Spatial and Temporal Damping of Fluid Perturbation at a Distance

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Abstract

A recent article by Cozzi and Kelliher [3] has demonstrated a sense of "locality" to the incompressible Euler equations which mirrors physical fluid behavior. We further develop the corresponding notion of "damping," and identify both temporal and local damping behaviors. To represent this problem, we perturb the initial condition of a fluid system in $\mathbb{R}^d$ and bound the effects of this perturbation at later times. We extend the previous L^2 result to the incompressible Navier-Stokes equations with any viscosity. Based on this result in L^2, we derive a similar bound in L^∞. We discuss the analogies between these bounds and the contributions due to the perturbation to the total kinetic energy and maximum velocity within a compact region of interest.

Contents

1 Introduction

Fluid flow is an important topic of physical interest, with a wide range of applications in industry and science. In particular, the paradigm of continuum fluid flow, which treats fluids as uniform substances with smoothly varying properties such as velocity and pressure, has yielded the most useful predictive models fluids in the macro-scale. One such continuum model, the Navier-Stokes equations [1] for incompressible flow, is given by

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u - \nu \nabla^2 u &= -\nabla p \\ \nabla \cdot u &= 0 \\ u|_{t=0} &= u^0 \end{aligned} \tag{1.0.1}$$

This model relates the velocity u of the fluid, with initial value u^0, to the pressure p and viscosity ν of the fluid, and describes the change in the first two of these quantities over time. The second equation, which declares the divergence of u to be zero, gives the incompressibility of the fluid. The derivation of these relations follows from the application of Newtons Laws to infinitesimal elements of the fluid.

While the Navier-Stokes model is a useful approximation to physical fluids in many cases, the continuum model cannot perfectly predict physical behavior of fluids. This is due in part to the particle nature of real fluids, which yields certain discrete behaviors not replicated by continuous velocity and pressure fields. Indeed, in a real fluid, the concept of a velocity or pressure at every point is absurd at the micro-scale, where some points are far from any matter at all. However, the macro-scale utility of the Navier-Stokes model motivates a certain flavor of physically-inspired investigation. In this type of investigation, we identify a physically evident property of fluids, and test the extent to which it holds in the mathematical model, and the ways in which it manifests.

For this study, we construct the following problem: a fluid system is defined in $\mathbb{R}^d$, with a given solution after some initial time. A compact ball is constructed somewhere in the space, denoted the *region of interest*. We perturb the initial conditions for the solution, requiring that both the original solution and the new solution are sufficiently well-behaved everywhere in space. Then the following investigation revolves around the behavior of the fluid in the region of interest, with particular interest in analogues to the kinetic energy and maximum speed contributions due to the perturbation. We reserve physical discussions for the case when the initial perturbation is zero in the region of interest.

In a physical system, we expect to see "damping" of the perturbation in this case, of which we identify two types. First, we expect that the solution will exhibit temporal damping, where the effects of the perturbation are forced to grow slowly in magnitude, since physical fluid waves have limited speed. Second, we expect to observe spatial damping, where the maximal effects of the perturbation at a given time are lessened if the perturbation is further from the region of interest. Exploring these behaviors is the primary motivation of this study.

The current inquiry follows directly from a result sketched by Cozzi and Kelliher [3]. They present a bound on the so-called L^2 norm of the velocity (a kinetic energy analogue) in the region of interest, but only in the case of zero viscosity. It is our hope to extend this result to the general case of any viscosity, as well as to derive a bound on the L^∞ norm, analogous to the maximum speed contribution from the perturbation in the region of interest.

2 Methods

2.1 Lebesgue Spaces

An important concept related to this research is that of Lebesgue spaces, denoted L^p for $1 \le p \le \infty$. In this context, the L^p spaces for p finite are sets of functions for which certain integral expressions are defined and finite. Consider a scalar function f defined on $\mathbb{R}^d$, such as a pressure field defined everywhere in a fluid. We say that f is in L^p for some finite p if the quantity

$$\left\{ \int_{\mathbb{R}^d} |f|^p dx \right\}^{1/p} \tag{2.1.1}$$

is defined and finite [2]. This expression can be thought of as the total magnitude of the function in some sense, possessing the same properties as a vector norm. In fact, expression (1.3.1) is called the L^p-norm of f, denoted $\|f\|_{L^p}$.

In the case of $p = \infty$, we require a concept known as the supremum. The supremum of a set S is defined to be the least upper bound on S. If S is finite, or in general if S has a maximum element, then the supremum of S is just the maximum of S. However, arbitrary sets might be bounded above without possessing a maximum element. For example, the set of negative real numbers has no maximum, but is certainly bounded above. Any non-negative number serves as an upper bound on this set, and the least non-negative number is zero, so we say the supremum of the negative real numbers is zero. In general, for a subset S of $\mathbb{R}^d$, we define

$$\sup S = \min\{k \in \mathbb{R}^d : k \geq s \text{ for all } s \in S\} \tag{2.1.2}$$

A scalar function f is in L^∞ if it is bounded above and below, or equivalently if its absolute value $|f|$ is bounded above[1]. In this case, we simply define the L^∞ norm [2] to be

$$\|f\|_{L^\infty} = \sup\{|f(x)| : x \in \mathbb{R}^d\}. \tag{2.1.3}$$

We will also want to discuss the norms of vector-valued functions such as velocity fields, and the matrix-valued functions that arise from the gradient of vector-valued functions. In these cases, we say that the function is in L^p if each of its component functions is in L^p, and define the L^p-norm to be the maximum of the L^p-norms of the components.

