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Beginning with the basic equations from the theory of elasticity and employing a model of crack propagation based on the concept of a constant plastic energy absorption per unti of crack extension, the commonly accepted power law of fatigue crack propagation was derived. This derivation uses an extension of the Dugdale model for the calculation of plastic energy absorption into the zone of gross yielding preceeding the crack tip. This Dugdale model together with a superposition technique are employed to compute the total plastic energy over a number of cyclic loadings. The power law relationship which results from this development is compared with experimental crack growth data to evaluate the material constant, the crack extension force.

AN ANALYTICAL MODEL FOR FATIGUE CRACK PROPAGATION

by

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LIST OF SYMBOLS

Latin

a	Crack length to end of plastic zone.
B	Frost and Dugdale's Growth Law function.
C_1	Head's Growth Law function.
C_2	Head's constant (=Yield Strength).
C_3	Frost and Dugdales' Growth Law constant.
C_4	Liu's Growth Law constant.
C_5	Paris' Growth Law constant.
$C_6(\Delta\sigma)$	Anderson's Growth Law function. (for ΔK relation)
$d\ell/dN$	Differential crack growth per cycle.
E	Modulus of Elasticity.
$f(\xi)$	Solution function, found from boundary conditions.
G	Shear Modulus
I_p	Volume of the plastic zone
$J_0(x)$	Bessel function of the first kind, of order zero
ℓ	Physical crack length-open portion.
M	Anderson's Growth Law material fatigue constant.
n	Paris' exponential constant.
N	Number of cycles until crack growth initiation.
$P(x)$	Crack opening pressure distribution.
$q(\eta)$	Integral function of pressure distribution.
U_i	Displacement component.
U	x component of displacement.

V	y component of displacement.
W_i	Work done in plastic zone during initial cycle.
W_p	Crack extension force.
Y	Yield strength.

Greek

β	Elastic constant-function of Poisson's ratio.
ΔE	Energy input per cycle into plastic zone.
ΔK	Stress intensity factor range ($=\Delta\sigma\sqrt{\pi \ell}$).
ΔN	Difference in number of cycles between states 1 & 2.
$\Delta\sigma$	Stress range - loading.
ε_{xx}, ε_{yy}	Normal strains.
η	Dummy variable of integration.
λ	Lame's Elastic constant.
ν	Poisson's ratio.
ξ	Dummy variable of integration.
ρ	Dummy variable of integration.
σ	Loading stress.
σ_{xx}, σ_{yy}	Normal stresses.
τ_{xy}	Shear stress
ϕ	Potential function for solution of displacements.
ω	Plastic zone length.

AN ANALYTICAL MODEL FOR
FATIGUE CRACK PROPAGATION

Introduction

In the past 21 years there have been numerous laws of fatigue-crack propagation formulated. Most of these were empirical fits to various specific sets of data. Much of the data was obtained by cycling a wide plate of some metal, with a central starter crack, over some stress range and measuring the crack length at corresponding numbers of cycles. There are four of these laws that have been the subject of several reviews, and as such are noteworthy.

Chronologically the first was that of Head [1][1] in 1953. He employed a mechanical model which considered rigid-plastic work-hardening elements ahead of a crack tip, and elastic elements over the remainder of the infinite sheet. After extensive calculations and deductions he obtained

$$\frac{d\ell}{dN} = \frac{C_1 \Delta\sigma^3 \ell^{3/2}}{(C_2-\sigma) w^{\frac{1}{2}}} \qquad (1)$$

where: $\Delta\sigma$ = stress range

w = plastic zone size

C_1 = f(strain-hardening modulus, modulus of elasticity, yield stress)

C_2 = yield stress

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1 Numbers in brackets are references listed on page 37.

Head assumed that the plastic zone size remained constant and was independent of crack length, but experimental work carried out later by Dugdale [2] proved this assumption to be incorrect. When this law was checked against a broader range of data it was found not to agree with the experimental findings and was discarded.

In 1958 Frost and Dugdale [3] using dimensional analysis argued that the cyclic growth rate should be directly proportional to the crack length. Therefore they proposed a propagation law of the form

$$\frac{d\ell}{dN} = B\ell \tag{2}$$

where B is a function of the applied stress and the material constants. After experimental study of fatigue crack growth in wide sheets of mild steel, aluminum, and copper, they concluded that the function for B was

$$B = \frac{\Delta\sigma^3}{C_3}$$

where C_3 is a constant depending on the material and the mean stress. This leads to a growth law of

$$\frac{d\ell}{dN} = \frac{\Delta\sigma^3 \ell}{C_3} \tag{3}$$

A short time later Liu [4] arrived at the same conclusion as Frost and Dugdale regarding the linearity of the growth law, Eq. (2), using a more elegant form of dimensional analysis. He observed that

B, which he called the crack propagation factor, was a function of the applied stress, and thru extensive analysis of his experimental data arrived at a rather complicated function for B. Subsequently Liu [5] used a model of an idealized elastic-plastic material and a concept of total hysteresis energy absorption to failure, to arrive at the stress function

$$B = C_4 \Delta\sigma^2$$

where C_4 was a material constant. This combines with Eq. (2) to give Liu's law as

$$\frac{d\ell}{dN} = C_4 \Delta\sigma^2 \ell \qquad (4)$$

At the time of his publication [5] in 1963 Paris and Erdogan called his attention to a broad range of data which did not support his growth law and Liu withdrew the proposed law.

Paris and Erdogan [6] argued that the small amount of data being used to provide validation for the previous crack-propagation laws, could be in agreement with several of the contradictory laws. They found by plotting the experimental results in the form of stress intensity factor range versus crack growth rate, that data from several sources could be plotted to obtain a single curve. From an empirical fit of data in this form they arrived at what was referred to as the Power Law of fatigue crack propagation.

$$\frac{d\ell}{dN} = C_5 \Delta K^n \qquad (5)$$

where: C_5 is a constant for a particular material/environment

n is a constant for a particular material/environment which usually has a value of about 4.

$\Delta K = \Delta\sigma\sqrt{\pi\ell}$ is the stress-intensity factor range.

This law is generally accepted as the one that consistently provides the best correlation over the widest range of conditions. It also seems to be the latest of the growth laws to be proposed and accepted. Some attempts have been made to derive this law using dimensional analysis [7] or in model form [8] but these attempts have been lacking in generality.

This paper will deal with a derivation of the Power Law starting with the basic equations of elasticity and employing a simple model of crack propagation based on the concept of total hysteresis plastic energy absorption to failure.

CHAPTER 2

The Elastic Crack

Consider a crack of length 2ℓ in an infinite sheet of some elastic material. Also consider that the crack is subjected to a stress applied at infinity, which may be a function of time. In the case to be investigated the crack will be assumed to be quasistatic, i.e. propagating, but so slowly that dynamic effects are not significant. The direction of propagation is assumed in the x direction, with the applied stress perpendicular in the y direction[2].

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2 Refer to Figure 1.

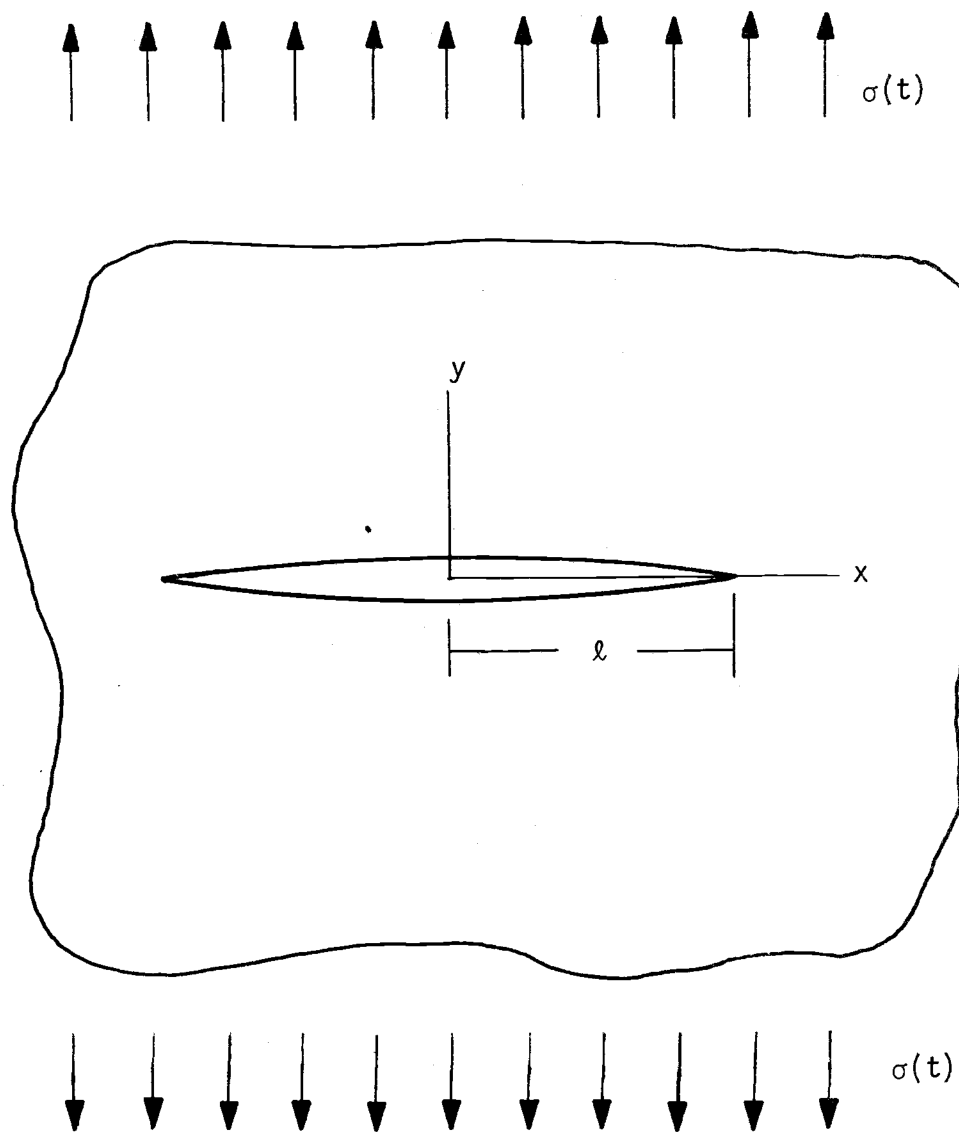

FIGURE 1

Crack configuration for elastic model.

2.1 Basic Equations

2.1.1 - Displacements

This is a mixed boundary value problem which we know from Elasticity can be described by the Navier-Cauchy equations

$$G\, U_{i,jj} + (\lambda+G)\, U_{j,ji} = 0$$

where U_i are displacement components

G is the shear modulus

λ is Lame's constant

For cases of plane strain or plane stress these equations reduce to

$$G\nabla^2 U + (\lambda+G)\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right] = 0$$

$$G\nabla^2 V + (\lambda+G)\frac{\partial}{\partial y}\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right] = 0$$

Dividing by G gives

$$\nabla^2 U + (\beta^2-1)\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right] = 0 \qquad (6)$$

$$\nabla^2 V + (\beta^2-1)\frac{\partial}{\partial y}\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right] = 0 \qquad (7)$$

where U is the displacement in the x direction

V is the displacement in the y direction

$$\beta^2 = \frac{2(1-\nu)}{(1-2\nu)} \quad (\beta^2-1) = \frac{1}{(1-2\nu)}$$

Using a potential formulation it has been shown [9] that the solutions to Eq. (6 & 7) are

$$U = \frac{1}{G}\left\{\frac{\partial\phi}{\partial x} + (\beta^2-1)\, y\, \frac{\partial^2\phi}{\partial x \partial y}\right\} \tag{8}$$

$$V = \frac{1}{G}\left\{-\beta^2\, \frac{\partial\phi}{\partial y} + (\beta^2-1)\, y\, \frac{\partial^2\phi}{\partial y^2}\right\} \tag{9}$$

where ϕ is a potential function, yet to be found, which satisfies Laplace's equation

$$\nabla^2\phi = 0$$

By using Hooke's Law in two dimensional form the stresses can be written in terms of the displacements and thus in terms of the potential function.

