polynomial order index, namely $k=k-1$.

Under these assumptions, we have for $\mathbf{x} \in D, \omega \in \Omega$, the electron mobility satisfies
$\mu_{e}=a_{2}(y+w / 2)^{2}+\sqrt{\lambda_{2}} \phi_{2}(\mathbf{x}) \psi_{2}(\omega)+a_{1}(y+w / 2)+\sqrt{\lambda_{1}} \phi_{1}(\mathbf{x}) \psi_{1}(\omega)+a_{0}+\sqrt{\lambda_{0}} \phi_{0}(\mathbf{x}) \psi_{0}(\omega)$.
Examples of the covariance functions described by $\phi_{k}$ above, the individual random processes, $\mu_{e, k}^{2}, k=0,1,2$, and a sample electron mobility function, see Figure 6.6. To see examples of the collocation sample points used, i.e. examples of the functions $\mu_{e}$, under the Fejér random grid and quadratic and beta distributions, see Figure 6.7. Finally, for an example of the deterministic solution for one of the collocation points, see Figure 6.8

Following the ideas from Section 6.2.1, we attempt to recover the optimal $a_{0}, \lambda_{0}, a_{1}, \lambda_{1}, a_{2}, \lambda_{2}$ that describes some 'true' set of deterministic coefficients and stochastic eigenvalues. We must also choose how we sample the random domain. To do this, we follow the work described in [9], using a Fejér grid of level $=3$. Note that the Fejèr grid is chosen to have no support on the boundaries, as the support of the beta distribution is centered around the mean, typically with little variance. Although there is no function to return the number of collocation points, we let $M_{Y, k}$ denote the number of sample points for $\mu_{e}$ in the $k^{\text {th }}$ direction. Thus, the number of samples taken in the random direction, $M$, is given by

$$
M=\sum_{Y \in\left\{\sigma, \mu_{e}\right\}} \sum_{k=1}^{N} M_{Y, k} .
$$

TABLE 6.4: Designated 'true' deterministic coeffecients and eigenvalues, for use in the numerical implementation of (DRUID)

| $a_{0}$ | $\lambda_{0}$ | $a_{1}$ | $\lambda_{1}$ | $a_{2}$ | $\lambda_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $10 / 6$ | 0.5 | 0 | 0.02 | 0.5 | 0.016 |

TABLE 6.5: Shape parameters for the random variables with an assumed beta distribution function.

| $Y_{0}$ | $Y_{1}$ | $Y_{2}$ |
| :---: | :---: | :---: |
| $(2,2)$ | $(3,1)$ | $(4,2)$ |

To calculate the expected value of the solution, we use the stochastic collocation averaging technique laid out in the previous section. To generate the artificial data, we set the 'true' deterministic coeffecients and eigenvalues to be as described in Table 6.4. The eigenfunctions can be seen in Figure 6.6. This figure also displays the terms of the KLE corresponding to each random process in the sum, as well as a single realization, the mean, of the quadratic $\mu_{e}$.

Figure 6.7 demonstrates every sample $\mu_{e}$, under the Fejér grid of level 3, i.e. it is a plot of every sample $\mu_{e}$, under the values described in Table 6.4, used in computing the SC average of the solutions. Finally, Figure 6.8 is the deterministic solution for the mean of the KLE for the quadratic $\mu_{e}$. Compare this directly with the deterministic examples in Figure 6.2 as well as those in Section 5.1.4.

We now outline the steps for the numerical solver. To begin, we generate 'true data,' by solving (4.2) using a spatially-fine grid using the numerical model. Examples of the spatial mesh can be seen in Figure 5.2. Also, note that we use the flexible generalized minimum residual method, integrated within COMSOL, to solve the numerical equivalent of the kinematic MHD equations. Furthermore, we make use of the SC method described in the previous section to include the distribution of each $\mu_{e, k}^{2}$ in our data. The collocation method returns the expected value and variance of our
solutions. Examples of some solutions distributions are seen above.
Thus, we have generated a 'true distribution' of data. However, it is unrealistic to expect that data gathered from an operational MHD system to be completely accurate, as all measurement instruments are subject to noise. Therefore, we corrupt the 'true data with noise, adding a vector of samples from a Gaussian distribution of mean-zero, variance given by a scalar factor of the norms of the solutions. We denote this scalar value the noise-level, and vary it to demonstrate the uncertain parameter recovery's sensitivity to noise. Furthermore, it is also unrealistic to expect to gather data from every point in the domain Thus, we examine the difference between the distribution recovery across the whole domain, versus a more realistic subset located solely within a single resistor. 9 However, in all cases, we refer to the corrupted data as $\mathcal{U}$.