When considering the norm of the velocity field u of a fluid, the L^2 and L^∞ norms of u are of the most physical interest. The L^2 norm is significant because it involves the integral of velocity squared, and so the norm is analogous to an upper bound on the total kinetic energy of a region of space. The L^∞ norm is even more directly significant, as an upper bound on the speed of the solution anywhere in the given region.

Within the field of partial differential equations, so-called "energy methods" make problems more tractable in the L^2 norm, so we often start there. However, one can construct fields which have a low magnitude in most of $\mathbb{R}^n$ but attain very high magnitudes small regions. The integral in the L^2 norm hides the high-magnitude regions in the integration, and so the L^2 norm might not give the best information when we make claims about the behavior of the velocity field. Hence, we seek wherever possible to find both L^2 and L^∞ bounds, as the combination of both results gives a more complete picture of the fluid behavior.

2.2 Smooth Bump Function

When discussing local behavior of either a vector- or scalar-valued function near a given point z, we need a mathematical tool to discount behavior away from z. More specifically, we need to apply some operation to a given function f that results in a new function f' with two properties. The first property is that close to z, say within a neighborhood of radius R, we have $f' = f$. The second property is that far away from z, say outside radius $2R$, we have $f' = 0$.

The simplest way to satisfy these requirements is to let $f' = fg$, where $g(x)$ is a scalar function of the distance from x to z. If $g = 1$ inside the neighborhood of radius R and $g = 0$ outside the neighborhood of radius $2R$, then the properties are satisfied. However, this still leaves us with quite a bit of freedom. We have not put any restrictions on the behavior of g for radii between R and $2R$, and a large family of functions satisfies these properties. We can imagine g as a step function with a discontinuity in the intermediate region, or we could choose some polynomial subject to boundary conditions at R and $2R$. We could even use a scaled and translated sinusoid for a smoother transition.

However, all of these options have a fundamental flaw. The step function is discontinuous, which will cause problems with infinity should we try to consider any derivatives of f'. Similarly, each of the other listed choices is only smooth to a certain extent; differentiating a few times will yield a discontinuous function. These discontinuities will affect the norms we take, and thus we need to avoid them.

[1] This is actually a stronger requirement than the standard definition of L^∞. Rigorously, the absolute value $|f|$ must only be bounded *almost everywhere* in $\mathbb{R}^d$, a term with a rigorous meaning in measure theory. We have chosen to suppress this nuance for the sake of clarity, and because the restriction that the functions (u, p) satisfy equations (1.1.1) prevents such edge cases from manifesting in our work.

Instead, we introduce the concept of a smooth bump function, φ_R. We are more interested in the existence of the bump function than its specific form. The existence of such functions is well-established, and one particular example takes the form

$$\varphi_R(x) = \begin{cases} 1 & \|x\| \leq R \\ \exp\left(\frac{1}{R} - \frac{R}{R^2-(\|x\|-R)^2}\right) & R < \|x\| < 2R \\ 0 & \|x\| \geq 2R \end{cases} . \tag{2.2.1}$$

This function can be shown to be an element of C^∞, the class of infinitely differentiable functions. This is an important requirement for smooth bump functions, ensuring that we are safe to take derivatives without worrying about discontinuities.

2.3 Hölder's Inequality

A frequent problem when working in Lebesgue spaces is deriving an expression in one L^p space given an expression in a different space. Our primary tool for managing such situations in this research is Hölder's inequality, which is presented formally as

Theorem 2.1. *Let f and g be measurable functions on a space S. Let p and q in $[1, \infty]$ define Lebesgue spaces L^p and L^q on S, and require $1/p + 1/q = 1$. Then*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} \tag{2.3.1}$$

Proof. See [4]. □

Put simply, Hölder's inequality relates the L^1 norm of a product to L^p and L^q norms of its factors. For our purposes, the only pairs p, q of relevance are $p = q = 2$ and $p = 1, q = \infty$, where we have defined for the purpose of this inequality that $1/\infty = 0$. The concept of a measurable function has roots in measure theory, and a formal definition is omitted from this discussion. However, all of our relevant functions are integrable, which implies their measurability.

