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A large number of results concerning the sums of certain infinite series involving Legendre functions (including conical functions) are derived. The generating principle chosen here is based upon the application of Integral transforms of the Fourier, Hankel and Meijer type of finite and infinite character. This particular choice leads to a group of three different types of Legendre series: Fourier series type, Cardinal series type and addition theorem type. The presented material is believed to be new and reduces, upon specialization of certain parameters, in a number of cases to already known relations. In view of the scope of the results, it seemed best to give their derivation in a rather condensed form; and for the sake of conciseness, frequently using tables as a method of arrangement. A list of notations and definitions of

the occuring functions is given in an appendix.

APPLICATIONS OF INTEGRAL TRANSFORMS IN GENERATING SUMMABLE SERIES INVOLVING HIGHER TRANSCENDENTAL FUNCTIONS

by

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APPLICATIONS OF INTEGRAL TRANSFORMS IN GENERATING SUMMABLE SERIES INVOLVING HIGHER TRANSCENDENTAL FUNCTIONS

INTRODUCTION

Earlier investigations concerning series or other relations involving the Legendre functions $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$, $\mathcal{P}_\nu^\mu(z)$, $\mathcal{Q}_\nu^\mu(z)$ (for their definition see apendix) were mostly restricted not only to the first two functions, for which the argument x is real and between -1 and $+1$, but also to the case when μ is an integer and ν an integer or half an odd integer. This was due to the fact that these functions occured in potential wave propagation and heat conduction problems of one-valued character. Later investigations were extended to unrestricted parameters and unrestricted arguments. Numerous expansion theorems and other relations for these functions have been found and represent standard material now. (See, for instance, [8, vol. 1, Ch. III], [15], [37], [38].) From the standpoint of mathematical physics, addition theorems for spherical wave functions are of particular interest. The classical Gegenbauer addition theorem [41] has been extended in recent times to hypershperical and spherical vector wave functions [3], [7], [13], [39]. Furthermore, the investigation of multivalent physical problems led to Legendre functions of non-discreet order. Finally, the consideration of conical structures led to the conical functions, for example, Legendre functions of the order $-\frac{1}{2}+ix$ with real x. The first classical contributions [29], [31], were vastly extended under the influence of microwave physics. Of special importance were the investigations of problems centering around electrostatic, electromagnetic and acoustic problems involving finite conical structures. This led to a number of results concerning conical functions [10], [18], [32]. Also they have been recognized as the kernel of an integral inversion formula (Mehler transform) [29], [8, vol. 1, p. 174] whose generalization (generalized Mehler transform) [24], [22], has become important in certain boundary value problems. All in all there has been added recently much information of great variety. It is attempted here to investigate the possibility of finding suitable "methods" for obtaining groups of results, rather than isolated ones, concerning expansions as mentioned before. Investigation has revealed that integral transform methods fulfill this requirement.

The use of integral transform methods for the summation of infinite series or for the transformation of such into one which converges more rapidly is widespread. The best known, and perhaps the earliest, example of such a series transformation is Poisson's summation formula

$$\text{(i)} \quad F(v) = \sum_{n=-\infty}^{\infty} e^{ia(v+nb)} f(v+nb) = \frac{1}{b} \sum_{m=-\infty}^{\infty} g\left(a+\frac{2\pi m}{b}\right) e^{-i2\pi mv/b}$$

where

$$\text{(ii)} \qquad g(u) = \int_{-\infty}^{\infty} f(t) e^{iut} dt$$

represents the "exponential" Fourier transform of the function $f(x)$ involved in the summation. This equation obviously represents the expansion of the periodic function $f(v)$ with period b as defined by the first sum in (i) and in the form of its Fourier series as expressed by the second sum in (i). The well-known classical transformation formula for the elliptic theta function is a simple example for (i). Other applications, for instance, yield the functional equation of the Riemann and of the Lerch zeta function and Hurwitz series for Hurwitz's zeta function [35]. Of special interest is the class of those functions $f(x)$ occuring in the summation of (i) which are such that their Fourier transform $g(u)$ given by (ii) vanishes identically outside a final interval of the variable u. In this case the second series in (i) reduces to a finite number of terms and the left hand side series is summed.

A similar but quite different approach to utilize integral transforms for the purpose of transforming or summing series was used by Mellin (see for instance [16],

[28]). Here the relation analogous to (i) is (under certain conditions)

$$\text{(iii)} \qquad \sum_{n=0}^{\infty} f(n+a) = \sum \text{Residue } [\zeta(s,a)g(s)]$$

with

$$\text{(iv)} \qquad g(s) = \int_0^{\infty} f(t)t^{s-1}dt,$$

the Mellin transform of $f(x)$ and $\zeta(s,a)$ is Hurwitz's zeta function. The summation at the right hand side of (iii) has to be taken over all singularities of the function $h(s) = \zeta(s,a)g(s)$ which are located in a certain left half of the complex s-plane. If the summation function $f(x)$ is such that its Mellin transform $g(s)$ contains only a finite number of singularities in this half plane then the summation (iii) is carried out. It is reminded that $\zeta(s,a)$ contains only one singularity in the finite s-plane, a pole of the first order of residue one. However, this second possibility, which was just outlined, shall not be pursued. Regarding the use of Poisson's summation formula (i), for the suggested purpose it is clear that the requirements with respect to the function f(x) are: firstly, $f(x)$ must involve either a single or a combination of Legendre functions where the variable x may occur in the argument or in one of the two parameters; secondly, its Fourier transform $g(u)$ must vanish identically outside a finite interval of u, where $a < u < b$, for example. Then, since (ii) is inverted by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{-iut}du = \frac{1}{2\pi}\int_{a}^{b} g(u)e^{-iut}du,$$

such functions $f(t)$ which involve Legendre functions and which can be represented in the form of a finite Fourier integral can be used. It is clear, of course, that this method applies for functions other than Legendre functions. Thus, we generate infinite series of the Fourier type whose coefficients involve Legendre functions with the summation index, either occuring in the argument or in one of the parameters. Series of this type will be considered in Part I.

The investigation in Part II is based upon the application of finite integral transform results whose kernel functions can be expressed in the form of a cardinal series [11], [12], [42].

$$\text{(v)} \qquad K(st) = \frac{\sin[\pi(s-c)]}{\pi}\sum_{r=-\infty}^{\infty}(-1)^r \frac{K[t(c+r)]}{s-c-r} .$$

Multiplying both sides of (v) by a suitable function $f(t)$, integrating both sides over an interval (a,b) in which (v) is valid, and assuming that term by term integration is permissible, one obtains

$$\int_a^b f(t)K(st)dt = \frac{\sin[\pi(s-c)]}{\pi} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{s-c-r} \int_a^b f(t)K[t(c+r)]dt.$$

Let

$$h(s) = \int_a^b f(t)K(st)dt.$$

Then

$$h(s) = \frac{\sin[\pi(s-c)]}{\pi} \sum_{r=-\infty}^{\infty} \frac{(-1)^r h(c+r)}{s-c-r}.$$

Hence, the function $h(s)$ is expressed as a cardinal series. The possibility of a repeated application of this procedure is obvious. The kernel function $K(y)$ chosen here is the Fourier cosine and the Fourier sine kernel. For the purpose pursued here such functions $f(t)$ can be used whose finite Fourier cosine and sine transforms lead to Legendre functions and hence to a group of cardinal type Legendre series. To simplify the operations under Part I and Part II a list of finite Fourier transforms has been provided. Part of the quoted results have been taken from existing sources [33] and some results are new.