With this fabricated data available, we then can implement any optimization scheme, under the guise of (DRUID), i.e. minimizing the difference between the expected value of the solutions, given some $\mu_{e}$, and the fabricated data. As mentioned prevoiusly, we use the native Matlab software lsqnonlin, specifically the bounded form of trust-region-reflective algorithm, as outlined in [29]. This is a local minimization method, and thus requires an initial guess to the distributions of the random variables. To demonstrate the robustness of the scheme, we give an initial error somewhere between $9 \%$ and $11 \%$ from the true values for both the mean and standard deviation. We now examine the results from this numerical implementation.

### 6.2.4 Numerical Experiments

We begin with the ideal investigation, the the full-domain problem. In Figure 6.9. we demonstrate the results for the uncertain parameter estimations scheme for varying noise levels. Note that we are under the assumption that the shape of the

[^7]covariance function has corresponding random variables with beta distributions, i.e. that $\psi \in \beta(a, b)$, for some $a, b$ shape parameters, and hence the shape of the recovered distributions. These different plots correspond to the random variable of the recovered distribution of each term $\left\{Y_{k}\right\}_{k=0}^{2}$, at the center of the channel. The corresponding shape parameters can be seen in Table 6.5. The results are better than expected. Under the assumption of knowing the entire domain, the recovery of the distributions under noise was remarkable. The error between the recovered distribution and true distribution is negligible. Noting this, we move onto a numerical demonstration of a more realistic example, when the solutions are known in only part of the domain.

It could be expected to know the electric current density and electric potential within the resistor, as this is external to the MHD channel itself! Thus, this numerical example is much more telling of the effectiveness of the parameter estimation scheme we have implemented thus far. We now present analogous plots to the full domain problem, with the results seen in Figure 6.10. We inspect the distribution of the random variables for each summand of the random process, $\left\{Y_{k}\right\}_{k=0}^{2}$. The results again are better than expected. The error for the constant term is again negligible, showing little sensitivity to the noise level. Interestingly, the insensitivity of the parameter recovery scheme applies to the quadratic term as well, although the recovery of the distributions is not as precise. This should be expected, as the 'impact' on the solutions of the quadratic term is less than that of the constant term, as seen in Figure 6.6

Finally, we inspect the results for the uniform partial domain uncertain parameter estimation scheme. These results are noted in Figure 6.11. It is immediately apparent that the recovery in the uniform case does not perform as well. In both the constant and quadratic terms, recovery of the mean (noted with an $\times$ ) is mildly sensitive to noise when compared to the recovery of the standard deviation with noise. Clearly, the estimation scheme here recovers more narrow distributions than the true case, which is also seen in the quadratic terms of the partial beta recoveries. However,

TABLE 6.6: Relative entropy for the average recovered beta distributions and the true distributions for the random parameters $Y_{0}, Y_{2}$ in the KLE for the UPE. The distributions for $Y_{0}, Y_{2}$, which were used to calculate these values can be seen in Figure 6.10.

| Noise Level | $D_{\mathrm{KL}}$ (rec. \||true), $Y_{0}$ | $D_{\mathrm{KL}}$ (true\\|rec.), $Y_{2}$ |
| :---: | :---: | :---: |
| 0.00 | 0.0009 | 6.63 |
| 0.05 | 0.0195 | 5.37 |
| 0.25 | 0.0563 | 6.64 |

this effect is noted in the uniform case for both the constant and quadratic terms. This is perhaps best explained when examining the results from the forward problem. Figure 6.4 displays the distribution of the solution values of the electric potential for both the beta distribution and uniform distribution at three select points in the domain. It follows that the distribution of the electric potential is less sensitive to the variance of the uniform distribution, as compared to the beta distribution, as the solution's distribution is narrower and with less variance, despite the two parameters' distributions having the same variance. Thus, the higher error between the true and recovered distributions are not surprising.

To measure these differences quantitatively, we again employ the $K L$ divergence, i.e. relative entropy, to give some statistical distance between the two distributions. These results are displayed in Table 6.6 for the partial-domain beta-distribution recovery, and Table 6.7 for the partial-domain uniform-distribution recovery. These values confirm the qualitative results from inspecting the distributions, that the beta distribution recovery outperformed that of the uniform, and that, in general, the $Y_{0}$ recovery outperformed that of $Y_{2}$. The most notable exception to the last remark is the uniform, noise-level 0.25 result. Further testing is required to validate these initial results.

Overall, the introduction of uncertainty to the deterministic problems was a success. In the forward problem, we demonstrated not only that the expected solutions

TABLE 6.7: Relative entropy for the average recovered uniform distributions and the true distributions for the random parameters $Y_{0}, Y_{2}$ in the KLE for the UPE. The distributions for $Y_{0}, Y_{2}$, which were used to calculate these values can be seen in Figure 6.10.

| Noise Level | $D_{\mathrm{KL}}($ rec. $\\|$ true $), Y_{0}$ | $D_{\mathrm{KL}}$ (true $\\|$ rec.),$Y_{2}$ |
| :---: | :---: | :---: |
| 0.00 | 9.81 | 21.4 |
| 0.05 | 6.79 | 20.67 |
| 0.25 | 23.18 | 26.45 |

differed significantly from deterministic solutions, we also demonstrated that the shape of the distribution is a factor in the shape of the solutions' distributions. This further confirms the need of accurate modeling and inclusion of uncertainty in any MHD model. The same can be said for the inverse problem. Comparing these results under noise to those of the deterministic equivalent, we can see that the expected recovery contains much more usable information. Finally, the shape of the distributions also played a critical role in the effectiveness of the recovery scheme.