2.4 Grönwall's Lemma

In the course of working with Lebesgue norms, we often need to work with complicated integral relations. Grönwall's lemma, sometimes called Grönwall's inequality, is a tool that allows us to simplify integral inequalities of a certain form. The lemma has both an integral and differential form, but only the integral form is relevant to this research. Grönwall's lemma in this form can be stated as

Theorem 2.2. *Let u, α, and β be functions defined on I, an interval in the real line, and let I be of the form $[a, b]$, $[a, b)$, or $[a, \infty)$. Let α be non-decreasing, and let u and β be continuous. Then if β is non-negative and the inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds \tag{2.4.1}$$

holds for all $t \in I$, then

$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(s)ds\right) \tag{2.4.2}$$

for $t \in I$.

Proof. See [5]. □

In practice, the integral of just β is often more tractable than the original integral. Moreover, the original inequality has u on both the left and right side, while the result has eliminated u from the right-hand side of the equation. In this research, the exponential result is a form well-suited to the increasing time bounds we hope to find.

3 Results

3.1 Inviscid Flow in L^2

A common approach to working with the Navier-Stokes equations is to first consider the inviscid case, where the viscosity of the fluid vanishes. Often, the problematic behavior of the Navier-Stokes model arises from the viscosity, so many problems are more tractable for inviscid flow. Regardless, the inviscid case has fewer terms to account for, and is thus simpler. The Navier-Stokes equations with $\nu = 0$ are known as the Euler equations. Only a single equation changes form, namely

$$\partial_t u + (u \cdot \nabla)u = -\nabla p \tag{3.1.1}$$

Cozzi and Kelliher [3] sketch a derivation for a locality bound on the Euler equations, which we now reproduce in full detail.

Theorem 3.1. *Let $B_r(x)$ be the ball of radius $R > 0$ centered at the point $x \in \mathbb{R}^d$. Let $\varphi(x) \in C^\infty(\mathbb{R}^d)$ be a smooth bump function, with $\varphi = 0$ outside $B_2(0)$ and $\varphi = 1$ on $B_1(0)$. Define $\varphi_R(x) = \varphi(x/R)$. Let (u_1, p_1) and (u_2, p_2) be solutions to the Euler equations in $\mathbb{R}^d$, and let $w = u_1 - u_2$, $\overline{p} = p_1 - p_2$. Let these solutions be well-behaved, with sufficient decay of velocity and vorticity to ensure that*

$$A \equiv \|\nabla u_2\|_{L^\infty([0,t]\times\mathbb{R}^d)} + \sum_{n=1}^{2} \Big(\|u_n\|_{L^\infty([0,t]\times\mathbb{R}^d)} + \|u_n\|_{L^\infty([0,t];L^2)} + \|p_n\|_{L^\infty([0,t];L^2)} \Big) < \infty. \tag{3.1.2}$$

Then for any $z \in \mathbb{R}^d$, there exists some constant C such that

$$\|\varphi_R(x-z)w(t)\|_{L^2}^2 \leq \left[\left\|\varphi_R(x-z)w^0\right\|_{L^2}^2 + \frac{C\max\{A^2, A^3\}}{R} t \right] e^{2At}. \tag{3.1.3}$$

Proof. To begin, we subtract the first Euler equation for u_2 from the first equation for u_1. This yields

$$\partial_t u_1 - \partial_t u_2 + u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 = -\nabla p_1 + \nabla p_2. \tag{3.1.4}$$

Use $w = u_1 - u_2$ and $\overline{p} = p_1 - p_2$ to simplify to

$$\partial_t w + u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 = -\nabla \overline{p}. \tag{3.1.5}$$

If we add to the left side $u_1 \cdot \nabla u_2 - u_1 \cdot \nabla u_2$, we observe that linearity of the inner product and the ∇ operator gives

$$u_1 \cdot \nabla u_1 - u_1 \cdot \nabla u_2 = u_1 \cdot \nabla w \tag{3.1.6}$$

and

$$u_1 \cdot \nabla u_2 - u_2 \cdot \nabla u_2 = w \cdot \nabla u_2. \tag{3.1.7}$$

Thus, we have

$$\partial_t w + u_1 \cdot \nabla w + w \cdot \nabla u_2 = -\nabla \overline{p}. \tag{3.1.8}$$

Let $\Phi = \varphi_R(x-z)$. We take the dot product with $\Phi^2 w$ and integrate over time and space to obtain

$$\int_0^t \int_{\mathbb{R}^d} \left[\frac{1}{2}\partial_t \left(\Phi^2 |w|^2\right) - (\Phi\nabla\Phi \cdot u_1)|w|^2 + (\Phi w \cdot \nabla u_2) \cdot (\Phi w) \right] = \int_0^t \int_{\mathbb{R}^d} 2\overline{p}\Phi\nabla\Phi \cdot w. \tag{3.1.9}$$

The first term in the left integrand can be isolated and rewritten as

$$\frac{1}{2}\int_0^t \partial_t \|\Phi w\|_{L^2}^2, \tag{3.1.10}$$

which begs for the application of the Fundamental Theorem of Calculus. Applying the FTC, we find

$$\|\Phi w(t)\|_{L^2}^2 = \left\|\Phi w^0\right\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^d} \left[2(\Phi\nabla\Phi \cdot u_1)|w|^2 - 2(\Phi w \cdot \nabla u_2) \cdot (\Phi w) + 4\overline{p}\Phi\nabla\Phi \cdot w\right]. \tag{3.1.11}$$