2.1.2 Stresses

Using Hooke's law the stresses can be written in the form

$$\sigma_{xx} = 2G\, \frac{\partial U}{\partial x} + \lambda\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)$$

$$\sigma_{yy} = 2G\frac{\partial V}{\partial y} + \lambda\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)$$

$$\tau_{xy} = G\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

Substitution of Eq. (8 & 9) and their derivatives into the above stress-strain relations results in the following relations for stress in terms of the potential function.

$$\sigma_{xx} = -2(\beta^2-1)\frac{\partial^2\phi}{\partial x^2} - 2(\beta^2-1)\ y\frac{\partial^3\phi}{\partial y^3} \qquad (10)$$

$$\sigma_{yy} = -2(\beta^2-1)\frac{\partial^2\phi}{\partial y^2} + 2(\beta^2-1)\ y\frac{\partial^3\phi}{\partial y^3} \qquad (11)$$

$$\tau_{xy} = 2(\beta^2-1)\ y\frac{\partial^3\phi}{\partial y^2\partial x} \qquad (12)$$

For crack problems I. N. Sneddon [10] has assumed a potential function of the form

$$\phi = \frac{1}{2(\beta^2-1)}\int_0^\infty \frac{f(\xi)}{\xi}\cos(\xi x)\ e^{-\xi y}\ d\xi + \frac{1}{2(\beta^2-1)}\int_0^\infty \frac{g(\xi)}{\xi}\sin(\xi x)\ e^{-\xi y}\ d\xi \qquad (13)$$

where the first term is symmetrical and the second is axisymmetric, and $f(\xi)$ and $g(\xi)$ are unknown functions to be determined using the boundary conditions. At this time consideration will be restricted to symmetrical loading and therefore only the first term in the potential function is needed. Thus

$$\phi = \frac{1}{2(\beta^2-1)} \int_0^\infty \frac{f(\xi)}{\xi} \cos(\xi x) e^{-\xi y} \, d\xi \qquad (14)$$

2.2 Superposition

Return now to the original problem which will be split into two problems, the superposition of which results in the original problem.[3] It is readily seen that the first part of the problem (which will be referred to as problem A) is simply a known uniform stress distribution and as such is uninteresting. Therefore it will be ignored for now with the note to add it to the final answer obtained for the second part of the problem (which will be referred to as problem B). $P(x)$ is called the crack opening pressure distribution, which is known.

The solution of this problem will therefore give the displacement of the crack face for $|x| \leq \ell$ and the stress distribution for $|x| \geq \ell$.

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3 See Figure 2.

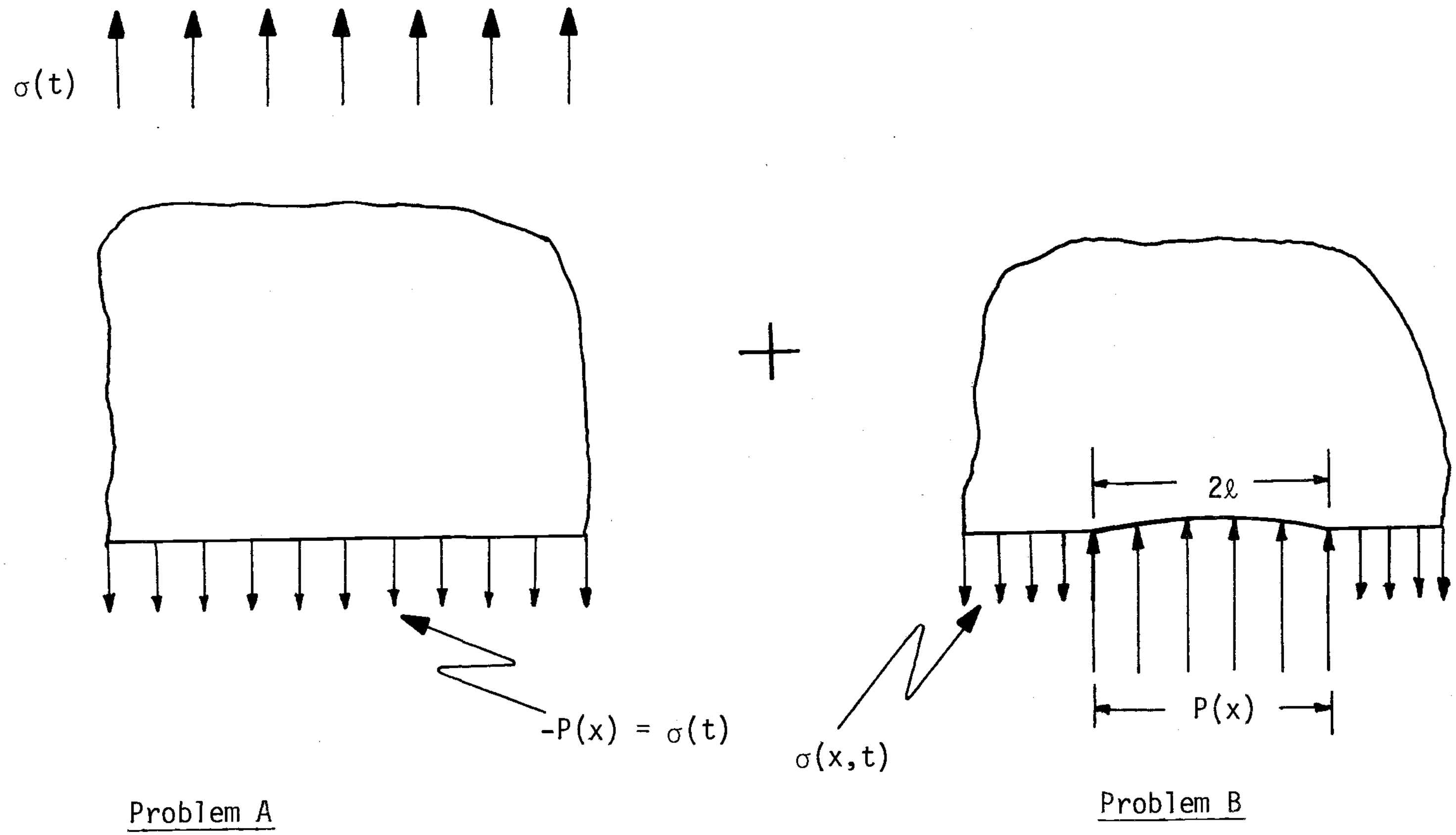

FIGURE 2

The separation of elastic problem into two new problems, the superposition of which results in the original problem.

2.3 Boundary Conditions

The boundary conditions for problem B are

$$\tau_{xy}\Big|_{y=0} = 0 \qquad \text{everywhere} \tag{15a}$$

$$\sigma_{yy}\Big|_{y=0} = -P(x) \qquad |x|\leq \ell \tag{15b}$$

$$V\Big|_{y=0} = 0 \qquad |x|\geq \ell \tag{15c}$$

$$\sigma_{yy}\Big|_{y=\infty} = 0 \qquad \text{all } x \tag{15d}$$

$$\sigma_{xx}\Big|_{y=0} = 0 \qquad \text{all } x \tag{15e}$$

2.4 Boundary Conditions - Potential Function

Using the potential formulation and Eq. (9, 11, & 12) the boundary conditions can be written in terms of the potential function as

$$\frac{\partial^2\phi}{\partial y^2}\Big|_{y=0} = \frac{P(x)}{2(\beta^2-1)} \qquad |x|\leq \ell \tag{16a}$$

$$\frac{\partial\phi}{\partial y}\Big|_{y=0} = 0 \qquad |x|\geq \ell \tag{16b}$$

2.5 Sample Problem

Consider the case when the loading is a uniform static stress distribution (i.e. $P(x) = P = \sigma = \text{constant}$). Applying the boundary conditions, Eq. (16a & b), to the potential function, Eq. (14), and using the above assumption results in the following dual integral equations.

$$\int_0^\infty f(\xi)\cos(\xi x)d\xi = 0$$

$$\int_0^\infty \xi f(\xi)\cos(\xi x)d\xi = \frac{P}{2(\beta^2-1)}$$

These dual integral equations have been solved for the unknown function $f(\xi)$ by Sneddon [11] giving

$$f(\xi) = \frac{2}{\pi}\int_0^\ell \eta J_0(\xi\eta)\int_0^\eta \frac{Pd\rho}{\sqrt{\eta^2 - \rho^2}}\,d\eta \tag{17}$$

Substituting this function into the potential function as given by Eq. (14) results in

$$\phi = \frac{1}{2(\beta^2-1)} \frac{2}{\pi} \int_0^\infty \frac{1}{\xi} \{ \int_0^\ell \eta\, J_o(\xi\eta) \int_0^\eta \frac{P d\rho}{\sqrt{\eta^2-\rho^2}} \} \cos(\xi x) e^{-\xi y}\, d\xi$$

This function can now be substituted into the expressions for the stresses, Eq. (10), and displacements, Eq. (8), and the integrals evaluated. This has been done in Appendix A which gives the following

$$\sigma_{yy}\Big|_{y=0} = P\{ \frac{x}{\sqrt{x^2-\ell^2}} - 1\} \qquad |x|>\ell \qquad (19a)$$

$$V\Big|_{y=0} = \frac{2(1-\nu^2)P}{E} \sqrt{\ell^2-x^2} \qquad |x|\leq\ell \qquad (19b)$$

CHAPTER 3

The Dugdale Crack

3.1 Plasticity Considerations

Note in Eq. (19a) that as $x \to \ell$ (i.e. at the physical crack tip) the stress approaches infinity. This is a result of assuming that the material remains elastic at all times. We know that it is physically impossible for the stresses to be infinite and therefore this model must be modified. In 1959 D.S. Dugdale [2] did this by assuming that there was a line plastic zone at the tip of the crack. Experimental data at that time showed that this was a reasonable assumption for the case of slits in a thin sheet of steel. More recent work has shown that the plastic zone looks considerably different than this model predicts and extends a considerable distance from the crack tip. However for the considerations involved in this study, which will involve energy computations, the Dugdale model allows direct analytical computations to be performed that are in keeping with the spirit of Irwin's original energy balance.

Including the plastic zone at the crack tip has the effect of maintaining finite stresses throughout the infinite sheet. The ratio of the open crack length to the crack length including the plastic zone (ℓ/a) can be found by requiring that the stress intensity factor be zero at the end of the plastic zone ($x=a$). This is equivalent to requiring that the stresses remain finite and will be referred to as

