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Abstract - Conditions are prescribed on a family of kernels K on R^n, $n \geq 2$, that ensure that the maximal singular operator, $\sup |K * f|$, is bounded in $L^p(R^n)$ where the supremum is over the family of kernels.

On A Class Of Singular Integrals

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ON A CLASS OF SINGULAR INTEGRALS

I. INTRODUCTION

In this section some results of singular integrals are presented. Some fundamental definitions of different concepts will not be given here, which will be presented in the next section.

As it is well-known that a basic example which lies at the source of the singular integrals is given by the Hilbert transform,

$$\lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y} dy.$$

The above operator is bounded on $L^p, 1 < p < \infty$. It was proved by M. Riesz, using complex function.

Let's introduce those operators which not only commute with translations but also with dilations. Then we need to define the kernel as

$$K(x) = \frac{\Omega}{|x|^n},$$

with Ω homogeneous of degree 0, i.e. $\Omega(\epsilon x) = \Omega(x)$, $\epsilon > 0$. The definition implies $K(x)$ is homogeneous of degree $-n$. The definition on Ω is equivalent with the fact that it is constant on rays emanating from the origin. So Ω is completely determined by its restriction to the unit sphere S^{n-1}. We have the following result.

Let Ω be homogeneous of degree 0, and suppose that Ω satisfies the cancellation property,

$$\int_{S^{n-1}} \Omega(x) d\sigma = 0,$$

and, the smoothness property, i.e. if

$$\sup_{|x-x'|<\delta} |\Omega(x) - \Omega(x')| = \omega(\delta), \qquad |x| = |x'| = 1$$

then

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty.$$

For $1 < p < \infty$, and $f \in L^p(R^n)$, let

$$T_\epsilon(f)(x) = \int_{|y|\geq\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy, \qquad \epsilon > 0.$$

(a) Then there exists a number A_p (independent of f or ϵ) so that

$$\|T_\epsilon(f)\|_p \leq A_p \|f\|_p.$$

(b) $\lim_{\epsilon\to o} T_\epsilon(f) = T(f)$ exists in L^p norm, and

$$\|T(f)\|_p \leq A_p \|f\|_p.$$

The natural counterpart of this result is that of convergence almost everywhere. As in other questions involving almost everywhere convergence, it is best to consider the corresponding maximal function. The following theorem gives the answer.

Suppose that Ω satisfies the conditions of the previous theorem. For $f \in L^p(R^n)$, $1 \leq p < \infty$, consider

$$T_\epsilon(f)(x) = \int_{|y|\geq\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy, \qquad \epsilon > 0.$$

(The integral converges absolutely for every x.)

(a) $\lim_{\epsilon\to o} T_\epsilon(f)$ exists for almost x.

(b) Let $T^*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)|$. If $f \in L^1(R^n)$, then the mapping $f \to T^* f$ is of weak type (1, 1).

(c) If $1 \leq p < \infty$, then $\|T^*(f)\|_p \leq \|f\|_p$.

More results have been given in the series papers by Chen (see [2], [3], [9]).

Let $K(x)$ be the truncated Calderón-Zygmund kernel on R^n times an arbitrary bounded radial function, i.e.

$$K(x) = \chi_{|x|>\epsilon} h(|x|) \frac{\Omega(x)}{|x|^n}.$$

Then the maximal singular operator $\sup_\epsilon \left| \int_{R^n} K(y) f(x-y) dy \right|$ is bounded on $L^p(R^n)$, $1 < p < \infty$. However the above conclusion is not true for the case of $n = 1$, which is a very surprising result.

In [3], Chen and Lin showed that if the class, M, is the set with Calderón-Zygmund kernel times the radial function h satisfying

$$\left(\int_0^\infty |h(r)|^S \frac{dr}{r} \right)^{1/S} \leq 1$$

then the above maximal singular operator is bounded on $L^p(R^n)$ for $Sn/(Sn-1) < p < \infty, 1 \leq S \leq 2$, where the region of p is the best possible.

