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In this paper we develop many of the construction techniques that are widely used by homotopy theorists and use them in two problems that come out of geometric topology. The first problem is a generalization of the topological dunce hat. The crucial properties of the dunce hat are that it is contractible but not collapsible, in the sense of J. H. C. Whitehead. We describe for each non-negative integer $n = 0, 1, 2, \ldots$, a polyhedral space D^n of dimension n where D^2 is the topological dunce hat, and prove that for all even dimensions $2n$, the space D^{2n} is contractible but not collapsible. We consequently call each D^{2n}, for $n = 0, 1, 2, \ldots$, a 2n-dimensional dunce hat.

The other problem is somewhat unusual in the sense that it was originally an attempt to prove an old conjecture of R. D. Edwards that was announced in 1978. We present a careful development of an intricate construction technique that apparently became for several researchers a methodolgy for approaching Edward's conjecture. In chapter four we give strong evidence that this methodology is flawed and cannot be used to give the desired result.

Homotopy Construction Techniques applied to the Cell Like Dimension Raising Problem and to Higher Dimensional Dunce Hats

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1. DIMENSION THEORIES

The problems addressed in this dissertation are deeply involved with the concept of dimension, and so we begin by including a brief summary of the theory of topological dimension.

The first definition presented here is the large inductive dimension first published by Čech in 1931.

1.1.1. Definition. For every normal space X, the *large inductive dimension* of X is an integer n where n is greater than or equal to -1, denoted $IndX = n$, or is infinite, denoted $IndX = \infty$, and is assigned according to the following rules:

1) $IndX = -1$ if and only if $X = \emptyset$.

2) $IndX \leq n$ for $0 \leq n < \infty$ if for each closed set $A \subset X$ and every open set $U \subset X$ with $A \subset U$ there exists an open set $V \subset X$ such that $A \subset V \subset \bar{V} \subset U$ with the $Ind(BdV) \leq n-1$.

3) The $IndX = n$ if the $IndX \leq n$ and the $IndX > n-1$.

4) The $IndX = \infty$ if the $IndX > n$ for each $n \in \{-1, 0, 1, \ldots\}$.

A closely related definition of dimension is the *small inductive dimension* of a space X, denoted $indX$, first presented by Menger and Urysohn.

The definition of small inductive dimension is identical to that of the large inductive dimension except the arbitrary closed subset A in the second rule is replaced by an arbitrary point $a \in X$.

The second dimension theory which will now be presented, known as covering dimension, has its roots in an early paper by Lebesgue in 1911, and was formally defined by Čech in 1933.

1.1.2. Definition. Let $\mathcal{U}$ be a collection of subsets from a set X. For any subset $A \subset X$ the *order of A in $\mathcal{U}$* will be the largest number n of elements of $\mathcal{U}$ which contain some point $x \in A$, and will be denoted by the $ord_A\mathcal{U} = n$. If $A = \{x\}$ for some $x \in X$ then the *order of x in $\mathcal{U}$* will be denoted by the $ord_x\mathcal{U} = n$. If no such largest integer exists, then A will be said to have infinite order in $\mathcal{U}$, and will be denoted by the $ord_A\mathcal{U} = \infty$. The *order of $\mathcal{U}$* will be defined and denoted by the $ord\mathcal{U} = sup\{ord_x\mathcal{U} | x \in X\}$.

1.1.3. Definition. For every paracompact space X, the *covering dimension* of X is an integer $n \geq -1$, denoted by the $dimX = n$ or is said to be infinite, denoted by the $dimX = \infty$, which is assigned according to the following rules:

1) The $dimX = -1$ if and only if the space $X = \emptyset$.

2) The $dimX \leq n, n \in \{0, 1, 2 \ldots\}$, if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$, with the $ord\mathcal{V} \leq n + 1$, which also covers X.

3) The $dimX = n$ if the $dimX \leq n$ and if the $dimX > n - 1$.

4) The $dimX = \infty$ if the $dimX > n$ for each $n \in \{-1, 0, 1, \ldots\}$.

1.1.4. The Coincidence Theorem. [Katětov (1952), Morita (1954)] *For every separable metric space X, the $indX = IndX = dimX$.*

The proof of The Coincidence Theorem can be found in [E].

A third dimension theory uses the concepts of essential families, and agrees with inductive dimension and covering dimension on finite dimensional spaces. The dimension theory based on essential families however has the property that it can separate infinite dimensional spaces into different categories.

1.1.5. Definition. [R-S-W] Let X be a separable metric space and Γ be an indexing set. A family $F = \{(A_k, B_k)|k \in \Gamma\}$ is *essential* in X if for all $k \in \Gamma$, (A_k, B_k) is a pair of disjoint closed sets in X such that if S_k separates A_k and B_k in X, then $\bigcap_{k\in\Gamma} S_k \neq \emptyset$. We say F is an α *-essential family* if card$(\Gamma) = \alpha$.

1.1.6. Theorem. *The $dimX \geq n$ if and only if X has an n-essential family.*

Proof. Assume $dimX \leq n-1$ and if $F = \{(A_k, B_k)|k = 1, \ldots, n\}$ is a set of n disjoint closed sets in X, then there exists a partition L_1 such that L_1 separates (A_1, B_1) and $dim(L_1 \cap X) \leq (n-1) - 1 = n-2$. This follows from the second separation theorem [E]. Thus there exists a partition L_2 such that L_2 separates (A_2, B_2) and $dim(L_2 \cap (L_1 \cap X)) \leq (n-2) - 1 = n-3$.

Continuing inductively, there exists partitions $L_1, \ldots, L_{n+1}$ such that $dim(L_1 \cap L_2 \cap \cap \ldots \cap L_{n+1}) \leq (n-1) - n = -1$. Hence $\bigcap_{i=1}^{n} L_i = \emptyset$. Thus if X has an n-essential family then $dimX > n-1$ and so $dimX \geq n$.

Now assume for all families $F = \{(A_k, B_k)\}|k = 1, \ldots, n\}$ of pairs of disjoint closed sets, there exists a collection of closed sets L $= \{L_i|i = 1, \ldots, n\}$ where L_i separates A_i from B_i and $\bigcap_{i=1}^{n} L_i = \emptyset$. Consider an open cover $\{U_i|i = 1, \ldots, n+1\}$ of the space X. The space X has a closed shrinkage $\{B_i|i = 1, \ldots, n+1\}$ [En] (i.e. there exists a family of closed sets $\{B_i|i = 1, \ldots, n+1\}$ such that $B_i \subset U_i$ for all I and $\{B_i|i = 1, \ldots, n+1\}$ is a cover of the space X).

Let $A_i = X - U_i$ for $i = 1, \ldots, n$. The sequence $(A_1, B_1), \ldots, (A_n, B_n)$ consists of n pairs of disjoint closed subsets of X. Hence there exists separators $L_1, \ldots, L_n$ such that L_i separates A_i and B_i and $\bigcap L_i = \emptyset$.

Now consider the open sets $V_i, W_i \subset X$ such that $A_i \subset V_i$ and $B_i \subset W_i$ and $V_i \cap W_i = \emptyset$, and $X - L_i = V_i \cup W_i$ for $i = 1, \ldots, n$.

Notice that

$$(\bigcup_{i=1}^{n} V_i) \bigcup (\bigcup_{i=1}^{n} W_i) = \bigcup_{i=1}^{n} (V_i \cup W_i) = \bigcup_{i=1}^{n} (X - L_i) = X - \bigcup_{i=1}^{n} L_i = X.$$

We have $B_{n+1} \subset U_{n+1}$. Thus

$$(\bigcup_{i+1}^{n} W_i)\bigcup[U_{n+1}\cap(\bigcup_{i=1}^{n} V_i)] = [(\bigcup_{i+1}^{n} W_i)\cup U_{n+1}]\bigcap[(\bigcup_{i+1}^{n} W_i)\bigcup(\bigcup_{i=1}^{n} V_i)] \supset \bigcup_{i+1}^{n} B_i = X.$$

Thus the family $\{W_i | i = 1, \ldots, n+1\}$ with $W_{n+1} = U_{n+1} \cap (\bigcup_{i=1}^{n} V_i)$ is an open shrinkage of the cover $\{U_i | i = 1, \ldots, n+1\}$. Thus

$$\bigcap_{i=1}^{n+1} W_i = (\bigcap_{i+1}^{n+1} W_i)\bigcap[U_{n+1} \cap (\bigcap_{i+1}^{n} V_i)] \subset (\bigcap_{i=1}^{n+1} W_i)\bigcap(\bigcap_{i+1}^{n} V_i) = \emptyset.$$

Hence $dimX \leq n-1$ [En]. Thus if $dimX \geq n$ then X has an n-essential family. ■

1.1.7. Corollary. *The $dimX = \infty$ if and only if there exists an n-essential family for all $n \in \{1, 2, 3, \ldots\}$.*

1.1.8. Definition. A space X is *Strongly Infinite Dimensional* (S.I.D.) if X has an ω-family, where ω represents the cardinality of the natural numbers.

1.1.9. Definition. An infinite dimensional space X is *Weakly Infinite Dimensional* (W.I.D.) if X is not Strongly Infinite Dimensional.

1.1.10. Definition. A space X is *Countable Dimensional* (C.I.D.) if X is the countable union of finite dimensional spaces.

For the purpose of this dissertation another definition of dimensions is more appropriate. This definition is identical with covering dimension for metric spaces.

1.1.11. Definition. If S^n is the sphere with covering dimension n (i.e. $S^n = \frac{I^n}{\partial I^n}$ for $I = [0,1]$), then

1) a space X has dimension less than or equal to n written $dimX \leq n$, provided each map $\alpha : A \longrightarrow S^n$ from a closed subset into the n-sphere, extends to a map $\tilde{\alpha} : X \longrightarrow S^n$,

2) $dimX = n$ provided $dimX \leq n$ and $dimX > n-1$,

3) $dimX = \infty$ provided $dimX \geq n$ for all n.

1.1.12. Theorem. *Definition 1.11 is equivalent to covering dimension.*

Theorem 1.1.12. is found in Hurewicz and Wallman [H-W] as theorem VI4.

The purpose of this characterization of dimension is for easy comparison with cohomological dimension which we now present.

We represent the k^{th} homotopy group of a space X by $\pi_k(X,*)$. A brief discussion of homotopy groups is found in Section 2.3.

1.1.13. Definition. An *Eilenberg-MacLane space* of order n, designated K_n is a space satisfying

$$\pi_k(K_n,*) = \begin{cases} \pi_k(S^n,*) & \text{if } k \leq n \\ 0 & \text{if } k \geq n+1. \end{cases}$$

Eilenberg-MacLane spaces can be constructed by attaching cells of appropriate dimension using a mapping cone. Mapping cones are discussed in Section 2.4.

1.1.14. Definition. If K_n is an Eilenberg-MacLane space of order n, then

1) a space X has cohomological dimension $\leq n$ written $c-dimX \leq n$, provided each map $\alpha : A \longrightarrow K_n$ from a closed subset into an Eilenberg-MacLane space, extends to a map $\tilde{\alpha} : X \longrightarrow K_n$,

2) $c-dimX = n$ provided $c-dimX \leq n$ and $c-dimX > n-1$,

3) $c-dimX = \infty$ provided $c-dimX \geq n$ for all n.

1.1.15. Theorem.

1) For any space X, $c-dimX \leq dimX$.

2) For any space X with $dimX < \infty$, $\quad c-dimX = dimX$.

Theorem 1.1.15. and its proof is found in [W] as theorem 3.2.

2. PRELIMINARY DEFINITIONS AND RESULTS

In this chapter we present some preliminary definitions and results.

2.1. Homotopy.

We will use the convention that a *map* is a continuous function.

2.1.1. Definition. Two maps $f, g : X \longrightarrow Y$ are *homotopic* ($f \simeq g$) if there exists a map $F : X \times I \longrightarrow Y$, $I = [0,1]$, where $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for all $x \in X$. If a map f is homotopic to a constant map we say it is *inessential*, otherwise we say it is *essential.*

We remark that (following Spanier) we will be considering the *homotopy category* of (pointed) topological spaces where the objects are pointed topological spaces and the morphisms are homotopy classes of maps.

2.1.2. Definition. Two topological spaces X and Y are *homotopy equivalent*, or of the same *homotopy type*, if there exist maps $h : X \longrightarrow Y$ and $h' : Y \longrightarrow X$ where $h' \circ h : X \longrightarrow X$ and $h \circ h' : Y \longrightarrow Y$ are each homotopic to the identity. I.e. $h' \circ h \simeq id : X \longrightarrow X$ and $h \circ h' \simeq id : Y \longrightarrow Y$. We designate homotopy equivalent spaces by $X \simeq Y$.

2.1.3. Definition. If $X \simeq W$ and $Y \simeq Z$, then the maps $f : X \longrightarrow Y$ and $g : W \longrightarrow Z$ are said to be *homotopy equivalent* if $k \circ f \simeq g \circ h$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle k} \\ W & \xrightarrow{g} & Z \end{array} \tag{2.1}$$

where the vertical maps are homotopy equivalences. In this case we say that the diagram in (2.1) *commutes homotopically.*

From a category point of view, the objects are the morphisms $X \xrightarrow{f} Y$ (of the homotopy category) and its morphisms with domain $X \xrightarrow{f} Y$ and range $W \xrightarrow{f} Z$ are pairs of morphisms $h : X \longrightarrow W$ and $k : Y \longrightarrow Z$ such that diagram 2.1 commutes homotopically. This is called the *category of morphisms* of the homotopy category.

We see that homotopic maps are also homotopy equivalent since we can let the vertical maps be the identities as in (2.1).

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ X & \xrightarrow{g} & Y \end{array} \tag{2.2}$$

If h and k are homeomorphisms and $k \circ f = g \circ h$, we say the maps f and g are *topologically equivalent.* Clearly topological equivalence implies homotopy equivalence.

2.2. Cones and Suspensions.

2.2.1. Definition. Let I be the closed interval $[0,1]$ and let $(X,*)$ be a pointed topological space. The *reduced cone of* X, CX, is the space $X \times I$ where the subset $(X \times \{1\}) \cup (\{*\} \times I)$ is identified to a single point. Throughout the remainder of this paper we shall refer to CX as simply the cone of X.

We can formally define CX by $CX = \frac{X \times I}{R_c}$ whereR_c is the equivalence relation defined by $(x,t)R_c(x',t')$ iff $(x,t) = (x',t')$, or $t = t' = 1$, or $x = x' = *$.

We denote the equivalence class of (x,t) with respect to R_c by $[x,t]_c$.

2.2.2. Definition. The *reduced suspension of a pointed space* $(X,*), SX$, is the space $X \times I$ where the subset $(X \times \{1\}) \cup (X \times \{0\}) \cup (\{*\} \times I)$ is identified to a single point. Again throughout the remainder of this paper we shall refer to SX as simply the suspension of X.

We formally define SX by $SX = \frac{X \times I}{R_s}$ where R_s is the equivalence relation defined by

$$(x,t)R_s(x',t') \quad iff \quad (x,t) = (x',t'),$$

$$or \quad t = t' = 0, \quad or \quad t = t' = 1, \quad or \quad x = x' = *.$$

Notice that $SX = \frac{CX}{R'}$ where $[x,t]_c R'[x',t']$ iff $[x,t]_c = [x',t']_c$ or $t = t' = 0$.

Let S^n be the n-sphere and e^n be the n-cell $\{x \in E^n | \|x\| \leq 1\}$. It is well known that $SS^{n-1} \cong S^n$ and $CS^{n-1} \cong e^n$ and $\frac{e^n}{\partial e^n} \cong S^n$.

If we wish to embed X into SX as the 'equator', then an alternate but equivalent definition of SX is more convenient. Let J be the closed interval $[-1,1]$, and let $SX = \frac{X \times J}{R_{\bar{s}}}$ where

$$(x,t)R_{\bar{s}}(x',t') \quad iff \quad (x,t) = (x',t'),$$

$$or \quad t = t' = 1, \quad or \quad t = t' = -1, \quad or \quad x = x' = *.$$

We denote the equivalence class of (x,t) with respect to R_s or $R_{\bar{s}}$ as $[x,t]_s$ and the natural embedding of X into SX is given by $h(x) = [x,0]_s$.

2.2.3. Definition. For $f : X \longrightarrow Y$, we define the *cone map* $Cf : CX \longrightarrow CY$ and the *Suspension map* $Sf : SX \longrightarrow SY$ by $Cf([x,t]_c) = [f(x),t]_c$ and $Sf([x,t]_s) = [f(x),t]_s$, respectively.

2.2.4. Definition. A space X is a *suspension* if it is homotopy equivalent to SW for some space W.

2.2.5. Definition. A map $f : X \longrightarrow Y$ is said to be a *suspension map* if f is homotopy equivalent to the suspension of a map $F : W \longrightarrow Z$, i.e. f is homotopy equivalent to $SF : SW \longrightarrow SZ$ for some map $F : W \longrightarrow Z$.

Taking a suspension can be viewed as a functor on the homotopy category.

2.2.6. Lemma. *If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are suspensions, then $g \circ f$ is a suspension.*

Proof. Consider the following diagram where the vertical arrows are homotopy equivalences, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ are maps where the diagram commutes homotopically.

$$(2.3)\qquad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} \\ SX & \xrightarrow{SF} & SY & \xrightarrow{SG} & SZ \end{array}$$

We have $SG \circ SF([x,t]_s) = SG([F(x),t]_s) = [G \circ F(x), t]_s$, hence $SG \circ SF = S(G \circ F)$ and thus the composition of suspensions is a suspension.

