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TITLE: GROUP AUTOMORPHISMS OF THE N-TORUS: A REPRESENTATION THEOREM AND SOME APPLICATIONS

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It is known that for a continuous automorphism of a compact abelian group G having a point x whose orbit separates $\hat{G}$, the dual of G, the adjoint $\tau^*:\hat{G} \to \hat{G}$ is algebraically isomorphic to the shift transformation restricted to some shift-invariant subgroup of the infinite torus $\hat{X} = \left(\bigtimes_{-\infty}^{\infty} K\right)_d$, K being the 1-torus or circle group. This is extended to show that the adjoint τ^* of any continuous epimorphism τ of a compact abelian group G having a point x such that $orb^+x = \{\tau^n x : n \in Z^+\}$ separates $\hat{G}$ can be represented via an algebraic isomorphism as the shift transformation restricted to a subgroup of $\hat{Y} = \left(\bigtimes_{0}^{\infty} K\right)_d$. It is shown that each group automorphism of the n-torus, K^n, satisfies these two properties, and a procedure for finding such points is obtained. For

type 1 automorphisms of K^n (those automorphisms of K^n whose associated unimodular matrices have distinct characteristic roots) we obtain a formula for embedding K^n into $\hat{X}$. This information is used to characterize the maximal ergodic subgroup for a given type 1 automorphism, and the characterization is then used to produce an example of an automorphism of K^3 not having the E-Z decomposition property. Finally we obtain a characterization of entropy classes of ergodic automorphisms of the two and three torus.

GROUP AUTOMORPHISMS OF THE N-TORUS: A REPRESENTATION THEOREM AND SOME APPLICATIONS

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I. INTRODUCTION

1. Basic Definitions and Terminology

Let (X_i, Σ_i, μ_i), where $i = 1, 2$, be measure spaces. A mapping $\phi: X_1 \to X_2$ is called measure-preserving if for each $A \varepsilon \Sigma_2$, $\phi^{-1}A \varepsilon \Sigma_1$ and $\mu_1(\phi^{-1}A) = \mu_2(A)$. If ϕ^{-1} exists and is also measure-preserving, ϕ is said to be invertible. Two measure spaces (X_i, Σ_i, μ_i), $i = 1, 2$, are said to be isomorphic if there exists an intvertible measure-preserving transformation θ mapping X_1 onto X_2. We will be almost exclusively interested in nonatomic Lebesgue spaces - i.e. measure spaces isomorphic to the unit interval $[0, 1]$ with the Lebesgue structure. Suppose (X, Σ, μ) is a probability space (Σ is a σ-algebra, $\mu \geq 0$ and $\mu(X) = 1$) and $\phi: X \to X$ is measure-preserving; then the quadruple $\Phi = (X, \Sigma, \mu, \phi)$ is often referred to as a dynamical system.

Let $\Phi = (X, \Sigma, \mu, \phi)$ be a dynamical system. ϕ is said to be ergodic if for $A \varepsilon \Sigma$, $\mu(A \Delta \phi^{-1}A) = 0$ implies $\mu A = 0$ or $\mu(X - A) = 0$ ($S_1 \Delta S_2$ denotes the symmetric difference of S_1 and S_2). An important consequence of ergodicity is the recurrence property [Halmos, p.10]; namely, if A and B are measurable subsets of positive measure then there exists an integer $k \geq 0$ such that $\mu(\phi^{-k}A \cap B) \neq 0$.

In the discussion that follows all relations are assumed to hold almost everywhere (mod 0). Let ϕ_1 and ϕ_2 be dynamical systems. If there exists a measure-preserving transformation $\Pi: X_1 \to X_2$ such that

(1-1) $$\Pi\phi_1 = \phi_2\Pi,$$

we say Φ_2 is a <u>factor</u> (metric factor) of Φ_1. Φ_1 and Φ_2 are called <u>weakly isomorphic</u> if Φ_2 is a factor of Φ_1 and Φ_1 is a factor of Φ_2. If in (1-1) Π is invertible, then

$$\phi_1\Pi^{-1} = \Pi^{-1}\phi_2,$$

and we say Φ_1 and Φ_2 are <u>isomorphic</u> (metrically isomorphic). When there is no possibility of confusion, we will omit the reference to a particular dynamical system but will speak of its mapping ϕ or of its underlying space X instead. For example, we will say ϕ_2 is a factor of ϕ_1, or X_2 is a factor of X_1, instead of saying Φ_2 is a factor of Φ_1.

Let τ_1 and τ_2 be continuous epimorphisms of the locally compact abelian topological groups G_1 and G_2 respectively. Then τ_2 is an <u>algebraic factor</u> of τ_1

if there exists a continuous homomorphism θ from G_1 onto G_2 such that $\theta\tau_1 = \tau_2\theta$. If τ_1 is also an algebraic factor of τ_2 we say τ_1 and τ_2 are <u>weakly algebraically isomorphic</u>. Suppose $\theta: G_1 \to G_2$ is a bicontinuous isomorphism and $\theta\tau_1 = \tau_2\theta$ (hence $\tau_1\theta^{-1} = \theta^{-1}\tau_2$), then τ_1 and τ_2 are called <u>algebraically isomorphic</u>. <u>Topological factor</u> and <u>weak topological isomorphism</u> are defined exactly as above except θ is a continuous map instead of a continuous homomorphism. Likewise for <u>topological isomorphism</u> we replace bicontinuous isomorphism with homeomorphism.

Suppose G_1 and G_2 are compact abelian groups, and let μ_1 and μ_2 be the normalized Haar measures defined on the Borel subsets of G_1 and G_2 respectively. It will be shown later (Chapter II) that each continuous homomorphism of G_1 onto G_2 is measure-preserving (with respect to μ_1 and μ_2). In particular, the continuous automorphisms of a compact abelian group G are a class of invertible measure-preserving transformations. We will be concerned primarily with dynamical systems of the type $(G, \mathcal{B}, \mu, \tau)$ where G is a compact abelian group, μ is the normalized Haar measure defined on $\mathcal{B}$ the Borel subsets of G, and τ is a continuous epimorphism or automorphism of G. It follows that for continuous epimorphisms of compact abelian groups, algebraic factor implies metric factor, and algebraic isomorphism implies metric isomorphism. In [1] Adler and Palais have shown that for automorphisms of the torus, algebraic and topological isomorphism coincide; so, in this case topological isomorphism also implies metric isomorphism.

2. Summary of the Problems and Main Results

T.L. Seethoff [14] has given an algebraic characterization of zero entropy automorphisms of a compact abelian metrizable group. In so doing, he proved that for each automorphism τ of such group G, there exists a unique maximal ergodic subgroup - i.e. a closed τ-invariant subgroup E of G on which τ is ergodic and the entropy of $\tau|E$ is equal to the entropy of τ. Seethoff then raises the question: Is it possible to decompose G into a direct sum, $G = E \oplus H$, where E is the maximal ergodic subgroup and H is a closed τ-invariant subgroup for which the entropy of $\tau|H$ is zero? Since the property of zero entropy for automorphisms of compact abelian metrizable groups can be regarded as an algebraic property, what can be said about automorphisms having positive entropy? In particular, to what extent is entropy a complete invariant under weak algebraic isomorphisms of automorphisms of the n-torus? Adler and Weiss [2] have shown that entropy is a complete metric invariant for ergodic automorphisms of the 2-torus.

In [5], J.R. Brown showed that for any ergodic automorphism τ of a compact abelian metrizable group G, the dual $\hat{G}$ is algebraically isomorphic to some shift-invariant subgroup of the infinite torus,

$$\bigtimes_{-\infty}^{\infty} K,$$

(K is the circle group or 1-torus) with the discrete topology; in fact, he showed that any group automorphism for which there exists a point in the group whose orbit separates the dual has this property. The question arises: For an arbitrary (not necessarily ergodic) automorphism of the n-torus, K^n, does there exist $a \in K^n$ whose orbit separates Z^n the dual of K^n? If so, it would follow that τ^*, the adjoint of τ, can be represented (via an algebraic isomorphism) as the shift transformation restricted to some shift-invariant subgroup of the infinite torus with the discrete topology. If this is possible, does there exist an explicit formula for such a representation? What about an analagous representation theorem for group epimorphisms? Finally, it would be interesting to know if the above representation could be applied to solve some of the problems mentioned in the preceeding paragraph.

Chapter II begins with an introduction of entropy.

Then the properties of group epimorphisms as well as automorphisms of the n-torus that are needed later are presented herein. Zero entropy automorphisms of a compact abelian metrizable group are then characterized. Finally, the existence and uniqueness of the maximal ergodic subgroup is discussed along with the E-Z decomposition problem.

Chapter III develops a representation theorem for a class of group automorphisms; namely, those for which there exists a point whose orbit separates the dual. A Universal model for ergodic automorphisms having this property is given herein. Then the representation theorem is modified to include a class of group epimorphisms.

In Chapter IV it is shown that the representation theorems for group automorphisms as well as group epimorphisms presented in Chapter III can be applied to automorphisms of the n-torus. In fact, a specific formula for finding a point whose orbit separates the dual is obtained. For type 1 automorphisms (those whose characteristic roots are distinct), we obtain the shift-invariant subgroup of the infinite torus to which the dual of K^n is algebraically isomorphic. This information is then used to characterize the maximal ergodic subgroup corresponding to a given type 1 automorphism.

Finally, the first section of Chapter V is devoted to an example which shows that the E-Z decomposition conjecture is false. In the second section, the results of Chapter IV are used to obtain a characterization of entropy classes of ergodic automorphisms of the two and three torus.

CHAPTER II

ENTROPY AND GROUP AUTOMORPHISMS

1. Entropy

We will briefly review the notion of entropy. For a detailed discussion the reader is referred to [4], [8] or [11].

Let (X, Σ, μ, ϕ) be a dynamical system. By a partition of X we mean a countable collection $\xi = \{A_1, A_2, \ldots\}$ of measurable subsets of positive measure such that

$$\bigcup_{i=1}^{\infty} A_i = X$$

and $A_i \cap A_j$ is empty for $i \neq j$ (all relations hold modulo null sets). For two partitions ξ and η of X, we define their refinement or product, written $\xi \vee \eta$, to be the partition consisting of the sets $\{A \cap B : A \in \xi, B \in \eta\}$. The product of a finite number of partitions

$$\bigvee_{i=1}^{n} \xi_i$$

is defined analagously. Now

$$\bigvee_{i=1}^{\infty} \xi_i$$

is defined to be the σ-algebra generated by

$$\bigcup_{i=1}^{\infty} \xi_i,$$

while $\hat{\xi}$ denotes the σ-algebra generated by ξ. We say that $\xi \leq \eta$ if each element of ξ is a union of elements from η. It follows that $\xi \leq \eta$ if and only if $\xi \vee \eta = \eta$. Observe that for any partition ξ of X, $\phi^{-1}\xi$ is also a partition, and if $\xi_1, \ldots, \xi_n$ are partitions then

$$\phi^{-1} \bigvee_{i=1}^{n} \xi_i = \bigvee_{i=1}^{n} \phi^{-1}\xi_i. \tag{2.1}$$

For a partition ξ of X set

$$H(\xi) = -\sum_{A\varepsilon\xi} \mu(A)\log\mu(A). \tag{2.2}$$

By convention $0 \log 0 = 0$. Next, the <u>entropy of ϕ with respect to ξ</u> is computed by

$$h(\phi, \xi) = \overline{\lim_{n}}\, \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^{-i}\xi\right). \tag{2.3}$$

Finally, the <u>entropy</u> of ϕ is given by

$$h(\phi) = \sup \{H(\phi, \xi) : \xi \text{ is a partition of } X\}. \tag{2.4}$$

Parry [11, p. 25] shows that (2.4) is the same regardless of whether finite or countable partitions are used (finite partitions being defined analagously) and that in (2.3) the limit actually exists.

Suppose ϕ_2 is a factor of ϕ_1 so that by (1.1) there is a measure-preserving transformation $\Pi:X_1 \to X_2$ such that $\Pi\phi_1 = \phi_2\Pi$. It follows that for $k \geq 1$

$$\Pi\phi_1^k = \phi_2^k\Pi$$

and, hence,

(2.5) $$\phi_1^{-k}\Pi^{-1} = \Pi^{-1}\phi_2^{-k} \quad \text{(these are set mappings).}$$

Replacing ξ by $\phi^{-1}\xi$ does not change the value of (2.2), so

(2.6) $$H(\xi) = H(\phi^{-1}\xi)$$

From (2.1), (2.5) and (2.6) it follows that

$$H\left(\bigvee_{i=0}^{n-1} \phi_2^{-i}\xi\right) = H\left(\bigvee_{i=0}^{n-1} \Pi^{-1}\phi_2^{-i}\xi\right)$$

$$= H\left(\bigvee_{i=0}^{n-1} \phi_1^{-i}\Pi^{-1}\xi\right).$$

Thus $h(\phi_2, \xi) = h(\phi_1, \Pi^{-1}\xi)$ so that by (2.4) $h(\phi_1) \geq h(\phi_2)$.

Thus, to each measure-preserving transformation ϕ of a probability space we can associate a number $h(\phi) \in [0, +\infty]$ called entropy which is invariant under isomorphisms (as well as weak isomorphisms).

Another way to compute (2.4) is described next. Let $f \in L_1(X)$, and let C be a sub-σ-algebra of Σ. Define a measure ν on C by

$$\nu(c) = \int_c f\,d\mu \qquad \text{(for each } c \in C).$$

Note that ν is absolutely continuous with respect to $\mu|C$ so by the Radon-Nikodym Theorem there exists an essentially unique C-measurable function $E(f/C) = d\nu/d(\mu|C)$ such that for each $c \in C$

$$\nu(c) = \int_c E(f/C)\,d\mu.$$

$E(f/C)$ is called the <u>conditional expectation</u> of f given C.

Suppose ξ is a partition of X, and let $A \in \xi$. Consider

$$E\left(\chi_A \Big/ \bigvee_{i=1}^{\infty} \phi^{-i}\xi\right)$$

which we will simply write

$$E\left(A \Big/ \bigvee_{i=1}^{\infty} \phi^{-i}\xi\right).$$

If we interpret $t \log t = 0$ at $t = 0$, then (2.3) can also be evaluated by

$$(2.7) \qquad h(\xi, \phi) = \int -\sum_{A\varepsilon\xi} E\left(A \Big/ \bigvee_{i=1}^{\infty} \phi^{-i}\xi\right) \log E\left(A \Big/ \bigvee_{i=1}^{\infty} \phi^{-i}\xi\right).$$

(2.7) can be computed more simply:

$$(2.8) \qquad h(\xi, \phi) = \int -\sum_{A\varepsilon\xi} \chi_A \log E\left(A \Big/ \bigvee_{i=1}^{\infty} \phi^{-i}\xi\right)$$

(see 11, p.9). Now $h(\phi)$ is again computed as in (2.4)

If $\xi \leq \eta$ or if

$$\bigvee_{i=0}^{\infty} \phi^{-i}\xi \leq \bigvee_{i=0}^{\infty} \phi^{-i}\eta,$$

then $h(\xi, \phi) \leq h(\eta, \phi)$. It follows that if

$$\bigvee_{i=0}^{\infty} \phi^{-i}\xi = \sum \quad (\text{mod } 0),$$

then $h(\xi, \phi) = h(\phi)$. In this case we call ξ a <u>strong generator</u>. If ϕ is invertible and

$$\bigvee_{-\infty}^{\infty} \phi^{-i}\xi = \sum \quad (\text{mod } 0)$$

we say ξ is a <u>generator</u>, and it follows that $h(\xi, \phi) = h(\phi)$. The role of a generator is fundamental in computing the entropy of a specific example. A necessary and sufficient condition that $h(\phi) = 0$ is that

$$\hat{\xi} \leq \bigvee_{i=1}^{\infty} \phi^{-i}\xi$$

for each partition ξ of X.

ϕ is said to have completely positive entropy or is called a K-automorphism (Kolomogorov) if for each nontrivial partition ξ of X, $h(\xi, \phi) > 0$. It is easy to show that ϕ is a K-automorphism if and only if each nontrivial factor of ϕ has positive entropy.

2. Topological Groups

In this section we will review those properties of topological groups which will be needed later. For a more complete discussion the reader is referred to [13].

The unit circle K in the complex plane is a topological group under multiplication which will be called the circle group. With the exception of the circle group, all groups will be written additively. The additive group of integers will be denoted by Z. If G is a topological group we will adopt the following terminology:

Hom (G) denotes the set of all continuous homomorphisms of G into G;

Epi (G) denotes the set of $\phi \varepsilon$ Hom (G) such that ϕ is a surjection;

Mon (G) denotes the set of all injective $\phi \varepsilon$ Hom (G);

Aut (G) denotes the set of all $\phi \varepsilon$ Epi (G) $\cap$ Mon (G) such that $\phi^{-1} \varepsilon$ Hom (G).

It is clear that Aut (G) is a subset of both Epi (G) and Mon (G).

With each locally compact abelian group G we can associate a nonnegative regular Borel measure μ, called the Haar measure, which is translation invariant and positive on open sets. Furthermore, μ is unique up to a multiplicative constant (A nonabelian group still admits unique left-invariant and right-invariant Haar measures, but the two are not necessarily the same). When dealing with compact abelian groups G, we will only be interested in the normalized Haar measure $(\mu G = 1)$.

Proposition 2.1: Let G_1 and G_2 be compact abelian groups and ϕ a continuous homomorphism from G_1 onto G_2. Then ϕ is measure-preserving.

Proof: Clearly ϕ is measurable because it is continuous. Set $\nu B = \mu_1 \phi^{-1}\{B\}$ for each Borel subset B of G_2. Then ν is a Borel measure on G_2 which is positive on open sets (by continuity). Let $\eta_\phi = \phi^{-1}\{0\}$ denote the kernel of ϕ. If $x \varepsilon \phi^{-1}\{y\}$ then $\phi^{-1}\{y + B\} = x + \eta_\phi + \phi^{-1}\{B\} = x + \phi^{-1}\{B\}$ since $\eta_\phi + \phi^{-1}\{B\} = \phi^{-1}\{B\}$. Thus $\nu\{y + B\} = \mu_1\{x + \phi^{-1}\{B\}\}$ $= \mu_1 \phi^{-1}\{B\} = \nu B$, so ν is translation invariant. Since $\nu(G_2) = \mu_1 \phi^{-1}\{G_2\} = \mu G_1 = 1$, it follows that $\nu = \mu_2$, and ϕ is measure-preserving.

In particular each $\phi \varepsilon \text{Aut}(G)$ where G is compact abelian is an example of an invertible measure-preserving transformation. Henceforth, unless otherwise specified, all automorphisms and epimorphisms of a topological group are assumed to be continuous.

Suppose G is a locally compact abelian group. Let $\hat{G}$ denote the set of all continuous homomorphisms (characters) from G into K, the circle group. $\hat{G}$ is called the dual (character) group of G. If $\gamma \varepsilon \hat{G}$ we will sometimes write $\langle x, \gamma\rangle$ instead of $\gamma(x)$. Since K is written multiplicatively we see that $\langle x + y, \gamma\rangle = \langle x, \gamma\rangle\langle y, \gamma\rangle$, and by definition $\langle x, \gamma + \delta\rangle = \langle x, \gamma\rangle\langle x, \delta\rangle$. $\hat{G}$ endowed with the compact-open topology is itself a locally compact abelian group, and by the Pontryagin Duality Theorem, the dual of $\hat{G}$ is G. G is compact if and only if $\hat{G}$ is discrete, and if G is compact, then G is metrizable if and only if $\hat{G}$ is countable. The set of characters $\hat{G}$ of a compact abelian group G forms a complete orthonormal basis for $L_2(G)$.

The direct sum $G = G_1 \oplus G_2 \oplus \ldots \oplus G_n$ of a finite number of locally compact abelian groups is a locally compact abelian group, and if $\hat{G}_i$ is the dual of G_i $(1 \leq i \leq n)$, then $\hat{G} = \hat{G}_1 \oplus \hat{G}_2 \oplus \ldots \oplus \hat{G}_n$.

Let $\tau \varepsilon \text{Hom}(G)$. Then τ induces a mapping $\tau^*:\hat{G} \to \hat{G}$ by $(\tau^*\gamma)(x) = \gamma(\tau x)$ called the <u>adjoint</u> of τ.