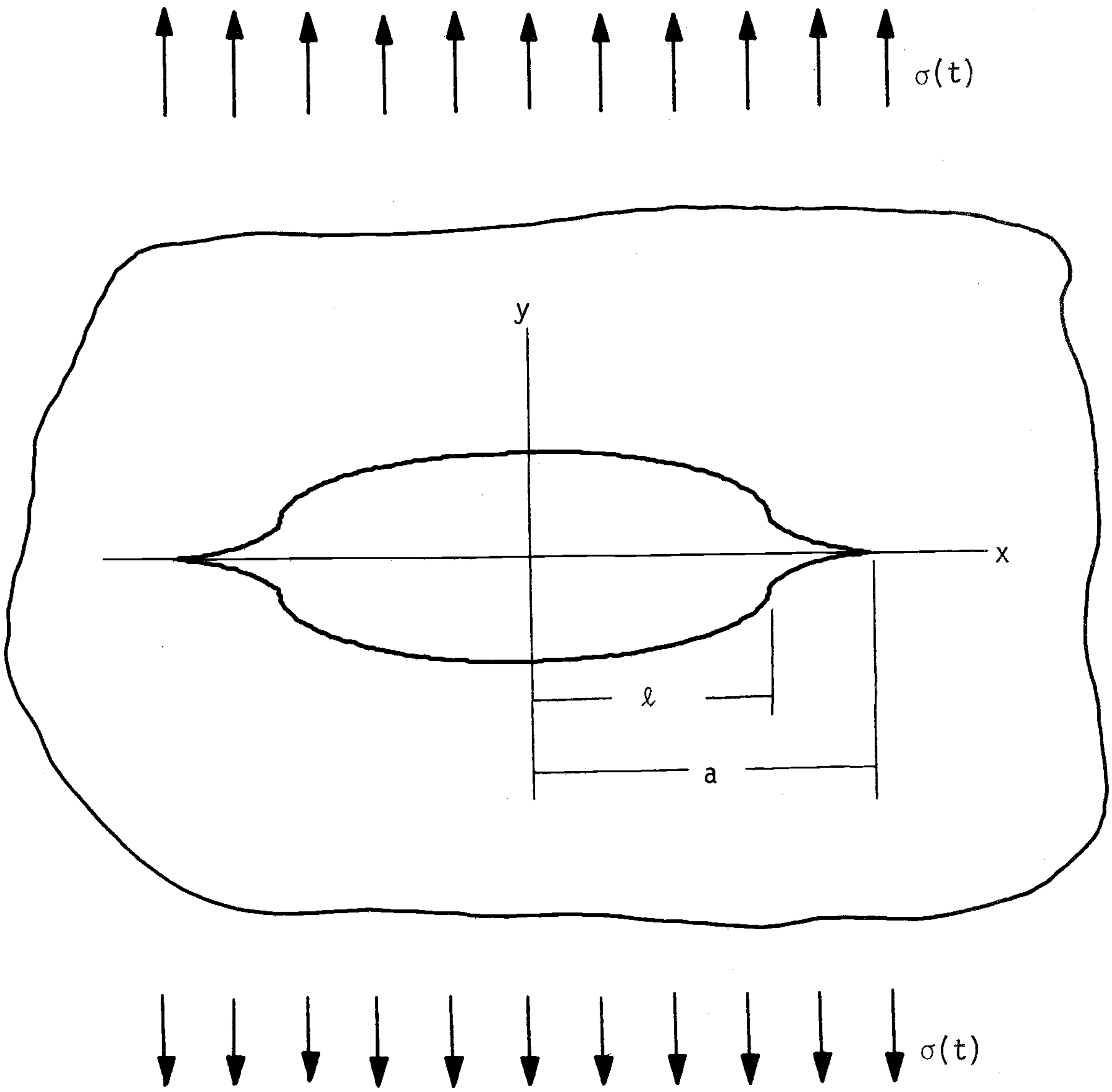

FIGURE 3

Crack configuration for Dugdale model.

the finiteness condition. Doing this results in[4]

$$\ell/a = \cos\left[\frac{\sigma\pi}{Y2}\right] \qquad (20)$$

This is the same relation that Dugdale [2] found using a series form of a stress function which was derived by Muskhelishvili [12].

3.2 Boundary Conditions

Using this model results in essentially the same problem as was first considered, only with a different pressure distribution. Therefore the boundary condition can be written as

$$\sigma_{yy}\Big|_{y=0} = 0 \qquad 0 \le x \le \ell \qquad (21a)$$

$$\sigma_{yy}\Big|_{y=0} = -Y \qquad \ell < x \le a \qquad (21b)$$

$$V\Big|_{y=0} = 0 \qquad x > a \qquad (21c)$$

3.3 Pressure Distribution

Again we split the problem into two problems, with two pressure distributions

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4 See Appendix F - Eq. (F5).

Problem A $\qquad P(x) = -\sigma$

$$\text{Problem B} \qquad P(x) = \begin{cases} \sigma & 0<x\leq \ell \\ \sigma - Y & \ell<x\leq a \end{cases} \tag{22}$$

3.4 Solution-Displacements and Stresses

Rewriting the potential function with the limits of integration including the plastic zone gives

$$\phi = \frac{1}{2(\beta^2-1)} \frac{2}{\pi} \int_0^\infty \frac{1}{\xi} \left\{ \int_0^a \eta \, J_0(\xi\eta) \int_0^\eta \frac{P(\rho)d\rho}{\sqrt{\eta^2-\rho^2}} \, d\eta \right\} \cos(\xi x) e^{-\xi y} d\xi \tag{23}$$

Introducing the notation

$$q(\eta) = \int_0^\eta \frac{P(\rho)d\rho}{\sqrt{\eta^2-\rho^2}}$$

results in a potential function of

$$\phi = \frac{1}{2(\beta^2-1)} \frac{2}{\pi} \int_0^\infty \frac{1}{\xi} \left\{ \int_0^a \eta \, J_0(\xi\eta) \, q(\eta) \, d\eta \right\} \cos(\xi x) e^{-\xi y} \, d\xi \tag{24}$$

Using this form of the potential function the stresses and displacements can be given as[5]:

$$
V\Big|_{y=0}^{6} = \frac{4(1-\nu^2)Y\ell}{\pi E}\begin{cases} \dfrac{\sigma\pi}{Y2\ell}\sqrt{\ell^2-x^2} + \log(a/\ell) & 0\le x\le \ell \\[2em] \tfrac{1}{2}\{\dfrac{x}{\ell}\log\left(\dfrac{x\sqrt{a^2-\ell^2}+\ell\sqrt{a^2-x^2}}{x\sqrt{a^2-\ell^2}-\ell\sqrt{a^2-x^2}}\right) \\[2em] \quad -\log\left(\dfrac{\sqrt{a^2-\ell^2}+\sqrt{a^2-x^2}}{\sqrt{a^2-\ell^2}-\sqrt{a^2-x^2}}\right)\} & \ell<x<a \end{cases}
$$

(25)

$$
\sigma_{yy}\Big|_{y=0} = Y\{1 - \frac{2}{\pi}\tan^{-1}(\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}})\} \quad x\ge a \qquad (26)
$$

Notice that the stresses as given by the Dugdale model remain bounded and are therefore a more reasonable representation of the stress state than the first model tried. The stress distribution

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5 See Appendices B and C.

6 Note that the term $(1-\nu^2)/E$ is for plane strain and must be replaced by $1/E$ for plane stress.

represented by Eq. (26) is plotted in Figure 4 for a loading pressure of $P = \sigma = Y/2$.

3.5 Residual Stress Distribution

It is desired to know what the residual stress-displacement field in the material would be after a loading-unloading cycle. This is accomplished by first computing the stress-displacement field using the Dugdale model. The unloading stress and displacement fields are then approximated by superimposing the stresses and displacements for the identical crack geometry but loaded with $-\sigma$, and noting that in the yielded area the material will behave elastically until the local elastic increment of stress reaches a value of $-2Y$ and only thereafter again flow plastically. Therefore the superposition of the stresses can be represented by the sum of the stress functions

$$(\sigma_{yy})_1 = f\{x,\ell,a_1,\sigma,Y\} \tag{27a}$$

$$(\sigma_{yy})_2 = f\{x,\ell,a_2,-\sigma,2Y\} \tag{27b}$$

where the function f is given by Eq. (26). From Eq. (20) it can be seen that one result of replacing Y with 2Y is that of the unloading plastic zone being smaller than the initial plastic zone and therefore the total crack length a_1 in Eq. (27a) must be replaced with a_2 in Eq. (27b). Another result of this shorter plastic subzone is that when the material is taken through a stress cycle all of the plastic work, at least after the initial cycle, is done in this subzone. This

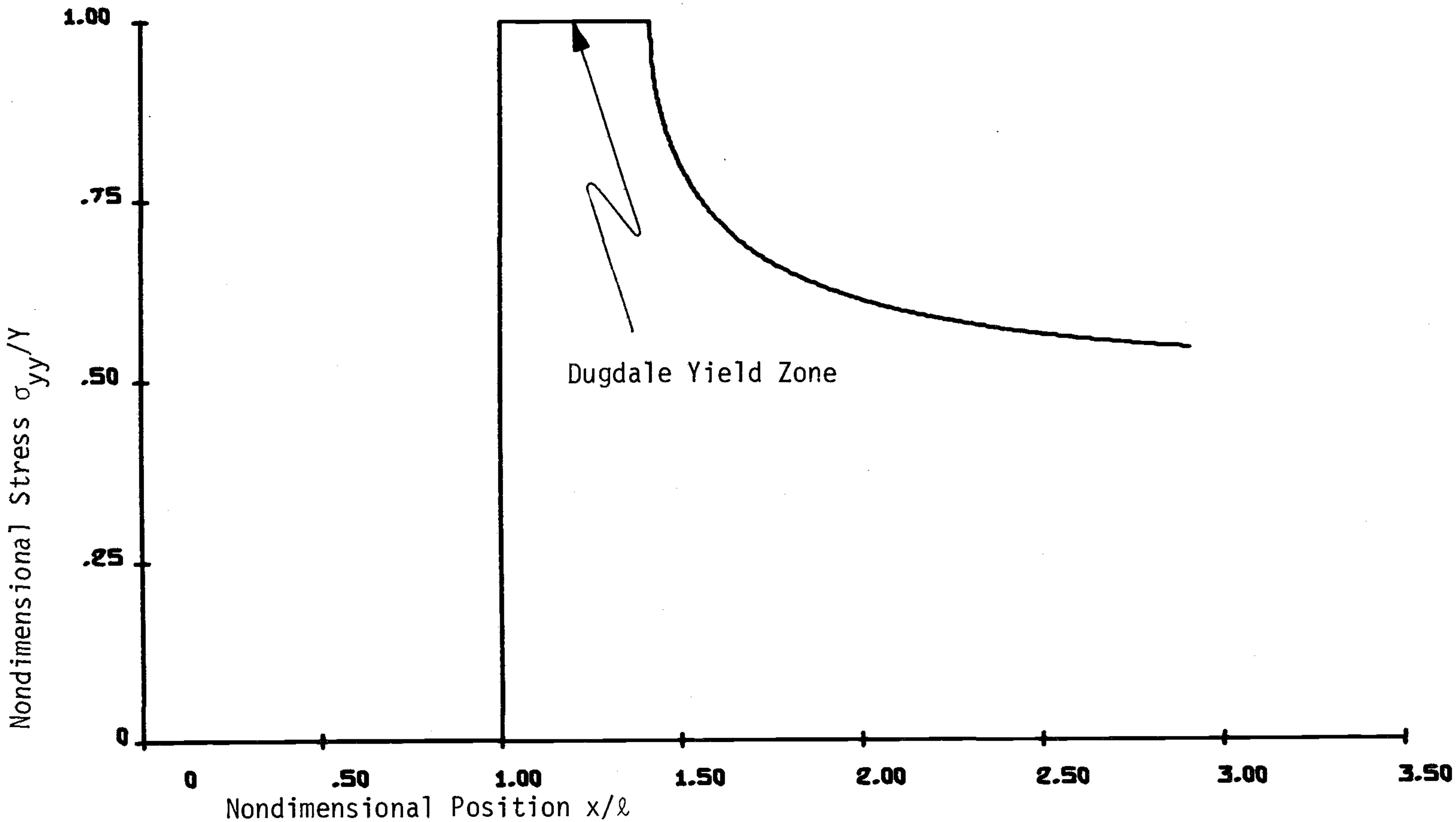

FIGURE 4

Stress distribution for Dugdale model ($\sigma = \frac{1}{2}Y$)

concept will be returned to later in the modelling for the growth law. An example of the superposition resulting in the residual stress-displacement fields is plotted in Figure 5 for $P = \sigma = Y/2$.

3.6 Discussion of Solutions

Even though functions supposed to represent the stress distribution and the residual stresses have been developed and plotted, no claim is made that these distributions could be verified by experiment. Nor is it claimed that the displacement function as given by Eq. (25) represents the actual shape of the plastic zone, although Dugdale [2] observed gross yielding zones of this shape in very thin sheets. What is claimed is that when these results are used to compute cyclic energy inputs for crack growth they will give a reasonable approximation to the actual energy. Irwin [13] used similar reasoning in his modelling of brittle crack propagation when he discussed the transfer of energy from elastic strain energy to surface energy, and with considerable success.