Since vertical arrows are homotopy equivalences the diagram commutes homotopically and thus $g \circ f$ is a suspension. ■

2.2.7. Lemma. *If $f : X \longrightarrow Y$ is inessential where X and Y are suspensions, then f is a suspension map.*

Proof. The map f is inessential and thus is homotopic to a constant map $f_0 : X \longrightarrow Y$. Let $y_0 \in Y$ be the image of f_0. The spaces X and Y are suspensions, thus there exist homotopy equivalences h and k such that $h : X \longrightarrow SZ$ and $k : Y \longrightarrow SW$ for some spaces Z and W. We define the map $g : SZ \longrightarrow SW$ as $k \circ f_0 \circ h'$ where h' is a homotopy inverse of h.

$$(2.4)\qquad \begin{array}{ccc} X & \xrightarrow{f_0} & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ SZ & \xrightarrow{g} & SW \end{array}$$

We are now left to show that g is homotopic to a suspension map.

If $k(y_0) = *$, then g is a constant map and $g([z,t]) = * = [*,t] = [G(z),t]$ where G is the constant map $G : Z \longrightarrow *$. Hence g is a suspension.

If $k(y_0) \neq *$, then $k(y_0) = [w_0, t_0]$ where $w_0 \neq *$ and $t_0 \in (0,1)$. Let $k(y) = (w,t)$ and define a homotopy K of k by

$$K(y,s) = \begin{cases} [w,0] & \text{if } t \leq st_0 \\ [w, \frac{t-st_0}{1-st_0}] & \text{if } t \geq st_0. \end{cases}$$

We have $K(y,0) = [w,t] = k(y)$ and

$$K(y,1) = \begin{cases} [w,0] & \text{if } t \leq t_0 \\ [w, \frac{t-t_0}{1-t_0}] & \text{if } t \geq t_0. \end{cases}$$

Hence $K(y_0,1) = [w_0,0] = *$. Let $g_0 = K_1 \circ f_0 \circ h'$ where $K_1(y) = K(y,1)$. We thus conclude that g is homotopic to the constant map $g_0 : SZ \longrightarrow *$ and hence is a suspension map. ■

2.3. Homotopy Groups.

Let X and Y be topological spaces and let A be the subset of $X \times I$ defined by $A = (X \times \{0\} \cup (X \times \{1\}) \cup (\{*\} \times I)$. It is convenient and accurate to let the space of all pointed maps from SX to Y, denoted $\mathfrak{C}(SX,Y)$, be viewed as the collection of all maps $f : (X \times I, A) \longrightarrow (Y, y_0)$.

If $f, g \in \mathfrak{C}(SX,Y)$, we define the map $h \in \mathfrak{C}(SX,Y)$ by

$$h(x,t) = \begin{cases} \text{f(x,2t)}, & \text{if } 0 \leq t \leq 1/2, \\ \text{g(x,2t-1)}, & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

and we write $h = f + g$.

Let $[SX,Y]$ represent the collection of all homotopy classes of maps in $\mathfrak{C}(SX,Y)$. Then $[SX,Y]$ is a group with the group operation defined as follows: For $[f],[g] \in [SX,Y]$ $\;[f]+[g] = \;[f+g] = [h]$. The identity is $[k] \in [SX,Y]$ where k is the constant map $k(x,t) = y_0$. Define $-f \in \mathcal{C}(SX,Y)$ by $-f(x,t) = f(x,1-t)$, then $[-f]$ designated $-[f]$ is the inverse of $[f]$.

If X is a suspension and Y an arbitrary pointed set, then the set of equivalence classes of homotopic maps $\{[f] | f : X \longrightarrow Y\}$ is a group where the group structure is induced by the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \\ SZ \arrow[ur, "f \circ h'"] & \end{tikzcd} \tag{2.5}$$

The group operation is defined by $[f]+[g]=[(f\circ h'+g\circ h')\circ h]$.

We call this group the *X-homotopy group* of Y designated $\pi_X(Y,y_0)$. When X is an n-sphere we simply call the group the *n-homotopy group* of Y, designated $\pi_n(Y,y_0)$. If n = 1, we call the group the *fundamental group* of Y.

Let X and Y be suspensions and Z arbitrary. If $g : X\longrightarrow Y$ and $f : Y\longrightarrow Z$, then the notation $[f]\circ[g]$ means $[f\circ g]$.

2.3.1. Theorem. **(Left distributivity)** *If $Sg : SX\longrightarrow SY$ and $f_1, f_2 : SY\longrightarrow Z$ then $([f_1]+[f_2])\circ[Sg]=[f_1\circ Sg]+[f_2\circ Sg]$.*

Proof. We view the suspension map $Sg : SX\longrightarrow SY$ as defined by $Sg([x,t]_s)=[g(x),t]_s$.

Thus,

$$(f_1+f_2)\circ Sg[x,t]=(f_1+f_2)[g(x),t]=\begin{cases} f_1[g(x),2t], & \text{if } 0\le t\le 1/2\\ f_2[g(x),2t-1], & \text{if } 1/2\le t\le 1\end{cases}$$

$$=\begin{cases} f_1\circ Sg[x,2t], & \text{if } 0\le t\le 1/2\\ f_2\circ Sg[x,2t-1], & \text{if } 1/2\le t\le 1\end{cases}$$

$$=\big((f_1\circ Sg)+(f_2\circ Sg)\big)[x,t]. \quad \blacksquare$$

We of course may infer the following:

For $g : X\longrightarrow Y$ a suspension map and $f_1, f_2 : Y\longrightarrow Z$ arbitrary maps, we have $([f_1]+[f_2])\circ[g]=([f_1\circ g])+([f_2\circ g])$.

2.4. Wedges.

2.4.1. Definition. If $(X,*)$ is a pointed space, then the *k-wedge of X* denoted $\bigvee^k X$ is the disjoint union of k copies of X with the k points $*$ identified.

We formally define $\bigvee^k X$ by

$$\bigvee^k X = \frac{\bigcup\limits_{i=0}^{k-1} (\{i\} \times X)}{R_\vee}$$

where

$$(i,x) R_\vee (i',x') \quad if \quad \begin{cases} (i,x) = (i',x') \\ or \quad x = x' = *. \end{cases}$$

We denote the equivalence class of (i,x) as $[i,x]_\vee$.

As an example we see that $\bigvee^k S^n$ is a CW-Complex (see section 2.8) consisting of k n-cells and *one* 0-cell.

2.4.2. Definition. We define the *wedge* of arbitrary pointed spaces (X, x_0) and (Y, y_0), denoted $X \bigvee Y$, by

$$X \bigvee Y = \frac{(\{0\} \times X) \cup (\{1\} \times Y)}{R_\vee}$$

where $(0,x) R_\vee (1,y)$ if $x = x_0$ and $y = y_0$. Without loss of generality we can consider $X \bigvee Y$ as $X \cup Y$ where $X \cap Y = \{*\}$.

2.4.3. Definition. If $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$, then the *wedge of f and g*, denoted $f \bigvee g$, is a map $f \bigvee g : X \bigvee X' \longrightarrow Y \bigvee Y'$ and is defined by

$$(f \bigvee g)(x) = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in X'. \end{cases}$$

2.4.4. Definition. If $f : (X,*) \longrightarrow (Y,*)$, then the *$k$-wedge of f*, denoted $\bigvee^k f$, is a map $\bigvee^k f : \bigvee^k X \longrightarrow \bigvee^k Y$ defined by

$$\bigvee^k f([i,x]_\vee) = [i, f(x)]_\vee \quad \text{for } i = 0, \ldots, k-1.$$

2.4.5. Definition. If $f : X \longrightarrow Y$ is a function from a set X to the set Y then the *equivalence relation on X induced by f*, denoted R_f is defined by

xR_fx' iff $f(x) = f(x')$. I.e. R_f partitions X into the equivalence classes of point inverses.

2.4.6. Lemma. *Let X, Y, Z be compact metric spaces and let $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$ be surjections. If $R_f = R_g$, then there exists a homeomorphism $h : Y \longrightarrow Z$ such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd} \tag{2.6}$$

Proof. Let $h = g \circ f^{-1}$. Since $R_f = R_g$ we have h is a function and one-to-one, since f and g are surjections we have h is a surjection. To see that h is continuous consider $h^{-1}(K)$, where K is closed. Then $h^{-1}(K) = f \circ g^{-1}(K)$ and $g^{-1}(K)$ is closed since g is continuous. Then $f(g^{-1}(K))$ is closed since X and Y are compact and hence f is a closed map. Thus h is a homeomorphism and $g = h \circ f$.■

2.4.7. Lemma. *1) If (X, x_0) and (Y, y_0) are pointed spaces, there is a homeomorphism $h_1 : S(X \bigvee Y) \longrightarrow (SX) \bigvee (SY)$ induced by the diagram below, and*
2) If $f : (X, x_0) \longrightarrow (X', x_0')$ and $g : (Y, y_0) \longrightarrow (Y', y_0')$ are pointed maps, there exists a homeomorphism $h_2 : S(X' \bigvee Y') \longrightarrow (SX') \bigvee (SY')$ such that the maps $S(f \vee g) : S(X \bigvee Y) \longrightarrow S(X' \bigvee Y')$ and $(Sf) \vee (Sg) : (SX) \bigvee (SY) \longrightarrow (SX') \bigvee (SY')$ are topologically equivalent by h_1 and h_2.

Proof. Assume X and Y are disjoint, and let $\biguplus$ be their topological sum. Consider the following diagram:

$$\begin{tikzcd} (X \biguplus Y) \times I \arrow[r, "k"] \arrow[d, "P_\vee \times id"] & (X \times I) \biguplus (Y \times I) \arrow[d, "P_\vee"] \\ (X \bigvee Y) \times I \arrow[d, "P_s"] & (X \times I) \bigvee (Y \times I) \arrow[d, "P_s \bigvee P_s"] \\ S(X \bigvee Y) \arrow[r, "h_1"] \arrow[d, "S(f \vee g)"] & (SX) \bigvee (SY) \arrow[d, "(Sf) \vee (Sg)"] \\ S(X' \bigvee Y') \arrow[r, "h_2"] & S(X') \bigvee S(Y') \end{tikzcd} \tag{2.7}$$

The map k is the natural homeomorphism and $P_\vee$ and P_s are the projection maps onto the wedge space and suspension space respectively. The equivalence relation R_1 on $(X \biguplus Y) \times I$ induced by $P_s \circ (P_v \times id)$ is generated by $(x,t)R_1(y,t')$ if and only if ($x = x_0$ and $y = y_0$), or ($t = t' = 0$) or ($t = t' = 1$). It is easy to see that the equivalence relation R_2 on $(X \biguplus Y) \times I$ induced by $(P_s \vee P_s) \circ P_v \circ k$ is the same. Then h_1 exists by the above lemma. The equivalence relations R_3 and R_4 on $(X \biguplus Y) \times I$ induced by $S(f \vee g) \circ P_s \circ (P_v \times id)$ and $(Sf) \vee (Sg) \circ (P_s \vee P_s) \circ P_v \circ k$, respectively are likewise equal and are generated by $(x,t) = (y,t')$ if and only if ($f(x) = x_0'$ and $f(y) = y_0'$), or ($t = t' = 0$), or ($t = t' = 1$). The result follows. ■

2.4.8. Corollary. *If $f : X \longrightarrow Y$ is a suspension map, then $\bigvee^k f : \bigvee^k X : \longrightarrow \bigvee^k Y$ is a suspension map.*

Proof. This follows immediately from Corollary 2.4.7. using a finite induction on k. ■

2.5. Mapping Cones.

2.5.1. Definition. Let X and Y be two disjoint spaces and let A be a closed subset of X, and let $f : A \longrightarrow Y$. In $X \biguplus Y$ we generate an equivalence relation R_f by $a \sim f(a)$ for each $a \in A$. the quotient space $\dfrac{X \biguplus Y}{R_f}$ is called the *attachment of X and Y by f* and is denoted $X \bigcup_f Y$ and called an adjunction space.

We formally define $X \bigcup_f Y \equiv \dfrac{X \biguplus Y}{R_f}$ where

$$xR_fx' \quad \text{if} \quad \begin{cases} & x = x' \\ \textit{or} & f(x) = f(x') \\ \textit{or} & f(x) = x' \\ \textit{or} & f(x') = x. \end{cases}$$

We denote the equivalence class of x as $[x]_f$. If $x \in X - A$, it will be convenient to let $x = [x]_f$.

2.5.2. Definition. Let X be compact and $f : X \longrightarrow Y$. The *mapping cone* of f designated M_f is defined by

$$M_f = CX \bigcup_f Y$$

where X is regarded as a closed subset of CX by embedding it in CX as $p(X \times \{0\})$ where $p : X \times I \longrightarrow CX$ is athe projection map.

2.5.3.Lemma. *If X is compact and if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are surjective maps, then there are maps $\hat{f}$ and $\hat{g}$ such that the following diagram commutes, where the vertical arrows are injections:*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow & & \downarrow \\ CX & \xrightarrow{\hat{f}} & M_f & \xrightarrow{\hat{g}} & M_{g \circ f} \end{array} \tag{2.8}$$

Proof. The map $\hat{f}$ is defined by

$$\hat{f}([x,t]) = \begin{cases} [f(x)] & \text{if } t = 0 \\ [x,t] & \text{if } t > 0 \end{cases}$$

and the map $\hat{g}$ is similarly defined by

$$\hat{g}(\hat{f}[x,t]) = \begin{cases} [g(f(x))] & \text{if } t = 0 \\ [x,t] & \text{if } t > 0\ . \end{cases}$$

We notice that $\hat{g}(\hat{f}([x,t])) = \widehat{g \circ f}([x,t])$ where $\widehat{g \circ f} : CX \longrightarrow M_{g \circ f}$. ∎

For the remainder of section 2.5 let $I = [0,1]$, and $J = [-1,1]$.

2.5.4. Lemma. *If R_1, R_2 are equivalence relations on $X \times J \times I$ defined by*

$$(x,s,t)R_1(x',s',t') \quad \text{iff} \begin{cases} & (x,s,t) = (x',s',t') \\ \textit{or} & x = x' = * \\ \textit{or} & s = s' = 1 \text{ and } t = t' \end{cases}$$

and

$$(x,s,t)R_2(x',s',t') \quad \text{iff} \begin{cases} & (x,s,t) = (x',s',t') \\ \textit{or} & x = x' = * \\ \textit{or} & s = s' = 1, \end{cases}$$

then $R_1 = R_2$.

For convenience we will henceforth use the terminology; R_1 is generated by $(x = x' = *)$ or $(s = s' = 1$ and $t = t')$ and R_2 is generated by $(x = x' = *)$ or $(s = s' = 1)$.

Proof. We will use the classical definition of equivalence relation to show that R_1 and R_2 are equal sets in $(X \times J \times I) \times (X \times J \times I)$.

It follows automatically that if $\xi = ((x, s, t), (x', s', t')) \in R_1$, then $\xi \in R_2$ and hence $R_1 \subset R_2$. Now let $\xi = ((x, s, t), (x', s', t')) \in R_2$. If $s \neq 1$ or $s' \neq 1$, then $(x, s, t) = (x', s', t')$ or $x = x' = *$; and hence $\xi \in R_1$. Now consider the case where $s = s' = 1$. We have $(x, 1, t)R_1(*, 1, t)$ and $(*, 1, t)R_1(*, 1, t')$ and $(*, 1, t')R_1(x', 1, t')$ and thus by transitivity $(x, 1, t)R_1((x', 1, t')$ or $\xi = ((x, 1, t), (x', 1, t')) \in R_1$.

Thus $R_2 \subset R_1$ and hence $R_1 = R_2$. ■

2.5.5. Lemma. *There is a naturally defined homeomorphism* $\alpha : CSX : \longrightarrow SCX$ *and hence* CSX *is a suspension.*

Proof. Let $P_c : X \times I \longrightarrow CX$ and $P_s : X \times J \longrightarrow SX$ be the projection maps and let $h : X \times J \times I \longrightarrow X \times I \times J$ be the natural homeomorphism defined by $h(x, s, t) = (x, t, s)$. Consider the following diagram.

(2.9)
$$\begin{array}{ccccc} X \times J \times I & \xrightarrow{P_s \times id} & SX \times I & \xrightarrow{P_c} & CSX \\ \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle \alpha} \\ X \times I \times J & \xrightarrow{P_c \times id} & CX \times J & \xrightarrow{P_s} & SCX \end{array}$$

To produce α it is sufficient by Lemma 2.4.6 to show that the equivalence relations induced on $X \times J \times I$ by $P_c \circ (P_s \times id)$ and $P_s \circ (P_c \times id) \circ h$ are equal.

First, the equivalence relation on $X \times J \times I$ induced by $P_c \circ (P_s \times id)$ is generated by $(s = s' = -1$ and $t = t')$ or $(s = s' = 1$ and $t = t')$ or $(x = x' = *$ and $t = t')$ or $(t = t' = 1)$ or $(x = x' = *)$. By Lemma 2.5.4. this is equivalent to

(I) $(s = s' = -1)$ or $(s = s' = 1)$ or $(x = x' = *)$ or $(t = t' = 1)$.

We now consider the bottom of the diagram. Since h is a homeomorphism we need only check the equivalence relation on $X \times I \times J$ induced by $P_s \circ (P_c \times id)$. The equivalence relation on $X \times I \times J$ induced by $P_s \circ (P_c \times id)$ is generated by $[(t = t' = 1$ or $x = x' = *)$ and $s = s']$ or $(s = s' = -1)$ or $(s = s' = 1)$ or $(x = x' = *)$, which is equivalent to $(t = t' = 1$ and $s = s')$ or $(x = x' = *$ and $s = s')$ or $(s = s' = -1)$ or $(s = s' = 1)$ or $(x = x' = *)$. This by Lemma 2.5.4. reduces to:

(II) $\quad (t = t' = 1)$ or $(s = s' = -1)$ or $(s = s' = 1)$ or $(x = x' = *)$.