### 6.3 Notation

### 6.3.1 Stochastic Collocation

| Term | Definition | Usage or Notes |
| :---: | :---: | :---: |
| Random | $\sigma, \mu_{e}, \mu_{i}, \mathbf{u}$ |  |
| Parameters $\left\{m_{k}\right\}$ | Collection of all random variables in the system | Assume all uniform on $[0,1]$ |
| $M$ $\Gamma_{k}$ | Number of random directions $m_{k}(\Omega)$ |  |
| $\Gamma$ | $\Pi m_{k}(\Omega)$ | Product of images of events under random variables |
| $\rho$ | Joint probability density for Prob. Space ( $\Gamma, H, \rho$ ) |  |
| $\widetilde{V} / \widetilde{W}$ | Random solution spaces for new probability space |  |
| $V_{h}, W_{h}$ | Spatial-FD representations of V/W | The $h$ dependence comes from the mesh. |
| $t_{h}$ | Spatial mesh used in SpatialFD reps |  |
| $P_{N_{k}}$ | Span of polynomials in direction $\Gamma_{k}$ up to order $N_{k}$ | Approximation to the random function space $L_{\rho}^{2}\left(\Gamma_{k}\right)$ |
| $r_{k}^{j}$ | Chebyshev polynomial of order $j$, acting on $\Gamma_{k}$ |  |



| Term | Definition | Usage or Notes |
| :---: | :---: | :---: |
| $\mathcal{I}_{V, N} / \mathcal{I}_{W, N}$ | Interpolation operators, map from continuous functions on $\Gamma \times$ the app. subspace. ( $V$ or $W$ ) |  |
| $w_{j}^{m_{j}}$ | Weights of polynomial of order $m_{j}$ on $\gamma_{j} \mathbf{g}(\mathbf{x}, y)$ | $\underline{\underline{\sigma}}(\mathbf{x}, y)(\mathbf{u}(\mathbf{x}, y) \times \mathbf{B}(\mathbf{x}))$ |
| $\partial_{n}^{k}$ | $\frac{\partial^{k}}{\partial y_{n}^{k}}$ | Notational convenience only |
| $\mathcal{A} / \mathcal{B}$ | Deterministic equivalent to $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}$ |  |
| $G^{k}(\psi)$ | $\int_{D} \partial_{y_{n}}^{k} \mathbf{g} \cdot \nabla \psi$ |  |
| C | $\frac{1}{a b}\left(\\|\mathcal{A}\\|_{\mathcal{L}\left(V, V^{\prime}\right)}+a\right)$ |  |
| $\alpha_{1, k} /$ | Constants used in Lemma | $k$ corresponds to the $k^{\text {th }}$ |
| $\alpha_{2, k} /$ | 6.1.6 to ensure conver- | derivative of $\mathbf{g}, \underline{\bar{\sigma}}^{-1}, n$ de- |
| $\gamma_{n}$ | gence | pendence is on the direction $\Gamma_{n}$ |
| $C_{1, k} / C_{2, k}$ | Constants used for bounding derivatives of $\mathbf{J}_{i}, \mathcal{V}$ |  |
| $C_{0}$ | $C\\|\mathbf{g}\\|_{W}$ | Just for better notation really |
| $\Sigma\left(\Gamma_{k}, \gamma_{k}\right)$ | $\begin{aligned} & z \quad y_{1}+i y_{2} \quad \epsilon \\ & \Sigma\left(\Gamma_{k}, \gamma_{k}\right) \Longleftrightarrow y_{1} \epsilon \\ & \Gamma_{k},\left\|y_{2}\right\| \leq \gamma_{k} \end{aligned}$ | Region of the complex plane, extending from the subset of the $\mathbb{R}, \Gamma_{k}$ |