In this paper, we are interesting in finding a class of kernels M such that the following maximal operator is bounded on $L^p(R^n)$ for some p.

$$\sup_{K\in M} \left| p.v. \int_{R^n} K(y) f(x-y) dy \right|.$$

II. NOTATIONS AND PRELIMINARIES

The Hardy-Littlewood maximal operator assigns to each function $g \in L^p(R^n)$, $1 \le p < \infty$, its Hardy-Littlewood maximal function $Mg(x)$ defined at $x \in R^n$ by

$$Mg(x) = \sup_{r>0} \frac{1}{\Omega_n r^n} \int_{|t|\le r} |f(x-r)|dt$$

where Ω_n is the Lebesgue measure of the solid unit sphere. It is well-known that Mg is bounded $L^p(R^n), 1 < p \le \infty$, i.e. $\|Mg\| \le C\|f\|_p$.

The dyadic decomposition is a canonical decomposition of R^n into rectangles. In the case of R^1 it is decomposed as the union of the "disjoint" interval (i.e. whose interior are disjoint) $[2^k, 2^{k+1}]$, $-\infty < k < \infty$, and, $[-2^k, -2^{k+1}]$, $-\infty < k < \infty$. This double collection of intervals, one collection for the positive half-line, the other for the negative halfline, will be the dyadic decomposition of R^1. For the case of R^n it is decomposed as the union of "disjoint" rectangles, which rectangles are products of the intervals which occurs for the dyadic decomposition of each of the axes.

The partial sums operator is defined as the following.

Let l denote an dyadic rectangle in R^n. A rectangle may be a possibly infinite rectangle with sides parallel to the axes. The partial sum operator is, say for l, the multiplier operator with $m = \chi_l$ = characteristic function of the rectangle l, i.e.

$$\widehat{S_l f} = \chi_l \hat{f}, \qquad f \in L^2(R^n) \cap L^p(R^n).$$

Let Δ denote the family of resulting rectangles of the dyadic decomposition of R^n. Suppose $f \in L^p(R^n), 1 < p < \infty$. Then

$$(\sum_{l\in\Delta} |S_l f(x)|^2)^{1/2} \in L^p(R^n),$$

and the ratio

$$\|(\sum_{l\in\Delta} |S_l f(x)|^2)^{1/2}\|_p / \|f\|_p$$

is contained between two bounds (independent of f).

Let H_k denote the linear space of homogeneous polynomials of degree k which are harmonic: the solid spherical harmonics of degree k. Let R, S be two functions defined on S^{n-1}. The inner product of the two functions on the unit sphere S^{n-1} is defined as

$$(R, S) = \int_{S^{n-1}} R(x)\bar{S}(x) d\sigma(x).$$

If P_k denote the linear space of all homogeneous polynomials of degree k, then $P_k = \underline{H}_k + |x|^2 P_{k-2}$. Let's restrict $\underline{H}_k$ to the unit sphere, say H_k then the elements of H_k are the surface spherical harmonics of degree k and

$$L^2(S^{n-1}) = \sum_{k=0}^{\infty} H_k,$$

i.e., if $f \in L^2(S^{n-1})$, then f can be represented as

$$f(x) = \sum_{k=0}^{\infty} Y_k(x), \qquad Y_k \in H_k$$

where the convergence is in the $L^2(S^{n-1})$ norm, and

$$\int_{S^{n-1}} |f(x)|^2 d\sigma(x) = \sum_k \int_{S^{n-1}} |Y_k(x)|^2 d\sigma(x).$$

Here, soppose $\{Y_{k,j}\}_{j=1}^{d_k}$ be a basis of H_k. Then, the function $L^2(S^{n-1})$ has the decomposition

$$\sum_{k=0}^{\infty} \sum_{j=1}^{d_k} a_{k,j} Y_{k,j}$$

in the $L^2(S^{n-1})$ space and $\|f\|_2 = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} |a_{k,j}|^2$.

Let T be a mapping from $L^p(R^n)$ to $L^q(R^n)$, $1 \le p \le \infty$, $1 \le p \le \infty$. Then T is of type (p, q) if

$$\|T(f)\|_q \le A\|f\|_p, \qquad f \in L^p(R^n)$$

where A does not depend on f. Similarly T is of weak-type (p,q) if

$$m\{x : |Tf(x)| > \alpha\} \le (\frac{A\|f\|_p}{\alpha})^q,$$

where $q < \infty$ and A does not depend on f or $\alpha, \alpha > 0$.