Since the relations I and II are the same, by Lemma 2.4.6., there exists a homeomorphism $\alpha : CSX \longrightarrow SCX$ such that the diagram commutes. ■

Recall that if $f : X \longrightarrow Y$, then $Sf : SX \longrightarrow SY$ is defined by $Sf([x, s]_s) = [f(x), s]_s$ and $\hat{f} : CX \longrightarrow CX \cup_f Y$ is defined by

$$\hat{f}([x,t]_c) = \begin{cases} [x,t]_c & \text{if } t > 0 \\ [f(x)]_f & \text{if } t = 0\,. \end{cases}$$

Thus $\widehat{Sf} : CSX \longrightarrow CSX \cup_{Sf} SY = M_{Sf}$ is defined by

$$\widehat{Sf}([[x,s]_s,t]_c) = \begin{cases} [[x,s]_s,t]_c & \text{if } t > 0 \\ [Sf[x,s]_s]_{Sf} & \text{if } t = 0 \end{cases} = \begin{cases} [[x,s]_s,t]_c & \text{if } t > 0 \\ [f(x),s]_s]_{Sf} & \text{if } t = 0\,, \end{cases}$$

and $S\hat{f} : SCX \longrightarrow S(CX \cup_f CY) = SM_f$ is defined by

$$S\hat{f}([[x,t]_c,s]_s) = [\hat{f}([x,t]_c),s]_s = \begin{cases} [[x,t]_c,s]_s & \text{if } t > 0 \\ [[f(x)]_f,s]_s & \text{if } t = 0\,. \end{cases}$$

2.5.6. Lemma *If $f : X \longrightarrow Y$ is a surjection between compact metric spaces, then the maps $\widehat{Sf} : CSX \longrightarrow M_{Sf}$ and $S\hat{f} : SCX \longrightarrow SM_f$ are topologically equivalent and hence $\widehat{Sf}$ is a suspension.*

Proof. Let P_c, P_s and h be the maps as defined as above. Consider the following diagram.

$$\text{(2.10)} \qquad \begin{array}{ccccccc} X \times J \times I & \xrightarrow{P_c \circ (P_s \times id)} & CSX & \xrightarrow{\widehat{Sf}} & CSX \cup_{Sf} SY & = & M_{Sf} \\ \downarrow h & & \downarrow j & & \downarrow k & & \\ X \times I \times J & \xrightarrow{P_s \circ (P_c \times id)} & SCX & \xrightarrow{S\hat{f}} & S(CX \cup_f Y) & = & SM_f \end{array}$$

From (I) in Lemma 2.5.5. we see that the equivalence relation on $X \times J \times I$ induced by $\widehat{Sf} \circ P_c \circ (P_s \times id)$ is generated by $(s = s' = -1)$ or $(s = s' = 1)$ or $(x = x' = *)$ or $(t = t' = 1)$ or $(f(x) = f(x')$ and $s = s'$ and $t = t' = 0)$. From II in the proof of Lemma 2.5.5. we can see that the equivalence relation on $X \times J \times I$ induced by $S\hat{f} \circ P_s \circ (P_c \times id) \circ h$ is the same and hence by Lemma 2.4.6. there exists a homeomorphism $k : M_{Sf} \longrightarrow SM_f$ such that the diagram commutes. ■

From the definition of $Cf : CX \longrightarrow CY$ and $Sf : SX \longrightarrow SY$ we see that the maps $CSf : CSX \longrightarrow CSY$ and $SCf : SCX \longrightarrow SCY$ are defined, respectively, as follows:

$$CSf([[x,s]_s,t]_c) = [Sf[x,s]_s,t]_c = [[f(x),s]_s,t]_c \text{ and}$$

$$SCf([[x,t]_c,s]_s) = [Cf[x,t]_c,s]_s = [[f(x),t]_c,s]_s.$$

2.5.7. Lemma. *The maps $CSf : CSX \longrightarrow CSY$ and $SCf : SCX \longrightarrow SCY$ are topologically equivalent and hence CSf is a suspension map.*

Proof. Consider the diagram;

$$\begin{array}{ccccc} X \times J \times I & \overset{P_c \circ (P_s \times id)}{\longrightarrow} & CSX & \overset{CSf}{\longrightarrow} & CSY \\ \downarrow h & & \downarrow j & & \downarrow l \\ X \times I \times J & \overset{P_s \circ (P_c \times id)}{\longrightarrow} & SCX & \overset{SCf}{\longrightarrow} & SCY . \end{array} \tag{2.11}$$

From (I) in Lemma 2.5.5. we conclude that the equivalence relation on $X \times J \times I$ induced by $CSf \circ P_c \circ (P_s \times id)$ is generated by $(s = s' = -1)$ or $(s = s' = 1)$ or $(x = x' = *)$ or $(t = t' = 1)$ or $(f(x) = f(x')$ and $s = s'$ and $t = t')$, and from (II) in Lemma 2.5.5 we have the equivalence relation on $X \times I \times J$ induced by $SCf \circ P_s \circ (P_c \times id)$ is the same and hence by Lemma 2.4.6., there exists a homeomorphism β such that the diagram commutes. ■

2.5.8. Corollary. $M_{Sf} \bigvee M_{Sg} \cong SM_f \bigvee SM_g \cong S(M_f \bigvee M_g)$.

2.5.9. Corollary. *If $F = S^n f : S^n X \longrightarrow S^n Y$ is a multiple suspension, then $M_F = M_{S^n f} \cong S^n M_f$.*

2.6. Homotopy Addition.

Let $f, g : SX \longrightarrow Y$ be maps. Recall that homotopy addition is defined by:

$$[f] + [g] = [h]$$

where h is defined by

$$h([s, x]) = \begin{cases} f([x, 2s]) & \text{if } s \in [0, 1/2] \\ g([x, 2s-1]) & \text{if } s \in [1/2, 1]. \end{cases}$$

We can conveniently write the sum of k maps $f_0, \ldots, f_{k-1} : SX \longrightarrow Y$ by:

$$\sum_{i=0}^{k-1} [f_i] = \Big[\sum_{i=0}^{k-1} f_i \Big]$$

where $\sum_{i=0}^{k-1} f_i$ is defined by:

$$\sum_{i=0}^{k-1} f_i([x, t]) = f_i([x, kt - i]) \quad \text{if} \quad t \in [\frac{i}{k}, \frac{i+1}{k}].$$

Let P_X be the projection map $P_X : SX \longrightarrow \bigvee^k SX$, where P_X is defined as follows:

$$P_X([x, t]_s) = (i, [x, kt - i]_s) \quad \text{for} \quad t \in [\frac{i}{k}, \frac{i+1}{k}]$$

$$i = 0, 1, 2, \ldots, k-1.$$

P_X is commonly referred to as a *pinch map.*

2.6.1. Lemma. *If X is a suspension, then the pinch map P_X is a suspension map. In particular, the maps $P_{SX} : SSX \longrightarrow \bigvee^k SSX$ and $SP_X : SSX \longrightarrow S(\bigvee^k SX)$ are topologically equivalent.*

Proof. In the diagram,

$$\begin{tikzcd}
X\times J\times I \arrow[r, "P_s\times id"] \arrow[d, "h"] & SX\times I \arrow[r, "P_s"] & SSX \arrow[r, "P_{SX}"] \arrow[d, "j"] & \bigvee^k SSX \arrow[d, "l"] \\
X\times I\times J \arrow[r, "P_s\times id"] & SX\times J \arrow[r, "P_s"] & SSX \arrow[r, "SP_X"] & S(\bigvee^k SX)
\end{tikzcd} \tag{2.12}$$

the maps h and j are from corollary 2.5.6. where $h(x,s,t) = h(x,t,s)$. The map P_{SX} is defined by

$$P_{SX}([[x,s],t]) = [i,[[x,s],kt-i]]_\vee,$$

and the map SP_X is defined by

$$SP_X([[x,t],s]) = [[i,[x,kt-i]]_\vee, s].$$

The corresponding equivalence relations induced on $X\times J\times I$ are equal and, for $(x,s,t),(x',s',t') \in X\times J\times I$, are induced by $(s=s'=\pm 1)$ or $(t=t'=i/k,)$ for some $i=0,1,\ldots,k)$ or $(x=x'=*)$. ∎

2.6.2. Definition. Let $f_0,\ldots,f_{k-1}$ be maps from SX to $(Y,*)$. We define the map $(f_i) : \bigvee^k SX \longrightarrow (Y,*)$, $\quad i=0,\ldots,k-1$ by

$$(f_i)(j,[x,s]_s) = f_j([x,s]_s).$$

This simply applies f_j to the j^{th} copy of SX in $\bigvee^k SX$.

We now give a more geometric version of homotopy addition.

To add the maps $f_0,\ldots,f_{k-1} : SX \longrightarrow (Y,*)$ in homotopy, we will first pinch SX to form a wedge of k copies of SX and then apply each f_j to a different copy.

2.6.3. Lemma. $[(f_i)\circ P] = \sum_{i=0}^{k-1}[f_i]$.

Proof. The proof is actually done in the topological category. For $t \in [\frac{i}{k}, \frac{i+1}{k}]$ and for $i = 0,1,2,\ldots,k-1$, we have

$$(f_i)\circ P([x,s]) = (f_i)(i,[x,ks-i]_s) = f_i([x,ks-i]) = \sum_{i=0}^{k-1} f_i([x,s]). \quad ∎$$

Thus we see homotopy addition can be factored as $[(f_i) \circ P]$.

2.6.4. Definition. If $f : SX \longrightarrow (Y, *)$ is a pointed map, we define $-f : SX \longrightarrow (Y, *)$ by

$$-f([x, s]) = f([x, 1 - s]),$$

and we define $(-1)^m f$ by:

$$(-1)^m f = \begin{cases} f & \text{if } m \text{ is even} \\ -f & \text{if } M \text{ is odd.} \end{cases}$$

It is clear and a standard result in homotopy theory that $f + (-f)$ is inessential.

In the suspension space SX, $X \times \{1/2\}$ is called the *equator* of SX, and X is called the *desuspension* of SX. In general the equator and the desuspension are unique, however if a space Y is a multiple suspension, i.e. $Y = S^n X$ then there are multiple equators and desuspensions.

E.g. If $Y = SSX$, then $Y = \{[[x, s_1]_s, s_2]_s\} \cong \{[[x, s_2,]_s, s_1]_s\}$.

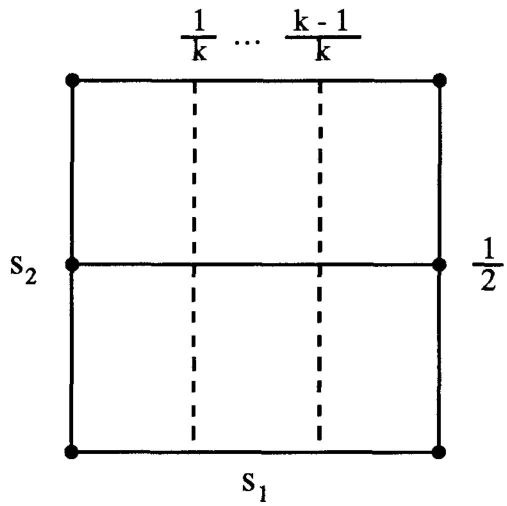
The desuspension are thus $\{[x, s_1]_s\}$ and $\{[x, s_2]_s\}$, respectively. And the equators are $\{[[x, s_1]_s, 1/2]_s\}$ and $\{[[x, s_2]_s, 1/2]_s\}$.

In the case of multiple suspensions the pinch map can be defined to run transverse to the equator.

E.g. if $\{[[x, s_1]_s, 1/2]_s\}$ is chosen as the equator we define the pinch map by

$$P([[x, s_1]_s, s_2]_s) = [i, [[x, ks_1 - i]_s, s_2]]_\vee \quad \text{for} \quad s_1 \in [\frac{i}{k}, \frac{i+1}{k}]$$

$$\text{and} \quad i = 0, 1, \ldots, k-1.$$



Pinch Map

Fig. 2.1

Thus the pinch map restricted to the desuspension is still a pinch map and the equator is mapped on the equators of each $(SX)_i$ in $\bigvee^k SX$.

The following is a reformulation of Left Distributivity (see Theorem 2.3.1).

2.6.5. Lemma. *If $f : SX \longrightarrow Y$ is a suspension map and $d_i : Y \longrightarrow Z$ are arbitrary maps for $i = 0, \ldots, k-1$, and P is the pinch map $P : SX \longrightarrow \bigvee^k SX$, then*

$$(d_i) \circ \bigvee^k f \circ P = \left(\sum_{i=0}^{k-1} d_i \right) \circ f.$$

Proof. Since f is a suspension map Y is a suspension, and consequently $\sum_{i=0}^{k-1} d_i$ is well defined. We have the maps $\bigvee^k f : \bigvee^k SX \longrightarrow \bigvee^k Y$ and $(d_i) : \bigvee^k Y \longrightarrow Z$ and $P : SX \longrightarrow \bigvee^k SX$.

Thus we have the following composition of maps;

$$SX \xrightarrow{P} \bigvee^k SX \xrightarrow{\bigvee^k f} \bigvee^k Y \xrightarrow{(d_i)} Z$$

which is equivalent to

$$SX \xrightarrow{P} \bigvee^{k} SX \xrightarrow{(d_i \circ f)} Z$$

which is equivalent to

$$SX \xrightarrow{\sum_{i=0}^{k-1} d_i \circ f} Z$$

and thus

$$(d_i) \circ \bigvee^{k} f \circ P = \sum_{i=0}^{k-1} d_i \circ f = \left(\sum_{i=0}^{k-1} d_i \right) \circ f$$

by left distributivity (Theorem 2.3.1). ■

The following commutative diagram is a summary of this result:

(2.13)
$$\begin{tikzcd}
SX \arrow[r, "P"] \arrow[dr, "f"'] & \bigvee^{k} SX \arrow[r, "\bigvee f"] & \bigvee^{k} Y \arrow[r, "(d_i)"] & Z \\
& Y \arrow[ur, "P"] \arrow[rr, "\sum d_i"] & & Z \arrow[u, "id"']
\end{tikzcd}$$

2.7. Simplices and Face Maps.

We now recall some facts about simplices. Let $\Delta^n = < v_0, v_1, \ldots, v_n >$. Then Δ^n is an n-cell and $Bd\Delta^n = S^{n-1}$. Let $(\Delta^n)^{(k)}$ be the polyhedron consisting of the union of all faces of Δ^n with dimension $\leq k$. It is easy to see $(\Delta^n)^{(n-1)} = S^{n-1}$.

In the following, $\partial\Delta^n$ is a historically motivated representation of $(\Delta^n)^{(n-1)}$ where orientation of the faces are consistantly preserved.

2.7.1. Definition. We define the boundary operator, $\partial\Delta^n$, by

$$\partial\Delta^n = \sum_{i=0}^{n} (-1)^n < v_0, \ldots, \hat{v}_i, \ldots, v_n >$$

where $\hat{v}_i$ represents a deleted vertex. The notation $+ < v_{i_k} >$ and $- < v_{i_k} >$ represent even and odd permutations of $< v_{i_k} >$.

2.7.2. Definition. Let $\Delta^n = < v_0, \ldots, v_n >$ and $\Delta^{n+1} = < u_o, \ldots, u_{n+1} >$. We define $n+2$ simplicial maps $d_i : \Delta^n \longrightarrow \Delta^{n+1}$, $i = 0, \ldots, n+1$, by $d_i(< v_0, \ldots, v_n >) = < u_0, \ldots, \hat{u}_i, \ldots, u_{n+1} >$, where $\hat{u}_i$ is a deleted vertex. The maps d_i are called face maps. I.e. d_i maps Δ^n linearly onto the face opposite the i^{th} vertex preserving the order of the vertices.

E.g. $d_2 : \Delta^3 \longrightarrow \Delta^4$ is defined by

$$d_2(< v_0, v_1, v_2, v_3 >) = < u_0, u_1, u_3, u_4 > .$$

Now consider d_i restricted to an $n-1$ face of Δ^n.

$$d_i(< v_0, \ldots, \hat{v}_j, \ldots, v_n >) = \begin{cases} < u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_{j+1}, \ldots, u_{n+1} > & \text{if } i \leq j \\ < u_0, \ldots, \hat{u}_j, \ldots, \hat{u}_i, \ldots, u_{n+1} > & \text{if } i > j. \end{cases}$$

E.g. $d_2(< v_0, v_1, v_2, \hat{v}_3 >) = < u_0, u_1, \hat{u}_2, u_3, \hat{u}_4 > = < u_0, u_1, u_3 >$ and $d_2(< v_0, \hat{v}_1, v_2, v_3 >) = < u_0, \hat{u}_1, \hat{u}_2, u_3, u_4 > = < u_0, u_3, u_4 >$.

We will have use for the following simple version of the Homotopy Addition Theorem. We will prove it here for completeness and for motivating generalizations of it that we will prove later.

We now notice that d_i restricted to $\partial\Delta^n$ is a map $d_i : S^{n-1} \longrightarrow (\Delta^{n+1})^{(n-1)}$, and hence represents an element of $\pi_{n-1}\big((\Delta^{n+1})^{(n-1)}\big)$. We thus have the notion of homotopy addition.

2.7.3. Theorem. (Homotopy Addition Theorem)
If $d_i : (\Delta^n)^{(n-1)} \longrightarrow (\Delta^{n+1})^{(n-1)}, i = 0, 1, \ldots, n$ are the face maps, then $\sum_{i=0}^{n} (-1)^i d_i \simeq 0$.