It is clear that $\tau^*\gamma$ is a character. Furthermore, since the topology of $\hat{G}$ is the compact-open topology, it follows that τ^* is continuous, so $\tau^* \in \text{Hom}(\hat{G})$. If $\tau \in \text{Epi}(G)$ then $\tau^* \in \text{Mon}(\hat{G})$ and $\tau^* \in \text{Epi}(\hat{G})$ implies $\tau \in \text{Mon}(G)$. $((\tau^*)^* = \tau)$.

More generally, if (X, Σ, μ, ϕ) is a dynamical system, ϕ induces a unitary operator, T_ϕ, on $L_2(X)$ by $(T_\phi f)(x) = f(\phi x)$, $f \in L_2(X)$. Recall that ϕ is ergodic means that $\phi^{-1}B = B$ (mod 0) implies $\mu B = 0$ or $\mu(X - B) = 0$. The following propositions whose proofs are found in [7, pp.25,53] give alternate criteria for ergodicity.

<u>Proposition 2.2</u>: Let (X, Σ, μ, ϕ) be a dynamical system. ϕ is ergodic if and only if for $f \in L_2(X)$, $T_\phi f = f$ a.e. implies f is constant a.e.

<u>Proposition 2.3</u>: Let G be a compact abelian group and $\tau \in \text{Aut}(G)$. Then τ is ergodic if and only if τ^* has no finite orbits in $\hat{G} - \{0\}$. Equivalently, τ is ergodic if and only if

$$\tau^{*k}\gamma = \gamma$$

for some $k \geq 1$ implies $\gamma = 0$.

Let G be a locally compact abelian group and H a subgroup of G. The closure of H, $\overline{H}$, is also a subgroup of G. If E is a closed subgroup of G, then

$$E^{\perp} = \{\gamma \in \hat{G} : \ker \gamma \supset E\}$$

is called the annihilator of E. $E^{\perp}$ is a closed subgroup of $\hat{G}$ and $(E^{\perp})^{\perp} = E$. The following theorem is found in [14, p.10].

Theorem 2.1: Let G be a compact abelian group, $\tau \in \text{Hom}\,(G)$ and G_1 be a closed subgroup of G such that $\tau G_1 \subset G_1$. Let $G_2 = G/G_1$, $\tau_1 = \tau|G$ and τ_2 be the homomorphism induced by τ in G_2. Set $H = G_1^{\perp}$, the annihlator of G_1. Define $\Phi : G_2 \to \hat{G}$ by

$$(\Phi\gamma)(x) = \gamma(x + G_1), \quad \gamma \in \hat{G}_2,\ x \in G.$$

Define $\Theta : \hat{G}/H \to \hat{G}_1$ by

$$[\Theta(\gamma + H)](x_1) = \gamma(x_1),\ \gamma \in \hat{G},\ x_1 \in G_1.$$

Then

i) Φ is a continuous monomorphism

ii) $\Phi(G_2) = H$ (i.e. $\widehat{G/G_1} \approx H$)

iii) Θ is a continuous isomorphism (i.e. $\hat{G}/H \approx \hat{G}_1$)

iv) $\Phi\tau_2^* = \tau^*\Phi$

v) $\Theta\tau_H^* = \tau_1^*\Theta$ where τ_H^* is the homomorphism induced by τ^* in $\hat{G}/H$.

($\approx$ denotes algebraic isomorphism)

Proof: Θ is well-defined because $H = G_1^{\perp} = \{\gamma \in \hat{G} : \gamma|G_1 = 0\}$. For the proofs of the first three assertions see [13, p.35].

For (iv) observe that for $\gamma \in \hat{G}_2$, $x \in G$,

$$\begin{aligned}(\Phi\tau_2^*\gamma)(x) &= \tau_2^*\gamma(x + G_1)\\ &= \gamma(\tau_2(x + G_1))\\ &= \gamma(\tau x + G_1)\\ &= \tau^*(\Phi\gamma)(x).\end{aligned}$$

To prove (v), let $\gamma \in \hat{G}$, $x_1 \in G_1$.

$$\begin{aligned}(\Theta\tau_H^*(\gamma + H))(x_1) &= (\Theta(\tau^*\gamma + H))(x_1)\\ &= \tau^*\gamma(x_1)\\ &= \gamma(\tau x_1)\\ &= \gamma(\tau_1 x_1)\\ &= (\tau_1^*\Theta(\gamma + H))(x_1).\end{aligned}$$

<u>Theorem 2.2</u>: Let G_1 and G_2 be compact abelian groups. Let $\tau_1 \in \text{Epi}(G_1)$ and $\tau_2 \in \text{Epi}(G_2)$. Suppose $\Theta^*:\hat{G}_2 \to \hat{G}_1$ is a continuous homomorphism such that $\Theta^*\tau_2^* = \tau_1^*\Theta^*$. Then Θ^* induces a continuous homomorphism $\Theta:G_1 \to G_2$ such that $\Theta\tau_1 = \tau_2\Theta$. Moreover, if Θ^* is one to one, Θ is onto, and if Θ^* is onto, Θ is one to one.

<u>Proof</u>: Define $\Theta:G_1 \to G_2$ by

$$\langle \Theta x, \gamma\rangle = \langle x, \Theta^*\gamma\rangle, \quad x \in G_1, \ \gamma \in \hat{G}_2$$

(Recall $x \varepsilon \hat{\hat{G}}_1$). Additivity follows from the additivity of Θ^* and x.

To prove continuity, note that for an arbitrary group G, the sets

$$N(C, \epsilon) = \{x \varepsilon G: |\langle x, \gamma\rangle - 1| < \epsilon \text{ for all } \gamma \varepsilon C\},$$

where $C \subset \hat{G}$ is compact and $\epsilon > 0$ is arbitrary, and their translates form a base for the topology of G [13, p.24]. Choose $C \subset \hat{G}_2$ compact and $\epsilon > 0$. Θ^* continuous implies $\Theta^*(C) \subset \hat{G}_1$ is compact. Let $x \varepsilon N(\Theta^*(C), \epsilon)$. Then for $\gamma \varepsilon C$,

$$|\langle \Theta x, \gamma\rangle - 1| = |\langle x, \Theta^*\gamma\rangle - 1| < \epsilon,$$

so $\Theta x \varepsilon N(C, \epsilon)$ and continuity follows.

Let $x \varepsilon G_1$. Then for $\gamma \varepsilon \hat{G}_2$,

$$\begin{aligned}\langle \Theta\tau_1(x), \gamma\rangle &= \langle \tau_1(x), \Theta^*\gamma\rangle \\ &= \langle x, \tau_1^*\Theta^*\gamma\rangle \\ &= \langle x, \Theta^*\tau_2^*\gamma\rangle \\ &= \langle \Theta(x), \tau_2^*\gamma\rangle \\ &= \langle \tau_2\Theta(x), \gamma\rangle\end{aligned}$$

Hence, $\Theta\tau_1 = \tau_2\Theta$.

Suppose Θ^* is one to one. Write $H = \Theta^*(\hat{G}_2)$. Then, since $\hat{G}_1$ is discrete, H is a closed subgroup of $\hat{G}_2$. Let $y \varepsilon G_2$ be arbitrary. Define a character x' on H by

$$\langle x', \gamma\rangle = \langle y, \Theta^{*-1}\gamma\rangle, \ \gamma \varepsilon H.$$

Then x' is well-defined since Θ^* is injective. Since H is closed, x' can be extended to a character x of $\hat{G}_1$ [13, p.36]. Note that $x \in G_1$. Next for $\delta \in \hat{G}_2$,

$$\begin{aligned}\langle \Theta x, \delta \rangle &= \langle x, \Theta^*\delta \rangle \\ &= \langle y, \Theta^{*-1}\Theta^*\delta \rangle \\ &= \langle y, \delta \rangle\end{aligned}$$

(The last equality follows since Θ^* is injective.) Therefore, $\Theta x = y$ so Θ is onto.

The proof of the last assertion is straightforward and will be omitted.

Corollary 2.2a: Let G be a compact abelian group and $\tau \in \text{Epi}(G)$. Then τ^* is one to one if and only if τ is onto and τ^* is onto if and only if τ is one to one.

Proof: The first assertion follows by setting $\Theta^* = \tau^*$ in Theorem 2.2.

For the second assertion, set $H = \tau G$. Then it follows by compactness and the continuity of τ that H is a closed subgroup of G. Now proceed as in the theorem.

Corollary 2.2b: Let G be compact abelian with $\tau \in \text{Aut}(G)$. Suppose τ^* is algebraically isomorphic to $\delta^* \in \text{Aut}(\hat{X})$ where $\hat{X}$ is a compact abelian group. Then τ is algebraically isomorphic to $\delta \in \text{Aut}(X)$.

<u>Corollary 2.2c</u>: Let $\tau_1 \in \text{Aut}(G_1)$ and $\tau_2 \in \text{Aut}(G_2)$ where G_1 and G_2 are compact abelian groups. Suppose τ_1^* and τ_2^* are algebraically isomorphic to $\delta_1^* \in \text{Aut}(\hat{X}_1)$ and $\delta_2^* \in \text{Aut}(\hat{X}_2)$ respectively, and suppose $\Phi^*:\hat{X}_2 \to \hat{X}_1$ is a continuous one to one homomorphism such that $\Phi^*\delta_2^* = \delta_1^*\Phi^*$. Then Φ^* induces a continuous one to one homomorphism $\Theta^*:\hat{G}_2 \to \hat{G}_1$ such that $\Theta^*\tau_2^* = \tau_1^*\Theta^*$, and it follows from Theorem 2.2 that τ_2 is an algebraic factor of τ_1.

<u>Proof</u>: Let ψ_1^* and ψ_2^* be algebraic isomorphisms of $\hat{G}_1$ onto $\hat{X}_1$ and $\hat{G}_2$ onto $\hat{X}_2$ respectively. Define $\Theta^*:\hat{G}_2$ into $\hat{G}_1$ by

$$\Theta^*(\gamma) = (\psi_1^{*-1}\Phi^*\psi_2^*)(\gamma), \ \gamma \in \hat{G}_2.$$

Then Θ^* is continuous since ψ_1^{*-1}, Φ and ψ_2^* are continuous. ψ_1^* and ψ_2^* are isomorphisms, so Φ^* one to one implies Θ^* is one to one. The commutativity of the diagram $\Theta^*\tau_2^* = \tau_1^*\Theta^*$ can be shown by a straightforward calculation and will be omitted here.

We shall need the following theorem in section 4 of this chapter; for a proof see [Yuzvinskii S.A.].

<u>Theorem 2.3</u>: Let G be a compact metrizable group, $\tau \in \text{Epi}(G)$ and H a closed normal subgroup of G such that $\tau H = H$. Then

$$h(\tau) = h(\tau|H) + h(\tau_H)$$

where τ_H is the epimorphism induced in G/H by τ. The entropy of τ, $\tau|H$ and τ_H is computed with respect to the normalized Haar measure of G, H and G/H respectively.

3. Automorphisms of the n-torus.

For a more general discussion of the ideas presented in this section see [3]. The product of n-copies of the circle group, K^n, is itself a compact abelian group and is called the n-torus (K is the 1-torus). A more useful formulation of K^n is R^n/Z^n - i.e. two points of R^n are identified if their cooriinates differ by integers (R denotes the real numbers). Addition is the usual vectorial addition in $\mathbb{R}^n$ modulo (1, 1, ..., 1). For an arbitrary vector $(x_1, x_2, \ldots, x_n) \in R^n$, we will refer to the point $(x_1, x_2, \ldots, x_n)$ mod(1, 1, ..., 1) $\in R^n/Z^n$ as the projection of $(x_1, x_2, \ldots, x_n)$ onto the n-torus.

The dual of K^n is Z^n, and if $(x_1, x_2, \ldots, x_n) \in R^n/Z^n$ and $(k_1, k_2, \ldots, k_n) \in Z^n$, then

$$(2.9) \qquad \langle (x_1, \ldots, x_n), (k_1, \ldots, k_n) \rangle = \exp 2\pi i \{(x_1, \ldots, x_n) \cdot (k_1, \ldots, k_n)\}$$

$$= \exp 2\pi i \left\{ \sum_{j=1}^{n} k_j x_j \right\}$$

[13, pp.13,37].

Each continuous epimorphism τ of K^n is determined by a nonsingular linear transformation of R^n into itself whose matrix A has integer entries. In particular, the matrix A associated with a continuous automorphism τ of K^n is unimodular - i.e. A has integer entries and determinent equal to ± 1 [3, p.67]. Observe that for $x = (x_1, x_2, \ldots, x_n) \in R^n/Z^n$,

$$x \longmapsto xA \mod(1, 1, \ldots, 1).$$

Note that we have used the left-hand notation and that x is used to denote both an n-tuple and its row matrix. The following proposition gives a useful criterian for testing a toral automorphism for ergodicity. See [6, p.34] for a proof.

<u>Proposition 2.4</u>: Suppose $\tau \in \mathrm{Aut}\,(K^n)$ with A its $n \times n$ unimodular matrix. Then τ is ergodic if and only if A has no characteristic roots which are roots of unity.

If turns out that the entropy of an automorphism of the n-torus is also determined by the characteristic roots of its associated unimodular matrix.

<u>Proposition 2.5</u>: Let A be the $n \times n$ unimodular matrix associated with an automorphism τ of K^n. Suppose $\lambda_1, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A. Then

$$h(\tau) = \sum_{|\lambda_i| \geq 1} \log|\lambda_i|$$

<u>Proof</u>: See [3] and [6].

Let $\tau \in \mathrm{Aut}\ (K^n)$ with A its associated $n \times n$ unimodular matrix. Then $\tau^* \in \mathrm{Aut}\ (Z^n)$, and a unimodular matrix A^t, the transpose of A, can be associated with τ^* such that for $m = (k_1, k_2, \ldots, k_n) \in Z^n$,

$$\tau^*(m) = (m)A^t$$

(Again we used m to denote both an n-tuple and its row matrix.)

4. Quasiperiodic Spectrum and Zero Entropy

Suppose τ is an automorphism of a compact abelian group G. In section 1 of this chapter it was pointed out that $h(\tau) = 0$ if and only if for any partition ξ of G,

$$\hat{\xi} \subset \bigvee_{i=1}^{\infty} \tau^{-i}\xi.$$

If, in addition, G is metrizable, Seethoff [14, p.36] has given a purely algebraic characterization of zero entropy; namely $h(\tau) = 0$ if and only if τ has quasi-periodic spectrum which is determined by the action of τ^* on $\hat{G}$. The results of this section are found in [14].

Definitions: Let G be a compact abelian group and $\tau \in \text{Epi}\ (G)$. Define subsets of $\hat{G}$ as follows:

(2.10) $$\Delta_0 = \{0\},$$

$$\Delta_1 = \bigcup_{k \geq 1} (\tau^{*k} - I)^{-1} \Delta_0$$

where I denotes the identity transformation (note that $\tau_1, \tau_2 \in \text{Hom}\ (G)$ implies $\tau_1^k \in \text{Hom}\ (G)$ for $k \geq 1$ and $\tau_1 + \tau_2 \in \text{Hom}\ (G)$). Inductively define

(2.11) $$\Delta_{n+1} = \bigcup_{k \geq 1} (\tau^{*k} - I)^{-1} \Delta_k.$$

Finally, set

(2.12) $$\Delta_\tau = \bigcup_{n \geq 0} \Delta_n$$

We shall say that τ has <u>periodic</u> spectrum if $\Delta_1 = \hat{G}$, τ has <u>quasiperiodic spectrum</u> if $\Delta_\tau = \hat{G}$ and τ^* is <u>aperiodic</u> if $\Delta_1 = \{0\}$.

Note that $\gamma \in \Delta_k$ if and only if there exists a polynomial in τ^* of the form

$$P(\tau^*) = \prod_{i=1}^{k} \left(\tau^{*^{p_i}} - I\right)$$

such that $P(\tau^*)\gamma = 0$.

If f is a mapping of some set X into itself we shall say that $A \subset X$ is <u>f-invariant</u> if $f^{-1}(A) = A$. Note that if f is onto, $f^{-1}(A) = A$ implies $f(A) = A$.

The following proposition summarizes some important properties of the Δ_k.

<u>Proposition 2.6</u>: Let G be a compact abelian group, $\tau \in \text{Epi}\,(G)$, and for $k \geq 0$ define Δ_k as in (2.10) and (2.11). Then the Δ_k form an increasing sequence of subgroups of $\hat{G}$ such that $\tau^*(\Delta_k) = \Delta_k$ $(k \geq 0)$.

<u>Proof</u>: We will first show that for $k \geq 0$

(2.13) $$\tau^*\Delta_k \subset \Delta_k.$$

Proceeding inductively, (2.13) clearly holds for $k = 0$. Assume (2.13) is true for $k = \ell$. Let $\gamma \in \Delta_{\ell+1}$. Then there exists $p \geq 0$ such that $\tau^{*p}\gamma - \gamma \in \Delta_\ell$. Our inductive hypothesis tells us that

$$\tau^*(\tau^{*p}\gamma - \gamma) = \tau^{*p}(\tau^*\gamma) - (\tau^*\gamma) \in \Delta_\ell.$$

Hence $\tau^*\gamma \in \Delta_{\ell+1}$.

Suppose $\gamma \in \Delta_k$. There exists a polynomial in τ^* of the form

$$P(\tau^*) = (\tau^{*P_1} - I)(\tau^{*P_2} - I) \ldots (\tau^{*P_k} - I)$$

such that $P(\tau^*)\gamma = 0$. $P(\tau^*)(-\gamma) = -P(\tau^*)\gamma = 0$, so $-\gamma \in \Delta_k$.

Next we must show $\Delta_k (k \geq 0)$ is closed under addition. The assertion clearly holds for $k = 0$. For $k = 1$ suppose $\gamma, \delta \in \Delta_1$. There exists $m_1, m_2 \geq 1$ such that $\tau^{*m_1}\gamma = \gamma$ and $\tau^{*m_2}\delta = \delta$. Since $\tau^{*m_1m_2}\gamma = \gamma$ and $\tau^{*m_2m_1}\delta = \delta$, we have

$$\tau^{*m_1m_2}(\gamma + \delta) = \tau^{*m_1m_2}(\gamma) + \tau^{*m_1m_2}(\delta)$$
$$= \gamma + \delta.$$

Thus $\gamma + \delta \in \Delta_1$. Now suppose that for $0 \leq k \leq \ell$, Δ_k is closed under addition. Let $\gamma, \delta \in \Delta_{\ell+1}$. Then there exists $m_1, m_2 \geq 1$ such that $\tau^{*m_1}\gamma - \gamma \in \Delta_\ell$ and $\tau^{*m_2}\delta - \delta \in \Delta_\ell$.

$$(2.14) \qquad \tau^{*m_1m_2}(\gamma + \delta) - (\gamma + \delta) = \sum_{i=1}^{m_2} \tau^{*m_1 i}(\gamma) - \sum_{i=1}^{m_2} \tau^{*m_1(i-1)}(\gamma)$$

$$= \sum_{j=1}^{m_1} \tau^{*jm_2}(\delta) - \sum_{j=1}^{m_1} \tau^{*(j-1)m_2}(\delta)$$

$$= \sum_{i=1}^{m_2} \tau^{*m_1(i-1)}(\tau^{*m_1}\gamma - \gamma)$$

$$+ \sum_{j=1}^{m_1} \tau^{*(j-1)m_2} (\tau^{*m_2}\delta - \delta).$$

If follows from the inductive hypothesis and (2.13) that (2.14) is an element of Δ_ℓ; hence $\gamma + \delta \in \Delta_{\ell+1}$.

