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This thesis considers one of the classical problems in the actuarial mathematics literature, the decay of the probability of ruin in the collective risk model. The claim number process $N(t)$ is assumed to be a renewal process, the resulting model being referred as the Sparre Andersen risk model. The inter-claim times form a sequence of independent identically distributed random variables. The additional non-classical feature is that the company invests in an asset with stochastic returns. A very general integro-differential equation is derived for expected values of functions of this renewal risk model with stochastic returns. Moreover, as a particular case, an integro-differential equation is derived for the probability of ruin, under very general conditions regarding the claim sizes, claim arrivals and the returns from investment. Through this unified approach, specific integro-differential equations of the ruin probability may be written for various risk model scenarios, allowing the asymptotic analysis of the ruin probabilities.

Renewal Risk Processes with Stochastic Returns on Investments - A Unified Approach and Analysis of the Ruin Probabilities

by

Corina D. Constantinescu

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Chair of the Department of Mathematics

Dean of the Graduate School

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Corina D. Constantinescu, Author

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1. INTRODUCTION

1.1. The Question

One of the basic questions of the actuarial science is the following: “What is the fair premium to be charged so that the insurance company may cover the risk of future losses and stay solvent, at the same time?”

Recently, because of the rising interaction of financial instruments and insurance products a more sophisticated question may be posed: “ What is the optimal investment strategy so that for a given premium rate the company may cover future losses and stay solvent at the same time?” .

In this thesis an “all or nothing” investment strategy is considered. Surprisingly, it can be shown that investing all the surplus in a risky asset is more dangerous than not investing at all. Here dangerous refers to the possibility of “ruin”, which refers to the capital of the insurance company becoming negative for the first time. This unfortunate event is quantified by the ruin probability $\Psi(u)$, where u is the initial capital of the insurance business.

Since it is difficult to calculate the probability of ruin exactly, one looks at the asymptotic decay of the ruin probability, as the initial capital gets really large. The desired scenario is that the probability of ruin will decay extremely fast to zero as u gets extremely large.

It is known from actuarial risk theory that in the case of no-investments the probability of ruin decays exponentially fast as u grows to infinity. In this thesis it is shown that, if the insurance company starts with an initial capital, u, and invests everything in a stock, whose price follows a geometric Brownian motion, the probability of ruin, $\Psi(u)$, will go to zero more slowly than if no investments are made. Therefore, a company will be better off choosing not to invest anything into a risky asset than choosing to invest everything into it.

1.2. Brief history of the collective risk model

The collective risk model was introduced by Cramér and Lundberg in early 1900's (Lundberg, 1909; Cramér, 1930). It is a compound Poisson model, as basic as initial capital plus premium collected minus paid claims. Assumptions of independence of claim times and sizes are made, in the sense that the time waited until another claim occurs is not related to the severity of that claim. Also, a basic "net profit condition" is necessary, saying that on average the incoming premium rate is bigger than the outgoing paid claim rate. This is a standard model for non-life insurance, simple enough to calculate probabilities of interest, but too simple to be realistic. Over time, the model was improved by considering randomness on the incoming premiums, time dependence or interest earned on the surplus. But the classical model is still relevant because it explains the two major causes of big losses: frequent claims and large claims. Most of the techniques developed for the Cramér-Lundberg model are useful for the more realistic renewal risk model.

The probability of ruin, $\Psi(u)$, is the subject of analysis of the insurance risk model. However, it is generally difficult to determine the function $\Psi(u)$ explicitly, therefore, bounds and approximations to the probability of ruin $\Psi(u)$ are investigated.

Initially, the primary focus of investigation was on specific conditions on the claim size distribution (Cramér, 1930; Gerber, 1973). For example, in the classical risk model under the Cramér-Lundberg condition regarding the claim sizes, the ruin probability presents an exponential decay as the initial capital $u \to \infty$ (Cramér, 1930). If the Cramér-Lundberg condition is weakened, the asymptotic behavior of the ruin probability changes dramatically. For instance, in the case of sub-exponentially distributed claim sizes, the ruin is asymptotically determined by a large claim (Embrechts et al., 1997).

Later on, some studies analyzed the effect of adding a perturbation modeled by a diffusion on the classical model. These perturbations model uncertainties in the rate at which premiums are being collected, or in the rate of returns on the investments performed by the insurance company (Paulsen and Gjessing, 1997; Paulsen, 1998a; Ma and Sun, 2003; Yuen et al., 2004; Pergamenshchikov and Zeitouny, 2006). Except for the case of very heavy-tailed negative jumps (Paulsen, 2002), the perturbations influence the asymptotic behavior of the ruin probability. For example, if the perturbation is a Brownian motion, the ruin probability still presents an exponential decay (Schmidli, 1995). On the other hand, if the risk model is altered by a geometric Brownian motion, then the asymptotic decay rate of the ruin probability is at best algebraic. This latter perturbation may be interpreted as the price of the risky asset bought with the capital at hand (Frolova et al., 2002; Constantinescu, 2003). The need of an optimal investment strategy that minimizes the probability of ruin is the subject of very recent studies (Gayer et al., 2003).

In 1957, Sparre Andersen suggested a more realistic model, by considering non-exponential inter-arrival times. The focus switched towards the choice of the distribution of the inter-arrival times and a new theory developed for the case "where the intensity for occurrence of claims depends on the space of time which has passed since the last claim." (Andersen, 1957)

Sparre Andersen model permits non-exponential inter-arrival times but retains the Cramér-Lundberg assumptions on the claim sizes. Bounds and asymptotics on the time value of ruin were derived under this new assumption regarding the model. For instance, it was shown that for any inter-arrival time distribution, the probability of ruin still has an exponential bound (Andersen, 1957). Also Lundberg type bounds were obtained for the joint distribution of the surplus immediately before and at ruin, by constructing an exponential martingale (Ng and Yang, 2005). For particular distributions for the inter-arrival times, the papers of Dickson and Hipp (1998); Dickson (2002); Li and Garrido (2004); Gerber and Shiu (2005) analyze either the asymptotic behavior of the probability of ruin or the moments of the time of ruin.

The subject of the most recent analyses is the effect of a perturbation introduced in the Sparre Andersen model. If the model is perturbed by an independent diffusion process and the inter-arrival time is $Erlang(n)$ a generalization of the defective renewal equation is presented in Li and Garrido (2003). This thesis sets up the ground for an analysis of the asymptotic behavior of the probability of ruin for a Sparre Andersen model perturbed by a non-negative stochastic process. More specifically, a very comprehensive equation is derived for the probability of ruin under very general conditions regarding the distributions of the inter-arrival times and claim sizes as well as the investments model. This equation can be extended to expected values of functions of the time of ruin, penalty and severity of ruin, the so-called Gerber-Shiu functions (Gerber and Shiu, 1998).

In this thesis, a particular case is considered, namely the investment price is modeled by a geometric Brownian motion. The decay of the probability of ruin is analyzed for the inter-arrival times distributed sum of exponentials with identical or different parameters . It is shown that the decay rate is slower than the exponential rate of the no investments case. More precisely the decay rate is at best algebraic, for the inter-arrival times distributions mentioned.

1.3. Brief history of the mathematical tools

Nowadays, the calculation of the probability of ruin is a powerful concept used in pricing insurance products and in optimal control problems.

In the analysis of the probability of ruin a multitude of tools have been used, closely following the trends from applied mathematics. For instance, the first estimate for an upper bound for probability of ruin was derived by deep complex variables methods (Cramér, 1930). Nowadays there is a very elegant derivation, for the same result, using martingales methods (Rolski et al., 1999).

The relationship between the diffusion processes and the partial differential equations brought new insights in the analysis of the problems. Hence, most of the original proofs appearing in this thesis use real analysis results and partial differential equations combined with probabilistic arguments. For instance, one of the results presented in this thesis, the well-known exponential decay of the probability of ruin in the classical Poisson compound model, may be derived by means of completely monotone functions and basic properties of Laplace transforms alone. The proof relies on an analysis of the solutions of the integro-differential equation verified by the ruin probability.

The later developments of the financial mathematics tools impacted the development of the actuarial field. The natural query of investing the capital arise. The first investments considered were in risk-less assets. Hence, together with premiums the company receives deterministic returns on investments (Teugels and Sundt, 1995). Recently, a great deal of the actuarial literature considers diffusion processes in the model (Paulsen and Gjessing, 1997; Paulsen, 1998a,b; Ma and Sun, 2003; Yuen et al., 2004; Pergamenshchikov and Zeitouny, 2006), representing returns from a risky investments.

In the case of the classical compound Poisson model combined with a geometric Brownian motion, a generalization of the (Frolova et al., 2002) result is obtained by means of Laplace transforms and Karamata-Tauberian arguments. The generalization consists in relaxing the conditions on the claim size distribution (Constantinescu, 2003). Following the same steps, an analysis of the decay of the Laplace transform of the time of ruin, and implicitly of the probability of ruin in finite time, for a classical model embedded into a geometric Brownian motion is presented in this paper.

1.4. Insurance risk models

However, the focus of this PhD thesis is first in deriving a general integro-differential equation for the probability of ruin and secondly in the analysis of the asymptotic behavior of the probability of ruin under various scenarios. The different scenarios, may be considered varying one of the random processes involved in the model.

Under a constant stream of premiums, c, for a given initial capital u, the event of ruin may be caused by any of the following three factors: large claim, frequent claims and “poor” returns from the investments. Mathematically this is translated in model incorporating three processes:

1. $(X_k)_k$ the claim sizes process

2. $(\tau_k)_k$ the inter-arrival times or $(N(t))_t$ the claim arrival process

3. $(Z_t^u)_{t\geq 0}$ the worth of a portofolio that invest the initial capital u and the incoming premiums over time t in a risky asset. Throughout this thesis, this is referred as the investment process.

The analysis requires some conditions for each of these random variables, more precisely on their distributions.

First, the claim amounts should be "light" or have "well-behaved" distributions. This condition means that large claims are not impossible, but the probability of their occurrence decreases exponentially fast to zero as the threshold x becomes larger and larger. Heavy-tailed distributions or sub-exponential distributions will not be considered in this thesis.

Second, the density f_τ considered for the time in between claims $(\tau_k)_{k\geq 0}$ satisfies an ordinary differential equation with constant coefficients, formally denoted by

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0,$$

with $\mathcal{L}^*(\frac{d}{dt})f_\tau(t)$ denoting the formal adjoint of the linear operator $\mathcal{L}$. Examples of densities satisfying such a equation are the exponential density,

$$f_\tau(t) = \lambda e^{-\lambda t} \quad \text{with} \quad (\frac{d}{dt} + \lambda)f_\tau(t) = 0, \quad \text{i.e.} \quad \mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt} + \lambda),$$

and the sum of exponentials, the so-called $Erlang(n, \beta)$ distributions with $n \in N$,

$$f_\tau(t) = \frac{\beta^n}{\Gamma(n)} t^{n-1} e^{-\beta t} \quad \text{with} \quad (\frac{d}{dt} + \beta)^n f_\tau(t) = 0, \quad \text{i.e.} \quad \mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt} + \beta)^n.$$

The third assumption regards the investment strategy. It is assumed that the company invests all its initial surplus and incoming premiums continuously into a risky asset with a price modeled by a non-negative stochastic process.

The main idea of this new setting is that the ruin may occur only at the time of a claim. Therefore, a discretization of the model is possible. There is a discrete time Markov process embedded in the model, irrespective of the investment strategy or the inter-arrival time distribution.The Cramér-Lundberg assumptions regarding the independence of inter-arrival times and claim sizes still hold.

Moreover, a very general integro-differential equation for the probability of ruin is derived in this case of a renewal process with inter-arrival time densities that verify a ordinary differential equation with constant coefficients. It can be shown that many integro-differential equations well-known in the literature are particular cases of this equation. Some applications, for particular cases of time distributions, will be discussed and the asymptotic behavior of the probability of ruin will be analyzed for the newly introduced models. It is shown that the rate is at best algebraic for inter-arrival times distributions that are sum of exponentials with identical or different parameters.

Specifically, using the integro-differential equation derived for the ruin probability, by Karamata-Tauberian arguments, it is possible to analyze the decay of the probability of ruin in the Sparre Andersen model with investments into a stock modeled by a geometric Brownian motion. The inter-arrival times considered are sum of exponentials with various parameters or $Gamma(n, \beta)$ distributed, where n is a natural number. The exponential time distribution is a be a particular case, $Gamma(1, \beta)$. Asymptotic bounds for the ruin probabilities are derived through probabilistic arguments or through the analysis of the asymptotic decays of the solutions of the integro-differential equations considered.

1.5. Summary of the thesis

Throughout this thesis, all the stochastic quantities are defined on a filtered complete probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and all the stochastic processes to be defined in this thesis are adapted.

The second chapter introduces the assumptions and some of the basic properties for both the Cramér-Lundberg and the Sparre-Andersen risk insurance models. The classical derivation of the equation for the probability of ruin is presented here and then applied to a few particular cases.

The third chapter presents the derivation of the transition operator for the surplus model under some given assumptions regarding the claim amounts, the inter-arrival times and the returns from investments. A relation between this and the probability of ruin allows the derivation of a general integro-differential equation for the probability of ruin.

The fourth chapter is dedicated to the Cramér-Lundberg model with and without investments. For the no-investments model the well-known results regarding the exponential decay of the probability of ruin in finite $\Psi(u, T)$ and infinite time $\Psi(u)$ are re-derived by means of Laplace transforms properties only. This chapter concludes with the analysis of a Cramér-Lundberg model with investments in a stock modeled by a geometric Brownian motion. After revisiting the asymptotic behavior of probability of ruin, $\Psi(u)$, of Constantinescu and Thomann (2005), the asymptotic decay of the probability of ruin in finite time, $\Psi(u, T)$, is examined. For investments in a risky asset with small volatility the ruin probabilities have algebraic decay rates.

The well-known result of Andersen (1957) presents the exponential asymptotic decay of the ruin probability. Here it is shown that for a Sparre Andersen model with investments in a stock modeled by a geometric Brownian motion, the asymptotic decay of the ruin probability is at best algebraic. It is surprisingly found that the decay has the same algebraic rate no matter what distribution is chosen for the inter-arrival times. In fact the decay rate depends on the parameter of the investments only.

The sixth chapter presents conclusions and future research projects. The last chapter is an appendix, containing the concepts and theorems used throughout the thesis.

The main results of this thesis are summarized in the following:

1. A general integro-differential equation for functions of the risk process.

2. An equation satisfied by the ruin probability in a general setup.

3. A comparison of ruin probabilities, for different inter-arrival time distributions.

4. Results on the decay of the ruin probabilities under various scenarios.

5. A unified approach for the asymptotic analysis: Karamata-Tauberian theorems.

2. INSURANCE RISK MODELS- THE CASE OF NO INVESTMENTS

In this chapter the collective risk model is reviewed together with some of the main results from the actuarial literature. The goal is to derive a differential equation for the probability of ruin. For some particular cases the equation has an explicit form solution but, in general, it allows only an asymptotic analysis of the ruin probability as a function of the surplus. For given inter-arrival time distributions these equations are derived on a case by case basis in the actuarial literature. The main strategy used in these examples is conditioning on the time and size of the first claim followed by differentiation.

2.1. The risk model -no investments

The basic insurance risk model goes back to the work of Filip Lundberg and Harald Cramér. Filip Lundberg laid the foundation of the actuarial risk theory in early 1900. Later on, Harald Cramér incorporated Lundberg's ideas into the theory of stochastic processes. The structure of this model referred to as the Cramér-Lundberg model is the following (Embrechts et al., 1997):

Definition 1. *The Cramér-Lundberg model is given by conditions (1)-(5):*

1. *The claim arrival process: the number of claims in the interval* $[0, t]$ *is denoted by*

$$N(t) = sup\{n \geq 1 : T_n \leq t\}, \quad t \geq 0,$$

where, by convention, $\sup \emptyset = 0$.

2. *The claim times: the claims occur at random instants of time*

$$0 < T_1 < T_2 < \cdots \quad a.s.$$

3. *The inter-arrival times:*

$$\tau_1 = T_1, \quad \tau_k = T_k - T_{k-1}, \quad k = 2, 3, \cdots \tag{2.1}$$

are iid exponentially distributed with finite mean $E\tau_1 = \frac{1}{\lambda}$.

4. *The claim size process: the claim sizes* $(X_k)_{k \in N}$ *are positive i.i.d.r.v. having a common distribution* F *with finite mean* $\mu = EX_1$, *and variance* $\sigma^2 = Var(X_1) \leq \infty$.

5. *The sequences* (X_k) *and* (τ_k) *are independent of each other.*

Clearly the Cramér-Lundberg model is a compound Poisson process. The claim arrivals process $N(t)$ is a homogeneous Poisson process with parameter $\lambda > 0$. Hence,

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \cdots$$

The renewal model is a generalization of the Cramér-Lundberg model, and in the actuarial literature is referred to as the Sparre Andersen model.

Definition 2. *The Sparre Andersen model is given by conditions (1), (2), (4), (5) from the Cramér-Lundberg model definition together with:*

3'. The inter-arrival times:

$$\tau_1 = T_1, \quad \tau_k = T_k - T_{k-1}, \quad k = 2, 3, \cdots \tag{2.2}$$

are independent, identically distributed with finite mean $E\tau_1 = \frac{1}{\lambda}$.

The claim arrivals process in a Sparre Andersen model is a renewal counting process, meaning:

Definition 3. *A process* $\{N(t), t \geq 0\}$ *is called a* **counting process** *if for all* $t, h \geq 0$ *the following three conditions are satisfied:*

- $N(0) = 0$
- $N(t) \in N$
- $N(t) \leq N(t+h)$

In the risk theory literature, the renewal counting process $\{N(t)\}$ is called the **claim counting process**, where $N(t+h) - N(t)$ models the number of claims occurring in the interval $(t, t+h.]$ The realizations of any counting process are monotonically decreasing and right-continuous functions.

Sometimes, it may be assumed that the probability of the first claim occurrence is different than that of the subsequent ones. In that case, the process is called, "delayed":

Definition 4. *Assume that* $T_1, T_2, \cdots$ *is a sequence of independent nonnegative random variable and that* $T_2, T_3, \cdots$ *are identically distributed with distribution* F_T*. The random variable* T_1 *may have an arbitrary distribution* F_1*, which need not be equal to* F_T*. Then* $\{\sigma_n, n \geq 1\}$ *with* $\sigma_n = T_1 + T_2 + \cdots T_n$ *is called a* **delayed renewal point process.** *The process* $\{N(t), t \geq 0\}$*, is called a* **delayed renewal counting process.**

The surplus process $(U(t))_{t\geq 0}$ is defined as

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \tag{2.3}$$

where $u \geq 0$ denotes the initial capital and $c > 0$ is the incoming premium rate. In order to measure the solvency of an insurance company, the following two measures are introduced.

Definition 5. *The probability of ruin in finite time, or before a given time T is denoted by $\Psi(u, T)$ and is defined as the probability that the ruin happens before the time T,*

$$\Psi(u, T) = P(T_u < T \mid U(0) = u).$$

By T_u is denoted the first time the surplus of the insurance company starting with an initial capital u goes below zero:

$$T_u = \inf_{t \geq 0}\{U(t) < 0 \mid U(0) = u\}.$$

T_u is called the time of ruin.

Definition 6. *The probability of ruin with infinite horizon, or simply the probability of ruin, is denoted by $\Psi(u)$ and is defined as the probability that the time of ruin is finite,*

$$\Psi(u) = P(T_u < \infty \mid U(0) = u).$$

Obviously, $\Psi(u) = \Psi(u, \infty)$.

Intuitively, the premium rate c should be chosen so that for a given u and a given T, $\Psi(u, T)$ will be small. A suitable premium rate c should first exceed the average paid claims as seen in the following results:

Lemma 1. *For the Cramér-Lundberg model,*

$$EU(t) = u + ct - \lambda\mu t. \tag{2.4}$$

For the renewal model,

$$EU(t) = u + ct - \mu EN(t). \tag{2.5}$$

Proof. See e.g. Embrechts et al. (1997), page 24. ☐

In either cases, using the elementary renewal theorem (see e.g. Mikosch (2004), page 62), one has

$$\lim_{t\to\infty}\frac{EU(t)}{t}=\lim_{t\to\infty}\frac{u}{t}+c-\mu\lim_{t\to\infty}\frac{EN(t)}{t}=c-\lambda\mu.$$

A natural necessary condition for solvency is the "net profit condition:"

$$c-\mu\lambda>0.$$

Moreover, one has the following:

Proposition 1. *On the event $\{T_u=\infty\}$, the model (2.3) has the following property:*

$$U(t)\to\infty \quad as \quad t\to\infty.$$

Proof. Since the ruin doesn't occur, $U(t)>0$ for any $t>0$. Thus, using the Law of Large Numbers and the Key Renewal theorem:

$$\begin{aligned}\lim_{t\to\infty}\frac{U(t)}{t} &= \lim_{t\to\infty}\frac{u}{t}+c-\frac{\sum_{k=1}^{N(t)}X_k}{t}\\ &= c-\lim_{t\to\infty}\frac{\sum_{k=1}^{N(t)}X_k}{N(t)}\lim_{t\to\infty}\frac{N(t)}{t}\\ &= c-\lambda\mu>0.\end{aligned}$$

In other words as $t\to\infty$, $U(t)\to\infty$. ☐

2.2. The integro-differential equation of the ruin probability

Definition 7. *Suppose that $\{X_t : t\geq 0\}$ is a* **continuous-parameter** *stochastic process. The process has the Markov property if for each $s<t$, the conditional*

distribution of X_t given $\{X_u, u \leq s\}$ is the same as the conditional distribution of X_t given X_s. Such a process is called a **continuous- parameter Markov process.**

The Cramér Lundberg risk process $U(t)$ is an example of a continuous Markov process. Between the jumps, caused by the claims, the process is deterministic. It is possible to show that the process is Markov using the lack of memory property of the exponential distribution.

Definition 8. *Recall the* **transition operator** T_t *of a Markov process $U(t)$ is given by*

$$T_t f(u) = \mathbf{E}[f(U(t)) \mid U(0) = u] \tag{2.6}$$

provided $\mathbf{E}[|f(U(t))|] < \infty$.

Definition 9. *The infinitesimal generator of $\{T_t, t > 0\}$, or of the Markov process X_t, is the linear operator $\mathbf{A}$ defined by:*

$$\mathbf{A}g(x) = \lim_{h \to 0} \frac{T_h g(x) - g(x)}{h}$$

for all real-valued, bounded, Borel measurable functions g defined on S, $g \in \mathbf{B}(S)$ such that the right side converges uniformly in x to some function. The class of all such functions g comprises the **domain $\mathcal{D}_A$ of A.**

The following is a fundamentally important result about the Markov processes.