Proof. We first compute

$$\begin{aligned} d_i(\partial\Delta^n) &= d_i \sum_{j=0}^{n} (-1)^j < v_0, \ldots, \hat{v}_j, \ldots, v_n > \\ &= \sum_{j=0}^{i-1} (-1)^j < u_o, \ldots, \hat{u}_j, \ldots, \hat{u}_i, \ldots, u_{n+1} > \\ &+ \sum_{j=i}^{n} (-1)^j < u_o, \ldots, \hat{u}_i, \ldots, \hat{u}_{j+1}, \ldots, u_{n+1} > . \end{aligned}$$

We now compute the alternating sum of $(-1)^i d_i, \quad i = 0, \ldots, n+1$ and show that the sum is 0.

$$\sum_{i=0}^{n+1}((-1)^i d_i(\partial < v_0, \ldots, v_n >)$$

$$= \sum_{i=0}^{n+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j < u_0, \ldots, \hat{u}_j, \ldots, \hat{u}_i, \ldots, u_{n+1} > \right.$$

$$\left. + \sum_{j=i}^{n} (-1)^j < u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_{j+1}, \ldots, u_{n+1} > \right)$$

$$= \sum_{i=0}^{n+1} \left(\sum_{j=0}^{i-1} (-1)^{i+j} < u_0, \ldots, \hat{u}_j, \ldots, \hat{u}_i, \ldots, u_{n+1} > \right.$$

$$\left. + \sum_{j+1}^{n} (-1)^{i+j} < u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_{j+1}, \ldots, u_{n+1} > \right).$$

It can be seen that when the simplex $< u_0, \ldots, \hat{u}_j, \ldots, \hat{u}_i, \ldots, u_{n+1} >$ in the first sum has the same vertices as the simplex $< u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_{n+1} >$ in the second sum then j in the first sum equals i in the second sum and i in the first sum equals $j+1$ in the second sum and thus replacing the i and j of the second sum in the coeficient $(-1)^{i+j}$ with the equivalent i and j of the first sum we have $(-1)^{i+j} = (-1)^{j+i-1} = -(-1)^{i+j}$. And thus the terms of the comprehensive sum cancels in pairs and the sum is 0. ■

Now consider $d_i : \Delta^n \longrightarrow (\Delta^{n+1})^{(n)}$ and $d_j : \Delta^{n+1} \longrightarrow (\Delta^{n+2})^{n+1}$.

$$d_i(\Delta^n) = < v_0, \ldots, \hat{v}_i, \ldots, v_{n+1} >$$

$$d_j(\Delta^{n+1}) = < u_0, \ldots, \hat{u}_j, \ldots, u_{n+2} >$$

and thus

$$d_j \circ d_i(\Delta^n) = d_j(< v_0, \ldots, \hat{v}_i, \ldots, v_{n+1} > = < u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots u_{n+2} > .$$

I.e. $d_j \circ d_i$ is a map that maps Δ^n linearly onto the n-face of Δ^{n+2} that does not contain the vertices u_i and u_j.

We can extend this to the composition of any number of maps.

$$d_{i_1} : \Delta^n \longrightarrow \Delta^{n+1}, \ldots, d_{i_k} : \Delta^{n+k-1} \longrightarrow \Delta^{n+k}$$

$$i_k < i_l \quad \text{if} \quad k < l.$$

Then $d_{i_k} \circ \ldots \circ d_{i_1} : \partial\Delta^n \longrightarrow (\Delta^{n+k})^{(n-1)}$ is a map that maps $\partial\Delta^n$ onto the boundary of the n-face of Δ^{n+k} that does not contain the vertices $v_{i_1}, \ldots v_{i_k}$.

The following diagram demonstrates the composition of maps:

$$(2.14) \qquad \begin{array}{ccccccc} & & \Delta^{n+1} & \xrightarrow{d_j} & \partial\Delta^{n+2} & = & (\Delta^{n+2})^{(n+1)} \\ & & \uparrow & & \uparrow & & \\ \Delta^n & \xrightarrow{d_i} & \partial\Delta^{n+1} & \xrightarrow{d_j} & (\Delta^{n+2})^{(n)} & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \partial\Delta^n & \xrightarrow{d_i} & (\Delta^{n+1})^{(n-1)} & \xrightarrow{d_j} & (\Delta^{n+2})^{(n-1)} & & \end{array}$$

The diagram can be extended up and to the right indefinetly. For convenience we abbreviate $d_{i_k} \circ \ldots, \circ d_{i_1} = d_{i_1,\ldots,i_k}$.

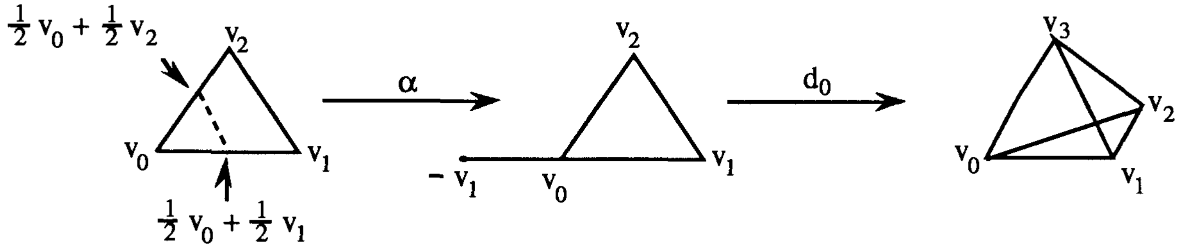
For all maps d_i, $\quad i \neq 0, \quad d_i(v_0) = u_0$. To preserve the base points for all maps d_i we want $d_0(v_0) = u_0$. To do this we alter d_0. We define $d_o^* : \Delta^n \longrightarrow \Delta^{n+1}$ by the composition of maps $d_0^* = \tilde{d}_0 \circ \alpha : \Delta^n \longrightarrow \Delta^{n+1}$ where α is the *linear map* $\alpha : \Delta^n \longrightarrow \Delta^n \cup < -v_1, v_0 >$ defined by;

$$\alpha : v_0 \mapsto -v_1$$

$$\alpha :< 1/2v_0 + 1/2v_1, \ldots, 1/2v_0 + 1/2v_n > \longrightarrow v_0$$

$$\alpha : v_n \mapsto v_n \quad \forall n \neq 0$$

and $\tilde{d}_0$ is defined by $\tilde{d}_0(x) = d_0(x)$ if $x \in \Delta^n$ and if $x \in < -v_1, v_0 >$, then $x = \lambda(-v_1) + (1-\lambda)v_0$ and then $\tilde{d}_0(x) = \lambda v_o + (1-\lambda)v_1$.



Face Map

Fig. 2.2

When α is restricted to $\partial\Delta^n$, α maps cells onto $< -v_1, v_0 >$ in pairs with opposites orientation and hence the inessentiality of $\sum d_i$ is preserved.

2.8. CW-Complexes.

We conclude this chapter with a section on CW-complices. The lemmas are given without proof as the proofs of these results can be found in [Sw].

2.8.1. Definition. [Sw 5.1.] A *Cell Complex* K on a space X is a collection $K = \{e^n_\alpha | n = 0, 1, 2, \ldots, \alpha \in J_n\}$ of subsets of X indexed by $n \in \{0, 1, \ldots\}$ and for each n by α running through some index set J_n. The set e^n_α is called a *cell of dim n*. K must satisfy the following conditions;

i) $X = \bigcup_{n,\alpha} e^n_\alpha = |K|$,

ii) if $bd\ e^n_\alpha \cap bd\ e^m_\beta$, then $n = m$ and $\alpha = \beta$,

iii) for each cell e^n_α there is a map $f^n_\alpha : (D^n, S^{n-1}) \longrightarrow (e^n_\alpha, bde^n_\alpha)$ which is surjective and maps $int\ D^n$ homeomorphically onto $int\ e^n_\alpha$.

2.8.2. Definition. [Sw 5.3.] A *CW-Complex* K on a space X is a cell complex K on X satisfying:

C) K is *closure finite*, i.e. each cell has only a finite number of faces.

W) X has the *Weak Topology* induced by K, i.e. A subset $S \subset X$ is closed if and only if $S \cap e^n_\alpha$ is closed in e^n_α for each n, α.

We easily see that any simplicial complex is a CW-complex on its underlying polyhedron.

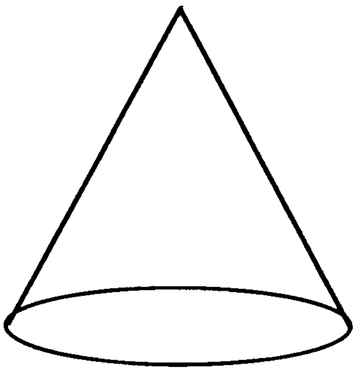
The following theorem follows directly from Lemmas 6.4, 6.5, 6.6 of [Sw].

2.8.3. Theorem. *If (X, A) is a relative CW-Complex and A is contractible, then the projection $p : (X, A) \longrightarrow (X/A, *)$ is a homotopy equivalence.*

3. DUNCE HATS

3.1. Introduction.

The traditional dunce hat, which was supposedly worn by grammar school pupils of an era gone by for failure to grasp the lessons of their teacher, was a triangular piece of material where two sides were sewn together. The third side formed a circle that fit around the head and thus formed a hat in the shape of a cone.



Traditional Dunce Hat

Fig. 3.1

The topological dunce hat is the simplicial abstraction of the traditional dunce hat where the seam is sewn to the circular base. The topological dunce hat is the simplest example of a polyhedron that is contractible in the sense of homotopy, but not collapsible in the sense of J. H. C. Whitehead. We shall henceforth omit any reference to traditional or topological dunce hat, and it will be understood that dunce hat will mean topological dunce hat.

In this chapter we construct a sequence of spaces D^n that are n-dimensional dunce hats in the sense of Thomas [T] and prove the following three properties:

1. D^n is not collapsible.

2. For $n = 0, 1, 2, \ldots$, D^{2n} is contractible.

3. For $n = 0, 1, 2, \ldots$, D^{2n+1} is homotopically equivalent to S^{2n+1}.

In a preprint by Marjanovic and Schori [M-S] a homology proof is given for the second property. In this paper a homotopy proof is given.

3.2. An n-dimensional dunce hat.

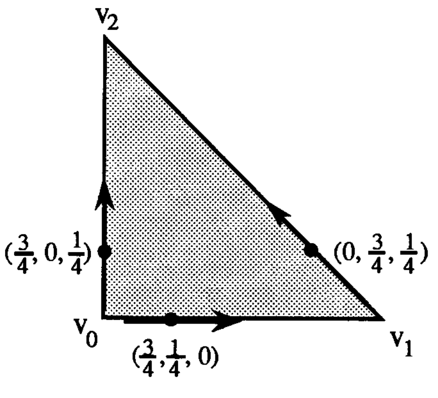
In this section we will define the 2-dimensional dunce hat and generalize that definition to n-dimensional dunce hats for n = 0,1,.... We also show that the n-dimensional dunce hat is not collapsible for n = 1,2,....

3.2.1. Definition. Let $(\lambda_1, \lambda_2, \lambda_3)$ represent the barycentric coordinates of a point in Δ^2 and identify points according to the following;

$$(1,0,0) \equiv (0,1,0) \equiv (0,0,1)$$

$$(a,b,0) \equiv (a,0,b) \equiv (0,a,b)$$

then the resulting space is called the *dunce hat.*



Dunce Hat

Fig. 3.2

In Figure 3.2, the dunce hat is formed by identifying the three vertices to a single point, and by identifying the edges as indicated by the arrows.

We use the following definition by Thomas [T] to define the higher dimensional dunce hats.

3.2.2. Definition. Let Δ^n be the standard n-simplex. The *n-dimensional dunce hat* D^n is the quotient space of Δ^n where points are identified as follows: If $p, q \in \Delta^n$, then $p \equiv q$ if and only if the ordered r-tuples, $1 \leq r \leq n+1$, of non-zero barycentric coordinates of p and q are equal.

We illustrate this definition by observing that for Δ^2, the points $(3/4, 1/4, 0) \equiv (0, 3/4, 1/4) \equiv (3/4, 0, 1/4)$ as seen in Figure 3.2.

To see that the $n-$dimensional dunce hat is not collapsible we follow the polyhedral definition of collapsing in Zeeman [Z].

3.2.3. Definition. Let X be a polyhedron and Y a subpolyhedron. There is an *elementary collapse* from X to Y if for some n there is an $n-$ball B^n with face B^{n-1} such that

$$X = Y \cup B^n$$

$$B^{n-1} = Y \cap B^n.$$

We describe the elementary collapse from X to Y by saying *collapse across B^n onto B^{n-1}*, or *collapse across B^n from B_*^{n-1}* where B_*^{n-1} is the complementary face of B^n. We say *X collapses to Y*, written $X \searrow Y$ if there is a sequence of elementary collapses

$$X = X_0 \searrow X_1 \searrow \dots \searrow X_n = Y.$$

If Y is a point we call X *collapsible* and write $X \searrow 0$.

3.2.4. Theorem. *The n-dimensional dunce hat D^n is collapsible if and only if $n = 0$.*

Proof. For any polyhedron X to be collapsible there must exist a cell $B \subset X$ with a free face on which to begin the collapse. In the dunce hat all faces have been identified so as to eliminate all free faces and thus D^n is not collapsible for any $n \neq 0$. For $n = 0$, D^0 is a single point and thus is trivially collapsible. ■

3.3. Symmetric Products.

The higher dimensional dunce hats can be developed through the notion of symmetric products. We will use the symmetric product characterization of dunce hats to prove the second two properties. In particular we will need to define the quotient map, q_n, from the n-simplex Δ^n to the n-dimensional dunce hat D^n. This is easily done through the notion of symmetric products.

3.3.1. Definition. Let X be a topological space. We designate the set of all nonempty closed subsets of X by 2^X, i.e. $2^X = \{A | A$ is a non-empty closed subset of $X\}$.

We define a topology on 2^X by: If $G_1, \ldots, G_n$ are open sets in X, let $U(G_1, \ldots, G_n) = \{A \in 2^X | A \subset \bigcup_{i=1}^{n} G_i \quad \& \quad A \cap G_i \neq \emptyset$ for each $i = 1, \ldots, n\}$. The set of all such $U(G_1, \ldots, G_n)$ is a basis for a topology on 2^X called the *Vietoris (finite) topology on* 2^X.

3.3.2. Definition. If A is a subset of a metric space X let $N_\epsilon A$ represent the set of all points in X whose distance to A is less than ϵ. Let (X, d) be a bounded metric space. We define the metric on 2^X as follows: If $A, B \in 2^X$, then

$$D(A, B) = inf\{\epsilon > 0 | A \subset N_\epsilon B \quad \& \quad B \subset N_\epsilon A\}.$$

The topology induced by this metric is called the*Hausdorff metric topology.*

3.3.3. Definition. If X is T_1 and $n \geq 1$, then the *n-fold symmetric product* of X is $X(n) = \{A \in 2^X | card\ A \leq n\}$.

We define a map $f : X^n \longrightarrow X(n)$ by $f(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$. If $x_i = x_j$, then they represent the same element in $\{x_1, \ldots, x_n\}$. E.g. $\{0, 1/2, 1/2, 1\} = \{0, 1/2, 1\}$.

We define an equivalence relation , $\equiv$, on X^n by

$$(x_1 \ldots, x_n) \equiv (y_1, \ldots, y_n) \Longleftrightarrow \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}.$$

E.g. $(1,0,0) \equiv (0,1,0) \equiv (0,0,1) \equiv (1,1,0) \equiv (1,0,1) \equiv (0,1,1)$ in I^3.

From [S] we have the following two results:

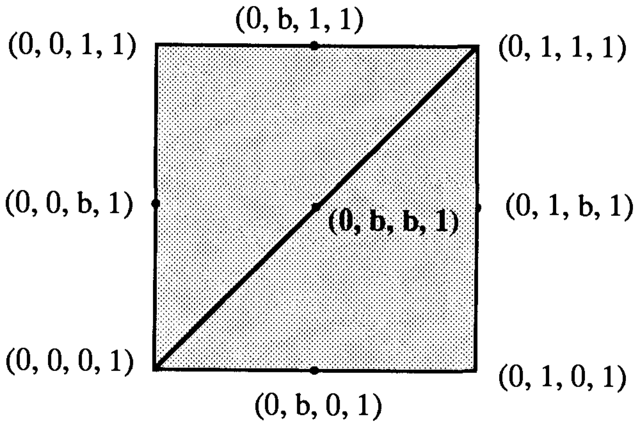
3.3.4. Theorem. *If X is a compact metric space, then the Vietoris topology on 2^X is equivalent to the Hausdorff metric topology on 2^X.*

3.3.5. Theorem. *If X is T_1, then $X(n) = X^n/\equiv$.*

3.3.6. Definition. For $I = [0,1]$, we shall let

$$I_0(n) = \{A \in I(n) | 0 \in A\} \text{ and}$$
$$I_0^1(n) = \{A \in I(n) | 0, 1 \in A\}.$$

For example, consider $I_0^1(4)$. In view of theorem 3.3.5., $I_0^1(4)$ is the projection of $E = \{(a,b,c,d) | a,b,c,d \in I$, where at least one coordinate is 0 and one coordinate is 1$\}$ under the quotient map. See Figure 3.3 for a picture of one 'leaf' of E.



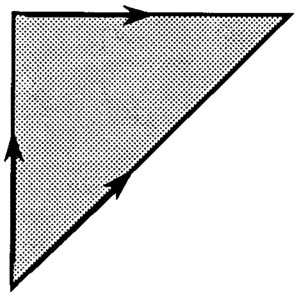
Leaf of E

Fig. 3.3

Notice that $(0,0,0,1) \equiv (0,0,1,1) \equiv (0,1,0,1) \equiv (0,1,1,1)$

and $(0,b,0,1) \equiv (0,0,b,1) \equiv (0,b,1,1) \equiv (0,1,b,1) \equiv (0,b,b,1)$.

Figure 3.3 reduces under the equivalence relation to the following figure where the edges are identified as indicated and all vertices are identified to a single point.



Quotient of E

Fig. 3.4

This is the usual representation for the 2-dimensional dunce hat.