Part III contains series expansions of the "addition theorem" type based on the addition theorems of Gegenbauer [41] and some of its specializations. One of them, for instance, is

$$w^{-\nu}J_\nu(w) = (\tfrac{1}{2}zZ)^{-\nu}\,\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)J_{\nu+n}(z)J_{\nu+n}(Z)C_n^\nu(\cos\varphi),$$

$J_\nu(z)$ being the Bessel function of order ν, and $C_n^\nu(\cos\varphi)$ the Gegenbauer polynomial. (For definitions see appendix.)

By the application of a suitable integral transform operation to both sides of the equation above, other relations can be obtained which are either similar to or generalizations of already known results.

I. LEGENDRE FUNCTION SERIES OF THE FOURIER TYPE

Poisson's summation formula is used here in a slightly different form [40, p. 60],

$$(1)\qquad \sum_{n=-\infty}^{\infty} e^{inx} f(v+nd) = \frac{1}{d}\sum_{m=-\infty}^{\infty} g\left(\frac{2\pi m+x}{d}\right) e^{-i\frac{v}{d}(2\pi m+x)}$$

with

$$(2)\qquad g(u) = \int_{-\infty}^{\infty} f(t) e^{iut} dt.$$

This formula is now applied to such functions $f(t)$ involving Legendre functions and for which $g(u)$ vanishes identically outside a finite u interval. The left side of (1) represents a Fourier series regarding x as variable whose coefficients $f(v+nd)$ are Legendre functions. The (terminating) series on the right side represents then the sum of this Fourier series. The summation formula (1) is applied here to an even function $f(t)$.
Then

$$(3)\qquad g(u) = 2\int_{0}^{\infty} f(t)\cos(ut)dt = 2G_c(u) = g(-u)$$

and $G_c(u)$ is the Fourier cosine transform of $f(t)$. If $f(t)$ is such that $G_c(u)$ is zero for $u > a > 0$ then (1) becomes

(4) $$\sum_{n=-\infty}^{\infty} e^{inx} f(v+nd) = \frac{2}{d} \sum_{m=m_1}^{m_2} G_c\left(\frac{2\pi m+x}{d}\right) e^{-i\frac{v}{d}(2\pi m+x)} .$$

The limits m_1 and m_2 in (4) are given by

(5) $$m_1 = -\left[\frac{ad+x}{2\pi}\right], \qquad m_2 = \left[\frac{ad-x}{2\pi}\right]$$

where the symbol $[y]$ means the largest positive number smaller or equal to y. When $\frac{1}{2}[ad+x]$ or (and) $\frac{1}{2}[ad-x]$ is an integer, then the term of the sum corresponding to m_1 or (and) m_2 has to be halved.

In the following Table I, a list of Fourier cosine transforms involving Legendre functions which vanish identically for $u > a$ is given. The first 2 pairs are known [33, p. 21, 35]. The remaining pairs have been evaluated by this author. (For a derivation see appendix.) The letter k denotes the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} (1-k^2 \sin^2 t)^{-1/2} dt.$$

Integrals (as those occuring in the following Table I) with respect to one of the parameters of Legendre functions have been investigated by MacRobert [25], [26], [27]. But only the first integral in the table below is given. Others, of a type different from those here, are not given in closed form but in the form of infinite series.

$f(x)$	$G_c(y) = \int_0^\infty f(x)\cos(xy)\,dx$	
$P^{\mu}_{-\frac{1}{2}+x}(\cos a)$	$\frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\mu)}(\sin a)^{\mu}(\cos y-\cos a)^{-\mu-\frac{1}{2}}$	$0<y<a$
$0<a<\pi$	0	$y>a$
$\mathcal{P}^{\mu}_{-\frac{1}{2}+ix}(\cosh a)$	$\frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\mu)}(\sinh a)^{\mu}(\cosh a-\cosh y)^{-\mu-\frac{1}{2}}$	$y<a$
$a>0$	0	$y>a$
$P_{\nu}^{\frac{a}{\pi}x}(b)P_{\nu}^{-\frac{a}{\pi}x}(b)$	$\frac{\pi}{2a}P_{\nu}[1-2(1-b^2)\cos^2(\frac{\pi}{2a}y)]$	$y<a$
$0<b<1$	0	$y>a$
$\mathcal{P}_{\nu}^{\frac{a}{\pi}x}(b)\mathcal{P}_{\nu}^{-\frac{a}{\pi}x}(b)$	$\frac{\pi}{2a}\mathcal{P}_{\nu}[1+2(b^2-1)\cos^2(\frac{\pi}{2a}y)]$	$y<a$
$b>1$	0	$y>a$
$\cos(ax)\mathcal{Q}_{-\frac{1}{2}+\frac{a}{\pi}x}(b)\cdot\mathcal{Q}_{-\frac{1}{2}-\frac{a}{\pi}x}(b)$	$\frac{\pi^2}{2a}[b^2-\sin^2(\frac{\pi}{2a}y)]^{-\frac{1}{2}}\cdot K[\cos(\frac{\pi}{2a}y)(b^2-\sin^2\frac{\pi y}{2a})^{-\frac{1}{2}}]$	$y<a$
$b>1$	0	$y>a$

Table I

(cont. on following page)

$f(x)$	$G_c(y) = \int_0^\infty f(x) \cos(xy)\, dx$	
$\mathcal{P}^{\mu}_{-\frac{1}{2}+\frac{1}{2}ix}(\cosh a) \cdot \mathcal{P}^{-\mu}_{-\frac{1}{2}+\frac{1}{2}ix}(\cosh a)$	$(\sinh a)^{-1} P_{\mu-\frac{1}{2}}\left(\frac{2\sinh^2 y}{\sinh^2 a} - 1\right)$	$y<a$
	0	$y>a$

Table I

(cont. from page 10)

The application of this table to Poisson's formula (4) gives then, when the abbreviation

(6) $$\alpha(m) = e^{-i\frac{v}{d}(2\pi m+x)}$$

is introduced,

(7) $$\sum_{n=-\infty}^{\infty} e^{inx} P^{\mu}_{-\frac{1}{2}+v+nd}(\cos a) = \frac{(2\pi)^{\frac{1}{2}}}{d\Gamma(\frac{1}{2}-\mu)}(\sin a)^{\mu} \cdot$$
$$\cdot \sum_{m_1}^{m_2} \alpha(m)[\cos(\frac{2\pi m+x}{\alpha}) - \cos a]^{-\mu-\frac{1}{2}},$$

$0 < a < \pi$, Re $\mu < \frac{1}{2}$; Re $\mu < -\frac{1}{2}$ when $x = \pm ad$.

(8) $$\sum_{n=-\infty}^{\infty} e^{inx} \mathcal{P}_{-\frac{1}{2}+i(v+nd)}(\cosh a) = \frac{(2\pi)^{\frac{1}{2}}}{d\Gamma(\frac{1}{2}-\mu)}(\sinh a)^{\mu} \cdot$$
$$\cdot \sum_{m_1}^{m_2} \alpha(m)[\cosh a - \cosh(\frac{2\pi m+x}{d})]^{-\mu-\frac{1}{2}},$$

Re $\mu < \frac{1}{2}$; Re $\mu < -\frac{1}{2}$ when $x = \pm ad$.

