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This paper defines four function space topologies, characterizes two of them in terms of more familiar concepts, and compares the four topologies. Then in the cases of the two less familiar topologies we have considered several common properties of topological spaces and attempted to answer the following question: If the range space has a given property, does the function space necessarily have the same property?

FOUR FUNCTION SPACE TOPOLOGIES

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CHAPTER I

Introduction

In Chapter II we shall introduce a method for topologizing a space of functions and consider four topologies given by this method. We then characterize two of these topologies in terms of more familiar concepts and compare the four topologies. The composition map is also considered in relation to all four topologies.

In Chapter III we consider the two less familiar function space topologies in regard to the following natural question. What properties of the range space carry over to the function space? Generally some of the separation properties are inherited by the function space and in one case some subspaces of the function space are shown to inherit a certain property when the whole space does not. For the other properties considered, examples are given to show that the range space can have the property without the function space having it. A knowledge of basic topological concepts is assumed; these topics are discussed in the references listed in the Bibliography.

CHAPTER II

FOUR TOPOLOGIES FOR FUNCTION SPACES

1. General Procedure

If X and Y are sets let $F(X,Y)$ denote the set of all functions mapping X into Y. There are many ways of topologizing $F(X,Y)$, however in this paper we will consider only the following method. Given a topological space $(Y,\mathfrak{J})$ and a collection $\mathfrak{A}$ of subsets of X then the corresponding topology on $F(X,Y)$ has for a subbase the collection of all sets of the form $\{f \in F(X,Y) | f[A] \subseteq G\}$ where $A \in \mathfrak{A}$ and $G \in \mathfrak{J}$. It will be convenient to denote $\{f \in F(X,Y) | f[E] \subseteq H\}$ by (E,H) where E is understood to be a subset of X and H a subset of Y. Since any collection of subsets is a subbase for some topology, this method does define a topology on $F(X,Y)$.

2. Four Specific Topologies

In this paper we are going to be concerned with four different collections $\mathfrak{A}$ of subsets of X. Letting $\mathfrak{A}_1 = \left\{\{x\} | x \in X\right\}$ we have that an arbitrary subbasic open set in the topology corresponding to $\mathfrak{A}_1$ will consist of all those functions in $F(X,Y)$ which map a given point $x_0 \in X$ into a given open set $G_0 \subseteq Y$. For this reason the topology on $F(X,Y)$ corresponding to $\mathfrak{A}_1$ is called the point-open topology. $\mathfrak{A}_2$ will be the collection of all infinite subsets of X; the corresponding topology is then called the infinite-open topology. Now suppose that X is a set with a topology, and let $\mathfrak{a}_3$ be the set of all compact subsets and let $\mathfrak{a}_4$ be the set of all closed subsets of X. The topologies corresponding to $\mathfrak{a}_3$ and $\mathfrak{a}_4$ are known as the compact-open and the closed-open, respectively. For convenience we may write P.O., I.O., C.O. and Cl.O. as abreviations for point-open, infinite-open, compact-open and closed-open, respectively.

3. Characterizations and Comparisons of these Topologies

The compact-open and the point-open topologies have been studied extensively and most of the results concerning them can be found in the texts listed in the Bibliography. The point-open topology has two equivalent characterizations one of which is given in the following definition and theorem.

Definition 2.1. The topology of pointwise convergence on $F(X,Y)$ is that topology obtained by defining convergence in the following manner: A net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $F(X,Y)$ converges to $f \in F(X,Y)$ iff for every $x \in X$ the net $\{f_\lambda(x)\}_{\lambda\in\Lambda}$ converges to $f(x)$ in Y.

Theorem 2.1. The point-open topology is equivalent to the topology of pointwise convergence on $F(X,Y)$.

Proof: Suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is any net of functions in $F(X,Y)$ which converges to f in the P.O.-topology. Let $x_0 \in X$ and any open set G in Y containing $f(x_0)$ be arbitrarily given. So $(\{x_0\}, G)$ is an open set in the P.O.-topology and $f \in (\{x_0\}, G)$ hence there exists a $\lambda_0 \in \Lambda$ for which $\lambda > \lambda_0$ implies $f_\lambda \in (\{x_0\}, G)$, or $f_\lambda(x_0) \in G$. Therefore $\{f_\lambda\}_{\lambda\in\Lambda}$ converges to f in the topology of pointwise convergence.

Now suppose $\{f_\lambda\}_{\lambda\in\Lambda}$ converges to f in the topology of pointwise convergence, and let H be any open set in the P.O.-topology which contains f. Then there exists a basic P.O.-open set B for which $f \in B \subseteq H$. B has the form $(\{x_1\}, B_1) \cap \cdots \cap (\{x_p\}, B_p)$ where the B_i are open sets in Y. Since the convergence is pointwise, corresponding to each x_i, $1 \leq i \leq p$, there is a $\lambda_i \in \Lambda$ such that $\lambda > \lambda_i$ implies $f_\lambda(x_i) \in B_i$. Now since Λ is a directed set there exists a $\lambda_0 \in \Lambda$ such that $\lambda > \lambda_0$ implies $\lambda > \lambda_i$ for $1 \leq i \leq p$. Therefore $\lambda > \lambda_0$ implies $f_\lambda(x_i) \in B_i$ for $1 \leq i \leq p$ which implies $f_\lambda \in B \subseteq G$, hence $\{f_\lambda\}_{\lambda\in\Lambda}$ converges to f in the P.O.-topology.

The set $F(X,Y)$ is the same as the Cartesian product of Y with itself with each element of X giving rise to one factor in the product. Associated with a product space is a natural map called the projection map. The projection map corresponding to an $x \in X$ is denoted π_x and is defined by $\pi_x(f) = f(x)$ for all $f \in F(X,Y)$. As we know the product topology is the smallest topology on $F(X,Y)$ with respect to which the projection maps are all continuous. Hence the class of all inverse images of open sets is a subbase for the product topology. But if G is any open set in Y then $\pi_x^{-1}[G] = \{f \in F(X,Y) | f(x) \in G\} = (\{x\}, G)$ for all $x \in X$. Hence we have proved

Theorem 2.2. The point-open topology is equivalent to the product topology on $F(X,Y)$.

We shall now compare the point-open topology with the other three topologies we have described. Note that if Y has the indiscrete topology then given any collection $\mathfrak{a}$ of subsets of X and any $A \in \mathfrak{a}$ the only subbasic open sets corresponding to A are $(A,\phi) = \phi$ and $(A,Y) = F(X,Y)$, so the topology on $F(X,Y)$ corresponding to $\mathfrak{a}$ is the indiscrete topology. Hence for comparison purposes we shall avoid the case where Y has the indiscrete topology. For a similar reason we shall only consider the case where X is infinite in dealing with the infinite-open topology. If X were finite the subbasis would be empty and using the convention that the empty union is the empty set and the empty intersection is the whole space we obtain the indiscrete topology on $F(X,Y)$.

Theorem 2.3. The compact-open topology always contains the point-open topology and coincides with it iff the only compact subsets of X are finite.

Proof: Since singletons are compact every subbasic open set in the P.O.-topology is also a subbasic open set in the C.O.-topology which proves the first assertion. Now suppose that the only compact subsets of X are finite and (K,G) is an arbitrary subbasic open set of the C.O.-topology. Since K is finite we can write $K = \{x_1, x_2, \ldots, x_q\}$. So $(K,G) = (\{x_1, \ldots, x_q\}, G) = \bigcap_{i=1}^{q} (\{x_i\}, G)$ which is an open set in the point-open topology. Now suppose there is a compact subset of X which is not finite, call it K_0. Then let G be an open subset of Y which is not all of Y. We claim the set (K_0, G) is not a point-open open set. If it were then for any $f \in (K_0, G)$ there would exist $x_1, \ldots, x_p \in X$ and $G_1, \ldots, G_p \subseteq Y$ such that the G_i are open in Y and $f \in (\{x_1\}, G_1) \cap \cdots \cap (\{x_p\}, G_p) \subseteq (K_0, G)$. But since K_0 is infinite there is some $x_0 \in K_0$ such that $x_0 \neq x_i$ for $1 \leq i \leq p$ and there is some point $y_0 \in Y - G$ so that the function $f_0(x) = \begin{cases} y_0, & x = x_0 \\ f(x), & x \neq x_0 \end{cases}$ is in $(\{x_1\}, G) \cap \cdots \cap (\{x_p\}, G_p)$ but not in (K_0, G) which is a contradiction. Hence we have proved the second assertion.

Theorem 2.4. Except in the degenerate cases of Y having the indiscrete topology or X being finite, the

P.O.-topology and the I.O.-topology are not comparable.

Proof: First we shall exhibit a set which is open in the P.O.-topology but not open in the I.O.-topology. Let G be any open set in Y such that $\phi \neq G \neq Y$ and let x_0 be any point of X. Then $(\{x_0\}, G)$ is the set we seek. Pick $y_0, y_1 \in Y$ such that $y_0 \in G$ and $y_1 \in Y - G$, then define $f(x) = \begin{cases} y_0, & x = x_0 \\ y_1, & x \neq x_0 \end{cases}$ so $f \in (\{x_0\}, G)$. Now suppose B is a basic open set in the I.O.-topology such that $f \in B \subseteq (\{x_0\}, g)$. By definition of a basic open set there exist infinite sets $F_i \subseteq X$ and open sets $G_i \subseteq Y$ such that $B = (E_1, G_1) \cap \cdots \cap (E_q, G_q)$. At least one of the E_i must contain x_0 or else B will not be contained in $(\{x_0\}, G)$, so let $E_1, \ldots, E_s$, possibly by renumbering, be the ones containing x_0. Since $f \in B \subseteq (\{x_0\}, G)$ we must have $f(x_0) \in G_1 \cap \cdots \cap G_s \subseteq G$. Since $y_1 \notin G$ there exists a p such that $1 \leq p \leq s$ and $y_1 \notin G_p$. But $f \in B$ implies $f \in (E_p, G_p)$ which is a contradiction since E_p is infinite but the only point mapped inside G_p by f is x_0. Therefore $(\{x_0\}, G)$ is not open in the I.O.-topology.

On the other hand with G as above let (E,G) be any subbasic open set in the I.O.-topology, and let f be any point of (E,G). Suppose H is a basic open set in the point-open topology such that $f \in H \subseteq (E,G)$. Then H has the form $(\{x_1\}, H_1) \cap \cdots \cap (\{x_q\}, G_q)$ and since E is infinite there exists an $x_0 \in E$ such that $x_0 \notin \{x_1,\ldots,x_q\}$. Now consider the function

$$g(x) = \begin{cases} f(x), & x \neq x_0 \\ y_1, & x = x_0 \end{cases}$$

, then $g \in H$ since $f \in H$ yet $g \notin (E,G)$ hence $H \not\subseteq (E,G)$ which is a contradiction. Therefore (E,G) is not open in the P.O.-topology.