3.3.7. Theorem. $I_0^1(n+2) \cong D^n$.

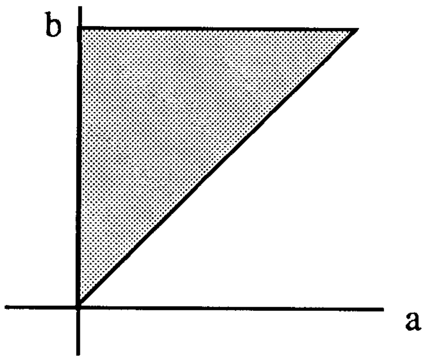
Proof. We first give an alternate presentation for the standard n-simplex, $\Delta^n = \{\sum_{i=0}^{n} \lambda_i v_i | \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1\}$,

where $v_0 = (0,0,\ldots,0)$, $v_1 = (1,0,\ldots,0), \ldots$ $v_n = (0,0,\ldots,1)$.

We define a 'nonstandard' n-simplex, $\underline{\Delta}^n$, by

$$\underline{\Delta}^n = \{(a_1,\ldots,a_n) | 0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq 1\}.$$

$\underline{\Delta}^2 = \{(a,b) | 0 \leq a \leq b \leq 1\}$



Non Standard n-Simplex

Fig. 3.5

We define the homeomorphism from the standard simplex Δ^n represented in barycentric coordinates to the nonstandard simplex $\underline{\Delta}^n$ represented in rectangular coordinates by

$$h(\sum_{i=0}^{n} \lambda_i v_i) = (\lambda_0, \lambda_0 + \lambda_1, \ldots, \lambda_0 + \lambda_1 + \ldots + \lambda_{n-1})$$

and its inverse is given by

$$h^{-1}(a_1, \ldots, a_n) = a_1 v_0 + (a_2 - a_1)v_1 + \ldots + (a_n - a_{n-1})v_{n-1} + (1 - a_n)v_n.$$

We now define the projection map $q_n : \underline{\Delta}^n \longrightarrow I_0^1(n+2)$ from the nonstandard n-simplex, $\underline{\Delta}^n$, to $I_0^1(n+2)$ by

$$q_n((a_1, a_2, \ldots, a_n)) = \{0, a_1, a_2, \ldots, a_n, 1\}.$$

Let $P_n : \Delta^n \longrightarrow D^n$ be the projection map from the standard n-simplex, Δ^n, to the dunce hat, D^n, as defined in definition 3.2.2.. Let $x = (\lambda_0, \ldots, \lambda_n)$ and $y = (\kappa_0, \ldots, \kappa_n)$ be barycentric coordinates of points in Δ^n.

We need only show that $P_n(x) = P_n(y)$ if and only if $q_n \circ h(x) = q_n \circ h(y)$ to induce a homeomorphism $f : D^n \longrightarrow I_0^1(n+2)$.

$$\begin{array}{lccc} & \Delta^n & \xrightarrow{h} & \underline{\Delta}^n \\ (3.1) & \downarrow{\scriptstyle P_n} & & \downarrow{\scriptstyle q_n} \\ & D^n & \xrightarrow{f} & I_0^1(n+2) \end{array}$$

Now, $P_n(x) = P_n(y)$ if and only if the ordered r-tuples of non-zero barycentric coordinates of x and y are equal, and this is true if and only if $(\lambda_0, \lambda_0 + \lambda_1, \ldots, \lambda_0 + \ldots + \lambda_{n-1})$ and $(\kappa_0, \kappa_0 + \kappa_1, \ldots, \kappa_0 + \ldots + \kappa_{n-1})$ have the same coordinate values in the same order disregarding duplication, and with the possible exceptions of the first and last coordinates where one may have the entry 0 or 1, respectively, and the others first or last entry is equal to the first or last non 0 or non 1 entry of the former.

This is true if and only if the following equations hold:

$$\begin{aligned} q_n \circ h(x) &= q_n(\lambda_0, \lambda_0 + \lambda_1, \ldots, \lambda_0 + \ldots + \lambda_{n-1}) \\ &= \{0, \lambda_0, \lambda_0 + \lambda_1, \ldots, \lambda_0 + \ldots + \lambda_{n-1}, 1\} \\ &= \{0, \kappa_0, \kappa_0 + \kappa_1, \ldots, \kappa_0 + \ldots + \kappa_{n-1}, 1\} \\ &= q_n(\kappa_0, \kappa_0 + \kappa_1, \ldots, \kappa_0 + \ldots + \kappa_{n-1}) \\ &= q_n \circ h(y). \quad \blacksquare \end{aligned}$$

E. g. consider the points indicated in Figure 3.2.

$$\begin{array}{ccccc} (3/4, 0, 1/4) & \xrightarrow{h} & (3/4, 3/4) & \xrightarrow{q_2} & \{0, 3/4, 1\} \\ (3/4, 1/4, 0) & \xrightarrow{h} & (3/4, 1) & \xrightarrow{q_2} & \{0, 3/4, 1\} \\ (0, 3/4, 1/4) & \xrightarrow{h} & (0, 3/4) & \xrightarrow{q_2} & \{0, 3/4, 1\} \end{array}$$

3.3.8. Corollary. $D^{n-1} \subset D^n$.

Proof. $I_0^1(n+1) \subset I_0^1(n+2)$. $\blacksquare$

3.4. The relationship between D^{n-1} and D^n.

Since D^{n-1} and D^n are quotient spaces of Δ^{n-1} and Δ^n respectively, the relationship between D^{n-1} and D^n is closely related to the relationship between Δ^{n-1} and Δ^n.

We begin by recalling the definition of the canonical face maps $d_i^n : \Delta^n \longrightarrow \partial\Delta^{n+1}$ between standard simplices.

3.4.1. Definition. The canonical *face maps* $d_i^n : \Delta^n \longrightarrow (\Delta^{n+1})^{(n)}$, for $i = 0, 1, \ldots, n+1$ are defined by

$$d_i^n\left(\sum_{j=0}^{n} \lambda_j v_j\right) = \sum_{j=0}^{i-1} \lambda_j v_j + 0 v_i + \sum_{j=i+1}^{n+1} \lambda_{j-1} v_j.$$

We can now define the face maps between the nonstandard simplices in terms of these face maps.

3.4.2. Definition. Let h be the homeomorphism between standard and nonstandard simplices. The canonical face map between nonstandard simplices is defined as $\underline{d}_i^n = h \circ d_i^n \circ h^{-1}$.

Let $(a_1, \ldots, a_n) \in \underline{\Delta}^n$. We now directly compute $\underline{d}_i^n(a_1, \ldots, a_n)$.

For $i \neq 0, n+1$ we have

$$\begin{aligned} h \circ d_i^n \circ h^{-1}(a_1, \ldots, a_n) &= h \circ d_i^n((a_1 v_0 + (a_2 - a_1)v_1 + \ldots \\ &\quad + (1 - a_n)v_n) \\ &= h(a_1 v_0 + (a_2 - a_1)v_1 + \ldots + (a_i - a_{i-1})v_{i-1} \\ &\quad + 0v_i + (a_{i+1} - a_i)v_{i+1} + \ldots + (1 - a_n)v_{n+1}) \\ &= (a_1, a_2, \ldots, a_i, a_i, a_{i+1}, \ldots, a_n). \end{aligned}$$

It is easily checked that for $i = 0$ and $i = n+1$ we have

$$\underline{d}_0((a_1, \ldots, a_n)) = (0, a_1, \ldots, a_n) \quad \text{and}$$

$$\underline{d}_{n+1}((a_1, \ldots, a_{n-1})) = (a_1, \ldots, a_n, 1).$$

For the remainder of the paper simplices and face maps will be understood to be nonstandard and thus we will omit the underline symbol to designate them, and simply use the symbols Δ^n and d_i^n.

Recall that $q_n : \Delta^n \longrightarrow D^n = I_0^1(n+2)$ is defined by

$$q_n(a_1, \ldots, a_n) = \{0, a_1, a_2, \ldots, a_n, 1\}.$$

We let $\bar{q}_n : (\Delta^n)^{(n-1)} \longrightarrow D^{n-1}$ denote the restriction of q_n to $(\Delta^n)^{(n-1)}$.

3.4.3. Lemma. *For each* $i = 0, 1, \ldots, n,\quad \bar{q}_{n+1} \circ d_i^n = q_n : \Delta^n \longrightarrow D^n$.

Proof. We have

$$\begin{aligned} \bar{q}_{n+1} \circ d_i^n((a_1, a_2, \ldots, a_n)) &= \bar{q}_{n+1}((a_1, a_2, \ldots, a_i, a_i, a_{i+1}, \ldots, a_n)) \\ &= \{0, a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n-1}, 1\} \\ &= q_n(a_1, a_2, \ldots, a_n). \quad \blacksquare \end{aligned}$$

Diagram (3.2) is a commutative diagram showing the relationships of these maps.

$$
(3.2) \qquad
\begin{tikzcd}
\Delta^n \arrow[r, "q_n"] \arrow[d, "d_i^n"] & D^n \\
(\Delta^{n+1})^{(n)} \arrow[ur, "\bar{q}_{n+1}"] &
\end{tikzcd}
$$

3.4.4. Lemma. *For each $n \geq 1$, $D^n \cong \Delta^n \cup_{\bar{q}_n} D^{n-1}$.*

Proof. Since $\bar{q}_n : (\Delta^n)^{(n-1)} \longrightarrow D^{n-1}$ is onto we have the naturally defined quotient map $\alpha : \Delta^n \longrightarrow \Delta^n \cup_{\bar{q}_n} D^{n-1}$. The map $q_n : \Delta^n \longrightarrow D^n$ is one-to-one on the interior of Δ^n and is equal to $\bar{q}_n$ on $(\Delta^n)^{n-1}$ and hence q_n and α induce the same equivalence relation on Δ^n. Thus there exists a homeomorphism
$h : \Delta^n \cup_{\bar{q}_n} D^{n-1} \longrightarrow D^n$.

3.4.5. Corollary. *For each $n \geq 1$, $D^n / D^{n-1} \cong S^n$.*

Proof. The composition of maps $\Delta^n \xrightarrow{\alpha} \Delta^n \cup_{\bar{q}_n} D^{n-1} \xrightarrow{h} D^n \xrightarrow{p} D^n/D^{n-1}$ clearly shows that $D^n/D^{n-1} \cong \Delta^n/(\Delta^n)^{(n-1)} \cong S^n$. ∎

The following lemma and its corollary follow directly from Theorem 2.8.3.

3.4.6. Lemma. ([Wh] Cor. 5.12 chap. 1) *Let A, B, X be CW-Complexes. If $A \subset X$, $h : A \longrightarrow B$ is a homotopy equivalence, then $f : X \longrightarrow X \bigcup_h B$ is also a homotopy equivalence, where f is defined by*

$$f(x) = \begin{cases} h(x) & \text{if } x \in A \\ x & \text{if } x \in X - A. \end{cases}$$

3.4.7. Corollary. *For A, X CW-Complexes, if A is a contractible closed subset of X, then the identification map $p : X \longrightarrow X/A$ defined by*

$$p(x) = \begin{cases} * & \text{if } x \in A \\ x & \text{if } x \in X - A \end{cases}$$

is a homotopy equivalence.

Proof. The map $p : A \longrightarrow *$ is a homotopy equivalence since A is contractible, and $X/A = X \bigcup_p *$. ■

3.4.8. Corollary. *If D^{n-1} is contractible, then $D^n \simeq S^n$.*

The following commutative diagram exemplifies the relationships of the simplices, dunce hats, and maps.

$$
\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & \\
\Delta^{n-1} \arrow[r, "q_{n-1}"] \arrow[d, "d_i^{n-1}"] & D^{n-1} \arrow[r, "p"] \arrow[dd, "inc"] & D^{n-1}/D^{n-2} \cong S^{n-1} \\
\partial\Delta^n \arrow[ur, "\bar{q}_n"] \arrow[d, "inc"] & & \\
\Delta^n \arrow[r, "q_n"] \arrow[d, "d_i^n"] & D^n \arrow[r, "p"] \arrow[dd, "inc"] & D^n/D^{n-1} \cong S^n \\
\partial\Delta^{n+1} \arrow[ur, "\bar{q}_{n+1}"] \arrow[d, "inc"] & & \\
\Delta^{n+1} \arrow[r, "q_{n+1}"] \arrow[d] & D^{n+1} \arrow[r, "p"] \arrow[d] & D^{n+1}/D^n \cong S^{n+1} \\
{} & {} &
\end{tikzcd}
\tag{3.3}
$$

3.5. The Main Result.

To complete the proofs of properties 2 and 3 we need two technical lemmas and the Homotopy Addition theorem.

3.5.1. Lemma. *If $f, g : X \longrightarrow Y$ are maps and $h : Y \longrightarrow Z$ is a homotopy equivalence where $h \circ f \simeq h \circ g$, then $f \simeq g$.*

Proof. $f \simeq h' \circ h \circ f \simeq h' \circ h \circ g \simeq g$. ■

3.5.2. Lemma. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are homotopy equivalences, then $g \circ f$ is a homotopy equivalence.*

Proof. Let f' and g' be the respective homotopy inverses. Then we have $f' \circ g' \circ g \circ f \simeq id \simeq g \circ f \circ f' \circ g'$. ∎

We will use the Homotopy Addition Theorem as found in Hu [Hu]. This is a more general version than Theorem 2.7.3.

3.5.3. Homotopy Addition Theorem. *For any map* $f : ((\Delta^{n+1})^{(n)}, (\Delta^{n+1})^{(n-1)}) \longrightarrow (X, x_0)$, *the homotopy class of* $f, [f] \in \pi_n(X, x_0)$, *and for* $n \geq 2$ *we always have* $[f] = \sum_{i=0}^{n+1} (-1)^i [f \circ d_i^n]$, *where the* $d_i^n : \Delta^n \longrightarrow \partial\Delta^{n+1}$ *are the face maps, and for* $n = 1$ *we have* $[f] = [f \circ d_2^1] \cdot [f \circ d_0^1] \cdot [f \circ d_1^1]^{-1}$.

We are now ready to complete the proofs. We need one final lemma.

3.5.4. Lemma. *If* D^{n-1} *is contractible and* n *is odd, then* D^{n+1} *is contractible.*

Proof. Let $p : D^n \longrightarrow S^n$ be the homotopy equivalence of Corollary 3.4.7.

Consider the map $p \circ \bar{q}_{n+1} : (\partial\Delta^{n+1}, (\Delta^{n+1})^{(n-1)}) \longrightarrow (S^n, *)$.

By virtue of the Homotopy Addition Thereom we have the following result:

For $n = 1$,

$$\begin{aligned}[p \circ \bar{q}_2] &= [p \circ \bar{q}_2 \circ d_2^1] \cdot [p \circ \bar{q}_2 \circ d_0^1] \cdot [p \circ \bar{q}_2 \circ d_1^1]^{-1} \\ &= [p \circ q_1] \cdot [p \circ q_1] \cdot [p \circ q_1]^{-1} = [p \circ q_1]\end{aligned}$$

and for $n \geq 2$, we have

$$[p \circ \bar{q}_{n+1}] = \sum_{i=0}^{n+1} (-1)^i [p \circ \bar{q}_{n+1} \circ d_i^n] = \sum_{i=0}^{n+1} (-1)^i [p \circ q_n].$$

Since n is odd, we have an odd number of maps each of which is the same except for sign, and thus $[p \circ \bar{q}_{n+1}] = [p \circ q_n]$. Thus, $[p \circ \bar{q}_{n+1}]$ as an element of $\pi_n(S^n, *)$ is represented by $p \circ q_n : (\Delta^n, \partial\Delta^n) \longrightarrow (S^n, *)$, which represents the identity element of $\pi_n(S^n)$ since the restriction of $p \circ q_n$ to the interior of Δ^n is a homeomorphism.

Consequently, $p \circ \bar{q}_{n+1} : \partial\Delta^{n+1} \longrightarrow S^n$ is homotopic to a homeomorphism and is therefore a homotopy equivalence.

By hypothesis, D^{n-1} is contractible and consequently $p : D^n \longrightarrow D^n/D^{n-1}$ is a homotopy equivalence by Corollary 3.4.7. Therefore if p' is a homotopy inverse of p, then $\bar{q}_{n+1} \simeq p' \circ p \circ \bar{q}_{n+1} : \partial\Delta^{n+1} \longrightarrow D^n$ is a homotopy equivalence. It follows directly from Lemmas 3.4.4 and 3.4.6 that $D^{n+1} \cong \Delta^{n+1} \cup_{\bar{q}_{n+1}} D^n \simeq \Delta^{n+1}$, which is contractible. ∎

3.5.5. Theorem. *D^{2n} is contractible for $n = 0, 1, \ldots$.*

Proof. We have shown that if $n-1$ is even and D^{n-1} is contractible, then D^{n+1} is contractible. We observe that D^0 is a single point and thus contractible. Thus, we conclude that D^{2n} is contractible for $n = 0, 1, \ldots$. ∎

3.5.6. Corollary. $D^{2n+1} \simeq S^{2n+1}$ for $n = 0, 1, \ldots$.

Proof. Since D^{2n} is contractible, we have $D^{2n+1} \simeq S^{2n+1}$ for $n = 0, 1, \ldots$. ∎

4. CONSTRUCTION TECHNIQUES APPLIED TO THE CELL LIKE DIMENSION RAISING MAP PROBLEM

4.1. Introduction.

This section is motivated by a long and important history in topology. The original and historical problem was due to Alexandrov and the second became widely known in the 1960's as an important problem in infinite dimensional topology, manifold topology, and homotopy theory.

Problem 1. (Alexandrov) Does there exist an infinite-dimensional compactum that has finite cohomological dimension?

Problem 2. (Cell-Like Dimension Raising Mapping Problem) Does there exist a cell-like map between compacta that raises dimension?

In 1978, Robert D. Edwards [Ed] announced two results that had a significant impact on the area. The first, was written up and publilshed by John Walsh [W] and shows one direction in the proof of the equivalence of Problems 1 and 2.