Theorem 1. *Let $U(t)$ be a right-continuous Markov process on a metric space S, and $f \in \mathcal{D}_A$. Assume that $s \to (Af)(U(s))$ is right-continuous for all sample paths. Then the process Z_t given by*

$$Z_t := f(U(t)) - \int_0^t (Af)(U(s))ds \quad t \geq 0,$$

is a $\{\mathcal{F}_t\}$-martingale, with $\mathcal{F}_t := \sigma\{U(x) : 0 \leq x \leq t\}$.

Proof. See e.g. Bhattacharya and Waymire (1990), page 375. □

Consequence of this theorem: $Ag(u) = 0$ implies that $g(U(t))$ is a martingale, $E[g(U(t)|U(0) = u] = g(u)$.

Paulsen and Gjessing (1997) introduces a relationship between the infinitesimal generator of a function of the the risk process and the probability of ruin. Actually they show that a function g that satisfies the equation $Ag(u) = 0$ together with some boundary conditions, is the ruin probability. The following theorem is an adapted form of their theorem to the Cramér-Lundberg case with no investments. Their original theorem together with the proof will be presented in the 4th chapter of this thesis.

Theorem 2. *Assume $g(u)$ is a bounded, differentiable function on $u \geq 0$, with a bounded derivative. If $g(u)$ satisfies*

$$Ag(u) = 0$$

together with the boundary conditions

$$g(u) = 1, \quad \textit{for} \quad u < 0,$$

$$\lim_{u \to \infty} g(u) = 0$$

then $g(u)$ is the probability of ruin, in other words

$$g(u) = P(T_u < \infty).$$

Proof. The detailed proof of the theorem from Paulsen and Gjessing (1997) can be found in the next chapter. Here is presented a sketch of the proof.

Let $T_u = inf\{t : U(t) < 0\}$, then $g(U(T_u)) = 1$ since $U(T_u) < 0$. By hypothesis, $Ag(u) = 0$ thus, $g(U(t))$ is a martingale. Since T_u is a stopping time,

$$\begin{aligned} g(u) &= E_u[g(U(T_u \wedge t))] \\ &= E_u[g(U_{T_u \wedge t})1_{\{T_u<t\}}] + E_u[g(U(T_u \wedge t))1_{\{T_u>t\}}] \\ &= E_u[g(U(T_u))]P(T_u < t) + E_u[g(U(t))]P(T_u > t) \end{aligned}$$

The result follows by letting $t \to \infty$ and using the boundary conditions and proposition 1,

$$g(u) = 1 * P(T_u < \infty) + 0 * P(T_u = \infty) = P(T_u < \infty).$$

□

As a consequence, the integro-differential equation satisfied by the ruin probability may be obtained if the generator of the process is calculated. For the Cramér-Lundberg risk process

$$U(t) = u + ct - S(t),$$

the infinitesimal generator is given by

$$Ag(u) = cg'(u) + \lambda \int_0^\infty g(u-x)dF_X(x) - \lambda g(u) = 0$$

where F_X is the distribution of the claim amounts X_k. Hence the integro-differential equation for the ruin probability is:

$$\Psi'(u) = \frac{\lambda}{c}\Psi(u) - \frac{\lambda}{c}\int_0^\infty \Psi(u-x)dF_X(x). \tag{2.7}$$

This equation was previously obtained by a "renewal argument" (Feller, 1971), page 183. Since the Poisson process is a renewal process and since ruin cannot occur in before the first claim arrival T_1, then the probability ruin $\Psi(u)$ satisfies the following relation:

$$\begin{aligned}
\Psi(u) &= E[\Psi(U(T_1)) \mid U_0 = u] \\
&= E[\Psi(u + cT_1 - X_1)] \\
&= \int_0^\infty \lambda e^{-\lambda s} \int_0^\infty \Psi(u + cs - x) dF_X(x) ds \\
&= \int_0^\infty \lambda e^{-\lambda s} \int_0^{u+cs} \Psi(u + cs - x) dF_X(x) ds + \int_0^\infty \lambda e^{-\lambda s} \int_{u+cs}^\infty dF_X(x) ds
\end{aligned}$$

As in Grandell (1991), the change of variables $y = u + cs$ leads to

$$\Psi(u) = \frac{\lambda}{c} \int_u^\infty e^{-\lambda y} c \int_0^y \Psi(y - x) dF_X(x) dy + \frac{\lambda}{c} \int_u^\infty e^{-\lambda y} c \int_y^\infty dF_X(x) dy. \quad (2.8)$$

Consequently, Ψ is differentiable and differentiation leads to

$$\Psi'(u) = \frac{\lambda}{c} \Psi(u) - \frac{\lambda}{c} \int_0^\infty \Psi(u - x) dF_X(x).$$

with the boundary condition

$$\lim_{u \to \infty} \Psi(u) = 0.$$

Thus, either through the infinitesimal generator, or through probabilistic arguments, the same integro-differential equation (2.7) may be derived for the probability of ruin in the classical Cramér-Lundberg model.

The Sparre Andersen model is a discrete Markov process (See Appendix A2), therefore one needs to define its generator, instead of an infinitesimal generator. This generator and the theorems relating it to the ruin probability will be the subject of the next chapter. However, once the inter-arrival time distribution is given, through similar probabilistic arguments (conditioning and differentiation), the probability of ruin equations may be obtained.

Examples form the literature of equations for different inter-arrival times, in a Sparre Andersen model are:

1. *Erlang*(2) (Dickson and Hipp, 1998; Dickson, 2002). Let $f_\tau(t) = \beta^2 t e^{-\beta t}$, for $t > 0$, be the density of the inter-arrival times. Then, by conditioning on the time and the amount of the first claim, the ruin probability satisfies

$$\Psi(u) = \int_0^\infty f_\tau(t) \int_0^{u+ct} \Psi(u+ct-x) f_X(x) dx dt + \int_0^\infty f_\tau(t) \int_{u+ct}^\infty f_X(x) dx dt.$$

After the change of variable, $s = u + ct$, differentiating the equation twice, it becomes:

$$c^2 \Psi''(u) - 2\beta c \Psi'(u) + \beta^2 \Psi(u) = \beta^2 \int_0^\infty \Psi(u-x) f_X(x) dx. \qquad (2.9)$$

2. *Erlang*(n) (Li and Garrido, 2004). Let $f_\tau(t) = \beta^n t^{n-1} \frac{e^{-\beta t}}{(n-1)!}$, for $t \geq 0$, be the inter-arrival times density. Then the equation satisfied by the ruin probability is derived to be:

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ n-k \end{pmatrix} \Psi^{(k)}(u) [\frac{-\beta}{c}]^{n-k} = [\frac{-\beta}{c}]^n \int_0^\infty \Psi(u-x) f_X(x) dx, \qquad (2.10)$$

where $\begin{pmatrix} n \\ k \end{pmatrix}$ represents the number of possible combinations of n objects taken k at a time.

3. Sum of exponentials (Gerber and Shiu, 2005). The inter-claim time random variables $\{\tau_k\}$ are the sum of n independent, exponentially distributed random variables. Denote $\tau := \tau_1$. Then

$$\tau = W_1 + W_2 + \cdots + W_n,$$

where W_i are n independent exponentially distributed random variables with $EW_i = \frac{1}{\lambda_i}$. Then, by conditioning in the value of $W_1+W_2+\cdots+W_j-t$, $t \geq 0$ (because the conditional distribution of this random variable is identical to the exponential distribution of W_{j+1}), one obtains the following recurrence:

$$\Psi_j = \lambda_{j+1}\int_0^\infty e^{-\lambda_{j+1}t}\Psi_{j+1}(u+ct)dt, \quad j = 0, 1, 2, \cdots, n-2.$$

Using the change of variables $z = u + ct$ one gets,

$$\Psi_{n-1}(u) = \frac{\lambda_n}{c}\int_u^\infty e^{-\lambda_n(z-u)/c}\int_0^z \Psi(z-x)f_X(x)dx + \frac{\lambda_n}{c}\int_u^\infty e^{-\lambda_n(z-u)/c}\int_z^\infty f_X(x)dx.$$

Differentiating with respect to u the appropriate number of times, one gets the desired integro-differential equation:

$$\gamma(D)\Psi(u) = \int_0^u \Psi(u-x)f_X(x)dx + \int_u^\infty f_X(x)dx, \quad u > 0, \tag{2.11}$$

where D denotes the differentiation operator and

$$\gamma(D) = \prod_{j=1}^{n}[(1 - \frac{c}{\lambda_j}D].$$

For $n = 2$, the distribution is referred to as a sum of two exponentials,

$$f_\tau(t) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

and the ruin probability equation has the following form,

$$(1 - \frac{c}{\lambda_1}\frac{d}{du})(1 - \frac{c}{\lambda_2}\frac{d}{du})\Psi(u) = \int_0^u \Psi(u-x)f_X(x)dx + \int_u^\infty f_X(x)dx, \quad u > 0,$$

equivalent to

$$c^2\Psi''(u) - (\lambda_1+\lambda_2)c\Psi'(u) + \lambda_1\lambda_2\Psi(u) = \lambda_1\lambda_2\int_0^\infty \Psi(u-x)f_X(x)dx. \tag{2.12}$$

To summarize, this chapter introduced the basic risk model with its assumptions and properties. Also, for a few particular inter-arrival time distributions, the equations for the probability of ruin were derived using the classical approach used in risk theory. Namely, calculate first the conditional expectation of the probability of ruin as a function of the process immediately after the first claim, given the initial surplus, then perform the necessary number of differentiations. These equations were written based on information about inter-arrival time distributions given apriori, when the only incoming flow is the constant premium rate.

In the next chapter, the equation for the probability of ruin will be written for a special class of distributions for the inter-arrival times, when the incoming flow consists of premiums and stochastic returns from investments.

3. INSURANCE RISK MODELS - THE CASE OF INVESTMENTS

This chapter contains the main result of this thesis. It derives an integro-differential equation for the probability of ruin, when the surplus process is a renewal risk process with stochastic returns from investments. It is referred to as the Sparre Andersen process with investments. The equation is written for a large class of inter-arrival time distributions with the option of stochastic returns from investments (if any). Many of the processes considered in the actuarial literature are particular cases of this one.

The result is derived through the following steps, each on its own being an original result.

First, for a Sparre Andersen surplus model with investments, a very general integro-differential equation is derived for the semigroup operator or, in other words, for the conditional expectation given the initial value of the process of a function of the surplus process immediately after the first claim. The equation derived here is valid for any claim sizes distribution, F_X, and any non-negative stochastic investment process. The only constraint of the model considered is that the density of the inter-arrival times needs to satisfy an ordinary differential equation with constant coefficients. The class of such distributions is referred to as mixture of Erlangs or phase type distributions.

The second result is the key result of this thesis. It connects the general equation developed for the transition operator of a function with the probability of ruin. Under a given assumption regarding the asymptotic behavior of the process at infinity, a function that satisfying the derived equation together with some specific boundary conditions is the probability of ruin.

Given specific values for the inter-arrival times density, and the investment process, one may obtain many of the well-known integro-differential equations satisfied by the probability of ruin.

3.1. The risk model - with investments

An insurance risk model with investments, is a classical surplus model (2.3),

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k$$

perturbed by a non-negative stochastic process. Recall that $U(t)$ represents the surplus at time t, u represents the initial surplus, c is the constant premium rate, $N(t)$ is a random variable representing the number of claims incurred up to time t, X_k are random variables representing the claim sizes, and the non-negative stochastic process represents the price of the risky asset where the surplus is invested.

The perturbed model is considered to model the surplus for an insurance company investing all its money, continuously into a risky asset with a price which follows a non-negative stochastic process. The non-negativity condition is imposed so that the ruin will not occur due to the investment only.

Since the ruin may occur only at the claim times, T_k, the surplus process may be discretized. The discrete version

$$U_k := U(T_k),$$

is a discrete time Markov process. The process U_k immediately after the payment of the $k-$th claim X_k may be written

$$U_k = Z_{\tau_k}^{U_{k-1}} - X_k, \tag{3.1}$$

where $Z_{\tau_k}^{U_k-1}$ represents the worth of a portfolio that results from investing the capital U_{k-1} (immediately after the payment of the $k-1$ claim) and the premiums collected over the time τ_k, into a risky asset.

Some example of this model equation for specific investment strategies are the following:

1. If there are no investments, then the surplus is the basic insurance risk process (2.3):
$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k.$$
2. If the price of the risky asset follows a geometric Brownian motion with drift a and volatility σ^2, then the equation of the surplus process is
$$U(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_s - \sum_{k=1}^{N(t)} X_k. \tag{3.2}$$
3. If the price of the risky asset is modeled by a diffusion, then the equation of the surplus model is:
$$U(t) = u + ct + \int_0^t a(s, U(s))ds + \int_0^t \sigma(s, U(s))dW_s - \sum_{k=1}^{N(t)} X_k, \tag{3.3}$$
where W_s is a standard Brownian motion $a(s, U(s))$, is the infinitesimal drift function and $\sigma(s, U(s))$ is the infinitesimal variance, provided that $\sigma^2(t, x) \geq 0$ for any $t \geq 0$ and $x \in E$, where E is the state space of $U(t)$.

Remark 1. *If $N(t)$ is Poisson distributed, the process (2.3) is a compound Poisson process and refers back to the classical Cramér-Lundberg model. Under this assumption, the process (3.1) is referred to as the Cramér-Lundberg model with investments.*

If $N(t)$ is a renewal process, then the process (2.3) is called the Sparre Andersen model and respectively (3.2) is referred to as the Sparre Andersen model with investments.

Lemma 2. *For the Cramér-Lundberg model with investments into an asset whose price is modeled by a geometric Brownian motion (3.2),*

$$\mathbf{E}U(t) = \frac{c - \mu\lambda}{a}(e^{at} - 1). \tag{3.4}$$

Proof. Since

$$\mathbf{E}U(t) = u + ct + a\int_0^t \mathbf{E}U(s)ds + \sigma\int_0^t \mathbf{E}U(s)dW_s - \mathbf{E}\sum_{k=1}^{N(t)} X_k,$$

where $\int_0^t \mathbf{E}U(s)dW_s = 0$ and for the Cramér-Lundberg model $\mathbf{E}\sum_{k=1}^{N(t)} X_k = \mu\lambda t$, then denoting $f(t) = \mathbf{E}U(t)$, one has

$$f(t) = u + (c - \lambda\mu)t + a\int_0^t f(s)ds.$$

Taking the derivative of f, one obtains the following ordinary differential equation:

$$f'(t) = (c - \lambda\mu) + af(t).$$

Using e^{-at} as integrating factor, the solution of the ODE is exactly (3.4)

$$\mathbf{E}U(t) = \frac{c - \lambda\mu}{a}(e^{at} - 1).$$

$\square$

This lemma shows that a solvency condition is $c - \lambda\mu > 0$ and also $a > 0$. In other words a positive drift of the investments together with the "net profit condition" introduced in the classical Cramér-Lundberg model.

The claims occur at random times, $T_1, T_2, \cdots$, where the time in between claims are denoted by $\tau_1 = T_1, \tau_2 = T_2 - T_1, \cdots \tau_n = T_n - T_{n-1}, \cdots$, and the assumption regarding the independence of the processes $\{\tau_k\}_k$ and $\{X_k\}_k$ still holds. This means that the size of a claim does not depend on the time elapsed since the previous claim.

Under renewal considerations, the specific assumptions introduced for the model to be analyzed here are with regard to the three main variables of the model: the claim sizes X_k, the inter-arrival times τ_k and the value of the investment Z_t^u.

First assumption. The claim amounts $\{X_k\}_k$ are said to be "light" or to have a "well-behaved " distribution F_X, i.e. exponentially bounded tail distribution

$$1 - F_X(x) \leq ce^{-ax}$$

for some positive a and c and for all $x \geq 0$.

Second assumption. The density of the inter-arrival times $(\tau_k)_k$, f_τ satisfies an ordinary differential equation with constant coefficients, formally denoted by

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0,$$

with $\mathcal{L}^*$ denoting the formal adjoint of the linear operator $\mathcal{L}$. In general, the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = \sum_{j=0}^{n} \alpha_j \frac{d^j}{dt^j} f_\tau(t),$$

with the adjoint $\mathcal{L}^*$,

$$\mathcal{L}^*(\frac{d}{dt})f_\tau(t) = \sum_{j=0}^{n} (-1)^j \alpha_j \frac{d^j}{dt^j} f_\tau(t).$$

Third assumption. The company receives premium at a constant rate c and invests all its money continuously into a stock with a price which follows a non-negative stochastic process. The investment process Z_t^u is a continuous Markov process with infinitesimal generator A. For instance, for an investment into a stock modeled by a geometric Brownian motion, the infinitesimal generator is

$$A = (c + au)\frac{d}{du} + \frac{\sigma^2}{2}\frac{d^2}{du^2},$$

whether if no investments are made, the "infinitesimal generator" is

$$A = c\frac{d}{du}.$$

By a method to be introduced here, a general integro-differential equation for the probability of ruin is derived and later on analyzed. Many particular cases of the equation have been already analyzed in the actuarial literature, but new higher-order integro-differential equations may be written for investment scenarios never considered before.

First an integro-differential equation is derived for the transition operator associated to the risk process.

3.2. The integro-differential equation of the transition operator

Definition 10. *For the discrete Markov process $U_0, U_1, U_2, \cdots$, where $U_0 = u$, $U_k = U_{T_k}$, on the set of all real-valued, bounded, Borel-measurable functions h, define the transition operator $Th : R \to R$,*

$$Th(u) := \mathbf{E}(h(U_1) \mid U_0 = u) = \int_0^\infty f_\tau(t) \int_0^\infty \mathbf{E}(h(Z_t^u - x) \mid U(0) = u) f_X(x)\,dx\,dt.$$

Definition 11. *The generator of the time discrete Markov process* $\{(U_k)_{k\leq 0} \mid U_0 = u\}$ *is,*

$$A_U g(u) = (T - I)g(u).$$

Remark 2. *Note that* A_U *denotes the generator for the process* U *and* A *the infinitesimal generator for the process* Z_t^u. $\mathcal{D}_{A_U}$ *is the domain of the operator* A_U *and respectively,* $\mathcal{D}_A$ *is the domain of the operator* A.

Proposition 2. $M_n = f(U_n) - \sum_{k=0}^{n-1} A_U f(U_k) = f(U_n) - \sum_{k=0}^{n-1}(T - I)f(U_k)$ *is a martingale.*

Proof.

$$\begin{aligned} \mathbf{E}(M_{n+1} \mid \mathcal{F}(U_0, U_1, \cdots U_n)) &= \mathbf{E}(f(U_{n+1}) \mid U_0, U_1, \cdots U_n) - \sum_{k=0}^{n}(T - I)f(U_k) \\ &= Tf(U_n) - Tf(U_n) + f(U_n) - \sum_{k=0}^{n-1}(T - I)f(U_k) \\ &= M_n \end{aligned}$$

□

One of the main results of this thesis shows that a certain integro-differential can be written for expected values of the function of both the risk process and the claim sizes process.

Theorem 3. *Let* $g \in \mathcal{D}_A$. *If* f_τ *satisfies the ordinary differential equation with constant coefficients*

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0$$

and

1. $f^{(k)}(0) = 0$, *for* $k = 0, \ldots, n - 2$

2. $\lim_{x\to\infty} f^{(k)}(x) = 0$*, for* $k = 0, \ldots, n-1$

then

$$\mathcal{L}^*(A)Tg(u,0) = f_\tau^{(n-1)}(0)\mathbf{E}g(u, X_1).$$

Proof. Recall that the surplus process at the time of the $k-$th claim, T_k, is $U_k = Z_{\tau_k}^{U_{k-1}} - X_k$. Consider the process $(Z_{\tau_k}^{U_{k-1}}, X_k)_{k\geq 0}$, with the first two states $(Z_0 = u, X_0 = 0)$, $(Z_{\tau_1}^u, X_1)$. Then the transition operator Tg of this process equals:

$$Tg(u,0) = \int_0^\infty \int_0^\infty \mathbf{E}(g(Z_t, x) \mid Z_0 = u) f_\tau(t) f_X(x) dx dt.$$

Since by assumption,

$$f_\tau(t) = -\frac{1}{\alpha_0} \sum_{j=1}^n \alpha_j \frac{d^j}{dt^j} f_\tau(t)$$

then it follows that

$$\begin{aligned}
Tg(u,0) &= \int_0^\infty \int_0^\infty (-\frac{1}{\alpha_0} \sum_{j=1}^n \alpha_j \frac{d^j}{dt^j} f_\tau(t)) \mathbf{E}(g(Z_t, x) \mid Z_0 = u) f_X(x) dx dt \\
&= -\frac{1}{\alpha_0} \sum_{j=1}^n \alpha_j \int_0^\infty \int_0^\infty \frac{d^j}{dt^j} f_\tau(t) \mathbf{E}(g(Z_t, x) \mid Z_0 = u) f_X(x) dx dt
\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}
Tg(u,0) &= \int_0^\infty f_X(x)[-\frac{1}{\alpha_0} \sum_{j=1}^n \alpha_j [\sum_{k=0}^j \frac{d^k}{dt^k} \mathbf{E}(g(Z_t, x) \mid Z_0 = u) f_\tau^{(j-1-k)}(t) \mid_{t=0}^\infty \\
&+ (-1)^j \int_0^\infty \frac{d^j}{dt^j} \mathbf{E}(g(Z_t, x) \mid Z_0 = u) f_\tau(t)) dt] dx.
\end{aligned}$$

Due to the hypothesis on the derivatives of f_τ, it further equals:

$$\begin{aligned}
Tg(u,0) &= \frac{1}{\alpha_0} \int_0^\infty f_X(x) \mathbf{E}(g(Z_0, x) \mid Z_0 = u) f_\tau^{(n-1)}(0) dx \\
&- \int_0^\infty \int_0^\infty \frac{1}{\alpha_0} \sum_{j=1}^n \alpha_j (-1)^j \frac{d^j}{dt^j} \mathbf{E}(g(Z_t, x) \mid Z_0 = u) f_\tau(t)) dt f_X(x) dx.
\end{aligned}$$

Using Kolmogorov' backward equation and the fact that $\mathbf{E}[g(Z_t,x)|Z_0=u]$ and A commute on $\mathcal{D}_A$, one has

$$\frac{d}{dt}\mathbf{E}[g(Z_t,x)|Z_0=u] = A\mathbf{E}[g(Z_t,x)|Z_0=u] = \mathbf{E}[Ag(Z_t,x)|Z_0=u].$$