Finally to show $\tau^*(\Delta_k) = \Delta_k$, it remains to show that $\tau^*\Delta_k \supset \Delta_k$ for $k \geq 0$. Again using induction, the assertion is clearly true for $k = 0$. Assume it is true for $k = \ell$. Let $\gamma \in \Delta_{\ell+1}$. There exists $m \geq 1$ such that $\tau^{*m}\gamma - \gamma \in \Delta_\ell$. By hypothesis there exists $\delta \in \Delta_\ell$ such that $\tau^*\delta = \tau^{*m}\gamma - \gamma$. It follows that

$$\begin{aligned} \gamma &= \tau^{*m}\gamma - \tau^*\delta \\ &= \tau^{*m}\gamma + \tau^*(-\delta) \\ &= \tau^*(\tau^{*m-1}\gamma - \delta). \end{aligned}$$

We have already shown that Δ_k $(k \geq 0)$ is closed under sums and additive inverses. Moreover, $m \geq 1$ so (2.13) implies $\tau^{*m-1}\gamma \in \Delta_{\ell+1}$. Since $\Delta_{\ell+1} \supset \Delta_\ell$, the assertion follows.

<u>Corollary 2.4</u>: Let $\tau \in \mathrm{Epi}\ (G)$ where G is compact abelian. Define Δ_τ as in (2.12). Then Δ_τ is a subgroup of $\hat{G}$, and $\tau^*(\Delta_\tau) = \Delta_\tau$.

<u>Proposition 2.7</u>: Suppose G is a compact abelian group and $\tau \in \mathrm{Aut}\ (G)$. Then τ is ergodic if and only if τ^* is aperiodic (i.e. $\Delta_1 = \{0\}$).

Proof: From Proposition (2.3) we see that τ is ergodic if and only if $\tau^{*k}\gamma = \gamma$ for some $k \geq 1$ implies $\gamma = 0$.

Note that $\Delta_1 = \{0\}$ implies $\Delta_k = \{0\}$ for $k \geq 1$ and, hence $\Delta_\tau = \{0\}$. Thus Δ_τ is nontrivial if and only if $\Delta_1 \neq \{0\}$. The following theorem characterizes those automorphisms of a compact abelian metric group which have zero entropy. For a proof see [14, p.36].

Theorem 2.4: Let G be a metrizable compact abelian group, and let $\tau \in \text{Aut}(G)$. Then $h(\tau) = 0$ if and only if τ has quasiperiodic spectrum (in other words $h(\tau) = 0$ if and only if $\Delta_\tau = \hat{G}$).

5. The E-Z Decomposition Problem

Let G be compact, abelian and metrizable. In view of Proposition 2.7 and Theorem 2.3 a question naturally arises: What happens if $\{0\} \subset \Delta_\tau \subset \hat{G}$ (strict inclusion)? Then τ would be a nonergodic automorphism of G having positive entropy. The following theorem gives an interesting property of such automorphisms.

Theorem 2.5: Let G be a compact metrizable abelian group and $\tau \in \text{Aut}(G)$. Then there exists a unique closed subgroup $E \subset G$ such that

i) $\tau E = E$,

ii) $\tau|E$ is ergodic,

iii) $h(\tau_E) = 0$ where τ_E is the epimorphism induced in G/E by τ,

iv) $h(\tau) = h(\tau|E)$.

Proof: The assertions clearly hold if either $\Delta_\tau = \hat{G}$ or $\Delta_\tau = \{0\}$; for we simply set $E = \{0\}$ or $E = G$ accordingly. Let

$$E = \Delta_\tau^\perp \tag{2.15}$$

the annihilator of Δ_τ. Then E is a closed subgroup of G, and $E^\perp = \Delta_\tau$. Let $x \in E$. For each $\gamma \in \Delta_\tau$,

$$\langle \tau^{-1}x, \gamma\rangle = \langle x, \tau^{*-1}\gamma\rangle = 1$$

since $\tau^*(\Delta_\tau) = \Delta_\tau$. Hence, $\tau(E) = E$.

(ii) and (iii) follow from the fact that $\hat{E}$ and $\widehat{G/E}$ are algebraically isomorphic to $\hat{G}/\Delta_\tau$ and Δ_τ respectively (Theorem 2.1); so $(\tau|E)^*$ is aperiodic and τ_E has quasiperiodic spectrum.

From Theorem 2.2 we have

$$h(\tau) = h(\tau|E) + h(\tau_E);$$

thus

$$h(\tau|E) = h(\tau).$$

For a proof of uniqueness see [14, p.40].

Thus, according to the above theorem we can separate off the ergodic part of τ leaving a factor automorphism with zero entropy. Seethoff [14] has considered the following group theoretic version of Pinskers conjecture (every ergodic automorphism of a Lebesgue space factors into a direct product of a K-automorphism and a zero entropy automorphism) without the assumption of ergodicity. Let G be a compact metrizable abelian group and $\tau \in \text{Aut}(G)$. Do there exist subgroups G_i $(i = 1, 2)$ such that $G = G_1 \oplus G_2$ where $\tau G_i = G_i$ and $\tau|G_1$ is ergodic while $\tau|G_2$ has zero entropy? If so, we say τ has the E-Z decomposition property. Is such a decomposition always possible?

Suppose G is compact abelian and metrizable, and suppose $\tau \in \text{Aut}(G)$ has the E-Z decomposition property. Now G/G_1 is algebraically isomorphic to G_2, and since $h(\tau) = h(\tau|G_1) + h(\tau_{G_1})$ (Theorem 2.2), $h(\tau) = h(\tau|G_1)$. By the uniqueness of E (theorem 2.5), and (2.15) it follows that $G_1 = E = \Delta_\tau^\perp$. Thus, in order to determine whether a given automorphism of a compact metrizable abelian group G has the E-Z decomposition property, it is only necessary to find a subgroup F such that $\tau F \subset F$ and $G = E \oplus F$. The following theorem is a reformulation of the E-Z decomposition problem.

Theorem 2.6: Let G be a compact metrizable abelian group and $\tau \in \text{Aut}(G)$. Then τ has the E-Z decomposition property if and only if there exists a subgroup

$H \subset \hat{G}$ such that $\tau^*H \subset H$ and $\hat{G} = H \oplus \Delta_\tau$.

Proof: See [14, p.42].

In Chapter V we will return to the E-Z decomposition problem producing an example of an automorphism of the 3-torus which does not have the E-Z decomposition property. It will first be necessary to characterize the subgroup $E = \Delta_\tau$, which we will call the maximal ergodic subgroup (the role of τ being understood). To do so we will need the aid of a representation theorem for such automorphisms which will be developed in the following two chapters.

CHAPTER III

A REPRESENTATION THEOREM FOR GROUP EPIMORPHISMS

1. A Representation Theorem and Universal Model for a Class of Group Automorphisms

In his paper [5], J.R. Brown presented a representation theorem for a certain class of group automorphisms from which he develops a universal model for the subclass consisting of ergodic automorphisms. Section I of this chapter develops some of these results which will be needed later, and Section II gives some extensions to include a class of group epimorphisms.

Define

(3.1) $$\hat{X} = \left(\bigtimes_{-\infty}^{\infty} K \right)_d ;$$

that is, $\hat{X}$ is the product of countably many copies of the circle group, and $\hat{X}$ is given the discrete topology. Let us write $\hat{X}$ additively. Then if $\{x_k\}, \{y_k\} \in \hat{X}$, it follows that

(3.2) $$\{x_k\} + \{y_k\} = \{x_k y_k\}.$$

Thus X is a discrete abelian topological group (the identity element is $\hat{0} = \{e_k\}$ where $e_k = 1$ for each k).

Define a one to one mapping σ^* of $\hat{X}$ onto itself by

$$(3.3) \qquad \sigma^*(\{x_k\}) = \{y_k\}, \quad \text{where} \quad y_k = x_{k+1}.$$

σ^* is called the <u>shift transformation</u> on $\hat{X}$, the terminology being clear from (3.3). Suppose $\{x_k\}, \{y_k\} \in \hat{X}$. Then

$$\begin{aligned} \sigma^*(\{x_k\} + \{y_k\}) &= \sigma^*(\{x_k y_k\}) \\ &= \{x_{k+1} y_{k+1}\} \\ &= \sigma^*(\{x_k\}) + \sigma^*(\{y_k\}). \end{aligned}$$

Since σ^* leaves constant sequences fixed, it maps the identity onto itself, so $\sigma^* \in \text{Aut}(X)$.

Let X be the dual of $\hat{X}$, and let $\sigma = (\sigma^*)^*$. Then X is a compact abelian group and $\sigma \in \text{Aut}(X)$. Note that X is not metrizable since $\hat{X}$ is not coutable.

Recall that if G_1 and G_2 are locally compact abelian groups, then $\tau_1 \in \text{Aut}(G_1)$ and $\tau_2 \in \text{Aut}(G_2)$ are algebraically isomorphic if there exists a bicontinuous isomorphism $\psi: G_1 \to G_2$ such that $\psi\tau_1 = \tau_2\psi$ and, hence, $\tau_1\psi^{-1} = \psi^{-1}\tau_2$. If $\tau \in \text{Aut}(G)$, then for $a \in G$ we define

$$(3.4) \qquad \text{orb}(a) = \{\tau^n a : n \in Z\}.$$

A subset $A \subset \hat{X}$ is said to be shift-invariant if it has the property $\sigma^* A = A$.

<u>Theorem 3.1</u>: Let G be a compact abelian group and $\tau \in \mathrm{Aut}(G)$. Suppose there exists $a \in G$ such that orb(a) separates $\hat{G}$. Then τ^* is algebraically isomorphic to $\sigma^*|\hat{A}$ where $\hat{A}$ is some shift-invariant subgroup of $\hat{X}$. Conversely, any shift-invariant subgroup $\hat{B}$ of $\hat{X}$ is the dual of some compact abelian group having the above property.

<u>Proof</u>: Define a mapping ψ^* from $\hat{G}$ into $\hat{X}$ by

$$\psi^*(\gamma) = \{\tau^{*k}\gamma(a)\} = \{\gamma(\tau^k a)\}. \tag{3.5}$$

Since orb(a) separates $\hat{G}$, ψ^* is injective. By (3.5),

$$\begin{aligned}\psi^*(\gamma - \delta) &= \{(\gamma - \delta)(\tau^k a)\}\\ &= \{\gamma(\tau^k a)[\delta(\tau^k a)]^{-1}\}\\ &= \{\gamma(\tau^1 a)\} - \{\gamma(\tau^k a)\}.\end{aligned}$$

Hence,

$$\psi^*(\gamma - \delta) = \psi^*(\gamma) - \psi^*(\delta) \qquad (\gamma, \delta \in \hat{G}). \tag{3.6}$$

Also,

$$\begin{aligned}\psi^*(\tau^*\gamma) &= \{\tau^*\gamma(\tau^k a)\}\\ &= \{\tau^{*k+1}\gamma(a)\}\\ &= \sigma^*\psi^*(\gamma),\end{aligned}$$

so

$$\psi^*\tau^* = \sigma^*\psi^*. \tag{3.7}$$

Let $\hat{A} = \psi^*\hat{G}$. Then (3.6) together with the facts that ψ^* is injective and that $\hat{G}$ and $\hat{A}$ are discrete imply ψ^* is a bicontinuous isomorphism of $\hat{G}$ onto $\hat{A}$. From (3.7) it follows that τ^* is algebraically isomorphic to $\sigma^*|\hat{A}$. Finally, from (3.7) we obtain $\sigma^* = \psi^*\tau^*\psi^{*-1}$, and since ψ^*, ψ^{*-1} and τ^* are bijections, $\hat{A}$ is shift-invariant.

For the converse let $\hat{B}$ be a shift-invariant subgroup of $\hat{X}$. $\hat{B}$ is descrete, so its dual B is a compact abelian group. Let τ be the adjoint of $\sigma^*|\hat{B}$; then $\tau \in \mathrm{Aut}(B)$. Define a mapping $\hat{a}:\hat{B} \to K$ by

(3.8) $$\hat{a}\{x_k\} = x_0.$$

Then (3.2) implies $\hat{a}$ is a homomorphism, and since $\hat{B}$ is descrete, $\hat{a}$ is continuous. Therefore $\hat{a} \in B$.

Now suppose $\{x_k\} \in \hat{B}$ is nontrivial. Then there exists $m \in Z$ such that $x_m \neq 1$.

$$\begin{aligned} \langle \{x_k\}, \tau^m(\hat{a}) \rangle &= \langle \sigma^{*m}\{x_k\}, \hat{a} \rangle \\ &= \langle \{x_{k+m}\}, \hat{a} \rangle \\ &= x_m \end{aligned}$$

(the last equality is from (3.8)). Hence, the orbit of $\hat{a}$ (under τ) separates $\hat{B}$. Finally, it follows from (3.8) that

$$\begin{aligned} \langle \tau^r\hat{a}, \{x_k\} \rangle &= \langle \hat{a}, \sigma^{*r}\{x_k\} \rangle \\ &= \langle \hat{a}, \{x_{k+r}\} \rangle \end{aligned}$$

$$= x_r.$$

Therefore, (3.5) implies

$$\psi^*\hat{B} = \hat{B}. \tag{3.9}$$

Thus, the adjoints of all automorphisms of compact abelian groups having a point whose orbit separates the dual can be represented by σ^* restricted to some shift-invariant subgroup of $\hat{X}$. It is natural to ask: what sort of group automorphisms satisfy this condition? This question will be taken up later, but first we will identify the subclass consisting of those automorphisms which are ergodic.

Let $\Delta = \Delta_\sigma = \hat{X}$ be defined as in (2.12). Then Δ is a shift-invariant subgroup of $\hat{X}$ (Corollary 2.4). It is clear that Δ is nontrivial since Δ_1 (see (2.11)) contains all constant sequences in $\hat{X}$. Hence σ is not ergodic (Proposition 2.7). It will be shown later that σ does not have quasiperiodic spectrum either (i.e. $\Delta \neq \hat{X}$). The following corollary will be needed in the next chapter.

<u>Corollary 3.1a</u>: Let G, τ and $\hat{A} = \psi^*\hat{G}$ be as in the above theorem. Define $\Delta = \Delta_\sigma \subset \hat{X}$ and $\Delta_\tau \subset \hat{G}$ as in (2.12). Then

$$\psi^*\Delta_\tau = \hat{A} \cap \Delta$$

or

$$\Delta_\tau = \psi^{*-1}\{\hat{A} \cap \Delta\} .$$

In other words, $\gamma \varepsilon \Delta_\tau$ if and only if $\psi^*\gamma \varepsilon \Delta$ for $\gamma \varepsilon \hat{G}$.

<u>Proof</u>: From the above theorem, $\sigma^*|\hat{A} = \psi^*\tau^*\psi^{*-1}$.
Hence on $\hat{A}$,

$$\sigma^{*m} = \psi^*\tau^{*m}\psi^{*-1} \qquad (m \geq 1).$$

Consider the following polynomial in σ^*:

$$\prod_{j=1}^{N}\left(\sigma^{*p_j} - I\right) = \prod_{j=1}^{N}\left(\psi^*\tau^{*p_j}\psi^{*-1} - \psi^*\psi^{*-1}\right)$$

$$= \prod_{j=1}^{N} \psi^*\left(\tau^{*p_j} - I\right)\psi^{*-1}$$

$$= \psi^*\left[\prod_{j=1}^{N}\left(\tau^{*p_j} - I\right)\right]\psi^{*-1}.$$

Then

$$\left[\prod_{j=1}^{N}\left(\sigma^{*p_j} - I\right)\right](\psi^*\gamma) = \left\{\left[\prod_{j=1}^{N}\left(\tau^{*p_j}\gamma - \gamma\right)\right](\tau^k a)\right\} . \tag{3.10}$$

Since orb(a) separates $\hat{G}$ it follows that the left-hand side of (3.10) is $\hat{0}$ if and only if

$$\prod_{j=1}^{N}\left(\tau^{*p_j}\gamma - \gamma\right) = 0,$$

which proves the corollary.

Corollary 3.1b: Let τ be an ergodic automorphism of a compact abelian group G in which there exists a point "a" whose orbit separates $\hat{G}$. Then τ^* is algebraically isomorphic to σ^* restricted to a shift-invariant subgroup $\hat{A}$ of $\hat{X}$ such that $\hat{A} \cap \Delta$ is trivial (Δ is defined in the preceeding corollary). Conversely, any shift-invariant subgroup $\hat{B}$ of $\hat{X}$ whose intersection with Δ is trivial is the dual of some compact abelian group having the above property.

Proof: From Proposition 2.7 $\{0\} = \Delta_1 \subseteq \hat{G}$, so (2.11) and (2.12) imply $\Delta_\tau = \{0\}$. The first part of the corollary now follows from Theorem 3.1 and Corollary 3.1a.

For the converse, let B and $\tau = (\sigma^*|\hat{B})^* \in \mathrm{Aut}(B)$ be as in Theorem 3.1. Since $\psi^*\hat{B} = \hat{B}$ (3.9) and since $\hat{B} \cap \Delta = \{\hat{0}\}$, it follows that Δ_τ (hence Δ_1) is trivial (Corollary 3.1a). Therefore τ is ergodic (Proposition 2.7).

Proposition 3.1: For each ergodic automorphism τ of a compact abelian metrizable group G, there exists a point $a \in G$ whose orbit separates $\hat{G}$.

Proof: Since G is compact metric, there exists a countable base $\{U_n\}$ $(n \geq 1)$ for the topology of G. Let

$$F_n = \{x \in G : \mathrm{orb}(x) \cap U_n \text{ is empty}\} \quad (n \geq 1).$$

First we will show that F_n is closed. Let $\{x_k\}$ $(k \geq 1)$ be a sequence of points in F_n which converges to x. Suppose $\mathrm{orb}(x) \cap U_n$ is nonempty, say $\tau^p x \in U_n$ for some $p \in Z$. By continuity there exists a neighborhood V of x such that $\tau^p(V) \subset U_n$. Choose $N \in Z^+$ so that $r \geq N$ implies $x_r \in V$. This means that for $r \geq N$, $\tau^p x_r \in U_n$ or $\mathrm{orb}(x_r) \cap U_n$ is nonempty, a contradiction.

Moreover, F_n is nowhere dense. Suppose F_n^0, the interior of F_n, is nonempty. The Haar measure μ is positive on open sets, so $\mu F_n^0 > 0$ and $\mu U_n > 0$. Since τ is ergodic, U_n is recurrent; namely, there exists a nonnegative $k \in Z$ with $\mu(F_n^0 \cap \tau^{-k} U_n) > 0$. Thus, there exists $x \in F_n^0 \cap \tau^{-k} U_n$. But then $\mathrm{orb}(x) \cap U_n$ is non-empty, a contradiction.

By the Baire Category Theorem

$$G - \bigcup_{n=1}^{\infty} F_n$$

is nonempty, so there exists $a \in G$ with a dense orbit. By continuity, it follows that $\gamma(\tau^n a) = 1$ for all n implies $\gamma = 0$, or the orbit of a separates $\hat{G}$.

Actually for a given ergodic $\tau \in \mathrm{Aut}(G)$, the set S of all points whose orbit separates $\hat{G}$ is of the second category since it is as least as large as

$$G - \bigcup_{n-1}^{\infty} F_n.$$

As we shall see later, there are many examples of automorphisms of compact abelian metrizable groups, so it follows that $\hat{X} \neq \Delta$. We will also give several examples of nonergodic automorphisms of metrizable compact abelian groups having a point whose orbit separates the dual. In fact, no examples are known to the author which do not have this property. The following is an example of a nonergodic automorphism of a nonmetrizable abelian group with this property.

Example: Let X be the dual of $\hat{X}$ (3.1) and $\sigma = (\sigma^*)^*$ be the adjoint of the shift transformation (3.3). It was mentioned earlier that X is not metrizable and σ is not ergodic. Define a mapping $\hat{a}$ of $\hat{X}$ into K by

$$\hat{a}\{x_k\} = x_0.$$

The same argument used in proving the second part of Theorem 3.1 applied here shows that $\hat{a} \varepsilon X$. Furthermore, for $r \varepsilon Z$,

$$\begin{aligned} \langle \sigma^r \hat{a}, \{x_k\} \rangle &= \langle \hat{a}, \sigma^{*r}\{x_k\} \rangle \\ &= \langle \hat{a}, \{x_{k+r}\} \rangle \\ &= x_r. \end{aligned}$$

It follows that $\text{orb}(\hat{a})$ separates $\hat{X}$.