Suppose that $1 < r \le \infty$. If T is a sub-additive mapping from $L^1(R^n) + L^r(R^n)$ to the space of measurable functions on R^n which is simultaneously of weak type $(1,1)$ and the weak type (r,r), then T is also of type (p,p), for all p such that $1 < p < r$. More explicitly: Suppose that for all f, $g \in L^1(R^n) + L^r(R^n)$

$$|T(f+g)(x)| \le |Tf(x)| + |Tg(x)|$$

$$m\{x : |Tf(x)| > \alpha\} \le \frac{A_1\|f\|_1}{\alpha}, \qquad f \in L^1(R^n)$$

$$m\{x : |Tf(x)| > \alpha\} \le (\frac{A_r\|f\|_r}{\alpha})^r,$$

(if $r < \infty$; when $r = \infty$ we assume that $\|T(f)\|_\infty \le A\|f\|_\infty$). Then

$$\|T(f)\|_p \le A_p\|f\|_p, \qquad f \in L^p(R^n)$$

for all $1 < p < r$, where A_p depends only on A_1, A_r, and r.

D is the space of all indefinitely differentiable functions on R^n, each with compact support. Let $\alpha = (\alpha_1 \cdot \ldots \cdot \alpha_n)$ and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1 \cdot \ldots \cdot \alpha_n}}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}}$$

be a differential monomial, whose total order is $|\alpha| = \alpha_1 + ... + \alpha_n$.

Suppose we are given two locally integrable functions on R^n, f and g. Then we say that

$$\partial^\alpha f = g,$$

if

$$\int_{R^n} f(x) \frac{\partial^\alpha \phi}{\partial x^\alpha}(x) dx = (-1)^{|\alpha|} \int_{R^n} g(x)\phi(x)dx,$$

for all $\phi \in D$.

For any non-negative integer k, the Sobolev space $L^p_K(R^n) = L^p_K$ is defined as the space of functions f, with $f \in L^p(R^n)$ and where all $\frac{\partial^\alpha f}{\partial x^\alpha}$ exist and $\frac{\partial^\alpha f}{\partial x^\alpha} \in L^p(R^n)$ in the above sense, whenever $|\alpha| \leq K$. This space of functions can be normed by the expression

$$\|f\|_{L^p_K} = \sum_{|\alpha| \leq K} \|\frac{\partial^\alpha f}{\partial x^\alpha}\|_p,$$

where $\frac{\partial^0 f}{\partial x^0} = f$.

III. FORMULATING THE PROBLEM

Let $H^\alpha(S^{n-1})$ denote Sobolev spaces on the sphere S^{n-1} (see [6]). In this paper, we consider the class $M = L^2\Big(R^+, H^\alpha(S^{n-1}), dr/r\Big)$, where $\alpha > (n-1)/2$. That is to say, the member in M, Ω is defined on $R^+ \times S^{n-1}$ and satisfies

$$\int_0^\infty \|\Omega(r,\cdot)\|^2_{H^\alpha(S^{n-1})} \frac{dr}{r} < \infty$$

where $\| \ \|_{H^\alpha}$ denotes the norm in Sobolev space. For α a real number, a function u on S^{n-1} is in the Sobolev space $H^\alpha(S^{n-1})$ if and only if the coefficients $\{a_{k,j}\}$ in its expansion in spherical harmonics satisfy

$$\|u\|^2_{H^\alpha(S^{n-1})} = \sum_{k,j} |a_{k,j}|^2 k^{2\alpha} < \infty$$

where for simplicity, when $k = 0$, we take $k^{2\alpha}$ to be 1. (see [6] page 258 - 259). Therefore, one can write $\Omega(r,\xi) = \sum_1^\infty \sum_1^{d_k} a_{k,j}(r) Y_{k,j}(\xi)$ where $Y_{k,j}$ is a spherical harmonic of degree k, $\{Y_{k,j}\}$ denotes a complete system of normalized spherical harmonics and $d_k = O(k^{n-2})$ as n large. Hence, the kernel, $\Omega(r,\xi)$, in the class of M we consider, satisfies

$$\int_0^\infty \sum_{k,j} |a_{k,j}(r) k^\alpha|^2 \frac{dr}{r} < \infty$$

where $\alpha > (n-1)/2$.

