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Title: Nonlinear Stability of Couette Flows of Micropolar Fluids

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After deriving the closed-form solution for steady, laminar plane Couette and rotational Couette flows of a micropolar fluid, these two basic flows are altered by a finite two-dimensional and a finite axisymmetric disturbance, respectively. Disturbance equations are derived, from which the disturbance energy integrals are found. Then, utilizing the solutions of the linearized disturbance equations, the amplitude equations are derived, in accordance with the procedures of the Stuart energy method. An expression for the marginal stability surface is formulated, and expressions for the critical numbers R_c, R_{gc}, and R_{kc} are given.

An elucidation of the flow mechanisms, induced to deal with the energies imparted by the disturbance on the basic flow, is given. The new concepts of swirl, microenergy of rotation, and mean couple stress, are explained during the physical interpretation of the disturbance energy equations. Also, inequalities describing flow stability (or instability) are presented.

The numerical procedures, needed to quantitatively sustantiate the qualitative non-linear stability analysis of this thesis, are outlined.

Nonlinear Stability of Couette Flows
of Micropolar Fluids

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I dedicate this thesis to my parents, Christopher and Rosina, to my brothers, Angelo and Damon, and to my sister, Lisa.

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I. INTRODUCTION

The immediate objective of the theory of hydrodynamic stability is to understand the mechanisms of instability in laminar flow and to obtain criteria for its occurence. A more fundamental objective is to understand how, and under what circumstances, turbulence may arise from laminar instability. All the possible transitions of a flow profile, from the placid patterns of simple laminar flows, to the chaotic complexity of highly turbulent flows, should be elucidated. Thus, because of the inherent non-linearity of the equations of motion governing a hydrodynamical system, this ambitious pursuit should employ nonlinear stability analysis.

The mathematical problem of nonlinear hydrodynamic stability[1] can be formulated, by taking a given steady-state solution of the equations of motion, and superimposing a disturbance of a suitable kind; this results in a set of nonlinear 'disturbance' equations which govern the behavior of the disturbance. If the solution of the equations shows that any disturbance ultimately decays to zero, the flow is said to be temporally stable; whereas if the disturbance can be permanently different from zero, the flow is unstable. Note that instability of a laminar flow does not always imply turbulent motion, but very often results in another (possibly more complex) form of laminar motion.

Some preliminary insight is gained when infinitesimal disturbances are considered. For these disturbances of small amplitude, the solution of the disturbance differential equations is simplified (in fact, the governing equations are linearized). However, the initial growth of the disturbance only can be determined in most problems. On the basis of this linearized theory, it is possible to consider disturbances which contain an exponential time factor of the form exp(kt), t being the time. The boundary conditions on the disturbance equations require the vanishing on the boundaries of all disturbance quantities like disturbance velocity components and disturbance microgyration components, relative to the boundaries. Consequently, the boundary conditions are homogeneous, and there arises an eigenvalue problem for the determination of possible eigenvalues, k. In this (linear) case, if k has a positive real part, the flow is unstable; otherwise, the flow is stable.

To comprehend more than just the initial growth of the disturbance, requires that the disturbance be of finite amplitude, in that, a finite disturbance. Features of the nonlinear terms of the equations of motion can now be studied. Furthermore, a clarification of the connection between linear and nonlinear theories can be assessed.

In cases of instability of fluid flow, the disturbance is usually periodic in at least one spatial direction. Thus, it is convenient to take averages with respect to one of the spatial dimensions, and also to separate the flow into a mean part and a disturbance part (with zero mean).

Now, consider a flow, with local non-dimensional parameters (classically, a Reynolds number) that do not vary, as in the case of flows between parallel plates or coaxial, rotating cylinders. A synopsis of nonlinear stability then reports as follows. Initially, a disturbance, superimposed on a given laminar flow, grows exponentially with time according to the linear theory; but eventually it reaches such a size that the transport of momentum and microinertia by the finite flucuations become appreciable and the associated mean stress (Reynolds stress) and the associated mean couple stress then have a significant effect on the mean flow. This distortion of the mean flow modifies the rate of transfer of energies from the mean flow to the disturbance, and since this energy transfer is the cause of the growth of the disturbance, there is a modification of the rate of growth of the disturbance.