Let $\Delta = \Delta_\sigma \subset \hat{X}$ be defined as in (2.12), and let $E = \Delta^\perp \subset X$ be the annihilator of Δ. Consider $\Gamma = \hat{X}/\Delta$. By Theorem 2.1, $\hat{E}$ is isomorphic to Γ. Suppose τ is an ergodic automorphism of a compact abelian group G, and suppose there exists an $a \in G$ whose orbit separates $\hat{G}$. Define a mapping ρ^* of $\hat{G}$ into Γ as follows:

$$\rho^*(\gamma) = \psi^*(\gamma) + \Delta$$

where ψ^* is the mapping defined in (3.5). By Corollary 3.1, $\psi^*(\hat{G}) \cap \Delta$ is trivial so ρ^* is a continuous monomorphism. Let $\rho: E \to G$ be the induced transformation. It follows that ρ is a continuous epimorphism. Furthermore, since $\psi^*\tau^* = \sigma\psi^*$ (3.7), it follows that $\rho^*\tau^* = \sigma_\Delta\rho^*$ where σ_Δ is the automorphism of $\hat{X}/\Delta$ induced by σ. By duality, $\tau\rho = \rho(\sigma^*|E)$ so G is an algebraic factor of $\sigma^*|E$. For a more detailed discussion of the following theorem the reader is referred to [5].

<u>Theorem 3.2</u>: Let τ be an ergodic automorphism of a compact abelian group G, and suppose there exists an $a \in G$ whose orbit separates $\hat{G}$. Then τ is an algebraic factor of $\sigma^*|E$ where σ^* is the adjoint of the shift transformation on $\hat{X}$ and $E = \Delta^\perp$.

2. A Representation Theorem for a Class of Group Epimorphisms

Let G be a compact abelian group and $\tau \in \mathrm{Epi}(G)$. Let $a \in G$, and set

$$\mathrm{orb}^+(a) = \{\tau^n a : n \geq 0\}.$$

Define

$$\hat{Y} = \left(\bigtimes_0^\infty K \right)_d ;$$

that is, $\hat{Y}$ is the product of countably many copies of the circle group (indexed by the nonnegative integers), and $\hat{Y}$ is given the discrete topology. Writing $\hat{Y}$ additively (see (3.2)), it is a discrete abelian topological group (the identity element is the constant sequence $\hat{0}$ consisting of ones).

Define a shift transformation σ^* of $\hat{Y}$ into itself as in (3.3). Note that

$$\{x_k\} = (-1, 1, 1, \ldots)$$

is mapped by σ^* into

$$\{x_{k+1}\} = (1, 1, \ldots) = \hat{0}.$$

Hence $\ker \sigma^*$ is nontrivial so σ^* is not injective. On the other hand, it is clear that σ^* is onto. The proof for additivity is the same as in the previous seciton, and σ^* leaves constant sequences fixed (hence $\sigma^*(\hat{0}) = \hat{0}$). Therefore $\sigma^* \in \mathrm{Epi}(\hat{Y})$. Again we will use the terminology shift-invariant to mean: $A \subset \hat{Y}$ is shift-invariant if $\sigma^* A = A$.

<u>Theorem 3.3</u>: Let τ be an epimorphism of a compact abelian group G. Suppose there exists $a \in G$ such that $\mathrm{orb}^+(a)$ separates $\hat{G}$. Then τ is algebraically isomorphic to σ^* restricted to a subgroup $\hat{A}$ of $\hat{Y}$ where $\sigma^*\hat{A} \subset \hat{A}$. Conversely, any subgroup $\hat{B}$ of $\hat{Y}$ for which $\sigma^*\hat{B} \subset \hat{B}$, is the dual of some compact abelian group having the above property.

<u>Proof</u>: Since $\mathrm{orb}^+(a)$ separates $\hat{G}$, the mapping

$$\psi^*(\gamma) = \{\tau^{*k}\gamma(a)\} = \{\gamma(\tau^k a)\}$$

is injective. Setting $\hat{A} = \psi^*\hat{G}$ the proof that τ^* is algebraically isomorphic to $\sigma^*|A$ is the same as in Theorem 3.1. It remains to show $\sigma^*\hat{A} \subset \hat{A}$. To this end, note that since ψ^* is a bijection, τ^* is an injection and

$$\sigma^* = \psi^*\tau^*\psi^{*-1} \quad \text{on} \quad \hat{A} \qquad \text{(see (3.7))},$$

it follows that $\sigma^*|A$ is one to one.

The proof of the converse is also the same as in Theorem 3.1.

<u>Corollary 3.3</u>: Let G, τ and $\hat{A} = \psi^*\hat{G}$ be as in the above theorem. Define $\Delta = \Delta_\sigma \subset \hat{Y}$ and $\Delta_\tau \subset \hat{G}$ as in (2.12). Then

$$\psi^*\Delta_\tau = \hat{A} \cap \Delta$$

or

$$\Delta_\tau = \psi^{*-1}\{\hat{A} \cap \Delta\}.$$

In other words, $\gamma \in \Delta_\tau$ if and only if $\psi^*\gamma \in \Delta$ for $\gamma \in \hat{G}$.

Proof: Proceed exactly as in Corollary 3.1a.

Replacing orb(x) with $orb^+(x)$ in the proof of Proposition 3.1 we obtain:

Proposition 3.2: For each ergodic epimorphism τ of a compact abelian metrizable group G, there exists $a \in G$ such that $orb^+(a)$ separates $\hat{G}$.

Note that if $\tau \in Aut(G)$ has the property that orb^+a separates $\hat{G}$ for some $a \in G$, then τ^* can be embedded onto a shift-invariant subgroup of $\hat{Y}$ (i.e. $\hat{A} = \psi^*(Z^n)$ is shift-invariant).

CHAPTER IV

A REPRESENTATION THEOREM FOR AUTOMORPHISMS OF THE TORUS

1. Type 1 Automorphisms

For the number theoretic results needed in this chapter, see [9] and [12], and for the matrix theory results used herein, see [9] and [10]. Let $Z[\lambda]$ denote the ring of polynomials in the indeterminate λ with integer coefficients and $F[\lambda]$ denote the field of polynomials in the indeterminate λ with rational coefficients, F being the field of rational numbers. Let $P(\lambda) \in Z[\lambda]$ be irreducible over Z. If, in addition, $P(\lambda)$ is monic (has leading coefficient 1), then $P(\lambda)$ is also irreducible over F [9, p.120]; moreover, the roots of $P(\lambda)$ are distinct [12, p.36].

Let λ_0 be a root of $P(\lambda) \in Z[\lambda]$ which is monic and irreducible over Z. Let $Q(\lambda) \in F[\lambda]$ be the minimal polynomial of λ_0 (i.e. the monic polynomial of lowest degree with rational coefficients having λ_0 as a root). By the division algorithm, there exists $S(\lambda), R(\lambda) \in F[\lambda]$ such that $P(\lambda) = Q(\lambda)S(\lambda) + R(\lambda)$ where deg $R(\lambda)$ < deg $Q(\lambda)$ [12, p.25]. $P(\lambda_0) = Q(\lambda_0) = 0$ implies $R(\lambda_0) = 0$, and since deg $R(\lambda)$ < deg $Q(\lambda)$, it follows that $R(\lambda)$ is the zero polynomial (otherwise $1/a\ R(\lambda)$ where a is the leading coefficient of $R(\lambda)$ would be a monic polynomial with rational coefficients having λ_0 as a root and deg$(1/a\ R(\lambda))$ < deg $Q(\lambda)$) [12, pp.35-36]. Since $p(\lambda)$ is also irreducible over F, it follows that $P(\lambda) = Q(\lambda)$. Now let $Q(\lambda) \varepsilon Z[\lambda]$, and let $Q(\lambda_0) = 0$. Then $P(\lambda)$ is a factor of $Q(\lambda)$ [12, p.36]; thus, we have proved the following lemma.

Lemma 4.1a: Let $P(\lambda) \varepsilon Z[\lambda]$ be irreducible and monic, and let $Q(\lambda) \varepsilon Z[\lambda]$. If $P(\lambda)$ and $Q(\lambda)$ have a common root λ_0, then each root of $P(\lambda)$ is also a root of $Q(\lambda)$.

Let $\tau \varepsilon \text{Aut}(K^n)$, and let A be its associated $n \times n$ unimodular matrix. The characteristic polynomial of A will be denoted by $\text{Ch}(A)$ $(\text{Ch}(A) = \det(A - \lambda I))$. Then $\text{Ch}(A) \varepsilon Z[\lambda]$, and since $\text{Ch}(A)$ has leading coefficient $(-1)^n$, it factors into a product

$$\text{Ch}(A) = (-1)^n P_1(\lambda) P_2(\lambda) \ldots P_r(\lambda)$$

where $P_j(\lambda) \varepsilon Z[\lambda]$ is an irreducible monic polynomial $(1 \leq j \leq r)$; moreover, the $P_j(\lambda)$ are uniquely determined except for order [9, pp.117-121] and [12, p.26]. Since

$$n = \deg Ch(A) = \sum_{j=1}^{r} \deg P_j(\lambda),$$

we have

(4.1) $$Ch(A) = \prod_{j=1}^{r} (-1)^{d_j} P_j(\lambda)$$

where $d_j = \deg P_j(\lambda)$, $1 \leq j \leq r$. If these factors are distinct $(P_i(\lambda) \neq P_j(\lambda),\ i \neq j)$, we say τ (or A) is of <u>type 1</u>. It follows by Lemma (4.1a) that distinct irreducible monic polynomials in $Z[\lambda]$ can have no common roots, so τ of type 1 means A has distinct characteristic values.

Let A be an $n \times n$ unimodular matrix. Then A can be viewed as a linear transformation of C^n into itself (C = complex numbers); that is, for $z = (z_1, z_2, \ldots, z_n) \in C^n$,

$$zA = \left(\sum_{i=1}^{n} z_i a_{i1}, \ldots, \sum_{i=1}^{n} z_i a_{in} \right) \in C^n$$

(Again we use z to denote both an n-tuple and its row matrix.) Therefore, if A is of type 1, its characteristic vectors span C^n.

Now let $x = (x_1, x_2, \ldots, x_n) \in R^n$ and $m = (m_1, m_2, \ldots, m_n) \in Z^n$. Then

$$m \cdot x = \sum_{j=1}^{n} m_j x_j$$

denotes the ordinary scaler (dot) product of m and x. Suppose $y = (y_1, y_2, \ldots, y_n) \in R^n$ and $y = x$ mod$(1, 1, \ldots, 1)$. Then there exists $k = (k_1, k_2, \ldots, k_n) \in Z^n$ such that $x = y + k$. Hence,

$$\begin{aligned} \exp(2\pi i(m \cdot x)) &= \exp(2\pi i(m \cdot (y + k))) \\ &= \exp(2\pi i(m \cdot y))\exp(2\pi i(m \cdot k)) \\ &= \exp(2\pi i(m \cdot y)) \end{aligned}$$

since $m \cdot k \in Z$. Therefore, if v is the projection of $x = (x_1, x_2, \ldots, x_n) \in R^n$ onto the n-torus, then (2.9) implies

(4.2) $$\langle v, m \rangle = \exp 2\pi i(m \cdot x).$$

Lemma 4.1b: Let A be an $n \times n$ unimodular matrix of type 1. Suppose $P(\lambda) \in Z[\lambda]$ is an irreducible monic factor of Ch(A). Let λ_j, $1 \leq j \leq q$, be the roots of $P(\lambda)$ and v_j, $1 \leq j \leq q$, be corresponding nontrivial eigenvectors. If for some $m \in Z^n$ and for some j_0 between 1 and q we have $m \cdot v_{j_0} = 0$, then $m \cdot v_j = 0$ for $1 \leq j \leq q$.

Proof: Write $A = (a_{ij})_{n \times n}$. Let λ_j, $1 \leq j \leq n$, be the characteristic roots of A (for $1 \leq j \leq q$, the λ_j are the roots of $P(\lambda)$). Consider the following homogeneous system where λ is arbitrary:

$$(4.3) \qquad \begin{bmatrix} a_{11}-\lambda & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & \cdots & a_{n2} \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $(A^t - \lambda I)X^t = 0$ if and only if $X(A - \lambda I) = 0$ ($X = 1 \times n$ row matrix), to solve for an eigenvector associated with λ_j $(1 \leq j \leq n)$, one substitutes λ_j for λ in (4.3) and solves the resulting system.

Substitute λ_1 for λ in (4.3). Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, the resulting coefficient matrix will have rank $n-1$. Thus, there exists a cofactor $C_{r,s}(\lambda_1)$ $(C_{r,s}(\lambda_1) = (A^t - \lambda_1 I)_{r,s})$ with nonvanishing determinant [10, p.153]. Consider $C_{r,s}(\lambda) = (A^t - \lambda I)_{r,s}$. Then $\det C_{r,s}(\lambda) \in Z[\lambda]$. Since $\det C_{r,s}(\lambda_1) \neq 0$, lemma 4.1a implies $\det C_{r,s}(\lambda_j) \neq 0$ for $1 \leq j \leq q$. Hence, by eliminating the rth row from the above system, (4.3), and replacing λ by λ_j, the remaining $n-1$ rows will be linearly independent for $1 \leq j \leq q$. Solving this system for j between 1 and q will yield nontrivial eigenvector solutions corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$, respectively. Using Kramer's rule, the solutions will have the form:

(4.4) $$u_j' = \frac{1}{D(\lambda_j)}(Q_1'(\lambda_j),\ Q_2'(\lambda_j),\ \ldots,\ Q_n'(\lambda_j))$$

where $D(\lambda)$, $Q_i'(\lambda) \in Z[\lambda]$, $1 \leq j \leq q$ and $1 \leq i \leq n$. Using the fact that any scaler multiple of an eigenvector is an eigenvector, we can clear fractions in (4.4) to obtain

(4.5) $$u_j = (Q_1(\lambda_j),\ Q_2(\lambda_j),\ \ldots,\ Q_n(\lambda_j))$$

where $Q_i \in Z[\lambda]$, $1 \leq j \leq q$ and $1 \leq i \leq q$. We will call (4.5) a <u>polynomial form</u> of an eigenvector u_j associated with λ_j. Note that $v_1, v_2, \ldots, v_q$ are scaler multiples of $u_1, u_2, \ldots, u_q$ respectively.

Now suppose

$$(m_1, \ldots, m_n)\cdot(Q_1(\lambda_{j_0}), \ldots, Q_n(\lambda_{j_0})) = 0.$$

This means that λ_{j_0} is a root of the polynomial

(4.6) $$\sum_{i=1}^{n} m_i Q_i(\lambda) \in Z[\lambda].$$

Since λ_j is a root of $P(\lambda)$ which is irreducible and monic, Lemma 4.1a implies λ_j $(1 \leq j \leq q)$ must also be roots of (4.6). Using the fact that v_j is a scalar multiple of u_j $(1 \leq j \leq q)$, it follows that $m\cdot v_{j_0} = 0$ for some j_0 between 1 and q implies $m\cdot v_j = 0$ for all j between 1 and q.

Let Ch(A) be the characteristic equation of a type 1 unimodular matrix A. Then each irreducible monic factor $P(\lambda)$ of $Ch(A)$ gives rise to a polynomial form from which eigenvectors corresponding to the roots of $P(\lambda)$ can be formed by evaluating the coordinate polynomials at the respective roots. It may happen that the system (4.3) may be solved for $\lambda = \lambda_j$ $(1 \leq j \leq q)$ by eliminating some row other than the rth row; so the polynomial form, (4.5), of the corresponding eigenvectors may not be unique.

Corollary 4.1a: Let A be as in the lemma with $P(\lambda)$ an irreducible monic factor of $Ch(A)$ having λ_j, $1 \leq j \leq q$, as roots. Let $u_j = (Q_1(\lambda), \ldots, Q_n(\lambda))$ be a polynomial form of the eigenvectors corresponding to λ_j, $1 \leq j \leq q$. Suppose $m \in Z^n$. Then there exists j_0 between 1 and q such that $m \cdot u_{j_0} \in Z$ if and only if $m \cdot u_j \in Z$ for $1 \leq j \leq q$.

Proof: Suppose $m = (m_1, \ldots, m_n)$. Then

$$m \cdot u_j = \sum_{i=1}^{n} m_i Q_i(\lambda_{j_0}).$$

Hence, $m \cdot u_{j_0} = k \in Z$ if and only if λ_{j_0} is a root of

$$\sum_{i=1}^{n} m_i Q_i(\lambda) - k \in Z[\lambda].$$

The corollary follows from Lemma 4.1a.

For $(x_1, x_2, \ldots, x_n) \in C^n$, let $\mathrm{Re}(x_1, x_2, \ldots, x_n)$ denote $(\mathrm{Re}\, x_1, \mathrm{Re}\, x_2, \ldots, \mathrm{Re}\, x_n)$ where $\mathrm{Re}\, x$ is the real part of $x \in C$. Similarly, let $\overline{(x_1, x_2, \ldots, x_n)} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ where $\bar{x}$ is the complex conjugate of $x \in C$. Note that

$$\mathrm{Re}(x_1, x_2, \ldots, x_n) = \frac{1}{2}(x_1, x_2, \ldots, x_n) + \frac{1}{2}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in R^n.$$

If $m \in Z^n$ and $v \in C^n$, then $\mathrm{Re}(m \cdot v) = m \cdot \mathrm{Re}\, v$.

<u>Corollary 4.1b</u>: Let $\tau \in \mathrm{Aut}(K^n)$ be of type 1 with A its $n \times n$ unimodular matrix. Let $P(\lambda)$ be an irreducible factor of $\mathrm{Ch}(A)$ having λ_j $(1 \leq j \leq q)$ as roots. Let $u_j = (Q_1(\lambda_j), \ldots, Q_n(\lambda_j))$ be a polynomial form of the eigenvectors corresponding to λ_j, and let v_j be the projection of $\mathrm{Re}\, u_j$ onto the n-torus, $1 \leq j \leq q$. Suppose $m \in Z^n$, and suppose for some j_0 between 1 and q, λ_{j_0} is real. Then $\langle m, v_{j_0} \rangle = 1$ implies $\langle m, v_j \rangle = 1$ for $1 \leq j \leq q$.

<u>Proof</u>: Since λ_{j_0} is real, $\mathrm{Re}\, u_{j_0} = u_{j_0}$, so (4.2) implies

$$\langle m, v_{j_0} \rangle = \exp 2\pi i (m \cdot u_{j_0}).$$

Thus, $\langle m, v_{j_0} \rangle = 1$ if and only if $m \cdot u_{j_0} \in Z$, and the corollary follows from Corollary 4.1a.

Before proceeding with the representation theorem, we will mention a few facts pertaining to the field of algebraic numbers - i.e. complex numbers which are roots of polynomials with rational coefficients (see [9, pp. 173-175] and [12, pp. 35-46]). Now $F[\lambda]$ is countable, so the algebraic numbers are countable, and since they are a field, the complex numbers form an infinite dimensional vector space over the algebraic numbers; in fact, for any $n \geq 1$ we can find real numbers $\alpha_1, \ldots, \alpha_n$ such that $\{1, \alpha_1, \ldots, \alpha_n\}$ is linearly independent. Any complex number which is not a root of a rational polynomial is called transcendental. It follows that $\alpha_1, \ldots, \alpha_n$ are transcendental because 1 is algebraic. Note that since complex roots of rational polynomials occur in conjugate pairs, z algebraic implies $\text{Re } z$ is algebraic ($\text{Re } z = 1/2\, z + 1/2\, \bar{z}$). Similarly, $\text{Im } z$ is also algebraic.

<u>Theorem 4.1</u>: Let $\tau \in \text{Aut}(K^n)$ be of type 1 with A its associated $n \times n$ unimodular matrix. Let $\text{Ch}(A) = (-1)^n P_1(\lambda) P_2(\lambda) \ldots P_r(\lambda)$, where $P_j(\lambda) \in Z[\lambda]$ is irreducible and monic $(1 \leq j \leq r)$. For each j between 1 and r, let λ_j be a root of $P_j(\lambda)$ with

$$u_j = (Q_1^j(\lambda_j), \ldots, Q_n^j(\lambda_j))$$

a corresponding eigenvector in polynomial form (4.5).