Definition 2.2. A topological space $(X,\mathfrak{T})$ is a T_1 space iff for any $x, y \in X$, $x \neq y$ there exists a nbhd. of x not containing y and a nbhd. of y not containing x.

It is easy to show that a space is T_1 iff all singletons are closed sets and that there are spaces which are not T_1. So since singletons are not necessarily closed and closed sets are not necessarily finite the point-open topology and the closed-open topology are not in general comparable. However the following result does hold:

Theorem 2.5. If X is a T_1 space, the Cl.O-topology contains the point-open topology and they coincide iff X is a finite T_1 space.

Proof: If X is a T_1 space then singletons are closed and any subbasic open set in the P.O.-topology is also a subbasic open set in the Cl.O.-topology which proves the first part. Now suppose X is a finite T_1 space.

Let (C,G) be any arbitrary subbasic open set in the Cl.O.-topology. Then since X is finite C must be finite so suppose $C = \{x_1,\ldots,x_r\}$, hence $(C,G) = (\{x_1\}, G) \cap \cdots \cap (\{x_r\}, G)$ which is open in the P.O.-topology so the topologies coincide. In the discussion preceding this theorem we ruled out the case where X was not T_1 so to prove the "only if" part suppose that X is not finite. Then if G is an open set in Y such that $\phi \neq G \neq Y$, then (X,G) is open in the Cl.O.-topology but not open in the P.O.-topology as was shown in the proof of Theorem 2.4.

We shall now characterize the compact-open topology in terms of a useful concept of analysis, namely that of uniform convergence on compact subsets.

<u>Definition 2.3</u>. Let (Y,d) be a metric space and let X be an arbitrary topological space. A sequence $\{f_n\}_{n=1}^{\infty}$ in $F(X,Y)$ is said to converge to $f \in F(X,Y)$ uniformly on every compact subset if for every compact $K \subseteq X$ and for every $\varepsilon > 0$ there is an integer $N = N(K,\varepsilon)$ such that $d(f(c), f_n(c)) < \varepsilon$ for every $n \geq N$ and every $c \in K$.

We shall let $C(X,Y)$ denote the subset of $F(X,Y)$ consisting of the continuous functions. Now $C(X,Y)$ is to be given the relative topology and a subbase for the relative topology on $C(X,Y)$ consists of all intersections of

$C(X,Y)$ with elements of a subbase for the topology on $F(X,Y)$. Therefore for convenience we shall write $(K,G)'$ for $(K,G) \cap C(X,Y)$. Restricting attention to this subspace the following theorem describes convergence in the C.O.-topology.

Theorem 2.6. Let (Y,d) be a metric space and let X be any topological space. A sequence $\{f_n\}_{n=1}^{\infty}$ in $C(X,Y)$ converges to an $f \in C(X,Y)$ uniformly on each compact subset iff $f_n \to f$ in the compact-open topology on $C(X,Y)$.

For the proof we shall need

Lemma 2.1. If (X,d) is a metric space and $A,B \subseteq X$ such that A is compact, B is closed and $A \cap B = \phi$, then $d(A,B) > 0$.

Proof: Suppose $d(A,B) = 0$ then given $\varepsilon = \frac{1}{n}$ there exist points $x_n \in A$ and $y_n \in B$ such that $d(x_n,y_n) < \frac{1}{n}$. Now since a compact subset of a metric space is countably compact we have that the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence which converges to a point $x \in A$. Claim x is also a limit point of B, since given $\varepsilon > 0$, we can choose n so large that $d(x,x_n) < \frac{\varepsilon}{2}$ and $\frac{1}{n} \leq \frac{\varepsilon}{2}$ so that $d(x,y_n) \leq d(x,x_n) + d(x_n,y_n) < \frac{\varepsilon}{2} + \frac{1}{n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence x is a limit point of B so, since B is closed, we have $x \in B$. Therefore $A \cap B \neq \phi$ which is a contradiction, so $d(A,B) > 0$.

Proof of Theorem 2.6: Suppose $f_n \to f$ uniformly on each compact subset of X. Let $(K,G)'$ be any subbasic open set in the C.O.-topology which contains f. Since f is continuous $f[K]$ is compact and $f[K] \subseteq G$ so $f[K]$ and $Y - G$ are a compact set and a disjoint closed set. So Lemma 2.1 tells us that there is an $\varepsilon > 0$ such that $d(f[K], Y - G) = \varepsilon$. Now pick N so that $n \geq N$ implies $d(f(c), f_n(c)) < \varepsilon$ for all $c \in K$; then clearly $f_n \in (K,G)'$ for all $n \geq N$. Therefore $f_n \to f$ in the compact-open topology.

Now suppose $f_n \to f$ in the C.O.-topology and let K be any compact set and $\varepsilon > 0$ be given. We want to show that $f_n \to f$ uniformly on K. Again since f is continuous $f[K]$ is compact so there exists a finite number of points x_i, $1 \leq i \leq n$ in K such that $\{y \in Y | d(y, f(x_i)) < \frac{\varepsilon}{3}\}$, $1 \leq i \leq n$, cover $f[K]$. Let

$K_i = K \cap f^{-1}\left[\overline{\{y \in Y | d(y, f(x_i)) < \frac{\varepsilon}{3}\}}\right]$ and let

$G_i = \{y \in Y | d(f(x_i), y) < \frac{2\varepsilon}{3}\}$. Now since f is continuous K_i is a closed subset of a compact set and hence is compact. We want to show that $f \in \bigcap_{i=1}^{n} (K_i, G_i) = B$, and that $g \in B$ implies $d(f(c), g(c)) < \varepsilon$ for all $c \in K$.

Now $f[K_i] \subseteq \overline{\{y \in Y | d(f(x_i), y) < \frac{\varepsilon}{3}\}}$ $\subseteq \{y \in Y | d(f(x_i), y) < \frac{2\varepsilon}{3}\}$ for $1 \leq i \leq n$ hence $f \in B$.

Let g be any point of B. Since the K_i cover K, if c is any element of K there exists an index p, $1 \leq p \leq n$ such that $c \in K_p$. Thus $d(f(c), f(x_p)) \leq \frac{\varepsilon}{3}$ and $g(c) \in G_p = \{y \in Y | d(f(x_p), y) < \frac{2\varepsilon}{3}\}$. So $d(f(x_p), g(c)) < \frac{2\varepsilon}{3}$ and the triangle inequality yields $d(f(c), g(c)) \leq d(f(c), f(x_p)) + d(f(x_p), g(c)) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$. Therefore, for any $\varepsilon > 0$ there exists an N such that $n \geq N$ implies $f_n \in B$ which implies $d(f(c), f_n(c)) < \varepsilon$ for all $c \in K$ which implies $f_n \to f$ uniformly on K.

There is a cannonical map from $F(X,Y) \times X$ to Y called the evaluation map. We shall denote this map by φ and it is defined by $\varphi(f,x) = f(x)$. If $\mathfrak{J}$ is a given topology on $F(X,Y)$, then $\mathfrak{J}$ is said to be admissible in case the evaluation map is continuous where $F(X,Y) \times X$ is given the product topology. Preceding the next theorem we need some definitions.

Definition 2.4. A topological space $(X,\mathfrak{J})$ is said to be locally compact if every point in X has some compact nbhd.

Definition 2.5. A subset S of a topological space is said to be relatively compact if its closure, $\overline{S}$, is compact.

In a locally compact Hausdorff space X the following two conditions are equivalent to local compactness:

(1) For each $x \in X$ and each nbhd. H of x there exists a relatively compact open set G for which $x \in G \subseteq \overline{G} \subseteq H$, and

(2) For each compact set C and open set H containing C there is a relatively compact open set G with $C \subseteq G \subseteq \overline{G} \subseteq H$.

Theorem 2.7. If X is a locally compact Hausdorff space, then the compact-open topology on $C(X,Y)$ is admissible and it is the smallest of all admissible topologies on $C(X,Y)$.

Proof: Suppose we are arbitrarily given $f \in C(X,Y)$, a point $x \in X$, and an open set H in Y containing $f(x)$. By continuity, $f^{-1}[H]$ is an open set in X containing x. By the remark preceding this theorem there exists a relatively compact open set G such that $x \in G \subseteq \overline{G} \subseteq f^{-1}[H]$. Then $B = (\overline{G},H)$ is a subbasic open set in the C.O.-topology containing f. To show that φ is continuous it will suffice to show (1) $B \times G$ is a nbhd. of (f,x) and (2) $\varphi[B \times G] \subseteq H$. (1) is clear by definition of the product topology and (2) holds since if $(g,y) \in B \times G$ then $g \in B$ which implies $g[\overline{G}] \subseteq H \Rightarrow g[G] \subseteq H \Rightarrow \varphi(g,y) \in H$. Hence φ is continuous and the compact-open topology is admissible.

To prove the second assertion let $\mathfrak{T}$ be any admissible topology on $C(X,Y)$. If (K,H) is any subbasic open set in the compact-open topology we want to show that $(K,H) \in \mathfrak{T}$. So take f any point in (K,H), then by continuity of φ, for any $x \in K$ there exists an open nbhd. G_x of $x \in X$ and a $\mathfrak{T}$-open nbhd. B_x of $f \in C(X,Y)$ such that $\varphi(B_x \times G_x) \subseteq H$. Since K is compact there exist points $x_1,\ldots,x_n$ in K such that $K \subseteq G_{x_1} \cup \cdots \cup G_{x_n}$. Let $B = B_{x_1} \cap \cdots \cap B_{x_n}$, then $f \in B$ and B is a $\mathfrak{T}$-open set. So it only remains to show that $B \subseteq (K,H)$. For this, suppose that $g \in B$ then $g[K] = \varphi[\{g\} \times K] \subseteq H$ since if $x \in K$ then there exists an index p, $1 \leq p \leq n$, such that $x \in G_{x_p}$, now $g \in B$ implies $g \in B_{x_p}$ which implies $\varphi(g, x) \in \varphi[B_{x_p} \times G_{x_p}] \subseteq H$ hence $\varphi[\{g\} \times K] \subseteq H$. Therefore (K,H) is $\mathfrak{T}$-open and the C.O.-topology is smaller than $\mathfrak{T}$.

We now search for some conditions under which we can compare the compact-open topology with the infinite-open and closed-open topologies.