| Term | Definition | Usage or Notes |
| :---: | :---: | :---: |
| $y_{n}^{*}$ | Set of values for $y$ in all directions but $y_{n}$ |  |
| $\mathbf{J}_{i}\left(z, y_{n}^{*}, \mathbf{x}\right) /$ | Analytic extensions of |  |
| $\mathcal{V}\left(z, y_{n}^{*}, \mathbf{x}\right)$ | $\mathbf{J}_{i}, \mathcal{V}$ in the complex region $\Sigma\left(\Gamma_{n}, \gamma_{n}\right)$ |  |
| $H$ | Any general Banach function space |  |
| $\mathcal{I}_{H}^{N_{n}}$ | Polynomial interpolation of order $N_{n}$, using sparse grid, of a function continuous on $\Gamma_{n}$ and in the function space $H$. |  |
| $\Gamma_{k^{*}}$ | Tensor product space of all directions except $\Gamma_{k}$ | Consistent with Babuska notation |
| ${ }^{\circ}{ }^{*}$ | General format for everything but the $k^{\text {th }}$ direction. |  |
| $\gamma$ | $\left[\gamma_{1}, \ldots, \gamma_{M}\right]$ | Collection of the decay constants from regularity section |
| $\Sigma(\Gamma, \gamma)$ | $\prod_{j=1}^{M} \Sigma\left(\Gamma_{j}, \gamma_{j}\right)$ | This is the tensor product of all the areas of the analytic extensions. |

### 6.3.2 Uncertain Parameter Estimation

| Term | Definition | Usage or Notes |
| :---: | :---: | :---: |
| $K_{V_{k^{*}}}$ | Constants from Lemma $6.1 .5$ | used in Lemma 6.1.6 |
| $K_{1}$ | $\max \left(K_{j}\right)$ |  |
| $Y$ | Any random process |  |
| $Q_{Y}$ | Function space which $Y$ acts upon |  |
| $\mathcal{C}(\mathrm{t}, \mathrm{s})$ | $\operatorname{Cov}(Y(\mathbf{t}, \omega), Y(\mathbf{s}, \omega)$ | Covariance function of the random process |
| $\left\{\lambda_{j}, \phi_{j}\right\}_{j=1}^{\infty}$ | Eigenvalues/Eigenfunctions (eigenpairs) of $\mathcal{C}$ |  |
| $\left\{\psi_{j}\right\}_{j=1}^{\infty}$ | Uncorrelated, mean-zero, unit-variance $\quad \mathbb{R}$-valued random varibales |  |
| $Y^{N}(\mathrm{x}, \omega)$ | Finite-dimensional approximation to the random process. |  |
| $Y_{k}^{N}(\mathbf{x}, \omega)$ | Random process whose sum from $k=1, \ldots, N$ is $Y^{N}$ |  |
| $P_{Y^{N}}$ | Probability distribution of $Y^{N}$. |  |
| U | Corrupted data |  |
| $\mathcal{U}^{\prime}$ | 'True' (simulated) data |  |



FIGURE 6.2: (a) is the deterministic solution of the means, (b) is the expected solution computed using stochastic collocation, while (c) is the difference between the first two plots. All are samples corresponding to the middle (in $z$ ) of a segmented Faraday channel, under the distributions described in Section 6.1.5.2.


FIGURE 6.3: Results from numerical experiments. (Left) beta distributions. Exp. $\mathcal{V}$ is the expected value of $\mathcal{V}$, solved for using the SC method at a level of 5 . Det. $\mathcal{V}$ is the deterministic $\mathcal{V}$, solved for with the means of each random parameter. (Inset plot) Relative error defined as the normalized difference between the Exp. $\mathcal{V}$ and Det. $\mathcal{V}$. (Right) Difference between Exp. $\mathcal{V}$ and Det. $\mathcal{V}$, with variances included. Solid line indicates uniform distribution results, dashed line indicates beta distribution results.


FIGURE 6.4: Distributions of the random variable $\mathcal{V}\left(\mathrm{x}^{*}, \omega\right)$ for $\mathrm{x}^{*}$ as in Table 6.3. labelled accordingly. .


FIGURE 6.5: Fabricated Data examples for the UPE. Each noise level is marked by differing colors.


FIGURE 6.6: (Left) Eigenfunctions for exponential covariance, i.e. $\phi_{k}$, for $k=0,1,2$. (Right) Individual random processes and quadratic $\mu_{e}$.


FIGURE 6.7: Collocation sample points of the $\mu_{e}$ under the KLE, with an Fejér grid of level= 3 .


FIGURE 6.8: Determinsitic solutions, (colored) $\mathcal{V}$, (streamlines) $\mathbf{J}$, for the expected value of $\mu_{e}$, under the quadratic KLE, with distributions as defined in Table 6.4.


FIGURE 6.9: Demonstrations of Distributional recovery, at $\mathbf{x}=(0.05,0.5,0.05)$, for quadratic $\mu_{e}$, with an assumed beta distribution covariance structure, and with the full domain available for data. Noise-levels of corrupted data, from left to right, are $0,0.05,0.25$


FIGURE 6.10: Demonstrations of Distributional recovery, at $\mathbf{x}=(0.05,0.5,0.05)$, for quadratic $\mu_{e}$, with an assumed beta distribution covariance structure, and with the partial domain available for data. Here $Y=\mu_{e}$, (left) is the constant term, and (right) is the quadratic term.