Inductively, the equality becomes:

$$\begin{aligned} Tg(u,0) &= \frac{1}{\alpha_0}\int_0^\infty f_X(x)\mathbf{E}(g(Z_0,x) \mid Z_0=u)f_\tau^{(n-1)}(0)dx \\ &- \int_0^\infty\int_0^\infty \frac{1}{\alpha_0}\sum_{j=1}^{n}\alpha_j(-1)^j\mathbf{E}A^{(j)}(g(Z_t,x) \mid Z_0=u)f_\tau(t))dtf_X(x)dx \end{aligned}$$

where $A^{(j)}g = A(A(A\ldots))g$, j times. Adding and and subtracting to the right-hand side of the equality

$$\begin{aligned} Tg(u,0) &= \mathbf{E}[g(Z_{T_1},X_1) \mid Z_0=u] \\ &= \int_0^\infty\int_0^\infty \frac{1}{\alpha_0}\alpha_0\mathbf{E}A^{(0)}(g(Z_t,x) \mid Z_0=u)f_\tau(t))dtf_X(x)dx \end{aligned}$$

one obtains:

$$\begin{aligned} Tg(u,0) &= Tg(u,0) + \frac{1}{\alpha_0}\int_0^\infty f_X(x)\mathbf{E}(g(Z_0,x) \mid Z_0=u)f_\tau^{(n-1)}(0)dx \\ &- \int_0^\infty\int_0^\infty \frac{1}{\alpha_0}\sum_{j=0}^{n}\alpha_j(-1)^j\mathbf{E}[A^{(j)}(g(Z_t,x) \mid Z_0=u)]f_\tau(t))dtf_X(x)dx \\ &= Tg(u,0) + \frac{1}{\alpha_0}\int_0^\infty f_X(x)\mathbf{E}(g(Z_0,x) \mid Z_0=u)f_\tau^{(n-1)}(0)dx \\ &- \int_0^\infty\int_0^\infty \frac{1}{\alpha_0}\mathbf{E}[\sum_{j=0}^{n}\alpha_j(-1)^jA^{(j)}(g(Z_t,x) \mid Z_0=u)]f_\tau(t))dtf_X(x)dx \\ &= Tg(u,0) + \frac{1}{\alpha_0}\int_0^\infty f_X(x)\mathbf{E}(g(Z_0,x) \mid Z_0=u)f_\tau^{(n-1)}(0)dx \\ &- \int_0^\infty\int_0^\infty \frac{1}{\alpha_0}\mathbf{E}\mathcal{L}^*(A)(g(Z_t,x) \mid Z_0=u)f_\tau(t))dtf_X(x)dx \\ &= Tg(u,0) + \frac{1}{\alpha_0}f_\tau^{(n-1)}(0)\mathbf{E}(g(u,X_1) - \frac{1}{\alpha_0}\mathbf{E}\mathcal{L}^*(A)(g(Z_{T_1},X_1) \mid Z_0=u) \end{aligned}$$

Since T and A commute in $\mathcal{D}_A$, i.e. $TA = AT$, then by the linearity property of the expected value, inductively, $\mathbf{E}\mathcal{L}^*(A) = \mathcal{L}^*(A)\mathbf{E}$. Thus,

$$Tg(u,0) = Tg(u,0) + \frac{1}{\alpha_0} f_\tau^{(n-1)}(0)\mathbf{E}g(u,X_1) - \frac{1}{\alpha_0}\mathcal{L}^*(A)\mathbf{E}(g(Z_{T_1},X_1) \mid Z_0 = u)$$

In other words, for any function g in the domain of A,

$$\frac{1}{\alpha_0} f_\tau^{(n-1)}(0)\mathbf{E}g(u,X_1) - \frac{1}{\alpha_0}\mathcal{L}^*(A)\mathbf{E}(g(Z_{T_1},X_1) \mid Z_0 = u) = 0,$$

or

$$\mathcal{L}^*(A)Tg(u,0) = f_\tau^{(n-1)}(0)\mathbf{E}g(u,X_1).$$

□

Corollary 1. *If the same conditions regarding the inter-arrival times distribution hold and moreover*

$$Tg(u,0) = \mathbf{E}[g(Z_{T_1},X_1) \mid Z_0 = u] = g(u,0)$$

then

$$\mathcal{L}^*(A)g(u,0) = f_\tau^{(n-1)}(0)\mathbf{E}g(u,X_1).$$

Corollary 2. *Let $h \in \mathcal{D}_A$. Then under the same conditions regarding the inter-arrival times distribution h satisfies the equation:*

$$\mathcal{L}^*(A)h(u) = f_\tau^{(n-1)}(0)\int_0^\infty h(u-x)f_X(x)dx. \tag{3.5}$$

Proof. For the proof of corollaries, one has that if,

$$\mathbf{E}[g(Z_{T_1},X_1) \mid Z_0 = u] = g(u,0),$$

equivalent to $Tg(u,0) = g(u,0)$ then

$$\mathcal{L}^*(A)g(u,0) = f_\tau^{(n-1)}(0)\mathbf{E}g(u,X_1).$$

Also, for $h(u-x) = g(u,x)$ one has

$$\mathcal{L}^*(A)h(u) = f_\tau^{(n-1)}(0)\int_0^\infty h(u-x)f_X(x)dx.$$

$\square$

3.3. Relation to the probabilities of ruin

As mentioned before, the mathematical assignment is the asymptotic analysis of the ruin probability of an insurance company that receives premiums, pays claims and invests its capital.

Recall that T_u denotes the first time the surplus of the insurance company starting with an initial capital u goes below zero:

$$T_u = \inf_{t\geq 0}\{U(t) < 0 \mid U(0) = u\}$$

and is called the time of ruin. The event of ruin refers to the first time the surplus is negative and the probability of ruin is defined as $\Psi(u) = P(T_u < \infty \mid U(0) = u)$.

Let T_k be the time of the $k-$ th claim. Then, the risk process

$$U_k = Z_{T_k}^{U_{k-1}} - X_{T_k}$$

is a discrete Markov process. Moreover, the claim amount X_k and and the time in between claims $\tau_k = T_{k+1} - T_k$ are independent, meaning that the severity of the claim doesn't depend on the amount of time elapsed since the last claim.

Theorem 4. *Assume that on the event $\{T_u = \infty\}$, $U_t \to \infty$ as $t \to \infty$. Assume that $P(T_u < \infty) = \Psi(u) \in \mathcal{D}_A$. Then the following are equivalent:*

1. *$g \in \mathcal{D}_A$ satisfies*

$$A_U g(u) = (T - I)g(u) = 0$$

together with the boundary conditions

$$g(u) = 1, \quad on \quad u < 0,$$

$$\lim_{u\to\infty} g(u) = 0$$

2. *$g(u)$ is the probability of ruin, in other words*

$$g(u) = \Psi(u).$$

From $A_U g = (T - I)g = 0$ one has $Tg = g$. Thus, a corollary of the theorem is:

Corollary 3. *Assume that $\Psi(u) \in \mathcal{D}_{A_U}$. Then the following are equivalent:*

1. *A function $g \in \mathcal{D}_{A_U}$ satisfies*

$$\mathcal{L}^*(A)g(u) = f_\tau^{(n-1)}(0) \int_0^\infty g(u - x) f_X(x)dx \tag{3.6}$$

together with the boundary conditions

$$g(u) = 1, \quad on \quad u < 0,$$

$$\lim_{u\to\infty} g(u) = 0$$

2. *$g(u)$ is the probability of ruin, in other words*

$$g(u) = \Psi(u).$$

Proof. The theorem is proved first.

First part "(2) $\implies$ (1)."

Since the process U_k is a renewal process and since ruin cannot occur in the the interval $(0, T_1)$, where T_1 represents the time of the first claim, then the probability of ruin, $g(u)$, satisfies the renewal equation,

$$g(u) = \mathbf{E}(g(U_1) \mid U_0 = 0) =: Tg(u).$$

It is proved in the previous theorem that Tg satisfies the equation for any $g \in \mathcal{D}_A$. Since, $Tg = g$ it follows that g satisfies the equation. Since g is the probability of ruin, it also satisfies the boundary conditions.

Second part, "(1) $\implies$ (2)."

Let $\mathbf{E}_u g(U_k) := \mathbf{E}(g(U(T_k)) \mid U_0 = u)$. Since $g \in \mathcal{D}_{A_U}$ with

$$A_U g(u) = 0,$$

then $g(U_k) = g(U(T_k)$ is a martingale, i.e for any k,

$$g(u) = \mathbf{E}_u g(U(T_k)).$$

The time of ruin T_u is a stopping time, thus

$$g(u) = \mathbf{E}_u g(U(T_u))$$

and moreover

$$\begin{aligned} g(u) &= \mathbf{E}_u g(U(T_u \wedge T_k)) \\ &= \mathbf{E}_u[g(U(T_u \wedge T_k))1_{\{T_u < T_k\}}] + \mathbf{E}_u[g(U(T_u \wedge T_k))1_{\{T_u > T_k\}}] \\ &= \mathbf{E}_u g(U(T_u))P(T_u < T_k) + \mathbf{E}_u g(U(T_k))P(T_u > T_k) \end{aligned}$$

The result thus follows by letting $t \to \infty$ and using the boundary conditions,

$$g(u) = 1 * P(T_u < \infty) + 0 * P(T_u = \infty) = P(T_u < \infty).$$

$\square$

Remark 3. *The general assumption made in the theorem that on the event* $\{T_u = \infty\}$, $U_t \to \infty$ *as* $t \to \infty$, *can be reformulated as a condition on the investment process* Z_t^u.

Example 1. *For a renewal risk process with no investments, it is true that on the event* $\{T_u = \infty\}$, $U_t \to \infty$ *as* $t \to \infty$, *through a law of large number type argument, as mentioned in the previous chapter (Proposition 1).*

Example 2. *For a renewal risk process with investments in a stock modeled by a geometric Brownian motion, the following are known:*

1. *Using Lemma (2), one has that for* $c > \lambda\mu$ *and* $a > 0$,

$$\mathbf{E}U(t) \to \infty \quad as \quad t \to \infty.$$

2. *If the process* $U(t)$ *is a geometric Brownian motion with no jumps (no claims) and no premiums, then, according to Oksendal (1998), page 65, for* $\frac{2a}{\sigma^2} > 1$,

$$U(t) \to \infty \quad as \quad t \to \infty.$$

The derivation of the exact conditions between the parameters of a Sparre Andersen model with investments in a stock priced by a geometric Brownian motion will be subject of future research.

3.4. The integro-differential equation of the ruin probability

Using theorem 4, the ruin probability satisfies the general integro-differential equation:

$$\mathcal{L}^*(A)\Psi(u) = f_\tau^{(n-1)} \int_0^\infty \Psi(u-x)f_X(x)dx \tag{3.7}$$

together with the boundary conditions:

1. $\lim_{u\to\infty} \Psi(u) = 0$,

2. (BC)

where (BC) stands for boundary conditions and n represents the degree of the ordinary differential equation satisfied by the density of the inter-arrival times. The boundary conditions (BC) may be derived from "compatibility" conditions assuming that the integro-differential equation and its derivatives hold at zero. For instance, if the investment considered is a geometric Brownian motion then the equation has order "$2n$". If the average time in between claims is $\frac{1}{\lambda}$, then the boundary conditions will be:

1. $\lim_{u\to\infty} \Psi(u) = 0$,

2. $c\Psi'(0) - \lambda\Psi(0) + \lambda = 0$,

3. $k-$th derivative of the equation at zero, $k = 1, 2, \cdots, n-2$.

Many well-known equations are a particular form of the equation (3.7). For instance, in the Cramér-Lundberg model with no investments, the density of the inter-arrival times is exponential, $f_\tau(t) = \lambda e^{-\lambda t}$, for $t > 0$. Hence, it satisfies the following ODE with constant coefficients:

$$(\frac{d}{dt} + \lambda) f_\tau(t) = 0.$$

Reconciling with the notation introduced here, one has

$$\mathcal{L}(\frac{d}{dt}) f_\tau(t) = (\frac{d}{dt} + \lambda) f_\tau(t),$$

with the adjoint

$$\mathcal{L}^*(\frac{d}{dt}) f_\tau(t) = (-\frac{d}{dt} + \lambda) f_\tau(t).$$

Since there are no investments, just a constant flow of premiums at a rate c, the infinitesimal generator of the "investment " process is

$$A = c\frac{d}{du}.$$

Therefore the integro-differential equation obtained from the general form (3.7) is:

$$(-c\frac{d}{du} + \lambda)\Psi(u) = \lambda \int_0^\infty \Psi(u-x) f_X(x) dx \tag{3.8}$$

equivalent to the equation (2.7) and with the same boundary condition,

$$\lim_{u\to\infty} \Psi(u) = 0.$$

Also, for the Sparre Andersen with no investments, the equations and their boundary conditions can be derived for different inter-arrival times.

1. *Erlang*(2). Using (3.7) the same equation can be derived for the ruin probability as in (Dickson and Hipp, 1998; Dickson, 2002):

   $$(-c\frac{d}{du} + \beta)^2 \Psi(u) = \beta^2 \int_0^\infty \Psi(u-x) f_X(x) dx,$$

   equivalent to (2.9), with the boundary conditions:

(a) $\lim_{u\to\infty}\Psi(u) = 0$,

(b) $c^2\Psi''(0) - 2\beta c\Psi'(0) + \beta^2\Psi(0) = \beta^2$.

2. *Erlang*(n). Using (3.7) the same equation can be derived for the ruin probability as in (Li and Garrido, 2004):

$$(-c\frac{d}{du} + \beta)^n\Psi(u) = \beta^n \int_0^\infty \Psi(u-x)f_X(x)dx,$$

equivalent to (2.10), with the boundary conditions:

(a) $\lim_{u\to\infty}\Psi(u) = 0$,

(b) $(-c\frac{d}{du} + \beta)^n\Psi(0) = \beta^n$,

(c) the first $n-2$ derivatives of the equation (2.10) evaluated at zero.

3. Sum of two exponentials. Using (3.7) the same equation can be derived for the ruin probability as in (Gerber and Shiu, 2005):

$$(-c\frac{d}{du} + \beta_1)(-c\frac{d}{du} + \beta_2)\Psi(u) = \beta_1\beta_2 \int_0^\infty \Psi(u-x)f_X(x)dx,$$

equivalent to (2.12), with the boundary conditions:

(a) $\lim_{u\to\infty}\Psi(u) = 0$,

(b) $c^2\Psi''(0) - c(\beta_1+\beta_2)\Psi'(0) + \beta_1\beta_2\Psi(0) = \beta_1\beta_2$.

3.5. Remarks

Remark 4. *The model (3.2) applies even if only a part $\eta \in (0,1]$ of the capital is invested in the risky asset. In such a case, one should replace the parameters a, σ, with ηa, respectively $\eta\sigma$.*

Remark 5. *A very similar derivation of the transition operator Tg can be done if the inter-arrival times density f_τ satisfies a ordinary differential equation with polynomial coefficients. The equation obtained is*

$$\mathbf{E}\mathcal{L}^*(A)(g(Z_{T_1}, X_1) \mid Z_0 = u) = f_\tau^{(n-1)}(0)\mathbf{E}(g(u, X_1),$$

where this time $\mathcal{L}$ is an operator with coefficients not necessarily constants. Therefore, in general $\mathbf{E}$ and $\mathcal{L}^*(A)$ will not commute. Thus, this is the general equation that can be obtained for the case of polynomial coefficients.

Remark 6. *One of the difficult parts in applying Theorem 4 is determining the domain of the operator $\mathcal{D}_A$. It will have to be studied on a case by case basis.*

For instance, Paulsen and Gjessing (1997) considered the case of exponentially distributed inter-arrival times and defined the operator A_U acting on twice continuously differentiable functions as follows:

$$A_U g(u) = \frac{\sigma^2}{2} u^2 g''(u) + (c + au)g'(u) + \lambda \int_0^\infty (g(u-x) - g(x))dF_X(x)$$

where λ is the intensity of the number of claims process N and F_X is the distribution of the claim amounts X_k. Their theorem is written for this special operator and it will be presented here, in the following paragraph. Also, the ruin probability equation for some special cases of claim sizes distributions are reproduced from their paper, to show the technique used in determining the boundary conditions for the ruin probability equation.

Theorem 5.

1. *Assume $g(u)$ is a bounded and twice continuously differentiable function on $u \geq 0$ with a bounded first derivative. If $g(u)$ solves*

$$Ag(u) = 0 \quad on \quad u > 0,$$

together with the boundary conditions

$$g(u) = 1 \quad onu < 0,$$

$$\lim_{u \to \infty} g(u) = 0,$$

then

$$g(u) = P(T_u < \infty).$$

2. *Assume* $q_\alpha(u), \alpha \geq 0$ *is a bounded and twice continuous differentiable function on* $u \geq 0$ *with a bounded first derivative. If* $q_\alpha(u)$ *solves*

$$Aq_\alpha(u) = \alpha q_\alpha(u) \quad on \quad u > 0,$$

together with the boundary conditions

$$q_\alpha(u) = 1 \quad on \quad u > 0,$$

$$\lim_{u \to \infty} q_\alpha(u) = 0,$$

then

$$q_\alpha(u) = \mathbf{E}[e^{-\alpha T_u}].$$

Proof. See Paulsen and Gjessing (1997). □

Paulsen and Gjessing (1997) concluded that the boundary conditions together with the boundedness assumptions are sufficient to determine g and q_α uniquely, provided the solutions exist. They also observed that for special claim size distributions the relevant integro-differential equations may be differentiated once or several times and thus remove the integral operator. In this case, one may retain the boundary value information from the original equation. To be specific, consider the equation

$$Aq_\alpha(u) - \alpha q_\alpha(u) = 0 \tag{3.9}$$

since the equation $Ag(u) = 0$ is obtained from this by setting $\alpha = 0$ and $q_\alpha = g$. Then letting $u \to 0$ in this equation and using the boundary conditions from the theorem one finds,

$$cq'_\alpha(0) - (\lambda + \alpha)q_\alpha(0) + \lambda = 0. \tag{3.10}$$

If (3.9) needs to be differentiated twice, then letting $u \to 0$ in

$$\frac{d}{du}(Aq_\alpha(u) - \alpha q_\alpha(u)) = 0$$

and assuming $f_X = F'_X$ exists and in continuous in an interval $[0, \epsilon)$, one gets the following additional boundary condition,

$$cq''_\alpha(0) + (a + \frac{c\lambda}{\alpha + \lambda} f_X(0) - \lambda - \alpha)q'_\alpha(0) - \frac{\alpha\lambda}{\alpha + \lambda} f_X(0) = 0. \tag{3.11}$$

Since the integral term involves the distribution of the claim amounts, in order to illustrate the choice of boundary conditions, consider the following three examples:

Example 3. *Consider the case when the claim sizes X_k are exponentially distributed with expectation $\frac{1}{\mu}$ and the inter-arrival times are exponentially distributed with expectation $\frac{1}{\lambda}$. Then from the fact that the derivative of an exponential is also an exponential, taking the derivative of (3.9), one finds:*

$$\frac{d}{du}(Aq_\alpha(u) - \alpha q_\alpha(u)) + \mu(Aq_\alpha(u) - \alpha q_\alpha(u)) = 0$$

Thus, $q_\alpha(u)$ solves

$$\frac{\sigma^2}{2}u^2 q'''_\alpha(u) + (\frac{\sigma^2}{2}\mu u^2 + (a+\sigma^2)u + c)q''_\alpha(u) + (\mu a u + (\mu c + a - \sigma - \lambda))q'_\alpha(u) - \alpha\mu q_\alpha(u) = 0, \tag{3.12}$$

with the boundary conditions

$$q_\alpha(u) = 1 \quad on \quad u < 0,$$

$$\lim_{u\to\infty} q_\alpha(u) = 0,$$

$$cq'_\alpha(0) - (\lambda + \alpha)q_\alpha(0) + \lambda = 0.$$

This implies that the ruin probability $\Psi(u)$ *solves the equation*

$$\frac{\sigma^2}{2}u^2\Psi'''(u)+(\frac{\sigma^2}{2}\mu u^2+(a+\sigma^2)u+c)\Psi''(u)+(\mu a u+(\mu c+a-\sigma-\lambda))\Psi'(u) = 0, \quad (3.13)$$

with the boundary conditions

$$\Psi(u) = 1 \quad on \quad u < 0,$$

$$\lim_{u\to\infty} \Psi(u) = 0,$$

$$c\Psi'(0) - \lambda\Psi(0) + \lambda = 0.$$

Moreover, if the volatility $\sigma = 0$ *then the surplus process* $U(t)$ *is given by*

$$U(t) = u + ct + a\int_0^t U(s)ds - \sum_{k=1}^{N(t)} X_k$$

and the ruin probability is the solution of the equation

$$(c + au)\Psi''(u) + (\mu a u + \mu c + a - \lambda)\Psi'(u) = 0 \quad (3.14)$$

with the boundary conditions

$$\lim_{u\to\infty} \Psi(u) = 0,$$

$$c\Psi'(0) - \lambda\Psi(0) + \lambda = 0$$

and it has an exact form:

$$\Psi(u) = \frac{\int_u^\infty e^{-\mu x}(1+\frac{ax}{c})^{\frac{\lambda}{a}-1}dx}{\frac{c}{\lambda} + \int_0^\infty e^{-\mu x}(1+\frac{ax}{c})^{\frac{\lambda}{a}-1}dx},$$

a result that does back to Segerdahl (1942).