An equilibrium state may be possible, in which the rate of transfer of energies from the distorted mean flow to the disturbance, balances precisely the rate of viscous dissipations of the energy of the disturbance. In such an equilibrium state, the disturbance will have a definite finite amplitude and the mean flow will be distorted from its original laminar form. An example of an equilibrium state of this kind occurs between coaxial, rotating cylinders, when the instability is observed to take the form of cellular, toroidal vortices (Taylor vortices) spaced regularly along the axes of the cylinders (Coles, 1965). Elucidation of this curious phenomenon of Taylor vortices will be a prize for nonlinear hydrodynamic stability theory.

The philosophy of purpose for this thesis is encoded in the following quotation.

> "Nonlinear hydrodynamic stability theory is really concerned, ultimately, with phenomena such as transition to turbulence. In practice, however, that phenomenon is so complex as to defy rational understanding at the present time. A

> more limited objective is that of gaining some understanding of nonlinear processes in fluid mechanics, perhaps with reference to the early, relatively-simple stages of the evolution of laminar flow to turbulence. Even then, the mathematical problems posed are challenging enough." (Stuart,1977)

This thesis employs micropolar fluid dynamics to the problems of flows of micropolar fluids between two parallel plates and between two coaxial, rotating cylinders. Closed-form solutions to these two problems are obtained for laminar flow, and are presented in chapters II and III. An understanding of nonlinear processes in fluid mechanics is gained, with reference to the early, relatively-simple stages of the evolution of laminar flow to turbulence.

Chapters IV and V deal, respectively, with two-dimensional disturbance plane Couette flow and axisymmetric disturbance Couette flow. These chapters study the basic flow of an incompressible viscous fluid that is altered by a finite disturbance flow. The resulting flow must satisfy the equations of motion and the same boundary conditions as the basic flow, but the disturbance flow is otherwise arbitrary. The analysis employs the procedures of the Stuart energy method to study the time-rate of change of the disturbance energies. By determining what becomes of the energy imparted to the basic flow by the disturbance flow, we begin to unravel such complex phenomena as transition to turbulence.

The linear theory of micropolar fluid dynamics,for plane Couette and rotational Couette flows, is briefly pursued, so that the micropolar analog of the Orr-Sommerfeld energy equations can be derived. The solution to these equations is assumed to be the spatial form (shape) of the superimposed nonlinear disturbance.

Returning to nonlinear stability analysis, the disturbance energy equations are derived; and a physical interpretation of the mechanisms induced to deal with the energies of the disturbance flow are discussed. The new concepts of swirl, microenergy of rotation, and mean couple stress are introduced, because of micropolar theory, into the discussion.

Amplitude equations are next derived from the corresponding nonlinear disturbance energy equations. Theoretical predictions concerning nonlinear stability[2] are finally presented, which include implicit values for the critical numbers, and marginal stability surfaces. Consequently, we establish the threshold of nonlinear stability of Couette flows of micropolar fluids.

In the sequel, equations are labelled as (f.g.h), where f corresponds to the chapter, g to the section of the f:th chapter, and h to the h:th equation in the g:th section. (Terms in the text, with a superscript number a, will indicate that further elucidations can be found in the endnote with that number a.)

The covariant derivative of a contravariant vector is given by

$$v^k_{;l} = \partial v^k/\partial x^l + v^s \left\{ {k \atop s\ l} \right\} .$$

$\left\{ {k \atop s\ l} \right\}$ is a Christoffel symbol of the second kind relative to the spatial curvilinear coordinate system x^l.