4.1.1. Theorem. (Edwards-Walsh) *If Y is a compact metric space with $c - dim\ Y \leq n$, then Y is the cell-like image of a compact metric space X with $dim\ X \leq n$.*

Thus, if Y is an infinite-dimensional compactum with finite-cohomological dimension, then Y is the image of a cell-like dimension raising map. The other direction is a consequence of the Vietoris-Begle mapping theorem. A complete discussion of this equivalence is found in Walsh [W].

The validity of the other result announced by Edwards in [Ed] is the specific motivation of this section. We list it here as a conjecture.

4.1.2. Conjecture. (Edwards [Ed]) *There exists an infinite-dimensional compactum with finite cohomological dimension if there exists an inverse sequence of spheres*

$$S^{m_0} \xleftarrow{f_1'} S^{m_1} \xleftarrow{f_2'} S^{m_2} \longleftarrow \ldots, \qquad \text{where } m_i < m_j \quad \text{for} \quad i < j$$

and where all of the bonding maps have been suspended at least once and all finite compositions of the maps are essential.

This conjecture has been worked on seriously by at least a hand full of topologists including Edwards, Walsh, Coppola, West, and Schori. A methodology evolved for solving this conjecture and it is the purpose of this section to elucidate this methodology and show that in its current form it will not work. In the process we have taken construction techniques involving cones, mapping cones, suspensions, and wedges that are in the tool box of homotopy theorists for use in the homotopy category and we have formulated and proved many results in the topological category. Apparently several of our formulations are new and should be of particular interest to point-set and geometric topologists.

To complete the history discussed in this introduction we remark that in 1988 A.N. Dranishnikov [D] announced the following result:

4.1.3. Theorem *There exists an infinite dimensional compact metric space with the integral cohomological dimension equal to three.*

This result answers both problems listed above in the affirmative but does not settle conjecture 4.1.2.

4.2. Definitions and General Strategy.

The notion of cell-likeness is equivalent to the shape theory concept of having the shape of a point. A convenient setting for cell-like sets is in the generalization of simplicial complexes called absolute neighborhood retracts. A metric space X

is an ***absolute neighborhood retract*** (ANR) if for each closed embedding of X in a metric space Y, there is a neighborhood of X in Y that retracts to X.

4.2.1. Definition. A compact metric space C is *cell-like* if for each embedding h of C into an ANR Y and for each neighborhood U of $h(C)$ in Y, $h(C)$ can be contracted to a point in U.

A mapping $f : X \longrightarrow Y$ is a *proper map* if the inverse image of each compact subset of Y is compact.

4.2.2. Definition. A map $f : X \longrightarrow Y$ is *cell-like* if it is proper and surjective and $f^{-1}(y)$ is a cell-like space for each y in Y.

The methodology developed for proving Edward's conjecture was based largely on a remark by J. J. Walsh in the paper *Dimension, Cohomological Dimension, and Cell-Like Mappings*.

4.2.3. Walsh's Remark. "The adjunction space $L_j \bigcup_{g_{jk}} L_k^{(n)}$ has a 'natural' structure of a finite CW-complex having no cells in dimension $n+1$, $n+2, \ldots, n+i$. A consequence (for example see [MS]) is that a compactum X having cohomological dimension $\leq n$ is the limit of a sequence $\{L_q, f_q\}$ satisfying:

1) Each L_q is a finite CW-complex.

2) For each q there is an integer $i_{(q)}$ such that $L_q^{(n)} = L_q^{(n+1)} = \ldots = L_q^{(n+i_{(q)})}$ and $\lim_{q \to \infty} i_{(q)} = \infty$.

3) For each q, there is an $\varepsilon_{(q)} > 0$ such that $sup\{diam \quad f_{\infty q}^{-1}(e) | e$ is a cell of $L_q\} < \varepsilon_{(q)}$ and $\lim_{q \to \infty} \varepsilon_{(q)} = 0$."

Assuming the existence on an inverse sequence of spheres of increasing dimension with essential bonding maps and all compositions essential, create a sequence of spaces and maps satisfying Walsh's comment but whose inverse limit has infinite dimension. The existence of such an inverse sequence of spheres is unknown.

Assume the existence of an inverse sequence of spheres

$$S^{m_0} \xleftarrow{f_1'} S^{m_1} \xleftarrow{f_2'} S^{m_2} \longleftarrow \ldots, \qquad \text{where } m_i < m_j \quad \text{for} \quad i < j$$

where all maps are suspensions and all finite compositions $f_i = f_1' \circ \ldots \circ f_i'$ are essential. The simplex Δ^{m_0+2} is a CW-complex with cells of dimension $0, 1, \ldots m_0 + 2$. If $r_1 = 2$, and $r_k = (m_{k-1} + r_{k-1}) - m_0$, we want to construct spaces:

$$\begin{array}{rl} L_2 \text{ with cells of dimension} & 0, 1, \ldots, m_0, m_1 + 1, m_1 + r_1, \\ L_3 \text{ with cells of dimension} & 0, 1, \ldots, m_0, m_2 + 1, m_2 + 2, \ldots, m_2 + r_2, \\ & \vdots \\ L_{k+1} \text{ with cells of dimension} & 0, 1, \ldots, m_0, m_k + 1, m_k + 2, \ldots, m_k + r_k, \\ & \vdots \end{array}$$

The technique for constructing these spaces will be inductive and use the mapping cone $M_f = CX \bigcup_f Y$ where the map $f : X \longrightarrow Y$ is essential and a suspension. We require f to be a suspension so that we may use the Homotopy Addition Theorem to construct a finite sequence of maps and spaces for the construction of each L_i.

To guarantee criteria 2 we construct the spaces L_k such that $m_i < m_j$, if $i < j$. This creates larger gaps in the dimension of the cells as higher dimensional CW-complexes are constructed.

To guarantee criteria 3 we create L_{k+1} from L_k by taking a sufficiently fine subdivision of L_k and 'replace' cells of lower dimension with cells of higher dimension.

To guarantee $dim\ L = \infty$, L is constructed so that $c - dim\ L < dim\ L_1$ and the bonding maps are constructed so that they are essential and all compositions are essential. This will imply that $dim\ L \geq dim\ L_1 > c - dim\ L$ and thus by a theorem of Kozlowski [K] $dim\ L = \infty$.

We let $L = \varprojlim\{L_q, f_q\}$ and we notice that if Walsh's conditions are satisfied $c - dimL \leq m_0$.

It was hoped that the dimension m_0 would have a finite upper bound, however the construction technique developed in this paper leads one to the conclusion that as the higher dimensional complexes are constructed the dimension m_0 must also rise and thus this strategy to construct the desired space must fail. As stated before we believe the construction techniques developed in this paper are of value.

In this paper we will focus on constructing spaces 'up to homotopy type' and maps that are 'homotopy equivalent'.

4.3. Some construction Techniques.

If we have an essential map $f : S^m \longrightarrow S^n$ (e.g. the Hopf map $h : S^3 \longrightarrow S^2$), then the mapping cone $M_f = CS^m \bigcup_f S^n = e^{m+1} \bigcup_f S^n$ is a CW-complex consisting of an $m+1$ cell and an n cell, and furthermore the boundary of the $m+1$ cell, in M_f, is an n sphere. In the Hopf map example $M_h = e^4 \bigcup_h S^2$ is known to be the complex projective 2-space CP^2.

4.3.1. Definition. Diagrams of maps and spaces are said to be *homotopy equivalent* if the corresponding maps and spaces are homotopy equivalent.

We decompose the m+1 sphere into the three closed subsets, the northern hemisphere, the southern hemishere, and equator. We indicate this decomposition with the following diagram where the vertical arrows are inclusions.

$$S^{m+1} = \begin{cases} CS^m \\ \uparrow \\ S_m \\ \downarrow \\ CS^m \end{cases} \tag{4.1}$$

We will often use the following standard result in homotopy theory.

4.3.2. Theorem. *A map $f : S^m \longrightarrow X$ is inessential if and only if f can be extended to a map $\phi : CS^m \longrightarrow X$.*

We now combine these ideas.

4.3.3. Theorem. *The diagram*

$$S^{m+1} = \left\{ \begin{tikzcd} CS^m \arrow[r, "\hat{f}"] & M_f \\ S^m \arrow[u, "in"'] \arrow[d, "in"] \arrow[r, "f"] & X \arrow[u, "in"'] \\ CS^m \arrow[ur, "\phi"] & \end{tikzcd} \right. \tag{4.2}$$

defines a suspension map $g : S^{m+1} \longrightarrow M_f$ *if the map* $f : S^m \longrightarrow X$ *is the suspension of an inessential map.*

Proof. Let $F : S^{m-1} \longrightarrow T$ be an inessential map where $f \simeq SF$. Then F extends to $\phi' : CS^{m-1} \longrightarrow T$, and we have the following diagram.

$$S^{m} = \left\{ \begin{tikzcd} CS^{m-1} \arrow[r, "\hat{F}"] & M_F \\ S^{m-1} \arrow[u, "in"'] \arrow[d, "in"] \arrow[r, "F"] & T \arrow[u, "in"'] \\ CS^{m-1} \arrow[ur, "\phi'"] & \end{tikzcd} \right. \tag{4.3}$$

This defines a map $G : S^m \longrightarrow M_F$, the suspension of which, $SG : S^{m+1} \longrightarrow SM_F$ is defined by the diagram:

$$SS^{m} = \left\{ \begin{tikzcd} SCS^{m-1} \arrow[r, "S\hat{F}"] & SM_F \\ SS^{m-1} \arrow[u, "in"'] \arrow[d, "in"] \arrow[r, "SF"] & ST \arrow[u, "in"'] \\ SCS^{m-1} \arrow[ur, "S\phi'"] & \end{tikzcd} \right. \tag{4.4}$$

which is homotopy equivalent to:

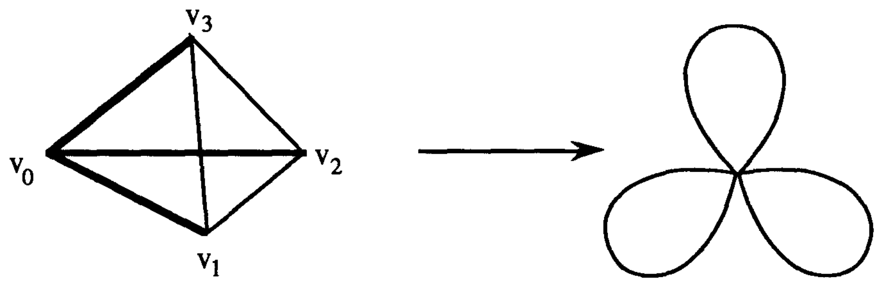
$$S^{m+1} = \left\{ \begin{tikzcd} CS^m \arrow[r, "\hat{f}"] & M_f \\ S^m \arrow[u, "in"'] \arrow[r, "f"] \arrow[d, "in"'] & X \arrow[u, "in"'] \\ CS^m \arrow[ur, "\phi"] & \end{tikzcd} \right. \tag{4.5}$$

■

It is known that the Hopf map provides an example of a suspension map $Sf : SX \longrightarrow SY$ that is inessential where the map on the equators $f : X \longrightarrow Y$ is essential. This example is given in 4.4.2.

4.3.4. Lemma. *$(\Delta^{n+2})^{(n)} \simeq \bigvee^{k-1} S^n$ where $k = \binom{n+3}{n+2} = n+3$, the number of $n+1$ cells in Δ^{n+2}.*

Proof. The n-cells containing the vertex v_0 contracts to v_0 yielding a wedge of $(k-1)$ n-spheres. ■



Wedge of (k-1) n-spheres

Fig. 4.1

Our next job is to generalize the following version of the Homotopy Addition Theorem which was proved in Chapter 2.

We restate theorem 2.7.3.

Theorem 4.3.5. (Homotopy Addition Theorem) If the maps $d_i : \partial\Delta_n \longrightarrow (\Delta^{n+1})^{(n-1)}$ are the restrictions of the standard face maps to $\partial\Delta^n$, then

$$\sum_{i=0}^{n+1}(-1)^i d_i = 0.$$

We make the following remarks as a motivation for generalizing this version of the HAT.

Remark 1. As discussed in Section 2.6, the summation statement in Theorem 4.3.5 can be represented geometrically as

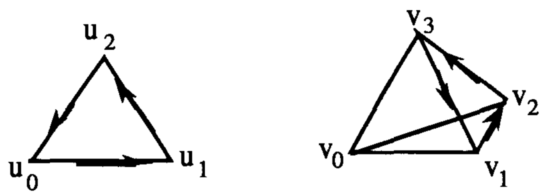
$$S^{n-1} \xrightarrow{P} \bigvee^{n+2} S^{n-1} \xrightarrow{d} (\Delta^{n+1})^{(n-1)}$$

where $P = P^{n+2}$ is the (n+2) pinch map and $d = ((-1)^i d_i)$.

By Lemma 4.3.4, $(\Delta^{n+1})^{(n-1)}$ is homotopy equivalent to $\bigvee^{n+1} S^{n-1}$ which facilitates the following remark.

Remark 2. The face map $d_0 : \partial\Delta^n \longrightarrow (\Delta^{n+1})^{(n-1)}$ is homotopy equivalent to the pinch map $P^{n+1} : S^{n-1} \longrightarrow \bigvee^{n+1} S^{n-1}$ and for each $i = 1, \ldots, n+1$, the map $(-1)^i d_i : \partial\Delta^n \longrightarrow (\Delta^{n+1})^{(n-1)}$ is homotopy equivalent to the map $-id_i : S^{n-1} \longrightarrow \bigvee^{n+1} S^{n-1}$ which injects S^{n-1} onto the i^{th} copy of S^{n-1} in the wedge with the orientation reversing homeomorphism.

We illustrate this remark in the case for $n = 2$ as in Figure 4.2.



Face Map of Δ^2 into Δ^3

Fig. 4.2

Let $\Delta^2 =< u_0, u_1, u_2 >$ and $\Delta^3 =< v_0, v_1, v_2, v_3 >$. Then

$$\begin{aligned} d_0(\partial < u_0, u_1, u_2 >) &= d_0(< u_1, u_2 > - < u_0, u_2 > + < u_0, u_1 >) \\ &=< v_2, v_3 > + < v_3, v_1 > + < v_1, v_2 > . \end{aligned}$$

The orientation of these image simplices are the same as the original orientations. This along with the fact that under the contraction mentioned in the proof of Lemma 4.3.4, the edges of $(\Delta^3)^{(1)}$ that contain v_0 will be collapsed to a point yielding a wedge of three S^1's with orientations consistent with the orientations induced by the pinch map P^{n+1}.

To illustrate the $(-1)^i d_i$ map, note that

$$\begin{aligned} -d_1(\partial < u_0, u_1, u_2 >) &= -\partial(< v_0, v_2, v_3 > \\ &= - < v_2, v_3 > + < v_0, v_3 > - < v_0, v_2 > . \end{aligned}$$

In this case the simplices containing v_0 are shrunk to a point leaving the $< v_2, v_3 >$ simplex with the opposite orientation than was originally assigned. This corresponds precisely to the injection of S^1 onto the 1_{st} copy of S^1 in the wedge of three S^1's with a negative orientation.

Now we are ready to state a more general result.

4.3.6. Theorem. (Generalized Homotopy Addition Theorem) *Let X be a compact metric space and consider the diagram*

$$SX \xrightarrow{P} \bigvee^{k} SX \xrightarrow{d} \bigvee^{k-1} SX$$

where $P = P^k$ is a k-fold pinch map and $d = P^{k-1} \vee \bigvee^{k-1} (-id)$. Then $d \circ P \simeq 0$.

Proof. In the range space $\bigvee^{k-1} SX$, with each copy of SX is associated an image of a copy of SX under the id map and the $(-id)$ map. In homotopy this is equivalent to $id + (-id) \simeq 0$ and consequently the result follows (essentially by definition of the maps involved). ∎

4.3.7. Definition. We call the map $d = d_X : \bigvee^{k} SX \longrightarrow \bigvee^{k-1} SX$ the *addition map*.

4.3.8. Theorem. (Left Distributivity) *If $f : SX \longrightarrow SY$ is a suspension map, then the composition of maps in the diagram*

$$SX \xrightarrow{P} \bigvee^{k} SX \xrightarrow{\bigvee^{k} f} \bigvee^{k} SY \xrightarrow{d} \bigvee^{k-1} SY$$

$d \circ \bigvee^{k} f \circ P$ is inessential.

Proof. The result follows directly from Lemma 2.6.5 and Theorem 4.3.5. ■

4.3.9. Corollary. *If $S^{m+1}f : S^{m+1}X \longrightarrow S^{m+1}Y$ is an $(m+1)-$suspension map, then $d \circ \bigvee^{k} Sf \circ P$ is inessential.*

Proof. The result follows from the preceeding theorem by a finite inductive argument. ■

4.3.10. Lemma. *The addition map $d_{SX} : \bigvee^{k} S^2X \longrightarrow \bigvee^{k-1} S^2X$ defined by $d_{SX} = P_X^{k-1} \vee \bigvee^{k-1}(-id)$ is a suspension. In particular, the map $d_{SX} : \bigvee^{k} S^2X \longrightarrow \bigvee^{k-1} S^2X$ is topologically equivalent to $Sd_X : S(\bigvee^{k} SX) \longrightarrow S(\bigvee^{k-1} SX)$ where $d_X = P_X^{k-1} \vee \bigvee^{k-1}(-id)$.*

Proof. By Lemma 2.4.8, the wedge of suspension maps is a suspension.

4.3.11. Theorem. *If $f : SX \longrightarrow SY$ is a suspension map, then the composition of maps in the diagram*

$$S^2X \xrightarrow{P} \bigvee^{k} S^2X \xrightarrow{\bigvee^{k} Sf} \bigvee^{k} S^2Y \xrightarrow{d} \bigvee^{k-1} S^2Y$$

$d \circ \bigvee^{k} Sf \circ P : S^2X \longrightarrow \bigvee^{k-1} S^2Y$ is a suspension map. In fact

$d \circ \bigvee^{k} f \circ P : S^2X \longrightarrow \bigvee^{k-1} S^2Y$ is topologically equivalent to
$S(d \circ \bigvee^{k} f \circ P : S(SX) \longrightarrow S(\bigvee^{k-1} SY)$.