Let $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers which are linearly independent over the algebraic numbers. Set

$$(4.7) \qquad a = \sum_{j=1}^{r} \alpha_j \operatorname{Re} u_j \quad \operatorname{mod}(1, 1, \ldots, 1).$$

Then $\operatorname{orb}(a)$ separates $\widehat{(K^n)} = Z^n$, and τ^* is algebraically isomorphic to σ^* restricted to some shift-invariant subgroup $\hat{A}$ of $\hat{X}$ via the mapping

$$(4.8) \qquad \psi^*(m) = \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \operatorname{Re}(\lambda_j^k (m \cdot u_j)) \right) \right\} ;$$

hence,

$$(4.9) \qquad \hat{A} = \psi^*(Z^n) = \left\{ \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \operatorname{Re}(\lambda_j^k (m \cdot u_j)) \right) \right\} : m \in Z^n \right\}.$$

<u>Proof</u>: Let $m = (m_1, \ldots, m_n) \in Z^n$. Then for $k \in Z$,

$$\langle \tau^{*k} m, a \rangle = \langle m, \tau^k a \rangle.$$

Equation (4.2) implies

$$\langle m, \tau^k a \rangle = \exp 2\pi i \left(m \cdot \left(\left(\sum_{j=1}^{r} \alpha_j \operatorname{Re} u_j \right) A^k \right) \right) .$$

Observe that since A has real entries,

$$(\operatorname{Re} u_j) A^k = \operatorname{Re}(\lambda_j^k u_j).$$

Therefore,

$$\langle m, \tau^k a\rangle = \exp 2\pi i\left(m\cdot\left(\sum_{j=1}^{r}\alpha_j \operatorname{Re}(\lambda_j^k u_j)\right)\right) ;$$

so, since m has real components,

$$(4.10) \qquad \langle m, \tau^k a\rangle = \exp 2\pi i\left(\sum_{j=1}^{r}\alpha_j \operatorname{Re}(\lambda_j^k (m\cdot u_j))\right) .$$

Note that (4.10) implies that for each $k \in Z$, $\langle m, \tau^k a\rangle = 1$ if and only if

$$(4.11) \qquad \sum_{j=1}^{r}\alpha_j \operatorname{Re}(\lambda_j^k (m\cdot u_j)) \in Z.$$

The λ_j $(1 \leq j \leq r)$ are algebraic numbers since they are roots of a polynomials with integer coefficients. From (4.5) and the fact that the algebraic numbers form a field, it follows that $m\cdot u_j$ is algebraic. Hence $\lambda_j^k(m\cdot u_j)$ is algebraic implying $\operatorname{Re}(\lambda_j^k(m\cdot u_j))$ is algebraic, $k \in Z$, $1 \leq j \leq r$. Since $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers which are linearly independent over the algebraic numbers, zero is the only possible integer value of (4.11), and for each $k \in Z$, this will only occur if $\operatorname{Re}(\lambda_j^k(m\cdot u_j)) = 0$ for each j between 1 and r.

Now, if $m\cdot u_j = 0$ for $1 \leq j \leq r$, it follows by Lemma 4.1b that m is orthogonal to a nontrivial eigenvector associated with each distinct characteristic value of A. But, the characteristic values of A are distinct

(A is of type 1). This means m is orthogonal to a spanning set of C^n implying $m = 0$. Hence, if $m \neq 0$, there exists j_0 between 1 and r such that $m \cdot u_{j_0} \neq 0$.

Suppose $\mathrm{Re}(m \cdot u_{j_0})$ is zero, meaning $m \cdot u_{j_0}$ is pure imaginary. But then $\mathrm{Re}(\lambda_{j_0}(m \cdot u_{j_0})) = 0$ if and only if λ_{j_0} is real. Since $\lambda_{j_0} \neq 0$, this would imply $m \cdot u_{j_0}$ is real (by (4.5)), a contradiction. Therefore, $\mathrm{Re}(m \cdot u_{j_0}) = 0$ implies $\mathrm{Re}(\lambda_{j_0}(m \cdot u_{j_0})) \neq 0$.

We have shown $m \neq 0$ implies there exists $k \in Z$ ($k = 0$ or $k = 1$) such that (4.11) $\notin Z$. From (4.10) it follows that $\mathrm{orb}(a)$ separates Z^n (In fact, we have shown $\mathrm{orb}^+(a)$ separates Z^n).

By Theorem 3.1, τ^* is algebraically isomorphic to σ^* restricted to some shift-invariant subgroup $\hat{A}$ of $\hat{X}$. From (3.5) we have for $m \in Z^n$, $\psi^*(m)$ is the bisequence $\{\langle m, \tau^k a\rangle\}$; it follows from (4.10) that

$$\psi^*(m) = \left\{\exp 2\pi i\left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^k(m \cdot u_j))\right)\right\},$$

so (4.8) and (4.9) follow.

Any polynomial $P(\lambda) \in Z[\lambda]$ having leading coefficient ± 1 and constant term ± 1 will be called unimodular. Suppose A is an $n \times n$ unimodular matrix associated with some $\tau \in \mathrm{Aut}(K^n)$. Let $P(\lambda) \in Z[\lambda]$ be a factor of

Ch(A). Since 0 is not a root of $P(\lambda)$ (A is non-singular), $P(\lambda)$ has constant term ± 1. It also follows that $P(\lambda)$ has leading coefficient ± 1, so $P(\lambda)$ is unimodular (hence Ch(A) is unimodular).

Let $P(\lambda)$ be unimodular - i.e.

$$P(\lambda) = a_m\lambda^m + a_{m-1}\lambda^{m-1} + \ldots + a_0,$$

where $a_i \in Z$ $(0 \leq i \leq m)$, $a_0 = \pm 1$, $a_m = \pm 1$. $P(\lambda)$ can be rewritten as

$$P(\lambda) = a_m(\lambda^m - c_{m-1}\lambda^{m-1} - \ldots - c_0),$$

where $c_i = -a_i/a_m$, $0 \leq i \leq m - 1$. Set

$$(4.12) \qquad P'(\lambda) = (-1)^m(\lambda^m - c_{m-1} - \ldots - c_0).$$

Then $P'(\lambda)$ is itself the characteristic polynomial of an automorphism of K^n defined by the companion matrix of $P'(\lambda)$; namely,

$$\begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \vdots & & & \vdots \\ c_0 & c_1 & c_2 \ldots & c_{m-1} \end{bmatrix}.$$

Note that this is not necessarily the only automorphism of K^n having characteristic polynomial $P'(\lambda)$ (such examples are found in Chapter V). Hence, we have shown:

Proposition 4.1: Any unimodular polynomial $P(\lambda)$ is equal to $\pm P'(\lambda)$ where $P'(\lambda)$ is the characteristic polynomial of an automorphism of K^m where m = degree $P(\lambda)$.

Corollary 4.1a: Let $\tau \in \mathrm{Aut}(K^n)$ with A its associated $n \times n$ unimodular matrix. Then

(4.13) $$Ch(A) = P_1'(\lambda)P_2'(\lambda) \ldots P_r'(\lambda)$$

where $P_j'(\lambda) \in Z[\lambda]$ is the characteristic polynomial of some automorphism of K^{m_j}, $m_j = \deg P_j'(\lambda)$, and $P_j'(\lambda)$ is irreducible over Z.

Proof: The corollary follows from (4.1), (4.12) and the above proposition (Note that $P_j'(\lambda) = (-1)^{m_j}P_j(\lambda)$, $1 \leq j \leq r$).

Now suppose $P(\lambda)$ is an irreducible (over Z) unimodular polynomial. We will say $P(\lambda)$ is ergodic if it has no roots which are roots of unity (see Proposition 2.4). Let A be an $n \times n$ unimodular matrix, and let $Ch(A) = P_1'(\lambda) \ldots P_s'(\lambda)P_{s+1}'(\lambda) \ldots P_r'(\lambda)$ be a factorization of $Ch(A)$ as in (4.13), where $P_j'(\lambda)$ is ergodic for $1 \leq j \leq s$, and $P_j(\lambda)$ is not ergodic for $s+1 \leq j \leq r$. Thus, $Ch(A)$ can be factored into two parts $P_1'(\lambda) \ldots P_s'(\lambda)$ and $P_{s+1}'(\lambda) \ldots P_r'(\lambda)$, which we will call the ergodic part and nonergodic part, respectively.

Observe that since $P'_j(\lambda) = (-1)^{m_j} P_j(\lambda)$ where $m_j = \deg P_j(\lambda)$ $(1 \le j \le r)$, it follows from (4.12) and Proposition 4.1 that both the ergodic and nonergodic parts of $Ch(A)$ may be regarded as characteristic polynomials of some automorphisms of K^{n_1} and K^{n_2} respectively where

$$n_1 = \sum_{j=1}^{s} m_j$$

and

$$n_2 = \sum_{j=s+1}^{r} m_j .$$

Before proceeding with the next proposition, we must first consider the relationship of $\psi^*(Z^n)$ (4.9) to $\Delta = \Delta_\sigma \subset \hat{X}$ (as defined in (2.12), (3.1) and (3.3)) for a given $\tau \in \mathrm{Aut}(K^n)$ of type 1. By Corollary 3.1a, $m \in \Delta_\tau \subset K^n$ if and only if $\psi^*(m) \in \Delta$. But, $\psi^*(m) \in \Delta$ if and only if there exists a polynomial in σ^* of the form

$$R(\sigma^*) = \prod_{i=1}^{N} (\sigma^{*p_i} - I), \quad p_i \in Z^+,$$

such that $R(\sigma^*)(\psi^* m) = \hat{0}$.

Let $p \ge 1$ and consider $(\sigma^{*p} - I)$. Then by (4.8),

$$(\sigma^{*p} - I)\psi^*(m) = \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k+p}(m \cdot u_j)) \right) \right\}$$

$$- \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k}(m \cdot u_j)) \right) \right\}$$

$$= \left\{ \left[\exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k+p}(m \cdot u_j)) \right) \right] \right.$$

$$\left. \left[\exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k}(m \cdot u_j)) \right) \right]^{-1} \right\}$$

$$= \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k+p}(m \cdot u_j)) - \sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k}(m \cdot u_j)) \right) \right\}$$

$$= \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}((\lambda_j^{k+p} - \lambda_j^{k})(m \cdot u_j)) \right) \right\}$$

$$= \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}(\lambda_j^{k}(\lambda_j^{p} - 1)(m \cdot u_j)) \right) \right\}.$$

It follows inductively that for $p_i \in Z^+$ $(1 \leq i \leq N)$,

(4.14) $$\prod_{i=1}^{N} (\sigma^{*p_i} - I)\psi^*(m)$$

$$= \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j \ \mathrm{Re}\left(\lambda_j^{k} \prod_{i=1}^{N} (\lambda_j^{p_i} - 1)(m \cdot u_j) \right) \right) \right\}.$$

Proposition 4.2: Suppose τ is a nonergodic automorphism of K^n with A its associated $n \times n$ unimodular matrix. Suppose Ch(A) is irreducible over Z. Then each root of Ch(A) is a root of unity.

Proof: Since Ch(A) is irreducible, τ is of type 1. Let λ be a root of Ch(A). Let α be a nonzero real trancendental number, and let u be a polynomial form of a nontrivial eigenvector associated with λ. Then according to Theorem 4.1, the mapping

$$\psi^*(m) = \{\langle m, \tau^k a\rangle\}$$

where $a = \alpha \mathrm{Re}\, u$ is an algebraic isomorphism from Z^n onto $\hat{A} \subset \hat{X}$, the roles of τ^* and σ^* being understood ($\hat{A}$ is given in (4.9)).

Since τ is not ergodic, there exists a nonzero $m \varepsilon \Delta_\tau \subset Z^n$ (Proposition 2.7). By Corollary 3.1a there exists a polynomial

$$\prod_{i=1}^{N} (\sigma^{*p_i} - I), \quad p_i \varepsilon Z^*,$$

such that

$$\prod_{i=1}^{N} (\sigma^{p_i} - I)\psi^*(m) = \hat{0}.$$

By (4.14) we have for each $k \varepsilon Z$,

$$\text{(4.15)} \quad \exp 2\pi i\left(\alpha \ \mathrm{Re}\left(\lambda^k \prod_{i=1}^{N} (\lambda^{p_i} - 1)(m\cdot u)\right)\right) = 1.$$

Since α is nonzero real and trancendental, (4.15) holds if and only if the quantity

$$\text{Re}\left(\lambda^k \prod_{i=1}^{N} (\lambda^{p_i} - 1)(m\cdot u)\right) \tag{4.16}$$

is zero for each $k \in Z$ ((4.16) is algebraic).

Since Ch(A) is irreducible, from Lemma 4.1b, it follows that $m\cdot u \neq 0$ (otherwise m would be orthogonal to a spanning set of C^n). If λ is real, (4.16) is simply

$$\lambda^k \prod_{i=1}^{N} (\lambda^{p_i} - 1)(m\cdot u)$$

which is zero only if $\lambda^{p_i} = 1$ for some i between 1 and N. On the other hand if λ is complex, then

$$\text{Re}\left(\lambda^k \prod_{i=1}^{N} (\lambda^{p_i} - 1)(m\cdot u)\right) = 0$$

for $k = 0,1$ only if $\prod_{i=1}^{N} (\lambda^{p_i} - 1) = 0$, which again implies $\lambda^{p_i} = 1$ for some i between 1 and N (see the proof of Theorem 4.1). Therefore, each root of Ch(A) is a root of unity.

Let τ be as in the above proposition. Choosing $N = 1$ and $p = \text{ord}\,\lambda$, it follows from (4.14) that

$$(\sigma^{*p} - I)\psi^*(m) = \hat{0}$$

for each $m \varepsilon Z^n$, so $\sigma^*|\hat{A}$ $(\hat{A} = \psi^*(Z^n))$ has periodic spectrum. Consequently, Corollary 3.1a and Theorem 2.4 imply $h(\tau) = 0$.

Suppose $\tau \varepsilon \mathrm{Aut}(K^n)$ and $P(\lambda)$ is an irreducible monic factor of $\mathrm{Ch}(A)$, where A is the $n \times n$ unimodular matrix associated with τ. Then Propositions 4.1 and 4.2 imply each root of $P(\lambda)$ is a root of unity.

Lemma 4.2: Let E and F be subgroups of a topological group G. Then $E^{\perp} = F^{\perp}$ implies $\bar{E} = \bar{F}$.

Proof: It follows by continuity that $E^{\perp} = (\bar{E})^{\perp}$ and $F^{\perp} = (\bar{F})^{\perp}$. Since $\bar{E}$ and $\bar{F}$ are closed subgroups of G,

$$((\bar{E})^{\perp})^{\perp} = \bar{E},$$

and

$$((\bar{F})^{\perp})^{\perp} = \bar{F}.$$

Therefore,

$$\begin{aligned}\bar{E} &= ((\bar{E})^{\perp})^{\perp} \\ &= (E^{\perp})^{\perp} \\ &= (F^{\perp})^{\perp} \\ &= ((\bar{F})^{\perp})^{\perp} \\ &= \bar{F}\end{aligned}$$

as we wished to show.

For any vector $(x_1, x_2, \ldots, x_n) \in C^n$, let $\mathrm{Im}(x_1, x_2, \ldots, x_n) = (\mathrm{Im}\, x_1, \mathrm{Im}\, x_2, \ldots, \mathrm{Im}\, x_n)$ where $\mathrm{Im}\, x$ denotes the imaginary part of $x \in C$. If S is a subset of R^n, let $sp\{S\}$ denote the linear span of S in R^n ($sp\{S\}$ is a subspace of R^n).

Theorem 4.2: Suppose $\tau \in \mathrm{Aut}(K^n)$ is of type 1 with A its associated $n \times n$ unimodular matrix. Let $\mathrm{Ch}(A) = (P_1(\lambda) \ldots P_s(\lambda))(P_{s+1}(\lambda) \ldots P_r(\lambda))$ be a factorization of $\mathrm{Ch}(A)$ as in (4.13) with $(P_1(\lambda) \ldots P_s(\lambda))$ the ergodic part. For each j between 1 and s let λ_j be a root of $P_j(\lambda)$ with v_j an associated nontrivial eigenvector. Then

(4.17) $\Delta_\tau = \{m \in Z^n : m \cdot v_j = 0 \text{ for } 1 \leq j \leq s\}$.

Furthermore, if T is the projection of $sp\{\mathrm{Re}\, v_1, \mathrm{Im}\, v_1, \ldots, \mathrm{Re}\, v_s, \mathrm{Im}\, v_s\}$ onto K^n, then T is a subgroup of K^n whose closure $\overline{T}$ is the maximal ergodic subgroup of K^n (i.e. $\overline{T} = E$).

Proof: Let λ_j be a root of $P_j(\lambda)$ for $s+1 \leq j \leq r$. For $1 \leq j \leq r$, let u_j be a nontrivial eigenvector in polynomial form associated with λ_j. Note that v_j is a constant (complex) multiple of u_j $(1 \leq j \leq s)$.

Let $1, \alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers which are linearly independent over the algebraic numbers. By Theorem 4.1 the mapping

$$\psi^*(m) = \left\{ \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j (\mathrm{Re}\ \lambda_j^k (m \cdot u_j)) \right) \right\}$$

for each $m \in Z^n$ is an algebraic isomorphism from Z^n onto $\hat{A}$, a shift-invariant subgroup of $\hat{X}$ (see (4.9)). By Corollary 3.1a,

$$\Delta_\tau = \psi^{*-1}(\hat{A} \cap \Delta).$$

But, $\psi^* m \in \Delta$ if and only if there exist positive integers $p_1, p_2, \ldots, p_N$ such that

$$\prod_{i=1}^{N} (\sigma^{*p_i} - I)\psi^*(m) = \hat{0},$$

and from (4.14) this will occur if and only if

$$(4.18)\quad \exp 2\pi i \left(\sum_{j=1}^{r} \alpha_j\ \mathrm{Re} \left(\lambda_j^k \prod_{i=1}^{N} (\lambda_j^{p_i} - 1)(m \cdot u_j) \right) \right)$$

is one for each $k \in Z$. Since

$$(4.19)\qquad \mathrm{Re}\ \lambda_j^k \prod_{i=1}^{N} (\lambda_j^{p_i} - 1)(m \cdot u_j)$$

is algebraic for each $k \in Z$ and $1 \leq j \leq r$, (4.18) is one if and only if (4.19) is zero, $k \in Z$, $1 \leq j \leq r$.

Rewrite (4.18) as follows:

$$\exp 2\pi i \left(\sum_{j=1}^{s} \alpha_j \operatorname{Re}\left(\lambda_j^k \prod_{i=1}^{N} (\lambda_j^{p_i} - 1)(m\cdot u_j)\right) + \sum_{s+1}^{r} \alpha_j \operatorname{Re}\left(\lambda_j^k \prod_{i=1}^{N} (\lambda_j^{p_i} - 1)(m\cdot u_j)\right)\right) .$$

Let j be between 1 and s, so λ_j is not a root of unity; consequently (4.19) cannot be zero for every $k \varepsilon Z$ unless $m\cdot u_j = 0$ $(1 \leq j \leq s)$. On the other hand, choosing $N = 1$ and p_1 = least common multiple of $\{t_{s+1}, \ldots, t_r\}$, t_i = order λ_i $(s+1 \leq i \leq r)$, forces the sum

$$\sum_{j=s+1}^{r} \alpha_j \operatorname{Re}(\lambda_j^k(\lambda_j^{p_i} - 1)(m\cdot u_j))$$

to be zero for each $k \varepsilon Z$. We have shown that

$$\sum_{j=1}^{s} \alpha_j \operatorname{Re}\left(\lambda^k \prod_{i=1}^{N} (\lambda_j^{p_i} - 1)(m\cdot u_j)\right)$$

is never zero for each $k \varepsilon Z$ unless $m\cdot u_j = 0$ $(1 \leq j \leq s)$. Therefore, $\psi^*(m) \varepsilon \Delta$ if and only if $m\cdot u_j = 0$ for $1 \leq j \leq s$, and since $v_j = \beta_j u_j$, β_j a nonzero complex number, (4.17) follows.