Similarly, the partial derivative of a contravariant vector is denoted by

$v^k_{,l} = \partial v^k/\partial x^l$, and so forth for higher order partial derivatives.

The material time derivative of a spatial vector $f^m(\underset{\sim}{x},t)$ is defined as

$$\dot{f}^m = \frac{Df^m}{Dt}(\underset{\sim}{x},t) = \partial f^m/\partial t + v^k f^m_{;k} .$$

I.1 Why Micropolar Theory

The point is that, it is not a point. Non-polar continuum theories embellish this approximation that the constituent objects being mathematically modelled are "material points". These zero-dimensional points, comprising a theoretical material, enable a non-polar theory to conveniently ignore the existence of body couples and couple stresses[3]. Such ignorance waned, when A. C. Eringen published a microcontinuum theory (Eringen,1964).

Eringen derived the basic equations of microcontinuum theory, with deformable vectors, now assigned to each material point. This theory, he called micromorphic theory. Thus, in the micromorphic theory, each material point can translate, independently rotate, and/or deform.

In 1966, Eringen elucidated a special case of the micromorphic theory, called the micropolar theory (Eringen,1966). The micropolar theory allows each material point the freedom to translate, independently rotate, but not to deform. In this theory the 'material points' are considered to be 'geometrical points' that possess properties similar to rigid particles. Moreover, this polar theory can recognize the existence of body couples and couple stresses.

A large class of real materials of great physical importance are known to be composed of a substructure with tiny aggregates of molecules which can be considered very nearly rigid. Some examples of such micropolar materials include fibrous and granular media like wood and wood composites, solid rocket propellant grains, colloidal suspensions, animal blood, liquid crystals, and polymeric fluids.

We begin our study by stating the balance laws and constitutive equations of the micropolar theory, and then deriving the governing field equations for micropolar fluid dynamics.

I.2 Balance Laws of Micropolar Theory

The balance laws of micropolar media (Eringen,1976) are given locally as follows.

Conservation of Mass: Conservation of mass for the body is stated as usual by

$$\partial\rho/\partial t + (\rho v^k)_{;k} = 0, \qquad (1.2.1)$$

where t is time, ρ is mass density per unit volume, and v^k is the velocity vector.

Conservation of Micromoment of Inertia (Microinertia): Conservation of microinertia is an entirely new balance law which is stated as

$$\partial j^{kl}/\partial t + j^{kl}_{\ ;m} v^m - j^{km} \nu^l_{\ m} - j^{ml} \nu^k_{\ m} = 0, \qquad (1.2.2)$$

where j^{kl} is the microinertia tensor, and ν^{kl} is the gyration tensor.

Conservation of Linear Momentum: The time-rate of change of the momentum of a material body is equal to the total force acting on the body. Mathematically, this is expressed as

$$t^{lk}_{\ ;l} + \rho(f^k - \dot{v}^k) = 0, \qquad (1.2.3)$$

where t^{lk} is the stress tensor and f^k is the body force per unit mass.

Conservation of Moment of Momentum: The time-rate of change of moment of momentum of the body is equal to the total torque acting on the body. This is expressed as

$$m^{rk}{}_{;r} + e^{klm}\, t_{lm} + \rho(\ell^{k} - \dot{\sigma}^{k}) = 0, \tag{1.2.4}$$

where m^{rk} is the couple stress tensor, e^{klm} is the alternating tensor, ℓ^{k} is the body couple per unit mass, and $\dot{\sigma}^{k}$ is the inertial spin vector. ($\dot{\sigma}^{kl}$ is the inertial spin tensor.)

Conservation of Energy: The time-rate of change of the total energy of the body is equal to the rate of working of the external loads and the heat energy. Mathematically,

$$\rho\dot{\varepsilon} = t^{kl}(v_{l;k} + \nu_{kl}) + m^{kl}\, \nu_{l;k} + q^{k}{}_{;k} + \rho h, \tag{1.2.5}$$

where ε is the internal energy density per unit mass, ν_l is the angular velocity vector, q_k is the heat vector directed out of the body, and h is the heat source per unit mass.