Let us define

$$T_\Omega f(x) = \int_{R^n} \frac{\Omega(|y|, y/|y|)}{|y|^n} f(x-y)\,dy$$

where Ω satisfies

(1) $\Omega \in L^2\Big(R^+, \Big(H^\alpha(S^{n-1})\Big), \frac{dr}{r}\Big)$, $\alpha > (n-1)/2$;

(2) $\Omega(r, \lambda\xi) = \Omega(r, \xi), \qquad \lambda > 0;$

(3) $\int_{S^{n-1}} \Omega(r, \xi) d\sigma(\xi) = 0$ for $r > 0$.

Here we should remark that the proof of the theorem follows the same ideas as in [3].

We should prove the following theorem.

Theorem: Let $2n/(2n-1) < p < \infty$. Suppose f is a smooth function (f is in Schwarz class). Then the operator $\sup_{\Omega \in M} |T_\Omega f|$ is bounded on $L^p(R^n)$, i.e.

$$\| \sup_{\Omega \in M} |T_\Omega f| \|_p \le C_p \|f\|_p.$$

IV. RESULT AND PROOF

Before we prove the theorem, we need the following lemma.

Lemma: Suppose Y_k is a spherical harmonic of degree k. Let $x \in R^n$. Then

$$\int_1^2 \left| \int_{\xi \in S^{n-1}} \exp(irx \cdot \xi)\ Y_{k,j}(\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \leq C \min\{|x|^2,\ |x|^{-\beta}\}$$

where $0 < \beta < 1/2$ and $d\sigma$ is the induced Euclidean measure on S^{n-1}.

Proof: Applying the cancellation of $Y_{k,j}$ (i.e. the mean of $Y_{k,j}$ on the sphere is 0), it is easy to see that the integral is bounded by $C|x|^2$. On the other hand, by the second mean value theorem, it suffices to show that the integral

$$\int_1^2 \left| \int_{S^{n-1}} \exp(irx \cdot \xi) Y_{k,j}(\xi) d\sigma(\xi) \right|^2 dr$$

is dominated by $|x|^{-2\beta}$, for some $0 < \beta < 1/2$. By the Fubini theorem, the above expression is equal to

$$(1) \qquad \int_{S^{n-1}} Y_{k,j}(\eta) \int_{S^{n-1}} \bar{Y}_{k,j}(\xi) \int_1^2 \exp(irx \cdot (\xi - \eta)) dr d\sigma(\xi) d\sigma(\eta)$$

where $\bar{Y}_{k,j}$ denotes the conjugate of $Y_{k,j}$. The integral

$$\int_1^2 \exp(irx \cdot (\xi - \eta)) dr$$

is bounded by 1 and $[x \cdot (\xi - \eta)]^{-1}$. Therefore, it is less than $[x \cdot (\xi - \eta)]^{-\beta}$ for some positive β, $0 < \beta < 1$. By Schwarz inequality, (1) is bounded by

$$C \int_{S^{n-1}} Y_{k,j}(\eta) \left(\int_{S^{n-1}} [x \cdot (\xi - \eta)]^{-2\beta} d\sigma(\xi) \right)^{1/2} d\sigma(\eta) \leq C|x|^{-\beta},$$

if $0 < \beta < 1/2$.

Proof of theorem: Let us write $T_\Omega f(x)$ as

$$
\begin{aligned}
T_\Omega f(x) &= \int_0^\infty \int_{S^{n-1}} \Omega(r,\xi) f(x - r\xi) d\sigma(\xi) \frac{dr}{r} \\
&= \int_0^\infty \sum_{k,j} a_{k,j}(r) k^\alpha \int_{S^{n-1}} \frac{1}{k^\alpha} Y_{k,j}(\xi) f(x - r\xi) d\sigma(\xi) \frac{dr}{r}.
\end{aligned}
$$

Applying Schwarz inequality, the absolute value of the operator, $T_\Omega f$, is dominated by

$$
\begin{aligned}
&\left(\int_0^\infty \sum_{k,j} \left| a_{k,j}(r) k^\alpha \right|^2 \frac{dr}{r} \right)^{1/2} \\
&\quad \cdot \left(\int_0^\infty \sum_{k,j} \left| \frac{1}{k^\alpha} \int_{S^{n-1}} Y_{k,j}(\xi) f(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2} \\
&\leq C \left(\int_0^\infty \sum_{k,j} \frac{1}{k^{2\alpha}} \left| \int_{S^{n-1}} Y_{k,j}(\xi) f(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2}.
\end{aligned}
$$