(9) $$\sum_{n=-\infty}^{\infty} e^{inx} P_v^{\frac{a}{\pi}(v+nd)}(b)\, P_v^{-\frac{a}{\pi}(v+nd)}(b) =$$
$$= \frac{\pi}{ad} \sum_{m_1}^{m_2} \alpha(m) P_v\{1 - 2(1-b^2)\cos^2[\frac{\pi}{2ad}(2\pi m+x)]\},$$

$0 < b < 1$, $x \neq \pm ad$.

$$(10)\quad \sum_{n=-\infty}^{\infty} e^{inx} \mathcal{P}_\nu^{\frac{a}{\pi}(v+nd)}(b)\mathcal{P}_\nu^{-\frac{a}{\pi}(v+nd)}(b) =$$

$$= \frac{\pi}{ad}\sum_{m_1}^{m_2} \alpha(m)\mathcal{P}_\nu\{1 + 2(b^2-1)\cos^2[\tfrac{\pi}{2ad}(2\pi m+x)]\},$$

$$b > 1, \quad x \neq \pm ad .$$

$$(11)\quad \sum_{n=-\infty}^{\infty} e^{inx}\cos[a(v+nd)]\,\mathcal{Q}_{-\frac{1}{2}+\frac{a}{\pi}(v+nd)}(b)\mathcal{Q}_{-\frac{1}{2}-\frac{a}{\pi}(v+nd)}(b) =$$

$$= \frac{\pi^2}{ad}\sum_{m_1}^{m_2} \alpha(m)\{b^2 - \sin^2[\tfrac{\pi}{2ad}(2\pi m+x)]\}^{-\frac{1}{2}} \cdot$$

$$\cdot\, K\{\cos[\tfrac{\pi}{2ad}(2\pi m+x)][b^2 - \sin^2(\tfrac{2\pi^2 m+\pi x}{2ad})]^{-\frac{1}{2}}\},$$

$$b > 1.$$

$$(12)\quad \sum_{n=-\infty}^{\infty} e^{inx}\mathcal{P}_{-\frac{1}{2}+\frac{1}{2}i(v+nd)}(\cosh a)\mathcal{P}_{-\frac{1}{2}-\frac{1}{2}i(v+nd)}(\cosh a) =$$

$$= \frac{2}{d}(\sinh a)^{-1}\sum_{m_1}^{m_2} \alpha(m)P_{\mu-\frac{1}{2}}\Big[\frac{2\sinh^2(\frac{2\pi m+x}{d})}{\sinh^2 a} - 1\Big],$$

$$x \neq 0, \quad x \neq ad.$$

The restrictions secure the convergence of the left hand side series. This follows from the asymptotic behavior of the Legendre functions for large parameters [8, p. 162].

All these relations seem to be new. MacRobert [26] and Kendall [19] gave the relations

$$\operatorname{cosec}(a/2) = d\sum_{0}^{\infty} \varepsilon_n P_{nd-\frac{1}{2}}(\cos a),$$

$$0 < a < \pi, \quad 0 < d < 2\pi/a.$$

This is clearly a very special case of (7). Put there $x = 0$, $\mu = 0$, $\nu = 0$ and $d < 2\pi/a$, then by (5), $m_1 = m_2 = 0$ and the right hand side of (7) reduces to just one term $(m = 0)$ and

$$\sum_{-\infty}^{\infty} P_{-\frac{1}{2}+nd}(\cos a) = \frac{1}{d}\,\frac{1}{\sin(\frac{a}{2})}\,.$$

But, since

$$P_\nu(x) = P_{-\nu-1}(x),$$

one has

$$P_{-\frac{1}{2}+nd}(\cos a) = P_{-\frac{1}{2}-nd}(\cos a)$$

upon which

$$\sum_{0}^{\infty} \varepsilon_n P_{-\frac{1}{2}+nd}(\cos a) = \frac{1}{d}\operatorname{cosec}(\tfrac{a}{2}).$$

Further series of a form similar to those (7) to (12) have also been given by MacRobert [26]. But the results are given in the form of an infinite integral and not explicit as it is the case here.

II. CARDINAL TYPE LEGENDRE FUNCTION SERIES

As mentioned in the introduction, the finite Fourier cosine and sine transform is used here. These kernels admit the representation in the form of cardinal series by means of the well-known formulas:

$$(13)\quad \cos(\lambda x) = -\frac{\lambda}{\pi}\sin(\lambda\pi)\sum_{n=0}^{\infty}(-1)^n \varepsilon_n \frac{\cos(nx)}{n^2-\lambda^2}, \qquad -\pi \leqq x \leqq \pi,$$

$$(14)\quad \cos(\lambda x) = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{(n+\frac{1}{2})\cos[(n+\frac{1}{2})x]}{(n+\frac{1}{2})^2-\lambda^2}, \qquad -\pi < x < \pi,$$

$$(15)\quad \sin(\lambda x) = -\frac{2}{\pi}\sin(\lambda\pi)\sum_{n=1}^{\infty}(-1)^n \frac{n\sin(nx)}{n^2-\lambda^2}, \qquad -\pi < x < \pi,$$

$$(16)\quad \sin(\lambda x) = 2\frac{\lambda}{\pi}\cos(\lambda\pi)\sum_{n=0}^{\infty}\frac{(-1)^n\sin[(n+\frac{1}{2})x]}{(n+\frac{1}{2})-\lambda^2}, \qquad -\pi \leqq x \leqq \pi,$$

These formulas can also be regarded as the decomposition into partial fractions of the respective meromorphic functions of λ, $\cos(\lambda x)/\sin(\lambda\pi)$, $\cos(\lambda x)/\cos(\lambda\pi)$, $\sin(\lambda x)/\sin(\lambda\pi)$, $\sin(\lambda x)/\cos(\lambda\pi)$. Since regarding as functions of y both sides in the above equations involve even or odd functions, it is sufficient to restrict oneself to the y-interval between 0 and π. A multiplication of both sides with a function $f(x)$ and subsequent integration between the limits 0 and a, $(0<a<\pi)$ yields the equations:

$$(17)\quad G_c(\lambda) = -\frac{\lambda}{\pi}\sin(\lambda\pi)\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{G_c(n)}{n^2-\lambda^2},$$

$$(18)\quad G_c(\lambda) = \frac{2}{\pi}\cos(\lambda\pi)\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{G_c(n+\frac{1}{2})}{(n+\frac{1}{2})^2-\lambda^2},$$

$$(19)\quad G_s(\lambda) = -\frac{2}{\pi}\sin(\lambda\pi)\sum_{n=1}^{\infty}(-1)^n\, n\,\frac{G_s(n)}{n^2-\lambda^2},$$

$$(20)\quad G_s(\lambda) = 2\frac{\lambda}{\pi}\cos(\lambda\pi)\sum_{n=0}^{\infty}(-1)^n\frac{G_s(n+\frac{1}{2})}{(n+\frac{1}{2})^2-\lambda^2}.$$