Theorem 2.8. The compact-open topology is never smaller than the infinite-open topology and they are comparable iff every subset of X is compact in which case the C.O.-topology is strictly larger than the I.O.-topology.

Proof: Suppose that there existed spaces X and Y such that the C.O.-topology was smaller than the I.O.-topology on $F(X,Y)$, then by Theorem 2.3 we would have the P.O.-topology smaller than the C.O.-topology smaller than the I.O.-topology which would contradict Theorem 2.4. So we have the first part of the theorem.

Now suppose every subset of X is compact and let (A,G) be any subbasic I.O.-open set, then $A \subseteq X$ implies A is compact hence (A,G) is a C.O.-open set which shows that the C.O.-topology is greater than or equal to the I.O.-topology. Note that since singletons are compact the C.O.-topology is strictly larger than the I.O.-topology.

To prove the "only if" part we shall assume that there exists an $A \subseteq X$ such that A is not compact and show that the assumption leads to a contradiction if the C.O.-topology is larger than the I.O.-topology. Let G be any open set in Y such that $\phi \neq G \neq Y$. Pick any $y_1 \in G$ and $y_2 \in Y - G$ and define $f(x) = \begin{cases} y_1, & x \in A \\ y_2, & x \in X - A. \end{cases}$

First we treat the case where $A = X$. Since A is not compact, A must be infinite so (A,G) is open in the I.O.-topology and $f \in (A,G)$. By assumption there exists a basic C.O.-open set B such that $f \in B \subseteq (A,G)$. Now $B = (K_1,H_1) \cap \cdots \cap (K_n,H_n)$ with K_i compact in X and H_i open in Y. Clearly $\bigcup_{i=1}^{n} K_i = A$ or else $B \nsubseteq (A,G)$

but also $\bigcup_{i=1}^{n} K_i \neq A$ since A is not compact and we have a contradiction. Now we treat the general case where A is properly contained in X. As before (A,G) is open in the I.O.-topology, and $\bigcup_{i=1}^{n} K_i \supseteq A$ or else $B \nsubseteq (A,G)$ and also $\bigcup_{i=1}^{n} K_i \neq A$ because A is not compact. So let $K_1, \ldots, K_s$ be the ones which intersect $X - A$. This implies that $\bigcup_{i=s+1}^{n} K_i$ is properly contained in A so there exists an $x_0 \in A - \bigcup_{i=s+1}^{n} K_i$. Hence the function f_0 defined by $f_0(x) = \begin{cases} f(x), & x \neq x_0 \\ y_2, & x = x_0 \end{cases}$ is in B but not in (A,G), a contradiction, and the theorem is proved.

The following result is easily established and gives some conditions under which we can compare the C.O.-topology and the Cl.O.-topology.

Theorem 2.9. If X is a Hausdorff space then the C.O.-topology is smaller than the Cl.O.-topology. The Cl.O.-topology is smaller than the C.O.-topology iff X is a compact space. And if X is a compact Hausdorff space then the C.O.-topology and the Cl.O.-topology coincide.

Proof: The first statement holds since any compact subset of a Hausdorff space is closed so that the subbasis of the C.O.-topology is contained in the subbasis of the Cl.O.-topology.

For the second statement, if X is compact then every closed subset is compact hence the subbase for the Cl.O.-topology is contained in the subbase for the C.O.-topology. On the other hand suppose that the Cl.O.-topology is smaller than the C.O.-topology. Since X is closed being the whole space (X,G) is open in the Cl.O.-topology where we choose $\phi \neq G \neq Y$. By assumption then (X,G) is a C.O.-open set so there exists a basic C.O.-open set $B \neq \phi$ which is contained in (X,G). B has the form $(K_1,H_1) \cap \cdots \cap (K_n,H_n)$ where the K_i are compact subsets of X and the H_i are open subsets of Y. Since $B \subseteq (X,G)$ we must have $\bigcup_{i=1}^{n} B_i = X$ which implies that X is compact, being the finite union of compact sets.

The third statement is clear by the first two since if the C.O.-topology is both larger and smaller than the I.O.-topology, then it must coincide with it.

The last remaining comparison is between the two topologies on which we are going to center our attention in Chapter III.

Theorem 2.10. If X is a T_1 space then the Cl.O.-topology is larger than the I.O.-topology iff X has the discrete topology.

Proof: Certainly if X has the discrete topology then every subset is closed so every subbasic open set in the I.O.-topology is also a subbasic open set in the Cl.O.-topology which proves the "if" part.

For the "only if" part suppose X is a T_1 space and that the I.O.-topology is contained in the Cl.O.-topology. Let G be an open subset of Y such that $\phi \neq G \neq Y$. Pick any $y_1 \in G$ and $y_2 \in Y - G$ and define

$$f(x) = \begin{cases} y_1, & x \in E \\ y_2, & x \in X - E \end{cases},$$ where E is chosen as follows.

If we suppose that X does not have the discrete topology then there exists a subset $E \subseteq X$ such that $\phi \neq E \neq X$ and E is not closed. By assumption there exists a basic Cl.O.-open set B such that $f \in B \subseteq (E,G)$. B has the form $(K_1,H_1) \cap \cdots \cap (K_n,H_n)$ with K_i closed subsets of X and H_i open subsets of Y. Since $B \subseteq (E,G)$ we have $E \subseteq \bigcup_{i=1}^{n} K_i$ and we cannot have equality since the K_i are closed. So let $K_1,\ldots,K_s$ be the ones which intersect $X - E$. This implies that $\bigcup_{i=s+1}^{n} K_i$ is properly contained in E so there exists an $x_0 \in E - \bigcup_{i=s+1}^{n} K_i$. Hence the function f_0 defined by $f_0(x) = \begin{cases} f(x), & x \neq x_0 \\ y_2, & x = x_0 \end{cases}$ is in B but not in (E,G). This is a contradiction so X must have the discrete topology.

4. The Composition Map

There is another canonical map associated with function spaces called the composition map. If $F(X,Y)$ and $F(Y,Z)$ are function spaces then the composition map, ω, maps $F(X,Y) \times F(Y,Z)$ into $F(X,Z)$ and is defined by $\omega(f,g) = g \circ f$ where $f \in F(X,Y)$ and $g \in F(Y,Z)$. We shall investigate the continuity of ω in relation to the topologies which we are considering in this paper on $C(X,Y)$, $C(X,Z)$ and $C(Y,Z)$ where the product $C(X,Y) \times C(Y,Z)$ is given the product topology.

Definition 2.5. Let X, Y, and Z be topological spaces and let f be a function mapping $X \times Y$ into Z. We say that f is continuous in the first variable if for any fixed $y_0 \in Y$ the map $f_1\colon X \to Z$ defined by $f_1(x) = f(x,y_0)$ is continuous. Continuity in the second variable is defined similarly and f is said to be continuous in both variables if it is continuous with respect to the product topology on $X \times Y$.

Lemma 2.2. If $f\colon X \times Y \to Z$ is continuous in both variables then it is continuous in each variable separately.

Proof: Suppose f is continuous in both variables we shall show that it is continuous in the first variable. Let y_0 be any fixed point in Y and define

$f_1: X \to Z$ by $f_1(x) = f(x,y_0)$. Now according to Definition 1.5 we must show that f_1 is continuous. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be any convergent net in X and suppose that it has limit x_0. Now $\{x_\lambda\}_{\lambda\in\Lambda}$ converging to x_0 implies that the net $\{(x_\lambda, y_0)\}_{\lambda\in\Lambda}$ converges to (x_0,y_0) in $X \times Y$. By the continuity of f we know that the net $\{f(x_\lambda, y_0)\}_{\lambda\in\Lambda}$ converges to $f(x_0, y_0)$ which means by definition of f_1 that $\{f_1(x_\lambda)\}_{\lambda\in\Lambda}$ converges to $f_1(x_0)$. Therefore f_1 is continuous. Continuity in the second variable is proved in a similar manner.

Theorem 2.11. If $C(X,Y)$, $C(Y,Z)$ and $C(X,Z)$ are given the point-open topology then ω is continuous in the first and second variables separately but not necessarily continuous in both variables.

Proof: Following the convention adopted in Theorem 2.6 we let $(F,G)' = (F,G) \cap C(X,Y)$. Fix $g_1 \in C(Y,Z)$ and let $(\{x_0\}, G)'$ be an arbitrary subbasic open set in $C(X,Z)$. As in Definition 2.5, $\omega_1: C(X,Y) \to C(X,Z)$ is defined by $\omega_1(f) = \omega(f,g_1)$. Then $\omega^{-1}_1[(\{x_0\}, G)'] =$

$\{f \in C(X,Y) | g_1 \circ f \in (\{x_0\}, G)'\}$

$= \{f \in C(X,Y) | f(x_0) \in g^{-1}_1[G]\} = (\{x_0\}, g^{-1}_1[G])'$.

Since g_1 is continuous, $g^{-1}_1[G]$ is open so we have that the inverse image of any subbasic open set under ω_1 is an open set. Therefore because set operations are preserved by inverse functions we have that $\omega_1^{-1}[H]$ is open for any open set H. Hence ω_1 is continuous or ω is continuous in the first variable.

Now fix $f_2 \in C(X,Y)$ and let $(\{x_0\}, G)'$ be an arbitrary subbasic open set in $C(X,Z)$, and let $\omega_2\colon C(Y,Z) \to C(X,Z)$ be defined by $\omega_2(g) = \omega(f_2, g)$. Then

$$\omega_2^{-1}[(\{x_0\}, G)'] = \{g \in C(Y,Z) \mid g \circ f_2 \in (\{x_0\}, G)'\}$$

$$= \{g \in C(Y,Z) \mid g(f_2(x_0)) \in G\}$$

$$= (\{f_2(x_0)\}, G)'$$

which is an open set in $C(Y,Z)$ hence ω_2 is continuous and so ω is continuous in the second variable.

The following example will show that with respect to the point-open topology ω is not necessarily continuous in both variables. Let $X = Y = Z$ be the real line with the usual topology, and consider the set $D = (\{1\}, (1,2))'$ which is open in $C(X,Z)$. Now if ω was continuous there would exist a nonempty basic open set $B \subseteq C(X,Y) \times C(Y,Z)$ such that $\omega[B] \subseteq D$. Since B is a basic open set in the product topology $B = G_1 \times G_2$ where G_1 and G_2 are open sets in $C(X,Y)$ and $C(Y,Z)$ respectively. Hence there exist nonempty basic open sets B_1 and B_2 for which $B_1 \subseteq G_1$ and $B_2 \subseteq G_2$ and $\omega[B_1 \times B_2] \subseteq \omega[G_1 \times G_2] \subseteq D$.