Example 4. *Assume that $\sigma = 0$ and the distribution of the claim amounts is a mixture of two exponentials:*

$$f_X(x) = (\theta\mu_1 e^{-\mu_1 x} + (1-\theta)\mu_2 e^{-\mu_2 x})1_{\{x\geq 0\}}.$$

Then the equation to be solved by the ruin probability is

$$\frac{d^2}{du^2}A_U\Psi(u) + (\mu_1+\mu_2)\frac{d}{du}A_U\Psi(u) + \mu_1\mu_2 A_U\Psi(u) = 0$$

i.e. the following differential equation:

$$(c+au)\Psi'''(u) + ((\mu_1+\mu_2)au + (2a + (\mu_1+\mu_2)c - \lambda))\Psi''(u)$$

$$+(\mu_1\mu_2 au + (\mu_1+\mu_2)a + \mu_1\mu_2 c - \lambda((1-\theta)\mu_1 + \theta\mu_2)))\Psi'(u) = 0$$

with the boundary conditions

$$\lim_{u\to\infty}\Psi(u) = 0,$$

$$c\Psi'(0) - \lambda\Psi(0) + \lambda = 0,$$

$$c\Psi''(0) + (a + c(\theta\mu_1 + (1-\theta)\mu_2) - \lambda)\Psi'(0) = 0.$$

Example 5. *Consider the case when the claim sizes X_k are exponentially distributed with expectation $\frac{1}{\mu}$ and the inter-arrival times are $Erlang(2,\beta)$ distributed with expectation $\frac{2}{\beta}$. Then taking the derivative of (3.7), one finds that $\Psi(u)$ solves a fifth order ordinary differential equation:*

$$\frac{d}{du}(-A+\beta)^2\Psi(u) = \beta^2[-\frac{\mu}{\beta^2}(-A+\beta)^2\Psi(u) + \mu\Psi(u)], \tag{3.15}$$

where, $A = \frac{\sigma^2}{2}u^2\frac{d^2}{du^2} + (c+au)\frac{d}{du}$, with the boundary conditions

1. $\Psi(u) = 1 \quad on \quad u < 0,$

2. $\lim_{u\to\infty} \Psi(u) = 0,$

3. $c^2\Psi''(0) + c(a - 2\beta)\Psi'(0) + \beta^2\Psi(0) - \beta^2 = 0,$

4. $c^2\Psi'''(0)+(2ac+2a\sigma^2+c\sigma^2+ac-2\beta)\Psi''(0)+(a-\beta)^2\Psi'(0)-\beta^2\mu\Psi(0)+\beta^2\mu = 0,$

5. $c^2\Psi^{(4)}(0) + (4c\sigma^2 + 5ac + 2a\sigma^2 - 2\beta c)\Psi'''(0) + (4a^2 + 2a\sigma^2 + \sigma^4 - 2\beta\sigma^2 - 4a\beta + \beta^2)\Psi''(0) - \beta^2\mu\Psi'(0) - \beta^2\mu^2 = 0.$

Remark 7. *If the density of the claim sizes satisfies an ODE with constant coefficients of order m, then the integro-differential equations may be reduced to homogeneous differential equations (after taking m derivatives). The boundary conditions may be obtained from the compatibility conditions.*

In view of equation (3.7) various integro-differential equations may be written for specific choices regarding the inter-arrival times, claim sizes or the investments process, leading to corresponding probabilities of ruin. A few of these cases will be presented in the next chapters, but many others can be considered.

4. RUIN PROBABILITIES IN THE CLASSICAL RISK MODEL

This chapter contains the analysis of the ruin probabilities for the Cramér-Lundberg model with and without investments. The surprising result is that investing everything in an asset with returns following a geometric Brownian motion leads faster to ruin than if no investments are made.

In the first section covering the classical case with no investments the well known exponential bound for the probability of ruin is derived through a modern, elegant martingale method (Rolski et al., 1999). Also, the well-known exponential decay of the probability of ruin is derived by means of Laplace transform and completely monotone function properties Feller (1971) .

The next section addresses the case of investments in an asset with stochastic returns modeled by a geometric Brownian motion. The asymptotic decay for both the ruin probability in finite and infinite time is analyzed using Karamata Tauberian methods.

For the probability of ruin with infinite horizon the surprising result is that the probability of ruin decays slower to zero than in the classical case without investments. Intuitively, this may be explained by the fact that a claim may occur when the market value of the asset is low and it is not possible to be cover the losses just by selling these assets (Frolova et al., 2002). Also surprising is that the decay of the ruin probability in case of investments (with small volatility) depends only on the investment parameters suggesting that the "insurance" part of the model does not influence the long term asymptotic behavior of the ruin probability. Investments in a stock with large volatility lead to ruin with probability one whatever is the initial capital.

The asymptotic behavior of the Laplace transform of the probability of ruin in finite time is shown to have an algebraic decay rate depending on the investments parameters and the Laplace transform coefficient.

4.1. Asymptotic analysis of the ruin probability-no investments

Recall that the classical risk model

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \tag{4.1}$$

is a compound Poisson process. The net profit conditions $c > \lambda\mu$ guarantees a positive incoming flow.

4.1.1 The classical martingale method

Lemma 3. *Consider* $\{U(t)_{t\geq 0}\}$ *a stochastic process with stationary and independent increments. Then a process of the form* $\{e^{-RU(t)}\}_{t\geq 0}$ *is a martingale iff*

$$cR - \lambda(\hat{f_X}(-R) - 1) = 0. \tag{4.2}$$

This is called the fundamental Lundberg equation.

Proof. Since the moment generating function of a compound Poisson process is

$$\mathbf{E}e^{-R\sum_{k=1}^{N(t)} X_k} = e^{\lambda t(\hat{f_x}(R)-1)}$$

then the conditional expected value is

$$\mathbf{E}(e^{-RU(t)} \mid U(0) = u) = exp(-Ru - Rct + \lambda t(\hat{f_X}(-R) - 1)).$$

Therefore, the martingale condition is equivalent to (4.2). □

This R has a crucial role in the asymptotic decay of the ruin probability.

Definition 12. *The exponent $R > 0$ for which the process is a martingale is called the* **Lundberg exponent** *or the* **adjustment coefficient**.

Theorem 6. *Under Cramér-Lundberg conditions, the following holds*

$$\Psi(u) \leq e^{-Ru},$$

where R is the Lundberg exponent.

Proof. For this martingale, Doob's inequality provides an easy and elegant way to prove the exponential upper bound of the ruin probability:

$$P(\sup_{0\leq v\leq t}(\sum_{k=1}^{N(v)} X_k - cv) \geq u) = P(\sup_{0\leq v\leq t} e^{R(\sum_{k=1}^{N(v)} X_k - cv)} \geq e^{Ru}) \leq \frac{\mathbf{E} e^{R(\sum_{k=1}^{N(t)} X_k - ct)}}{e^{Ru}} = e^{-Ru}$$

Let $t \to \infty$ to get

$$\Psi(u) = P(\inf_{t\geq 0}(u + ct - \sum_{k=1}^{N(t} X_k) \leq 0) = P(\sup_{t\geq 0}(\sum_{k=1}^{N(t} X_k - ct) \geq u) \leq e^{-Ru}.$$

□

4.1.2 The Karamata method

The following well-known theorem regarding the exponential decay of the ruin probability in the classical Cramér-Lundberg model will be proved using Laplace transforms properties only. Some of the completely monotone functions properties used here are presented in the Appendix D.

Theorem 7. *Consider the classical Cramér-Lundberg risk model (2.3),*

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k,$$

under the net profit condition, $\lambda\mu < c$. Assume that there exists an $R > 0$ the smallest positive solution of the Lundberg equation,

$$\int_0^\infty e^{Rx} dF_I(x) = \frac{c}{\lambda\mu},$$

where

$$F_I(x) = \frac{1}{\mu}\int_0^x (1 - F_X(y))dy \, x \geq 0.$$

Then

$$\lim_{u\to\infty} e^{-Ru}\Psi(u) = \frac{\lambda\mu - c}{c + \lambda \hat{f_X}'(-R)}. \tag{4.3}$$

Proof. The integro-differential equation for the probability of ruin is

$$\Psi'(u) = \frac{\lambda}{c}\Psi(u) - \frac{\lambda}{c}\int_0^\infty \Psi(u-x)F(dx).$$

Splitting the integral from 0 to u and from u to ∞,

$$\Psi'(u) = \frac{\lambda}{c}\Psi(u) - \frac{\lambda}{c}\int_0^u \Psi(u-x)F(dx) - \frac{\lambda}{c}\int_u^\infty \Psi(u-x)F(dx),$$

and recalling that $\Psi(x) = 1$ for $x < 0$ we have

$$\Psi'(u) = \frac{\lambda}{c}\Psi(u) - \frac{\lambda}{c}\int_0^u \Psi(u-x)F(dx) - \frac{\lambda}{c}(1 - F(u)),$$

since $F(\infty) = 1$. Taking the Laplace transform and using its properties, one obtains the equation

$$s\hat{\Psi}(s) - \Psi(0) = \frac{\lambda}{c}\hat{\Psi}(s) - \frac{\lambda}{c}\hat{\Psi}(s)\hat{f}(s) - \frac{\lambda}{c}(\frac{1}{s} - \frac{\hat{f}(s)}{s})$$

with the solution

$$\begin{aligned}
\hat{\Psi}(s) &= \frac{c\Psi(0) - \frac{\lambda}{s}(1 - \hat{f}(s))}{cs - \lambda(1 - \hat{f}(s))} \\
&= \frac{cs\Psi(0) - \lambda(1 - \hat{f}(s))}{cs - \lambda(1 - \hat{f}(s))}\frac{1}{s} \\
&= \frac{cs\Psi(0) + [cs - \lambda(1 - \hat{f}(s))] - cs}{cs - \lambda(1 - \hat{f}(s))}\frac{1}{s} \\
&= (1 + \frac{cs\Psi(0) - cs}{cs - \lambda(1 - \hat{f}(s))})\frac{1}{s} \\
&= (1 + \frac{\Psi(0) - 1}{1 - \frac{\lambda}{c}\frac{(1-\hat{f}(s))}{s}})\frac{1}{s}.
\end{aligned}$$

Since $\hat{f}$ is completely monotone, that implies that $\frac{1-\hat{f}(s)}{s}$ is also completely monotone (see the Appendix) . Therefore

$$1 + \frac{\Psi(0) - 1}{1 - \frac{\lambda}{c}\frac{(1-\hat{f}(s))}{s}}$$

is the Laplace Stieltjes transform of a measure P.

The denominator of the function $\hat{\Psi}(s)$ is the quantity appearing in the Lundberg equation(4.2), thus it vanishes at $-R$. Since the numerator doesn't vanish at $-R$, $-R$ is a pole of $\hat{\Psi}(s)$. Rewrite

$$\hat{\Psi}(s) = (1 + \frac{\Psi(0) - 1}{1 - \frac{\lambda}{c}\frac{(1-\hat{f}(s))}{s}})\frac{s+R}{s}\frac{1}{s+R},$$

where R is the adjustment coefficient. The function $h(s) = s$ is completely monotone, therefore $(h(s))^{-1} = \frac{1}{s}$ is completely monotone and also $h(s+R)$ and $(h(s+R))^{-1}$ are completely monotone. Since a product of completely monotone functions is completely monotone, that means that

$$1 + \frac{\Psi(0) - 1}{1 - \frac{\lambda}{c}\frac{(1-\hat{f}(s))}{s}}\frac{s+R}{s} =: \hat{G}(s)$$

is completely monotone and thus the Laplace Stieltjes transform of a measure Q. Hence, the function $\hat{\Psi}(s)$ can be written as

$$\begin{aligned}\hat{\Psi}(s) &= \frac{1}{s+R}\hat{G} \\ &= \frac{1}{s+R}(\hat{G}(s) - \Psi(0) + \Psi(0)) \\ &= \frac{1}{s+R}\hat{H}(s) + \frac{1}{s+R}\Psi(0),\end{aligned}$$

where $\hat{H}(s) = \hat{G}(s) - \Psi(0)$, for all s in the domain of $\hat{G}$. The function $\hat{G}(s)$ is the Laplace transform of a function $G(u)$ and $\hat{H}(s)$ the Laplace transform of the corresponding $H(u)$. As the Laplace transform of a product of two functions is the Laplace transform of the convolution of the given functions, by the uniqueness of the Laplace transform it follows that:

$$\Psi(u) = \int_0^u e^{-(u-t)R} H(t)dt + e^{-Ru}\Psi(0). \tag{4.4}$$

Passing to the limit,

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = \int_0^\infty e^{-(-R)t} H(t)dt + \Psi(0) = \hat{H}(-R) + \Psi(0),$$

which is equivalent to

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = \hat{G}(-R) - \Psi(0) + \Psi(0) = \hat{G}(-R).$$

Recall, by definition, $\hat{G}(s) = \hat{\Psi}(s)(s+R)$, thus

$$\hat{\Psi}(s) = \frac{\hat{G}(s)}{s+R} = \frac{c\Psi(0) - \frac{\lambda}{s}(1-\hat{f}(s))}{cs - \lambda(1-\hat{f}(s))} := \frac{a(s)}{b(s)}$$

This set of equalities requires that

$$\hat{G}(s) = \frac{a(s)}{\frac{b(s)}{s+R}} = \frac{a(s)}{\frac{b(s)-b(-R)}{s+R}}, \tag{4.5}$$

where $b(-R) = 0$, since $b(s) = 0$ is the Lundberg equation. Under these same considerations, $cs/\lambda = (1 - \hat{f}(s))$, for $s = -R$. This implies

$$\hat{G}(-R) = \frac{a(-R)}{b'(-R)} = \frac{c\Psi(0) - \frac{\lambda}{-R}\frac{c}{\lambda}(-R)}{c + \lambda \hat{f}'(-R)} = \frac{\Psi(0) - 1}{1 + \frac{\lambda}{c}\hat{f}'(-R)}.$$

The initial condition, $\Psi(0) = \frac{\lambda\mu}{c}$, leads to

$$\hat{G}(-R) = \frac{\lambda\mu - c}{c + \lambda \hat{f}'(-R)}.$$

Thus,

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = \frac{\lambda\mu - c}{c + \lambda \hat{f}'(-R)}. \tag{4.6}$$

□

Remark 8. *If the claim sizes are exponentially distributed, the conditions on the Laplace transform of the ruin probability are trivially satisfied, since $\hat{H}(s) = 0$. In general, the existence of the function H depends upon the tail of the distribution of the claim sizes.*

4.2. Asymptotic analysis of the ruin probabilities - investments

If the surplus is continuously invested in a risky asset the asymptotic behavior of the probability of ruin changes dramatically. Namely, if the insurance company invests the capital in an asset with a price that follows a geometric Brownian motion with drift a and volatility σ, then the ruin probability has an algebraic decay rate or equals one, depending only on the parameters a and σ of the asset.

The integro-differential equation satisfied by the probability of ruin is a particular case of the equation (3.7) derived in the previous chapter. The asymptotic analysis uses Karamata-Tauberian theorems and permits a generalization of the result from Frolova et al. (2002). In the cited paper the result is established only for exponentially distributed claim sizes with a method of proof relying on special properties of the exponential functions. A generalization of the result for distributions of the claim sizes having moment generating functions defined on a neighborhood of the origin is possible.

4.2.1 Asymptotic decay of the probability of ruin

Recall the surplus process with investments in a stock modeled by a geometric Brownian motion

$$U(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dWs - \sum_{k=1}^{N(t)} X_k, \tag{4.7}$$

is a continuous Markov process. Here W_s denotes the standard Brownian motion. The infinitesimal generator of the process $U(t)$ is denoted by A_U. The investments process satisfies

$$dZ_t = (c + aZ_t)dt + \sigma Z_t dW_t,$$

with

$$A = (c + au)\frac{d}{du} + \frac{\sigma^2}{2}\frac{d^2}{du^2}.$$

Using the equation introduced in the previous chapter, the probability of ruin satisfies

$$\mathcal{L}^*(A)\Psi(u) = \int_0^\infty \Psi(u - x)f_X(x)dx$$

since the exponential distributed inter-arrival times satisfy $\mathcal{L}^* = (-\frac{d}{dt} + \lambda)$. Hence, in this case the equation is

$$(-(c + au)\frac{d}{du} - \frac{\sigma^2}{2}u^2\frac{d^2}{du^2} + \lambda)\Psi(u) = \lambda\int_0^\infty \Psi(u - x)f_X(x)dx. \tag{4.8}$$

Lemma 4. *Let $\Psi \in \mathcal{D}_A$ such that*

$$A_U \Psi(u) = 0$$

with the boundary conditions

$$\lim_{u\to\infty} \Psi(u) = 0,$$

$$c\Psi'(0) - \lambda\Psi(0) + \lambda = 0.$$

Then the Laplace transform of the equation has the form

$$\frac{\sigma^2 s^2}{2}\hat{\Psi}''(s)+(2s\sigma^2-as)\hat{\Psi}'(s)+(cs-\lambda+\lambda\hat{f}_X(s)+\sigma^2-a)\hat{\Psi}(s) = c\Psi(0)-\frac{\lambda}{s}(1-\hat{f}_X(s)). \tag{4.9}$$

Proof. For the model (4.7) the following is true:

$$A_U\Psi(u) = \frac{\sigma^2}{2}u^2\Psi''(u) + (au+c)\Psi'(u) + \lambda\int_0^\infty (\Psi(u-y) - \Psi(u))\, dF_X(y).$$

Because, by definition, $\Psi(u-y) = 1$ for any $u < y$, and $\int_0^\infty dF_X(y) = 1$, the equation is equivalent to

$$\frac{\sigma^2}{2}u^2\Psi''(u) + (au+c)\Psi'(u) + \lambda\int_0^u \Psi(u-y)\, dF_X(y) + \lambda(1 - F_X(u)) - \lambda\Psi(u) = 0.$$

The Laplace transform of this equation is

$$\frac{\sigma^2}{2}\frac{d^2(s^2\hat{\Psi}(s))}{ds^2} - a\frac{d(s\hat{\Psi}(s))}{ds} + cs\hat{\Psi}(s) - \lambda\hat{\Psi}(s) + \lambda\hat{\Psi}(s)\hat{f}_X(s) + \frac{\lambda}{s}(1-\hat{f}_X(s)) = c\Psi(0),$$

and after differentiation becomes

$$\frac{\sigma^2 s^2}{2}\hat{\Psi}''(s)+(2s\sigma^2-as)\hat{\Psi}'(s)+(cs-\lambda+\lambda\hat{f}_X(s)+\sigma^2-a)\hat{\Psi}(s) = c\Psi(0)-\frac{\lambda}{s}(1-\hat{f}_X(s)).$$

$\square$

Lemma 5. *Consider the same model (4.7), with $a, \sigma^2, \lambda, \mu, c$ defined as before. Let $p, q, g : R \to R$ holomorphic functions defined by*

$$p(s) = p_0 = \frac{2(2\sigma^2 - a)}{\sigma^2}$$
$$q(s) = q_0 + q_1(s) = \frac{2(\sigma^2 - a)}{\sigma^2} + q_1(s)$$
$$g(s) = g_0 + g_1(s) = \frac{2(c\Psi(0) - \lambda\mu)}{\sigma^2} + g_1(s)$$

where q_1, g_1 are also holomorphic. Then the equation (4.9) has the form

$$s^2 y'' + p(s)sy' + q(s)y = g(s), \tag{4.10}$$

and its solution has the form

$$y = c_1 s^{-1}\gamma_1(s) + c_2 s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s) + c_3\gamma_3(s), \tag{4.11}$$

under the condition $\frac{2a}{\sigma^2} < 2$, with c_1, c_2, c_3 real constants, γ_1, γ_2 and γ_3 holomorphic functions and $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$.

Proof. The homogeneous part of the equation (4.10) is

$$s^2 y'' + p(s)sy' + q(s)y = 0.$$

It has $s = 0$ as a regular singular point. Thus, the solution of the homogeneous equation has the form

$$y(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k = \sum_{k=0}^{\infty} c_k s^{\rho+k}, \tag{4.12}$$

where the coefficients satisfy the recurrence system of equations $c_0 = 1$ and

$$c_k f(\rho + k) + c_{k-1} f_1(\rho + k - 1) + \cdots + c_0 f_k(\rho) = 0,$$

with

$$f(\rho) = \rho(\rho - 1) + p_0\rho + q_0,$$

and

$$f_k(\rho) = \rho p_k + q_k$$

as in Fedoryuk (1991). The first of these equations $c_0 f(\rho) = 0$ is equivalent to

$$\rho^2 + \frac{3\sigma^2 - 2a}{\sigma^2}\rho + \frac{2\sigma^2 - 2a}{\sigma^2} = 0.$$

If $2\sigma^2 \neq a$, the solutions of the homogeneous equation are of the form

$$y_1(s) = s^{-1}\gamma_1(s) \quad y_2(s) = s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s),$$

where $\gamma_1(0) = \gamma_2(0) = 1$ with c_1, c_2, c_3 real constants, γ_1, γ_2 and γ_3 holomorphic functions and $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$..

Using the method of variation of parameters, the particular solution of the non-homogeneous equation is obtained using the general form from Boyce and DiPrima (2005), (page 239):

$$y_p(s) = \sum_{m=1}^{n} y_m(s) \int_{s_0}^{s} \frac{g(t)W_m(t)}{W(t)} dt \tag{4.13}$$

where n is the order of the differential equation, s_0 is arbitrary, $g(s) = \frac{g(s)}{s^2}$ is the non-homogeneous part of the equation, $W(s)$ is the determinant of coefficients $W(y_1, y_2)$ and it is nowhere zero since y_1, y_2 are linearly independent solutions of the homogeneous equation. Here W_m is the determinant obtained from W by replacing the m-th column by the column $(0, 1)$.

$$W(s) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} s^{\rho_1}\gamma_1(s) & s^{\rho_2}\gamma_2(s) \\ s^{\rho_1 - 1}\gamma_1^1(s) & s^{\rho_2 - 1}\gamma_2^1(s) \end{vmatrix} = s^{\rho_1 + \rho_2 - 1}\gamma(s),$$

where $\gamma_i^1 = \rho_i\gamma_i + \gamma_i'$ and $\gamma = \gamma_1\gamma_2^1 - \gamma_2\gamma_1^1$. Also, the W_m determinants look like

$$W_1(s) = W(I, y_2) = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2 = -s^{\rho_2}\gamma_2(s)$$

and

$$W_2(s) = W(y_1, I) = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1 = s^{\rho_1}\gamma_1(s)$$

Thus, the particular solution looks like

$$\begin{aligned} y_p(s) &= y_1(s)\int_0^s \frac{g(t)W_1(t)}{t^2W(t)}dt + y_2(s)\int_0^s \frac{g(t)W_2(t)}{t^2W(t)}dt \\ &= s^{\rho_1}\int_0^s \frac{g(t)(-t^{\rho_2}\gamma_2(t))}{t^2t^{\rho_1+\rho_2-1}\gamma(t)}dt + s^{\rho_2}\int_0^s \frac{g(t)(t^{\rho_1}\gamma_1(t))}{t^2t^{\rho_1+\rho_2-1}\gamma(t)}dt \\ &= -s^{\rho_1}\int_0^s g_1(t)t^{-\rho_1-1}dt + s^{\rho_2}\int_0^s g_2(t)t^{-\rho_2-1}dt \\ &= -s^{\rho_1}s^{-\rho_1}g_1^1(s) + s^{\rho_2}s^{-\rho_2}g_2^1(s) \\ &= \gamma_3(s) \end{aligned}$$

where $g_1, g_2, g_1^1, g_2^1, \gamma_3(s)$ are holomorphic. In order to integrate within the process, the following conditions should be imposed

$$-\rho_1 - 1 > -1 \quad \text{and} \quad -\rho_2 - 1 > -1$$

in other words

$$\rho_1 < 0 \quad \text{and} \quad \rho_2 < 0.$$

Since $\rho_1 = -1 < 0$ for any a, σ but $\rho_2 < 0$ imposes the condition $\frac{2a}{\sigma^2} < 2$.