The abbreviations G_c and G_s represent the finite Fourier cosine or Fourier sine transform of $f(x)$ respectively

$$(21)\qquad G_c(y) = \int_0^a f(x)\cos(xy)\,dx,$$

$$(22)\qquad G_s(y) = \int_0^a f(x)\sin(xy)\,dx.$$

Substituting x by $\pi - x$ in (13) to (16) and performing the same operation again the following results are obtained:

$$(23)\quad \cos(\lambda\pi)G_c(\lambda) + \sin(\lambda\pi)G_s(\lambda) = -\frac{\lambda}{\pi}\sin(\lambda\pi)\sum_{n=0}^{\infty}\frac{G_c(n)}{n^2-\lambda^2},$$

$$0 < a < 2\pi,$$

(24) $\cos(\lambda\pi)G_c(\lambda) + \sin(\lambda\pi)G_s(\lambda)$

$$= \frac{2}{\pi}\cos(\lambda\pi)\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\frac{G_s(n+\frac{1}{2})}{(n+\frac{1}{2})^2-\lambda^2},$$

$$0 < a < 2\pi,$$

(25) $$\sin(\lambda\pi)G_c(\lambda) - \cos(\lambda\pi)G_s(\lambda) = \frac{2}{\pi}\sin(\lambda\pi)\sum_{n=1}^{\infty} n\frac{G_s(n)}{n^2-\lambda^2},$$

$$0 < a < 2\pi,$$

(26) $\sin(\lambda\pi)G_c(\lambda) - \cos(\lambda\pi)G_s(\lambda)$

$$= 2\frac{\lambda}{\pi}\cos(\lambda\pi)\sum_{n=0}^{\infty}\frac{G_c(n+\frac{1}{2})}{(n+\frac{1}{2})^2-\lambda^2},$$

$$0 < a < 2\pi.$$

The right hand side series here are the same as under (17) to (20) but without the alternating sign. It is, however, necessary in this case to have both expressions (21) and (22); but the resulting relation holds in a wider interval for a. The Table II below shows a list of finite Fourier transforms. They are the inverse transforms of Table I.

$f(x)$	$G_c(y) = \int_0^a f(x)\cos(xy)dx$
$[\cos(\frac{c}{a}x) - \cos c]^{-\mu-\frac{1}{2}}$ $0 < c < \pi$, $\mathrm{Re}\,\mu < \frac{1}{2}$	$(\frac{1}{2}\pi)^{\frac{1}{2}} \frac{a}{c}\Gamma(\frac{1}{2}-\mu)(\sin c)^{-\mu} \cdot$ $\cdot P^{\mu}_{-\frac{1}{2}+\frac{a}{c}y}(\cos c)$
$[\cosh c - \cosh(\frac{c}{a}x)]^{-\mu-\frac{1}{2}}$ $\mathrm{Re}\,\mu < \frac{1}{2}$	$(\frac{1}{2}\pi)^{\frac{1}{2}} \frac{a}{c}\Gamma(\frac{1}{2}-\mu)(\sinh c)^{-\mu} \cdot$ $\cdot \mathcal{P}^{\mu}_{-\frac{1}{2}+i\frac{a}{c}y}(\cosh c)$
$P_\nu[1-2(1-b^2)\cos^2(\frac{\pi x}{2a})]$ $b < 1$	$aP_\nu^{\frac{a}{\pi}y}(b)\ P_\nu^{-\frac{a}{\pi}y}(b)$
$\mathcal{P}_\nu[1+2(b^2-1)\cos^2(\frac{\pi x}{2a})]$ $b>1$	$a\mathcal{P}_\nu^{\frac{a}{\pi}y}(b)\mathcal{P}_\nu^{-\frac{a}{\pi}y}(b)$
$P_{\mu-\frac{1}{2}}[2(\frac{\sinh c\frac{x}{a}}{\sinh c})^2 - 1]$	$\frac{1}{2}\pi\frac{a}{c}\sinh c \cdot$ $\cdot\mathcal{P}^{\mu}_{-\frac{1}{2}+i\frac{a}{2c}y}(\cosh c)\mathcal{P}^{-\mu}_{-\frac{1}{2}+i\frac{a}{2c}y}(\cosh c)$

Table II

(cont. on next page)

$f(x)$	$G_c(y) = \int_0^a f(x) \cos (xy) dx$
$[b^2 - \sin^2(\frac{\pi x}{2a})]^{-\frac{1}{2}} \cdot$ $\cdot\ K\{\cos(\frac{\pi x}{2a})[b^2 - \sin^2(\frac{\pi x}{2a})]^{-\frac{1}{2}}\}$ $b > 1$	$\frac{a}{\pi} \cos(ay)\, \mathcal{Q}_{-\frac{1}{2}+\frac{a}{\pi}y}(b) \cdot$ $\cdot\ \mathcal{Q}_{-\frac{1}{2}-\frac{a}{\pi}y}(b)$

Table II

(cont. from page 18)

It does not seem possible to obtain the corresponding finite Fourier sine transforms $G_s(y)$ for the functions $f(x)$ in the left column of the above table. These results applied to (17) and (18) yield then the 12 identities: (In the first 4 formulas $0 < k < 1$, $0 < c < \pi$; in the remaining eight, $k, c, > 0$; the case $k = 0$ is trivial in as much as these formulas become simply obvious identities)

(27) $$P^{\mu}_{-\frac{1}{2}+k\lambda}(\cos c) = -\frac{\lambda}{\pi}\sin(\pi\lambda)\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{P^{\mu}_{-\frac{1}{2}+kn}(\cos c)}{n^2-\lambda^2},$$

(28) $$P^{\mu}_{-\frac{1}{2}+k\lambda}(\cos c) = \frac{2}{\pi}\cos(\pi\lambda)\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{P^{\mu}_{-\frac{1}{2}+k(n+\frac{1}{2})}(\cos c)}{(n+\frac{1}{2})^2-\lambda^2},$$

(29) $$P_{\nu}^{k\lambda}(\cos c)P_{\nu}^{-k\lambda}(\cos c) = -\frac{\lambda}{\pi}\sin(\pi\lambda)\cdot$$
$$\cdot\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{P_{\nu}^{kn}(\cos c)P_{\nu}^{-kn}(\cos c)}{n^2-\lambda^2},$$

(30) $$P_{\nu}^{k\lambda}(\cos c)P_{\nu}^{-k\lambda}(\cos c) = \frac{2}{\pi}\cos(\pi\lambda)\cdot$$
$$\cdot\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{P_{\nu}^{k(n+\frac{1}{2})}(\cos c)P_{\nu}^{-k(n+\frac{1}{2})}(\cos c)}{(n+\frac{1}{2})^2-\lambda^2},$$

(31) $$\mathcal{P}^{\mu}_{-\frac{1}{2}+ik\lambda}(\cosh c) = -\frac{\lambda}{\pi}\sin(\pi\lambda)\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{\mathcal{P}^{\mu}_{-\frac{1}{2}+ikn}(\cosh c)}{n^2-\lambda^2},$$

(32) $$\mathcal{P}_{-\frac{1}{2}+ik\lambda}(\cosh c) = \frac{2}{\pi}\cos(\pi\lambda)\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{\mathcal{P}_{-\frac{1}{2}+ik(n+\frac{1}{2})}(\cosh c)}{(n+\frac{1}{2})^2-\lambda^2},$$

(33) $$\mathcal{P}_{\nu}^{k\lambda}(\cosh c)\mathcal{P}_{\nu}^{-k\lambda}(\cosh c) = -\frac{\lambda}{\pi}\sin(\pi\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{\mathcal{P}_{\nu}^{kn}(\cosh c)\mathcal{P}_{\nu}^{-kn}(\cosh c)}{n^2-\lambda^2},$$