Focusing on the first term in the integral, we note that an integral is maximized when the integrand is nonnegative. Thus,

$$\int_{\mathbb{R}^d} 2(\Phi\nabla\Phi \cdot u_1)|w|^2 dx \leq \int_{\mathbb{R}^d} \left|2(\Phi\nabla\Phi \cdot u_1)|w|^2\right| \tag{3.1.12}$$

Noting that the right side is equivalent to $\left\|2(\Phi\nabla\Phi \cdot u_1)|w|^2\right\|_{L^1}$, we apply Theorem 2.1 to find

$$\int_{\mathbb{R}^d} 2(\Phi\nabla\Phi \cdot u_1)|w|^2 dx \leq \|2\nabla\Phi \cdot u_1\|_{L^\infty} \left\|\Phi|w|^2\right\|_{L^1}. \tag{3.1.13}$$

The second term in this product is written in integral form as

$$\left\|\Phi|w|^2\right\|_{L^1} = \int_{\mathbb{R}_d} \left|\Phi|w|^2\right|. \tag{3.1.14}$$

Since both Φ and $|w|$ are non-negative, we can drop the outer absolute value from this integrand. Then, since the magnitude of Φ is bounded by 1, we have

$$\left\|\Phi|w|^2\right\|_{L^1} \leq \int_{\mathbb{R}_d} |w|^2. \tag{3.1.15}$$

Of course, this expression is the square of the L^2 norm of w, which is bounded by A. Thus, we conclude that

$$\left\|\Phi|w|^2\right\|_{L^1} \leq A^2. \tag{3.1.16}$$

The remaining factor is written explicitly as the supremum

$$\|2\nabla\Phi \cdot u_1\|_{L^\infty} = \sup_{\mathbb{R}^d} [2\nabla\Phi \cdot u_1] \tag{3.1.17}$$

Clearly, this quantity is maximized if both u_1 and $\nabla\Phi$ attain their maximum value at the same point with the same direction. Then

$$\|2\nabla\Phi \cdot u_1\|_{L^\infty} \leq 2\|\nabla\Phi\|_{L^\infty} \|u_1\|_{L^\infty} \tag{3.1.18}$$

The second norm in this product is bounded by A. We also apply the chain rule to the gradient of Φ, gaining a factor of $1/R$ to find

$$\|2\nabla\Phi \cdot u_1\|_{L^\infty} \leq \frac{2A}{R} \|\nabla\varphi(x-z)\|_{L^\infty}. \tag{3.1.19}$$

Finally, φ is a continuous function. Thus, its restriction to the ball of radius 2 about the origin, a compact space, is uniformly continuous. Being in C^∞, φ is also differentiable everywhere. Thus, its derivatives are all bounded on the ball of radius 2 about the origin. Finally, the derivatives are all zero outside the ball of radius 2, so every derivative of φ is bounded on all of $\mathbb{R}^d$. This bound depends only on the choice of φ, so is constant with respect to every other parameter. Then

$$\|2\nabla\Phi \cdot u_1\|_{L^1} \leq \frac{CA}{R}. \tag{3.1.20}$$

Combining equations (3.1.16) and (3.1.20), we conclude that

$$\int_{\mathbb{R}^d} 2(\Phi\nabla\Phi \cdot u_1)|w|^2 dx \leq \frac{CA^3}{R}. \tag{3.1.21}$$

The second spatial integral from equation (3.1.11) is bounded by the integral of the absolute value of the integrand, so

$$\int_{\mathbb{R}^d} -2(\Phi w \cdot \nabla u_2) \cdot (\Phi w) dx \leq \|2(\Phi w \cdot \nabla u_2) \cdot (\Phi w)\|_{L^1} . \tag{3.1.22}$$

Applying Theorem 2.1 and writing the inner product sums directly, we have

$$\int_{\mathbb{R}^d} -2(\Phi w \cdot \nabla u_2) \cdot (\Phi w) dx \leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left\|2\partial_j u_2^i\right\|_{L^\infty} \left\|(\Phi w^i)(\Phi w^j)\right\|_{L^1} . \tag{3.1.23}$$

Then, since $A \geq \|\nabla u_2\|_{L^\infty}$,

$$\int_{\mathbb{R}^d} -2(\Phi w \cdot \nabla u_2) \cdot (\Phi w) dx \leq 2A \sum_{i=1}^{d} \sum_{j=1}^{d} \left\|(\Phi w^i)(\Phi w^j)\right\|_{L^1} , \tag{3.1.24}$$

and applying the definitions of the L^1 and L^2 norms yields

$$\int_{\mathbb{R}^d} -2(\Phi w \cdot \nabla u_2) \cdot (\Phi w) dx \leq 2A \|\Phi w\|_{L^2}^2 , \tag{3.1.25}$$