One takes a smooth function $p(r)$ supported on $\{r \mid 1/2 < |r| < 2\}$ and $\sum_l p(2^l r) = 1$. Let us define the partial sum operators

$$
\widehat{S_l f} = p(2^l |x|) \widehat{f}(x).
$$

Since $f = \sum_i (S_{l+i} f)$ for any integer l, applying Minkowski's inequality,

$$
\begin{aligned}
&|T_\Omega f(x)| \\
&\leq C \left(\sum_{k,j} \sum_l \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \left| \sum_i \int_{S^{n-1}} Y_{k,j}(\xi) (S_{i+l} f)(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2} \\
&\leq C \sum_i \left(\sum_{k,j} \sum_l \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \left| \int_{S^{n-1}} Y_{k,j}(\xi) (S_{i+l} f)(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2} \\
&\equiv C \sum_i B_i f(x).
\end{aligned}
$$

In here, first we compute the L^2-norm of $B_i f(x)$. Since

$$\|B_i f\|_2^2 = \int_{R^n} \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \left| \int_{S^{n-1}} Y_{k,j}(\xi)(S_{i+l}f)(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} dx$$

$$= \sum_{k,j} \sum_{l} \int_{2^l}^{2^{l+1}} \int_{R^n} \frac{1}{k^{2\alpha}} \left| \int_{S^{n-1}} Y_{k,j}(\xi)(S_{i+l}f)(x - r\xi) d\sigma(\xi) \right|^2 dx \frac{dr}{r},$$

applying Plancherel's theorem, we have

$$\|B_i f\|_2^2$$
$$\leq \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \int_{1/2 \leq |2^{l+i}x| \leq 2} \left| \int_{S^{n-1}} \exp(irx \cdot \xi)\ Y_{k,j}(\xi) d\sigma(\xi) \right|^2$$
$$\cdot \left| \hat{f}(x) \right|^2 dx \frac{dr}{r}$$
$$= \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_1^2 \int_{1/2 \leq |2^{l+i}x| \leq 2} \left| \int_{S^{n-1}} \exp(i2^l|x|rx' \cdot \xi) Y_{k,j}(\xi) d\sigma(\xi) \right|^2$$
$$\cdot \left| \hat{f}(x) \right|^2 dx \frac{dr}{r}.$$

Applying the lemma to the last equality, the L^2-norm of $B_i f$ square, $\|B_i f\|_2^2$, is bounded by

$$C \sum_{k,j} \frac{1}{k^{2\alpha}} \sum_{l} \int_{2^{-i-1} \leq |2^l x| \leq 2^{-i+1}} (min\ \{(2^l|x|)^2,\ (2^l|x|)^{-2\beta}\}) |\hat{f}|^2 dx.$$

From the hypotheses, $\sum_{k,j} 1/k^{2\alpha} = \sum_k (1/k^{2\alpha}) d_k < \infty$ when $\alpha > (n-1)/2$. Therefore, we conclude

$$\|B_i f\|_2 \leq\ C\ min\{(2^i)^{-1}, (2^i)^{\beta/2}\} \|f\|_2, \quad 0 < \beta < 1/2. \tag{2}$$

Now we consider L^p-norm of $B_i f$ where $p > 2$. Let us write

$$\|B_i f\|_p^2 =$$
$$\left[\int_{R^n} \left| \left(\sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \left| \int_{S^{n-1}} Y_{k,j}(\xi)(S_{i+l}f)(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2} \right|^p dx \right]^{2/p}.$$

By duality, there exists a function, $g(x)$ in $L^{(p/2)'}$, such that

$$\|B_i f\|_p^2$$
$$= \int_{R^n} \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \left| \int_{S^{n-1}} Y_{k,j}(\xi)(S_{i+l}f)(x - r\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} g(x) dx.$$