We have listed above only the five balance laws of micropolar theory, with which we are concerned.

From the linear theory of isotropic micropolar fluids, the following constitutive equations are derived (Eringen,1976):

$$_D\hat{t}_{kl} = \lambda_v v^{m}{}_{;m}\, g_{kl} + (2\mu_v + \kappa_v) d_{kl} + \kappa_v (v_{l;k} - e_{klm}\nu^{m}) \tag{1.2.6}$$

$$\hat{m}_{kl} = \alpha e_{klm}\theta,^{m} + \alpha_v \nu^{m}{}_{;m}\, g_{kl} + \beta_v \nu_{k;l} + \gamma_v \nu_{l;k} \tag{1.2.7}$$

$$\hat{q}_{k} = \kappa\theta,_{k} + \beta e_{klm}\nu^{l;m} \tag{1.2.8}$$

λ_v, μ_v, κ_v, α_v, β_v, γ_v, κ, α, and β are the viscosity coefficients.

θ is absolute temperature, d_{kl} is the deformation rate tensor, and g_{kl} is the metric tensor. The carat ^ symbolizes a constitutive response functional. For example, the dissipative stress tensor $_{D}\hat{\underset{\sim}{t}} = -\pi \underset{\sim}{I} + \underset{\sim}{t}$ where $\underset{\sim}{I}$ is the identity tensor, $\underset{\sim}{t}$ is the stress tensor, and π is the thermodynamic pressure.

Eringen also derives that the viscosity coefficients must obey the following inequalities:

$$3\lambda_{v} + 2\mu_{v} + \kappa_{v} \geqq 0, \quad 2\mu_{v} + \kappa_{v} \geqq 0, \quad \kappa_{v} \geqq 0,$$
$$3\alpha_{v} + \beta_{v} + \gamma_{v} \geqq 0, \quad \gamma_{v} + \beta_{v} \geqq 0, \quad (\alpha - \beta/\theta)^2 \leqq 2\kappa(\gamma_{v} - \beta_{v})/\theta. \qquad (1.2.9)$$

Because of the classical limit, we also assume $\mu_{v} \geqq 0$ (Eringen,1966).

I.3 Field Equations — Micropolar Fluid Dynamics

Inserting the constitutive equations (1.2.6) - (1.2.8) into the balance laws yields the nine field equations of Micropolar Fluid Dynamics (Eringen, 1976) for constant viscosity coefficients, λ_v, μ_v, κ_v, α_v, β_v, γ_v, κ, α, and β.

$$\partial\rho/\partial t + \nabla\cdot(\rho\underset{\sim}{v}) = 0 \tag{1.3.1}$$

$$\partial j^{kl}/\partial t + j^{kl}{}_{;m}\, v^m + (e^{kmr}\, j^l{}_m + e^{lmr}\, j^k{}_m)\nu_r = 0 \tag{1.3.2}$$

$$-\nabla\pi + (\lambda_v + 2\mu_v + \kappa_v)\nabla(\nabla\cdot\underset{\sim}{v}) - (\mu_v + \kappa_v)\nabla\times\nabla\times\underset{\sim}{v} + + \kappa_v\nabla\times\underset{\sim}{\nu} + \rho(\underset{\sim}{f} - \dot{\underset{\sim}{v}}) = \underset{\sim}{0} \tag{1.3.3}$$

$$(\alpha_v + \beta_v + \gamma_v)\nabla(\nabla\cdot\underset{\sim}{\nu}) - \gamma_v\nabla\times\nabla\times\underset{\sim}{\nu} + \kappa_v\nabla\times\underset{\sim}{v} - - 2\kappa_v\underset{\sim}{\nu} + \rho(\underset{\sim}{l} - \dot{\underset{\sim}{\sigma}}) = \underset{\sim}{0} \tag{1.3.4}$$

$$\rho\theta\dot{\theta}\, \partial^2\Psi/\partial\theta^2 - \theta\partial\pi/\partial\theta\, \nabla\cdot\underset{\sim}{v} + {}_D\hat{t}^{kl}(v_{l;k} - e_{klm}\, \nu^m) + + m^{kl}\, \nu_{l;k} - \kappa\nabla^2\theta - \rho h = 0 \tag{1.3.5}$$