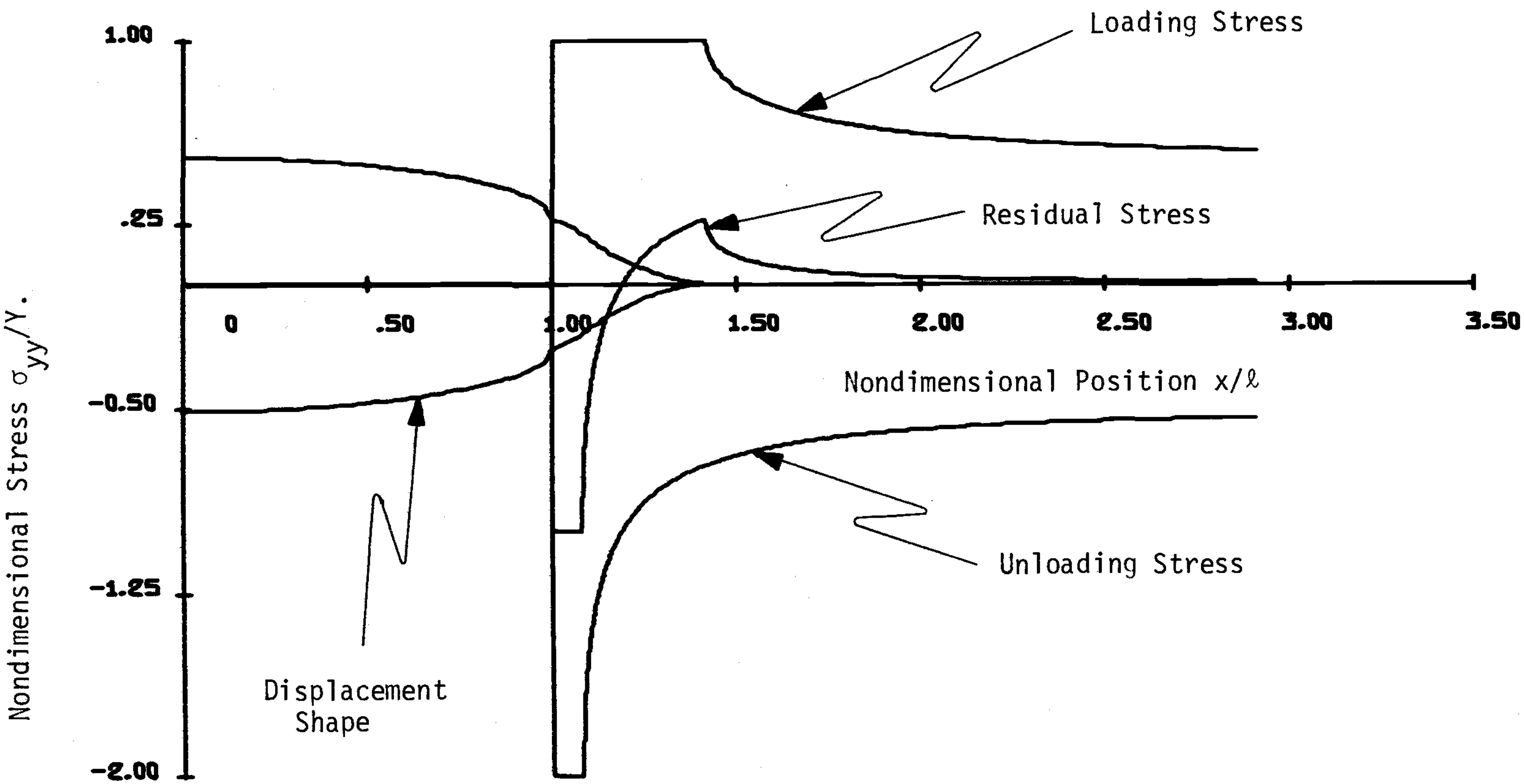

FIGURE 5

Residual stress distribution and displacement shape after unloading for the Dugdale model ($\sigma = \frac{1}{2}Y$).

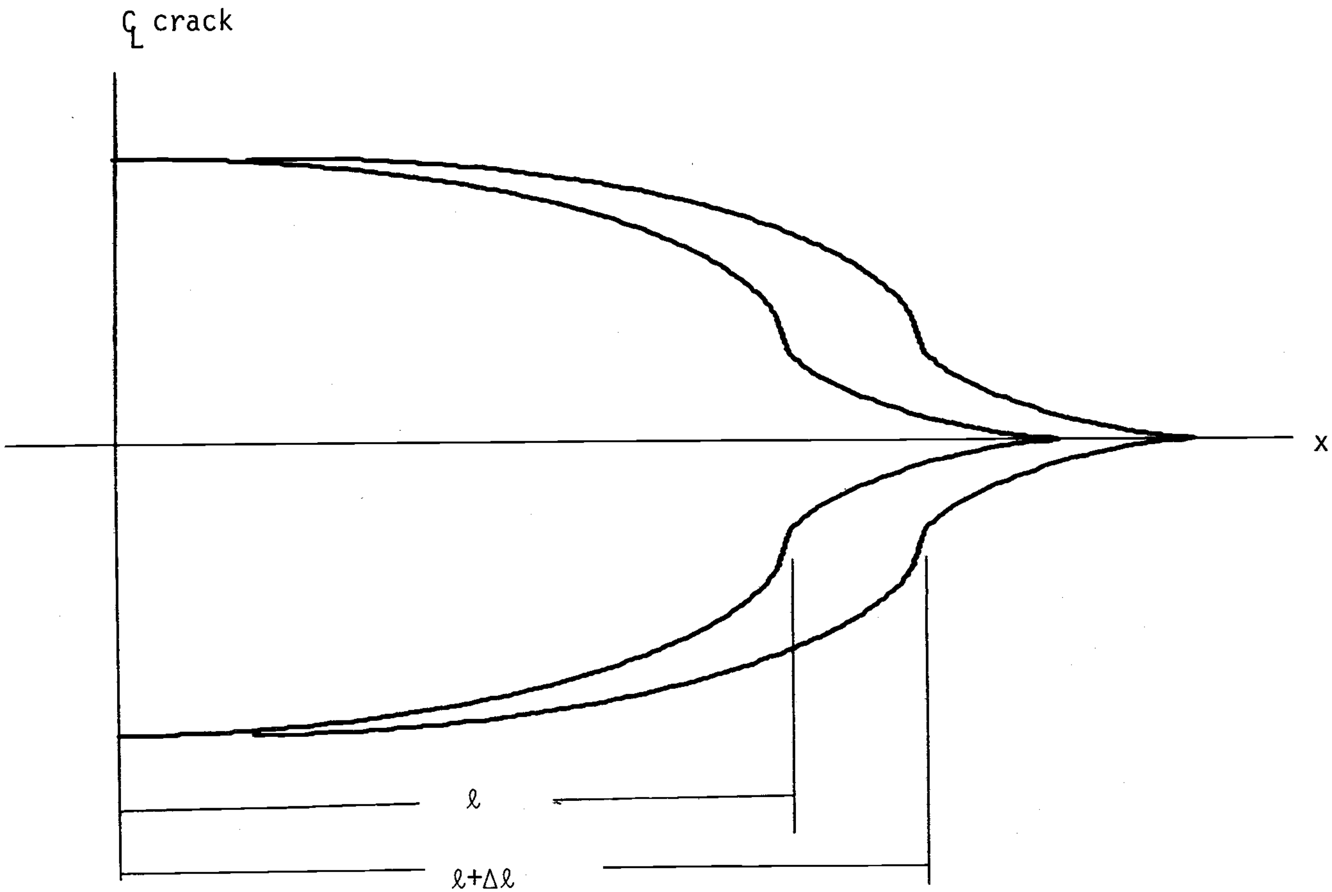

FIGURE 7

Configuration for crack extension

CHAPTER 4

A Model for the Growth Law

4.1 Postulates for Model

In attempting to formulate a theory of fatigue crack growth certain assumptions or postulates must be made. Therefore it is postulated that the energy required per unit of crack extension is a constant. Thus

$$\frac{\Delta E}{\Delta \ell} = \text{constant} = W_p \tag{28}$$

where W_p is a material property, which is sometimes referred to as the crack extension force. Irwin [13] used a similar constant, which he called the crack extension force, in his work with brittle fracture. In this particular case these constants need not numerically be identical, however they do have a similar physical interpretation.

4.2 Plastic Energy Absorption

The work done per cycle in plastic flow into the plastic subzone can be given as

$$\Delta E/\text{cycle} = 2(2Y)(2\text{ volume of subzone}) \tag{29}$$

$$= 8Y\ I_p(2Y)$$

$$= \frac{32(1-\nu^2)Y^2\ell^2}{\pi E}\{\gamma \tan\gamma - 2 \log \cos \gamma\} \tag{30}$$

where $\gamma = \frac{\Delta\sigma\pi}{Y4}$

The first factor of two in Eq. (29) is to account for the tension and compression portions of the cycle, and the last factor of two is present to account for the top and bottom halves which were not included in the volume as given by Eq. (G6).

4.3 The Growth Equation

Using the chain rule, the crack extension per cycle can be written as

$$\frac{\Delta \ell}{\Delta N} = \frac{\Delta E/\Delta N}{\Delta E/\Delta \ell} = \frac{\Delta E/\text{cycle}}{W_p} \quad . \tag{31}$$

where the term $\Delta E/\Delta \ell$ was replaced with the crack extension force from Eq. (28).

Replacing the energy input per cycle in Eq. (31) with that of Eq. (30) results in a growth law of

$$\frac{\Delta \ell}{\Delta N} = \frac{32(1-\nu^2)Y^2\ell^2}{\pi E W}\{\gamma \tan\gamma - 2 \log \cos\gamma\} \tag{32}$$

The term in the brackets can be expanded using the series expansions

$$\tan\gamma = \gamma + \frac{\gamma^3}{3} + \ldots. \qquad \gamma^2 < \frac{\pi^2}{4} \tag{33}$$

$$\log \cos\gamma = - \frac{\gamma^2}{2} - \frac{\gamma^4}{12} \qquad \gamma^2 < \frac{\pi^2}{4} \tag{34}$$

to give the result

$$\{\gamma \tan\gamma - 2 \log \cos\gamma\} \simeq 2\gamma^2 \qquad (35)$$

where only the first terms in the expansions were retained. If this is substituted into Eq. (32), (after replacing γ and writing in the differential form, the following result is obtained.

$$\frac{d\ell}{dN} \simeq \frac{4(1-\nu^2)\pi}{EW} \Delta\sigma^2 \ell^2 \qquad (36)$$

which is restricted to small values of γ. Since much of the data in the literature is given in the form of stress intensity factor range versus growth rate per cycle, the substitution of

$$\Delta K = \Delta\sigma \sqrt{\pi\ell}$$

or

$$\ell^2 = \left(\frac{\Delta K^4}{\pi^2 \Delta\sigma^4}\right)$$

will be made in Eq. (36) giving the final result of

$$\frac{d\ell}{dN} = \frac{4(1-\nu^2)}{\pi EW \, \Delta\sigma^2} \Delta K^4 \qquad (37)$$

$$= C_6(\Delta\sigma) \quad \Delta K^4 \qquad (38)$$

It should be noticed that except for a dependency of the function $C_6(\Delta\sigma)$ on the stress range, this is Paris' [6] noted power law with the exponent for ΔK equal to four.

4.4 Comparison with Experimental Data

Paris used, as an independent check for the comparison of various growth laws [6], the data of Martin and Sinclair [15], in which sheets of 24S-T3 aluminum alloy were cycled under different loadings, and the central crack length was measured at corresponding cycles. Table 1 lists the material properties of 24S-T3 as given by Martin and Sinclair. In Figure 8 the growth curve given by Paris' Power Law is fitted to the data for three different specimens tested by Martin and Sinclair at different stress levels. These are the same three specimens used by Paris and Erdogen [6] in their comparisons of growth laws. Fitting the data in Figure 8 with Paris' Law (where the exponent n=4 was used) results in a value for C_5 of

$$C_5 = 3.07 \times 10^{-22} \text{ in}^6/\text{lb}^4$$

Using this value for the function $C_6(\Delta\sigma)$ and the middle stress range of the three specimens allow the computation of a numerical value for W_p to be used in Eq. (37). This gives a value of

$$W_p = 6.47 \times 10^5 \text{ lb/in}$$

This value can be compared with the numerical value of crack extension force used by Irwin [13] for this material which is

$$G \simeq 490 \text{ lb/in}$$

These are then clearly not the same material property.