Now suppose $B_1 = (B_1^1, E_1^1) \cap \cdots \cap (B_n^1, E_n^1)$. If none of $B_1^1, \ldots, B_n^1$ are $\{1\}$ then $B_1[\{1\}] = \{f(1) | f \in B_1\} = Y$ which is infinite. If $B_1^1, \ldots, B_p^1$ are all equal to $\{1\}$ then $B_1[\{1\}]$ is $E_1^1 \cap \cdots \cap E_p^1$ which is a nonempty open set so is also infinite. In either case $H = \{f(1) | f \in B_1\}$ is an infinite subset of Y. Now if we are to have $\omega[B_1 \times B_2] \subseteq (\{1\}, (1,2))'$, B_2 must be a subset of $C(Y,Z)$ for which $f \in B_2$ implies $f[H] \subseteq (1,2)$. But this is a contradiction since as was shown in the proof of Theorem 2.4 sets of the form $(H, (1,2))'$ contain no non-empty basic open sets when H is infinite. Hence the composition map with respect to the point-open topology is continuous in each variable separately but not continuous in both variables.

Theorem 2.12. With respect to the compact-open topology ω is always continuous in each variable separately. In case X, Y, and Z are Hausdorff spaces and Y is locally compact then ω is continuous in both variables.

Proof: Let ω_1 be defined as before and let $(K,G)'$ be any subbasic open set in $C(X,Z)$. Then

$$\omega_1^{-1}[(K,G)'] = \{f \in C(X,Y) | g_1 \circ f \in (K,G)'\}$$

$$= \{f \in C(X,Y) | f[K] \subseteq g_1^{-1}[G]\}$$

$= (K, g_1^{-1}[G])'$ which is an open set in $C(X,Y)$ since the continuity of g_1 implies $g_1^{-1}[G]$ is an open set. Therefore ω is continuous in the first variable.

For proving continuity in the second variable let ω_2 be defined as before and let $(K,G)'$ be any subbasic open set in $C(X,Z)$. Then $\omega_2^{-1}[(K,G)'] =$

$\{g \in C(Y,Z) | g \circ f_1 \in (K,G)]\} = \{g \in C(Y,Z) | g \in (f_1[K], G)\}$

$= (f_1[K],G)'$ which is an open set in $C(Y,Z)$ since $f_1[K]$ is compact being the continuous image of a compact set. Hence ω is continuous in the second variable.

For proving continuity in both variables fix $f_1 \in C(X,Y)$ and $g_1 \in C(Y,Z)$ and let $(K,G)'$ be any subbasic open set containing $g_1 \circ f_1$. Since g_1 is continuous $g_1^{-1}[G] \subset Y$ is open and since f_1 is continuous $f_1[K]$ is compact and $g_1 \circ f_1 \in (K,G)'$ implies $f_1[K] \subseteq g_1^{-1}[G]$. Since Y is a locally compact Hausdorff space we know by the remarks preceding Theorem 2.7 that there exists a relatively compact open set H such that $f_1[K] \subseteq H \subseteq \overline{H} \subseteq g_1^{-1}[G]$. Therefore $(K,H)'$ and $(\overline{H}, G)'$ are subbasic open sets containing f_1 and g_1 respectively, and it is clear that $\omega[(K,H)' \times (\overline{H}, G)'] \subseteq (K,G)'$. Finally suppose that E is any open set containing $g_1 \circ f_1$ then there is a basic open set B such that $g_1 \circ f_1 \in B \subseteq E$

and $B = (K_1, G_1)' \cap \cdots \cap (K_p, G_p)'$ for some compact subsets K_i of X and open subsets G_i of Z. As above for each pair K_i, G_i there exists a relatively compact open H_i such that $\omega[(K_i, H_i)' \times (\overline{H}_i, G_i)'] \subseteq (K_i, G_i)'$ where $f_1 \in (K_i, H_i)'$ and $g_1 \in (\overline{H}_i, G_i)'$. Hence $A = \bigcap_{i=1}^{p}(K_i, H_i)'$ is an open set in $C(X,Y)$ containing f_1 and $B = \bigcap_{i=1}^{p}(\overline{H}_i, G_i)'$ is an open set in $C(Y,Z)$ containing g_1 and furthermore $\omega[\bigcap_{i=1}^{p}(K_i, H_i)' \times \bigcap_{i=1}^{p}(\overline{H}_i, G_i)']$

$$= \omega[\bigcap_{i=1}^{p}\{(K_i, H_i)' \times (\overline{H}_i, G_i)'\}] \subseteq \bigcap_{i=1}^{p}\omega[(K_i, H_i)' \times (\overline{H}_i, G_i)']$$

$\subseteq \bigcap_{i=1}^{p}(K_i, G_i)' = B \subseteq E$. Therefore given any open set $E \subseteq C(X,Z)$ containing $g_1 \circ f_1$ there exists a basic open set, namely $\bigcap_{i=1}^{p}(K_i, H_i)' \times \bigcap_{i=1}^{p}(\overline{H}_i, G_i)'$, containing (f_1, g_1) which is mapped inside E by ω, hence ω is continuous in both variables.

<u>Theorem 2.13</u>. With respect to the infinite-open topology ω is continuous in the first variable but not necessarily continuous in the second variable.

<u>Proof</u>: Let ω_1 be defined as before and let $(A, G)'$ be any subbasic open set in $C(X,Z)$. Then

$$\omega_1^{-1}[(A,G)'] = \{f \in C(X,Y) \mid g_1 \circ f \in (A,G)'\}$$

$$= \{f \in C(X,Y) \mid f[A] \subseteq g_1^{-1}[G]\}$$

$$= (A, g_1^{-1}[G])'$$ which is an open set in $C(X,Y)$ since $g_1^{-1}[G]$ is open by continuity of g. Hence ω is continuous in the first variable.

We now give an example to show that ω may not be continuous in the second variable. Let $X = Y = Z$ all be the real line with the discrete topology. Let $f_2 \in C(X,Y)$ be the function $f_2(x) = 0$ for all $x \in X$, and let G be any nonempty proper subset of Z. Then $(X,G)'$ is a subbasic open set in $C(X,Z)$, but $\omega_2^{-1}[(X, G)'] =$

$$\{g \in C(Y,Z) \mid g \circ f_2 \in (X, G)'\} = \{g \in C(Y,Z) \mid g \in (\{0\}, G)\}$$

$= (\{0\}, G)'$ which we showed in the proof of Theorem 2.4 is not an infinite-open open set. Hence ω_2 is not continuous or ω is not continuous in the second variable.

We note that in view of Lemma 2.2 ω will not, in general, be continuous in both variables.

Theorem 2.14. With respect to the closed-open topology ω is always continuous in the first variable but not necessarily continuous in the second variable.

Proof: Let ω_1 be defined as before and let $(F,G)'$ be any subbasic open set in $C(X,Z)$. Then $\omega_1^{-1}[(F,G)'] = \{f \in C(X,Y) | g_1 \circ f \in (F,G)'\} =$

$= \{f \in C(X,Y) | f[F] \subseteq g_1^{-1}[G]\} = (F, g_1^{-1}[G])'$ which is open in $C(X,Y)$ hence ω is continuous in the first variable.

The following example shows that ω may not be continuous in the second variable. Let $X = Y = Z$ all be the real line with the usual topology and let $f_2(x) = \dfrac{x^2}{x^2+1}$.

Then if we take $([0,\infty), (-1,1))'$ for our open set in $C(X,Z)$ the inverse image under ω_2 is

$$\{g \in C(Y,Z) | g \circ f_2 \in ([0,\infty), (-1,1))\}$$

$$= \{g \in C(Y,Z) | g \in (f_2[[0,\infty)], (-1,1))\}$$

$$= \{g \in C(Y,Z) | g \in ([0,1), (-1,1))\}.$$

We now claim that this set is not open in the closed-open topology, thus ω_2 is not continuous. Suppose $E = ([0,1), (-1,1))'$ was open in the closed-open topology. Then there would exist a basic open set B such that $(id) \in B \subseteq E$ where (id) denotes the identity function on the reals which is a point of E. B has the form $(C_1, G_1)' \cap \cdots \cap (C_k, G_k)'$ where the C_j are closed subsets of $Y = R^1$ and the G_j are open subsets of $Z = R^1$.

Clearly $\bigcup_{i=1}^{k} C_i$ contains $[0,1)$ and since $\bigcup_{i=1}^{k} C_i$ is a closed set at least one of the C_j must contain the point 1. If all of the C_j's contain 1 then let $\varepsilon = \frac{1}{2}$, if not then choose ε in the following manner: possibly by renumbering let $C_1, \ldots, C_p$ be the ones not containing 1. Then for C_j, $1 \leq j \leq p$ there exists an $\varepsilon_j > 0$ such that $(1 - \varepsilon_j, 1 + \varepsilon_j) \cap C_j = \phi$ in which case take $\varepsilon = \min\{\frac{1}{2}, \min_{1 \leq j \leq p} \{\varepsilon_j\}\}$. Now $C_{p+1}, \ldots, C_k$ all contain 1 hence $G_{p+1}, \ldots, G_k$ are all open sets containing 1 since $(id) \in B$. Therefore $\bigcap_{i=p+1}^{k} G_i$ is also an open set containing 1 and as such there exists a $\delta > 0$ for which $(1 - \delta, 1 + \delta) \subseteq \bigcap_{i=p+1}^{k} G_i$. Now let $\eta = \min\{\varepsilon, \delta\}$ and define h as the following function:

$$h(x) = \begin{cases} x, & x \leq 1 - \eta \text{ or } 1 \leq x \\ 2x + (\eta - 1), & 1 - \eta < x < 1 - \frac{\eta}{2} \\ 1, & 1 - \frac{\eta}{2} \leq x < 1 \end{cases}$$

We claim h is a continuous function which is contained in B but not contained in E. Continuity and the fact that $h \notin E$ are easily checked, so let us show that $h \in B$. The only values of x for which there is any question are those between $1 - \eta$ and 1 since for all other values of x, h agrees with the identity function which is in B. So suppose $x_0 \in (1 - \eta, 1)$. Since $\eta \leq \varepsilon$ we know that the only C_j

that x_0 can be in are those for which $p + 1 \leq j \leq k$, but since $\eta \leq \delta$ we have that $h(x_0) \in (1 - \eta, 1 + \eta) \subseteq (1 - \delta, 1 + \delta) \subseteq G_i$ for $P + 1 \leq i \leq k$ hence $h \in B$. Therefore $B \nsubseteq E$ and this contradiction proves the theorem.