Proof. The three maps of the composition are suspension maps, hence the composition is a suspension. ■

4.4. Constructing a suspension map.

In this section we construct an m-suspension map between an $m+2$-suspension and the mapping cone $M_{d\circ \bigvee^{k} f}$.

Let $f : S^{m+1}X \longrightarrow S^{m+1}Y$ be an $(m+1)$-suspension map.
Thus $f \simeq S^{m+1}F' : S^{m+1}X \longrightarrow S^{m+1}Y$. We desuspend f m times to get the map $SF' : SX \longrightarrow SY$. For convenience let $SF' = F$. Decompose S^2X by

$$(4.6) \qquad S^2X = \left\{ \begin{array}{c} CSX \\ \uparrow in \\ S^{m+1} \\ \downarrow in \\ CS^{X} \end{array} \right.$$

and consider the diagram:

$$(4.7) \qquad S^2X = \left\{ \begin{array}{ccccccc} CSX & \xrightarrow{CP} & \bigvee^{k} CSX & \xrightarrow{\widehat{\bigvee^{k} F}} & M_{\bigvee^{k} F} & \xrightarrow{\hat{d}} & M_{d\circ \bigvee^{k} F} \\ \uparrow in & & \uparrow in & & \uparrow in & & \uparrow in \\ SX & \xrightarrow{P} & \bigvee^{k} SX & \xrightarrow{\bigvee^{k} F} & \bigvee^{k} SY & \xrightarrow{d} & \bigvee^{k-1} SY \\ \downarrow in & & & & & & \uparrow \phi' \\ CSX & & & \xrightarrow{id} & & & CSX \end{array} \right.$$

By Theorem 4.3.8 the composition of maps $d \circ \bigvee^{k} F \circ P$ is inessential, and thus this composition can be extended to a map $\phi' : CSX \longrightarrow \bigvee^{k-1} SY$.

The diagram defines a map $G : S^2X \longrightarrow M_{d\circ\vee F}$. We suspend this diagram m times to get the following diagram;

(4.8)

$$S^{m+2}X = \left\{ \begin{array}{ccccccc}
S^mCSX & \xrightarrow{CP} & S^m\bigvee^k CSX & \xrightarrow{S^m\widehat{\bigvee^k F}} & S^mM_{\overset{k}{\vee} F} & \xrightarrow{\hat{d}} & S^mM_{d\circ\overset{k}{\vee} F} \\
\uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} \\
S^mSX & \xrightarrow{P} & S^m\bigvee^k SX & \xrightarrow{S^m\bigvee^k F} & S^m\bigvee^k SY & \xrightarrow{d} & S^m\bigvee^{k-1} SY \\
\downarrow{\scriptstyle in} & & & & & & \uparrow{\scriptstyle \phi} \\
S^mCSX & & & \xrightarrow{id} & & & S^mCSX
\end{array} \right.$$

which defines the map $S^mG : S^{m+2}X \longrightarrow S^mM_{d\circ\vee F}$.

The diagram in (4.8) is homotopy equivalent to the following diagram;

(4.9)

$$S^{m+2}X = \left\{ \begin{array}{ccccccc}
CS^{m+1}X & \xrightarrow{CP} & \bigvee^k CS^{m+1}X & \xrightarrow{\widehat{\bigvee^k f}} & M_{\overset{k}{\vee} f} & \xrightarrow{\hat{d}} & M_{d\circ\overset{k}{\vee} f} \\
\uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} \\
S^{m+1}X & \xrightarrow{P} & \bigvee^k S^{m+1}X & \xrightarrow{\bigvee^k f} & \bigvee^k S^{m+1}Y & \xrightarrow{d} & \bigvee^{k-1} S^{m+1}Y \\
\downarrow{\scriptstyle in} & & & & & & \uparrow{\scriptstyle \phi} \\
CS^{m+1}X & & & \xrightarrow{id} & & & CS^{m+1}X
\end{array} \right.$$

where $\phi \simeq S^m\phi'$.

We thus have the following result.

4.4.1. Theorem. *If $f : S^{m+1}X \longrightarrow S^{m+1}Y$ is an $(m+1)-$suspension map, then diagram (4.9) defines an $m-$suspension map $g : S^{m+2}X \longrightarrow M_{d\circ\overset{k}{\vee} f}$.*

4.4.2. Remark. It was hoped that given an $m+1$ suspension map the diagram of Theorem 4.4.1 would yield an $m+1$ suspension map. However this is possible only if $d \circ \bigvee^k F' \circ P$ is inessential. We observe here that $d \circ \bigvee^k F \circ P$

was inessential because the pinch map could be chosen to run transverse to the equator. If the desuspension of F, F', is not a suspension map the pinch map cannot run transverse to the equator and hence we cannot guarantee the inessentiality of $d \circ \bigvee^{k} F' \circ P$.

We provide here a key example which shows that if the suspension of a map Sf is inessential, that the map f need not be inessential.

It is well known that $\pi_3(S^2) \simeq \mathbb{Z}$ and that the Hopf map $p : S^3 \longrightarrow S^2$ represents a generator. It is also true, [Hu, p 328] that $\pi_4(S^3) \simeq \mathbb{Z}_2$ and the suspension of the Hopf map Sp represents a generator. Consequently, if $f = p + p$, then Sf is inessential whereas f is essential.

We note at the end of section 4.5 that this appears to be an insurmountable obstacle.

4.5. Constructing a replacement simplicial complex.

In the following construction we will have frequent use of the composition of face maps as shown in diagram 4.10.

(4.10)
$$\begin{tikzcd}
 & & {} & \\
 & & (\Delta^{n+3})^{(n+2)} \arrow[u, "in"'] \arrow[r, "d_k"] & {} \\
 & (\Delta^{n+2})^{(n+1)} \arrow[r, "d_j"] & (\Delta^{n+3})^{(n+1)} \arrow[u, "in"'] \arrow[r, "d_k"] & {} \\
(\Delta^{n+1})^{(n)} \arrow[r, "d_i"] & (\Delta^{n+2})^{(n)} \arrow[u, "in"'] \arrow[r, "d_j"] & (\Delta^{n+3})^{(n)} \arrow[u, "in"'] \arrow[r, "d_k"] & {} \\
(\Delta^{n+1})^{(n-1)} \arrow[u, "in"'] \arrow[r, "d_i"] & (\Delta^{n+2})^{(n-1)} \arrow[u, "in"'] \arrow[r, "d_j"] & (\Delta^{n+3})^{(n-1)} \arrow[u, "in"'] \arrow[r, "d_k"] & {}
\end{tikzcd}$$

Let $f : S^m \longrightarrow S^n$, $m \geq n$, be a suspension map. We first will show how to modify Δ^{n+2} to obtain a CW-complex with cells of dimension $m+2,\ m+1,\ n,$

$n-1, \ldots, 1, 0$. Let $k = n+3$, the number of $(n+1)$-faces of Δ^{n+2}, let $d = ((-1)^i d_i)$, where $d_i : (\Delta^{n+1})^n \longrightarrow (\Delta^{n+2})^n$, $i = 0, \ldots, n+2$, are the face maps, and observe that $(\Delta^{n+1})^n \cong S^n$. The composition of maps

$$\bigvee^k S^m \xrightarrow{\bigvee^k f} \bigvee^k S^n \xrightarrow{d} (\Delta^{n+2})^{(n)}$$

leads to the following mapping cone construction diagram:

$$\text{(4.11)} \qquad \begin{array}{ccc} \bigvee e^{m+1} & \xrightarrow{\widehat{d\circ\vee f}} & M_{d\circ\vee f} \\ \uparrow & & \uparrow \\ \bigvee S^m & \xrightarrow{d\circ\vee f} & (\Delta^{n+2})^{(n)} \end{array}$$

The map $d \circ \vee f$ induces the mapping cone $M_{d\circ\vee f} = \bigvee^k e^{m+1} \bigcup_{d\circ\vee f} (\Delta^{n+2})^{(n)}$ which can be thought of as the $(n+1)$-skeleton of Δ^{n+2} where the $(n+1)$-cells have been replaced by $(m+1)$-cells. We introduce the notation $M_{d\circ\vee f} \equiv (\Delta^{n+2})^{(n+1)}(\frac{m+1}{n+1})$ to emphasize this property.

Now we want to 'replace' the $n+2$ cell with an $m+2$ cell.

$$\text{Decompose } S^{m+1} \text{ as} \quad S^{m+1} = \left\{ \begin{array}{l} CS^m = e_+^{m+1} \\ \uparrow inc \\ S^m \\ \downarrow inc \\ CS^m = e_-^{m+1}. \end{array} \right.$$

Since $(\Delta^{n+2})^{(n)} \simeq \bigvee^{n+2} S^n$ and by Theorem 4.4.1, the diagram

$$\text{(4.12)} \qquad S^{m+1} = \left\{ \begin{array}{ccccccc} e_+^{m+1} & \xrightarrow{CP} & \bigvee^k e^{m+1} & \xrightarrow{\widehat{\bigvee^k f}} & M_{\vee^k f} & \xrightarrow{\hat{d}} & M_{d\circ\vee^k f} \\ \uparrow in & & \uparrow in & & \uparrow in & & \uparrow in \\ S^m & \xrightarrow{P} & \bigvee^k S^m & \xrightarrow{\bigvee^k f} & \bigvee^k S^n & \xrightarrow{d} & (\Delta^{n+2})^{(n)} \\ \downarrow in & & & & & & \uparrow \phi \\ e_-^{m+1} & & & \xrightarrow{id} & & & e_-^{m+1} \end{array} \right.$$

defines a map $g_1 : S^{m+1} \longrightarrow M_{d \circ \overset{k}{\vee} f}$ that is a 1-suspension, and

$$M_{g_1} = e^{m+2} \bigcup_{g_1} M_{d \circ \overset{k}{\vee} f} \equiv \Delta^{n+2}(\frac{m+2}{n+2}, \frac{m+1}{n+1}).$$

The mapping cone M_{g_1} is a CW-complex with cells of dimension $m+2$, $m+1, n, \ldots, 1, 0$. We view M_{g_1} as an $n+2$ simplex where the open $n+2$ and $n+1$ cells have been replaced by open $m+2$ and $m+1$ cells, respectively.

For simplification of notation we will let $\Delta(p,q)$ represent the $(n+q)$ skeleton of the $(n+p)$ simplex where the $n+1,\ n+2, \ldots, n+q$ faces have been 'replaced' with $m+1,\ m+2, \ldots, m+q$ faces, respectively. Notationally, we use

$$\Delta(p,q) = (\Delta^{n+p})^{(n+q)}(\frac{m+q}{n+q}, \ldots, \frac{m+2}{n+2}, \frac{m+1}{n+1}).$$

Thus in our construction above we have constructed $M_{d \circ \vee f} = \Delta(2,1)$ and $M_{g_1} = \Delta(2,2)$. We view this construction as the first step in an inductive construction of $\Delta(r,r)$ for each positive integer $r \geq 2$. For $r > 2$, we now describe the construction in steps. The steps will be referred to when describing the inductive steps of the construction.

Step 1). Given $\Delta(2,1)$, construct the spaces $\Delta(3,1), \ldots, \Delta(r,1)$ and maps $\hat{d}_j : \Delta(2,1) \longrightarrow \Delta(3,1)$, $\hat{d}_k : \Delta(3,1) \longrightarrow \Delta(4,1), \ldots$. Starting from diagram 4.12 construct the following diagram where $d = (d_i)$.

$$(4.13)\quad \begin{array}{ccccccccc} \bigvee e^{m+1} & \xrightarrow{\widehat{d \circ \vee f}} & \Delta(2,1) & \xrightarrow{\hat{d}_j} & \Delta(3,1) & \xrightarrow{\hat{d}_k} & \ldots & \xrightarrow{\hat{d}_l} & \Delta(r,1) \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\ \bigvee S^m & \xrightarrow{d \circ \vee f} & (\Delta^{n+2})^{(n)} & \xrightarrow{d_j} & (\Delta^{n+3})^{(n)} & \xrightarrow{d_k} & \ldots & \xrightarrow{d_l} & (\Delta^{n+r})^{(n)} \end{array}$$

The spaces $\Delta(3,1), \ldots, \Delta(r,1)$ are obtained in a natural way. The space $\Delta(3,1) = (\Delta^{n+3})^{(n+1)}(\frac{m+1}{n+1})$ is constructed as the mapping cone $M_{d \circ \vee f}$ in

$$(4.14)\quad \begin{array}{ccccc} \bigvee^{k_1} e^{m+1} & \longrightarrow & M_{d \circ \vee f} & = & \Delta(3,1) \\ \uparrow & & \uparrow & & \\ \bigvee^{k_1} S^m & \xrightarrow{d \circ \vee f} & (\Delta^{n+3})^{(n)} & & \end{array}$$

where $k_1 = \binom{n+4}{n+2}$ = the number of $(n+1)$-faces of Δ^{n+3}, and $d = (d_j \circ d_i)$ for all allowable combinations of i, j where d_i, d_j are the face maps from diagram 4.10.

To construct the maps $\hat{d}_j$, $\hat{d}_k, \ldots$ we use Lemma 2.5.3. which is illustrated by the diagram

$$
\begin{array}{ccccc}
CX & \xrightarrow{\hat{f}} & M_f & \xrightarrow{\hat{g}_1} & M_{g_1 \circ f} \\
\uparrow & & \uparrow & & \uparrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g_1} & Z
\end{array}
\tag{4.15}
$$

which provides for the existence of $\hat{g}_1 : M_f \longrightarrow M_{g_1 \circ f}$.

Step 2). Construction of $\Delta(3,2)$.

We construct $\Delta(3,2)$ as a mapping cone. The composition of maps

$$\bigvee^{k} S^{m+1} \xrightarrow{\vee g_1} \bigvee^{k} \Delta(2,1) \xrightarrow{\hat{d}} \Delta(3,1),$$

where $\hat{d} = (\hat{d}_j)$ and $k = n+4$, the number of $(n+2)$-faces of Δ^{n+3}, yields the diagram

$$
\begin{array}{ccccc}
\bigvee e^{m+2} & \longrightarrow & M_{\hat{d} \circ \vee g_1} & = & \Delta(3,2) \\
\uparrow & & \uparrow & & \\
\bigvee S^{m+1} & \longrightarrow & \Delta(3,1) & &
\end{array}
\tag{4.16}
$$

which gives $\Delta(3,2)$.

Step 3). Constructing the map $g_2 : S^{m+2} \longrightarrow \Delta(3,2)$. Consider the following diagram;

$$
S^{m+2} = \left\{
\begin{array}{ccccccc}
e_+^{m+2} & \xrightarrow{CP} & \bigvee^{k_2} e^{m+2} & \xrightarrow{\bigvee^{k_2} \hat{g}_1} & \bigvee^{k_2} M_{g_1} & \xrightarrow{\hat{d}} & \Delta(3,2) \\
\uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} \\
S^{m+1} & \xrightarrow{P} & \bigvee^{k_2} S^{m+1} & \xrightarrow{\bigvee^{k_2} g_1} & \bigvee^{k_2} \Delta(2,1) & \xrightarrow{\tilde{d}} & \Delta(3,1) \\
\downarrow{\scriptstyle in} & & & & & & \uparrow{\scriptstyle \phi} \\
e_-^{m+2} & & & \xrightarrow{id} & & & e_-^{m+2}
\end{array}
\right.
\tag{4.17}
$$

where $\hat{d} = ((-1)^j\hat{d}_j)$ and $\hat{\hat{d}} = ((-1)^j\hat{\hat{d}}_j)$, where the $\hat{d}_j$ maps come from diagram 4.13 and the $\hat{\hat{d}}_j$ maps come from an application of Lemma 2.5.3.

We have $((-1)^j\tilde{d}_j) \circ \bigvee^{k_2} g_1 \circ P = (\sum_{j=0}^{k_2}((-1)^j\tilde{d}_j)) \circ (g_1)$, and we claim $\sum_{j=0}^{k_2}(-1)^j\tilde{d}_j = 0$. If we let $\mathcal{U}_k$ represent the $(m+1)$-cell in $\Delta(2,1)$ that replaces the $(n+1)$-cell in $(\Delta^{n+2})^{(n+1)}$ we see that in the argument that

$$\sum_{j=0}^{k_3}(-1)^j d_j : (\Delta^{n+2})^{(n+1)} \longrightarrow (\Delta^{n+3})^{(n+1)}$$

is inessential, every cell $\mathcal{U}_k$ appears twice in the sum with opposite sign and hence the sum is 0, as in the $\sum(-1)^j d_j$ case.

Thus $((-1)^j\tilde{d}_j) \circ \bigvee^{k_2} g_1 \circ P : S^{m+1} \longrightarrow \Delta(3,1,)$ is inessential and thus by Theorem 4.4.1 and Lemma 2.5.6, if g_1 is a suspension map then the diagram defines a map $g_2 : S^{m+2} \longrightarrow \Delta(3,2)$ and $M_{g_2} = e^{m+3} \bigcup_{g_2} \Delta(3,2) \equiv (\Delta^{n+3})(\frac{m+3}{n+3}, \frac{m+2}{n+2}, \frac{m+1}{n+1}) \equiv \Delta(3,3)$, a CW-complex with cells of dimension $m+3, m+2, m+1, n, n-1, \ldots, 1, 0$. This completes Step 3).