Therefore, employing Hõlder's inequality

$$\|B_i f\|_p^2 \le \int_{R^n} \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \left| \left(\int_{S^{n-1}} |Y_{k,j}(\xi)|^2 d\sigma(\xi) \right)^{1/2} \right.$$
$$\left. \cdot \left(\int_{S^{n-1}} |(S_{i+l}f)(x - r\xi)|^2 d\sigma(\xi) \right)^{1/2} \right|^2 \frac{dr}{r} g(x) dx$$
$$\le \int_{R^n} \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} \int_{2^l}^{2^{l+1}} \int_{S^{n-1}} |(S_{i+l}f)(x - r\xi)|^2 d\sigma(\xi) \frac{dr}{r} g(x) dx.$$

After changing variables, we have

$$(3) \quad \|B_i f\|_p^2 \le \int_{R^n} \sum_{k,j} \sum_{l} \frac{1}{k^{2\alpha}} |(S_{i+l}f)(x)|^2 \int_1^2 \int_{S^{n-1}} g(x + 2^l r\xi) d\sigma(\xi) \frac{dr}{r} dx$$
$$\le \sum_{k,j} \frac{1}{k^{2\alpha}} \int_{R^n} \sum_{l} |(S_{i+l}f)(x)|^2 Mg(x) dx$$
$$\le \sum_{k,j} \frac{1}{k^{2\alpha}} \left\| \sum_{l} |(S_{i+l}f)(x)|^2 \right\|_{p/2} \|Mg(x)\|_{(p/2)'}$$
$$\le C \sum_{k,j} \frac{1}{k^{2\alpha}} \|f\|_p^2 \le C\|f\|_p^2$$

Here, Mg denotes the classical Hardy-Littlewood Maximal function. The last two inequalities are obtained by applying Schwarz inequality, Littlewood-Paley theorem (see [8], page 104), the classical maximal function Mg is bounded on $L^p(R^n)$ for $1 < p \le \infty$ and $\sum_{k,j} 1/k^{2\alpha} = \sum_k (1/k^{2\alpha}) d_k < \infty$ if $\alpha > (n-1)/2$. Interpolating between (2) and (3) and applying Minkowski's inequality, we get

$$\| \sup_{\Omega \in M} |T_\Omega f| \| \le C_p \|f\|_p$$

where $2 \leq p < \infty$.

Next let us compute the boundedness of the operator, $B_i f$, if $2n/(2n-1) < p < 2$. Let

$$E_{ikjl}f(x,r) = \int_{S^{n-1}} Y_{k,j}(\xi)(S_{i+l}f)(x - 2^l r\xi)d\sigma(\xi).$$

Then

$$B_i f(x) = \left(\sum_{k,j}\sum_{l} \frac{1}{k^{2\alpha}} \int_1^2 |E_{ikjl}f|^2 \frac{dr}{r}\right)^{1/2}.$$

Therefore in order to show that $B_i f \in L^p(R^n)$, it is equivalent to show

$$E_{ikjl}f(x,r) \in L^p\left(l^2\left(l^2\left(\left(L^2\left([1,2], \frac{dr}{r}\right), l\right), j\right), \frac{1}{k^{2\alpha}}\right), dx\right).$$

Again by duality, there exists a function

$$g(x,k,j,l,r) \in L^{p'}\left(l^2\left(l^2\left(l^2\left(L^2\left([1,2], \frac{dr}{r}\right), l\right), j\right), \frac{1}{k^{2\alpha}}\right), dx\right),$$

such that

$$\begin{aligned}
&\|B_i f\|_p \\
&= \int_{R^n} \sum_{k,j}\sum_{l} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} Y_{k,j}(\xi)(S_{i+l}f)(x - 2^l r\xi)d\sigma(\xi)g(x,k,j,l,r)\frac{dr}{r}dx \\
&= \int_{R^n} \sum_{l}\sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} Y_{k,j}(\xi)g(x + 2^l r\xi,k,j,l,r)(S_{i+l}f)(x)d\sigma(\xi)\frac{dr}{r}dx.
\end{aligned}$$

Repeatedly using Hölder's inequality, $\|B_i f\|_p$ is dominated by

$$\begin{aligned}
&\int_{R^n} \left[\sum_{l}\left(\sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} Y_{k,j}(\xi)g(x + 2^l r\xi,k,j,l,r)d\sigma(\xi)\frac{dr}{r}\right)^2\right]^{1/2} \\
&\qquad \cdot \left(\sum_{l} |(S_{i+l}f)(x)|^2\right)^{1/2} dx \\
&\leq \left\| \left[\sum_{l}\left(\sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} Y_{k,j}(\xi)g(x + 2^l r\xi,k,j,l,r)d\sigma(\xi)\frac{dr}{r}\right)^2\right]^{1/2} \right\|_{p'} \\
&\qquad \cdot \left\| \left(\sum_{l} |(S_{i+l}f)(x)|^2\right)^{1/2} \right\|_p .
\end{aligned}$$