Again Lemma 2.2 rules out continuity in both variables.

CHAPTER III

RELATIONSHIPS BETWEEN A TOPOLOGY FOR $F(X,Y)$ AND THE TOPOLOGIES OF X OR Y

We now ask the question as to how the properties of the topological spaces X and Y influence the space $F(X,Y)$ when $F(X,Y)$ is given the infinite-open or the closed-open topology. In this connection we specifically attempt to answer the question: If Y has a given property then does the function space $F(X,Y)$ also have this property?

1. Separation of Points

Theorem 3.1. $F(X,Y)$ with the infinite-open topology is a T_0 space iff Y is a T_1 space.

Proof: First suppose that Y is a T_1 space and $f, g \in F(X,Y)$ with $f \neq g$. Since $f \neq g$ there exists a point $x_0 \in X$ for which $f(x_0) \neq g(x_0)$. Now $\{f(x_0)\}$ and $\{g(x_0)\}$ are closed in Y since Y is T_1, therefore $\{f(x_0)\}^c$ and $\{g(x_0)\}^c$ are open sets in Y where $\{f(x_0)\}^c$ denotes $Y - \{f(x_0)\}$. We now have three cases:

Case 1: $S = \{x \in X | g(x) \in \{f(x_0)\}^c\}$ is an infinite set. Then since $x_0 \in S$, $(S,\{f(x_0)\}^c)$ is an infinite-open open set which contains g but not f.

<u>Case 2</u>: $T = \{x \in X | f(x) \in \{g(x_0)\}^c\}$ is infinite. In this case $(T, \{g(x_0)\}^c)$ is an open set containing f but not g.

<u>Case 3</u>: Both S and T are finite in which case consider the set $(S^c, \{g(x_0)\}^c)$. Since S is finite and X is infinite we have that S^c is infinite so this set is open. It contains g since if $x \in S^c$ then $g(x) = f(x_0)$ and $f(x_0) \in \{g(x_0)\}^c$. However this set does not contain f since f is equal to $g(x_0)$ except at a finite number of points and hence f cannot map any infinite set into $\{g(x_0)\}^c$. Therefore $F(X,Y)$ is a T_0 space.

On the other hand suppose that $F(X,Y)$ is a T_0 space with the infinite-open topology and that y_1 and y_2 are any two distinct points of Y. Claim we can find an open set containing y_2 but not y_1. Consider the two functions

$$f(x) = \begin{cases} y_1, & x = x_0 \\ y_2, & x \neq x_0 \end{cases} \quad \text{where } x_0 \text{ is any fixed point of } X$$

and $g(x) = y_2$ for all $x \in X$. We now show that there is no open set in $F(X,Y)$ containing f but not g. Suppose there was such an open set, say G, then there is a basic open set H such that $f \in H \subseteq G$. H being a basic open set has the form $(S_1,H_1) \cap \cdots \cap (S_n,H_n)$ where the S_j are infinite subsets of X and the H_j are open subsets of Y. Now if no S_j contains x_0 then clearly, $g \in H$ since

$g = f$ except at x_0, but this implies $g \in G$ which is a contradiction. So possibly by renumbering let $S_1, \ldots, S_p$ be the ones containing x_0, but each S_j is infinite so they must all contain points different from x_0. Since $f \in (S_j, H_j)$ for $1 \leq j \leq p$ the sets H_j, $1 \leq j \leq p$, must contain both y_1 and y_2, hence $g \in (S_j, H_j)$ for $1 \leq j \leq n$ which implies $g \in H$ or $g \in G$, a contradiction. Therefore since we have assumed that $F(X,Y)$ is T_0 there must exist an open set A containing g but not f. So there exists a basic open set $B = (P_1,B_1) \cap \cdots \cap (P_m,B_m)$ such that $g \in B \subseteq A$. Let $P_1, \ldots, P_r$ be the ones containing x_0, and $r \geq 1$ since $f \notin B$. Then $B_1, \ldots, B_r$ must be open sets containing y_2 such that $\bigcap_{i=1}^{r} B_i$ is an open set containing y_2 but not y_1 or else $f \in B$. Hence Y is T_1 as desired.

The above proof actually shows more. In showing that there is no open set containing f but not g no special properties of Y were used and since the two functions f and g can be constructed for any spaces X and Y, we have also proved

<u>Theorem 3.2</u>. $F(X,Y)$ with the infinite-open topology is never a T_1 space.

We now turn to the same question when $F(X,Y)$ is given the closed-open topology. Remembering that in order to define the closed-open topology it was necessary that X

be a topological space we might suspect that the properties of the topology on X might affect the results and we see that they do.

Theorem 3.3. $F(X,Y)$ with the closed-open topology is a T_0 space if X is a T_1 space and Y is a T_0 space. The condition that Y is a T_0 space is also necessary and if Y is T_0 but not T_1 then X being T_1 is both necessary and sufficient.

Proof: First assume that X is a T_1 space and Y is a T_0 space, and that f and g are any two distinct points of $F(X,Y)$. Then there exists an $x_0 \in X$ such that $f(x_0) \neq g(x_0)$, and since Y is T_0 there exists an open set $H \subseteq Y$ which contains say $f(x_0)$ but not $g(x_0)$. Since X is T_1, $\{x_0\}$ is closed in X so $(\{x_0\}, H)$ is an open set in $F(X,Y)$ which contains f but not g so that $F(X,Y)$ is T_0.

Second assume that $F(X,Y)$ is a T_0 space and let y_1, y_2 be any two distinct points of Y. Since $F(X,Y)$ is T_0 there exists an open set G which contains one of the functions $f(x) = y_1$ and $g(x) = y_2$ and not the other one. Suppose that G contains f, then there is a basic open set $B = (C_1, B_1) \cap \cdots \cap (C_p, B_p)$ such that $f \in B \subseteq G$. Now since $g \notin B$ at least one of $B_1, \ldots, B_p$ must not contain y_2, but all are open sets containing y_1 so Y is a T_0 space.

Now we want to show that if Y is T_0 but not T_1 then $F(X,Y)$ being T_0 implies X is T_1. Since Y is not T_1 there exist two points $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and any set containing y_1 also contains y_2. Now consider the two functions $f(x) = \begin{cases} y_1, & x \neq x_0 \\ y_2, & x = x_0 \end{cases}$ and

$g(x) = y_1$ where x_0 is any fixed point in X. Clearly there is no open set containing g and not f since y_1 cannot be separated from y_2 by open subsets of Y. Since $F(X,Y)$ is T_0 there is an open set G containing a basic open set $B = (C_1, B_1) \cap \cdots \cap (C_r, B_r)$ such that $f \in B \subseteq G$ and $g \notin G$. At least one of $B_1, \ldots, B_r$ must contain y_2 and not y_1, or else $g \in B \subseteq G$, say it is B_k. Then since $f \in (C_k, B_k)$ we must have that $C_k = \{x_0\}$ which means that $\{x_0\}$ is a closed set in X. Since x_0 was arbitrary, X is a T_1 space as we wanted to show.

The following example shows, however, that if Y is T_1 then $F(X,Y)$ can be T_0 without X being T_1.

Example 3.2. Let X be the set $\{a,b,c\}$ with the topology $\mathfrak{J} = \{X, \phi, \{b,c\}, \{a,c\}, \{c\}, \{b\}\}$. Then X is not T_1 since $\overline{\{c\}} = \{a,c\}$, but $\{a\}$ and $\{b\}$ are closed sets. If f and g are any two distinct points of $F(X,Y)$ then there is some point at which they differ. First suppose that either $f(a) \neq g(a)$ or $f(b) \neq g(b)$ in which case since Y is T_1 either $(\{a\},\{f(a)\}^c)$ or

$(\{b\}, \{f(b)\}^c)$ is an open set containing g but not f. Next suppose $f(a) = g(a)$ and $f(b) = g(b)$ so that $f(c) \neq g(c)$. Then either $f(b) = g(b) \in \{f(c)\}^c$ or $f(b) = g(b) \in \{g(c)\}^c$ or both so either $f \in (\{b,c\}, \{g(c)\}^c)$ which is an open set not containing g or $g \in (\{b,c\},\{f(c)\}^c)$ which is an open set not containing f. Therefore $F(X,Y)$ is T_0 and it is not possible to improve on Theorem 2.3.

Theorem 3.4. The function space $F(X,Y)$ is T_1 in the closed-open topology iff X is T_1 and Y is T_1.

Proof: Suppose first that X is T_1 and Y is T_1 and that f and g are any two distinct points of $F(X,Y)$. Since $f \neq g$ there exists an $x_0 \in X$ for which $f(x_0) \neq g(x_0)$. Furthermore, $\{x_0\}$ is a closed set in X and since Y is T_1, $\{f(x_0)\}^c$ is an open set in Y so that $(\{x_0\}, \{f(x_0)\}^c)$ is an open set in $F(X,Y)$ containing g but not f. By interchanging f and g we obtain the open set $(\{x_0\}, \{g(x_0)\}^c)$ which contains f but not g, so we have shown that $F(X,Y)$ is a T_1 space.

Second suppose that $F(X,Y)$ is a T_1 space. If Y is not T_1 then there exist y_1 and y_2 in Y such that $y_1 \neq y_2$ and any open set containing y_1 also contains y_2. Hence the function $f(x) = y_1$ cannot be separated from the function $g(x) = y_2$ so we have a contradiction. Now suppose that Y is T_1 but X is not T_1, so some point $x_0 \in X$ is such that $\{x_0\}$ is not closed. Pick any y_1, y_2

distinct points in Y and define the two functions

$$f(x) = \begin{cases} y_1, & x \neq x_0 \\ y_2, & x = x_0 \end{cases} \quad \text{and} \quad g(x) = y_2 \quad \text{for all} \quad x \in X.$$

Since $F(X,Y)$ is T_1 there is an open set, H, containing f but not g which contains a basic open set $B = (C_1, B_1) \cap \cdots \cap (C_s, B_s)$ with $f \in B \subseteq H$. Let $C_1, \ldots, C_p$ be the ones containing x_0; then at least one of $B_1, \ldots, B_p$ must contain y_2 but not y_1. But since $f \in B$ this implies that at least one of $C_1, \ldots, C_p$ must be $\{x_0\}$. This implies that $\{x_0\}$ is closed which is a contradiction.

<u>Theorem 3.5</u>. The space $F(X,Y)$ with the closed-open topology is a Hausdorff space iff Y is a Hausdorff space and X is a T_1 space.