(34) $$\mathcal{P}_{\nu}^{k\lambda}(\cosh c)\mathcal{P}_{\nu}^{-k\lambda}(\cosh c) = \frac{2}{\pi}\cos(\pi\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{\mathcal{P}_{\nu}^{k(n+\frac{1}{2})}(\cosh c)\mathcal{P}_{\nu}^{-k(n+\frac{1}{2})}(\cosh c)}{(n+\frac{1}{2})^2-\lambda^2},$$

(35) $$\mathcal{P}^{\mu}_{-\frac{1}{2}+ik\lambda}(\cosh c)\mathcal{P}^{-\mu}_{-\frac{1}{2}+ik\lambda}(\cosh c) = -\frac{\lambda}{\pi}\sin(\pi\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{\mathcal{P}^{\mu}_{-\frac{1}{2}+ikn}(\cosh c)\mathcal{P}^{-\mu}_{-\frac{1}{2}+ikn}(\cosh c)}{n^2-\lambda^2}$$

(36) $$\mathcal{P}^{\mu}_{-\frac{1}{2}+ik\lambda}(\cosh c)\mathcal{P}^{-\mu}_{-\frac{1}{2}+ik\lambda}(\cosh c) = \frac{2}{\pi}\cos(\pi\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{\mathcal{P}^{\mu}_{-\frac{1}{2}+ik(n+\frac{1}{2})}(\cosh c)\mathcal{P}^{-\mu}_{-\frac{1}{2}+ik(n+\frac{1}{2})}(\cosh c)}{(n+\frac{1}{2})^2-\lambda^2},$$

(37) $$\cos(\pi k\lambda)\,\mathcal{P}_{-\frac{1}{2}+k\lambda}(\cosh c)\,\mathcal{P}_{-\frac{1}{2}-k\lambda}(\cosh c) = -\frac{\lambda}{\pi}\sin(\pi\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\frac{\cos(\pi nk)\,\mathcal{P}_{-\frac{1}{2}+nk}(\cosh c)\,\mathcal{P}_{-\frac{1}{2}-nk}(\cosh c)}{n^2-\lambda^2},$$

(38) $$\cos(\pi k\lambda)\,\mathcal{P}_{-\frac{1}{2}+k\lambda}(\cosh c)\,\mathcal{P}_{-\frac{1}{2}-k\lambda}(\cosh c) = \frac{2}{\pi}\cos(\pi\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})\frac{\cos[\pi k(n+\frac{1}{2})]\,\mathcal{P}_{-\frac{1}{2}+k(n+\frac{1}{2})}(\cosh c)\,\mathcal{P}_{-\frac{1}{2}-k(n+\frac{1}{2})}(\cosh c)}{(n+\frac{1}{2})^2-\lambda^2}.$$

All the left side functions represent entire functions of λ.

The series (28) is a generalization of the well-known series by Dougall [6], [8, vol. 1, p. 167], [26]. The series (31) and (32) are similar to those (27) and (28), but they involve the so-called conical functions. In general, all these formulas represent either generalizations of known results or are of a similar form [8, vol. 1, pp. 167, 168], [25], [32].

III. LEGENDRE FUNCTION SERIES OF THE ADDITION THEOREM TYPE

The following addition formulas [8, vol. 2, p. 101], [41] are chosen from the well-known addition theorems of Gegenbauer to generate these Legendre function series.

$$(39)\quad w^{-\nu}J_\nu(w) = (\tfrac{1}{2}zZ)^{-\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)C_n^\nu(\cos\varphi)\cdot J_{\nu+n}(z)J_{\nu+n}(Z),$$

$$(40)\quad w^{-\nu}J_{-\nu}(w) = (\tfrac{1}{2}zZ)^{-\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(-1)^n(\nu+n)C_n^\nu(\cos\varphi)\cdot J_{-\nu-n}(z)J_{\nu+n}(Z),$$

$$(41)\quad w^{-\nu}K_\nu(w) = (\tfrac{1}{2}zZ)^{-\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)C_n^\nu(\cos\varphi)\cdot I_{\nu+n}(z)K_{\nu+n}(Z),$$

where

$$(42)\quad w = (z^2 + Z^2 - 2zZ\cos\varphi)^{\frac{1}{2}} \qquad \text{for } z \leq Z.$$

But if $z > Z$, interchange z and Z. These equations are also valid when z and Z are complex (for conditions, see [41]).

By performing certain operations on both sides of these equations, one obtains similar relations for other types of functions (see [1], [13] and [16]). Since the main concern here is for Legendre functions, only procedures which yield these types are given.

The addition theorem for the Legendre function $P_\nu(\cos\omega)$ with

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\psi$$

is [8, vol. 1, p. 168]

(43) $$P_\nu(\cos\omega) = P_\nu(\cos\theta)P_\nu(\cos\theta') + 2\sum_{m=1}^{\infty}(-1)^m P_\nu^{-m}(\cos\theta)P_\nu^m(\cos\theta')\cos(m\psi)$$

$$= P_\nu(\cos\theta)P_\nu(\cos\theta') + 2\sum_{m=1}^{\infty}\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(\cos\theta)P_\nu^m(\cos\theta')\cos(m\psi).$$

Similar formulas hold for $Q_\nu(\cos\omega)$, $\mathcal{P}_\nu(\cosh\gamma)$ and $\mathcal{Q}_\nu(\cosh\gamma)$, where

$$\cosh\gamma = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta\cos\psi,$$

[8, vol. 1, p. 168].

As is well-known, (43) has been established by Hobson [15] for arbitrary ν, θ and θ'. However, it has not been possible to derive an addition theorem for the P_ν^μ, Q_ν^μ, $\mathcal{P}_\nu^\mu$ and $\mathcal{Q}_\nu^\mu$ for arbitrary ν and μ (μ an integer). A number of interesting relations of the type above have been given by Cowling [2] and Henrici [14], essentially Legendre series involving products of Legendre functions.

But these expansions concern mostly finite series. It is shown in the following that addition theorems of the type [43] for arbitrary lower and upper script can be given for the case $\theta = \theta'$ or $\gamma = \gamma'$ respectively. The author encountered insurmountable difficulties in attempting the general case. The difficulty lies in the fact that it does not seem possible to obtain the Hankel or the Meijer transforms of certain required functions in a closed form.

Assuming $z = Z = x$ in (42), we obtain from (39) to (41)

(44) $$2^{-2\nu}(\sin \tfrac{\varphi}{2})^{-\nu}\, x^{\nu} J_{\nu}(2x \sin \tfrac{\varphi}{2}) = \Gamma(\nu) \sum_{n=0}^{\infty} (\nu+n) C_n^{\nu}(\cos \varphi) J_{\nu+n}^2(x),$$

(45) $$2^{-2\nu}(\sin \tfrac{\varphi}{2})^{-\nu}\, x^{\nu} J_{-\nu}(2x \sin \tfrac{\varphi}{2}) = \Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n (\nu+n) C_n^{\nu}(\cos \varphi) \cdot J_{-\nu-n}(x) J_{\nu+n}(x),$$

(46) $$2^{-2\nu}(\sin \tfrac{\varphi}{2})^{-\nu}\, x^{\nu} K_{\nu}(2x \sin \tfrac{\varphi}{2}) = \Gamma(\nu) \sum_{n=0}^{\infty} (\nu+n) C_n^{\nu}(\cos \varphi) I_{\nu+n}(x) \cdot K_{\nu+n}(x).$$

These formulas shall be used to produce the desired results. We apply to each of the formulas the Fourier cosine, the Hankel and the Meijer transforms with x as a variable and y as a transformation parameter. It is obvious that this can be accomplished only if the particular transformation is known for both sides. The formulas that shall be needed here are listed in the table below.