Thus, the solution of this equation (4.10) has the form

$$y = c_1s^{-1}\gamma_1(s) + c_2s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s) + c_3\gamma_3(s),$$

under the condition $\frac{2a}{\sigma^2} < 2$, with c_1, c_2, c_3 real constants, γ_1, γ_2 and γ_3 holomorphic functions and $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$.

□

Theorem 8. *Consider the model given by (4.7) and assume that* $\sigma > 0$. *Assume also that the distribution of the claims sizes* F *has a moment generating function defined on a neighborhood of the origin. Then:*

1. *If the ruin probability decays at infinity, then*

$$2a/\sigma^2 > 1.$$

2. *If* $1 < 2a/\sigma^2 < 2$, *then for some* $K > 0$,

$$\lim_{u \to \infty} \Psi(u) u^{-1+\frac{2a}{\sigma^2}} = K.$$

Proof. According to Karamata-Tauberian theorem, in order to find the asymptotic behavior of $\Psi(u)$ at infinity it is enough to analyze the asymptotic behavior of the Laplace transform $\hat{\Psi}$ at zero. Therefore, first is analyzed the asymptotic behavior at zero of the solution of equation (5.7). The leading term of this linear combination dictates the asymptotic behavior of the solution as $s \to 0$. Since

$$\rho_1 \leq \rho_2$$

two cases can be distinguished.

Case 1. If the leading term of the linear combination is $s^{\rho_1} = s^{-1}$ then

$$\hat{\Psi}(s) \sim c_1 \gamma_1(s) s^{-1} \quad \text{as} \quad s \to 0,$$

where $\gamma_1(0) = 1$. Applying Karamata Tauberian theorems

$$\Psi(u) \sim \frac{c_1}{\Gamma(2)}\gamma(\frac{1}{u}) \quad \text{as} \quad u \to \infty.$$

Hence

$$\lim_{u\to\infty} \Psi(u) = c_1,$$

where $c_1 \neq 0$ is a real constant. In other words, the ruin probability has a constant asymptotic behavior, as $u \to \infty$. Obviously, in this case, the function does not satisfy the boundary conditions of the equation the of the ruin probability. Thus, it is not a solution that can be considered.

Case 2. If $s^{\rho_2} = s^{-2+\frac{2a}{\sigma^2}}$ is the leading term, then

$$\hat{\Psi}(s) \sim c_2\gamma(s)s^{-2+\frac{2a}{\sigma^2}} \quad \text{as} \quad s \to 0.$$

Then the Proposition 10 for $l(s) = \gamma_2(s)$ implies

$$\Psi(u) \sim \frac{c_2(2-\frac{2a}{\sigma^2})}{\Gamma(3-\frac{2a}{\sigma^2})}\gamma(\frac{1}{u})u^{2-\frac{2a}{\sigma^2}-1}, \quad \text{as} \quad u \to \infty.$$

Since $\Psi(u)$ must decay, $\frac{2a}{\sigma^2}$ needs to satisfy the condition $2-\frac{2a}{\sigma^2}-1 < 0$ which proves the first part of the theorem $\frac{2a}{\sigma^2} > 1$. Thus, the asymptotic decay at infinity is

$$\lim_{u\to\infty} \Psi(u)u^{-1+\frac{2a}{\sigma^2}} = K, \quad \text{for} \quad 1 < \frac{2a}{\sigma^2} < 2,$$

where $K = \frac{c_2(2-\frac{2a}{\sigma^2})}{\Gamma(3-\frac{2a}{\sigma^2})}$, which proves the second part of the theorem. □

4.2.2 Asymptotic decay of the probability of ruin in finite time

Gerber and Shiu (1998) derive the decay for the Laplace transform of the probability of ruin in finite time, $\hat{\Psi}(u,t)$ via the Laplace transform of the time of ruin $\Phi_\lambda(u) = E_u(e^{-\lambda T_u})$. The relationship between the two is a simple integration by parts:

$$\begin{aligned}
\Phi_\lambda(u) &= E(e^{-\lambda T_u} \mid U(0) = u] \\
&= \int_0^\infty e^{-\lambda t}\frac{d}{dt}\Psi(u,t)dt \\
&= \lambda \int_0^\infty e^{-\lambda t}\Psi(u,t)dt \\
&= \lambda\hat{\Psi}(u,t).
\end{aligned}$$

In their paper, Gerber and Shiu show that $\Phi_\lambda(u)$ has an exponential decay at infinity given by

$$\Phi_\lambda(u) \sim \frac{\lambda}{-\lambda\hat{f}'_X(-R) - c}(\frac{1}{R} + \frac{1}{\rho})e^{-Ru}$$

and conclude that the decay of the Laplace transform of the ruin probability in finite time has an exponential decay as well

$$\hat{\Psi}(u,t) \sim \frac{1}{-\lambda\hat{f}'_X(-R) - c}(\frac{1}{R} + \frac{1}{\rho})e^{-Ru},$$

where R and ρ are solutions of the Lundberg equation.

The decay of the Laplace transform of the probability of ruin in finite time $\hat{\Psi}(u,t)$ may be obtained from the asymptotic decay of the Laplace transform of the time of ruin $\Phi_\lambda(u)$. Recall the theorem of Paulsen and Gjessing (1997) that gives an equation for the Laplace transform of the time of ruin:

Theorem 9. *Let Φ_λ be a bounded twice continuous differentiable on $u \geq 0$ with a bounded first derivative. If Φ_λ solves*

$$A\Phi_\lambda(u) = \lambda\Phi_\lambda(u)$$

together with the boundary conditions

$$\Phi_\lambda(u) = 1, \quad on \quad u < 0,$$

$$\lim_{u\to\infty} \Phi_\lambda(u) = 0$$

then

$$\Phi_\lambda(u) = E(e^{-\lambda T_u})$$

Obviously when $\lambda = 0$, $\Phi_\lambda = \Psi$.

Lemma 6. *Let $\Psi \in \mathcal{D}_A$ such that*

$$A\Phi_\lambda(u) = \lambda\Phi_\lambda(u)$$

with the boundary conditions

$$\lim_{u\to\infty} \Phi_\lambda(u) = 0,$$

$$c\Phi'_\lambda(0) - (\lambda + \alpha)\Phi_\lambda(0) + \lambda = 0.$$

Then the Laplace transform of the equation has the form

$$\frac{\sigma^2 s^2}{2}\hat{\Phi}''_\lambda(s) + (2s\sigma^2 - as)\hat{\Phi}'_\lambda(s) + (cs - \lambda + \lambda\hat{f}_X(s) + \sigma^2 - a - \lambda)\hat{\Phi}_\lambda s) = c\Phi_\lambda(0) - \frac{\lambda}{s}(1 - \hat{f}_X(s)). \tag{4.14}$$

Proof. The Laplace transform of this equation is

$$\hat{A}\hat{\Phi}_\lambda(s) = \lambda\hat{\Phi}_\lambda(s).$$

The left hand side is exactly the same as before, the only difference is a $\lambda\hat{\Phi}_\lambda(s)$ on the right hand side that it will combine with the other $\hat{\Phi}_\lambda(s)$ on the left proving the result. □

The equation (4.14) has the same form as before

$$s^2 y'' + p(s)sy' + q(s)y = g(s),$$

with p, q and g holomorphic functions of the form. The only difference appears in the q_0 term.

$$\begin{aligned}
p(s) &= p_0 = \frac{2(2\sigma^2 - a)}{\sigma^2} \\
q(s) &= q_0 + q_1(s) = \frac{2(\sigma^2 - a - \lambda)}{\sigma^2} + q_1(s) \\
g(s) &= g_0 + g_1(s) = \frac{2(c\Psi(0) - \lambda\mu)}{\sigma^2} + g_1(s).
\end{aligned}$$

Following the same regularity arguments in analyzing the homogeneous equation, the equation to be solved is

$$\rho(\rho - 1) + p_0\rho + q_0 = 0$$

with solutions

$$\begin{aligned}
\rho_1 &= -1 + \frac{(a - \frac{\sigma^2}{2}) - \sqrt{(a - \frac{\sigma^2}{2})^2 + 2\lambda\sigma^2}}{\sigma^2} \\
\rho_2 &= -1 + \frac{(a - \frac{\sigma^2}{2}) + \sqrt{(a - \frac{\sigma^2}{2})^2 + 2\lambda\sigma^2}}{\sigma^2}.
\end{aligned}$$

Thus, the solution of the non-homogeneous equation will be a linear combination of the solutions of the homogeneous and a particular solution. For the particular solutions, the method of variation of parameter will be used in the same manner as in the previous chapter. From (4.13) one has that $y_p(s) = \gamma_3(s)$, where γ_3 is a holomorphic function. As before, in order to integrate within the process, the following conditions should be imposed

$$\rho_1 < 0 \quad \text{and} \quad \rho_2 < 0.$$

It can be shown that $\rho_1 < 0$ for any a, σ, λ, but $\rho_2 < 0$ imposes the condition $\lambda < \sigma^2 - a$. For $\lambda = 0$ the condition derived from $\rho_2 < 0$ is $\frac{2a}{\sigma^2} < 2$.

Exactly like in the calculations of the Laplace transform of the probability of ruin, the equation for $\hat{\Phi}_\lambda(u)$ has the form:

$$\hat{\Phi}_\lambda(s) = c_1 s^{-1}\gamma_1(s) + c_2 s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s) + c_3\gamma_3(s),$$

under the condition $\lambda < \sigma^2 - a$, with c_1, c_2, c_3 real constants, γ_1, γ_2 and γ_3 holomorphic functions and $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$.

The asymptotic decay of the Laplace transform of the time of ruin will be given by the following theorem:

Theorem 10. *Consider the model given by (4.7) and assume that $\sigma > 0$. Assume also that the distribution of the claims sizes F has a moment generating function defined on a neighborhood of the origin. Define*

$$\Phi_\lambda(u) = \mathbf{E}(e^{-\lambda T_u}|U(0) = u)$$

and

$$\delta_\lambda = -\frac{(a - \frac{\sigma^2}{2}) + \sqrt{(a - \frac{\sigma^2}{2})^2 + 2\lambda\sigma^2}}{\sigma^2}.$$

If $\lambda < \sigma^2 - a$, then for some $K_\lambda > 0$,

$$\lim_{u\to\infty} \Phi_\lambda(u)u^{-\delta_\lambda} = K_\lambda.$$

Proof. Under the assumption $\frac{\sigma^2}{2} - a > 0$ there is the following ordering among the powers of the solutions

$$\rho_1 \leq -1 \leq \rho_2.$$

As discussed in the case of the decay of the probability of ruin, s^{ρ_1} will not produce decay on the u side. Therefore, the only candidate for the decay is

$$s^{\rho_2} = s^{-1+\frac{(a-\frac{\sigma^2}{2})+\sqrt{(a-\frac{\sigma^2}{2})^2+2\lambda\sigma^2}}{\sigma^2}}.$$

Then, by Proposition 10, the conclusion is

$$\lim_{u\to\infty} \Phi_\lambda(u) u^{\frac{(a-\frac{\sigma^2}{2})+\sqrt{(a-\frac{\sigma^2}{2})^2+2\lambda\sigma^2}}{\sigma^2}} = K_\lambda,$$

and the result follows. □

Corollary 4.

$$\lim_{u\to\infty} \hat{\Psi}(u,t) u^{-\delta_\lambda} = \frac{K_\lambda}{\lambda}$$

Remark 9. *Note that for* $\lambda = 0$ *the necessary condition for integrability is again* $1 < \frac{2a}{\sigma^2} < 2$ *and the asymptotic decay is*

$$\lim_{u\to\infty} \Phi_0(u) u^{\rho_\lambda} = \lim_{u\to\infty} \Phi_0(u) u^{-1+\frac{2a}{\sigma^2}} = K$$

where $\Phi_0 = \Psi$, *i.e. the decay for the ruin probability is re-derived.*

Surprisingly, investing everything in a risky asset leads faster to ruin than not investing anything. This result is proved for a Cramér -Lundberg model with investments is a stock with returns modeled by a geometric Brownian motion. More surprising is the fact that the decay rate depends on the parameters of the investment only, namely the drift a and the volatility σ. This suggests that the premium rate, the average claim size or the average waiting time are irrelevant in terms of the decay rate. This idea will prove instrumental in understanding the intuition of the next chapter result, where the asymptotic decay will be studied for inter-arrival times that are sum of exponentials with identical or various parameters. It will be shown that the asymptotic behavior has the same decay rate irrespective of the waiting time in between claims.

Karamata-Tauberian theorems are the key arguments in the analysis of the asymptotic behavior of the probability of ruin in both finite or infinite time. Also, this type of argument is proved to work in deriving the well-known exponential decay of the probability of ruin if no investments are made. The same steps will be taken in the next chapter to derive the asymptotic decay for the ruin probability for a discrete time Markov process, in other words for a Sparre Andersen surplus model with investments.

5. RUIN PROBABILITIES IN THE RENEWAL RISK MODEL

This chapter presents a comparison of the ruin probabilities when only the inter-arrival times distributions are different.

First, by probabilistic arguments an ordering of the ruin probabilities is presented. Next, since the probabilities or ruin are solutions of the newly introduced integro-differential equations, new comparisons are possible. The striking conclusion is that the probabilities of ruin will have the same asymptotic decay rate, irrespective of the inter-arrival times distribution as long as they are sums of exponentials with identical parameters. The case of a sum of exponentials with different parameters is conjectured to have the same asymptotic decay rate.

Note that the models considered have investments in a risky asset with prices modeled by a geometric Brownian motion. As before, the decay rate depends on the parameter a and σ of the investment. For small volatility $2a > \sigma^2$ the probability of ruin has the common algebraic decay rate irrespective of the inter-arrival times and claim sizes processes. The large volatility case is still an open question.

5.1. Ordering of the Ruin Probabilities

One of the possible methods in establishing the comparison between ruin probabilities is the sample path-wise domination.

Let

$$U^{(1)}(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_s - \sum_{k=1}^{N^{(1)}(t)} X_k \tag{5.1}$$

be a Cramér-Lundberg risk model with investments in a risky asset with a price which follows a geometric Brownian motion. The inter-arrival times $\{\tau_k^{(1)}\}_k$ are independent, $\exp(\beta)$ distributed random variables . The claims arrival process $N^{(1)}(t)$ is a Poisson process. The probability of ruin for this process will be denoted by

$$\Psi_1(u) = P(T_u^{(1)} < \infty).$$

Let

$$U^{(2)}(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_s - \sum_{k=1}^{N^{(2)}(t)} X_k$$

be a Sparre Andersen risk model with investments in the same risky asset as in (5.1), but with inter-arrival times $\{\tau_k^{(2)}\}_k$ independent, $Erlang(2,\beta)$ distributed random variables. The claim arrivals process $N^{(2)}(t)$ is a renewal process. The corresponding ruin probability is

$$\Psi_2(u) = P(T_u^{(2)} < \infty).$$

Recall that $\tau_1^{(1)} + \tau_2^{(1)}$ have the same distribution as $\tau_1^{(2)}$.

The comparison of the surplus processes under these different inter-arrival times distributions may be achieved by a coupling of both processes derived from the common underlying Brownian motion. To be precise, one uses

$$Z(t) = Z(0)\exp\{(a-\frac{\sigma^2}{a})t+\sigma W_t\}+c\int_0^t \exp\{(a-\frac{\sigma^2}{a})(t-u)+\sigma(W_t-W_u)\}du, \tag{5.2}$$

the explicit representation in terms of the Brownian motion of the solution of the stochastic differential equation governing the investment process

$$dZ = (aZ + c)dt + \sigma Z dW_t,$$

given in e.g. Thomann and Waymire (2003)). This can be thought of as a type of stochastic Duhamel principle which can be verified using Itô's lemma.

Lemma 7. *If $Z(t)$ satisfies the equation (5.2), then for any $0 \leq s \leq t$,*

$$Z(t) = Z(s)\exp\{(a - \frac{\sigma^2}{a})(t-s) + \sigma(W_t - W_s)\} + c\int_s^t \exp\{(a - \frac{\sigma^2}{a})(t-u) + \sigma(W_t - W_u)\}du$$

Proof.

$$\begin{aligned}
Z(t) &= Z(0)\exp\{(a - \frac{\sigma^2}{a})t + \sigma W_t\} + c\int_0^t \exp\{(a - \frac{\sigma^2}{a})(t-u) + \sigma(W_t - W_u)\}du \\
&= Z(0)\exp\{(a - \frac{\sigma^2}{a})(t - s + s) + \sigma(W_t - W_s + W_s)\} \\
&+ c\int_0^s \exp\{(a - \frac{\sigma^2}{a})[(t-s) + (s-u)] + \sigma[(W_t - W_s) + (W_s - W_u)]\}du \\
&+ c\int_s^t \exp\{(a - \frac{\sigma^2}{a})(t-u) + \sigma(W_t - W_u)\}du \\
&= \{Z(0)\exp\{(a - \frac{\sigma^2}{a})s + \sigma W_s\} + c\int_0^s \exp\{(a - \frac{\sigma^2}{a})(s-u) + \sigma(W_s - W_u)\}du\} \\
&\times \exp\{(a - \frac{\sigma^2}{a})(t-s) + \sigma(W_t - W_s)\} \\
&+ c\int_s^t \exp\{(a - \frac{\sigma^2}{a})(t-u) + \sigma(W_t - W_u)\}du \\
&= Z(s)\exp\{(a - \frac{\sigma^2}{a})(t-s) + \sigma(W_t - W_s)\} \\
&+ c\int_s^t \exp\{(a - \frac{\sigma^2}{a})(t-u) + \sigma(W_t - W_u)\}du.
\end{aligned}$$

□

Proposition 3. *For the processes $U^{(1)}$ and $U^{(2)}$ defined above, the probabilities of ruin have the following order:*

$$\Psi_1(u) \geq \Psi_2(u).$$

Proof. In order to compare the two ruin probabilities $\Psi_1(u)$ and $\Psi_2(u)$, one compares the two surplus processes $U^{(1)}$ and $U^{(2)}$ along each sample path of the Brownian motion. Both start with the same initial surplus u and have the same underling Brownian motion. Let $T_1^{(1)}$ denote the time of the first claim in the $U^{(1)}$ process.

Then for any $0 \le t < T_1^{(1)}$, according to the equation (5.2) one has

$$\begin{aligned} U^{(2)}(t) &= u\exp\{(a-\frac{\sigma^2}{a})t+\sigma W_t\}+c\int_0^t \exp\{(a-\frac{\sigma^2}{a})(t-u)+\sigma(W_t-W_u)\}du \\ &= U^{(1)}(t). \end{aligned}$$

At $t = T_1^{(1)}$,

$$\begin{aligned} U^{(2)}(T_1^{(1)}) &= u\{\exp[(a-\frac{\sigma^2}{2})T_1^{(1)}+\sigma W_{T_1^{(1)}}\} \\ &+ c\int_0^{T_1^{(1)}} \exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u)+\sigma(W_{T_1^{(1)}}-W_u\}du \\ &\geq u\{\exp[(a-\frac{\sigma^2}{2})T_1^{(1)}+\sigma W_{T_1^{(1)}}\} \\ &+ c\int_0^{T_1^{(1)}} \exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u)+\sigma(W_{T_1^{(1)}}-W_u\}du - X_1^{(1)} \\ &= U^{(1)}(T_1^{(1)}). \end{aligned}$$

For $T_1^{(1)} \le t < T_2^{(1)}$, according to Lemma 7

$$\begin{aligned} U^{(2)}(t) &= U^{(2)}(T_1^{(1)})\exp\{(a-\frac{\sigma^2}{a})(t-T_1^{(1)})+\sigma(W_t-W_{T_1^{(1)}}\} \\ &+ c\int_{T_1^{(1)}}^t \exp\{(a-\frac{\sigma^2}{a})(t-u)+\sigma(W_t-W_u)\}du \\ &\geq U^{(1)}(T_1^{(1)})\exp\{(a-\frac{\sigma^2}{a})(t-T_1^{(1)})+\sigma(W_t-W_{T_1^{(1)}}\} \\ &+ c\int_{T_1^{(1)}}^t \exp\{(a-\frac{\sigma^2}{a})(t-u)+\sigma(W_t-W_u)\}du \\ &= U^{(1)}(t) \end{aligned}$$

It follows by induction that $U^{(1)}(t) \le U^{(2)}(t)$ for any t. Therefore, $P(T_u^{(1)} < \infty) \le P(T_u^{(2)} < \infty)$ for any u. $\square$

Proposition 4. *For $m < n$, in the case of $Erlang(m, \beta), Erlang(n, \beta)$ risk processes*

$$\Psi_n(u) \leq \Psi_m(u).$$

Proof. Analogously to the previous proof, it can be shown that

$$\Psi_{n+1}(u) \leq \Psi_n(u).$$

Inductively, this means

$$\Psi_1(u) \geq \Psi_2(u) \geq \Psi_3(u) \geq \cdots \Psi_n(2) \cdots$$

for any $n \in N$. Thus, for an $m < n$, the result follows. $\square$

5.2. Asymptotic analysis of the ruin probability

5.2.1 Inter-arrival times $Erlang(2, \beta)$ distributed

Lemma 8. *Consider that the surplus model (3.2)) has inter-arrival times τ_k that are $Erlang(2, \beta)$, distributed with the density function*

$$f_\tau(t) = \beta^2 t e^{-\beta t}, \quad for \quad t \geq 0.$$

Then the equation for the probability of ruin is

$$(-A + \beta)^2 \Psi(u) = f_\tau^{(n-1)}(0) \int_0^\infty \Psi(u - x) f_X(x) dx \tag{5.3}$$

equivalent to

$$\begin{aligned}\beta^2\int_0^u \Psi(u-x)f_X(x)dx + \beta^2(1-F_X(u)) &= (\frac{\sigma^2}{2})^2 u^4 \Psi^{(4)}(u) \\ &+ \sigma^2 u^2(c+au+\sigma^2 u)\Psi'''(u) \\ &+ [(c+au)^2+\sigma^2 u[2au+\frac{\sigma^2}{2}u-\beta u+c)]\Psi''(u) \\ &+ (c+au)(a-2\beta)\Psi'(u) \\ &+ \beta^2\Psi(u)\end{aligned}$$

with the boundary conditions:

1. $\lim_{u\to\infty}\Psi(u)=0$,
2. $c^2\Psi''(0)+c(a-2\beta)\Psi'(0)+\beta^2\Psi(0)-\beta^2=0$,
3. $c^2\Psi'''(0)+92ac+2a\sigma^2+c\sigma^2+ac-2\beta)\Psi''(0)+(a-\beta)^2\Psi'(0)-\beta^2\Psi(0)f_X(0)+\beta^2 f_X(0)=0$,
4. $c^2\Psi^{(4)}(0)+(4c\sigma^2+5ac+2a\sigma^2-2\beta c)\Psi'''(0)+(4a^2+2a\sigma^2+\sigma^4-2\beta\sigma^2-4a\beta+\beta^2)\Psi''(0)-\beta^2 f_X(0)\Psi'(0)+\beta^2 f_X'(0)=0$.