In view of Lemma 4.2, to prove the second assertion it will suffice to show $T^{\perp} = \Delta_\tau$ since T is clearly a subgroup of K^n. To this end, suppose $m \not\varepsilon \Delta_\tau$. Then there exists j_0 between 1 and s such that $m \cdot v_{j_0} \neq 0$. Thus, either $Re(m \cdot v_{j_0}) \neq 0$ or $Im(m \cdot v_{j_0}) \neq 0$. We may assume without loss of generality that $Re(m \cdot v_{j_0}) \neq 0$. Find $\alpha \in R$ such that the projection of $Re\alpha v_{j_0} = \alpha Re\ v_{j_0}$ onto the n-torus is nontrivial and $\alpha Re(m \cdot v_{j_0}) \notin Z$. Then $Re(m \cdot \alpha v_{j_0}) = Re(\alpha(m \cdot v_{j_0})) = \alpha Re(m \cdot v_{j_0}) \notin Z$, so by (4.2), $m \notin T^{\perp}$. Therefore, $T^{\perp} \subset \Delta_\tau$.

On the other hand it is clear that $T^{\perp} \supset \Delta_\tau$, so $T^{\perp} = \Delta_\tau$ as we wished to show.

<u>Corollary 4.2</u>: Let $\tau \in Aut(K^n)$ be of type 1. Then $h(\tau) = 0$ if and only if all characteristic roots of A are roots of unity, A being the unimodular matrix associated with τ.

<u>Proof</u>: If each characteristic root of A is a root of unity, the ergodic part of Ch(A) is trivial (= +1), so by Theorem 4.2, $\Delta_\tau = Z^n$ meaning $h(\tau) = 0$ by Theorem 2.4.

Suppose some characteristic root λ_0 of A is not a root of unity. Let $Ch(A) = P_1(\lambda) \ldots P_r(\lambda)$ be a factorization of Ch(A) as in (4.13). Then there exists j between 1 and r such that λ_0 is a root of $P_j(\lambda)$. Since $P_j(\lambda)$ is irreducible over Z, Propositions 4.1

and 4.2 imply no roots of $P_j(\lambda)$ are roots of unity. Consequently, by Theorem 4.2, the maximal ergodic subgroup E is nontrivial.

$\tau|E$ is ergodic (Theorem 2.5), so E nontrivial implies $(\tau|E)^*$ is aperiodic (Proposition 2.7). Theorem 2.4 implies $h(E) > 0$, so by Theorem 2.5, $h(\tau) > 0$.

Let $\tau \in \mathrm{Aut}(K^n)$ with E its maximal ergodic subgroup. Then E is a compact abelian group being a closed subgroup of K^n. Furthermore, since $\hat{E} = Z^n/\Delta_\tau$, E is metrizable. Now $\tau|E$ is ergodic, so by Proposition 3.1 and Theorem 3.1 its dual, Z^n/Δ_τ, is algebraically isomorphic to some shift-invariant subgroup of $\hat{X}$. The next proposition shows that $\hat{E}$ can be embedded into $\psi^*(Z^n) \subset \hat{X}$ via an algebraic isomorphism.

<u>Proposition 4.3:</u> Let τ, A and $Ch(A)$ be as in Theorem 4.2. Suppose $1, \alpha_1, \alpha_2, \ldots, \alpha_s$ are real numbers linearly independent over the algebraic numbers. Let u_j be the polynomial form of a nontrivial eigenvector associated with λ_j, a root of $P_j(\lambda)$, $1 \leq j \leq s$. Let p = least common multiple of $\{t_{s+1}, t_{s+2}, \ldots, t_r\}$ where t_j = order λ_j for $s+1 \leq j \leq r$. Then

$$a_E = \sum_{j=1}^{s} \alpha_j \operatorname{Re}(\lambda_j^p - 1)u_j \qquad \mathrm{mod}(1, 1, \ldots, 1)$$

is an element of E whose orbit separates $\hat{E} = Z^n/\Delta_\tau$. Extend the set $\{\alpha_1, \ldots, \alpha_s\}$ to $\{\alpha_1, \ldots, \alpha_s, \alpha_{s+1}, \ldots, \alpha_r\}$ so that α_j, $0 \leq j \leq r$ $(\alpha_0 = 1)$, are real and linearly independent over the algebraic numbers. Let u_j be a nontrivial eigenvector in polynomial form associated with λ_j, a root of $P_j(\lambda)$, $s+1 \leq j \leq r$. Let $a \in K^n$ be as in (4.7), and let ψ^* be the corresponding map (4.8) which embeds Z^n into $\hat{X}$ (4.9). Similarly, let ψ_E^* be the embedding of $\hat{E}$ into $\hat{X}$ corresponding to a_E (see (3.5)). Then $\psi_E^*(\hat{E})$ is a shift-invariant subgroup of $\psi^*(Z^n)$ whose intersection with Δ is trivial.

<u>Proof</u>: Let $m + \Delta_\tau \in \hat{E} = Z^n/\Delta_\tau$ be nonzero. From (4.17) it follows that there exists j_0 between 1 and s such that $m \cdot u_{j_0} \neq 0$. Now for $k \in Z$, since $\tau E = E$ we have

$$\langle m + \Delta_\tau, \tau_E^k a_E \rangle = \langle m, \tau^k a_E \rangle,$$

and by (4.2)

$$\begin{aligned}\langle m, \tau^k a_E \rangle &= \exp 2\pi i \left(m \cdot \tau^k a_E \right) \\ &= \exp 2\pi i \left(m \cdot \sum_{j=1}^{s} \alpha_j \ \mathrm{Re}(\lambda_j^k u_j) \right) \\ &= \exp 2\pi i \left(\sum_{j=1}^{s} \alpha_j \ \mathrm{Re}(\lambda_j^k (m \cdot u_j)) \right) .\end{aligned}$$

Again if $\operatorname{Re}(m \cdot u_{j_0}) \neq 0$, it follows by the fact that $\operatorname{Re}(m \cdot u_j)$ is algebraic $(1 \leq j \leq s)$ that this last expression is nonzero for $k = 0$. On the other hand, if $\operatorname{Re}(m \cdot u_{j_0}) = 0$ then $\lambda_{j_0} \neq 0$ implies $\operatorname{Re}(\lambda_{j_0}(m \cdot u_{j_0})) \neq 0$, so that

$$\sum_{j=1}^{s} \alpha_j \operatorname{Re}(\lambda_j^k (m \cdot u_j)) \neq 0$$

which by the choice of the $\{\alpha_j\}$ $(1 \leq j \leq s)$ cannot also be an integer. Therefore, $\langle m, \tau^k a_E \rangle \neq 0$ for some $k \in Z$ which proves the first part.

For the second assertion, note that for $m \in Z^n - \Delta_\tau$, (4.8) implies

$$\text{(4.20)} \qquad \psi_E^*(m + \Delta_\tau) = \psi_E^*(m)$$

$$= \left\{ \exp 2\pi i \left(\sum_{j=1}^{s} \alpha_j \operatorname{Re}(\lambda_j^k (\lambda_j^p - 1)(m \cdot u_j)) \right) \right\},$$

and

$$\psi^*(m) = \left\{ \exp 2\pi i \left(\sum_{j=1}^{s} \alpha_j \operatorname{Re}(\lambda_j^k (m \cdot u_j)) + \sum_{j=s+1}^{r} \alpha_j \operatorname{Re}(\lambda_j^k (m \cdot u_j)) \right) \right\}.$$

Since $\hat{A} = \psi^*(Z^n)$ is shift-invariant, it follows by (4.14) that

$$(\sigma^{*p} - I)\psi^*(m) = \left\{\exp 2\pi i\left(\sum_{j=1}^{s}\alpha_j \mathrm{Re}(\lambda_j^k(\lambda_j^p - 1)(m\cdot u_j))\right.\right.$$

$$\left.\left. + \sum_{j=s+1}^{r}\alpha_j\ \mathrm{Re}(\lambda_j^k(\lambda_j^p - 1)(m\cdot u_j))\right)\right\}$$

is an element of $\hat{A}$. But,

(4.21) $(\sigma^{*p} - I)(\psi^*(m)$

$$= \left\{\exp 2\pi i\left(\sum_{j=1}^{s}\alpha_j\ \mathrm{Re}(\lambda_j^k(\lambda_j^p - 1)(m\cdot u_j))\right)\right\},$$

since λ_j is a pth root of unity for $s+1 \leq j \leq r$. Comparing (4.20) and (4.21) we conclude that $\psi_E^*(\hat{E})$ is a shift-invariant subgroup of $\psi^*(Z^n)$.

Since $\tau|E$ is ergodic, Corollary 3.1b implies $\psi_E^*(\hat{E}) \cap \Delta$ is trivial.

It is interesting to note that in the above proposition, $a_E = (\tau^p - I)a$, where a is defined in (4.7), and that $(\tau^{*p} - I)\Delta_\tau = 0$.

2. A Representation Theorem for Automorphisms of the n-torus

In this section we will consider arbitrary automorphisms of the n-torus, which in general are not of type 1. This means that their associated $n \times n$ unimodular matrices may have repeated characteristic roots so that the eigenvectors do not necessarily span C^n, a fact that was essential in the development of a representation theorem for type 1 automorphisms. To obtain a generalized representation theorem, we will make use of the theory of Jordan canonical forms. For a detailed discussion of canonical forms see [9] or [10].

Let λ be a complex number. Then the matrix

$$\begin{bmatrix} \lambda & 1 & 0 \ \dots & 0 \\ 0 & \lambda & 1 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 \ \dots & 1 \\ 0 & 0 & 0 \ \dots & \lambda \end{bmatrix}$$

is called a basic Jordan block belonging to λ. A matrix J is said to be in Jordan form if

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_\ell \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} B_{i1} & & & \\ & B_{i2} & & \\ & & \ddots & \\ & & & B_{ir_i} \end{bmatrix}$$

and where $B_{i1}, \ldots, B_{ir_i}$ are basic Jordan blocks belonging to λ_i. Note that the eigenvalues of J appear on the main diagonal.

Let $\tau \varepsilon \mathrm{Aut}(K^n)$ with A its associated $n \times n$ unimodular matrix. Let $\lambda_1, \ldots, \lambda_\ell$ be the distinct characteristic roots of A. Then A is similar to a matrix J in Jordan form where $B_{i1}, \ldots, B_{ir_i}$ are the basic Jordan blocks belonging to λ_i, $1 \leq i \leq \ell$ (In fact, J is unique up to a permutation of the λ_i and of the basic Jordan blocks within the corresponding subblocks J_i) [9, p. 258]. Hence there exists a basis of C^n: $\{\xi_{ijk}\}$, $1 \leq i \leq \ell$, $1 \leq j \leq r_i$, $1 \leq k \leq s_{ij}$ (where r_i is the multiplicity of the eigenvalue λ_i, and s_{ij} is the size of the basic Jordan block B_{ij}) such that the linear transformation defined by A on C^n has matrix J. Furthermore, if $\{e_t\}$ $(1 \leq t \leq n)$ denotes the usual Euclidean basis for C^n, then $e_t P = \xi_{ijk}$ where

$$t = \sum_{p=1}^{i} \sum_{q=1}^{r_p} s_{pq} + k$$

[9, p. 235]. Finally, since the entries of both A and J are algebraic numbers, we may assume that the entries of P are algebraic [9, p. 269]. It follows that the coordinates of each vector ξ_{ijk} $(1 \leq i \leq \ell, 1 \leq j \leq r_i, 1 \leq k \leq s_{ij})$ are algebraic numbers.

Fix i between 1 and ℓ and j between 1 and r_i. Then

$$\xi_{ij1}J = \lambda_i \xi_{ij1} + \xi_{ij2}$$

$$\xi_{ij2}J = \lambda_i \xi_{ij2} + \xi_{ij3}$$

$$\vdots$$

$$\xi_{ij(s_{ij}-1)}J = \lambda_i \xi_{ij(s_{ij}-1)} + \xi_{ijs_{ij}}$$

$$\xi_{ijs_{ij}}J = \lambda_i \xi_{ijs_{ij}} ,$$

so $\xi_{ijs_{ij}}$ is an eigenvector associated with λ_i. For $1 \leq i \leq \ell$, $1 \leq j \leq r_i$, we will call $u_{ij} = \xi_{ij1}$ the <u>generator associated with the basic Jordan block B_{ij}</u>.

<u>Theorem 4.3</u>: Let $\tau \in \text{Aut}(K^n)$ with A its associated $n \times n$ unimodular matrix, and suppose $\lambda_1, \ldots, \lambda_\ell$ are the distinct characteristic values of A. Let J be a Jordan form of A with u_{ij} a generator associated with the basic Jordan block B_{ij} $(1 \leq i \leq \ell,\ 1 \leq j \leq r_i)$. Let

$$1, \alpha_{11}, \beta_{11}, \ldots, \alpha_{1r_1}, \beta_{1r_1}, \ldots, \alpha_{\ell r_\ell}, \beta_{\ell r_\ell}$$

be real numbers which are linearly independent over the algebraic numbers. Set

$$\text{(4.22)} \qquad a = \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} (\alpha_{ij} \text{Re}\, u_{ij} + \beta_{ij} \text{Im}\, u_{ij}) \mod(1,1,\ldots,1).$$

Then $a \in K^n$ and $\text{orb}^+(a)$ separates Z^n. Hence, by Theorem 3.1, τ^* is algebraically isomorphic to σ^* restricted to some shift-invariant subgroup of $\hat{X}$ (moreover, Theorem 3.3 implies σ^* is algebraically isomorphic to σ^* restricted to some shift-invariant subgroup of $\hat{Y}$ (3.11); note that σ^* has a different meaning here).

Proof: Let $m = (m_1, \ldots, m_n)$ be a nonzero element in Z^n. Then if $\{\xi_{ijk}\}$ $(1 \le i \le \ell,\ 1 \le j \le r_i,\ 1 \le k \le s_{ij})$ is a basis of C^n for which the linear transformation defined by A on C^n has matrix J (recall that the coordinates of ξ_{ijk} are algebraic and $u_{ij} = \xi_{ij1}$, $1 \le i \le \ell,\ 1 \le j \le r_i$), then

$$m \cdot \xi_{i_0 j_0 k_0} \ne 0 \tag{4.23}$$

for some i_o, j_0 and k_0 where $1 \le i_0 \le \ell$, $1 \le j_0 \le r_{i_0}$ and $1 \le k_0 \le s_{i_0 j_0}$. We may assume, without loss of generality, that k_0 is the first integer between 1 and $s_{i_0 j_0}$ for which (4.23) holds.

Now,

$$u_{i_0 j_0} J = \xi_{i_0 j_0 1} J = \lambda_{i_0} \xi_{i_0 j_0 1} + \xi_{i_0 j_0 2}$$

and

$$u_{i_0 j_0} J^2 = \lambda_{i_0}^2 \xi_{i_0 j_0 1} + 2\lambda_{i_0} \xi_{i_0 j_0 2} + \xi_{i_0 j_0 3} \ .$$

It follows inductively that for $1 \leq p < s_{i_0 j_0}$,

$$(4.24) \qquad u_{i_0 j_0} J^p = \lambda_{i_0}^p \xi_{i_0 j_0 1} + p\lambda_{i_0}^{p-1} \xi_{i_0 j_0 2}$$

$$+ \frac{p(p-1)}{2!} \lambda_{i_0}^{p-2} \xi_{i_0 j_0 3} + \frac{p(p-1)(p-2)}{3!} \lambda_{i_0}^{p-3} \xi_{i_0 j_0 4}$$

$$+ \cdots + \xi_{i_0 j_0 (p+1)} .$$

Therefore,

$$u_{i_0 j_0} J^{k_0 - 1} = \lambda_{i_0}^{k_0 - 1} \xi_{i_0 j_0 1} + (k_0 - 1)\lambda_{i_0}^{k_0 - 2} \xi_{i_0 j_0 2}$$

$$+ \cdots + \xi_{i_0 j_0 k_0},$$

so $$m\cdot\left(u_{i_0 j_0} J^{k_0 - 1}\right) = m\cdot\xi_{i_0 j_0 k_0} .$$

From (4.22) and (4.24) it follows that

$$m\cdot(a J^{k_0 - 1}) = \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \Bigg(\alpha_{ij} \operatorname{Re}(m\cdot(u_{ij} J^{k_0 - 1}))$$

$$+ \beta_{ij} \operatorname{Im}(m\cdot(u_{ij} J^{k_0 - 1}))\Bigg) + \alpha_{i_0 j_0} \operatorname{Re}(m\cdot\xi_{i_0 j_0 k_0})$$

$$+ \beta_{i_0 j_0} \operatorname{Im}(m\cdot\xi_{i_0 j_0 k_0}) .$$

From the choice of α_{ij}, β_{ij} $(1 \leq i \leq \ell,\ 1 \leq j \leq r_i)$ and the fact that $m\cdot(u_{ij} J^{k_0 - 1})$ is algebraic, we conclude

$$m\cdot(a\,J^{k_0-1}) \notin Z.$$

Hence, (4.2) implies

$$\langle m, \tau^{k_0-1}(a)\rangle \neq 1,$$

so $orb^+(a)$ separates Z^n as we wished to show.

Corollary 4.3: Let $\tau \in Aut(K^n)$ with A its associated $n \times n$ unimodular matrix, and let $Ch(A) = (P_1(\lambda) \ldots P_s(\lambda))(P_{s+1}(\lambda) \ldots P_r(\lambda))$ be a factorization of $Ch(A)$ as in (4.13) with $(P_1(\lambda) \ldots P_s(\lambda))$ the ergodic part. Suppose $\lambda_1, \ldots, \lambda_h$ are the distinct roots of the ergodic part, while $\lambda_{h+1}, \ldots, \lambda_\ell$ are the distinct roots of the nonergodic part. Let J be a Jordan form of A with $\{B_{ijk}\}$, $1 \leq j \leq r_i$, the basic Jordan blocks associated with λ_i $(1 \leq i \leq \ell)$. If $\{\xi_{ijk}\}$ $(1 \leq i \leq \ell,\ 1 \leq j \leq r_i,\ 1 \leq k \leq s_{ij})$ is a basis of C^n each member of which has algebraic coordinates and for which the linear transformation defined by A on C^n has matrix J, then

$$\Delta_\tau \supset \{m \in Z^n\colon m\cdot\xi_{ijk} = 0 \text{ for } 1 \leq i \leq h,\ 1 \leq j \leq r_i,\ 1 \leq k \leq s_{ij}\}.$$

Proof: By Theorem 4.3, $orb^+(a)$ separates Z^n where $a \in K^n$ is given by (4.22), so Theorem 3.3 implies the map

$$\psi^*(m) = \{\langle \tau^{*p}(m), a\rangle\}$$

$$= \{\langle m, \tau^p a\rangle\},$$

$p \geq 0$, is an algebraic isomorphism of Z^n onto a shift-invariant subgroup $\hat{A}$ of $\hat{Y}$. Now, (4.2) implies

$$\psi^*(m) = \{\exp 2\pi i(m \cdot aJ^p)\},$$

so (4.22) implies

$$\psi^*(m) = \left\{\exp 2\pi i\left(m \cdot \left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \alpha_{ij} \operatorname{Re}(u_{ij} J^p) + \beta_{ij} \operatorname{Re}(u_{ij} J^p)\right)\right)\right\},$$

or

$$\psi^*(m) = \left\{\exp 2\pi i\left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \alpha_{ij} \operatorname{Re}(m \cdot (u_{ij} J^p)) + \beta_{ij} \operatorname{Im}(m \cdot (u_{ij} J^p))\right)\right\} . \tag{4.25}$$

Choose $m \in Z^n$ so that $m \cdot \xi_{ijk} = 0$ for $1 \leq i \leq h$, $1 \leq r \leq r_i$, $1 \leq k \leq s_{ij}$. From (4.24) we have for $1 \leq p < s_{ij}$,

$$u_{ij} J^p = \lambda_i^p \xi_{ij1} + p \lambda_i^{p-1} \xi_{ij2} + \frac{p(p-1)}{2!} \lambda_i^{p-2} \xi_{ij3} + \ldots + \xi_{ij(p+1)} .$$

Since $\xi_{ijs_{ij}}$ is an eigenvector associated with λ_i, it follows that for an arbitrary $p \geq s_{ij}$

$$u_{ij} J^p = \lambda_i^p \xi_{ij1} + Q_1(p) \lambda_i^{p-1} \xi_{ij2} + \ldots + Q_{s_{ij}-1}(p) \lambda_i^{p-(s_{ij}-1)} \xi_{ijs_{ij}} ,$$

where $Q_t(x) \in F[x]$ $(1 \leq t \leq s_{ij} - 1)$. Hence, $m\cdot(u_{ij}\ J^p) = 0$ for $p \geq 0$ and $1 \leq i \leq h$. Then (4.25) becomes

$$(4.26) \qquad \psi^*(m) = \left\{ \exp 2\pi i \left(\sum_{h+1}^{\ell} \sum_{j=1}^{r_i} \alpha_{ij}\ \mathrm{Re}(m\cdot(u_{ij}\ J^p)) + \beta_{ij}\ \mathrm{Im}(m\cdot(u_{ij}\ J^p)) \right) \right\} .$$

It follows from Corollary 1b and Proposition 4.2 that λ_i, $h + 1 \leq i \leq \ell$, is a root of unity. Let $q = \ell.c.m.\{\mathrm{ord}\ \lambda_i : h + 1 \leq i \leq \ell\}$. Observe that

$$\begin{aligned} u_{ij}\ J^{p+q} - u_{ij}\ J^p = {} & \lambda_i^p(\ {}_i^q - 1)\xi_{ij1} \\ & + (Q_1(p + q)\lambda_i^{p+q} - Q_1(p)\lambda^p)\xi_{ij2} \\ & + \dots + \left(Q_{s_{ij}-1}(p)\lambda_i^{p-(s_{ij}-1)+q} - Q_{s_{ij}-1}(p)\lambda_i^{p-(s_{ij}-1)}\right)\xi_{ijs_{ij}} \end{aligned}$$

where $Q_w(p) = 0$ if $p < w$. Since $(\lambda_i^q - 1) = 0$,

$$\begin{aligned} (u_{ij}\ J^{p+q} - u_{ij}\ J^p)(J^q - I) & = \lambda^p(\lambda_i^q - 1)(Q_1(p + q)\lambda_i^{p+q} - Q_1(p)\lambda^p)\xi_{ij2} \\ & + Q_2'(\lambda)\xi_{ij3} + \dots + Q_{s_{ij}-1}'(\lambda)\xi_{ijs_{ij}} \end{aligned}$$

where $Q_t'(x) \in F[x]$. It follows inductively that for

$h + 1 \leq i \leq \ell$ and $1 \leq j \leq r_i$,

$$(4.27) \qquad u_{ij}\left(J^p \prod_1^M (J^p - I)\right) = 0$$

where $M = \max\{s_{ij}: h + 1 \leq i \leq \ell,\ 1 \leq j \leq r_i\}$.