<u>Proof</u>: Suppose X is T_1 and Y is Hausdorff with f and g any two distinct points of $F(X,Y)$. Let x_0 be a point in X for which $f(x_0) \neq g(x_0)$ and since Y is a Hausdorff space there are disjoint open sets F and G with $f(x_0) \in F$ and $g(x_0) \in G$. Now since X is T_1 the sets $(\{x_0\}, F)$ and $(\{x_0\}, G)$ are subbasic open sets in $F(X,Y)$. Because $F \cap G = \phi$ we have $(\{x_0\},F) \cap (\{x_0\},G) = \phi$ which together with $f \in (\{x_0\}, F)$ and $g \in (\{x_0\}, G)$ implies that $F(X,Y)$ is a Hausdorff space.

Assume now that Y is not a Hausdorff space. So there are points $y_1 \neq y_2$ in Y which cannot be separated by disjoint open sets. We claim that the functions $f(x) = y_1$ and $g(x) = y_2$ cannot be separated by disjoint open sets in $F(X,Y)$. If they can be separated by disjoint open sets then they can be separated by disjoint basic open sets so suppose B and E are such that $B = (S_1,B_1) \cap \cdots \cap (S_p,B_p)$ and $E = (T_1,E_1) \cap \cdots \cap (T_q,E_q)$ and $f \in B$, $g \in E$ with $B \cap E = \phi$. Now $f \in (\bigcup_{i=1}^{p} S_i, \bigcap_{i=1}^{p} B_i) \subseteq B$ and

$g \in (\bigcup_{j=1}^{q} T_j, \bigcap_{j=1}^{q} E_j) \subseteq E$ but $\bigcap_{i=1}^{p} B_i$ and $\bigcap_{j=1}^{q} E_j$ are open

sets containing y_1 and y_2 respectively, so there exists a $y_0 \in [\bigcap_{i=1}^{p} B_i] \cap [\bigcap_{j=1}^{q} E_j]$. Hence the function $h(x) = y_0$

is in $B \cap E$ so $F(X,Y)$ is not a Hausdorff space.

We now assume that Y is a Hausdorff space but that X is not T_1 and show that assuming $F(X,Y)$ to be Hausdorff leads to a contradiction. There is some point $x_0 \in X$ for which $\{x_0\}$ is not a closed set. Now define

$f(x) = \begin{cases} y_1, & x \neq x_0 \\ y_2, & x = x_0 \end{cases}$ and $g(x) = y_1$ where y_1 and y_2

are any distinct points of Y. Since $F(X,Y)$ is a Hausdorff space there is a basic open set $B = (C_1,B_1) \cap \cdots \cap (C_s,B_s)$ with $f \in B$ and $g \notin B$. Suppose $C_1,\ldots,C_p$ are the ones containing x_0, then at least one of $B_1,\ldots,B_p$ must contain y_2 but not y_1. Since $f \in B$ at least one of $C_1,\ldots,C_p$ must be $\{x_0\}$ which implies that $\{x_0\}$ is closed which is a contradiction.

2. Separation Involving Sets

We now consider two of the separation properties involving sets and ask if they carry over from Y to $F(X,Y)$.

Definition 3.1. A topological space X is regular iff for any point x and any closed set F not containing x there exist disjoint open sets G and H such that $x \in G$ and $F \subseteq H$.

An alternate formulation which we shall find useful is that a space X is regular iff for every point x and every open set G containing x there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$.

In the next example we show that with the infinite-open topology $F(X,Y)$ may not be regular even if Y is a regular space.

Example 3.3. Let Y be the reals with the usual topology and let X be the set of real numbers. Consider the subbasic open set $([0,1], (-\frac{1}{2}, \frac{1}{2}))$ and the constant $f(x) = 0$. Then f and S^c are a point and a disjoint closed set in $F(X,Y)$. The function $g(x) = \begin{cases} 1, & x = \frac{1}{2} \\ 0, & x \neq \frac{1}{2} \end{cases}$ is in $\overline{\{f\}}$ since as we showed in the proof of Theorem 2.1 there is no open set containing g but not f. Since $\overline{\{f\}}$ is the smallest closed set containing f and

$g \in \overline{\{f\}} \cap S^c$ we see that $F(X,Y)$ is not regular.

Now we turn to the same question when $F(X,Y)$ is given the closed-open topology. The following example shows that Y being regular does not imply $F(X,Y)$ is regular. We will show more than is needed here as it will be useful in this form later.

Definition 3.2. A topological space X is normal iff for any two disjoint closed sets F_1 and F_2 there exist disjoint open sets G_1 and G_2 such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.

Another formulation which is easily seen to be equivalent is that a space X is normal iff for any closed set F and open set G containing F there is an open set H for which $F \subseteq H \subseteq \overline{H} \subseteq G$.

Example 3.4. We give an example in which Y is regular but $F(X,Y)$ with the closed-open topology is not regular. Let $X = \{1,2,3,\ldots\}$ with the discrete topology and $Y = \{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$, where Y is given the relative topology from the reals. First we show that Y is a normal space. Suppose F_1 and F_2 are any two disjoint closed sets. The point 0 cannot be in both F_1 and F_2 so suppose that $0 \notin F_1$. This implies that F_1 is an open set since each singleton except $\{0\}$ is an open set and F_1 is the union of the singletons contained in it. Therefore F_1 and F_1^c are disjoint open sets that $F_1 \subseteq F_1$ and

$F_2 \subseteq F_1^c$, hence Y is a normal space. Furthermore Y is a Hausdorff space being a subspace of a Hausdorff space which implies that Y is also regular. Now let $f \in F(X,Y)$ be defined by $f(n) = \frac{1}{n}$ for all $n \in X$, and let $S = (X, Y - \{0\})$. We claim that there does not exist any open set G such that $f \in G \subseteq \overline{G} \subseteq S$. This will prove that $F(X,Y)$ is not regular since S is open in the closed-open topology and $f \in S$. Suppose there was such a set G then it would contain a basic open set E which would also have the property $f \in E \subseteq \overline{E} \subseteq S$. And E has the form $(K_1, G_1) \cap \cdots \cap (K_s, G_s)$ where the K_i are closed sets in X and the G_i are open sets in Y. Since $E \subseteq S$ we know that $\bigcup_{i=1}^{s} K_i = X$. Now pick a subsequence of the sequence, $\langle 1,2,3,\ldots\rangle$, of positive integers which is either in or out of each K_i as follows: For the first step either $K_1 \cap \langle 1,2,3,\ldots\rangle$ is finite or infinite. If it is infinite take $N_1 = K_1 \cap \langle 1,2,3,\ldots\rangle$ and if it is finite take $N_1 = \langle 1,2,3,\ldots\rangle - K_1$, so N_1 is a subsequence of $\langle 1,2,3,\ldots\rangle$ which is either in or out of K_1. As the second step either $N_1 \cap K_2$ is finite or infinite. If it is finite take $N_2 = N_1 - K_2$ and if it is infinite take $N_2 = K_2 \cap N_1$, so N_2 is a subsequence of N_1 which is either in or out of K_2. And since N_2 is a subsequence of N_1 it is also either in or out of K_1. Continuing this process for s steps we obtain a subsequence $N = N_s$

which has the desired property. Now since the K_i, $1 \leq i \leq s$, cover X the subsequence N must be in at least one of the K_i. Relabel the K_i's if necessary so that N is in $K_1, \ldots, K_p$ and N is out of $K_{P+1}, \ldots, K_s$ where $1 < P \leq S$. Suppose $N = \langle x_1, x_2, x_3, \ldots \rangle$ then $f(x_m) = \frac{1}{x_m} \in G_1 \cap \cdots \cap G_p$, for $m = 1,2,3,\ldots$, since $x_j \in K_i$, $1 \leq i \leq p$, $i = 1,2,3,\ldots$, and $f \in E$. We now prove that $\bar{E} \nsubseteq S$. Define a new function g_k as follows:

$$g_k(x) = \begin{cases} f(x), & x \notin N \\ \frac{1}{x_k}, & x \in N \end{cases}.$$

We claim that $g_q[K_i] \subseteq G_i$ for $1 \leq i \leq S$ and $q = 1,2,3,\ldots$. In the first place if $x \in K_i$ and $x \notin N$ then $g_q(x) = f(x) \in G_i$ since $f \in (K_i, G_i)$. Secondly if $x \in N$, then i must be between 1 and p and then $\frac{1}{x_q} \in G_i$ so $g_q(x) \in G_i$. Let h be another function defined by $h(x) = \begin{cases} f(x), & x \notin N \\ 0, & x \in N \end{cases}$. We

claim that $g_k \to h$ in the closed-open topology on $F(X,Y)$; so let $B = \bigcap_{i=1}^{r} (L_i, H_i)$ be an arbitrary basic open set containing h. Suppose $L_1, \ldots, L_q$ are the ones intersecting N. Then $0 \in H_1 \cap \cdots \cap H_q = U$ which is an open set containing 0. If there is no j such that $L_j \cap N \neq \phi$ then take $U = (-\frac{1}{2}, \frac{1}{2})$. The sequence $\{\frac{1}{x_j}\}_{j=1}^{\infty}$ decreases monotonically to 0 so there exists an M such that $m > M$ implies $\frac{1}{x_m} \in U$, hence if $m > M$ then

$g_m \in \bigcap_{i=1}^{r} (L_i, H_i)$. Therefore $g_k \to h$ which implies $\overline{E} \cap S^c \neq \phi$ since $h \in \overline{E}$ and $h \in S^c$, and since G was arbitrary we know that there is no open set containing f whose closure is contained in S.

In particular we have shown that Y being regular does not imply F(X,Y) is regular. We can show, however, that under certain conditions F(X,Y) with the closed-open topology does have some regular subspaces. Let Cl(X,Y) denote the subspace of F(X,Y) consisting of the closed mappings. As before since a subbasis for the relative topology on Cl(X,Y) consists of the class of all $B = (C,G) \cap Cl(X,Y)$ where C is closed in X and G is open in Y, we shall denote B by $(C,G)^*$.

<u>Theorem 3.6</u>. If X is a T_1 space and Y is a normal space then Cl(X,Y) is a regular space.