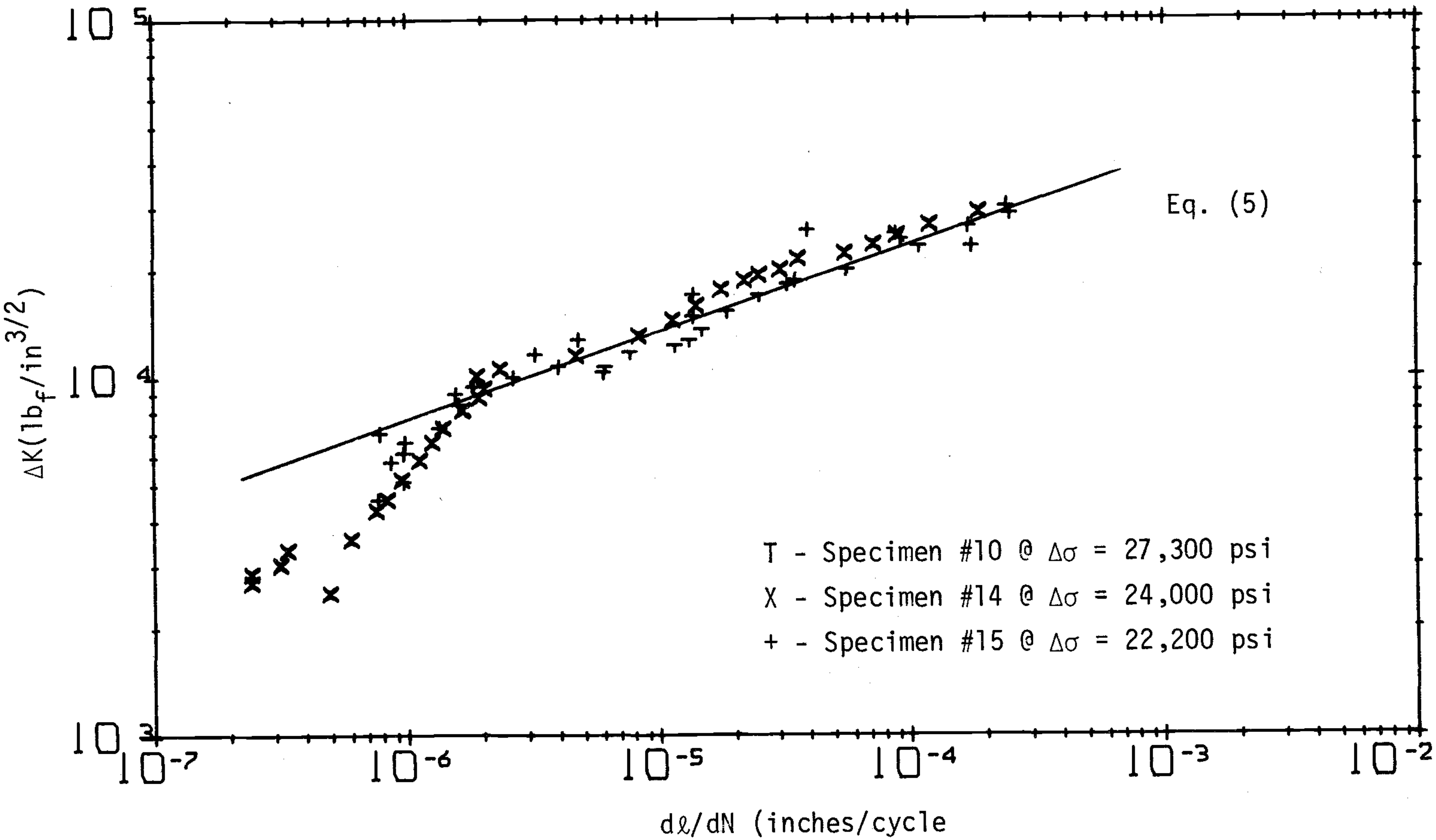

FIGURE 8

Stress intensity factor range versus crack growth rate - experimental data and fit to Eq. (5).

TABLE 1

MATERIAL PROPERTIES OF 24S-T3[7] ALUMINUM ALLOY

Tensile Strength	70,000 psi
Yield Strength @ 2% Offset	55,000 psi
Modulus of Elasticity	10,000,000 psi
Shear Modulus	3,800,000 psi
Poisson's Ratio[8]	0.316
Per Cent Elongation in 2 inches	13
Fracture Toughness[9]	8450 $\frac{\text{in-lb}}{\text{in}^3}$

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7 Modern designation is 2024-T3 Aluminum alloy.

8 Calculated from the Modulus of Elasticity and the Shear Modulus.

9 Estimated from an approximate stress-strain diagram.

CHAPTER 5

Summary and Conclusions

Paris' Power Law for fatigue crack propagation has been derived from first principles. It is believed that this is the first analytical derivation of the growth law that has been done with the generality of this derivation. The developed growth law is compared to the same experimental data which Paris used for verification when the Power Law was first proposed. The derived growth law does not account for either environmental effects or for the strain hardening effects of some materials.

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APPENDICES

APPENDIX A

Derivation of Elastic Displacements and Stresses

Evaluation of $\sigma_{yy}\Big|_{y=0}$ for $x>\ell$

Substituting the potential function as given by Eq. (18) into the stress function, Eq. (11) for y=0, $x\geq\ell$, gives

$$\sigma_{yy}\Big|_{y=0} = -2(\beta^2-1)\,\frac{\partial^2\phi}{\partial y^2}$$

$$= -\int_0^\infty \left\{ \frac{2}{\pi} \int_0^\ell \eta\, J_o(\xi\eta) \int_0^\eta \frac{P d\rho}{\sqrt{\eta^2-\rho^2}}\, d\eta \right\} \xi \cos(\xi x) d\xi \tag{A1}$$

Evaluating integrals

$$\int_0^\eta \frac{P d\rho}{\sqrt{\eta^2-\rho^2}} = P\left[\sin^{-1}(\rho/\eta)\right]\Bigg|_0^\eta = P\,\frac{\pi}{2}$$

Interchanging the order of integration on the other two terms

$$\sigma_{yy}\Big|_{y=0} = -P \int_0^{\ell} \eta \left\{ \int_0^{\infty} \xi \cos(\xi x)\, J_0(\xi\eta) d\xi \right\} d\eta$$

Observing that

$$\int_0^{\infty} \xi \cos(\xi x)\, J_0(\xi\eta) d\xi = \frac{d}{dx} \int_0^{\infty} \sin(\xi x)\, J_0(\xi\eta)\, d\xi$$

Watson [16] gives the solution of this integral as:

$$\int_0^{\infty} \sin(\xi x)\, J_0(\xi\eta) d\xi = \begin{cases} 0 & , \quad x<\eta \\ \infty & , \quad x=\eta \\ \dfrac{1}{\sqrt{x^2-\eta^2}} & , \quad x>\eta \end{cases}$$

Since for the stress being considered $|x| \geq \ell$, x must therefore be greater than η which is less than ℓ, and the last solution is the appropriate one. Therefore

$$\sigma_{yy}\Big|_{y=0} = -P \frac{d}{dx} \int_0^{\ell} \frac{\eta}{\sqrt{x^2-\eta^2}}\, d\eta$$

$$\sigma_{yy}\Big|_{y=0} = -P\,\frac{d}{dx}\left[x - \sqrt{x^2-\ell^2}\right]$$

$$\sigma_{yy}\Big|_{y=0} = P\left[\frac{x}{\sqrt{x^2-\ell^2}} - 1\right] \qquad |x|>\ell \qquad \text{(A2)}$$

<u>Evaluation of $V\big|_{y=0}$ for $x\leq\ell$</u>

Substitute the potential function, Eq. (18), into the displacement relation, Eq. (9).

$$V\Big|_{y=0} = \frac{1}{G}\,\{-\beta^2\frac{\partial\phi}{\partial y}\}$$

$$= \frac{1}{G}\,\{\frac{\beta^2}{2(\beta^2-1)}\int_0^\infty \{\frac{2}{\pi}\int_0^\ell \eta\, J_o(\xi\eta)\int_0^\eta \frac{P d\rho}{\sqrt{\eta^2-\rho^2}}\, d\eta\}\cos(\xi x)\, d\xi\}$$

(A3)

Evaluating Integrals

$$\int_0^\eta \frac{P d\rho}{\sqrt{\eta^2-\rho^2}} = P\,\frac{\pi}{2}$$

Interchanging the order of integration on the remaining two gives

$$V\Big|_{y=0} = \left\{ \frac{\beta^2}{2(\beta^2-1)} \frac{P}{G} \int_0^{\ell} \eta \left\{ \int_0^{\infty} J_0(\xi\eta) \cos(\xi x) d\xi \right\} d\eta \right\}$$

From Watson [16] the solution of the inner integral is given as:

$$\int_0^{\infty} J_0(\xi\eta) \cos(\xi x) d\xi = \begin{cases} 0 & 0<\eta<x \\ (\eta^2-x^2)^{-\frac{1}{2}} & \eta>x \end{cases}$$

Since there is no contribution from the integral when η is less than x, the limits on the outer integral must be changed such that

$$V\Big|_{y=0} = \left\{ \frac{\beta^2}{2(\beta^2-1)} \frac{P}{G} \int_x^{\ell} \frac{\eta}{\sqrt{\eta^2-x^2}} d\eta \right\}$$

$$V\Big|_{y=0} = \frac{\beta^2}{2(\beta^2-1)} \frac{P}{G} \sqrt{\ell^2-x^2}$$

since

$$\beta^2 = \frac{2(1-\nu)}{(1-2\nu)} \quad \rightarrow \quad \frac{\beta^2}{2(\beta^2-1)} = (1-\nu)$$

Therefore

$$V\Big|_{y=0} = \frac{(1-\nu)P}{G} \sqrt{\ell^2-x^2} = 2(1-\nu^2)\,\frac{P}{E}\,\sqrt{\ell^2-x^2} \qquad \text{(A4)}$$

APPENDIX B

Derivation of Dugdale Displacements

Beginning with the potential function given by Eq. (24) and the displacement function given by (evaluated at y=0) Eq. (9) we arrive at:

$$V\Big|_{y=0} = \frac{1}{G}\left[-\beta^2 \frac{\partial\phi}{\partial y}\right]$$

$$= \frac{1}{G}\left[\frac{\beta^2}{2(\beta^2-1)} \frac{2}{\pi} \int_0^\infty \{ \int_0^a \eta\, J_0(\xi\eta)q(\eta)d\eta\} \cos(\xi x)d\xi\right] \tag{B1}$$

where:

$$q(\eta) = \int_0^\eta \frac{P(\rho)d\rho}{\sqrt{\eta^2-\rho^2}}$$

Interchanging the order of integration results in

$$V\Big|_{y=0} = \frac{1}{G}\left[\frac{\beta^2}{2(\beta^2-1)} \frac{2}{\pi} \int_0^a \eta\, q(\eta)\, \{ \int_0^\infty J_0(\xi\eta) \cos(\xi x)d\xi\}\, d\eta\right] \tag{B2}$$

The inner integral is given by Watson [16] (for the case of $\eta>x$) as:

$$(\eta^2-x^2)^{-\frac{1}{2}}$$

Therefore Eq. (B2) can be rewritten (noting that because $\eta > x$ the limits must be changed) as

$$V\Big|_{y=0} = \frac{\beta^2}{2(\beta^2-1)} \frac{2}{\pi G} \int_x^a \frac{\eta\, q(\eta)}{\sqrt{\eta^2-x^2}}\, d\eta \tag{B3}$$

or in the form

$$V\Big|_{y=0} = \frac{4(1-\nu^2)}{\pi E} \int_x^a \frac{\eta\, q\,(\eta)}{\sqrt{\eta^2-x^2}}\, d\eta \tag{B4a}$$

$$q(\eta) = \int_0^\eta \frac{P(\rho)d\rho}{\sqrt{\eta^2-\rho^2}} \tag{B4b}$$

Using the pressure distribution as given by Eq. (22) the term $q(\eta)$ can be written as

$$q(\eta) = \begin{cases} \displaystyle\int_0^\eta \frac{\sigma}{\sqrt{\eta^2-\rho^2}}\, d\rho & o \le \eta \le \ell \\[2ex] \displaystyle\int_0^\ell \frac{\sigma}{\sqrt{\eta^2-\rho^2}}\, d\rho + \int_\ell^\eta \frac{(\sigma-Y)}{\sqrt{\eta^2-\rho^2}}\, d\rho & \ell \le \eta \le a \end{cases}$$