Therefore by (4.26) and (4.27),

$$\prod_1^M (\sigma^{*q} - I)\psi^*(m)$$

$$= \left\{\exp 2\pi i \left(\sum_{h+1}^{\ell} \sum_{j=1}^{r_i} \alpha_{ij} \operatorname{Re}\left(m \cdot u_{ij}\left(J^p \prod_1^M (J^q - I)\right)\right)\right.\right.$$

$$\left.\left. + \beta_{ij}\left(\operatorname{Im} m \cdot u_{ij}\left(J^p \prod_1^M (J^p - I)\right)\right)\right)\right\}$$

$$= \hat{0}$$

(the identity of $\hat{Y}$). Corollary 3.3 implies $m \in \Delta_\tau$ which completes the proof.

In the appendix it is shown Δ_τ is in fact equal to $\{m \in Z^n: m \cdot \xi_{ijk} = 0$ for $1 \leq i \leq h,\ 1 \leq j \leq r_i$, $1 \leq k \leq s_{ij}\}$, and a characterization of the maximal ergodic subgroup is then obtained.

CHAPTER V

APPLICATIONS

1. An Automorphism of the 3-torus Not Having the E-Z Decomposition Property

Before proceeding with the example, let $\tau \in \mathrm{Aut}(K^n)$ be of type 1 with E its maximal ergodic subgroup. By Proposition 4.3, $\hat{E}$ can be embedded via an algebraic isomorphism onto a subgroup of $\hat{A} = \psi^*(Z^n)$ (namely $(\sigma^{*p} - I)\hat{A}$ where $p = \ell.c.m.\{\text{ord } \lambda_i : s + 1 \leq i \leq r\}$, λ_i a characteristic root of A) whose intersection with Δ is trivial. In view of Theorem 2.6 the question arises: Does there exist a shift-invariant subgroup $H \subset \hat{X}$ such that $\psi^*(Z^n) = H \oplus [\psi^*(Z^n) \cap \Delta]$? If so, then τ would be algebraically isomorphic to

$$\sigma | \widehat{(\psi^* Z^n)}$$

which would, by Theorem 2.6, have the E-Z decomposition property. As we shall see in the following example, the answer is negative.

Example: Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -5 & 5 \end{bmatrix}$$

be an automorphism of the three torus. The characteristic equation of A is $Ch(A) = -(\lambda - 1)(\lambda^2 - 4\lambda + 1)$ whose roots are 1, $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Note that $-(\lambda - 1)$ is the nonergodic part, whereas $\lambda^2 - 4\lambda + 1$ is the ergodic part, and both are irreducible over $Z[\lambda]$. Solving for eigenvectors in polynomial form we obtain $(1, 1, 1)$, $(1, 2 + \sqrt{3}, 7 + 4\sqrt{3})$, $(1, 2 - \sqrt{3}, 7 - 4\sqrt{3})$ associated with 1, $2 + \sqrt{3}$, $2 - \sqrt{3}$ respectively.

Since $\lambda^2 - 4\lambda + 1$ is irreducible over Z having real roots only, it follows by Theorem 4.2 that the maximal ergodic subgroup E is the closure of the projection of the subspace $\{\alpha(1, 2 + \sqrt{3}, 7 + 4\sqrt{3}): \alpha \in R\}$ onto the 3-torus. Its annihilator, Δ_τ, is the subgroup of Z^3 generated by $(1, -4, 1)$. On the other hand, the annihilator of the closed subgroup H formed by the projection of $\{\alpha(1, 1, 1): \alpha \in R\}$ onto K^3 is the subgroup of Z^3 generated by $\{(1, -1, 0), (0, 1, -1)\}$.

In other words, defining $gp(S)$ to be the group generated by S where S is a subset of Z^n, we have

$$E = [gp\{(1, -4, 1)\}]^\perp,$$

and

$$H = [gp\{(1, -1, 0), (0, 1, -1)\}]^\perp;$$

but, $E \cap H \neq \{0\}$ because $\langle (1, -4, 1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\rangle = \exp 2\pi i(\frac{1}{2} - 2 + \frac{1}{2}) = 1$, so $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in E$. Note that this is the only nonzero point common to both.

Suppose $\overline{K}$ is a subgroup of K^3 such that $\tau\overline{K} = \overline{K}$, $E \cap \overline{K} = \{0\}$ and $E + \overline{K} = K^3$. Clearly $(\tau - I)E \subset E$ and $(\tau - I)\overline{K} \subset \overline{K}$. Set $v_1 = (1, 2 + \sqrt{3}, 7 + 4\sqrt{3})$, $v_2 = (1, 2 - \sqrt{3}, 7 - 4\sqrt{3})$ and $h = (1, 1, 1)$. Since v_1, v_2, h are a basis for R^3, each element $x \varepsilon K^3$ can be written $\alpha v_1 + \beta v_2 + \gamma h$, $\alpha, \beta, r \varepsilon R$ (All relations hold mod $(1, 1, 1)$ so that this representation is not necessarily unique).

Now $(\tau - I)x = \alpha(\lambda_1 - 1)v_1 + \beta(\lambda_2 - 1)v_2 + 0$. Thus, $(\tau - I)x \varepsilon E$, or $(\tau - I)K^3 \subset E$. This implies $(\tau - I)\overline{K} = \{0\}$. Again let $x = \alpha v_1 + \beta v_2 + \gamma h \varepsilon K^3$. Then

$$\begin{aligned}(\tau - I)x &= \alpha(\lambda_1 - 1)v_1 + \beta(\lambda_2 - 1)v_2 + 0 \\ &= \alpha(1 + \sqrt{3}, 5 + 3\sqrt{3}, 19 + 11\sqrt{3}) \\ &\quad + \beta(1 - \sqrt{3}, 5 - 3\sqrt{3}, 19 - 11\sqrt{3}).\end{aligned}$$

If $(\tau - I)x = 0$, it follows that

$$(\alpha + \beta) + \sqrt{3}(\alpha - \beta) \varepsilon Z, \tag{5.1}$$

and

$$5(\alpha + \beta) + 3\sqrt{3}(\alpha - \beta) \varepsilon Z. \tag{5.2}$$

Multiplying (5.1) by -3 and adding the result to (5.2) yields

$$2(\alpha + \beta) \varepsilon Z. \tag{5.3}$$

Similarly,

(5.4) $$2\sqrt{3}(\alpha - \beta) \in Z.$$

Now consider $\frac{1}{4}(1, 1, 1) = e + k$, $e = \alpha v_1 + \beta v_2 + rh \in E$ and $k = \alpha' v_1 + \beta' v_2 + \gamma' h \in \overline{K}$. First note that $\gamma = 0$ or $\frac{1}{2}$ because $e - \alpha v_1 - \beta v_2 = \gamma h \in E$. Secondly $(\tau - I)e = (\tau - I)k = 0$, so α, α' and β, β' each satisfy (5.3) and (5.4).

$$\begin{aligned}2\alpha v_1 + 2\beta v_2 &= 2\alpha(1, 2 + \sqrt{3}, 7 + 4\sqrt{3}) + 2\beta(1, 2 - \sqrt{3}, 7 - 4\sqrt{3}) \\ &= (2(\alpha + \beta), 4(\alpha + \beta) + 2\sqrt{3}(\alpha - \beta), \\ &\quad 14(\alpha + \beta) + 4\sqrt{3}(\alpha - \beta)) \\ &= 0 \bmod (1, 1, 1).\end{aligned}$$

Similarly $2\alpha' v_1 + 2\beta' v_2 = 0 \bmod (1, 1, 1)$. Thus,

$$\begin{aligned}2\tfrac{1}{4}(1, 1, 1) &= \tfrac{1}{2}(1, 1, 1) \\ &= 2(e + k) \\ &= 2\gamma h + 2\gamma' h \\ &= 0 + 2\gamma' h \\ &= 2k \\ &\in \overline{K} \cap E,\end{aligned}$$

a contradiction. Therefore A does not have the E-Z decomposition property.

2. A Characterization of Entropy Classes of Ergodic Automorphisms of the Two and Three Torus

In [2] Adler and Weiss give the following examples of two algebraically nonisomorphic ergodic automorphisms of the 2-torus with the same entropy:

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}.$$

However, there exist epimorphisms of K^2

$$\Theta_1 = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \Theta_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

such that $\theta_1 A = B\theta_1$ and $A\theta_2 = \theta_2 B$. This means that A and B are weakly algebraically isomorphic. Since any pair of weakly algebraically isomorphic automorphisms have the same entropy, the question arises: To what extent is entropy a complete invariant under weak algebraic isomorphisms of the n-torus?

In this section we will attempt to answer this question in stages. We will consider the question first for ergodic automorphisms of the 2-torus, then the three torus and then for higher dimensions. To do so, the following lemmas are needed. We will continue to use the notion of an automorphism of the n-torus and its associated $n \times n$ unimodular matrix interchangeably.

Lemma 5.1a: If A is an ergodic automorphism of K^n, then A has no rational characteristic values.

Proof: The characteristic polynomial of A is of the form

$$Ch(A) = (-1)^n \lambda^n + b_{n-1}\lambda^{n-1} + \dots + b_1\lambda + \det A$$

where $b_i \in Z$, $1 \leq i \leq n - 1$. If $Ch(A)$ has a rational root say p/q in lowest terms ($p, q \in Z$), then since $\det A = \pm 1$, both p and q must divide 1. Hence $|p/q| = 1$, contradicting the ergodicity of A.

Lemma 5.1b: Let A be an ergodic automorphism of the 2-torus. Then A has no complex characteristic values.

Proof: Write $A = (a_{ij})_{2\times 2}$. The characteristic polynomial of A is

$$Ch(A) = \lambda^2 - \operatorname{tr} A\lambda + \det A \tag{5.5}$$

whose roots are

$$\frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2} \tag{5.6}$$

($\operatorname{tr} A$ = trace of A). Observe that (5.6) is complex if and only if $\operatorname{tr} A < 2$ and $\det A = \pm 1$.

If $\operatorname{tr} A = 0$, (5.5) would then be $\lambda^2 + 1 = 0$, or $\lambda^2 = \pm i$. On the other hand if $\operatorname{tr} A = \pm 1$, then (5.6) would yield $(-1 + 3i)/2$ or $(1 - 3i)/2$ which are roots of unity. The lemma follows from Proposition 2.4.

Theorem 5.1: Let τ be an ergodic automorphism of the 2-torus. Then the weak algebraic isomorphism class of automorphism of K^2 generated by $\{\tau, -\tau\}$ consists of all ergodic automorphisms of the 2-torus with entropy $h(\tau)$.

<u>Proof</u>: Let $A = (a_{ij})_{2\times 2}$ be the unimodular matrix associated with τ. Let $B = (b_{ij})_{2\times 2}$ be an ergodic automorphism of K^2 having the same entropy as τ. By Lemma 5.1b, Ch(A) and Ch(B) have only real characteristic values. Since the only possible rational roots of Ch(A) and Ch(B) are ± 1 (Lemma 5.1a), both Ch(A) and Ch(B) are irreducible over Z meaning A and B are of type 1.

Let a_1, a_2 be the roots of Ch(A) and b_1, b_2 be the roots of Ch(B). Then $a_1, a_2, b_1, b_2 \in R$ and $|a_1 a_2| = |b_1 b_2| = 1$. We may assume without loss of generality that $|a_1| > 1$ and $|b_1| > 1$ (Hence $|a_2| < 1$ and $|b_2| < 1$). From Proposition 2.5,

$$\log|a_1| = h(A) = h(B) = \log|b_1|$$

This means that $|a_1| = |b_1|$ and, hence, $|a_2| = |b_2|$. We have two possibilities:

Case 1: $a_1 = b_1$ or $a_2 = b_2$,

Case 2: $a_1 = -b_1$ and $a_2 = -b_2$.

If, as in case 1, $a_1 = b_1$ we proceed as follows. Let α be a nonnegative real transcendental number. Set $a_1 = b_1 = \lambda$. Then a polynomial form of an eigenvector associated with λ will be

(5.7) $$u_A = (-a_{12}, a_{11} - \lambda),$$

and

(5.8) $$u_B = (-b_{12}, b_{11} - \lambda)$$

corresponding to A and B respectively. Notice that $b_{12} \neq 0$ because $b_{12} = 0$ would mean $\det(B - \lambda I) = (b_{11} - \lambda)(b_{22} - \lambda)$ which contradicts Lemma 5.1a since both b_{11} and b_{22} would be integer characteristic values of B.

For $M = (m, n) \in Z^2$, define mappings

(5.9) $$\psi_A^*(M) = \{\exp 2\pi i(\alpha\lambda^k(M\cdot u_A))\},$$

and

(5.10) $$\psi_B^*(M) = \left\{\exp 2\pi i\left(\frac{\alpha}{b_{12}}\lambda^k(M\cdot u_B)\right)\right\}$$

from Z^2 into $\hat{X}$. By Theorem 4.1 and (4.8), ψ_A^* and ψ_B^* are algebraic isomorphisms (with respect to A and B) from Z^2 onto shift-invariant subgroups of $\hat{X}$. Evaluate $M\cdot u_A$ and $M\cdot u_B$ in (5.9) and (5.10) respectively to obtain

(5.11) $$\psi_A^*(M) = \{\exp 2\pi i(\alpha\lambda^k(-ma_{12} + na_{11} - n\lambda))\},$$

(5.12) $$\psi_B^*(M) = \left\{\exp 2\pi i\left(\alpha\lambda^k\left(-m + n\frac{b_{11} - \lambda}{b_{12}}\right)\right)\right\}$$

Consider the map $\Theta_1 : \psi_A^*(Z^2) \to \psi_B^*(Z^2)$ defined by:

(5.13) $$\Theta_1\{\exp 2\pi i(\alpha\lambda^k(m, n)\cdot u_A))\} =$$

$$\left\{\exp 2\pi i\left(\frac{\alpha}{b_{12}}\lambda^k((a_{12}m - (a_{11} - b_{11})n, b_{12}n)\cdot u_B)\right)\right\}.$$

From (5.12) the right-hand side of (5.13) is the bisequence

$$\left\{\exp 2\pi i\left(\alpha\lambda^k\left(-(a_{12}m - (a_{11} - b_{11})n) + b_{12}n \frac{b_{11} - \lambda}{b_{12}}\right)\right)\right\}$$

which when simplified becomes

$$(5.14) \qquad \left\{\exp 2\pi i(\alpha\lambda^k(-ma_{12} + na_{11} - n\lambda))\right\} .$$

From (5.13) it is clear that Θ_1 is additive. Since α is transcendental and since λ cannot be a root of $(-ma_{12} + na_{11}) - n\lambda$ ($Ch(A)$ is irreducible over $Z[\lambda]$, and $\deg Ch(A) = 2$), (5.14) implies Θ_1 is injective.

Compare (5.11) with (5.14). We see that $\psi_A^*(Z^n)$ is a subgroup of $\psi_B^*(Z^n)$ and Θ_1 is just the identity. It follows that $\Theta_1^*\sigma_A^* = \sigma_B^*\Theta_1$, where $\sigma_A^* = \sigma^*|\psi_A^*(Z^n)$ and $\sigma_B^* = \sigma^*|\psi_B^*(Z^n)$. Since Θ_1 is one to one, we can apply Corollary 2.2b to conclude A is an algebraic factor of B.

In (5.9) replace α by α/a_{12}, and in (5.10) replace α/b_{12} by α. Then the above argument can be reversed to show that B is an algebraic factor of A. Therefore, A and B are weakly algebraically isomorphic.

If case 2 holds, then -A has characteristic roots $-a_1$ and $-a_2$, and we can proceed as in case 1.

<u>Theorem 5.2:</u> Let τ be an ergodic automorphism of teh 3-torus. Then the weak algebraic isomorphism class in

K^3 generated by $\{\tau, -\tau, \tau^{-1}, -\tau^{-1}\}$ consists of all ergodic automorphisms of the 3-torus with entropy $h(\tau)$.

<u>Proof</u>: Let $A = (a_{ij})_{3\times 3}$ be the unimodular matrix associated with τ. Let $B = (b_{ij})_{3\times 3}$ be the matrix of an ergodic automorphism of K^3 having entropy $h(\tau)$. Since $\deg Ch(A) = 3$, any factorization of $Ch(A)$ over Z must include a linear factor which, by Lemma 5.1a, is impossible. Hence, $Ch(A)$ and $Ch(B)$ are irreducible over Z, so A and B are of type 1.

$Ch(A), Ch(B) \in Z[\lambda]$, so complex roots must occur in conjugate pairs. It follows that both $Ch(A)$ and $Ch(B)$ have at least one real root. Let a_1, a_2, a_3 be the roots of $Ch(A)$ and b_1, b_2, b_3 be the roots of $Ch(B)$. From the relation $|a_1a_2a_3| = 1$ and by rearranging a_1, a_2, a_3 if necessary we have two possibilities; namely,

$$|a_1| < 1 \quad \text{while} \quad |a_2| \geq 1 \quad \text{and} \quad |a_3| \geq 1,$$

or

$$|a_1| > 1 \quad \text{while} \quad |a_2| \leq 1 \quad \text{and} \quad |a_3| \leq 1.$$

A similar situation exists for b_1. Set $a_1 = \lambda_A$ and $b_1 = \lambda_B$. Then λ_A and λ_B are real. From the relation (Proposition 2.5)

$$h(\tau) = \sum_{\substack{|a_i|>1 \\ 1\leq i\leq 3}} \log|a_i| = \sum_{\substack{|b_i| \\ 1\leq i\leq 3}} \log|b_i|$$

we obtain four cases:

Case 1: $\lambda_A = \lambda_B$,

Case 2: $\lambda_A = -\lambda_B$,

Case 3: $\lambda_A = 1/\lambda_B$,

Case 4: $\lambda_A = -1/\lambda_B$.