$f(x)$	$\int_0^\infty f(x)\cos(xy)\,dx$		
$x^{\nu}J_{-\nu}(ax)$	$(2a)^{\nu}\pi^{\frac{1}{2}}[\Gamma(\frac{1}{2}-\nu)]^{-1}(a^2-y^2)^{-\nu-\frac{1}{2}}$,	$y<a$	33, 62
Re $\nu < \frac{1}{2}$	0,	$y>a$	
$J_{\mu}(ax)J_{-\mu}(ax)$	$\frac{1}{2}a^{-1}P_{\mu-\frac{1}{2}}(\frac{y^2}{2a^2}-1)$,	$y<2a$	33, 65
$x^{\nu}K_{\nu}(ax)$,	$\frac{1}{2}\pi^{\frac{1}{2}}(2a)^{\nu}\Gamma(\frac{1}{2}+\nu)(a^2+y^2)^{-\nu-\frac{1}{2}}$,	$y<2a$	33, 87
Re $\nu > -\frac{1}{2}$	0,	$y>2a$	
$I_{\nu}(ax)K_{\nu}(ax)$	$(\frac{1}{2}ab)^{-\frac{1}{2}}\mathcal{Q}_{\nu-\frac{1}{2}}(1+\frac{y^2}{2a^2})$		33, 87
Re $\nu > -\frac{1}{2}$			

Table III. Fourier cosine transforms

$f(x)$	$\int_0^\infty f(x)J_\mu(xy)dx$
$x^\nu J_\nu(ax)$	$2^\nu\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\nu)[(\frac{1}{2}+\frac{1}{2}\mu-\nu)]^{-1}y^{-1}\ .$
$\mathrm{Re}(\mu+2\nu) > -1$	$\cdot\ (y^2-a^2)^{-\frac{1}{2}\nu}P^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}(1-2\frac{a^2}{y^2}),$
$\mathrm{Re}\ \nu < 1$	$y > a,$
$J_\nu^2(ax)$	$y^{-1}\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\nu)[\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\nu)]^{-1}y^{-1}\ .$
$\mathrm{Re}(\mu+2\nu) > -1$	$\cdot\ \{P^{-\nu}_{-\frac{1}{2}+\frac{1}{2}\mu}[(1-4\frac{a^2}{y^2})^{\frac{1}{2}}]\}^2$
$x^\nu J_{-\nu}(ax)$	$2^\nu y^{-1}(y^2-a^2)^{-\frac{1}{2}\nu}P_{\frac{1}{2}\mu-\frac{1}{2}}(1-2\frac{a^2}{y^2}),$
$\mathrm{Re}\ \mu > -1$	$y > a$
$\mathrm{Re}\ \mu < 1$	
$J_{-\nu}(ax)J_\nu(ax)$	$y^{-1}P^{\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1-4\frac{a^2}{y^2})^{\frac{1}{2}}]P^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1-4\frac{a^2}{y^2})^{\frac{1}{2}}]$
$\mathrm{Re}\ \mu > -1$	

Table IV. Hankel transforms

(cont. on next page)

$x^{\nu}K_{\nu}(ax)$	$2^{\nu}y^{-1}(a^{2}+y^{2})^{-\frac{1}{2}\nu}e^{-i\pi\nu}\mathcal{Q}^{\nu}_{\frac{1}{2}\mu-\frac{1}{2}}(1+2\frac{a^{2}}{y^{2}})$	
$\mathrm{Re}\ \nu > -\frac{1}{2}$		
$I_{\nu}(ax)K_{\nu}(ax)$	$y^{-1}\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\nu)[\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\nu)]^{-1}.$	
$\mathrm{Re}\ \nu > -\frac{1}{2}$	$\cdot\mathcal{P}^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1+4\frac{a^{2}}{y^{2}})^{\frac{1}{2}}]\mathcal{Q}^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1+4\frac{a^{2}}{y^{2}})^{\frac{1}{2}}]$	9, 20

Table IV. Hankel transforms

(cont. from page 27)

$f(x)$	$\int_0^\infty f(x)K_\mu(xy)\,dx$	
$x^\nu J_\nu(ax)$ $\mathrm{Re}(2\nu\pm\mu)>-1$	$2^{\nu-1}\Gamma(\nu+\frac{1}{2}+\frac{1}{2}\mu)\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\mu)y^{-1}\cdot$ $\cdot(a^2+y^2)^{-\frac{1}{2}\nu}\mathcal{P}^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}(1+2\frac{a^2}{y^2})$	
$J_\nu^2(ax)$ $\mathrm{Re}(2\nu\pm\mu)>-1$	$\frac{1}{2}\Gamma(\nu+\frac{1}{2}+\frac{1}{2}\mu)\Gamma(\nu+\frac{1}{2}-\frac{1}{2}\mu)y^{-1}\cdot$ $\cdot\{\mathcal{P}^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1+4\frac{a^2}{y^2})^{\frac{1}{2}}]\}^2$	9, 138
$x^\nu J_{-\nu}(ax)$	$2^{\nu-1}\pi\sec(\frac{1}{2}\pi\mu)(a^2+y^2)^{-\frac{1}{2}\nu}y^{-1}$ $\cdot\mathcal{P}^{\nu}_{\frac{1}{2}\mu-\frac{1}{2}}(1+2\frac{a^2}{y^2})$	
$J_{-\nu}(ax)J_\nu(ax)$ $-1<\mathrm{Re}\,\mu<1$	$\frac{1}{2}\pi y^{-1}\sec(\frac{1}{2}\pi\mu)\mathcal{P}^{\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1+4\frac{a^2}{y^2})^{\frac{1}{2}}]\cdot$ $\cdot\mathcal{P}^{-\nu}_{\frac{1}{2}\mu-\frac{1}{2}}[(1+4\frac{a^2}{y^2})^{\frac{1}{2}}]$	9, 138

Table V. Meijer transforms

The numbers in the last column refer to the source. So, for instance 9, 138 means [9, p. 138]. Open spaces mean that no reference is available.

a) Fourier transform results

If the Fourier cosine transform is taken on both sides of (45) and (46), it follows immediately after a slight change of the parameters that when $\operatorname{Re}\nu < \frac{1}{2}$,

(47) $$\Gamma(\nu)\sum_{n=0}^{\infty}(-1)^n(\nu+n)C_n^{\nu}(\cos\varphi)P_{\nu+n-\frac{1}{2}}(-x) =$$

$$= \frac{2\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)}(2x - 2\cos\varphi)^{-\nu-\frac{1}{2}}, \qquad \cos\varphi < x,$$

$$= 0, \qquad \text{otherwise.}$$

(48) $$\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)C_n^{\nu}(\cos\varphi)\mathcal{Q}_{\nu+n-\frac{1}{2}}(z) =$$

$$= \pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}+\nu)(2z - 2\cos\varphi)^{-\nu-\frac{1}{2}}, \qquad \operatorname{Re}\nu > -\tfrac{1}{2}.$$