FIGURE 6.11: Demonstrations of Distributional recovery, at $\mathbf{x}=(0.05,0.5,0.05)$, for quadratic $\mu_{e}$, with an assumed uniform distribution covariance structure, and with the partial domain available for data. Here $Y=\mu_{e}$, (left) is the constant term, and (right) is the quadratic term.

## 7 Two Stage Optimization

We now reintroduce the goal of this dissertation, an investigation into the feasibility of real-time optimization for an MHD generator. Other papers have considered optimization of design of the generator [18, 41, 56, 51]. The wide breadth of these optimization investigations are due to the fact that the optimal operating conditions of this generator depend upon many variables. There are many different optimization parameters one could consider under this situation. For instance, there are geometrical considerations, such as the angle of the walls, as investigated in [41], or some optimization of pressure and temperature based on the geometry of the nozzle accelerator [56]. There is also the choice of fluid to consider, as can be seen in [51]. Another, to be considered in future work, is the angle between paired electrodes within a diagonal Faraday geometry. For the purposes of this manuscript, wefocus on one optimization parameters, which greatly affects the efficiency of an MHD generator: the load on the channel. We choose this with the understanding that the inclusion of other optimization parameters would be developed further in the real-time optimization of an MHD generator. We now discuss how exactly the sensitivity of the power out to this optimization parameter, and further how the sensitive the optimization parameter is to the given state.

### 7.1 Optimal load

The idea of an optimal load is not new. Consider, for instance, any of the ideal pwoer equations presented in Section 5.1.3. Both took the form

$$
P=f(\text { geom }) K(1-K),
$$

where $f$ is the appropriately defined function for the given geometry geom. Note that this $f$ is somewhat simplistic, but can be found in the Section 5.1.3, and is also a function of the random parameters. Examining this, it is easily seen that the power
is a quadratic function of the load factor, with a maximum at $K=0.5$.
However, this load factor is inherently a theoretical number, as it is not feasible to know a-priori the 'true' max load of a generator, for a given set of operating conditions. Thus, choosing the optimal load value is not simple. Recall here that we replicate the load on the channel with a resistor of some given resistance and periodic boundary conditions. Thus, choosing the optimal load on the channel is equivalent to choosing the resistance of the resistor. For demonstration purposes, it is sufficient to include a resistance range of $10^{-2} \Omega$ to $10^{2} \Omega$, based on the model validation of Section 5.1 .3

It must be made known, however, that there is an obvious coupling between the optimization parameters and state parameters for the system, not just in one direction. As the velocity, and subsequently the other state parameters, are affected by a change in the operating conditions, there would be feedback within the system from a shift in the optimization parameters. This would result in a 'new,' system, implying a new optimal load and geometry. However, for the sake of computational complexity, we have thus far neglected this further coupling of the system by prescribing the fluid flow. We continue to do so in this optimization section, under the pretense that this is only one iteration in the optimization of power out from a Faraday MHD generator.

### 7.2 Sensitivity Experiments

We now move onto a numerical demonstration of the sensitivity of the powerout to these two optimization variables. First, we clarify the goal of the optimization scheme. As mentioned in the model validation section of Chapter 5, the electric power within the channel is defined as the dot-product of the current density, $\mathbf{J}$, and electric field, $\mathbf{E}$. The same holds true for the resistor. As this is representative of the load, the more power that remains within the resistor is the same power that the load would
receive. To this end, we define the 'power-out' of the MHD generator to be

$$
\begin{equation*}
\int_{\text {Resistors }}-\mathbf{J} \cdot \mathbf{E} d \mathbf{x} \tag{7.1}
\end{equation*}
$$

where Resistors refers to the section of the geometry corresponding to the resistors in the segmented Faraday geometry, Figure 2.3. We approximate this integral using standard numerical integration techniques [24]. To approximate the expected power out, we simply apply the same stochastic collocation procedure as with the uncertain forward problem of Chapter 6. For the entirety of this section, we assume a Fejér grid of level 3 .

It follows from the numerical demonstrations of the previous two chapters that the power out of the generator is sensitive to the state of the system, in both a deterministic and distributional sense. However, we have not yet shown that the optimal resistance is sensitive to the state of the system, in either sense! To verify this sensitivity, we use the simplified random model presented in the uncertain parameter estimation scheme, Section 6.2. Namely, this implies that the only random parameter within the system is the electron mobility, which we have estimated with a quadratic KL expansion. Thus, we use the beta distribution from the uncertain parameter estimation problem, as described in Table 6.4, acting as if these were the recovered distributions. For the random mesh, we again use a Fejèr grid with a given level of 3 , which for 3 random directions, results in 251 nodes 10 Under these conditions, a colormap of the power as a function of the order of the resistance and the electron mobility is shown in Figure 7.1. Note that here, the horizontal axis corresponds to the mean value of $\mu_{e}$ on $D$, while the vertical axis is the order of the resistance, i.e. if we denote this axis $y$,

$$
\text { Resistance }=5 \cdot 10^{y} \text {. }
$$

With this coarse of a mesh however, it is difficult to see a great sensitivity of the optimal resistance to the electron mobility. Thus, we inspect a second figure, the

[^8]

FIGURE 7.1: Expected power for system with beta distribution of constant $\mu_{e}$.