Finally, the last spatial integral is bounded by an L^1 norm,

$$\int_{\mathbb{R}^d} 4\overline{p}\Phi\nabla\Phi \cdot w \leq \|4\overline{p}\Phi\nabla\Phi \cdot w\|_{L^1} . \tag{3.1.26}$$

We apply Theorem 2.1 twice to find

$$\int_{\mathbb{R}^d} 4\overline{p}\Phi\nabla\Phi \cdot w \leq 4 \|\overline{p}\|_{L^2} \|w\|_{L^2} \|\Phi\nabla\Phi\|_{L^\infty} . \tag{3.1.27}$$

Together with the definition of A and the fact that $|\Phi| \leq 1$, this implies

$$\int_{\mathbb{R}^d} 4\overline{p}\Phi\nabla\Phi \cdot w \leq 4A^2 \|\nabla\Phi\|_{L^\infty} . \tag{3.1.28}$$

As has already been discussed, this final norm is bounded by $\frac{C}{R}$, so

$$\int_{\mathbb{R}^d} 4\overline{p}\Phi\nabla\Phi \cdot w \leq \frac{CA^2}{R} \tag{3.1.29}$$

Substituting equations (3.1.21), (3.1.25), and (3.1.29) into equation (3.1.11) gives

$$\|\Phi w(t)\|_{L^2}^2 \leq \left\|\Phi w^0\right\|_{L^2}^2 + \int_0^t \left[\frac{A^3 C}{R} + 2A \|\Phi w(s)\|_{L^2}^2 + \frac{A^2 C}{R}\right] ds. \tag{3.1.30}$$

Finally, we apply Theorem 2.2 to find

$$\|\Phi w(t)\|_{L^2}^2 \leq \left[\left\|\Phi w^0\right\|_{L^2}^2 + \frac{C \max\{A^2, A^3\}}{R} t\right] e^{2At}. \tag{3.1.31}$$

□

There are several features to the temporal and spatial dependence of this result which agree with the physical behaviors we hoped to observe. These are discussed in detail in Chapter 4.

3.2 The Viscosity Term in L^2

The result of Theorem 3.1 is achieved by removing the most complex term from the Navier-Stokes equations. The viscosity term scales with the Lapacian of the velocity, a second-order differential operator. However, Theorem 3.1 has a straightforward extension to the case of any viscosity.

Theorem 3.2. *Let w, $\overline{p}$, and φ_R be defined as in Theorem 3.1. Redefine A to also bound $\|\nabla u_1\|_{L^\infty}$, as*

$$A \equiv \sum_{n=1}^{2} \left(\|\nabla u_n\|_{L^\infty([0,t]\times\mathbb{R}^d)} + \|u_n\|_{L^\infty([0,t]\times\mathbb{R}^d)} + \|u_n\|_{L^\infty([0,t];L^2)} + \|p_n\|_{L^\infty([0,t];L^2)} \right) < \infty. \tag{3.2.1}$$

Then for any $z \in \mathbb{R}^d$, there exists some constant C such that

$$\|\varphi_R(x-z)w(t)\|_{L^2}^2 \leq \left[\left\|\varphi_R(x-z)w^0\right\|_{L^2}^2 + \frac{C\max\{A^2, A^3\}}{R} t \right] e^{2At}. \tag{3.2.2}$$

Proof. For simplicity, we assume that the viscosity $\nu = 1$. The inclusion of the Navier-Stokes viscosity term creates two additional terms in the original difference equation. By the linearity of the Laplacian, these reduce to a single term in w:

$$\Delta u_1 - \Delta u_2 = \Delta w. \tag{3.2.3}$$

Then this term generates another spatial integral in equation (3.1.11), of the form

$$\int_{\mathbb{R}^d} \Phi^2 w \cdot \Delta w. \tag{3.2.4}$$

The ith component of the vector Laplacian Δw is given by

$$\sum_{j=1}^{d} \partial_j^2 w^i, \tag{3.2.5}$$

so the dot product definition implies the integrand is equivalent to the sum

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \Phi^2 w^i \partial_j^2 w^i. \tag{3.2.6}$$

Considering only a single term of this double sum, we apply integration by parts to find

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i = \int_{\mathbb{R}^d} -(\partial_j w^i)\partial_j(\Phi^2 w^i). \tag{3.2.7}$$

Expanding the derivative of the product on the right side,

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i = \int_{\mathbb{R}^d} -(\partial_j w^i)[\Phi^2 \partial_j w^i + 2\Phi w^i \partial_j \Phi]. \tag{3.2.8}$$

Distributing the product and separating the integrals yields

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i = \int_{\mathbb{R}^d} -(\Phi \partial_j w^i)^2 + \int_{\mathbb{R}^d} -2\Phi w^i \partial_j w^i \partial_j \Phi. \tag{3.2.9}$$

The first integrand is nonpositive everywhere, so it only reduces the value of the right hand side. Thus,