Evaluating integrals results in:

$$\int_0^{\eta} \frac{\sigma d\rho}{\sqrt{\eta^2-\rho^2}} = \sigma \sin^{-1}(\rho/\eta) \Big|_0^{\eta} = \sigma \frac{\pi}{2}$$

$$\int_0^{\ell} \frac{\sigma d\rho}{\sqrt{\eta^2-\rho^2}} = \sigma \sin^{-1}(\ell/\eta)$$

$$\int_{\ell}^{\eta} \frac{(\sigma-Y)\, d\rho}{\sqrt{\eta^2-\rho^2}} = (\sigma-Y)\left[\frac{\pi}{2} - \sin^{-1}(\ell/\eta)\right]$$

Thus

$$q(\eta) = \begin{cases} \sigma \frac{\pi}{2} & o\leq\eta\leq\ell \\ (\sigma-Y)\frac{\pi}{2} + Y \sin^{-1}(\ell/\eta) & \ell<\eta\leq a \end{cases} \qquad \text{(B5)}$$

Substituting Eq. (B5) into Eq. (B4a) results in the crack displacement given by

$$V\Big|_{y=0} = \frac{4(1-\nu^2)}{\pi E}\begin{cases} \displaystyle\int_x^{\ell} \frac{\eta\sigma\frac{\pi}{2}}{\sqrt{\eta^2-x^2}}\,d\eta + \int_{x=\ell}^{a} \frac{\eta\left[(\sigma-Y)\frac{\pi}{2} + Y\sin^{-1}(\ell/\eta)\right]d\eta}{\sqrt{\eta^2-\ell^2}} & 0 \le x \le \ell \\ \displaystyle\int_x^{a} \frac{\eta\left[(\sigma-Y)\frac{\pi}{2} + Y\sin^{-1}(\ell/\eta)\right]}{\sqrt{\eta^2-x^2}}\,d\eta & \ell < x \le a \end{cases}$$

(B6)

Evaluating integrals

$$\int_x^{\ell} \sigma\,\frac{\pi}{2}\,\frac{\eta}{\sqrt{\eta^2-x^2}}\,d\eta = \sigma\,\frac{\pi}{2}\left[\sqrt{\eta^2-x^2}\right]\Bigg|_x^a = \sigma\,\frac{\pi}{2}\sqrt{\ell^2-x^2}$$

$$\int_{x=\ell}^{a} \frac{\eta\left[(\sigma-Y)\frac{\pi}{2} + Y\sin^{-1}(\ell/\eta)\right]}{\sqrt{\eta^2-\ell^2}}\,d\eta = \ell\,\log\frac{a}{\ell}$$ [10]

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10 See Appendix E.

$$\int_x^a (\sigma-Y)\frac{\pi}{2} \frac{\eta}{\sqrt{\eta^2-x^2}}\,d\eta = (\sigma-Y)\frac{\pi}{2}\left[\sqrt{\eta^2-x^2}\,\right]\Big|_x^a = (\sigma-Y)\frac{\pi}{2}\sqrt{a^2-x^2}$$

$$\int_x^a Y\,\frac{\eta\,\sin^{-1}(\ell/\eta)}{\sqrt{\eta^2-x^2}}\,d\eta = Y\sqrt{a^2-x^2}\quad \sin^{-1}(\ell/a)$$ [11]

$$+\,Y\frac{\ell}{2}\left\{\frac{x}{\ell}\log\left(\frac{x\sqrt{a^2-\ell^2}+\ell\sqrt{a^2-x^2}}{x\sqrt{a^2-\ell^2}-\ell\sqrt{a^2-x^2}}\right) - \log\left(\frac{\sqrt{a^2-\ell^2}+\sqrt{a^2-x^2}}{\sqrt{a^2-\ell^2}-\sqrt{a^2-x^2}}\right)\right\}$$

Thus the crack displacement can be written in the form

$$V\Big|_{y=0} = \frac{4(1-\nu^2)}{\pi E}\begin{cases} V_1(x) & 0\le x\le \ell \\ V_2(x) & \ell<x\le a\end{cases}$$

where

$$V_1(x) = \sigma\,\frac{\pi}{2}\sqrt{\ell^2-x^2} + Y\ell\,\log\frac{a}{\ell}$$

---

11 See Appendix D.

$$V_2(x) = (\sigma-Y)\frac{\pi}{2}\sqrt{a^2-x^2} + Y\sqrt{a^2-x^2}\,\sin^{-1}(\ell/a)$$

$$+ Y\frac{\ell}{2}\{\frac{x}{\ell}\log(\frac{x\sqrt{a^2-\ell^2} + \ell\sqrt{a^2-x^2}}{x\sqrt{a^2-\ell^2} - \ell\sqrt{a^2-x^2}}) - \log(\frac{\sqrt{a^2-\ell^2} + \sqrt{a^2-x^2}}{\sqrt{a^2-\ell^2} - \sqrt{a^2-x^2}})\}$$

The first two terms in $V_2(x)$ can be rewritten in the form

$$-Y\sqrt{a^2-x^2}\ \{(1-\frac{\sigma}{Y})\frac{\pi}{2} + \sin^{-1}(\ell/a)\}$$

which from Eq. (F3) we know to be equal to zero. Therefore the displacements can be written in the final form of:

$$V\Big|_{y=0} = \frac{4(1-\nu^2)Y}{\pi E}\,\ell \begin{cases} \frac{\sigma}{Y}\frac{\pi}{2\ell}\sqrt{\ell^2-x^2} + \log\frac{a}{\ell} & 0\le x\le\ell \\ \frac{1}{2}\{\frac{x}{\ell}\log(\frac{x\sqrt{a^2-\ell^2} + \ell\sqrt{a^2-x^2}}{x\sqrt{a^2-\ell^2} - \ell\sqrt{a^2-x^2}} \\ \quad -\log(\frac{\sqrt{a^2-\ell^2} + \sqrt{a^2-x^2}}{\sqrt{a^2-\ell^2} - \sqrt{a^2-x^2}})\} & \ell<x<a \end{cases}$$

(B7)

APPENDIX C

Derivation of Dugdale Stresses

Beginning with Eq. (A1)

$$\sigma_{yy}\Big|_{y=0} = - \int_0^\infty \{ \frac{2}{\pi} \int_0^\ell \eta \, J_o(\xi\eta) \int_0^\eta \frac{P(\rho)d\rho}{\sqrt{\eta^2-\rho^2}} d\eta\} \, \xi \cos(\xi\eta)d\xi$$

(A1)

Interchanging the order of integration and evaluating (see Appendix A) gives

$$\sigma_{yy}\Big|_{y=0} = - \frac{2}{\pi}\frac{d}{dx} \int_0^a \{ \frac{\eta}{\sqrt{x^2-\eta^2}} \int_0^\eta \frac{P(\rho)d\rho}{\sqrt{\eta^2-\rho^2}} \} \, d\eta$$

(C1)

Observing that the pressure distribution given by Eq. (22) can also be written as

$$P(x) = \begin{cases} \sigma & 0 \le x \le a \\ -Y & \ell \le x \le a \end{cases}$$

Substitute this into Eq. (C1)

$$\sigma_{yy} = -\frac{2}{\pi}\frac{d}{dx}\left\{ \int_0^a \left[\frac{\eta}{\sqrt{x^2-\eta^2}} \int_0^\eta \frac{\sigma d\rho}{\sqrt{\eta^2-\rho^2}} \right] d\eta \right.$$

$$\left. + \int_\ell^a \left[\frac{\eta}{\sqrt{x^2-\eta^2}} \int_0^\eta \frac{-Y d\rho}{\sqrt{\eta^2-\rho^2}} \right] d\eta \right\} \qquad \text{(C2)}$$

Evaluating the integrals over the variable ρ gives

$$\sigma_{yy} = -\frac{2}{\pi}\frac{d}{dx}\left\{ \sigma \int_0^a \frac{\eta}{\sqrt{x^2-\eta^2}} \left[\sin^{-1}(\rho/\eta) \right]_0^\eta d\eta \right.$$

$$\left. -Y \int_\ell^a \frac{\eta}{\sqrt{x^2-\eta^2}} \left[\sin^{-1}(\rho/\eta) \right]_\ell^\eta d\eta \right.$$

$$= -\frac{2}{\pi}\frac{d}{dx}\left\{ \sigma \int_0^a \frac{\eta}{\sqrt{x^2-\eta^2}} \left[\pi/2-0 \right] d\eta \right.$$

$$\left. -Y \int_\ell^a \frac{\eta}{\sqrt{x^2-\eta^2}} \left[\pi/2 - \sin^{-1}(\ell/\eta) \right] d\eta \right\}$$

(C3)

$$= -\frac{2}{\pi}\frac{d}{dx}\left\{ \sigma \frac{\pi}{2} \int_{o}^{a} \frac{\eta}{\sqrt{x^2-\eta^2}}\, d\eta \right.$$

$$\left. -Y \int_{\ell}^{a} \frac{\eta}{\sqrt{x^2-\eta^2}}\, d\eta + Y \int_{\ell}^{a} \frac{\eta \sin^{-1}(\ell/\eta)}{\sqrt{x^2-\eta^2}}\, d\eta \right\} \tag{C4}$$

where in the last equation the evaluated integrals in Eq. (C3) were broken into three separate integrals. Evaluating the integrals in Eq. (C4) results in

$$\sigma_{yy} = -\frac{2}{\pi}\frac{d}{dx}\left\{ \sigma \frac{\pi}{2} \left[-\sqrt{x^2-\eta^2} \right]_{o}^{a} - Y \frac{\pi}{2} \left[-\sqrt{x^2-\eta^2} \right]_{\ell}^{a} \right.$$

$$\left. +Y \int_{\ell}^{a} \frac{\eta \sin^{-1}(\ell/\eta)}{\sqrt{x^2-\eta^2}}\, d\eta \right\}$$

$$= -\frac{2}{\pi}\frac{d}{dx}\left\{ -\sigma \frac{\pi}{2} \left[\sqrt{x^2-a^2} - x \right] + Y\frac{\pi}{2} \left[\sqrt{x^2-a^2} - \sqrt{x^2-\ell^2} \right] \right.$$

$$+ Y \int_{\ell}^{a} \frac{\eta \sin^{-1}(\ell/\eta)}{\sqrt{x^2-\eta^2}}\, d\eta \tag{C5}$$

To evaluate the remaining integral first integrate by parts

$$\int_{\ell}^{a} \frac{\eta \sin^{-1}(\ell/\eta)}{\sqrt{x^2-\eta^2}} \, d\eta = \Big[- \sqrt{x^2-\eta^2} \sin^{-1}(\ell/\eta) \Big]_{\ell}^{a}$$

$$- \int_{\ell}^{a} \frac{\ell}{\eta} \sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}} \, d\eta \qquad \text{(C6)}$$

The remaining integral is given by Petit Bois [17] as:

$$\int_{\ell}^{a} \frac{1}{\eta} \sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}} \, d\eta = - \Big[\tan^{-1} \sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}} - \frac{x}{\ell} \tan^{-1}\Big(\frac{\ell}{x} \sqrt{\frac{x^2-\eta^2}{\eta^2-x^2}}\Big) \Big]_{\ell}^{a}$$

$$= - \Big[\tan^{-1} \sqrt{\frac{x^2-a^2}{a^2-\ell^2}} - \frac{x}{\ell} \tan^{-1}\Big(\frac{\ell}{x} \sqrt{\frac{x^2-a^2}{a^2-x^2}}\Big)$$

$$+ \frac{\pi}{2} \Big(\frac{x}{\ell} - 1\Big) \Big] \qquad \text{(C7)}$$

Substituting the evaluated integral into Eq. (C5) gives (after rearranging and cancelling terms) the stress distribution as

$$\sigma_{yy} = -\frac{2}{\pi}\frac{d}{dx}\left\{-\frac{\pi}{2}\sqrt{x^2-a^2}\left[(\sigma-Y)+\frac{2}{\pi}Y\sin^{-1}(\ell/a)\right]-\frac{\pi}{2}\sigma x\right.$$

$$\left.-Y\ell\{\sqrt{\frac{x^2-a^2}{a^2-\ell^2}}-\frac{x}{\ell}\tan^{-1}(\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-x^2}})+\frac{\pi}{2}(\frac{x}{\ell}-1)\}\right\} \tag{C8}$$