If $r = 3$, then $M_{g_2} = \Delta(3,3)$ and we are finished. If $r > 3$, we will repeat Steps 1) - 3) in abreviated form, but showing the major changes. The diagram for Step 1) becomes

$$\begin{array}{lccccccc} & \bigvee e^{m+2} & \longrightarrow & \Delta(3,2) & \xrightarrow{\hat{d}_k} & \Delta(4,2) & \ldots \xrightarrow{\hat{d}_l} & \Delta(r,2) \\ (4.18) & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & \bigvee S^{m+1} & \xrightarrow{\hat{d}\circ\vee g_1} & (\Delta(3,1) & \xrightarrow{\hat{d}_k} & \Delta(4,1) & \longrightarrow \ldots \xrightarrow{\hat{d}_l} & \Delta(r,1) \end{array}$$

where the spaces $\Delta(4,2), \ldots, \Delta(r,2)$ are constructed as before. For $\Delta(4,2)$, let $k_2 = \binom{n+5}{n+3}$ = the number of $(n+2)$-faces of Δ^{n+4}, and let $d = (\hat{d}_k \circ \hat{d}_j)$. Then the mapping cone $M_{d\circ\vee g_2}$ in the diagram

$$\begin{array}{lcccc} & \bigvee^{k_2} e^{m+2} & \longrightarrow & M_{d\circ\vee g_1} & = \Delta(4,2) \\ (4.19) & \uparrow & & \uparrow & \\ & \bigvee^{k_2} S^{m+1} & \xrightarrow{d\circ\vee g_1} & \Delta(4,1) & \end{array}$$

is the space $\Delta(4,2)$. The maps $\hat{\hat{d}}_k, \ldots, \hat{\hat{d}}_l$ are obtained with applications of Lemma 2.5.3.

To construct $\Delta(4,3)$, the diagram from Step 2) becomes

$$\bigvee^{k_3} S^{m+2} \xrightarrow{\vee g_2} \bigvee^{k_3} \Delta(3,2) \xrightarrow{\hat{\hat{d}}} \Delta(4,2)$$

where $\hat{\hat{d}} = (\hat{\hat{d}}_k)$ and $k_3 = n+5$ and then $\Delta(4,3) = M_{\hat{\hat{d}} \circ \vee g_2}$.

A diagram corresponding to the one in Step 3) yields $g_3 : S^{m+3} \longrightarrow \Delta(4,3)$ and $M_{g_3} = e^{m+3} \bigcup_{g_3} \Delta(4,3) = \Delta(4,4)$.

In general to construct $\Delta(r,r)$ given that we have constructed $\Delta(r-1, r-1)$ we construct, in order, the adjunction spaces, $\Delta(r, r-2)$ and $\Delta(r, r-1)$ and construct the map $g_{(r-1)} : S^{m+r-1} \longrightarrow \Delta(r, r-1)$ using the commutative diagram presented in this section. Then $M_{g_{(r-1)}} = e^{m+r} \bigcup_{g_{(r-1)}} \Delta(r, r-1) = \Delta(r,r)$.

We observe that the construction of $g_{(r-1)}$ will require the original map f to be an $r-1$ suspension map. Hence we have $f \simeq S^{r-1}F : S^m \longrightarrow S^n$ and thus we are forced to conclude $n \geq r$. This conclusion appears to be a fatal flaw in this construction technique. As we construct higher dimensional spaces for our inverse sequence we are also forced to increase the dimension of the initial space of the sequence, and hence defeats our goals.

We conclude this paper with the construction of the spaces of the inverse sequence and the construction of the bonding maps to show that the rest of the construction methodology for the problem works.

4.6. Constructing the spaces of an inverse sequence.

We assume there exists an inverse sequence of spheres of increasing dimension where the bonding maps and all compositions are essential and suspensions of sufficient order.

$$S^{m_0} \xleftarrow{f'_1} S^{m_1} \xleftarrow{f'_2} S^{m_2} \xleftarrow{f'_3} \ldots, \quad m_i < m_j \text{ for } i < j.$$

Let $f_i = f'_1 \circ f'_2 \circ \ldots f'_i : (S^{m_i}, *) \longrightarrow (S^{m_0}, *)$.

Let $f = f_1$, $n = m_0$ and construct the space $\Delta(r_0, r_0)$ where $r_0 = 2$.

Let $f = f_2$, $n = m_0$, and $r_1 = r_0 + (m_1 - m_0)$ and construct the space $\Delta(r_1, r_1)$. We iteratively construct spaces $\Delta(r_i, r_i)$ where $f = f_{i+1}$, $n = m_0$, and $r_i = r_0 + (m_i - m_0)$.

We now construct the spaces $L_1, L_2, \ldots$ of the inverse sequence. Let $L_1 = \Delta^{m_0+r_0}$, and let $L_2 = \Delta(r_0, r_0)$, a CW-complex with cells of dimension $0, 1, \ldots, m_0, m_1 + 1, \ldots m_1 + r_0$. We take a simplicial subdivision of L_2 to get a space L'_2 that has cells of all dimensions $0, 1, \ldots, m_1 + r_0$.

We now use the Δ-functor of R.F. Williams [Wi] to replace all $m_1 + r_0$-cells of L'_2 with $\Delta(r_1, r_1)$ to form L_3. L_3 is a CW-complex with cells of dimension $0, 1, \ldots, m_0, m_2 + 1, \ldots, m_2 + r_1$.

Again we take a simplicial subdivision to get a space L'_3 that has cells of all dimensions $0, 1, \ldots, m_2 + r_1$ and we use the Δ-functor of R.F. Williams to replace all $m_2 + r_1$ cells of L'_3 with $\Delta(r_2, r_2)$ to form L_4.

We continue iteratively in this fashion to construct the spaces $L_5, L_6, \ldots$.

By taking simplicial subdivisions we can guarantee the diameter of each cell of L_q decreases to zero as q goes to ∞. We also notice that the gap in dimensions increases as $q \to \infty$, and clearly each L_q is a finite CW-complex. Thus the three conditions of Walsh's remark are met.

We have yet to construct the bonding maps betwen the coordinate spaces.

4.7. Constructing the bonding maps.

To construct the bonding map $r_k : L_{k+1} \longrightarrow L_k$ we simply need to construct a map $r : \Delta(r, r) \longrightarrow \Delta^{n+r}$ where $\Delta(r, r)$ is a 'cell' in L_{k+1} and Δ^{n+r} is the corresponding cell in L_k and extend r to all cells in L_{k+1}.

4.7.1.Lemma. *If X is compact and if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are surjective maps, then there are maps $\hat{f}$ and $\bar{g}$ such that the following diagram commutes, where the vertical arrows are injections:*

$$
\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
\downarrow & & \downarrow & & \downarrow \\
CX & \xrightarrow{\hat{f}} & M_f & \xrightarrow{\bar{g}} & CZ
\end{array}
\tag{4.20}
$$

Proof. The map $\hat{f}$ is defined as in Lemma 2.5.3. and the map $\bar{g}$ is defined by

$$\bar{g}(\hat{f}[x,t]) = [(g \circ f(x), t]. \quad \blacksquare$$

Let $f : S^m \longrightarrow S^n$ be the map in section 4.5. In the diagram

$$
\begin{array}{ccccccc}
S^m & \xrightarrow{f} & S^n & \xrightarrow{id} & S^n & = & (\Delta^{n+1})^{(n)} \\
\downarrow & & \downarrow & & \downarrow & & \\
CS^m & \xrightarrow{\hat{f}} & M_f & \xrightarrow{r_1} & CS^n & = & \Delta^{n+1}
\end{array}
\tag{4.21}
$$

$r_1 : M_f \longrightarrow \Delta^{n+1}$ is determined by Lemma 4.7.1.

In the construction of $\Delta(2,1)$ we used the diagram

$$
\begin{array}{ccccc}
\bigvee e^{m+1} & \xrightarrow{\widehat{\bigvee f}} & \bigvee M_f & \xrightarrow{((-1)^i \hat{d}_i)} & \Delta(2,1) \\
\uparrow & & \uparrow & & \uparrow \\
\bigvee S^m & \xrightarrow{\bigvee f} & \bigvee S^n & \xrightarrow{((-1)^i d_i)} & (\Delta^{n+2})^{(n)}
\end{array}
\tag{4.22}
$$

where the maps d_i and $\hat{d}_i$, $i = 0, \ldots, n+2$, determine the map
$r_2' : \Delta(2,1) \longrightarrow (\Delta^{n+2})^{(n+1)}$ as in the diagram:

$$
\begin{array}{ccc}
\bigvee M_f & \xrightarrow{((-1)^i \hat{d}_i)} & \Delta(2,1) \\
\big\downarrow \bigvee r_1 & & \big\downarrow r_2' \\
\bigvee \Delta^{n+1} & \xrightarrow{((-1)^i d_i)} & (\Delta^{n+2})^{(n+1)}
\end{array}
\tag{4.23}
$$

In the construction of $\Delta(2,1)$ we constructed a map $g_1 : S^{m+1} \longrightarrow \Delta(2,1)$. We use g_1 in the diagram

$$\text{(4.24)} \qquad \begin{tikzcd} S^{m+1} \arrow[r, "g_1"] \arrow[d] & \Delta(2,1) \arrow[r, "r_2'"] \arrow[d] & (\Delta^{n+2})^{(n+1)} \arrow[d] \\ e^{m+2} \arrow[r, "\hat{g}_1"] & \Delta(2,2) \arrow[r, "r_2"] & \Delta^{n+2} \end{tikzcd}$$

to obtain $r_2 : \Delta(2,2) \longrightarrow \Delta^{n+2}$ by Lemma 4.7.1.

In the construction of $\Delta(3,2)$ we used the diagram

$$\text{(4.25)} \qquad \begin{tikzcd} \bigvee e^{m+2} \arrow[r, "\widehat{\bigvee g}"] & \bigvee M_{g_1} \arrow[r, "((-1)^j \hat{\hat{d}}_j)"] & \Delta(3,2) \\ \bigvee S^{m+1} \arrow[r, "\bigvee g"] \arrow[u] & \bigvee \Delta(2,1) \arrow[r, "((-1)^j \hat{d}_j)"] \arrow[u] & (\Delta(3,1) \arrow[u] \end{tikzcd}$$

where the maps $\hat{d}_j$ and $\hat{\hat{d}}_j$, are induced face maps. Thus, we have a uniquiely defined $r_3' : \Delta(3,2) \longrightarrow (\Delta^{n+3})^{(n+2)}$ as determined by the diagram

$$\text{(4.26)} \qquad \begin{tikzcd} \bigvee M_{g_1} \arrow[r, "((-1)^j \hat{\hat{d}}_j)"] \arrow[d, "\bigvee r_2"] & \Delta(3,2) \arrow[d, "r_3'"] \\ \bigvee \Delta^{n+2} \arrow[r, "((-1)^j d_j)"] & (\Delta^{n+3})^{(n+2)}. \end{tikzcd}$$

In the construction of $\Delta(3,2)$ we constructed a map $g_2 : S^{m+2} \longrightarrow \Delta(2,1)$. We use g_2 in the diagram

$$\text{(4.27)} \qquad \begin{tikzcd} S^{m+2} \arrow[r, "g_2"] \arrow[d] & \Delta(3,2) \arrow[r, "r_3'"] \arrow[d] & (\Delta^{n+3})^{(n+2)} \arrow[d] \\ e^{m+3} \arrow[r, "\hat{g}_2"] & \Delta(3,3) \arrow[r, "r_3"] & \Delta^{n+3} \end{tikzcd}$$

to obtain $r_3 : \Delta(3,3) \longrightarrow \Delta^{n+3}$ by Lemma 4.7.1.

We continue in this fashion constructing maps r_k' and r_k untill we have consttructed the map $r_r : \Delta(r,r) \longrightarrow \Delta^{n+r}$.

We can now define our map $r_k : L_{k+1} \longrightarrow L_k$ by $r_k = r_r$ on every $\Delta(r,r)$ cell in L_{k+1}.

4.8. Concluding Remarks and Summary.

We finish this chapter by summarizing the status of the Edward's Conjecture, 4.1.2. The methodology as laid out in Section 4.5 evolved over the years with several researchers as discussed earlier. However, no one has been successful in making the program work, and John Walsh of the University of California, Riverside, is reported to have suspected that the bonding maps between the spheres were being forced to be of higher and higher order of suspension. In this chapter, using some preliminaries from Chapter 2, we have carefully developed the necessary homotopy equivalences between maps and spaces, many of which turned out to be topological equivalences required to carry out this program. In the inductive construction of the spaces, a map $f : S^m \longrightarrow S^n$ was needed to be a suspension in order to construct a map $g : S^{m+1} \longrightarrow M_{d\circ \vee f}$ as in the diagram:

$$(4.28)\qquad S^{m+1} = \left\{ \begin{array}{ccccccc} e_+^{m+1} & \xrightarrow{CP} & \overset{k}{\bigvee} e^{m+1} & \xrightarrow{\overset{\widehat{k}}{\bigvee} f} & M_{\overset{k}{\vee} f} & \xrightarrow{\hat{d}} & M_{d\circ \overset{k}{\vee} f} \\ \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} & & \uparrow{\scriptstyle in} \\ S^m & \xrightarrow{P} & \overset{k}{\bigvee} S^m & \xrightarrow{\overset{k}{\bigvee} f} & \overset{k}{\bigvee} S^n & \xrightarrow{d} & (\Delta^{n+2})^{(n)} \\ \downarrow{\scriptstyle in} & & & & & & \uparrow{\scriptstyle \phi} \\ e_-^{m+1} & & & \xrightarrow{id} & & & e_-^{m+1} \end{array} \right.$$

The map g is then used in place of the map f in a higher dimensional version of this diagram which means that g must be a suspension which by our Theorem 4.4.1 requires f to be a double suspension. In this way, the map f is inductively forced to be a higher and higher order suspension which leads to a contradiction.

We summarize the specific homotopy issue that causes this problem. In diagram 4.28 if f is a suspension, then the whole upper part of the diagram is a suspension but not the map ϕ in the lower part of the diagram. For ϕ to be a suspension, the map $d \circ \vee f \circ P$, which is a suspension and inessential, must desuspend to an inessential map. This need not be the case as we have illustrated with an example constructed from the Hopf map, see 4.4.2. However, our case is quite special in the sense that $d \circ \vee f \circ P$ desuspends to a map of the form $d' \circ \vee f' \circ P'$ where $d = Sd'$, $f = Sf'$, and $P = SP'$, where $d' = (d'_i)$ and $\sum d'_i = 0$, and P' is a pinch map on S^{m-1}. Furthermore, $d' \circ \vee f' \circ P'$ is a 'geometric' version of $\sum(d'_i \circ f')$ and if we have Left Distributivity, $\sum(d'_i \circ f') = \left(\sum d'_i\right) \circ f'$, then we in fact have that this map is nullhomotopic. However, we only have Left Distributivity if f' (in this case) is a suspension; and this is the problem. If f' is a suspension, then f has to be a double suspension, etc. Finally, we point out that Left Distributivity has little chance of succeeding for specific functions unless the function is a suspension and in that case we know it works.

BIBLIOGRAPHY

[A] J.F. Adams, On the Groups J(X) - IV, Topology Vol. 5. (1966), 21 - 71.

[D] A.N. Dranishnikov, An Infinite Dimensional Compact Metric Space with Cohomological Dimension Three, Sbornik vol 135, 1988.

[D-S] A.N. Dranishnikov and E.B. Shchepin, Cell-like maps. The problem of raising dimension, Uspekhi Mat. Nauk 41:6, 1986.

[E] R. Engelking, Dimension Theory, North-Holland Publishing Company, New York, 1978.

[Ed] R.D. Edwards, A theorem and a question related to cohomological dimension and cell-like maps, Notices Amer. Math.Soc. 25, 1978, A-259.

[Hi] P.J. Hilton, An Introduction to Homotopy Theory, Cambridge University Press, London, 1961.

[Hu] S. Hu, Homotopy Theory, Academic Press, New York, 1959.

[H-W] Hurewitz and Wallman, Dimension Theory, Princeton University Press, 1941.

[K] G. Kozlowski, Images of ANR's, Preprint.

[L] R. C. Lacher, Cell-Like Mappings and their Generalizations, Bulletin of the American Mathematical Society, 1977.

[L-W] A. T. Lundell and S Weingram, The Topology of CW Complexes, Van Nostrand Reinhold, New York, 1969.

[M] S. Mardesic, Factorization Theorems for Cohomological Dimension, Preprint.

[M-S] M. Marjanovic and R. M. Schori, Higher Dimensional Dunce Hats, Preprint.

[R] D. Rohm, Alternative characteristics of Weak Infinite-Dimensionality and their relation to a problem of Alexandroff's, Ph.D. Dissertation, Oregon State University, 1987.

[R-S-W] L. Rubin, R. Schori and J. Walsh, New dimension theory techniques for constructing infinite dimensional examples, General Topology Appl., 1979, 93-102.

[S] R.M. Schori, Hyperspaces and symmetric products of topological spaces, Fundamenta Mathematicae LXIII, 1968.

[Sp] E.H. Spanier, Algebraic Topology, McGraw - Hill, NY, 1966.

[Sw] R. M. Switzer Algebraic Topology - Homotopy and Homology, Springer - Verlag, New York, 1975.

[St] N. E. Steenrod, A convenient category of topological spaces, Michigan Math. J. 14, 1967, 133-152.

[T] D. R. Thomas, Symmetric Products of Cubes, Ph.D. Dissertation, State University of New York at Binghampton, 1975.

[W] J. J. Walsh, Dimension, cohomological dimension, and cell-like mappings, Lecture Notes in Mathematics, # 882, Springer-Verlag, New York, 1981.

[Wh] B. W. Whitehead, Elements of Homotopy Theory, Springer - Verlag, New York, 1978.

[Wi] R. F. Williams, A Useful Functor and Three Famous Examples in Topology, Trans. Amer. Math. Soc., 106, 1963, 319-329.

[Z] E.C. Zeeman, On the Dunce Hat, Topology, Vol 2, 1964, 341-358.