FIGURE 7.2: Expected load factor for system with beta distribution of constant $\mu_{e}$.
sensitivity of the load factor, $K$, on the optimal resistance order. The optimal power out corresponds directly to the load factor of 0.5 . A colormap of the load factor is shown in Figure 7.2, with identical axes to Figure 7.1. It is immediately ascertained from this figure that the load factor becomes much less sensitive to resistance as the electron mobility increases. As well, the optimal load factor values clearly change as a function of the hall parameter, and thus we continue to use this as the representative random parameter in the optimization scheme.

TABLE 7.1: Results from the two stage optimization problem for a variety of levels, under the assumption that the random processes are beta distributions, and that the UPE was peformed on the partial grid. Results from the UPE that correlate to this run are seen in Figure 6.10.

| Noise Level | Optimal Resistance <br> of Resistor $[\Omega]$ | Expected Average <br> Resistor Power $[W]$ |
| :---: | :---: | :---: |
| 0 | 12.37 | 9.85 E 6 |
| 0.05 | 12.31 | 9.86 E 6 |
| 0.25 | 12.34 | 9.85 E 6 |

### 7.3 Optimization Method

We now move onto the demonstration of feasibility of optimization. We do so by focusing on the second of the 'two stage optimization cycle,' i.e. finding the optimal resistance of the resistor for some given fabricated data, in order to diminish the computational complexity. Thus, using the results from the partial domain parameter estimation problem for both distributions and all noise levels, we attempt to maximize the expected power out. To calculate the expected power out, we using the stochastic collocation method described in Section 6.1, as well as numerical integration within COMSOL to calculate the power within the resistors, as described in (7.1) for each deterministic run. To search through the optimal parameter space, we again make use of the trust-region algorithm of lsqnonlin [29], as done for the UPE. The results of this search for recovered beta distributions at each noise level are summarized in Table 7.1, while the analogous for the uniform distributions are in Table 7.2.

The results are as expected. As the uncertain parameter estimation scheme proved very effective at all three noise levels, the optimal resistance is similar among them, for both distributions. As well, it is notable that the optimal resistance seems less sensitive to the shape of the distributions than the kinematic MHD equations were, as can be seen in Figure 6.3. To test the sensitivity of the optimization scheme with respect to the variance of the function, the eigenvalues were all multiplied by

TABLE 7.2: Results from the two stage optimization problem for a variety of levels, under the assumption that the random processes are uniform distributions, and that the UPE was peformed on the partial grid. Results from the UPE that correlate to this run are seen in Figure 6.11.

| Noise Level | Optimal Resistance <br> of Resistor $[\Omega]$ | Expected Average <br> Resistor Power $[W]$ |
| :---: | :---: | :---: |
| 0 | 12.28 | 9.85 E 6 |
| 0.05 | 12.14 | 9.87 E 6 |
| 0.25 | 12.17 | 1.02 E 7 |

10, which corresponds to a new covariance function. Using this new distribution, the optimization scheme was run again. However, we did not recover the random parameters, but used the true values, i.e. only performing the second step of the optimization scheme. Doing so for a uniform distribution results in an optimal resistance of $12.34[\Omega$ ], only a $0.73 \%$ change from the uniform optimal resistance for the original distribution. Performing the same steps for a beta distributions resulted in an optimal resistance of $12.50[\Omega]$, which is roughly a $1 \%$ change. Thus, as an optimization variable, the load placed on the channel seems insensitive to uncertainty within the electron mobility. This is expected, given that it is insensitive to the electron mobility as a scalar term, as seen in Figure 7.1.

## 8 Conclusion

### 8.1 Conclusions

In this work, we developed several tools and techniques to apply to the real-time optimization of an MHD generator. While theoretical in nature, we focused our efforts on addressing two major concerns; model accuracy and the feasibility. These both are crucial in the determination of optimal operating conditions, and thus optimal power.

To address the model accuracy, several steps were taken. Firstly, we introduced how to include the ion-slip parameter, a term which characterizes the sub-optimal material interactions between the fluid and electromagnetic fields, within the kinematic MHD model. It was shown that the procedure for including this still results in a well-posed system. More notably was that the generalized Ohm's law resulted in an invertible matrix when non-negligible ion slip was assumed, and thus the previously established BBK theorem could be applied. We then turned to developing a deterministic parameter estimation scheme, investigating the approximations required to generate a numerically implementable algorithm. We showed that these approximations resulted in a function space parameter estimation convergent scheme, implying that we could recover not just a scalar parameter from the system, but a more realistic functional parameter set.