It should be observed from Appendix F, Eq. (F3), that because of the finiteness condition on the stress that the term in the [ ] brackets must equal zero. Therefore Eq. (C8) can be written as

$$\sigma_{yy} = \frac{d}{dx}\{\sigma x + Y\ell(\frac{x}{\ell}-1) + Y\ell\frac{2}{\pi}\left[\tan^{-1}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}}\right.$$

$$\left.-\frac{x}{\ell}\tan^{-1}(\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}})\right]\}$$

Carrying out the differentiation with the observation that

$$\frac{d}{dx}\left\{\tan^{-1}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}}\right\} = \frac{x}{x^2-\ell^2}\sqrt{\frac{a^2-\ell^2}{x^2-a^2}}$$

$$\frac{d}{dx}\{\tan^{-1}(\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}})\} = \left[\frac{\ell}{x^2-a^2}-\frac{\ell}{x^2}\right]\frac{x^2}{a^2}\frac{x^2-a^2}{x^2-\ell^2}\sqrt{\frac{a^2-\ell^2}{x^2-a^2}}$$

results in a stress distribution of

$$\sigma_{yy} = \{(\sigma+Y) - Y\frac{2}{\pi}\tan^{-1}(\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}})$$

$$+ Y\ell\frac{2}{\pi}\left[\frac{x}{x^2-\ell^2} - (\frac{x}{x^2-a^2} - \frac{1}{x})\frac{x^2}{a^2}\frac{x^2-a^2}{x^2-\ell^2}\right]\sqrt{\frac{a^2-\ell^2}{x^2-a^2}}\}$$

Expanding the term in the square brackets gives

$$\left[\frac{x}{x^2-\ell^2} - (\frac{x^2}{a^2}\frac{x}{x^2-\ell^2} - \frac{x}{a^2}\frac{x^2-a^2}{x^2-\ell^2})\right]\sqrt{\frac{a^2-\ell^2}{x^2-a^2}}$$

Factoring out the common $x/(x^2-\ell^2)$ term leaves

$$\left[1 - \frac{x^2}{a^2} + \frac{x^2-a^2}{a^2}\right]\frac{x}{x^2-\ell^2}\sqrt{\frac{a^2-\ell^2}{x^2-a^2}}$$

which we see by inspection is equal to zero. Therefore the solution for the stress distribution of Problem B is

$$\sigma_{yy} = \{(\sigma+Y) - Y\frac{2}{\pi}\tan^{-1}(\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}})\} \qquad \text{(C9)}$$

To get the final form for the stress distribution of the entire problem we need to add the uniform stress distribution for Problem A of $-\sigma$ resulting in

$$\sigma_{yy}(x) = Y \{ 1 - \frac{2}{\pi} \tan^{-1}(\frac{\ell}{x} \sqrt{\frac{x^2-a^2}{a^2-\ell^2}})\} \qquad x \geq a$$

(C10)

As a check of the reasonableness of this stress distribution lets look at the stresses at two points where the stresses are known.

1) $\sigma_{yy}(x=a) = Y$

2) $\sigma_{yy}(x=\infty) = \sigma$

Substituting x=a into Eq. (C10) results in $\sigma_{yy}(a) = Y$. Therefore Eq. (C10) gives the proper stress at x=a. To look at the stresses at x=∞ first rewrite Eq. (C10) in the form

$$\sigma_{yy}(x) = Y \{ 1 - \frac{2}{\pi} \tan^{-1}(\ell \sqrt{\frac{1-(a/x)^2}{a^2-\ell^2}})\}$$

After taking the limit as $x \to \infty$ and rearranging, gives

$$\sigma_{yy}(x=\infty) = Y \left\{ 1 - \frac{2}{\pi} \tan^{-1}(\frac{\ell}{\sqrt{a^2-\ell^2}}) \right\} \qquad \text{(C11)}$$

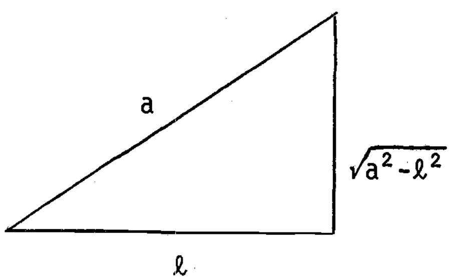
Referring to Figure 9 it is seen that the arctan can be rewritten in the form

$$\tan^{-1}\left(\frac{\ell}{\sqrt{a^2-\ell^2}}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{a^2-\ell^2}}{\ell}\right) = \frac{\pi}{2}\left(1-\frac{\sigma}{Y}\right)$$

When this is substituted into Eq. (C11) it is readily seen that the equation reduces to

$$\sigma_{yy}(x{=}\infty) = \sigma$$

which is the correct value.



$$\frac{\ell}{a} = \cos\left(\frac{\sigma\pi}{Y2}\right)$$

$$\frac{\sqrt{a^2-\ell^2}}{\ell} = \tan\left(\frac{\sigma\pi}{Y2}\right)$$

FIGURE 9

Triganometric relations

APPENDIX D

Evaluation of the Displacement Integral

$$I = \int_x^a \frac{\eta \sin^{-1}(\ell/\eta)}{\sqrt{\eta^2-x^2}}\, d\eta \qquad \text{(D1)}$$

Integrating by parts

$$U = \sin^{-1}(\ell/\eta) \qquad V = \sqrt{\eta^2-x^2}$$

$$dU = \frac{-\ell}{\eta\sqrt{\eta^2-\ell^2}} \qquad dV = \frac{\eta}{\sqrt{\eta^2-x^2}}\, d\eta$$

gives

$$I = \sqrt{\eta^2-x^2}\, \sin^{-1}(\ell/\eta) \Big|_x^a + \ell \int_x^a \frac{1}{\eta} \sqrt{\frac{\eta^2-x^2}{\eta^2-\ell^2}}\, d\eta$$

$$= \sqrt{a^2-x^2}\ \sin^{-1}(\ell/a) + \ell \int_x^a \frac{1}{\eta} \sqrt{\frac{\eta^2-x^2}{\eta^2-\ell^2}}\, d\eta \qquad \text{(D2)}$$

Letting Q represent the remaining integral or

$$Q = \int_x^a \frac{1}{\eta} \sqrt{\frac{\eta^2-x^2}{\eta^2-\ell^2}}\, d\eta \qquad \text{(D3)}$$

Multiplying the radical by the square root of -1 or,

$$\sqrt{\frac{\eta^2-x^2}{\eta^2-\ell^2}} \qquad = \qquad i\sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}}$$

Thus

$$Q = i\int_x^a \frac{1}{\eta}\sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}}\, d\eta$$

Petit Bois [17] gives the solution of this integral as:

$$Q = i\left[\tan^{-1}\sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}} - \frac{x}{\ell}\tan^{-1}\frac{\ell}{x}\sqrt{\frac{x^2-\eta^2}{\eta^2-\ell^2}}\right]_x^a$$

$$= i\left[\tan^{-1}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}} - \frac{x}{\ell}\tan^{-1}\frac{\ell}{x}\sqrt{\frac{x^2-a^2}{a^2-\ell^2}}\right]$$

Since $x < a$ the radicals must be imaginary or

$$Q = i\left[\tan^{-1}\left(i\sqrt{\frac{a^2-x^2}{a^2-\ell^2}}\right) - \frac{x}{\ell}\tan^{-1}\left(i\,\frac{\ell}{x}\sqrt{\frac{a^2-x^2}{a^2-\ell^2}}\right)\right]$$

From the calculus of complex variables we know that the inverse tangent of a purely imaginary number is

$$\tan^{-1}(iy) = \frac{i}{2}\log\frac{1+y}{1-y} = \frac{i}{2}\{\log(1+y) - \log(1-y)\}$$

Therefore,

$$Q = i\left[\frac{i}{2}\log\left(1 + \sqrt{\frac{a^2-x^2}{a^2-\ell^2}}\right) - \frac{i}{2}\log\left(1 - \sqrt{\frac{a^2-x^2}{a^2-\ell^2}}\right)\right.$$

$$\left. - \frac{x}{\ell}\,\frac{i}{2}\left\{\log\left(1 + \frac{\ell}{x}\sqrt{\frac{a^2-x^2}{a^2-\ell^2}}\right) - \log\left(1 - \frac{\ell}{x}\sqrt{\frac{a^2-x^2}{a^2-\ell^2}}\right)\right\}\right]$$

$$Q = \tfrac{1}{2}\left\{\frac{x}{\ell}\log\left(\frac{x\sqrt{a^2-\ell^2} + \ell\sqrt{a^2-x^2}}{x\sqrt{a^2-\ell^2} - \ell\sqrt{a^2-x^2}}\right) - \log\left(\frac{\sqrt{a^2-\ell^2} + \sqrt{a^2-x^2}}{\sqrt{a^2-\ell^2} - \sqrt{a^2-x^2}}\right)\right\} \qquad \text{(D4)}$$

Gathering terms to evaluate the original integral gives the result of

$$I = \sqrt{a^2-x^2}\,\sin^{-1}(\ell/a) + \frac{\ell}{2}\left\{\frac{x}{\ell}\log\left(\frac{x\sqrt{a^2-\ell^2} + \ell\sqrt{a^2-x^2}}{x\sqrt{a^2-\ell^2} - \ell\sqrt{a^2-x^2}}\right)\right.$$

$$\left. - \log\left(\frac{\sqrt{a^2-\ell^2} + \sqrt{a^2-x^2}}{\sqrt{a^2-\ell^2} - \sqrt{a^2-x^2}}\right)\right\} \qquad \text{(D5)}$$

APPENDIX E

Evaluation of Crack Tip Displacement (x=ℓ)

The displacement at x=ℓ is given, Eq. (B6a) by

$$V(\ell,0) = \frac{4(1-\nu^2)}{\pi E} \int_{x=\ell}^{a} \frac{\eta \left[(\sigma-Y)\frac{\pi}{2} + Y \sin^{-1}(\ell/\eta)\right]}{\sqrt{\eta^2-\ell^2}} \, d\eta \qquad \text{(E1)}$$

Evaluating the integrals,

$$\int_{\ell}^{a} (\sigma-Y)\frac{\pi}{2} \frac{\eta}{\sqrt{\eta^2-\ell^2}} \, d\eta = (\sigma-Y)\frac{\pi}{2} \left[\sqrt{\eta^2-\ell^2}\right]_{\ell}^{a} = (\sigma-Y)\frac{\pi}{2}\sqrt{a^2-\ell^2}$$

$$\int_{\ell}^{a} Y\frac{\eta \sin^{-1}(\ell/\eta)}{\sqrt{\eta^2-\ell^2}} \, d\eta = Y\sqrt{\eta^2-\ell^2} \sin^{-1}(\ell/\eta)\Big|_{\ell}^{a} + \ell Y \int_{\ell}^{a} \frac{1}{\eta} \, d\eta$$

$$= Y\sqrt{a^2-\ell^2} \sin^{-1}(\ell/a) + \ell Y \log\left(\frac{a}{\ell}\right)$$

combining terms,

$$V(\ell,0) = \frac{4(1-\nu^2)}{\pi} \frac{Y}{E}\ell \{(\frac{\sigma}{Y} -1)\frac{\pi}{2} \sqrt{a^2-\ell^2} + \sqrt{a^2-\ell^2} \sin^{-1}(\ell/a)+\log(a/\ell)\}$$

and rewriting, gives

$$V(\ell,0) = \frac{4(1-\nu^2)}{\pi} \frac{Y}{E}\ell \{\sqrt{a^2-\ell^2} \Big[(\frac{\sigma}{Y} -1)\frac{\pi}{2} + \sin^{-1}(\ell/a) \Big] + \log(a/\ell)\}$$

From Eq. (F3) the terms in the [ ] brackets is equal to zero.