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i \leq \int_{\mathbb{R}^d} -2\Phi w^i \partial_j w^i \partial_j \Phi. \tag{3.2.10}$$

This integrand is bounded above by by its absolute value, bounding the integral by the L^1 norm

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i \leq \left\|2\Phi w^i \partial_j w^i \partial_j \Phi\right\|_{L^1}. \tag{3.2.11}$$

We apply Theorem 2.1 to this norm to find

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i \leq 2 \left\| \Phi \partial_j \Phi \right\|_{L^\infty} \left\| w^i \partial_j w^i \right\|_{L^1}. \tag{3.2.12}$$

As in the preceding section, the L^∞ norm is bounded by $\frac{C}{R}$, yielding

$$\int_{\mathbb{R}^d} \Phi^2 w^i \partial_j^2 w^i \leq \frac{C}{R} \int_{\mathbb{R}^d} |w^i \partial_j w^i|. \tag{3.2.13}$$

Since $\nabla w = \nabla u_1 - \nabla u_2$ is bounded by A, so is each partial derivative of w. Since A also bounds the norm of w, it bounds each of the components. Thus all dependence on i or j disappears, and we conclude that

$$\int_{\mathbb{R}^d} \Phi^2 w \cdot \Delta w \leq \frac{A^2 C}{R}. \tag{3.2.14}$$

Adding this term into equation (3.1.30) only changes the constant factor on the term of order A^2, and the rest of the proof follows identically.

□

This is perhaps a surprising result. The viscosity, which often acts in Navier-Stokes problems to add complexity not seen in the inviscid case, has no significant effect on the bound. See Chapter 4 for more discussion of this point.

3.3 A General Approach to the L^∞ Bound

As discussed in Section 2.1, the L^2 bound is roughly analogous to the total kinetic energy in a given region. While this is a physically interesting quantity to know about, even a good bound in L^2 allows for unphysical behavior. For example, a very narrow spike in the velocity in one location might be hidden by an integral over a larger space. The L^∞ norm is roughly equivalent to the maximum speed of the fluid in a region. Having both the L^2 and L^∞ bounds provides an more complete sense of how extreme the perturbation is at any given time.

In general, it is not possible to derive an L^∞ bound from any other L^p bound. However, in this case, we have placed several constraints on the function whose norm we need; A bounds w in several senses, while the continuity and compact support of Φ restricts the scope of the calculations. We begin with a general proof regarding functions which are Lipschitz continuous, which we then expand to fit our case.

Lemma 3.1. *Let f_1 and f_2 be Lipschitz continuous functions on $\mathbb{R}^d$ with the same Lipschitz constant α. That is, assume there is some scalar α such that each f_i satisfies*

$$|f_i(x) - f_i(y)| \leq \alpha |x - y| \tag{3.3.1}$$

for all points x and y. Then

$$\|f_1 - f_2\|_{L^\infty} \leq C \alpha^{d/(d+2)} \|f_1 - f_2\|_{L^2}^{2/(d+2)}. \tag{3.3.2}$$

Proof. We first choose some point x_0 in $\mathbb{R}^d$ and define

$$\epsilon \equiv |f_1(x_0) - f_2(x_0)|. \tag{3.3.3}$$

Note that ϵ is not necessarily small. Next, construct the ball B of radius $\frac{\epsilon}{4\alpha}$ centered at x_0. Choose a point y from B, and consider the quantity $|f_1(y) - f_2(y)|$. By the reverse triangle inequality, we have

$$\begin{aligned}
|f_1(y) - f_2(y)| &= |f_1(x_0) - f_2(y) - (f_1(x_0) - f_1(y))| && (3.3.4)\\
&\geq |f_1(x_0) - f_2(y)| - |f_1(x_0) - f_1(y)| && (3.3.5)\\
&= |f_1(x_0) - f_2(x_0) - (f_2(y) - f_2(x_0))| - |f_1(x_0) - f_1(y)| && (3.3.6)\\
&\geq |f_1(x_0) - f_2(x_0)| - |f_1(x_0) - f_1(y)| - |f_2(x_0) - f_2(y)| && (3.3.7)\\
&= \epsilon - |f_1(x_0) - f_1(y)| - |f_2(x_0) - f_2(y)|. && (3.3.8)
\end{aligned}$$

Applying the equation (3.3.1), we have

$$|f_1(y) - f_2(y)| \geq \epsilon - 2\alpha|x_0 - y|. \tag{3.3.9}$$

But since y is in B, we have $|x_0 - y| \leq \frac{\epsilon}{4\alpha}$. Thus, we conclude that

$$|f_1(y) - f_2(y)| \geq \frac{\epsilon}{2} \tag{3.3.10}$$

for all y in B. Next, we consider the L^2 norm of $f_1 - f_2$ over all of $\mathbb{R}^d$. By definition of the norm,

$$\|f_1 - f_2\|_{L^2} = \left[\int_{\mathbb{R}^d} |f_1 - f_2|^2\right]^{1/2}. \tag{3.3.11}$$