Again, from Littlewood-Paley theorem, we have

$$\|B_i f\|_p$$

$$\leq \left\| \left[\sum_l \left(\sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} Y_{k,j}(\xi) g(x + 2^l r\xi, k, j, l, r) d\sigma(\xi) \frac{dr}{r} \right)^2 \right]^{1/2} \right\|_{p'} \|f\|_p$$

$$\equiv \|(D_i g)^{1/2}\|_{p'} \|f\|_p.$$

Since $\|(D_i g)^{1/2}\|_{p'} = \|D_i g\|^2_{p'/2}$, $p' > 2$, there exists a function h in $L_{(p'/2)'}(R^n)$, here $p'/2 > 1$, such that

$$\|D_i g\|_{p'/2}$$

$$= \int_{R^n} \sum_l \left(\sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} Y_{k,j}(\xi) g(x + 2^l r\xi, k, j, l, r) d\sigma(\xi) \frac{dr}{r} \right)^2 h(x) dx$$

$$\leq \int_{R^n} \sum_l \left[\sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \left(\int_{S^{n-1}} |Y_{k,j}(\xi)|^2 d\sigma(\xi) \right)^{1/2} \right.$$

$$\left. \cdot \left(\int_{S^{n-1}} |g(x + 2^l r\xi, k, j, l, r)|^2 d\sigma(\xi) \right)^{1/2} \frac{dr}{r} \right]^2 h(x) dx$$

As before, repeatedly using the Schwarz inequality, we get

$$\|D_i g\|_{p'/2}$$

$$\leq \int_{R^n} \sum_l \left[\sum_{k,j} \frac{1}{k^{2\alpha}} \left(\int_1^2 \int_{S^{n-1}} |g(x + 2^l r\xi, k, j, l, r)|^2 d\sigma(\xi) \frac{dr}{r} \right)^{1/2} \right]^2 h(x) dx$$

$$\leq \left(\sum_{k,j} \frac{1}{k^{2\alpha}} \right) \int_{R^n} \sum_l \sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} |g(x + 2^l r\xi, k, j, l, r)|^2 d\sigma(\xi) \frac{dr}{r} h(x) dx.$$

Let Nf denote the spherical maximal function, i.e.,

$$Nh(x) = \sup_{\epsilon > 0} \int_{\xi \in S^{n-1}} |h(x - \epsilon\xi)| d\sigma(\xi).$$

Then, from the last inequality,

$$\|D_i g\|_{p'/2} \leq C \int_{R^n} \sum_l \sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 \int_{S^{n-1}} h(x - 2^l r\xi) d\sigma(\xi) g^2(x, k, j, l, r) \frac{dr}{r} dx$$

$$\leq C \int_{R^n} \sum_l \sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 Nh(x) g^2(x, k, j, l, r) \frac{dr}{r} dx$$

$$\leq C \left\| \int_{R^n} \sum_l \sum_{k,j} \frac{1}{k^{2\alpha}} \int_1^2 g^2(x, k, j, l, r) \frac{dr}{r} \right\|_{p'/2} \|Nh(x)\|_{(p'/2)'}.$$

It is well known that the spherical maximal function is bounded on $L^p(R^n)$ when $p > n/(n-1)$ (see [7]). The other term is bounded by definition of $g(x,k,j,l,r)$. Combining all of these, we have

$$\|B_i f\|_p \leq C_p \|f\|_p \quad \text{if} \quad 2n/(2n-1) < p < 2.$$

Again interpolating between (2) and the above inequality and using Minkowski's inequality, we have

$$\| \sup_{\Omega \in M} |T_\Omega f| \|_p \leq C_p \|f\|_p$$

if $2n/(2n-1) < p < 2$. The proof is complete.

V. CONCLUSION AND FUTURE WORK

We have proved the maximal singular operator, $\sup|K * f|$, is bounded in $L^p(R^n)$ where the supremum is over the family of kernels. Future work includes potential applications of the theorem.

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