Thus,

$$V(\ell,0) = \frac{4(1-\nu^2)}{\pi} \frac{Y}{E} \ell \{\log(a/\ell)\} \qquad \text{(E2)}$$

APPENDIX F

Evaluation of Plastic Zone to Length Ratio - a/ℓ

It is postulated that, since the stresses must be finite everywhere, at the physical crack tip (including the plastic zone) the stress intensity factor must equal zero. If this is not so we see from Eq. (19a) that the stress goes to infinity at the crack tip. Therefore this condition will be used to find the physical crack length 'a'. The stress intensity factor is given by [18]:

$$K = \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} \sigma_y(x,0) \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx$$

where: $\sigma_y(x,0)$ is the stress distribution evaluated at $y=0$.

a is the physical crack length.

Let $\sigma_y(x,0)$ equal the pressure distribution term $P(x)$. Thus

$$K = \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} P(x) \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx \qquad \text{(F1)}$$

where $P(x)$ is given by Eq. (22). Substituting this pressure distribution into the above gives

$$P(x) = \begin{cases} \sigma & 0 \leq x \leq \ell \\ \sigma - Y & \ell < x \leq a \end{cases} \tag{22}$$

$$K = \frac{1}{\sqrt{\pi a}} \left\{ \int_{-a}^{-\ell} (\sigma - Y) \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx + \int_{-\ell}^{+\ell} \sigma \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx \right.$$

$$\left. + \int_{+\ell}^{+a} (\sigma - Y) \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx \right\} \tag{F2}$$

Evaluating integrals:

$$\int_{-a}^{-\ell} \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx = \left[a \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} \right]_{-a}^{-\ell}$$

$$= a\frac{\pi}{2} - a \sin^{-1}(\ell/a) - \sqrt{a^2 - \ell^2}$$

$$\int_{-\ell}^{+\ell} \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx = \left[a \sin^{-1}(x/a) - \sqrt{a^2 - x^2} \right]_{-\ell}^{+\ell}$$

$$= 2a \sin^{-1}(\ell/a)$$

$$\int_{+\ell}^{+a} \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx \quad = \quad a\frac{\pi}{2} + \sqrt{a^2-\ell^2} - a \sin^{-1}(\ell/a)$$

Substituting these into Eq. (F2) gives

$$K = \frac{1}{\sqrt{\pi a}} \quad \{(\sigma-Y) \Big[(a\frac{\pi}{2} - a \sin^{-1}(\ell/a) - \sqrt{a^2-\ell^2})$$

$$+ (a\frac{\pi}{2} - a \sin^{-1}(\ell/a) + \sqrt{a^2-\ell^2}) \Big]$$

$$+ \sigma(2a \sin^{-1}(\ell/a))\}$$

$$= \frac{1}{\sqrt{\pi a}} \{(\sigma-Y) \Big[a\pi - 2a \sin^{-1}(\ell/a) \Big] + \sigma \Big[2a \sin^{-1}(\ell/a) \Big] \}$$

$$= \frac{1}{\sqrt{\pi a}} \{(\sigma-Y)a\pi + Y\ 2a \sin^{-1}(\ell/a)\}$$

Let the stress intensity factor equal zero.

$$0 = \frac{1}{\sqrt{\pi a}} \quad \{(\sigma-Y)a\pi + Y\ 2a \sin^{-1}(\ell/a)\}$$

Cancelling terms and simplifying results in

$$\{(\sigma-Y)\frac{\pi}{2} + Y \sin^{-1}(\ell/a)\} \quad = 0 \qquad \text{(F3)}$$

Solving for the ratio ℓ/a gives

$$\ell/a = \sin\left[\left(1 - \frac{\sigma}{Y}\right)\frac{\pi}{2}\right] \qquad \text{(F4)}$$

Thru the use of trigonometric identities this can be transformed into the result

$$\ell/a = 1 - 2\sin^2\left[\frac{\sigma}{Y}\frac{\pi}{4}\right]$$

which is the result that Dugdale [2] derived in 1959 thru similar reasoning. This can also be transformed into

$$\ell/a = \cos\left[\frac{\sigma}{Y}\frac{\pi}{2}\right] \qquad \text{(F5)}$$

APPENDIX G

Evaluation of the Plastic Zone Volume

$$I_p = \int_{\ell}^{a} V(x)\, dx \qquad \text{(G1)}$$

$$= \int_{\ell}^{a} \left\{ \frac{4(1-\nu^2)}{\pi E} \int_{x}^{a} \frac{\eta\, q(\eta)}{\sqrt{\eta^2 - x^2}}\, d\eta \right\} dx \qquad \text{(G2)}$$

Where the displacement distribution $V(x)$ as given by Eq. (B4a) was used in the second equation. Interchanging the order of integration with the reminder that the limits of integration must also be changed gives

$$I_p = \frac{4(1-\nu^2)}{\pi E} \int_{\ell}^{a} \eta\, q(\eta) \int_{\ell}^{\eta} \frac{dx}{\sqrt{\eta^2 - x^2}}\, d\eta$$

$$= \frac{4(1-\nu^2)}{\pi E} \int_{\ell}^{a} \eta\, q(\eta) \left\{ \sin^{-1}(x/\eta) \Big|_{\ell}^{\eta} \right\} d\eta \qquad x^2 < \eta^2$$

$$I_p = \frac{4(1-\nu^2)}{\pi E} \int_{\ell}^{a} \eta\, q(\eta) \left\{ \pi/2 - \sin^{-1}(\ell/\eta) \right\} d\eta \qquad \text{(G3)}$$

Note that the value in brackets can also be written in terms of the inverse cosine as

$$\frac{\pi}{2} - \sin^{-1}(\ell/\eta) - \cos^{-1}(\ell/\eta)$$

Substituting the above identity into the function $q(\eta)$ as given by Eq. (B5) for $\ell < \eta \leq a$ results in

$$q(\eta) = (\sigma - Y)\frac{\pi}{2} + Y \sin^{-1}(\ell/\eta) \qquad \text{(B5)}$$

$$= \sigma\frac{\pi}{2} - Y \{\frac{\pi}{2} - \sin^{-1}(\ell/\eta)\}$$

$$= Y \left[\frac{\sigma}{Y}\frac{\pi}{2} - \cos^{-1}(\ell/\eta)\right]$$

Making the substitution for $q(\eta)$ as given above into the volume integral given by Eq. (G3) leads to

$$I_p = \frac{4(1-\nu^2)}{\pi E} \int_{\ell}^{a} \eta \; Y \left[\frac{\sigma\pi}{Y2} - \cos^{-1}(\ell/\eta)\right] \cos^{-1}(\ell/\eta) \, d\eta$$

$$I_p = \frac{4(1-\nu^2)Y}{\pi E} \int_{\ell}^{a} \{\frac{\sigma\pi}{Y2} \eta \cos^{-1}(\ell/\eta) - \eta \left[\cos^{-1}(\ell/\eta)\right]^2 \} d\eta$$

Breaking this integral into two integrals and evaluating (see Appendix H) results in

$$I_p = \frac{4(1-\nu^2)Y}{\pi E} \left[\frac{\sigma\pi}{Y2} \frac{1}{2} \{a^2 \cos^{-1}(\ell/a) - \ell\sqrt{a^2-\ell^2}\} \right.$$

$$\left. - \{\tfrac{1}{2}a^2 \left[\cos^{-1}(\ell/a)\right]^2 - \ell \cos^{-1}(\ell/a) \sqrt{a^2-\ell^2} - \ell^2 \log(\ell/a)\} \right]$$

$$I_p = \frac{2(1-\nu^2)Y}{\pi E} \left[\frac{\sigma\pi}{Y2} a^2 \cos^{-1}(\ell/a) - \frac{\sigma\pi}{Y2} \ell\sqrt{a^2-\ell^2} - a^2\left[\cos^{-1}(\ell/a)\right]^2 \right.$$

$$\left. + 2\ell \cos^{-1}(\ell/a) \sqrt{a^2-\ell^2} + 2\ell^2 \log(\ell/a) \right] \quad \text{(G4)}$$

Using the condition that the stress intensity factor must remain finite at the crack, as applied in Appendix F Eq. (F5)

$$\ell/a = \cos\left(\frac{\sigma\pi}{Y2}\right) \quad \text{(F5)}$$

therefore,

$$\cos^{-1}(\ell/a) = \frac{\sigma\pi}{Y2}$$

Substituting this condition into Eq. (G4) leads to

$$I_p = \frac{2(1-\nu^2)Y}{\pi E} \left\{ \frac{\sigma\pi}{Y2} \ell\sqrt{a^2-\ell^2} - 2\ell^2 \log(\ell/a) \right\} \quad \text{(G5)}$$

Referring to Figure 9 and using trigonometric relations it is possible to transform Eq. (G5) into the final form of

$$I_p = \frac{2(1-\nu^2)\ell^2 Y}{\pi E} \left\{\frac{\sigma\pi}{Y2} \tan\left(\frac{\sigma\pi}{Y2}\right) - 2 \log \left[\cos\left(\frac{\sigma\pi}{Y2}\right)\right]\right\} \qquad \text{(G6)}$$

APPENDIX H

Evaluation of the Volume Integrals

$$\int_{\ell}^{a} \eta \cos^{-1}(\ell/\eta)\, d\eta = I_1 \qquad \text{(H1)}$$

Making the substitution $x = \ell/\eta$ $\qquad d\eta = -\ell \dfrac{dx}{x^2}$

$$I_1 = -\ell^2 \int \frac{\cos^{-1} x}{x^3}\, dx$$

This integral is given by Dwight [19] as

$$I_1 = -\ell^2 \left[-\frac{1}{2x^2} \cos^{-1} x + \frac{\sqrt{1-x^2}}{2x} \right]$$

Replacing the original variable results in

$$I_1 = \tfrac{1}{2} \left[\eta^2 \cos^{-1}(\ell/\eta) - \ell\sqrt{\eta^2\ \ell^2} \right] \Bigg|_{\ell}^{a}$$

$$I_1 = \tfrac{1}{2} \left[a^2 \cos^{-1}(\ell/a) - \ell\sqrt{a^2-\ell^2} \right] \qquad \text{(H2)}$$

$$\int_{\ell}^{a} \eta \left[\cos^{-1}(\ell/\eta)\right]^2 d\eta = I_2 \qquad \text{(H3)}$$

Again making the substitution $x = \ell/\eta$ results in

$$I_2 = -\ell^2 \int \frac{\left[\cos^{-1} x\right]^2}{x^3} dx$$

Integrating by parts $\quad U = \left[\cos^{-1} x\right]^2 \qquad V = -\frac{1}{2x^2}$

$$dU = -\frac{2\cos^{-1} x}{\sqrt{1-x^2}} dx \qquad dV = \frac{dx}{x^3}$$

$$I_2 = -\ell^2 \left[-\frac{\left[\cos^{-1} x\right]^2}{2x^2} - \int \frac{\cos^{-1} x}{x^2\sqrt{1-x^2}} dx \right]$$

Integrating the remaining integral by parts

$$I_3 = \int \frac{\cos^{-1} x}{x^2 \sqrt{1-x^2}} dx \qquad U = \cos^{-1} x \qquad V = -\frac{\sqrt{1-x^2}}{x}$$

$$dU = \frac{-1}{\sqrt{1-x^2}} \qquad dV = \frac{dx}{x\sqrt{1-x^2}}$$

$$= -\cos^{-1} x \frac{\sqrt{1-x^2}}{x} - \int \frac{dx}{x}$$

$$I_3 = -\cos^{-1} x \frac{\sqrt{1-x^2}}{x} - \log x$$

Substituting into I_2 results in

$$I_2 = -\ell^2 \left[- \frac{\left[\cos^{-1} x\right]^2}{2x^2} + \cos^{-1} x \ \frac{\sqrt{1-x^2}}{x} + \log x \right]$$

Reverting to the original variable η and replacing limits gives

$$I_2 = \left[\tfrac{1}{2}\,\eta^2 \left[\cos^{-1}(\ell/\eta)\right]^2 - \ell \cos^{-1}(\ell/\eta)\sqrt{\eta^2-\ell^2} - \ell^2 \log(\ell/\eta) \right] \Bigg|_{\ell}^{a}$$

$$I_2 = \left[\tfrac{1}{2}\, a^2 \left[\cos^{-1}(\ell/a)\right]^2 - \ell \cos^{-1}(\ell/a)\sqrt{a^2-\ell^2} - \ell^2 \log(\ell/a) \right] \qquad \text{(H4)}$$