But the integrand is then guaranteed non-negative everywhere, so we can restrict the domain of integration to B without increasing value of the integral:

$$\|f_1 - f_2\|_{L^2} \geq \left[\int_B |f_1 - f_2|^2\right]^{1/2}. \tag{3.3.12}$$

Applying equation (3.3.10), we have

$$\|f_1 - f_2\|_{L^2} \geq \left[\int_B \frac{\epsilon^2}{4}\right]^{1/2}. \tag{3.3.13}$$

Now our integrand is constant, so we obtain

$$\|f_1 - f_2\|_{L^2} \geq \frac{\epsilon}{2} \left[\mu(B)\right]^{1/2}, \tag{3.3.14}$$

where $\mu(B)$ is the measure of B. This depends on the dimension d of the space, and is an indicator of the size of the set B. In $\mathbb{R}^2$, the measure is the area of the ball, and in $\mathbb{R}^3$, it is the volume of the ball. Then the measure must have the same dimensionality as the space, up to some constant, yielding

$$\|f_1 - f_2\|_{L^2} \geq C\epsilon \left[\frac{\epsilon}{\alpha}\right]^{d/2}. \tag{3.3.15}$$

Reducing further,

$$\epsilon^{d/2+1} \leq C\alpha^{d/2} \|f_1 - f_2\|_{L^2}. \tag{3.3.16}$$

Now we use equation (3.3.3) to bound the difference at x_0 as

$$|f_1(x_0) - f_2(x_0)| \leq C \left[\alpha^{d/2} \|f_1 - f_2\|_{L^2}\right]^{1/(d/2+1)}. \tag{3.3.17}$$

The right side is constant with respect to x_0, so we can freely take the supremum over all $\mathbb{R}^d$ to find

$$\|f_1 - f_2\|_{L^\infty} \leq C\alpha^{d/(d+2)} \|f_1 - f_2\|_{L^2}^{2/(d+2)}. \tag{3.3.18}$$

□

To utilize this result in our case, we need to show that $f_1 = \Phi u_1$ and $f_2 = \Phi u_2$ are Lipschitz continuous. This proof is the subject of the final theorem:

Theorem 3.3. *Let w, $\overline{p}$, Φ, and A be defined as in Theorem 3.2. Then there exist constants C_1 and C_2 such that*

$$\|\Phi w\|_{L^\infty} \leq C_2 \left\{A^d \left(\frac{R+1}{R}\right)^d \left[\left\|\Phi w^0\right\|_{L^2}^2 + \frac{C_1 \max\{A^2, A^3\}}{R} t\right] e^{2At}\right\}^{1/(d+2)}. \tag{3.3.19}$$

Proof. Let x and y be points in $\mathbb{R}^d$. We need to track the change in f_i, and we consider the scenario that maximizes this change. This worst-case bound can be represented using the L^∞ bound of the vector gradient ∇f_i to bound the rate of change, similar to arguments related to other differential operators. This bound is

$$|f_i(x) - f_i(y)| \leq \|\nabla f_i\|_{L^\infty} |x - y| \tag{3.3.20}$$

To decompose this gradient, we use a variant of the product rule which includes the tensor product, yielding

$$\nabla f_i = u_i \otimes \nabla \Phi + \Phi \nabla u_i. \tag{3.3.21}$$

Then, we have

$$|f_i(x) - f_i(y)| \leq CA\frac{R+1}{R}|x - y|, \tag{3.3.22}$$

so both f_1 and f_2 are Lipschitz continuous with Lipschitz constant $\alpha = CA\frac{R+1}{R}$. Application of Lemma 3.1 yields

$$\|\varphi_R w\|_{L^\infty} \leq CA^{d/(d+2)} \left(\frac{R+1}{R}\right)^{d/(d+2)} \|\varphi_R w\|_{L^2}^{2/(d+2)}. \tag{3.3.23}$$

Finally, we substitute in the result from Theorem 3.2 to find the final bound on the L^∞ norm:

$$\|\Phi w\|_{L^\infty} \leq C_2 \left\{ A^d \left(\frac{R+1}{R}\right)^d \left[\|\Phi w^0\|_{L^2}^2 + \frac{C_1 \max\{A^2, A^3\}}{R} t\right] e^{2At} \right\}^{1/(d+2)}. \tag{3.3.24}$$

□

This leaves us with a rough bound on the maximum speed contribution from the perturbation in the region of interest, which takes a similar form to the bound on the kinetic energy analogue. Further discussion can be found in Chapter 4.

4 Discussion

4.1 Analysis of Results

In this case, we consider two identical fluid systems who are momentarily in the same state in the ball of radius $2R$ about z. We note that this ball contains the region of interest, the ball of radius R centered at z. In general, as the system evolves with time, differences between the two systems propagate from outside the region of interest to cause differing behavior within. The L^2 norm of Φw bounds the integral of velocity squared inside the region of interest, and so represents the total kinetic energy contribution of the perturbation in this region. Likewise, the L^∞ norm bounds the maximum speed of the fluid in the region of interest. Both of these norms exhibit the "damping" behaviors discussed in Section 1.1.