<u>Proof</u>: Let f^* be a point of Cl(X,Y) contained in the subbasic open set $(F,G)^*$. Now since f^* is a closed mapping $f^*[F]$ is a closed set contained in G. By the normality of Y there exists an open set E such that $f^*[F] \subseteq E \subseteq \overline{E} \subseteq G$, hence $f^* \in (F,E)^*$. If $g \notin (F,\overline{E})^*$ then there exists an $x_0 \in F$ such that $g(x_0) \notin \overline{E}$, hence $(\{x_0\}, Y - \overline{E})^*$ is a closed-open open set which contains g and is disjoint from $(F,E)^*$. Thus we have shown that $\overline{(F,E)^*} = \overline{(F,E)}^* \subseteq (F,\overline{E})^* \subseteq (F,G)^*$. Now

let G be any open set containing f^*, hence G contains a basic open set $B = (F_1, G_1)^* \cap \cdots \cap (F_n, G_n)^*$ which contains f^*. To each G_i there corresponds an E_i as above and we set $H = (F_1, E_1)^* \cap \cdots \cap (F_n, E_n)^*$ and claim that $f^* \in H \subseteq \overline{H} \subseteq G$. We know that $f^* \in H$ since $f^* \in (F_i, E_i)^*$ for $1 \leq i \leq n$ as above and $H \subseteq \overline{H}$ is always true. Moreover, we find $\overline{H} = \overline{(F_1, E_1)^* \cap \cdots \cap (F_n, E_n)^*} \subseteq$

$\overline{(F_1, E_1)^*} \cap \cdots \cap \overline{(F_n, E_n)^*} = \overline{(F_1, E_1)}^* \cap \cdots \cap \overline{(F_n, E_n)}^* \subseteq$

$(F_1, \overline{E}_1)^* \cap \cdots \cap (F_n, \overline{E}_n)^* \subseteq (F, G_1)^* \cap \cdots \cap (F_n, G_n)^* = B \subseteq G$.

Thus the subspace $Cl(X,Y)$ is regular.

Let Y be a topological space with a distinguished point which we shall call 0. We let $C_0(X,Y)$ be the class of all continuous functions, f, from X to Y for which there exists a compact set $K \subseteq X$ with the property that $x \in K^c$ implies $f(x) = 0$.

Theorem 3.7. If X is a Hausdorff space and Y is a regular Hausdorff space then the subset $C_0(X,Y)$ is a regular subspace of $F(X,Y)$ in the relative closed-open topology.

Proof: For convenience we let $(F,G)_0 = (F,G) \cap C_0(X,Y)$ denote an arbitrary subbasic open set in the relative topology, where F is closed in X and G is open in Y. Let f^* be an arbitrary point in $C_0(X,Y)$ and let K be a compact set for which $x \in K^c$ implies $f^*(x) = 0$. Suppose $(F,G)_0$ is any subbasic open set containing f^*; we want to show that there is an open set E with $f^* \in E \subseteq \overline{E} \subseteq (F,G)_0$.

Case 1: $F \cap K^\circ = \phi$, where K° is the interior of the set K. First we want to show that $f^*[F] = \{0\}$, and for this it will do to show that if f^* is 0 off of K then it is also 0 off of K°. Suppose $x \in K - K^\circ$ then x is in the boundary of K, so every nbhd. of x intersects K^c. Hence we can choose a net of points in K^c which converges to x. Since f^* is continuous it must preserve this limit and since $f^*[K^c] = \{0\}$ we know that $f^*(x) = 0$. Now since $\{0\}$ is a closed set in Y and G is an open set containing 0 we know by regularity that there exists an open set $E \subseteq Y$ for which $0 \in E \subseteq \overline{E} \subseteq G$. So we have $f^* \in (F,E)_0 \subseteq \overline{(F,\ E)_0} \subseteq (F,\overline{E})_0 \subseteq (F,G)_0$.

Case 2: $F \cap (K^c) = \phi$. Here F is a closed subset of the compact set K and hence F is compact. Since $f^* \in (F,G)_0$ we know $f^*[F] \subseteq G$ so by regularity of Y, for each $y \in f^*[F]$ there exists an open set E_y such that $y \in E_y \subseteq \overline{E}_y \subseteq G$, and the E_y's form an open covering of $f^*[F]$. Since f^* is continuous $f^*[F]$ is compact and so some finite subcollection $E_{y_1},\ldots,E_{y_n}$ covers $f^*[F]$. Letting $E = E_{y_1} \cup \cdots \cup E_{y_n}$ we note that

$f^*[F] \subseteq E$ and $\overline{E} = \overline{E_{y_1} \cup \cdots \cup E_{y_n}} = \overline{E}_{y_1} \cup \cdots \cup \overline{E}_{y_n} \subseteq G$.

Now suppose that $g \in C_0(X,Y)$ and $g \notin (F,\overline{E})_0$, then there exists an $x_0 \in F$ for which $g(x_0) \notin \overline{E}$, therefore $(\{x_0\}, Y - \overline{E})_0$ is an open set containing g and which is disjoint from $(F,E)_0$. So if $g \notin (F,\overline{E})_0$ then $g \notin \overline{(F,E)_0}$, hence $\overline{(F,E)_0} \subseteq (F,\overline{E})_0 \subseteq (F,G)_0$.

<u>Case 3</u>: $F \cap K^\circ \neq \phi$ and $F \cap (K^c) \neq \phi$. Since K is a compact subset of a Hausdorff space it must be closed and since K° is open $(K^\circ)^c$ is also closed so we know that $F \cap K$ and $F \cap (K^\circ)^c$ are closed subsets of X. Furthermore $(F \cap K, G)_0 \cap (F \cap (K^\circ)^c, G)_0 = ((F \cap K) \cup (F \cap (K^\circ)^c, G)_0 = (F,G)_0$. Now since $(F \cap K) \cap (K^c) = \phi$ and $(F \cap (K^\circ)^c) \cap K^\circ = \phi$ cases 1 and 2 quarantee the existence of open sets E' and E'' in Y such that $f^* \in (F \cap K, E')_0 \subseteq \overline{(F \cap K, E')_0} \subseteq (F \cap K, \overline{E}')_0 \subseteq (F \cap K, G)_0$ and $f^* \in (F \cap (K^\circ)^c, E'') \subseteq \overline{(F \cap (K^\circ)^c, E'')_0} \subseteq (F \cap (K^\circ)^c, \overline{E}'')_0 \subseteq (F \cap (K^\circ)^c, G)_0$. Now suppose G is an arbitrary open set containing f^* so there exists a basic open set $B = (F_1, G_1)_0 \cap \cdots \cap (F_n, G_n)_0$ such that $f^* \in B \subseteq G$. Now rearrange the $(F_i, G_i)_0$'s if necessary, so that for $1 \leq k \leq p - 1$ we have $F_k \cap K^\circ = \phi$

and for $p \leq k \leq q - 1$ we have $F_k \cap K^c = \phi$

and for $q \leq k \leq n$ we have $F_k \cap K^\circ \neq \phi$ and $F_k \cap K^c \neq \phi$.

As above for k such that $q \leq k \leq n$ we can find E_k' and

E_k'' open sets in Y for which

$\overline{(F_k \cap K, E_k')_0 \cap (F_k \cap (K^\circ)^c, E_k'')_0} \subseteq \overline{(F_k \cap K, E_k')_0} \cap \overline{(F_k \cap (K^\circ)^c, E_k'')_0} \subseteq (F_k \cap K, G)_0 \cap (F_k \cap (K^\circ)^c, G)_0 = (F_k, G)_0.$

Now we claim that the open set

$$H = \left[\bigcap_{i=1}^{q-1} (F_i, E_i)_0\right] \cap \left[\bigcap_{i=q}^{n} \{(F_i \cap K, E_i')_0 \cap (F_i \cap (K^\circ)^c, E_i'')_0\}\right]$$

has the desired property which is $f^* \in H \subseteq \overline{H} \subseteq B \subseteq G$. That $f^* \in H$ and $H \subseteq \overline{H}$ are easily checked so we show $\overline{H} \subseteq B$ by the following computation:

$$\overline{H} = \overline{\left[\bigcap_{i=1}^{q-1} (F_i, E_i)_0\right] \cap \left[\bigcap_{i=q}^{n} \{(F_i \cap K, E_i')_0 \cap (F_i \cap (K^\circ)^c, E_i'')_0\}\right]}$$

$$\subseteq \overline{\left[\bigcap_{i=1}^{q-1} (F_i, E_i)_0\right]} \cap \overline{\left[\bigcap_{i=q}^{n} \{(F_i \cap K, E_i')_0 \cap (F_i \cap (K^\circ)^c, E_i'')_0\}\right]}$$

$$\subseteq \left[\bigcap_{i=1}^{q-1} \overline{(F_i, E_i)_0}\right] \cap \left[\bigcap_{i=q}^{n} \{\overline{(F_i \cap K, E_i')_0} \cap \overline{(F_i \cap (K^\circ)^c, E_i'')_0}\}\right]$$

$$\subseteq \left[\bigcap_{i=1}^{q-1} (F_i, \overline{E}_i)_0\right] \cap \left[\bigcap_{i=q}^{n} \{(F_i \cap K, \overline{E}_i')_0 \cap (F_i \cap (K^\circ)^c, \overline{E}_i'')_0\}\right]$$

$$\subseteq \left[\bigcap_{i=1}^{q-1} (F_i, G_i)_0\right] \cap \left[\bigcap_{i=q}^{n} (F_i, G_i)_0\right] = B \subseteq G.$$

Therefore $C_0(X,Y)$ is a regular subspace of $F(X,Y)$.

We now turn to the question of normality and see how Example 3.4 shows that Y being a normal space does not imply that F(X,Y) is normal either with the infinite-open or the closed-open topology.

Lemma 3.1. With X and Y defined as in Example 3.4 the function $f(n) = \frac{1}{n}$ is closed as a singleton in the infinite-open topology on F(X,Y).

Proof: Suppose $g \in F(X,Y)$ and $g \neq f$ so there exists an n_0 for which $g(n_0) \neq f(n_0) = \frac{1}{n_0}$. We now have two cases:

Case 1: $g^{-1}[\{\frac{1}{n_0}\}]$ is an infinite subset of X. In this case the set $(g^{-1}[\{\frac{1}{n_0}\}], \{\frac{1}{n_0}\})$ is an infinite-open open set containing g but not f.

Case 2: $g^{-1}[\{\frac{1}{n_0}\}]$ is finite which implies $g^{-1}[\{\frac{1}{n_0}\}^c]$ is infinite. Since $\{\frac{1}{n_0}\}^c$ is open the set $G = (g^{-1}[\{\frac{1}{n_0}\}^c], \{\frac{1}{n_0}\}^c)$ is an open set in F(X,Y). Clearly $g \in G$ but $f \notin G$ since $n_0 \in g^{-1}[\{\frac{1}{n_0}\}^c]$ because $g \neq f$ at n_0 and $f(n_0) = \frac{1}{n_0} \notin \{\frac{1}{n_0}\}^c$. Therefore $\{f\}$ is closed in the infinite-open topology on F(X,Y).