Here $\Psi = \varepsilon - \theta\eta$ represents the free energy. (η is the entropy density.)

Note that

$$\dot{v}^k = \partial v^k/\partial t + v^k{}_{;l}\, v^l \quad \text{and} \tag{1.3.6}$$

$$\dot{\sigma}^k = j^{kl}(\partial\nu_l/\partial t + \nu_{l;m}\, v^m) - e^{kmr}\, j^l{}_m\, \nu_r\, \nu_l . \tag{1.3.7}$$

Also of importance are the following basic relations of micropolar theory.

Symmetry of the microinertia tensor: $j^{kl} = j^{lk}$.

Skew-symmetry of gyration tensor: $\nu^{mk} = -\nu^{km} = e^{kmr}\nu_r$.

Spin momentum: $\sigma^k = j^{kl}\nu_l$ from which equation (1.3.7) is derived.

I.4 Basic Assumptions and the Surmised Field Equations

We now introduce the following simplifying assumptions into the theory:

(i) The fluid is isothermal, which implies that $\theta_{,k} = 0$.

(ii) The fluid is incompressible, homogeneous, and isotropic. As a result of incompressibility, the thermodynamic pressure π is replaced by the hydrostatic pressure p.

(iii) The fluid is assumed to be microisotropic; that is, $j^{kl} = j\,g^{kl}$ where j is a scalar which is taken as a constant here.

(iv) There are no body forces and body couples; that is, $f^k = 0 = \ell^k$.

These assumptions are not unrealistic since there exist a wide class of fluids, as listed previously, for which they are known to be valid. Furthermore, these assumptions are found to simplify the field equations to a more tractable form.

As a consequence of the above assumptions, the field equations (1.3.1) - (1.3.5) take the form:

$$\nabla\cdot\underset{\sim}{v} = 0 \tag{1.4.1}$$

$$-\nabla p + (\mu+\kappa)\nabla^2\underset{\sim}{v} + \kappa\nabla\times\underset{\sim}{\nu} - \rho\dot{\underset{\sim}{v}} = \underset{\sim}{0} \tag{1.4.2}$$

$$(\alpha+\beta)\nabla(\nabla\cdot\underset{\sim}{\nu}) + \gamma\nabla^2\underset{\sim}{\nu} + \kappa\nabla\times\underset{\sim}{v} - 2\kappa\underset{\sim}{\nu} - \rho\dot{\underset{\sim}{\sigma}} = \underset{\sim}{0} \tag{1.4.3}$$

It is these seven surmised field equations that we will be using for investigating the nonlinear stability of Couette flows of micropolar fluids.

II. BASIC PLANE COUETTE FLOW

II.1 Geometry of Plane Couette Flow

Plane Couette flow is defined to mean any flow, occurring between two parallel plates, that is caused by the two plates moving relative to each other.

We will use a rectangular Cartesian coordinate system (x,y,z), where x denotes the distance parallel to the plates, and z denotes the distance normal to the plates as measured from the channel center. The total distance between the plates always equals $2h$.

The plates are assumed to be of infinite extent in the xy-plane. For simplicity, the relative motion of the two plates will be chosen so that the upper plate is moving at constant velocity U (in the positive x-direction), and the lower plate is at rest. Also, the fluid flows to be considered are assumed to be under no external pressure gradients.

The velocity field will be $\underset{\sim}{v} = (u, v, w)$.