We also discussed the inclusion of uncertainty within the theoretical MHD framework. This uncertainty was introduced through the parameters, changing their definition to include dependence on some random domain. In turn, we viewed them not as functions, but as random variables with functional observations, i.e. random processes. We extended the well-posedness arguments of the deterministic equivalent to the now stochastic system, demonstrating again that we the forward problem was well-posed, and that the finite-dimensional approximation to the inverse problem method stable. We then turned to numerical implementations to further verify the newly developed theory.

For the numerical implementation, we made use of two different software, COMSOL as the deterministic solver, and Matlab as the 'controlling' software, handling any optimization and averaging required. COMSOL initially was alluring, due to the automated coupling of complex multi-physics systems. We also developed our numerical theory under the pretense of a mixed-Poisson system of electromagnetics, which the AC/DC module of COMSOL was set-up to use. Thus, implementation was straight forward. We then verified the numerical implementation under the assumption of two different Faraday geometries, the continuous and the segmented. We developed new ideal power equations for this model verification, again introducing the concept of the ion-slip parameter into previous theory regarding MHD generators. We further verified these new equations, using our numerical model. Under the deterministic scheme, we also implemented a numerical method for recovering parameters from simulated 'true' data. Furthermore, the results from these numerical tests again confirmed the need for uncertainty to be included, as recovery was not only sensitive to noise, but also asymmetric with respect to expected error.

To include the uncertainty of the parameters within the numerical method for the forward problem, we introduced stochastic collocation. As shown, this numerical method is an efficient way of computing the the discrete distributions of the solutions. However, in order to apply this method, the forcing function, i.e. the Lorenz force and random parameters, must satisfy some regularity properties. We then showed that this numerical method will converge to the true distributions as the approximations approach infinity, with an error analysis of the method. The numerical demonstrations confirmed the effectiveness of the method, showing convergence to a high-order approximation. We also explored the impacts of including uncertainty within the system, comparing the expected value of the solutions to some deterministic equivalent. Furthermore, we explored the impact of the shape of the distributions discretely, comparing the solutions' distributions both qualitatively and quantitatively, utilizing the Kulback-Leibler divergence, i.e. relative entropy, to measure the statistical differ-
ence between these. We then moved onto a numerical demonstration of the uncertain parameter estimation scheme, the first step in the two stage optimization method.

To validate the uncertain inverse problem theory developed in Chapter 4. we first developed further approximation theory to implement the problem numerically. This was done mainly in the way of the Karhunen-Loève expansion. Through this, and some assumptions regarding the covariance structure, we demonstrated that recovering the discrete distribution was analogous to recovering the coefficients of the mean and eigenvalues of the eigenfunctions of the covariance. Thus, we implemented a numerical scheme to recover these scalar values, in lieu of recovering the distribution values themselves. We then again qualitatively and quantitatively compared the recovered solution distributions, using again the relative entropy for the quantitative difference, and plots of the recovered distributions at a fixed $\mathbf{x} \in D$. In this, it is clear that the recovery of the distributions under noise can be expected to perform well, but better with the beta distributions over the uniform. This of course demonstrates some sensitivity of the two stage optimization problem to the shape of the distributions.

We concluded this discussion with an investigation into the two stage optimization problem. Although there are many different optimization variables to choose from, we focused solely on the load being placed on the channel. We demonstrate that the power out was clearly sensitive to this, as to be concluded from the ideal power equations as well. Finally, we performed the full two stage optimization scheme, demonstrating that from data, it is feasible to obtain an optimal load to place on the generator.

### 8.2 Future Work

There are multiple next steps available in regards to the demonstration of feasibility of the optimization of an MHD generator. Completing the optimization loop is notably one, i.e. introducing some feedback from the optimization parameters to the
uncertain parameters. As well, exploring the sensitivity of the optimization parameters to the shape of the distributions, outside of a fixed beta and uniform distribution, would allow for a deeper understanding of the relationship between the two. As mentioned previously, other optimization parameters have also been investigated, both geometrical or not. One such geometrical parameter is the angle between electrodes, which results in a new geometry, the diagonal MHD geometry.

### 8.2.1 A full Optimization Loop

In order to complete the coupling of the system, we must consider some way to introduce feedback from the state of the MHD system to the uncertain parameters, i.e. $\mathbf{u}, \sigma, \mu_{e}, \mu_{i}$. This can be done in many ways, such as with the inclusion of the fluid dynamics. This would allow for feedback from the optimization problem and allow for the two stage optimization problem to be run cyclically, as $\mathbf{u}$ will respond to changes in the electromagnetic fields. COMSOL does have a computational fluid dynamics module, similar to $\mathrm{AC} / \mathrm{DC}$ module, that runs a variety of two equation modules [33].