Proof. For an $Erlang(2,\beta)$ distribution,

$$\mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt}+\beta)^2,$$

thus the equation (3.7) is specifically:

$$(-A+\beta)^2\Psi(u) = f_\tau^{(n-1)}(0)\int_0^\infty \Psi(u-x)f_X(x)dx$$

equivalent to

$$(AA-2\beta A+\beta^2)\Psi(u) = \beta^2\int_0^u \Psi(u-x)f_X(x)dx + \beta^2\int_u^\infty f_X(x)dx \tag{5.4}$$

where

$$A = \frac{\sigma^2}{2}u^2\frac{d^2}{du^2} + (au+c)\frac{d}{du}$$

and

$$AA = \frac{\sigma^2}{2}u^2A'' + (au+c)A'$$

with

$$A'\Psi(u) = \frac{\sigma^2}{2}u^2\Psi'''(u) + \sigma^2 u\Psi''(u) + a\Psi'(u) + (c+au)\Psi''(u)$$

and

$$A''\Psi(u) = \frac{\sigma^2}{2}u^2\Psi^{(4)}(u) + (2\sigma^2 u + c + au)\Psi'''(u) + (2a+\sigma^2)\Psi''(u).$$

Thus, the explicit equation for the ruin probability in case on investments in a geometric Brownian motion with inter-arrival times $Erlang(2,\beta)$ distributed is a forth order integro-differential equation:

$$\begin{aligned}
\beta^2\int_0^u \Psi(u-x)f_X(x)dx + \beta^2(1-F_X(u)) \quad &= \quad (\frac{\sigma^2}{2})^2u^4\Psi^{(4)}(u) \\
&+ \quad \sigma^2u^2(c+au+\sigma^2u)\Psi'''(u) \\
&+ \quad [(c+au)^2 + \sigma^2u[2au + \frac{\sigma^2}{2}u - \beta u + c)]\Psi''(u) \\
&+ \quad (c+au)(a-2\beta)\Psi'(u) \\
&+ \quad \beta^2\Psi(u)
\end{aligned}$$

with the boundary conditions obtained from the fact that the equation holds at zero and so do the first two derivatives of the equation. □

Lemma 9. *The Laplace transform of equation (5.4) is of the form*

$$\hat{A}\hat{A}\hat{\Psi} - 2\beta\hat{A}\hat{\Psi} + \beta^2\hat{\Psi} = \beta^2\hat{\Psi}\hat{f_X}, \tag{5.5}$$

and its general solution have the form:

$$\hat{\Psi}(s) = c_1s^{\rho1}\gamma_1(s) + c_2s^{\rho_2}\gamma_2(s) + c_3s^{\rho_3}\gamma_3(s) + c_4s^{\rho_4}\gamma_4(s) + c_5\gamma_5(s),$$

under the condition $\frac{2a}{\sigma^2} < 2$, *with*

1. $\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} = \frac{\rho_2-1}{2} + \sqrt{(\frac{\rho_2+1}{2})^2 + \frac{4\beta}{\sigma^2}}$

2. $\rho_4 = -1 + \frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} = \frac{\rho_2-1}{2} - \sqrt{(\frac{\rho_2+1}{2})^2 + \frac{4\beta}{\sigma^2}}$

and c_1, c_2, c_3 *real constants,* γ_i, *holomorphic functions,* $\gamma_i(0) = 1$, *for* $i = 1, .., 5$.

Proof. The Laplace transform of equation (5.4) is of the form

$$\hat{A}\hat{A}\hat{\Psi} - 2\beta\hat{A}\hat{\Psi} + \beta^2\hat{\Psi} = \beta^2\hat{\Psi}\hat{f_X}, \tag{5.6}$$

where

$$\hat{A}\hat{\Psi}(s) = \frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] - a\frac{d}{ds}[s\hat{\Psi}(s)] + c(s\hat{\Psi}(s) - \Psi(0))$$

and

$$\begin{aligned}
\hat{A}\hat{A}\hat{\Psi}(s) &= (\frac{\sigma^2}{2})^2\frac{d^2}{ds^2}[s^2\frac{d^2}{ds^2}s^2\hat{\Psi}(s)] - a\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\frac{d}{ds}s\hat{\Psi}(s)] + c\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2(s\hat{\Psi}(s) - \Psi(0))] \\
&- \frac{\sigma^2}{2}a\frac{d}{ds}[s\frac{d^2}{ds^2}(s^2\hat{\Psi}(s))] + a^2\frac{d}{ds}[s\frac{d}{ds}(s\hat{\Psi}(s))] + ac\frac{d}{ds}(s^2\hat{\Psi}(s) - s\Psi(0)) \\
&+ \frac{\sigma^2}{2}cs\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] - acs\frac{d}{ds}[s\hat{\Psi}(s)] + c^2s(s\hat{\Psi}(s) - \Psi(0)) - cA\Psi(u)\mid_{u=0}.
\end{aligned}$$

All together,

$$\begin{aligned}
\beta^2\hat{\Psi}(s)\hat{f_X}(s) &= \frac{\sigma^4}{4}\frac{d^2}{ds^2}[s^2\frac{d^2}{ds^2}s^2\hat{\Psi}(s)] - \frac{\sigma^2}{2}a\frac{d^2}{ds^2}[s^2\frac{d}{ds}s\hat{\Psi}(s)] \\
&+ \frac{\sigma^2}{2}c\frac{d^2}{ds^2}[s^2(cs\hat{\Psi}(s) - \Psi(0))] - \frac{\sigma^2}{2}a\frac{d}{ds}[s\frac{d^2}{ds^2}(s^2\hat{\Psi}(s)] \\
&+ a^2\frac{d}{ds}[s\frac{d}{ds}(s\hat{\Psi}(s))] - ac\frac{d}{ds}[s^2\hat{\Psi}(s) - s\Psi(0)] \\
&+ \frac{\sigma^2}{s}cs\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] - acs\frac{d}{ds}[s\hat{\Psi}(s)] + c^2s^2\hat{\Psi}(s) - c^2s\Psi(0) \\
&- cA\Psi(u)\mid_{u=0} - 2\beta\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] + 2\beta a\frac{d}{ds}[s\hat{\Psi}(s)] \\
&- 2\beta c(s\hat{\Psi}(s) - \Psi(0)) + \beta^2\hat{\Psi}(s) + \beta^2(\frac{1}{s} - \frac{\hat{f}_X(s)}{s})
\end{aligned}$$

Thus, the equation to be analyzed in the Laplace side is a non-homogeneous ordinary differential equation of the form

$$s^4y'''' + p(s)s^3y''' + q(s)s^2y'' + r(s)sy' + m(s)y = n(s), \tag{5.7}$$

with p, q, r, m and n holomorphic functions. The regularity at zero of this ODE implies that the solution has the form

$$\hat{y}(s) = s^\rho\sum_{k=0}^{\infty}c_ks^k = \sum_{k=0}^{\infty}c_ks^{\rho+k}, \tag{5.8}$$

Since the equation is regular at zero, the coefficients of powers of s less or equal than ρ should be zero. The powers higher than ρ will be zero at zero, therefore their coefficients will not be classified.

Looking term by term at the Laplace transform equation, the different powers ρ of s and their coefficients are found as follows.

The terms that are the power ρ of s :

$$\begin{aligned}
T(s^\rho) &:= \frac{\sigma^4}{4}\frac{d^2}{ds^2}[s^2\frac{d^2}{ds^2}s^2\hat{\Psi}(s)] - \frac{\sigma^2}{2}a\frac{d^2}{ds^2}[s^2\frac{d}{ds}s\hat{\Psi}(s)] - \frac{\sigma^2}{2}a\frac{d}{ds}[s\frac{d^2}{ds^2}(s^2\hat{\Psi}(s)] \\
&+ a^2\frac{d}{ds}[s\frac{d}{ds}(s\hat{\Psi}(s))] - 2\beta\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] + 2\beta a\frac{d}{ds}[s\hat{\Psi}(s)] + \beta^2\hat{\Psi}(s)
\end{aligned} \tag{5.9}$$

with the equation in ρ to be solved:

$$[-(\rho+1)(\frac{\sigma^2}{2}(\rho+2)-a)+\beta]^2=\beta^2. \tag{5.10}$$

Denote $\delta := (\rho+1)[\frac{\sigma^2}{2}(\rho+2)-a]$ then the equation to be solved is of the form

$$(-\delta+\beta)^2=\beta^2. \tag{5.11}$$

This equation has two solutions

$$\delta_1 = 0$$

and

$$\delta_2 = 2\beta,$$

that lead to two second order equations in ρ:

$$(\rho+1)[\frac{\sigma^2}{2}(\rho+2)-a]=0$$

and

$$(\rho+1)[\frac{\sigma^2}{2}(\rho+2)-a]=2\beta.$$

The first equation produces the same solutions as in the exponential times case. The other equation produces solutions that depend on the parameter β of the time density. The solutions are:

1. $\rho_1 = -1$
2. $\rho_2 = -2 + \frac{2a}{\sigma^2}$
3. $\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} = \frac{\rho_2-1}{2} + \sqrt{(\frac{\rho_2+1}{2})^2+\frac{4\beta}{\sigma^2}}$
4. $\rho_4 = -1 + \frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} = \frac{\rho_2-1}{2} - \sqrt{(\frac{\rho_2+1}{2})^2+\frac{4\beta}{\sigma^2}}$

Note that the order of these solutions is the following:

$$\rho_4 \le \rho_1 \le \rho_2 \le \rho_3.$$

If $2\sigma^2 \neq a$, the solutions of the homogeneous equation are distinct:

$$y_1(s) = s^{\rho_1}\gamma_1(s) \quad y_2(s) = s^{\rho_2}\gamma_2(s) \quad y_3(s) = s^{\rho_3}\gamma_3(s), \quad y_4(s) = s^{\rho_4}\gamma_4(s)$$

where $\gamma_1(0) = 1$, for $i = 1, \ldots, 4$. Using the method of variation of parameters, the particular solution of the non-homogeneous equation is obtained using the general form from Boyce and DiPrima (2005), (page 239):

$$y_p(s) = \sum_{m=1}^{n} y_m(s) \int_{s_0}^{s} \frac{g(t)W_m(t)}{W(t)} dt$$

where, in this case, $n = 4$ is the order of the differential equation, s_0 will be considered zero, $g(s) = \frac{n(s)}{s^4}$ is the right-hand side of the equation. Proceeding in the same manner as in the previous chapter, one obtains:

$$W(s) = W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = s^{\rho_1+\rho_2+\rho_3+\rho_4-6}\gamma(s)$$

where $\gamma(s)$ is a holomorphic function. Here W_m is the determinant obtained from W by replacing the m-th column by the column $(0, 0, 0, 1)$. Consequently,

$$W_1(s) = W(I, y_2, y_3, y_4) = \begin{vmatrix} 0 & y_2 & y_3 & y_4 \\ 0 & y_2' & y_3' & y_4' \\ 0 & y_2'' & y_3'' & y_4'' \\ 1 & y_2''' & y_3''' & y_4''' \end{vmatrix} = \gamma^1(s)s^{\rho_2+\rho_3+\rho_4-3}$$

...

$$W_4(s) = W(y_1, y_2, y_3, I) = \begin{vmatrix} y_1 & y_2 & y_3 & 0 \\ y_1' & y_2' & y_3' & 0 \\ y_1'' & y_2'' & y_3'' & 0 \\ y_1''' & y_2''' & y_3''' & 1 \end{vmatrix} = \gamma^4(s) s^{\rho_1+\rho_2+\rho_3-3}$$

where γ^i are holomorphic, for $i = 1, 2, 3, 4$.

Thus, the particular solution is:

$$\begin{aligned} y_p(s) &= \sum_{m=1}^{4} y_m(s) \int_0^s \frac{n(t)W_m(t)}{t^4 W(t)} dt \\ &= y_1 \int_0^s \frac{n(t)W_1(t)}{t^4 W(t)} dt + y_2 \int_0^s \frac{n(t)W_2(t)}{t^4 W(t)} dt + y_3 \int_0^s \frac{n(t)W_3(t)}{t^4 W(t)} dt + y_4 \int_0^s \frac{n(t)W_4(t)}{t^4 W(t)} dt \\ &= s^{\rho_1} \int_0^s \frac{C_1 n(t) t^{\rho_2+\rho_3+\rho_4-3}}{C t^4 t^{\rho_1+\rho_2+\rho_3+\rho_4-6}} dt + s^{\rho_2} \int_0^s \frac{C_2 n(t) t^{\rho_1+\rho_3+\rho_4-3}}{C t^4 t^{\rho_1+\rho_2+\rho_3+\rho_4-6}} dt \\ &+ s^{\rho_3} \int_0^s \frac{C_3 n(t) t^{\rho_1+\rho_2+\rho_4-3}}{C t^4 t^{\rho_1+\rho_2+\rho_3+\rho_4-6}} dt + s^{\rho_4} \int_0^s \frac{C_4 n(t) t^{\rho_1+\rho_2+\rho_3-3}}{C t^4 t^{\rho_1+\rho_2+\rho_3+\rho_4-6}} dt \\ &= s^{\rho_1} \int_0^s \frac{C_1 n(t) t^{-\rho_1-1}}{C} dt + s^{\rho_2} \int_0^s \frac{C_2 n(t) t^{-\rho_2-1}}{C} dt \\ &+ s^{\rho_3} \int_0^s \frac{C_3 n(t) t^{-\rho_3-1}}{C} dt + s^{\rho_4} \int_0^s \frac{C_4 n(t) t^{-\rho_4-1}}{C} dt \\ &= s^{\rho_1} s^{-\rho_1} n_1(s) + s^{\rho_2} s^{-\rho_2} n_2(s) + s^{\rho_3} s^{-\rho_3} n_3(s) + s^{\rho_4} s^{-\rho_4} n_4(s) \\ &= \gamma_5(s) \end{aligned}$$

where $\gamma_5(s)$ is a holomorphic function. Note that in order to integrate the following restriction should be imposed on the powers ρ_is:

$$-1 - \rho_i > -1 \quad \text{or} \quad \rho_i < 0$$

for any $i = 1, 2, 3, 4$. Since some of the powers are negative for any a, σ, β and between the powers there is the following order

$$\rho_4 \leq \rho_1 = -1 \leq \rho_2 \leq \rho_3$$

the only conditions to be imposed are:

$$\rho_2 = -2 + \frac{2a}{\sigma^2} < 0$$

and

$$\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 4\beta\sigma^2}}{\sigma^2} < 0.$$

These translate into

$$\frac{2a}{\sigma^2} < 2$$

and

$$\frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 4\beta\sigma^2}}{\sigma^2} < 1$$

$$\sqrt{(\frac{\sigma^2}{2} - a)^2 + 4\beta\sigma^2} < \sigma^2 + \frac{\sigma^2}{2} - a$$

$$(\frac{\sigma^2}{2} - a)^2 + 4\beta\sigma^2 < \sigma^4 + 2\sigma^2(\frac{\sigma^2}{2} - a) + (\frac{\sigma^2}{2} - a)^2$$

i.e.

$$\beta < \frac{\sigma^2 - a}{2}.$$

Therefore, the solution of the Laplace transform equation is

$$\hat{\Psi}(s) = c_1 s^{\rho_1}\gamma_1(s) + c_2 s^{\rho_2}\gamma_2(s) + c_3 s^{\rho_3}\gamma_3(s) + c_4 s^{\rho_4}\gamma_4(s) + c_5\gamma_5(s),$$

under the conditions $\frac{2a}{\sigma^2} < 2$, and $\beta < \frac{\sigma^2 - a}{2}$ with c_i real constants and γ_i holomorphic functions with $\gamma_i(0) = 1$, for $i = 1, .., 5$.

□

Theorem 11. *Consider the model given by*

$$U_t = u + ct + a\int_0^t U_s ds + \sigma\int_0^t U_s dWs - \sum_{k=1}^{N(t)} X_k, \tag{5.12}$$

with positive volatility $\sigma > 0$. The distribution of the inter-arrival times F_τ is Erlang$(2, \beta)$ and the distribution of the claim sizes F_X is assumed to have a moment generating function defined on a neighborhood of the origin, with mean μ. Then as $u \to \infty$,

$$\Psi(u) \sim \begin{cases} k_1 u^{1-\frac{2a}{\sigma^2}} & \text{if } \; 1 < \frac{2a}{\sigma^2} < 2 \\ k_2 u^{-2\sqrt{\frac{\beta}{\sigma^2}}} & \text{if } \; \frac{2a}{\sigma^2} = 1 \\ k_3 u^{-\alpha} & \text{if } \; \frac{2a}{\sigma^2} < 1 \end{cases}$$

where $\alpha = \frac{-(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} > 0$ *when* $2a < \sigma^2$.

Proof. Similarly to the exponential inter-arrival times analysis (Constantinescu and Thomann, 2005), the main ingredient of the proof is the Karamata-Tauberian theorem.

Since the equation of the Laplace transform is regular at zero, it was established that the solution is a linear combination of powers of s, $\rho_1, \rho_2, \rho_3, \rho_4$. The equation for the coefficient of s^ρ will be a fourth order equation in ρ whose solutions are the powers of the Laplace transform of the ruin probability, $\hat{\Psi}(s)$. In other words, they will indicate the decay of $\hat{\Psi}(s)$ as $s \to 0$ and imply the decay of $\Psi(u)$ as $u \to \infty$. The asymptotic behavior of the solution as $s \to 0$, is determined by the smallest power of s. When written in increasing order,

$$\rho_4 \leq \rho_1 \leq \rho_2 \leq \rho_3$$

they allow an analysis of their individual potential to be the leading term of the decay.

Consider $\rho_4 = -1 + \frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2}$ the first candidate for the algebraic decay rate of $\hat{\Psi}(s)$. This means,

$$\tilde{U}(s) = \hat{\Psi}(s) \sim \tilde{k} s^{-1+\frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2}} \quad \text{as} \quad s \to 0$$

and the Karamata-Tauberian theorem implies that

$$\Psi(u) \sim \tilde{k} s^{\frac{(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2}} \quad \text{as} \quad u \to \infty,$$

where $\frac{(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2}$ is always positive. Thus, as $u \to \infty$ the probability of ruin has an algebraic growth, contradicting the fourth boundary condition that requires decay to zero at infinity. Hence, since both ρ_4 and ρ_1 are less or equal than -1, they cannot be leading terms. This implies that the linear combination of the other two solutions

$$\hat{\Psi}(s) = c_2 s^{\rho_2} \gamma_2(s) + c_3 s^{\rho_3} \gamma_3(s)$$

will determine the decay as $s \to 0$, and furthermore the decay as $u \to \infty$ of $\Psi(u)$. Depending on the drift and volatility of the risky asset, the following cases are analyzed.

Case 1. If $2a/\sigma^2 > 1$, then the asymptotic behavior is given by the linear combination of powers ρ_2 and ρ_3 of s. Since the decay is driven by the slowest one, the leading term is $\rho_2 = -2 + \frac{2a}{\sigma^2}$. By Karamata-Tauberian arguments

$$\Psi(u) \sim \frac{c_2(2-\frac{2a}{\sigma^2})}{\Gamma(3-\rho)} \gamma(\frac{1}{u}) u^{1-\frac{2a}{\sigma^2}}, \quad \text{as} \quad u \to \infty,$$

where $\gamma(0) = 1$. Letting

$$k_1 = \frac{c_2(2-\frac{2a}{\sigma^2})}{\Gamma(3-\rho)}$$

gives

$$\Psi(u) \sim k_1 u^{1-\frac{2a}{\sigma^2}}, \quad \text{as} \quad u \to \infty.$$

If the coefficient $c_2 = 0$ this implies the leading term is s^{ρ_3}, but this will imply that the probability of ruin decays faster in the Erlang(2) than in the Erlang(3) risk model, contradicting the ordering established at the beginning of this chapter. Therefore, $c_2 \neq 0$ which proves the first part of the theorem.

Case 2. If $2a/\sigma^2 = 1$, as before, $\rho_2 = -1$ cannot be considered. The leading term is

$$\rho_3 = -1 + 2\sqrt{\frac{\beta}{\sigma^2}}$$

and going through Karamata-Tauberian arguments, the asymptotic decay of the ruin probability is:

$$\Psi(u) \sim k_2 u^{-2\sqrt{\frac{\beta}{\sigma^2}}}, \quad \text{as} \quad u \to \infty.$$

Case 3. If $2a/\sigma^2 < 1$, ρ_2 is excluded because is not producing decay of the ruin probability. Again the leading power is

$$\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 4\beta\sigma^2}}{\sigma^2}.$$

The same Karamata-Tauberian, Monotone Function Theorem arguments imply that the asymptotic decay of the ruin probability is:

$$\Psi(u) \sim k_3 u^{-\frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 4\beta\sigma^2}}{\sigma^2}}, \quad \text{as} \quad u \to \infty$$

□

5.2.2 Inter-arrival times $Erlang(n, \beta)$ distributed

Theorem 12. *Consider the model given by*

$$U_t = u + ct + a\int_0^t U_s ds + \sigma \int_0^t U_s dWs - \sum_{k=1}^{N(t)} X_k, \tag{5.13}$$

with positive volatility $\sigma > 0$. The distribution of the inter-arrival times F_τ is Erlang(n, β) and the distribution of the claim sizes F_X is assumed to have a moment generating function defined on a neighborhood of the origin, with the n-th moment $E(X^n) = \mu_n$. Then

$$\lim_{u\to\infty} \Psi(u) u^{-1+\frac{2a}{\sigma^2}} = k_n \quad \textit{for} \quad \frac{2a}{\sigma^2} < 3.$$

Proof. As before, the probability of ruin satisfies the equation

$$\mathcal{L}^*(A)\Psi(u) = \beta^n \int_0^\infty \Psi(u-x) f_X(x) dx,$$

where for the *Erlang*(n, β) distributed inter-arrival times, $\mathcal{L}^*(\frac{d}{dt}) = (-\frac{d}{dt} + \beta)^n$. Hence the probability of ruin is a solution of the equation,

$$(-A+\beta)^n \Psi(u) = \beta^n \int_0^\infty \Psi(u-x) f_X(x) dx$$

with the boundary conditions:

1. $\lim_{u\to\infty} \Psi(u) = 0$,
2. $(-\frac{\sigma^2}{2}u^2\frac{d^2}{du^2} - (c+au)\frac{d}{du} + \beta)^n \Psi(0) = \beta^n$.