Although the result from Section 3.1 is not original, the structure of the result forms the basis for both original theorems. At the initial time $t = 0$, the time components on the right side vanish, and the inequality is satisfied by a tautological equivalence given $w^0 = w(t = 0)$. However, our primary interest is in tracking the effect of an initial perturbation far away from z, and due to the nature of Φ, the L^2 norm on the right side only contains information about the component of the perturbation near z. Thus, for the rest of the discussion, we will assume that Φw^0 is identically zero, such that this bound vanishes. The remaining term captures the damping behaviors we hoped to find.

The temporal damping behavior is apparent from structure of the time factor: the factor of t ensures that the differences within the region of interest vanish at time $t = 0$, while the overall structure of te^{2At} bounds the evolution by a smooth function. Notably, the constant A serves as a metric of the most extreme changes in w, and the bound reflects this; the bound is more restrictive for smaller A.The constant A also appears in the pre-factor, but its effect here is more restrained then in the temporal term. If $A \geq 1$, then the bound scales with a factor of A^3. However, if $A < 1$, then $A^3 < A^2$, so the bound only scales with a factor of A^2.

The spatial damping behavior is represented by the $1/R$ dependence of the bound. Since we are working under the assumption that the initial perturbation is outside the region of interest with radius $2R$, then increasing the distance from the initial perturbation to the center of the region allows an increase in R. Thus, spacing the perturbation further from the center of the region decreases the bound at any given time t. In fact, the square of the L^2 norm roughly corresponds to the to the total contribution from the perturbation to the fluid kinetic energy in the region of interest, so the kinetic energy density contribution would carry another factor of R^{-d}. Thus, the energy density due to perturbation actually decreases with $R^{-(d+1)}$.

One of the most significant results was the fact that viscosity has no effect on the structure of the bound. Viscosity is associated with some of the mathematically "problematic" behaviors of the Navier-Stokes model, including chaotic behavior such as turbulence. Significantly, the viscosity term did not effect the overall structure of the result. Further, one portion of the viscosity contribution was found to be strictly non-positive, meaning that viscosity may actually aid with damping. Using our methods, however, we are unable to draw any such conclusions with confidence.

Finally, we discuss the L^∞ bound we generated from the L^2 bound. This result is significant for its rough correspondence to the contribution from the perturbation to the maximum fluid speed in the region of interest. The bound has many of the same features as its L^2 counterparts, but with important distinctions. The $(d + 2)$th root of the temporal component initially grows from zero more quickly than does te^{2At} itself, but grows much more slowly after $t = 1$. We believe this corresponds to the property that the added speed in the region of interest can increase more quickly due to a small region of fast-moving fluid, but more fluid at that same speed will not change the bound. Contrast this with the kinetic energy analogue, which would scale based on how much fluid was at this high energy. The dependence on A in the pre-factor also changes, now taking on a value of $\max\{1, A^{(d+3)/(d+2)}\}$. This change represents an overall lesser bound for $A > 1$ and a greater bound for $A < 1$, as compared with the L^2 result.

The spatial damping changes in a more complex way. The $(d+2)$th root of $\frac{1}{R}$ results in a greater bound with increasing dimensions for small R, and a corresponding lesser bound for large R. However, there is an additional factor of $\left(\frac{R+1}{R}\right)^{d/(d+2)}$. This factor is strictly decreasing in R, approaching unity for large R. As R approaches zero, this factor blows up, behavior not present in the L^2 bound. This can be interpreted by imagining an arbitrarily large initial velocity perturbation, "aimed" directly at the region of interest. If the region is small, this perturbation will disperse very little before reaching the region of interest, creating a very quick rise in the velocity in the region. Thus, the very quick rise in maximum speed allowed by this term becoming large has a reasonable interpretation

4.2 Future Research

We believe that this concept of damping, and the locality associated with it, hold promise for several avenues of future research. Perhaps most ambitiously, the methods used here were fairly restrictive and simplistic by the standards of typical Navier-Stokes study. This required us to place a laundry list of restrictions on the well-behavedness of the velocity and pressure, decreasing the generality of our results. Future research could follow the approach of the primary theorems in Cozzi and Kelliher [3] for a wider class of solutions.

More closely related to our results, the effects of the viscosity are not entirely well-described by this method. In fact, we might see an increase in the damping behavior due to the viscosity terms in this study, though our methods do not allow us to pursue this possibility. If the viscosity of the fluid does in fact increase the damping of perturbations, this could lead to a stronger claim about the physicality of the Navier-Stokes model in this context.

Finally, there are other fluid quantities of interest which could be studied in a method which mirrored ours. The vorticity, the curl of the velocity of the fluid, is often studied in mathematical fluid dynamics. One interesting property of the vorticity is that it is conserved for each infinitesimal fluid element over time. We are optimistic that analogous results to our own could be shown for the change in vorticity of a fluid system under perturbation.

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