The microgyration field will be $\underset{\sim}{\nu} = (\zeta, \nu, \eta)$.

We will be non-dimensionalizing all equations, choosing as reference length, h, which represents one-half the distance between the parallel plates, and as reference velocity, U, which represents the constant velocity of the upper plate. These reference parameters, along with the constant density ρ, are used to provide a reference time, $t = h\bar{t}/U$; a reference pressure gradient, $\nabla p = \rho U^2 \bar{\nabla}\bar{p}/h$; a reference microinertia, $j = h^2\bar{j}$; and a reference microgyration, $\nu = U\bar{\nu}/h$. The over-bar denotes a dimensionless variable.

Four nondimensional numbers, for the micropolar theory, are defined as follows:

$$R = \frac{\rho h U}{\mu}; \quad R_k = \frac{\rho h U}{\kappa}; \quad R_g = \frac{\rho h^3 U}{\gamma}; \quad \text{and} \quad R_b = \frac{\rho h^3 U}{\alpha+\beta} . \qquad (2.1.1)$$

We define

$$1/M = 1/R + 1/R_k .$$

In accordance with the constitutive theory, given by equations (1.2.6) - (1.2.8), we present the following nomenclature for the viscosity coefficients: μ = dynamic viscosity; κ = gyrational viscosity; γ = gyrational-gradient viscosity (right); β = gyrational-gradient viscosity (left); and α = dilational-gyration viscosity. Also, do not confuse R with the classical Reynolds number (Re), even though,their definitions are very similar, with the only difference being that the μ listed here is from micropolar theory.

Physically, R represents the ratio of inertial force to viscous force,(inertial force being $\rho h^2 U^2$ and viscous force being μhU). Hence, R plays the same role as the classical Reynolds number, in that, the μ-viscosity is created by relatively translating volume elements.

Physically, R_k represents the ratio of inertial force to gyrational-viscous force,(gyrational-viscous force being κhU). Since the micropolar theory has the added feature of geometrical points rotating, the adjective 'gyrational' is inserted to emphasize that the κ-viscosity is created by relatively rotating volume elements.

Physically, R_g represents the ratio of inertial couple to gyrational-gradient couple, (gyrational-gradient couple being γU). The adjective 'gyrational-gradient' is inserted to emphasize that the γ-viscosity is created by relatively rotating and translating volume elements.

Physically, R_b represents the ratio of inertial couple to dilational-gyration couple, (dilational-gyration couple being $(\alpha+\beta)U$). The adjective 'dilational-gyration' is inserted to emphasize that the $(\alpha+\beta)$-viscosity is created by relatively rotating, translating, and dilating volume elements.

For reference purposes, the surmised field equations are presented in rectangular Cartesian coordinates.

II.2 Field Equations — Rectangular Coordinates

Listed below are the seven field equations (1.4.1) - (1.4.3) that were deduced in section I.4. These non-dimensionalized field equations, in rectangular Cartesian coordinates, are:

$$u_{,x} + v_{,y} + w_{,z} = 0 \tag{2.2.1}$$

$$-p_{,x} + (u_{,xx} + v_{,yy} + w_{,zz})/M + (\eta_{,y} - \nu_{,z})/R_k =$$
$$= u_{,t} + uu_{,x} + vu_{,y} + wu_{,z} \tag{2.2.2}$$

$$-p_{,y} + (v_{,xx} + v_{,yy} + v_{,zz})/M + (\zeta_{,z} - \eta_{,x})/R_k =$$
$$= v_{,t} + uv_{,x} + vv_{,y} + wv_{,z} \tag{2.2.3}$$

$$-p_{,z} + (w_{,xx} + w_{,yy} + w_{,zz})/M + (\nu_{,x} - \zeta_{,y})/R_k =$$
$$= w_{,t} + uw_{,x} + vw_{,y} + ww_{,z} \tag{2.2.4}$$