Suppose case 1 holds; i.e. $\lambda_A = \lambda_B$. Consider the system (4.3) with $\lambda = \lambda_A$ and $n = 3$. The coefficient matrix has rank 2 (λ_A is a simple root), and λ_A cannot be a root of any polynomial of degree less than 3. This means that the last row may be eliminated in order to solve for a nontrivial eigenvector associated with λ_A. The polynomial form of this eigenvector is

$$u_A = (a_{13}\lambda+\det A_{31},\ a_{23}\lambda-\det A_{32},\ \lambda^2-(a_n+a_{22})\lambda+\det A_{33})$$

where A_{ij} is the cofactor associated with a_{ij} $(1 \leq i, j \leq 3)$. Likewise,

$$u_B = (b_{13}\lambda+\det B_{31},\ b_{23}\lambda-\det B_{32},\ \lambda^2-(b_{11}+b_{22})\lambda+\det B_{33})$$

is the polynomial form of an eigenvector associated with $\lambda_B = \lambda$.

Notice that b_{13} and b_{23} cannot both be zero because $b_{13} = b_{23} = 0$ would mean $\det(B - \lambda I) = (b_{33} - \lambda)\det B_{33}$ implying $\lambda = b_{33} \in Z$ is a root of $Ch(B)$, and this would contradict Lemma 5.1a. Let α be a nonzero real transcendental number. By Theorem 4.1, orb(b) where

$$b = \alpha \text{Re } u_B \bmod(1,\ 1,\ 1)$$

separates Z^3. From (4.8) it follows that for $(m,\ n,\ \ell) \neq (0,\ 0,\ 0) \in Z^3$,

$$\text{Re } \lambda^k((m,\ n,\ \ell) \cdot u_B) \neq 0$$

for some $k \in Z$ implying $(m,\ n,\ \ell) \cdot u_B \neq 0$. In particular,

(5.15) $$(b_{23},\ -b_{13},\ 0) \cdot u_B = b_{23} \det B_{31} + b_{13} \det B_{32} \neq 0.$$

Note that for $(m,\ n,\ \ell) \in Z_3$,

$$(m,\ n,\ \ell) \cdot u_A = m(a_{13}\lambda + \det A_{31}) + n(a_{23}\lambda + \det A_{32})$$
$$+ \ell(\lambda^2 - (a_{11} + a_{22}) + \det A_{33}).$$

Consider the following systems of equations in the unknowns $x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2$.

$$b_{13}x_1 + b_{23}x_2 = a_{13}$$

$$\det B_{31}x_1 - \det B_{32}x_2 = \det A_{31}\ ,$$

$$b_{13}y_1 + b_{23}y_2 = a_{23}$$

$$\det B_{31}y_1 - \det B_{32}y_2 = \det A_{32}$$

and

$$b_{13}z_1 + b_{23}z_2 = -(a_{11} + a_{22}) + b_{11} + b_{22})$$

$$\det B_{31}z_1 - \det B_{32}z_2 = \det A_{33} - \det B_{33}.$$

Since their coefficient matrices are the same, the above three systems will have unique rational solutions because

$$\det\begin{bmatrix} b_{13} & b_{23} \\ \det B_{31} & -\det B_{32} \end{bmatrix}$$

is nonzero by (5.15).

Choose $p \in Z - \{0\}$ so that px_1, px_2, py_1, py_2, pz_1 and pz_2 are integers. Let

$$a = p\alpha \operatorname{Re} u_A \mod(1, 1, 1).$$

By Theorem 4.1, the orbit of a under τ separates Z^3. Let ψ_A^* and ψ_B^* be the embeddings (4.9) of Z^3 into $\hat{X}$ corresponding to A, a and B, b respectively. Then $\sigma^*|\psi_A^*(Z^3)$ and $\sigma^*|\psi_B^*(Z^3)$ are algebraically isomorphic to A^* and B^* respectively (Theorem 3.1 and Theorem 4.1).

Define a map $\Theta:\psi_A^*(Z^3) \to \psi_B^*(Z^3)$ by

(5.16) $$\Theta^*\psi_A^*(M) = \{\exp 2\pi i(\alpha\operatorname{Re}(\theta^*(M)\cdot u_B))\}$$

where $\theta^*(M) = \theta^*(m,n,\ell) = p(x_1m+y_1n+z_1\ell, x_2m+y_2n+z_2\ell, \ell)$.

Considering the above three systems we have

$$\begin{aligned} 1/p\theta^*(M)\cdot u_B &= (b_{13}x_1 + b_{23}x_2)m + (\det B_{31}x_1 - \det B_{32}x_2)m \\ &\quad + (b_{13}y_1 + b_{23}y_2)n + (\det B_{31}y_1 - \det B_{32}y_2)n \\ &\quad + (b_{13}z_1 + b_{23}z_2)\ell + (\det B_{31}z_1 - \det B_{32}z_2)\ell \\ &\quad + (\lambda^2 - (b_{11} + b_{22})\lambda + \det B_{33})\ell \\ &= (a_{13} + \det A_{31})m + (a_{23} - \det A_{32})n \end{aligned}$$

$$+ (\lambda^2 - (a_{11} + a_{22})\lambda + \det A_{33})\ell$$

$$= M\cdot u_A \ .$$

Therefore,

$$\theta^*(M)\cdot u_B = pM\cdot u_A. \tag{5.17}$$

Since by (4.9)

$$\psi_A^*(Z^3) = \{\{\exp 2\pi i(p\alpha \mathrm{Re}\lambda^k(M\cdot u_A)): M \varepsilon Z^3\}\} \tag{5.18}$$

and

$$\psi_B^*(Z^3) = \{\{\exp 2\pi i(\alpha \mathrm{Re}\lambda^k(M\cdot u_B)): M \varepsilon Z^3\}\}, \tag{5.19}$$

it follows that Θ^* is a one to one continuous homomorphism of $\psi_A^*(Z^3)$ into $\psi_B^*(Z^3)$. In fact (5.16), (5.17), (5.18) and (5.19) imply $\psi_A^*(Z^3)$ is a subgroup of $\psi_B^*(Z^3)$ and Θ^* is just the identity, so $\Theta^*\sigma_A^* = \sigma_B^*\Theta^*$.

Now apply Corollary 2.2b to obtain A is an algebraic factor of B. The above argument can be reversed to show B is also an algebraic factor of A, or A and B are weakly algebraically isomorphic.

Case 2 is treated by replacing B with $-B$ and proceeding as in case 1. For case 3 replace B by B^{-1} and proceed as in case 1. Replacing B by $-B^{-1}$ reduces case 5 to case 1.

We have shown that if $A, B \varepsilon \mathrm{Aut}(K^3)$ are ergodic and have the same entropy, then A is weakly algebraically isomorphic to one of the following $\{B, -B, B^{-1}, -B^{-1}\}$, and the theorem follows.

It is very likely that Theorems 5.1 and 5.2 can be proved using techniques from matrix theory. In this case, it is necessary to obtain a similarity relation say $\Theta A = B\Theta$ where Θ is a nonsingular 2×2 or 3×3 matrix (depending upon whether A, B are ergodic automorphisms of K^2 or K^3) having integer entries only. In fact since each continuous epimorphism of K^n is associated with a nonsingular $n \times n$ matrix with integer entries, it follows that $A, B \in \mathrm{Aut}(K^n)$ are weakly algebraically isomorphic only if A and B have the same characteristic equations.

The following examples show that the conclusion of Theorem 3 cannot be extended to ergodic automorphisms of K^n where $n \geq 4$.

<u>Example 1</u>: Let $A \in \mathrm{Aut}(K^3)$ be defined by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 0 \end{bmatrix}$$

It can be shown that A has only one real characteristic root which lies between -1 and 0. Hence, A is ergodic.

Consider

$$B = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} .$$

$Ch(B) = Ch(A) \cdot Ch(A^{-1})$ and $Ch(C) = Ch(A)^2$, so B and C are ergodic automorphisms of the six torus. Moreover, Proposition 2.5 implies

$$h(B) = h(C) = 2h(A).$$

Observe that since $Ch(B^{-1}) = Ch(B)$, C cannot be weakly algebraically isomorphic to either B or B^{-1} (their characteristic roots do not coincide). Similarly, C cannot be weakly algebraically isomorphic to $-B$ or $-B^{-1}$.

Note that Proposition 2.5 also implies $h(A^2) = 2h(A)$ which gives an example of an ergodic automorphism of the 3-torus having the same entropy as B, an ergodic automorphism of the 6-torus.

<u>Example 2:</u> Let

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

which is the example given at the beginning of this section. A is an ergodic automorphism of K^2. It follows that

$$B = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

are ergodic automorphisms of the 4-torus having the same entropy. Using an argument similar to that used in the previous example, it can be shown that B and C also do not satisfy the conclusion of Theorem 5.1.

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APPENDIX

SOME EXTENSIONS

Let $\tau \in \text{Aut}(K^n)$ with associated unimodular matrix A, and let $\text{Ch}(A) = (P_1(\lambda)\ldots P_s(\lambda))(P_{s+1}(\lambda)\ldots P_r(\lambda))$ be a factorization of $\text{Ch}(A)$ as in (4.13) with $(P_1(\lambda)\ldots P_s(\lambda))$ the ergodic part. Suppose $\lambda_1, \ldots, \lambda_h$ are the distinct roots of the ergodic part, while $\lambda_{h+1}, \ldots, \lambda_\ell$ are the distinct roots of the nonergodic part. In Corollary 4.3 it was shown that

$$\Delta_\tau \supset \{m \in Z^n\colon m\cdot\xi_{ijk} = 0,\ 1 \le i \le h,\ 1 \le j \le r_i,\ 1 \le k \le s_{ij}\}$$

where $\{\xi_{ijk}\}$ $(1 \le i \le \ell,\ 1 \le j \le r_i,\ 1 \le k \le s_{ij})$ is a basis of C^n for which the linear transformation defined by A on C^n has matrix J (a Jordan form of A). Here we will show that we actually have equality and will use this information to characterize the maximal ergodic subgroup. To do so, a slightly different version of Theorem 4.3 is needed.

<u>Theorem A:</u> Let $\tau \in \text{Aut}(K^n)$ with A its associated unimodular matrix. Let J be a Jordan form of A with $\{\xi_{ijk}\}$ a basis of C^n as defined in Corollary 4.3, and suppose $\{1, \{\alpha_{ijk}\}\}$ are real numbers linearly independent over the algebraic numbers $(1 \le i \le \ell,\ 1 \le j \le r_i,\ 1 \le k \le s_{ij})$. Set

$$a = \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \sum_{k=1}^{s_{ij}} \alpha_{ijk} \operatorname{Re} \xi_{ijk} \quad \bmod (1, 1, \ldots, 1).$$

Then $\mathrm{orb}^+(a)$ separates Z^n, so the map

$$\psi^*(m) = \left\{\exp 2\pi i \left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \sum_{k=1}^{s_{ij}} \alpha_{ijk} \operatorname{Re}(m \cdot (\xi_{ijk} J^p)) \right)\right\}$$

is an algebraic isomorphism of Z^n onto a shift-invariant subgroup of $\hat{Y}$ (Theorem 3.3).

<u>Proof</u>: Let $m \in Z^n$. By (4.2),

$$\langle \tau^p a, m \rangle = \exp 2\pi i \left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \sum_{k=1}^{s_{ij}} \alpha_{ijk} (m \cdot (\operatorname{Re}(\xi_{ijk} J^p))) \right)$$

$$= \exp 2\pi i \left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \sum_{k=1}^{s_{ij}} \alpha_{ijk} \operatorname{Re} (m \cdot (\xi_{ijk} J^p)) \right).$$

Since $\operatorname{Re}(m \cdot \xi_{ijk} J^p)$ is algebraic (the coordinates of ξ_{ijk} are algebraic), it follows from the choice of $\{\xi_{ijk}\}$ that $\langle \tau^p a, m \rangle = 1$ for $p \geq 0$ if and only if $\operatorname{Re}(m \cdot (\xi_{ijk} J^p)) = 0$, $1 \leq i \leq \ell$, $1 \leq j \leq r_i$, $1 \leq k \leq s_{ij}$.

If $m \neq o$ there exist i_0, j_0, k_0 where $1 \leq i_0 \leq \ell$, $1 \leq j_0 \leq r_{i_0}$, $1 \leq k_0 \leq s_{i_0 j_0}$, such that $m \cdot \xi_{i_0 j_0 k_0} \neq 0$. If $k_0 \neq s_{i_0 j_0}$ we may assume without loss of generality that $m\cdot\xi_{i_0j_0k} = 0$ for $k_0 < k \leq s_{i_0j_0}$. Consider

$$\mathrm{Re}(m\circ(\xi_{i_0j_0k_0}\ J^p)).$$

First set $p = 0$. If $\mathrm{Re}(m\cdot\xi_{i_0j_0k_0}) \neq 0$ we are finished. If not, set $p = 1$. If $k_0 = s_{i_0j_0}$ (i.e. $\xi_{i_0j_0k_0}$ is an eigenvector associated with λ_{i_0}), then

$$\mathrm{Re}(m\circ(\xi_{i_0j_0k_0}\ J)) = \mathrm{Re}\ \lambda_{i_0}(m\circ\xi_{i_0j_0k_0})$$

which cannot be zero if $\mathrm{Re}(m\circ\xi_{i_0j_0k_0})$ is zero. On the other hand for $k_0 < s_{i_0j_0}$ we have

$$\mathrm{Re}(m\circ(\xi_{i_0j_0k_0}\ J) = \mathrm{Re}(m\cdot(\lambda_{i_0}\xi_{i_0j_0k_0} + \xi_{i_0j_0(k_0+1)})$$

$$= \mathrm{Re}(\lambda_{i_0}(m\circ\xi_{i_0j_0k_0}))+\mathrm{Re}(m\circ\xi_{i_0j_0(k_0+1)})$$

$$= \mathrm{Re}(\lambda_{i_0}(m\cdot\xi_{i_0j_0k_0}))$$

which cannot be zero if $\mathrm{Re}(m\circ\xi_{i_0j_0k_0}) = 0$. Therefore, $m \neq 0$ implies that either $\langle a, m\rangle \neq 1$ or $\langle \tau a, m\rangle \neq 1$ as we wished to show.

<u>Theorem B</u>: Let $\tau \in \mathrm{Aut}(K^n)$ with A its associated unimodular matrix, and let $\mathrm{Ch}(A) = (P_1(\lambda)\ldots P_s(\lambda))(P_{s+1}(\lambda)\ldots P_r(\lambda))$ be a factorization of Ch(A) as in (4.13) with $(P_1(\lambda)\ldots P_s(\lambda))$ the ergodic part. Suppose $\lambda_1, \ldots, \lambda_h$ are the distinct roots of the ergodic part, while

$\lambda_{h+1}, \ldots, \lambda_\ell$ are the distinct roots of the nonergodic part. If $\{\xi_{ijk}\}$ is a basis of C^n as defined in Corollary 4.3, then

$$\Delta_\tau = \{m \in Z^n : m \cdot \xi_{ijk} = 0,\ 1 \leq i \leq h,\ 1 \leq j \leq r_i,\ 1 \leq k \leq s_{ij}\}.$$

Furthermore, if T is the projection of $sp\{\mathrm{Re}\ \xi_{ijk}, \mathrm{Im}\ \xi_{ijk}: 1 \leq i \leq h,\ 1 \leq j \leq r_i,\ 1 \leq k \leq s_{ij}\}$ onto the n-torus, then T is a subgroup of K^n whose closure $\overline{T}$ is the maximal ergodic subgroup (i.e. $\overline{T} = E$).

<u>Proof</u>: If $\{\xi_{ijk}\}$ $(1 \leq i \leq \ell,\ 1 \leq j \leq r_i,\ 1 \leq k \leq s_{ij})$ and $a \in K^n$ are defined as in Theorem A, then $orb^+(a)$ separates Z^n, and the map

$$\psi^*(m) = \left\{\exp 2\pi i \left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \sum_{k=1}^{s_{ij}} \alpha_{ijk}\ \mathrm{Re}(m \cdot (\xi_{ijk}\ J^p)) \right)\right\}$$

is an algebraic isomorphism of Z^n onto a shift-invariant subgroup of $\hat{Y}$ (J being the corresponding Jordan form of A).

For $t = 1, 2, \ldots, N$ let p_t be a positive integer. The proof of Corollary 4.3 can be modified to show that

$$\prod_{t=1}^{N} (\sigma^{*p_t} - I)\psi^*(m)$$

$$= \left\{ \exp 2\pi i \left(\sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \sum_{k=1}^{s_{ij}} \alpha_{ijk} \operatorname{Re}\left(m \cdot \left(\xi_{ijk} J^p \prod_{t=1}^{N} (J^{p_t} - I) \right)\right)\right)\right\} .$$

Since the coordinates of ξ_{ijk} as well as the entries of J are algebraic, it follows by the choice of $\{\xi_{ijk}\}$ that

$$\prod_{t=1}^{N} (\sigma^{*p_t} - I)\psi^*(m) = \hat{0}$$

if and only if for each $p \geq 0$

$$\text{(1)} \qquad \operatorname{Re}\left(m \cdot \left(\xi_{ijk} J^p \prod_{t=1}^{N} (J^{p_t} - I) \right)\right) = 0$$

$(1 \leq i \leq \ell,\ 1 \leq j \leq r_i,\ 1 \leq k \leq s_{ij})$.

Suppose $m \cdot \xi_{i_0 j_0 k_0} \neq 0$ where $1 \leq i_0 \leq h$, $1 \leq j_0 \leq r_{i_0}$, $1 \leq k_0 \leq s_{i_0 j_0}$. If $k_0 < s_{i_0 j_0}$ we may assume without loss of generality that $m \cdot \xi_{i_0 j_0 k} = 0$ for $k_0 < k \leq s_{i_0 j_0}$. If $k_0 < s_{i_0 j_0}$, evaluate (1) for $i = i_0$, $j = j_0$, $k = k_0$ to obtain

$$\operatorname{Re}\left(m \cdot \left(\lambda_{i_0}^{p} \prod_{y=1}^{N} (\lambda^{p_t} - 1)\xi_{i_0 j_0 k_0} + \sum_{k=k_{0+1}}^{s_{i_0 j_0}} Q_k(\lambda_{i_0})\xi_{i_0 j_0 k} \right)\right)$$

$$= \mathrm{Re}\left(\lambda_{i_0}^{p} \prod_{t=1}^{N} (\lambda^{p_t} - 1)(m\cdot\xi_{i_0j_0k_0}) + \sum_{k=k_0+1}^{s_{i_0j_0}} Q_k(\lambda_{i_0})(m\cdot\xi_{i_0j_0k})\right)$$

where $Q_k(\lambda) \in Z[\lambda]$ for $k_0 < k \leq s_{i_0j_0}$. Since $m\cdot\xi_{i_0j_0k} = 0$ for $k_0 < k \leq s_{i_0j_0}$, we have

$$\mathrm{Re}\left(m\cdot\left(\xi_{i_0j_0k_0}\, J^p \prod_{t=1}^{N} (J^{p_t} - I)\right)\right)$$

$$= \mathrm{Re}\left(\lambda_{i_0}^{p} \prod_{t=1}^{N} \left(\lambda_{i_0}^{p_t} - 1\right)\left(m\cdot\xi_{i_0j_0k_0}\right)\right)$$

which cannot be zero for both $p = 0$ and 1 since λ_{i_0} is not a root of unity. The same argument applies to the case where $k_0 = s_{i_0j_0}$. Therefore, $m\cdot\xi_{i_0j_0k_0} \neq 0$ for $1 \leq i_0 \leq h$, $1 \leq j_0 \leq r_{i_0}$, $1 \leq k_0 \leq s_{i_0j_0}$ implies $\psi^*(m) \notin \Delta$; hence, it follows from Corollary 3.3 that $m \notin \Delta_\tau$, which together with Corollary 4.3 proves the first assertion.

The second assertion is proved as in Theorem 4.2 using Lemma 4.2.

<u>Corollary</u>: Let $\tau \in \mathrm{Aut}(K^n)$. Then $h(\tau) = 0$ if and only if all characteristic roots of A are roots of unity, A being the unimodular matrix associated with τ.

<u>Proof</u>: See the proof of Corollary 4.2.