Now in terms of Example 3.4 the set $S = (X, Y - \{0\})$ is open in the infinite-open topology since X is an infinite set so $\{f\}$ and S are a closed set contained in an open set. When we showed that there was no open set G such that $\{f\} \subseteq G \subseteq \overline{G} \subseteq S$ we were working with the closed-open topology on $F(X,Y)$ and the discrete topology on X. Hence the closed subsets of X were all the subsets so the closed-open topology contained the infinite-open topology. Thus it is impossible to find such an open set G in the infinite-open topology, therefore $F(X,Y)$ may not be normal with the infinite-open topology even if Y is normal.

Example 3.4 also shows this result for the closed-open topology on $F(X,Y)$. With the discrete topology on X it is a T_1 space and recalling that Y is also a T_1 space, Theorem 3.4 says that $F(X,Y)$ is T_1 in the closed-open topology. Therefore $\{f\}$ is a closed set so the example shows that $\{f\}$ and S^c are disjoint closed sets which cannot be separated by disjoint open sets hence $F(X,Y)$ is not normal in the closed-open topology.

3. Compactness

We now give the following example to show that if Y is a compact topological space then $F(X,Y)$ with the infinite-open topology need not be compact.

Example 3.5. Let X be the set of positive integers and $Y = \{a,b\}$ with the discrete topology. Now for every $f \in F(X,Y)$ either one or both of the sets $f^{-1}[\{a\}]$ and $f^{-1}[\{b\}]$ must be an infinite subset of X. In case both

$f^{-1}[\{a\}]$ and $f^{-1}[\{b\}]$ are infinite sets define $W(f)$ as $(f^{-1}[\{a\}], \{a\}) \cap (f^{-1}[\{b\}], \{b\})$, if just $f^{-1}[\{a\}]$ is infinite then $W(f) = (f^{-1}[\{a\}], \{a\})$, and if just $f^{-1}[\{b\}]$ is infinite then $W(f) = (f^{-1}[\{b\}], \{b\})$. Clearly $f \in W(f)$ for every $f \in F(X,Y)$ so $U = \bigcup_{f \in F(X,Y)} W(f)$ is an open cover of $F(X,Y)$. It will suffice to show that U has no finite subcover so suppose that it does, say $F(X,Y) \subseteq W(f_1) \cup W(f_2) \cup \cdots \cup W(f_n)$. Possibly by rearranging let $S_1 = \{f_1,\ldots,f_p\}$ be the set of those f's which are b at only a finite number of points of X and $S_2 = \{f_{p+1},\ldots,f_q\}$ contain those f's which are a at only a finite number of points of X and let $S_3 = \{f_{q+1},\ldots,f_n\}$ contain those f's which are not in S_1 or S_2. At least one of these sets must be nonempty so suppose it is S_1. Let $B = f_1^{-1}[\{b\}] \cup \cdots \cup f_p^{-1}[\{b\}]$. Now B is a finite subset of X being the finite union of finite sets so there exists an $x_0 \in X - B$. We claim that the function f_0 defined by $f_0(x) = \begin{cases} b, & x = n_0 \\ a, & x \neq n_0 \end{cases}$ is not in $W(f_1) \cup \cdots \cup W(f_n)$. Certainly $f_0 \notin W(f_1) \cup \cdots \cup W(f_p)$ since none of $f_1,\ldots,f_p$ are b at n_0, thus $n_0 \in f_i^{-1}[\{a\}]$ for $1 \leq i \leq p$ and f_0 maps n_0 into $\{b\}$, not $\{a\}$. Also $f_0 \notin W(f_{p+1}) \cup \cdots \cup W(f_n)$ since f_0 is b at only one point. If $S_1 = \phi$ but $S_2 \neq \phi$ then we arrive at a contradiction in the same way with a and b switched.

If $S_1 = \phi$ and $S_2 = \phi$ then none of the $W(f_i)$ contain either constant function which again shows that $F(X,Y) \not\subseteq W(f_1) \cup \cdots \cup W(f_n)$. Therefore $F(X,Y)$ is not compact in the infinite-open topology.

The following similar example shows that compactness is not inherited from Y by $F(X,Y)$ in the case of the closed-open topology.

Example 3.6. Let X be the positive integers with the discrete topology and $Y = \{a,b\}$ also with the discrete topology. Then what we shall show is that the closed-open topology on $F(X,Y)$ is the discrete topology, so $F(X,Y)$ will not be compact. Let f be any point of $F(X,Y)$ then $f^{-1}[\{a\}]$ and $f^{-1}[\{b\}]$ are closed subsets of X since all subsets are closed. Thus the set $(f^{-1}[\{a\}], \{a\}) \cap (f^{-1}[\{b\}], \{b\})$ is open in the closed-open topology and is exactly $\{f\}$. Therefore $F(X,Y)$ has the discrete topology. We note that it does no good to consider the subclass of continuous functions since all functions are continuous when X is discrete.

4. Countability Properties

We now seek to find out if either first or second countability is carried over to $F(X,Y)$ from Y in case $F(X,Y)$ has either the infinite-open or closed-open topology. We shall use $\overline{\overline{X}}$ to denote the cardinal number of the set X.

Let $\aleph_0$ be the cardinal number of the set of natural numbers and let c be the cardinal number of the real numbers. Recall that $2^{\aleph_0} = c$ and if a is any cardinal number then $2^a > a$. Note that the set $F(X,Y)$ with $\overline{\overline{X}} = a$ and $Y = \{b,d\}$ has cardinal number 2^a. The next example shows that when Y is C_{II}, $F(X,Y)$ with the infinite-open topology may fail to be C_{II}.

Example 3.7. Let X be the set of real numbers and let $Y = \{a,b\}$ with the discrete topology. If $A = \{f \in F(X,Y) | f^{-1}[\{b\}]$ and $f^{-1}[\{a\}]$ are infinite subsets of $X\}$, then we claim that $\overline{\overline{A}} > c$. Let $B = \{f \in F(X,Y) | f^{-1}[\{b\}]$ is finite$\}$ and $D = \{f \in F(X,Y) | f^{-1}[\{a\}]$ is finite$\}$ so that $F(X,Y)$ is the disjoint union of A, B and D, and hence $\overline{\overline{F(X,Y)}} = \overline{\overline{A}} + \overline{\overline{B}} + \overline{\overline{D}}$. There is a one-to-one correspondence between the finite subsets of X and the functions in B or D so that $\overline{\overline{B}} = \overline{\overline{D}}$ is equal to the cardinality of the set of finite subsets of X. It is known that the cardinal number of the set of finite subsets of a set of cardinality c is again c. Therefore $2^c = c + c + \overline{\overline{A}} = c + \overline{\overline{A}}$ which shows that $\overline{\overline{A}} > c$. For every $f \in A$ we have that $\{f\} = (f^{-1}[\{a\}], \{a\}) \cap (f^{-1}[\{b\}], \{b\})$ is an open set in the infinite-open topology and so the cardinal number of the infinite-open topology on $F(X,Y)$ is greater than c. However, there is a theorem (3, p. 82) which states if

$(X,\mathfrak{J})$ is a C_{II} space then $\bar{\bar{\mathfrak{J}}} \leq c$ so we know that $F(X,Y)$ with the infinite-open topology is not C_{II}.

This same question can be answered more easily in the case of the closed-open topology.

Example 3.8. Let X be the positive integers with the discrete topology and let $Y = \{a,b\}$ again with the discrete topology. We showed in Example 3.6 that $F(X,Y)$ with the closed-open topology is a discrete space. This, together with the fact that $F(X,Y)$ is an uncountable set, since $F(X,Y) = 2^{\aleph_0} = c$, implies that $F(X,Y)$ is not C_{II}.

The final property that we are going to consider is first countability. Recall that Ω is the first uncountable ordinal and $[0, \Omega)$ is used to denote the set of all ordinals less than Ω. The next example proves that the C_I property is not inherited by $F(X,Y)$ from Y in the case of the infinite-open topology.

Example 3.9. Let $X = [0,\Omega)$ and $Y = [0,\Omega)$ both with the discrete topology. Suppose for purposes of obtaining a contradiction that $B = \{G_n | n = 1,2,3,...\}$ is a countable open base of the identity function which we shall again denote by (id). We can generate from B a countable open base of (id) which is made up of basic open sets, and suppose that $B^* = \{B_n | n = 1,2,3,...\}$ is such a base of (id). Then $B_n = (G_1^n, H_1^n) \cap \cdots \cap (G_{k(n)}^n, H_{k(n)}^n)$ for some G_i^n

infinite subsets of X and H_i^n open subsets of Y.

Now for each $n = 1,2,3,\ldots,$ consider the class of all intersections $\bigcap_{i \in P} \{H_i^n\}$ where P is any subset of $\{1,2,3,\ldots,k(n)\}$. With each intersection for a given n associate its least element and if the intersection is empty associate the element 0. For each n there are only a finite number of such intersections so let M_n be the maximum of the associated numbers. The set of all M_n's is then countable so it has a supremum, say M, which is less than Ω.

We claim that the open set $E = ([M + 1,\Omega), [M + 1,\Omega))$ contains (id) but contains no B_n, $n = 1,2,3,\ldots$. Let B_{n_0} be any member of B^*, and let x_0 be any point of $[M + 1,\Omega)$. Now $B_{n_0} = (G_1^{n_0}, H_1^{n_0}) \cap \cdots \cap (G_{k(n_0)}^{n_0}, H_{k(n_0)}^{n_0})$ so rearrange if necessary to make $G_1^{n_0},\ldots,G_q^{n_0}$ the ones containing x_0. If none of the $G_i^{n_0}$ contains x_0 then clearly $B_{n_0} \not\subseteq E$. The intersection $\bigcap_{j=1}^{q} H_j^{n_0}$ cannot be empty since $(id) \in B_{n_0}$, hence this intersection has a least element, say m, which is less than or equal to $M_{n_0} \leq M$. Therefore the function $\phi(x) = \begin{cases} x, & x \neq x_0 \\ m, & x = x_0 \end{cases}$ is in B_{n_0} but not in E, hence no B_n is contained in E so there is no countable base of the identity function. This means that $F(X,Y)$ is not necessarily C_I with the infinite-open topology if Y is C_I.

The same example shows that first countability in Y does not imply first countability in $F(X,Y)$ with the closed-open topology.

Example 3.10. Let $X = Y = [0,\Omega)$ both with the discrete topology. Now since the discrete topology on X implies that all subsets are closed, and since we used no properties of the sets G_i^n which appeared in Example 3.9 the same proof shows that $F(X,Y)$ with the closed-open topology is not C_I.

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