These two formulas are very similar to some already known [8, vol. 1, p. 166]. For the special case $\nu = 0$, by considering that

$$\lim_{\nu\to 0}\Gamma(\nu)C_{\alpha}^{\nu}(\cos\varphi) = \frac{2}{\alpha}\cos(\alpha\varphi), \qquad \alpha \neq 0,$$

they become equal.

b) Hankel transform results

Taking suitable Hankel transforms of both sides of

(44) - (46), and using the given Hankel transform table, one obtains the following:

$$(49)\quad \Gamma(\mu)\sum_{n=0}^{\infty}(\mu+n)\,\frac{\Gamma(1+\nu+\mu+n)}{\Gamma(1+\nu-\mu-n)}\,C_n^{\nu}(\cos\varphi)[P_{\nu}^{-\mu-n}(x)]^2 =$$

$$= (1-x^2)^{\frac{1}{2}\mu}(4\sin\tfrac{\varphi}{2})^{-\mu}\,\frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)}\;\cdot$$

$$\cdot\,[1-(1-x^2)\sin^2(\tfrac{\varphi}{2})]^{-\frac{1}{2}\mu}\,P_{\nu}^{-\mu}[1-2(1-x^2)\sin^2\tfrac{\varphi}{2}],$$

$$(50)\quad \Gamma(\mu)\sum_{n=0}^{\infty}(-1)^n(\mu+n)C_n^{\nu}(\cos\varphi)P_{\nu}^{\mu+n}(x)P_{\nu}^{-\mu-n}(x) =$$

$$= (1-x^2)^{\frac{1}{2}\mu}(4\sin\tfrac{\varphi}{2})^{-\mu}[1-(1-x^2)\sin^2(\tfrac{\varphi}{2})]^{-\frac{1}{2}\mu}\;\cdot$$

$$\cdot\;P_{\nu}^{\mu}[1-2(1-x^2)\sin^2\tfrac{\varphi}{2}],$$

$$(51)\quad \Gamma(\mu)\sum_{n=0}^{\infty}(\mu+n)C_n^{\mu}(\cos\varphi)\Gamma(\mu+1+\nu+n)\Gamma(\mu+n-\nu)[\mathcal{P}_{\nu}^{-\mu-n}(z)]^2 =$$

$$= (z^2-1)^{\frac{1}{2}\mu}(4\sin\tfrac{\varphi}{2})^{-\mu}[1+(z^2-1)\sin^2(\tfrac{\varphi}{2})]^{-\frac{1}{2}\mu}\cdot$$

$$\cdot\,\mathcal{P}_{\nu}^{\mu}[1+2(z^2-1)\sin^2\tfrac{\varphi}{2}],$$

$$(52)\quad \Gamma(\mu)e^{i\pi\mu}\sum_{n=0}^{\infty}(-1)^n(\mu+n)\,\frac{\Gamma(1+\nu+\mu+n)}{\Gamma(1+\nu-\mu-n)}\,\cdot$$

$$\cdot\; C_n^{\mu}(\cos\varphi)\mathcal{P}_\nu^{-\mu-n}(z)\mathcal{Q}_\nu^{-\mu-n}(z) =$$

$$= (z^2-1)^{\frac{1}{2}\mu}(4\sin\tfrac{\varphi}{2})^{-\mu}[1+z^2-1)\sin^2\tfrac{\varphi}{2}]^{-\frac{1}{2}\mu}\,\cdot$$

$$\cdot\,\mathcal{Q}_\nu^{\mu}[1+2(z^2-1)\sin^2\tfrac{\varphi}{2}],$$

$$(53)\quad \Gamma(\mu)\sum_{n=0}^{\infty}(-1)^n(\mu+n)C_n^{\mu}(\cos\varphi)\mathcal{P}_\nu^{\mu+n}(z)\mathcal{P}_\nu^{-\mu-n}(z) =$$

$$= (z^2-1)^{\frac{1}{2}\mu}(4\sin\tfrac{\varphi}{2})^{-\mu}[1+(z^2-1)\sin^2\tfrac{\varphi}{2}]^{-\frac{1}{2}\mu}\,\cdot$$

$$\cdot\,\mathcal{P}_\nu^{\mu}[1+2(z^2-1)\sin^2\tfrac{\varphi}{2}].$$

It is easily recognized that for $\mu = 0$, under the consideration that

$$\lim_{\mu\to 0} C_\alpha^{\mu}(\cos\varphi)\;\Gamma(\mu) = \frac{\cos\alpha\varphi}{\alpha}$$

for $(\alpha \neq 0)$, the formulas above yield the known classical results [8, vol. 1, p. 168].

For similar results see [2], [14], [20], [21], [22], [23], [30], [36], [43].

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APPENDICES

APPENDIX I.

List of notations

Legendre functions (general):

$$\mathcal{P}_\nu^\mu(z) = \frac{\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}}{\Gamma(1-\mu)}\, {}_2F_1(-\nu,\ \nu+1;\ 1-\mu;\ \frac{1-z}{2})$$

$$\mathcal{Q}_\nu^\mu(z) = \frac{2^{-\nu-1}e^{i\pi\mu}}{\Gamma(\frac{3}{2}+\nu)}\, \pi^{\frac{1}{2}}\Gamma(\mu+\nu+1)\, z^{-\mu-\nu-1}(z^2-1)^{\frac{1}{2}\mu} \cdot$$

$$\cdot\ {}_2F_1(\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu+2}{2};\ \nu+\frac{3}{2};\ \frac{1}{z^2})$$

z is a point of the complex z plane cut along the real axis from 1 to $-\infty$.

Legendre functions on the cut:

For a real argument x between -1 and $+1$ the following definition is used

$$P_\nu^\mu(x) = \frac{\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu}}{\Gamma(1-\mu)}\, {}_2F_1(-\nu,\ \nu+1;\ 1-\mu;\ \frac{1-x}{2})$$

$$Q_\nu^\mu(x) = \tfrac{1}{2}e^{-i\pi\mu}[e^{-\frac{1}{2}i\pi\mu}\mathcal{Q}_\nu^\mu(x+i0) + e^{\frac{1}{2}i\pi\mu}\mathcal{Q}_\nu^\mu(x-i0)],$$

For both, $-1 < x < 1$.