The other $2n-2$ boundary conditions regarding the values of the derivatives of Ψ at zero will not be used in the derivation of the asymptotic decay. The Laplace transform of this equation is

$$((-1)^n \hat{A}^n \hat{\Psi}(s) + \cdots \beta^n) = \beta^n \hat{\Psi}(s) \hat{f}_x(s) + \beta^n (\frac{1}{s} - \frac{\hat{f}_X(s)}{s}),$$

with $s = 0$ a regular singular point of the homogeneous equation. Thus, solving the homogeneous equation it reduces again to solving the indicial equation. Also, note that since the right-hand side of the equation is analytic, it can be shown that the particular solution of the non-homogeneous equation is analytic. For the asymptotic analysis to follow, this means that the particular solution will not be a candidate for the decay rate. Therefore, the relevant solutions are the ones coming from solving the homogeneous equation, which in this case means solving the indicial equation. Recall that the coefficient T_ρ of the s^ρ is as follows:

$$T_\rho(\hat{A}\hat{\Psi}) = [\frac{\sigma^2}{2}(\rho+2) - a](\rho+1) := \delta,$$

$$T_\rho(\hat{A}\hat{A}\hat{\Psi}) = [\frac{\sigma^2}{2}(\rho+2) - a]^2(\rho+1)^2 := \delta^2$$

and, by induction,

$$T_\rho(\hat{A}^{(n)}\hat{\Psi}) = [\frac{\sigma^2}{2}(\rho+2) - a]^n(\rho+1)^n := \delta^n.$$

Thus, the T_ρ coefficient in the Laplace transform equation is

$$T_\rho(equation) := (-\delta+\beta)^n - \beta^n.$$

This should be zero, leading to the conclusion that the solutions are of the form

$$\delta = \beta(1 - e^{\frac{2\pi ik}{n}}), \tag{5.14}$$

where $k = 0, 1, 2, \cdots, n-1$, and $\delta = (\rho+1)(\rho+2-\frac{2a}{\sigma^2})$. These solutions should be real, therefore, two cases can be distinguished, n odd and n even. But before discussing this particular cases, an important observation may be made about the complex solutions of this equation.

Proposition 5. *The real parts of the complex conjugate solutions of the equation (5.14) lie always outside the interval determined by* $\rho_1 = -1$ *and* $\rho_2 = -2 + \frac{2a}{\sigma^2}$.

Proof. The equation (5.14) is equivalent to

$$(\rho-\rho_1)(\rho-\rho_2) = \beta(1 - e^{\frac{2\pi ik}{n}}). \tag{5.15}$$

Consider the complex solutions $\rho = \alpha + ib$, where $b \neq 0$. Then the equation (5.15) may be written as

$$(\alpha - \rho_1 + ib)(\alpha - \rho_2 + ib) = \beta(1 - \cos(\frac{2\pi k}{n}) - i\sin(\frac{2\pi k}{n})),$$

with the real part satisfying the equation

$$(\alpha - \rho_1)(\alpha - \rho_2) - b^2 = \beta(1 - \cos(\frac{2\pi k}{n})).$$

This implies

$$(\alpha - \rho_1)(\alpha - \rho_2) = b^2 + \beta(1 - \cos(\frac{2\pi k}{n})) > 0,$$

i.e. the product $(\alpha - \rho_1)(\alpha - \rho_2)$ is always positive. Therefore, $(\alpha - \rho_1)$ and $(\alpha - \rho_2)$ have to have the same sign, in other words, α is either bigger than both ρ_1 and ρ_2 or is smaller than both. Thus the result follows. □

After analyzing the complex solutions, returning to the possible real solutions, one has the following two cases.

Case 1. For n odd the only two real solutions are

1. $\rho_1 = -1$

2. $\rho_2 = -2 + \frac{2a}{\sigma^2}$

where $\rho_1 \leq \rho_2$. This is exactly the same situation encounter in the exponential case. Since ρ_1 doesn't produce decay, ρ_2 is the leading term, i.e.

$$\lim_{s \to 0} \hat{\Psi}(s) = s^{2 - \frac{2a}{\sigma^2}} c_2$$

. The same Karamata Tauberian arguments used before imply that the decay of the probability of ruin is

$$\lim_{u\to\infty} \Psi(u)u^{-1+\frac{2a}{\sigma^2}} = k_n \quad \text{for} \quad 1 < \frac{2a}{\sigma^2} < 2.$$

Case 2. For n even there are four real solutions

1. $\rho_1 = -1$
2. $\rho_2 = -2 + \frac{2a}{\sigma^2}$
3. $\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} = \frac{\rho_2-1}{2} + \sqrt{(\frac{\rho_2+1}{2})^2 + \frac{4\beta}{\sigma^2}}$
4. $\rho_4 = -1 + \frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2} = \frac{\rho_2-1}{2} - \sqrt{(\frac{\rho_2+1}{2})^2 + \frac{4\beta}{\sigma^2}}$,

where $\rho_4 \le \rho_1 \le \rho_2 \le \rho_3$. The first two candidates, ρ_4 and ρ_1 do not produce decay. Therefore, $\hat{\Psi}(s) \sim c_2 s^{\rho_2} + c_3 s^{\rho_3}$ are to be considered. The decay will not be faster than the slowest one, so s_2^{ρ} is the leading term. This gives the same decay as for exponentials,

$$\lim_{u\to\infty} \Psi(u)u^{-1+\frac{2a}{\sigma^2}} = k_n \quad \text{for} \quad 1 < \frac{2a}{\sigma^2} < 2.$$

Suppose $c_2 = 0$. Then the leading term of the decay is s^{ρ_3}. This will imply

$$\lim_{u\to\infty} \Psi(u)u^{\frac{-(\frac{\sigma^2}{2}=k_n-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+4\beta\sigma^2}}{\sigma^2}(=-\frac{\rho_2-1}{2}+\sqrt{(\frac{\rho_2+1}{2})^2+\frac{4\beta}{\sigma^2}})} \quad \text{for} \quad 1 < \frac{2a}{\sigma^2} < 2.$$

a faster decay at infinity than the one of an exponential. Due to the ordering of the ruin probabilities established before, $\Psi_{n+1}(u) \le \Psi_n(u) \le \Psi_{n-1}(u)$, where $n-1$ and $n+1$ are odd, i.e. both Ψ_{n-1} and Ψ_{n+1} decay as slow as an exponential (Case 1) then $\Psi_n(u)$ cannot decay faster than an exponential, i.e. $c_2 \neq 0$.

Thus, for any n the probability of ruin has the same decay rate as in the case of inter-arrival times exponentially distributed

$$\lim_{u\to\infty} \Psi(u)u^{-1+\frac{2a}{\sigma^2}} = k_n \quad \text{for} \quad 1 < \frac{2a}{\sigma^2} < 2.$$

□

5.2.3 Inter-arrival times distributed as a sum of two exponentials

As a summary of the steps introduced here for the asymptotic analysis, the case of inter-arrival times that are distributed as a sum of two exponentials is presented in detail.

Theorem 13. *If in the Sparre Andersen model with investments in a stock with returns modeled by a geometric Brownian motion with drift a and positive volatility* σ*,*

$$U_t = u + ct + a\int_0^t U_s + \sigma \int_0^t U_s dW_s - \sum_{k=1}^{N(t)} X_k,$$

the inter-arrival times follow a mixture of exponentials type distribution,

$$f_\tau(t) = \frac{\beta_1\beta_2}{\beta_2 - \beta_1}(e^{-\beta_1 t} - e^{-\beta_2 t})$$

then the probability of ruin has an algebraic decay rate as u goes to infinity,

$$\Psi(u) \sim \begin{cases} k_1 u^{1-\frac{2a}{\sigma^2}} & \text{if } \; 1 < \frac{2a}{\sigma^2} < 2 \\ k_2 u^{-2\sqrt{\frac{\beta_1+\beta_2}{2\sigma^2}}} & \text{if } \; \frac{2a}{\sigma^2} = 1 \\ k_3 u^{-\alpha} & \text{if } \; \frac{2a}{\sigma^2} < 1 \end{cases}$$

where $\alpha = \frac{-(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+2(\beta_1+\beta_2)\sigma^2}}{\sigma^2} > 0$ *when* $2a < \sigma^2$.

Proof. Recall that the probability of ruin satisfies the integro-differential equation

$$\mathcal{L}^*(A)\Psi(u) = f_\tau^{(n-1)}(0)\int_0^\infty \Psi(u-x)f_X(x)dx.$$

In the case of inter-arrival times distributed as a mixture of exponentials, the density f_τ satisfies an ordinary differential equation of order $n = 2$, and the linear operator $\mathcal{L}$ is

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \beta_1)(\frac{d}{dt} + \beta_2)f_\tau(t) = 0,$$

thus

$$\mathcal{L}^*(A) = (-A + \beta_1)(-A + \beta_2).$$

Hence the integro-differential equation satisfied by the probability of ruin is:

$$(-A + \beta_1)(-A + \beta_2)\Psi(u) = f'_\tau(0)\int_0^\infty \Psi(u - x)f_X(x)dx$$

equivalent to

$$AA\Psi(u) - (\beta_1 + \beta_2)A\Psi(u) + \beta_1\beta_2\Psi(u) = \beta_1\beta_2[\Psi * f_X(u) + 1 - F_X(u)]$$

where F_X is the distribution of the claim amounts, $F'_X = f_X$. Since an exact solution of such an equation cannot be established without enough boundary conditions, an asymptotic analysis of the behavior at infinity is presented. For that purpose, it will be analyzed the asymptotic behavior of the Laplace transform of the solution around zero and then the conclusion regarding the asymptotic behavior of the probability of ruin as $u \to \infty$ will be made using Karamata-Tauberian theorems.

In other words, the equation is transformed using the Laplace transform and the result is an non-homogeneous ordinary differential equation (Step 1). Moreover, it is shown that the homogeneous part of the ODE is regular at zero, thus the solution of the equation is a power function s^ρ (Step 2). Using regularity arguments, the candidates of ρ are established (Step 3). Since the original function $\Psi(u)$ presents a decay at infinity, its Laplace transform $\hat{\Psi}(s)$ should have a decay at zero. Among all the possible candidates, it is argued that the only one possible is

$$\hat{\Psi}(s) \sim s^{-2+\frac{2a}{\sigma^2}}, \quad \text{as} \quad s \to 0$$

which by Karamata-Tauberian completes the proof (Step 4).

Step 1. The Laplace transform equation

$$\hat{A}\hat{A}\hat{\Psi}(s) - (\beta_1 + \beta_2)\hat{A}\hat{\Psi}(s) + \beta_1\beta_2\hat{\Psi}(s) = \beta_1\beta_2[\hat{\Psi}(s)\hat{f}_X(s) + \frac{1}{s} - \frac{\hat{f}_X(s)}{s}]$$

is equivalent to the following ordinary differential equation in $\hat{\Psi}(s)$:

$$\hat{A}\hat{A}\hat{\Psi}(s) - (\beta_1 + \beta_2)\hat{A}\hat{\Psi}(s) + \beta_1\beta_2\hat{\Psi}(s) - \beta_1\beta_2\hat{\Psi}(s)\hat{f}_X(s) = \beta_1\beta_2(\frac{1}{s} - \frac{\hat{f}_X(s)}{s})$$

where

$$\hat{A}\hat{\Psi}(s) = \frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] - a\frac{d}{ds}[s\hat{\Psi}(s)] + c(s\hat{\Psi}(s) - \Psi(0))$$

$$\begin{aligned}
\hat{A}\hat{A}\hat{\Psi}(s) &= (\frac{\sigma^2}{2})^2\frac{d^2}{ds^2}[s^2\frac{d^2}{ds^2}s^2\hat{\Psi}(s)] - a\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\frac{d}{ds}s\hat{\Psi}(s)] + c\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2(s\hat{\Psi}(s) - \Psi(0))] \\
&- \frac{\sigma^2}{2}a\frac{d}{ds}[s\frac{d^2}{ds^2}(s^2\hat{\Psi}(s))] + a^2\frac{d}{ds}[s\frac{d}{ds}(s\hat{\Psi}(s))] + ac\frac{d}{ds}(s^2\hat{\Psi}(s) - s\Psi(0)) \\
&+ \frac{\sigma^2}{2}cs\frac{d^2}{ds^2}[s^2\hat{\Psi}(s)] - acs\frac{d}{ds}[s\hat{\Psi}(s)] + c^2s(s\hat{\Psi}(s) - \Psi(0)) - cA\Psi(u)\mid_{u=0}.
\end{aligned}$$

The equation is of degree four, and has the form:

$$p_1(s)\hat{\Psi}''''(s) + p_2(s)\hat{\Psi}'''(s) + p_3(s)\hat{\Psi}''(s) + p_4(s)\hat{\Psi}'(s) + p_5(s)\hat{\Psi}(s) = p_6(s)$$

where the coefficient of the fourth order derivative is $p_1(s) = k_1s^4$ suggesting that $s = 0$ is a singular point.

Since

$$\lim_{s\to 0}(s-0)\frac{p_2(s)}{p_1(s)} = s\frac{k_2s^3}{k_1s^4} < \infty$$
$$\lim_{s\to 0}(s-0)^2\frac{p_3(s)}{p_1(s)} = s^2\frac{k_3s^2}{k_1s^4} < \infty$$
$$\lim_{s\to 0}(s-0)^3\frac{p_4(s)}{p_1(s)} = s^3\frac{k_4s^1}{k_1s^4} < \infty$$
$$\lim_{s\to 0}(s-0)^4\frac{p_5(s)}{p_1(s)} = s^4\frac{k_5s^0}{k_1s^4} < \infty$$

then the homogeneous equation is regular at zero, $s-0$ is a regular singular point. Therefore, it may be assumed that the equation has a solution of the form

$$\hat{\Psi}(s) = s^\rho.$$

In order to solve for ρ, the so called indicial equation has to be solved. This equation says that the coefficient of the s^ρ term should be zero.

Step 2. The regularity of the homogeneous part

Considering the term by term expansion, the coefficients of the s^ρ terms, will be denoted T_{s^ρ} and are identified to be as follows. For

$$\hat{A}\hat{\Psi}(s) = \frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2(s^\rho)] - a\frac{d}{ds}[s(s^\rho)] + c(s(s^\rho) - \Psi(0))$$

$$T_{s^\rho}(\hat{A}\hat{\Psi}) = \frac{\sigma^2}{2}(\rho+2)(\rho+1) - a(\rho+1) = (\rho+1)(\frac{\sigma^2}{2}(\rho+2) - a) := \delta$$

Analogously, it can be shown:

$$T_{s^\rho}(\hat{A}\hat{A}\hat{\Psi}) = \delta^2.$$

Also, obviously the coefficient of s^ρ is one for the following:

$$T_{s^\rho}(\hat{\Psi}) = 1$$

$$T_{s^\rho}(\hat{\Psi}\hat{f_X}) = 1$$

thus, the indicial equation is:

$$\delta^2 - (\beta_1 + \beta_2)\delta + \beta_1\beta_2 - \beta_1\beta_2 = 0$$

in other words,

$$\delta(\delta - (\beta_1 + \beta_2)) = 0$$

having as solutions

$$\delta = 0$$

or

$$\delta = \beta_1 + \beta_2.$$

The $\delta = (\rho+1)(\frac{\sigma^2}{2}(\rho+2) - a) = 0$ solution returns the same two solutions encounter before in the exponential time distribution (Cramér-Lundberg model), more specifically

$$\rho_1 = -1$$

and

$$\rho_2 = -2 + \frac{2a}{\sigma^2}$$

and then two other solutions from $\delta = \beta_1 + \beta_2$,

$$\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 2\sigma^2(\beta_1 + \beta_2)}}{\sigma^2} = \frac{\rho_2 - 1}{2} + \sqrt{(\frac{\rho_2 + 1}{2})^2 + \frac{2(\beta_1 + \beta_2)}{\sigma^2}}$$

$$\rho_4 = -1 + \frac{-(\frac{\sigma^2}{2} - a) - \sqrt{(\frac{\sigma^2}{2} - a)^2 + 2\sigma^2(\beta_1 + \beta_2)}}{\sigma^2} = \frac{\rho_2 - 1}{2} - \sqrt{(\frac{\rho_2 + 1}{2})^2 + \frac{2(\beta_1 + \beta_2)}{\sigma^2}}$$

Analogously with the case of inter-arrival times $Erlang(2, \beta)$ distributed, the method of variation of parameters provides the particular solution of the non-homogeneous equation according to the general form:

$$y_p(s) = \sum_{m=1}^{n} y_m(s) \int_{s_0}^{s} \frac{g(t)W_m(t)}{W(t)} dt.$$

Here $n = 4$, s_0 is chosen to be zero, and $g(s) = \frac{p_6(s)}{s^4}$. The determinant of the coefficients, $W(s) = W(y_1, y_2, y_3, y_4) = \gamma(s)s^{\rho_1+\rho_2+\rho_3+\rho_4-6}$ and the determinants $W_m = s^{\rho_{m-1}+\rho_{m+1}+\rho_{m+2}-3}\gamma_m(s)$ where γ, γ_m are holomorphic, for any $m = 1, \ldots, 4$.

Thus, the particular solution is, as before:

$$y_p(s) = \sum_{m=1}^{4} y_m(s) \int_{0}^{s} \frac{n(t)W_m(t)}{t^4 W(t)} dt = \gamma_5(s) \tag{5.16}$$

where $\gamma_5(s)$ is a holomorphic function. As before, due to integration, the restrictions on the powers ρ_is are

$$\rho_i < 0 \quad \text{for any} \quad i = 1, 2, 3, 4.$$

Since some of the powers are negative for any a, σ, β and between the powers there is the following order

$$\rho_4 \leq \rho_1 = -1 \leq \rho_2 \leq \rho_3$$

the only conditions to be imposed are: $\rho_2 = -2 + \frac{2a}{\sigma^2} < 0$, i.e.

$$\frac{2a}{\sigma^2} < 2$$

and $\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 2(\beta_1 + \beta_2)\sigma^2}}{\sigma^2} < 0$, leading to

$$\beta_1 + \beta_2 < \sigma^2 - a.$$

Step 3. The possible solutions, candidates for decay

The asymptotic behavior of the solution as $s \to 0$, is determined by the smallest power of s. When written in increasing order,

$$\rho_4 \leq \rho_1 \leq \rho_2 \leq \rho_3$$

they allow an analysis of their individual potential to be the leading term of the decay.

Step 4. The asymptotic behavior at zero, respectively at infinity

Consider $\rho_4 = -1 + \frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+2(\beta_1+\beta_2)\sigma^2}}{\sigma^2}$ the first candidate for the algebraic decay rate of $\hat{\Psi}(s)$. This means,

$$\tilde{U}(s) = \hat{\Psi}(s) \sim \tilde{k} s^{-1+\frac{-(\frac{\sigma^2}{2}-a)-\sqrt{(\frac{\sigma^2}{2}-a)^2+2(\beta_1+\beta_2)\sigma^2}}{\sigma^2}} \quad \text{as} \quad s \to 0$$

and the Karamata-Tauberian theorem implies that

$$\Psi(u) \sim \tilde{k} s^{\frac{(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+2(\beta_1+\beta_2)\sigma^2}}{\sigma^2}} \quad \text{as} \quad u \to \infty,$$

where $\frac{(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+2(\beta_1+\beta_2)\sigma^2}}{\sigma^2}$ is always positive. Thus, as $u \to \infty$ the probability of ruin has an algebraic growth, contradicting the fourth boundary condition that requires decay to zero at infinity. Hence, since both ρ_4 and ρ_1 are less or equal than -1, they cannot be leading terms. This implies that the linear combination of the other two solutions

$$\hat{\Psi}(s) = c_2 s^{\rho_2} \gamma_2(s) + c_3 s^{\rho_3} \gamma_3(s)$$

will determine the decay as $s \to 0$, and furthermore the decay as $u \to \infty$ of $\Psi(u)$. Depending on the drift and volatility of the risky asset, the following cases are analyzed.

Case 1. If $2a/\sigma^2 > 1$, then the asymptotic behavior is given by the linear combination of powers ρ_2 and ρ_3 of s. Since the decay is driven by the slowest one, the leading term is $\rho_2 = -2 + \frac{2a}{\sigma^2}$. By Karamata-Tauberian arguments

$$\Psi(u) \sim \frac{c_2 \gamma(\frac{1}{u})(2 - \frac{2a}{\sigma^2})}{\Gamma(3-\rho)} u^{1-\frac{2a}{\sigma^2}}, \quad \text{as} \quad u \to \infty.$$

Letting

$$k_1 = \frac{c_2 \gamma(\frac{1}{u})(2 - \frac{2a}{\sigma^2})}{\Gamma(3-\rho)}$$

gives

$$\Psi(u) \sim k_1 u^{1-\frac{2a}{\sigma^2}}, \quad \text{as} \quad u \to \infty.$$

If the coefficient $c_2 = 0$ this implies the leading term is s^{ρ_3}, but this will imply that the probability of ruin in the case of inter-arrival time distributed as a sum of exponentials with parameters (β_1, β_2) decays faster than an Erlang $(2, \max \beta_1, \beta_2)$, which is conjecture to be impossible. Therefore, $c_2 \neq 0$ which proves the first part of the theorem.