Another approach to completing this loop would be to model the chemistry behind the system. By computing species interactions, with some initial conductivity, electron mobility, and ion mobility, one could model the impacts of the electromagnetic field on these random parameters. Doing this, one could recover a much more realistic drop in the conductivity as well as the Hall and ion-slip parameters along the channel, given the load being extracted from the channel. With either form of coupling, a more robust model of an MHD generator would be generated, and allow for the feedback from the optimization step to affect the uncertain parameters.

### 8.2.2 Sensitivity of Optimal Operating Conditions

Another open question regarding the optimization of an MHD generator is the sensitivity of other the optimal operating conditions to the shape of the parameters. It immediately follows from the ideal equations and previous work in the deterministic framework that the optimal power is sensitive to the mean of the solutions. As well,
we investigated this sensitivity in Chapter 7 and Chapter 6, in the sense of comparing two different parameter distributional shapes, i.e. comparing the distribution of the solutions and optimal resistance to parameters with beta and uniform distributions. Although differences were noted, these are inherently only two data points in the infinite-dimensional space of probability distributions. As well, even the variance investigated yielded minimal results, as it was seen that the resistance of the resistor was not sensitive to the electron mobility. Thus, it shouldn't be expected that it would be sensitive to the distribution of the electron mobility. However, other parameters cannot be expected to behave the same way, due to the complex coupling within an MHD system. Thus, understanding how the optimal solutions change with this shape is important. One example approach for this would be hyper differential sensitivity analysis [49]. This is a local sensitivity analysis, investigating how much the optimal solution will change relative to some other given parameters; in our case, the shape of the distributions. Investigating the sensitivity of optimization parameters to different uncertain parameters will give a better understanding on when an inclusion of uncertainty will result in a dramatically more reliable power out.

### 8.2.3 Diagonal Geometry

Comparable to the segmented Faraday generator, the diagonal Faraday geometry has segmented electrodes placed along the channel walls, allowing current to flow parallel to the Lorenz force. However, it differs from the segmented Faraday geometry in two distinct ways. The first is that the load placed on the channel is no longer run in parallel along each pair of electrodes. Instead there are two resistors, upstream of one set of electrodes and connected downstream from the other. This allows for a more efficient transfer of current for the diagonal geometry. Next, stemming from the name, is that the paired electrodes are shifted downstream/upstream from one another. An example of the diagonal Faraday geometry is given in Figure 8.1.

The components match those within the segmented and continuous Faraday


FIGURE 8.1: Simple schematic of a Diagonal Faraday generator. Periodic boundary conditions are again noted with matching dashed colored lines.
generator geometries, as described in Section 2.1. As before, the geometry is completed with a system of periodic boundary conditions. Matching the schematic in Rosa, we have that the electrode pairs are connected with periodic boundary conditions, as are the resistors with a 'point of connection,' downstream. These are again colorcoded, with matched colors representing matched boundary conditions. All other boundaries obey the perfectly-electrically-insulating condition presented in Section 3.2. Theoretically, this electrode placement allows for the current density to flow in the tilted direction of the Lorenz force, and under ideal conditions, results in no Hall current (See [42]-Section 4.3). It follows that the ideal angle, for some given load factor and hall parameter, is given by

$$
\theta=\arctan \left(\frac{K}{\beta_{e}(1-K)}\right) .
$$

An obvious question arises from this however; how could one change the angle between paired electrodes in an operational MHD generator? Continuously, without physically moving the electrodes along the channel, this would prove difficult. However, if one connects each electrode to a switchboard, it could be realistically estimated that the angle between electrodes can be shifted by some discrete fixed degree amount. With this new geometry, model verification must be done. This implies a need for new ideal power equations. Most notably, the calculation of the load factor under this geometry is difficult. As well, this geometry is not verified, as we did with the
segmented and continuous in Section 5.1.3. In order to do so, accurate measurements of $K$ must be extrapolated from the system, which is not as simple as the approach for previous model verification. Additionally, a new ideal equation must be generated that incorporates the theory of this new geometry.

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[^0]:    ${ }^{1}$ Negligible Hall or ion-slip parameters

[^1]:    ${ }^{2}$ See Section 2.1

[^2]:    ${ }^{3}$ See Section 5.1.2 for examples of this mesh

[^3]:    ${ }^{4}$ See Section 2.1

[^4]:    ${ }^{5}$ Recall here that $\mathbf{J}_{i}=\left(\mathbf{J}_{i, x_{1}}, \mathbf{J}_{i, x_{2}}, \mathbf{J}_{i, x_{3}}\right)$

[^5]:    ${ }^{6}$ See Section 2.4

[^6]:    ${ }^{7}$ See Section 2.4.1

[^7]:    ${ }^{8}$ Recall the operator $\mathfrak{C}^{K}$ from the method-stability analysis of (DRUID).
    ${ }^{9}$ See Section 2.1.1

[^8]:    ${ }^{10}$ To see an example of a 2 dimensional Fejèr grid, see Figure 5.1