In their respective limits of their arguments they are linearly independent solutions of Legendre's differential equation

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + [\nu(\nu+1) - \frac{\mu^2}{1-z^2}]w = 0.$$

The conical function $\mathcal{P}^{\mu}_{-\frac{1}{2}+ix}(\cosh a)$ is immediately defined by the definition for the $\mathcal{P}^{\mu}_{\nu}(z)$. However, it is of an advantage to give a different form

$$\mathcal{P}^{\mu}_{-\frac{1}{2}+ix}(\cosh a) = (2\pi \sinh a)^{-\frac{1}{2}}\cdot$$

$$\cdot\{e^{-iax}\frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-\mu-ix)}\, {}_2F_1(\tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\mu;\ 1+ix;\ \frac{-e^{-a}}{2\sinh a}) +$$

$$+ e^{iax}\frac{\Gamma(ix)}{\Gamma(\frac{1}{2}-\mu+ix)}\, {}_2F_1(\tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\mu;\ 1-ix;\ -\frac{-e^{-a}}{2\sinh a})\}.$$

Gegenbauer's functions

The Gegenbauer functions are defined by

$$C^{\nu}_{\alpha}(z) = \frac{\Gamma(\alpha+2\nu)}{\Gamma(\alpha+1)\Gamma(\nu)}(2\pi)^{\frac{1}{2}}2^{-\nu}(z^2-1)^{\frac{1}{4}-\frac{1}{2}\nu}\mathcal{P}^{\frac{1}{2}-\nu}_{\alpha+\nu-\frac{1}{2}}(z)$$

$$= \frac{\Gamma(\alpha+2\nu)}{\Gamma(\alpha+1)\Gamma(2\nu)}\, {}_2F_1(\alpha+2\nu,\ -\alpha;\ \tfrac{1}{2}+\nu;\ \tfrac{1-z}{2}).$$

They are analytic and one-valued functions of z in the complex z-plane cut along the real z-axis from -1 to

∞. If $\alpha = n$, $(n = 1, 2, 3, \ldots .)$ they reduce to polynomials of degree n in z. (Gegenbauer polynomials.

$$C_\alpha^{\frac{1}{2}}(z) = \mathcal{P}_\alpha(z)$$

There exists, furthermore, the relation

$$\lim_{\nu \to 0} (\nu) C_\alpha^\nu(\cos\varphi) = \frac{2}{\alpha}\cos(\alpha\varphi), \quad \alpha \neq 0.$$

Bessel functions:

They are solutions of Bessel's differential equation:

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)w = 0,$$

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}, \qquad \text{Bessel function,}$$

$$Y_\nu(z) = J_\nu(z)\cot(\pi\nu) - \frac{J_{-\nu}(z)}{\sin(\pi\nu)},$$

Neumann function,

$$H_\nu^{(1),(2)}(z) = J_\nu(z) \pm iY_\nu(z),$$

Hankel functions.

Modified Bessel functions:

They are solutions of the modified Bessel differential equation:

$$\frac{d^2w}{dz^2} + \frac{1}{2}\frac{dw}{dz} - \left(1+\frac{\nu^2}{z^2}\right)w = 0$$

$$I_\nu(z) = e^{-i\frac{1}{2}\pi\nu} J_\nu(ze^{i\frac{\pi}{2}}) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)},$$

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)} [I_{-\nu}(z) - I_\nu(z)]$$

$$= \pm \frac{1}{2} i\pi e^{\pm \frac{1}{2} i\pi\nu} H_\nu^{\left\{\substack{(1)\\(2)}\right\}} (ze^{\pm i\frac{\pi}{2}}).$$

APPENDIX II

We shall prove here only two of the new Fourier transform formulas listed in Table 1.

(a) $$I_a = \int_0^\infty \mathcal{P}_\nu^{x}(z)\mathcal{P}_\nu^{-x}(z)\cos(xy)dx =$$

$$= \tfrac{1}{2}\mathcal{P}_\nu[1+2(z^2-1)\cos^2(\tfrac{1}{2}y)], \quad 0 < y < \pi,$$

$$= 0, \quad y > \pi.$$

(b) $$I_b = \int_0^\infty \mathcal{P}^{\mu}_{-\frac{1}{2}+ix}(z)\mathcal{P}^{-\mu}_{-\frac{1}{2}+ix}(z)\cos(xy)dx =$$

$$= \tfrac{1}{2}(z^2-1)^{-\frac{1}{2}}P_{\mu-\frac{1}{2}}[2(z^2-1)^{-1}\sinh^2(\tfrac{1}{2}y) - 1],$$

$$\sinh(\tfrac{1}{2}y) < (z^2-1)^{-\frac{1}{2}},$$

$$= 0, \quad \sinh(\tfrac{y}{2}) > (z^2-1)^{-\frac{1}{2}}.$$

The proof of the remaining ones are analogous. We replace part of the integrand by a suitable integral representation, thus transforming the single integral into a repeated one. Interchanging the order of integration, which is easily seen to be permissible in our case, and performing integration, we get the desired result.

Proof of (a): We use the relation [9, vol. 2, p. 138(21)]

(c) $\mathcal{P}_\nu^\mu(z)\mathcal{P}_\nu^{-\mu}(z) = -\frac{2}{\pi}\sin(\pi\nu)\cdot$

$$\cdot\int_0^\infty J_\mu(\tfrac{1}{2}t\sqrt{z^2-1})\; J_{-\mu}(\tfrac{1}{2}t\sqrt{z^2-1})K_{2\nu+1}(t)dt$$

which represents the product of two Legendre functions as a Meijer transform. Inserting (c) into (a), we obtain

$$I_a = -\frac{2}{\pi}\sin(\pi\nu)\int_0^\infty K_{2\nu+1}(t)\;[\int_0^\infty J_x(\tfrac{t}{2}\sqrt{z^2-1})\cdot$$

$$\cdot\; J_{-x}(\tfrac{t}{2}\sqrt{z^2-1})\cos(xy)\,dx]dt.$$

The inner integral is known [33, p. 84]

$$\int_0^\infty J_x(\tfrac{t}{2}\sqrt{z^2-1})\; J_{-x}(\tfrac{t}{2}\sqrt{z^2-1})\cos(xy)dx =$$

$$= -\frac{1}{2}J_0(t\sqrt{z^2-1}\cos\tfrac{y}{2}), \qquad y < \pi,$$

$$= 0, \qquad y > \pi.$$

Hence,

$$I_a = -\frac{1}{\pi}\sin(\pi\nu)\int_0^\infty K_{2\nu+1}(t)J_0(t\sqrt{z^2-1}\cos\tfrac{y}{2})dt, \quad y < \pi,$$

$$= 0, \qquad y > \pi.$$

This integral is also known [9, vol. 2, p. 145]

$$\int_0^\infty J_0(ax)K_\nu(xy)dx = \frac{\pi}{2y}\sec(\tfrac{1}{2}\pi\nu)\mathcal{P}_{\frac{\nu}{2}-\frac{1}{2}}(1+2\frac{a^2}{y^2}).$$

Thus, (a) follows.

Proof of (b): We substitute (c) in I_b when ν is replaced by $-\frac{1}{2}+ix$,

$$I_b = \frac{2}{\pi}\int_0^\infty J_\mu\left(\frac{t}{2}\sqrt{z^2-1}\right) J_{-\mu}\left(\frac{t}{2}\sqrt{z^2-1}\right)\cdot$$

$$\cdot\left[\int_0^\infty K_{i2x}(t)\cosh(\pi x)\cos(xy)\,dx\right]dt.$$

The inner integral is known [33, p. 99] and equals

$\frac{\pi}{4}\cos(t\sinh y)$, therefore,

$$I_b = \frac{1}{2}\int_0^\infty J_\mu\left(\frac{t}{2}\sqrt{z^2-1}\right) J_{-\mu}\left(\frac{t}{2}\sqrt{z^2-1}\right)\cos(t\sinh y)\,dt.$$

The right hand side integral I is also known [33, p. 65] and has the value

$$I = \frac{1}{2}a^{-1}P_{\nu-\frac{1}{2}}\left(\frac{y^2}{2a^2}-1\right), \qquad y < 2a,$$

$$= 0, \qquad y > 2a.$$

Consequently, assertion (b) follows.