Case 2. If $2a/\sigma^2 = 1$, as before, $\rho_2 = -1$ cannot be considered. The leading term is

$$\rho_3 = -1 + 2\sqrt{\frac{\beta_1 + \beta_2}{2\sigma^2}}$$

and going through Karamata-Tauberian arguments, the asymptotic decay of the ruin probability is:

$$\Psi(u) \sim k_2 u^{-2\sqrt{\frac{\beta_1+\beta_2}{2\sigma^2}}}, \quad \text{as} \quad u \to \infty.$$

Case 3. If $2a/\sigma^2 < 1$, ρ_2 is excluded because is not producing decay of the ruin probability. Again the leading power is

$$\rho_3 = -1 + \frac{-(\frac{\sigma^2}{2} - a) + \sqrt{(\frac{\sigma^2}{2} - a)^2 + 2(\beta_1 + \beta_2)\sigma^2}}{\sigma^2}.$$

The same Karamata-Tauberian, Monotone Function Theorem arguments imply that the asymptotic decay of the ruin probability is:

$$\Psi(u) \sim k_3 u^{-\frac{-(\frac{\sigma^2}{2}-a)+\sqrt{(\frac{\sigma^2}{2}-a)^2+2(\beta_1+\beta_2)\sigma^2}}{\sigma^2}}, \quad \text{as} \quad u \to \infty$$

$\square$

Remark 10. *As discussed in Fedoryuk (1991), page 8, if the powers* ρ_1, ρ_2 *of the solutions of a second order differential equation, that has* $s = 0$ *a singular point , differ by an integer, then the fundamental system of solutions has the form:*

$$y_1 = s^{\rho_1}\gamma_1(s) \quad y_2 = ay_1 lns + s^{rho_2}\gamma_2(s),$$

where a is a constant and γ_1, γ_2 *are holomorphic. Since in the cases considered here,* $\rho_1 = -1$ *and* $-1 < \rho_2 < 0$, *such a situation cannot occur.*

In all the three cases considered in this chapter the probability of ruin has the same algebraic decay rate as in the exponential arrivals case. Thus, whether the distribution of the inter-arrival times is exponential, or a general sum of exponentials, the ruin probability has the same asymptotic behavior (see Theorems 12, 13, 14). Intuitively, one expects that for less frequent claims of equal intensity the company will stay solvent a longer period of time, but these results show the opposite. The results hold for a drift-volatility ratio of the stock satisfying $1 < \frac{2a}{\sigma^2} < 2$. For the complementary case, ($\frac{2a}{\sigma^2} < 1$), the above mentioned theorems provide some insight, and a thorough asymptotic analysis will be the subject of future research.

6. CONCLUSIONS AND FUTURE RESEARCH

The classical approach in deriving equations satisfied by the probability of ruin is conditioning on the time of the first claim and its size, followed by differentiation. In contrast, the uniform approach of this thesis consists in deriving a general equation for the classical conditional expectation that relates to the probability of ruin via Theorem 4. The general integro-differential equation (3.7) derived in chapter 3 can be applied to the Cramér-Lundberg model (chapter 4) and Sparre-Andersen model (chapter 5) to obtain asymptotics of ruin probabilities.

The asymptotics of the ruin probabilities are also derived via a novel approach. The classical approach in deriving the asymptotics for the ruin probability is to differentiate the integro-differential equation until it becomes a differential equation with no integral term. The current thesis illustrates how Laplace transforms and Karamata Tauberian arguments can be used effectively in the analysis of the asymptotic behavior of the ruin probabilities. For example, the classical Cramér-Lundberg result can be obtained using elementary properties of the Laplace transform.

When investments with stochastic returns are modeled by a geometric Brownian motion the asymptotic behavior of the ruin probability can be derived using this methodology for inter-arrival times having as distribution a mixture of Erlangs. The surprising result is that irrespective of the waiting time between claims investing everything in a stock is more likely to lead to ruin than if no investments were made at all. An analysis of the optimal investment strategy in risky versus riskless assets will be the subject of future research.

6.1. Conclusions

Using probabilistic arguments, proposition 4 proves that the probability of ruin in the Erlang case is bounded from above by the probability of ruin with exponential time. The asymptotic analysis of the ODE solved by the Laplace transform of the probability of ruin gives a lower bound (Theorems 12, 13).

For any Erlang distributed times, in the case of investments with small volatility the algebraic decay rate of the probability of ruin is the same as the one for the exponential times $u^{1-\frac{2a}{\sigma^2}}$. If Ψ_n denotes the ruin probability for $Erlang(n)$ processes, then

$$\lim_{u\to\infty} \Psi_n(u) u^{\frac{2a}{\sigma^2}-1} = K_n$$

where K_n depends on n, in other words

$$\frac{\lim_{u\to\infty} \Psi_m(u)}{\lim_{u\to\infty} \Psi_n(u)} = \frac{K_m}{K_n}.$$

The comparison of different K_n will be subject of future research.

In the case of light claims and investments in stocks modeled by a geometric Brownian motion, the asymptotic decay rate of the ruin probability depends only on the parameters of the investments. Since the average time between claims is not a parameter in the decay rate, the results for different Erlangs leading to the same asymptotic behavior are not surprising. In other words, changing the expected time between two claims from β to 2β has no effect on the asymptotic decay.

In the classical Cramér-Lundberg model without investments, the equation is of order one, while in the case of Cramér-Lundberg with investments is of order two, as in the case of Sparre Andersen without investments. Depending on the distribution chosen for inter-arrival times the equation of the model without investment will have the same order as the order of the ODE satisfied by the density. If the investments considered are in a risky asset with a stock price following a geometric Brownian motion, the order of the equation will be twice the order of the ODE satisfied by the time density.

The general form of the integro-differential equation allows different non-negative stochastic processes as investments. Any combination of exponentials is a good candidates for the claims inter-arrival times. The equations obtained will be high order integro-differential equations.

6.2. Future research

The results obtained in this thesis raise further interesting questions and challenges.

1. Identify conditions on the investment strategy Z_t^u so that $U(t) \to \infty$ as $u \to \infty$.

2. Write explicitly the particular form of the equation (3.7) when also the density of the claim sizes satisfies an ODE with constant coefficients. In this case after a sufficient number of differentiations, the equation becomes an homogeneous ODE with given boundary conditions. Identity a general method in solving this equations exactly or at least asymptotically.

3. Calculate the decay of the Laplace transform of the time of ruin in the case of a Sparre Andersen model with investments.

There are three main venues of future research to pursue (that do not exclude each other).

1. *Gamma*(α, β) inter-arrival times. *Erlang*(n, β) is a Gamma distribution with the first parameter an integer. A natural extension of the problems presented is for inter-arrival times distributed *Gamma*(α, β), where α is real.

   One idea is to use fractional calculus. Through the natural conditioning on the time and size of the first claim, a fractional integro-differential equation may be derived for the probability of non-ruin $\Phi(u)$,

   $$\Phi(u) = \int_0^{\infty} \frac{(\beta)^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \int_0^{u+ct} \Phi(u + ct - x) p(x) dx dt.$$

   After few changes of variables and using some fractional integral properties the equation can be written:

   $$e^{(\frac{\beta}{c})(u)} \big(\frac{\beta}{c}\big)^{-\alpha} \mathbf{D}_{-}^{\alpha} [e^{(-\frac{\beta}{c})u} \Phi(u)] = \int_0^u \Phi(u - x) f_X(x) dx,$$

   for a Sparre Andersen model with no-investments.

   Another idea is to identify the ODE solved by the density $\gamma(\alpha, \beta)$ and proceed in a similar manner as in the case of Erlang distributions

2. Gerber-Shiu Functions

   In Gerber and Shiu (1998) the expected penalty at ruin function is introduced,

   $$\Phi(u) = E[w(U(T_{-}), \mid U(T) \mid) e^{-\delta T_u} 1_{T<\infty} \mid U(0) = u],$$

   where T_u is the time of ruin, $U(T_{-})$ represents the surplus immediately before ruin and $\mid U(T) \mid$ the surplus at ruin, often called the severity of ruin. This function solves the integro-differential derived under appropriate boundary conditions, allowing for the analysis of the penalty at ruin and severity of ruin.

3. Stochastic control.

   There is an extensive literature using the probability of ruin as a control measure. For instance, determine the investment strategy that would minimize the ruin probability. Or, determine the reinsurance policy that would minimize the ruin probability. These stochastic control problems could use the integro-differential equations derived in this thesis. Hamilton-Jacobi-Bellman type equations need to be solved in order to establish the optimal investment strategy or the recommended reinsurance policy.

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APPENDICES

A Stochastic Processes

All the processes and random variables are defined in a probability space $(\Omega, \mathcal{F}, P)$, where Ω is a nonempty set of all possible "outcomes" of an experiment, $\mathcal{F}$ is a set of "events", and P: $\mathcal{F} \to [0, 1]$ is a function that assigns probabilities to events. Most of the definitions from this section are from Bhattacharya and Waymire (1990).

Definition 1. *Given an index set I, a* ***stochastic process*** *indexed by I is a collection of random variables $\{X_\lambda : \lambda \in I\}$ on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a set S. The set S is called the state space of the process.*

Definition 2. *A stochastic process $X_0, X_1, \ldots, X_n, \ldots$ has the Markov property if, for each n and m, the conditional distribution of $X_{n+1}, \ldots, X_{n+m}$ given $X_0, X_1, \ldots, X_n$ is the same as its conditional distribution given X_n alone. A process having the Markov property is called a* ***discrete time Markov process****.*

If, in addition, the state space of the process is countable, then the Markov process is called a **Markov chain**.

B Diffusion Processes

Definition 1. *A* **Brownian motion with drift** a **and diffusion coefficient** σ^2 *is a stochastic process $\{X_t : t \geq 0\}$ having continuous sample paths and independent Gaussian increments. The increments $X_{t+s} - X_t$ have mean sa and variance $s\sigma^2$.*

Definition 2. *A Brownian motion with drift zero and diffusion coefficient of 1 is called* **standard Brownian motion.**

Definition 3. *Let* $X_t = X_0 + at + \sigma W_t$, $t \geq 0$ *where* W_t *is a standard Brownian motion starting at zero and independent of* X_0. *Then the process*

$$Z_t = Z_0 e^{(a - \frac{\sigma^2}{2})t + \sigma W_t}$$

with $Z_0 = e^{X_0}$ *is the* **geometric Brownian motion.**

Definition 4. *A Markov process* $\{X(t), t \geq 0\}$ *on the state space* $S = \{(a, b), -\infty \leq a < b \leq \infty\}$ *is said to be a* **diffusion with drift** $a(t, x)$ **and diffusion coefficient** $\sigma^2(t, x) > 0$, *if it has continuous sample paths, and the following relationships hold for all* $\epsilon > 0$:

$$E((X_{s+t} - X_s)1_{[|X_{t+s} - X_s| \leq \epsilon]}|X_s = x) = ta(t, x) + o(t)$$

$$E((X_{s+t} - X_s)^2 1_{[|X_{t+s} - X_s| \leq \epsilon]}|X_s = x) = t\sigma^2(t, x) + o(t)$$

$$P((|X_{s+t} - X_s| > \epsilon)|X_s = x) = o(t)$$

as $t \to 0^+$, *where* $a(t, x)$ *and* $\sigma^2(t, x) > 0$ *are continuous differentiable with bounded derivatives on* S. *Also,* σ'' *exists and is continuous, and* $\sigma^2 > 0$ *for all* x.

The stochastic differential equation

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t, \tag{B.1}$$

is equivalent to the equation

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t \sigma^2(s, X(s))dW_s, t \geq 0. \tag{B.2}$$

If the solution is unique, then the process $X(t)$ is called a diffusion process with infinitesimal drift function $a(t, x)$ and infinitesimal variance $\sigma^2(t, x)$ at (t, x), provided that $\sigma^2(t, x) > 0$ for all $t \geq 0$ and $x \in S$, where $S \subset \mathbf{R}$ is the state space of $X(t)$ (Rolski et al., 1999).

Lemma 10 (Itô's lemma)**.** *Let X_t be a process given by*

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t. \tag{B.3}$$

Let $g(t, x) \in \mathcal{C}^2([0, \infty) \times \mathbf{R})$, i.e. f is twice continuously differentiable in $[0, \infty) \times \mathbf{R}$. Then $Y_t = f(t, X_t)$ is also a diffusion process and (Oksendal, 1998)

$$dY_t = \frac{\rho f}{\rho t}(t, X_t)dt + \frac{\rho f}{\rho x}(t, X_t)dX_t + \frac{1}{2}\frac{\rho^2 f}{\rho x^2}(t, X_t)(dX_t)^2 \tag{B.4}$$

Proposition 6. *Let $\{X_t\}$ be a diffusion process on $S = (a, b)$:*

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t, \tag{B.5}$$

Then, for all twice continuously differentiable g, vanishing outside a closed bounded subinterval of S, and belonging to $\mathcal{D}_A$, the infinitesimal generator is given by

$$\mathbf{A}g(x) = a(t, x)g'(x) + \frac{1}{2}\sigma^2(t, x)g''(x). \tag{B.6}$$

Remark 11. *The drift of $Y_t = g(X_t)$ is $a(X_t)g'(X_t) + \frac{\sigma^2(X_t)}{2}g''(X_t)$, so $\mathbf{A}g(X_t)$ can be interpreted as the drift term of the function g of the diffusion process (X_t). In the special case of $a(t, X(t)) = aX(t)$ and $\sigma(t, X(t)) = \sigma X(t)$ the infinitesimal generator is given by*

$$\mathbf{A}g(x) = axg'(x) + \frac{\sigma^2}{2}x^2 g''(x), \tag{B.7}$$

C Stopping Times and Martingales

Consider a sequence of random variables $\{X_n : n = 0, 1, 2, \cdots\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$. Stopping times with respect to $\{X_n\}$, also called Markov times, are defined as follows.

Definition 1. *A stopping time τ for the process $\{X_n\}$ is a random variable taking non-negative integer values, including possibly the value $+\infty$ such that*

$$\{\tau \leq n\} \in \mathcal{F}_n \quad (n = 0, 1, \cdots).$$

Since $\mathcal{F}_n$ are increasing sigma-fields and τ is integer-valued, an equivalent definition is

$$\{\tau = n\} \in \mathcal{F}_n \quad (n = 0, 1, \cdots).$$

Definition 2. *The first passage time τ_B to a (Borel) set $B \in R$, is a stopping time, defined by*

$$\tau_B = \min\{n \geq 0 : X_n \in B\}.$$

If X_n doesn't lie in B for any n, one takes $\tau_B = \infty$.

Definition 3. *Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables, and $\{\mathcal{F}_n\}$ an increasing sequence of sigma-fields such that, for every n, $X_0, X_1, X_2, \cdots, X_n$ are $\mathcal{F}_n$-measurable. If $E \mid X_n \mid < \infty$ and*

$$E(X_{n+1} \mid \mathcal{F}_n) = 0$$

holds for all n, then $\{X_n : n = 0, 1, 2, \cdots\}$ is said to be a sequence of $\mathcal{F}_n$-martingale differences. The sequence of partial sums $\{S_n = X_0 + \cdots + X_n : n = 0, 1, 2, \cdots\}$ is then said to be a $\mathcal{F}_n$-martingale.

Theorem 14. *Doob's Inequality*

Let $\{X(t)\}$ be a submartingale. Then for each $x > 0$ and $t \geq 0$,

$$P(\sup_{0 \leq v \leq t} X(v) \geq x) \leq \frac{EX(t)}{x}.$$

Proof. See e.g. Rolski et al. (1999).

□

D Laplace Transforms

The following definitions and results help in proving the asymptotic decay of the ruin probability for a classical risk model using completely monotone functions and Laplace transforms properties only. In the classical Cramér Lundberg case the integro-differential equation of the probability of ruin becomes a linear equation in the Laplace transform side. Thus, the Laplace transform of the probability of ruin has a very simple form. The question arising is if the expression obtained for the Laplace transform of the probability of ruin can be proved to be the Laplace transform of a density function. Feller (1971) introduces methods to recognize if a given function is the Laplace transform of a probability density function. The following results are stated from (Feller, 1971) and they will be useful in the derivation of the exponential decay of the probability of ruin by Laplace transform properties only.

Definition 1. *Let F be a measure concentrated on $(0, \infty)$. If the integral*

$$\tilde{F}(\lambda) = \int_0^\infty e^{-\lambda x} F(dx) = \int_0^\infty e^{-\lambda x} dF(x)$$

converges for $\lambda > a$ then the function $\tilde{F}$ defined for $\lambda > a$ is called the ***Laplace Stieltjes transform of*** *F. If F has a density f, the Laplace Stieltjes transform of F is also called the* ***Laplace transform of*** *f,*

$$\tilde{F}(\lambda) = \int_0^\infty e^{-\lambda x} F(dx) = \int_0^\infty e^{-\lambda x} f(x)(dx) := \hat{f}(\lambda).$$

The Laplace Stieltjes transform will be denoted with “tilde” and the Laplace transforms with “hat”.

Definition 2. *A function φ on $(0, \infty)$ is* ***completely monotone*** *if it possesses* *derivatives* $\varphi^{(n)}$ *of all orders and*

$$(-1)^n\varphi^{(n)}(\lambda) \geq 0, \quad \lambda > 0.$$

The theorem that will help build the rationale is the following:

Theorem 15. *A function* φ *on* $(0, \infty)$ *is the Laplace Stieltjes transform of a probability distribution* F *(the Laplace transform of the density* $f = F'$*), iff it is completely monotone, and* $\varphi(0) = 1$.

The equivalent form of the theorem is:

Theorem 16. *The function* φ *on* $(0, \infty)$ *is completely monotone iff it is of the form*

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda x} F(dx) = \int_0^\infty e^{-\lambda x} f(x)dx),$$

$\lambda > 0$, *where* F *is not a necessarily finite measure on* $[0, \infty)$, *with* $f = F'$.

This theorem leads to simple tests to check whether a given function is the Laplace Stieltjes transform of a probability distribution F or the Laplace transform of the density $f = F'$. Other useful propositions regarding completely monotone functions are presented next.

Proposition 7. *If* φ *and* ψ *are completely monotone so is their product* $\varphi\psi$.

Proof. Using induction one can show that the derivatives of $\varphi\psi$ alternate in sign. Assume that for every pair φ, ψ of completely monotone functions the first n derivatives of $\varphi\psi$ alternate in sign. As $-\varphi'$ and $-\psi'$ are completely monotone, the induction hypothesis applies to the products $-\varphi'\psi$ and $-\varphi\psi'$, and from

$$-(\varphi\psi)' = -\varphi'\psi - \varphi\psi',$$

the conclusion is that the first $n + 1$ derivatives of $\varphi\psi$ alternate in sign. Since the hypothesis is trivially true for $n = 1$ the proposition is proved. □

Proposition 8. *If $\varphi > 0$ on $(0, \infty)$ and has a derivative φ' that is completely monotone there, then $\frac{1}{\varphi}$ is also completely monotone.*

Corollary 5. *If $\varphi < 1$ on $(0, \infty)$ and has a derivative φ' that is completely monotone there, then $\frac{1}{1-\varphi}$ is also completely monotone.*

Feller (1971) is able to show using Laplace transform properties that the integro-differential equation satisfied by the non-ruin probability Φ has a unique solution, and calculates its value at zero, $\Phi(0)$.

Proposition 9. *Let $\Phi(u)$ be the probability of non-ruin starting with an initial capital u. Then the integro-differential equation*

$$\Phi'(u) = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_0^u \Phi(u - x) f_X(dx),$$

with

$$\lim_{u \to \infty} \Phi(u) = 1,$$

has a unique solution Φ, where f_X is a probability density with finite expectation μ. Also,

$$\Phi(0) = 1 - \frac{\lambda\mu}{c}.$$

E Karamata-Tauberian Theorems

For the asymptotic analysis, the following Karamata-Tauberian theorems are used. A comprehensive reference is Bingham et al. (1987)

Definition 1. *Let l be a positive measurable function, defined in some neighborhood $[M, \infty)$ of infinity, and satisfying*

$$\frac{l(\lambda x)}{l(x)} \to 1, \quad as \quad x \to \infty, \forall \lambda > 0,$$

then l is said to be slowly varying in Karamata's sense (Bingham et al., 1987).

Theorem 17. *(Karamata Tauberian Theorem)*

Let U be a nondecreasing right continuous function on $\mathbf{R}$ with $U(x) = 0$ for all $x < 0$. If l varies slowly and $c \geq 0, \rho \geq 0$ the following are equivalent:

$$U(x) \sim \frac{cl(x)}{\Gamma(1+\rho)} x^{\rho}, \quad as \quad x \to \infty,$$

$$\tilde{U}(s) \sim cl(1/s)x^{-\rho}, \quad as \quad x \to \infty,$$

where $\tilde{U}(s) = \int_0^\infty e^{-su} dU(u)$ is the Laplace-Stieltjes transform of U.

Definition 2. *A function f is ultimately monotone if there exists y such that for any $x > y$, $f(x)$ is monotone.*

Theorem 18. *Monotone Density Theorem Let $U(x) = \int_0^x u(y)dy$. If*

$$U(x) \sim cl(1/s)x^{\rho}, \quad as \quad x \to \infty,$$

where $c \in R, l \in R_0$, and if u is ultimately monotone, then

$$u(x) \sim cl(1/s)x^{\rho-1}, \quad as \quad x \to \infty.$$

The following theorem relates the asymptotic behavior of a function at infinity with the asymptotic behavior at zero of its Laplace transform.

Proposition 10. *Given that the Laplace transform of a function behaves asymptotically like s^{ρ} at zero,*

$$\hat{\Psi}(s) \sim cl(s)s^{\rho} \quad as \quad s \to 0$$

then the function will behave asymptotically as $u^{-\rho-1}$ at infinity.

$$\Psi(u) \sim ku^{-\rho-1} \quad as \quad u \to \infty$$

Proof. Consider the function

$$U(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ \int_0^u \Psi(x)dx & \text{if} \quad u \geq 0. \end{cases}$$

Note that the Laplace transform of the function $\Psi(u)$, $\hat{\Psi}(s)$, equals the Laplace Stieltjes transform of the function $U(u)$, $\tilde{U}(s)$,

$$\hat{\Psi}(s) = \mathcal{L}(\Psi(u))(s) = \int_0^\infty e^{-su}\Psi(u)du = \int_0^\infty e^{-su}dU(u) = \tilde{U}(s).$$

Take the Laplace transform of $\Psi(u)$, and analyze its asymptotic behavior. Assume that $\hat{\Psi}(s)$ behaves at zero as $cl(s)s^\rho$. Thus,

$$\tilde{U}(s) = \hat{\Psi}(s) \sim cl(s)s^\rho \quad \text{as} \quad s \to 0$$

Then, using the Karamata Tauberian Theorem, $U(u)$ behaves asymptotically at infinity,

$$U(u) \sim \frac{c}{\Gamma(1-\rho)} l(\frac{1}{u})u^{-\rho} \quad \text{as} \quad u \to \infty$$

The result then follows from the Monotone Density Theorem applied to $U(u) = \int_0^u \Psi(y)dy$.

$\square$