$$(\zeta_{,xx} + \nu_{,xy} + \eta_{,xz})/R_b + (\zeta_{,xx} + \zeta_{,yy} + \zeta_{,zz})/R_g + (w_{,y} - v_{,z})/R_k =$$
$$= 2\zeta/R_k + j(\zeta_{,t} + u\zeta_{,x} + v\zeta_{,y} + w\zeta_{,z}) \tag{2.2.5}$$

$$(\zeta_{,yx} + \nu_{,yy} + \eta_{,yz})/R_b + (\nu_{,xx} + \nu_{,yy} + \nu_{,zz})/R_g + (u_{,z} - w_{,x})/R_k =$$
$$= 2\nu/R_k + j(\nu_{,t} + u\nu_{,x} + v\nu_{,y} + w\nu_{,z}) \tag{2.2.6}$$

$$(\zeta_{,zx} + \nu_{,zy} + \eta_{,zz})/R_b + (\eta_{,xx} + \eta_{,yy} + \eta_{,zz})/R_g + (v_{,x} - u_{,y})/R_k =$$
$$= 2\eta/R_k + j(\eta_{,t} + u\eta_{,x} + v\eta_{,y} + w\eta_{,z}) \tag{2.2.7}$$

The over-bars on nondimensional quantities have been omitted for convenience, and a concise notation for partial derivatives employed.

Note that assumption (iii) stated in section I.4 and the expression for spin momentum, allow the inertial spin vector to be expressed as:

$$\dot{\sigma}^k = j\,\frac{D\nu^k}{Dt} = j(\partial\nu^k/\partial t + \nu^k_{;m}\, v^m). \tag{2.2.8}$$

II.3 Laminar Plane Couette Flow

The plane Couette flow problem for steady, laminar micropolar fluid flows in the xz-plane is now presented and solved.

The laminar flow is assumed to be in the x- direction. For steady, laminar plane Couette flow, we prescribe the velocity and microgyration fields, respectively, to be:

$$\underset{\sim}{v} = \{u(z),\ 0,\ 0\} \quad \text{and} \quad \underset{\sim}{\nu} = \{0,\ \nu(z),\ 0\}. \tag{2.3.1}$$

The field equations (2.2.1) - (2.2.7), in accord with the prescription (2.3.1) and pertinent nondimensional numbers (2.1.1), reduce to:

$$-\frac{dp}{dx} + \frac{1}{M}\frac{d^2u}{dz^2} - \frac{1}{R_k}\frac{d\nu}{dz} = 0, \tag{2.3.2}$$

$$\frac{1}{R_g}\frac{d^2\nu}{dz^2} + \frac{1}{R_k}\frac{du}{dz} - \frac{2\nu}{R_k} = 0, \tag{2.3.3}$$

$$\partial p/\partial y = 0 = \partial p/\partial z. \tag{2.3.4}$$

Enforcing the strict adherence boundary conditions[4] gives, non-dimensionally,

$$u(-1) = 0 = \nu(\pm 1) \quad \text{and} \quad u(1) = 1. \tag{2.3.5}$$

We now solve equations (2.3.2-3) for $\nu(z)$ and $u(z)$. From equation (2.3.2), we obtain

$$-\frac{dp}{dx} = A = \frac{1}{R_k}\frac{d\nu}{dz} - \frac{1}{M}\, d^2u/dz^2 . \tag{2.3.6}$$

Note that since the right-hand side of (2.3.6) is a function of z, and the left-hand side is a function of x, it follows that A = constant. Integrating equation (2.3.6) with respect to z gives

$$Az + B_1 = \frac{\nu}{R_k} - \frac{1}{M}\frac{du}{dz}, \tag{2.3.7}$$

where B_1 is a constant of integration. Substituting for du/dz from (2.3.7) into (2.3.3) yields

$$d^2\nu/dz^2 - \lambda^2\nu = MR_g(Az + B_1)/R_k , \tag{2.3.8}$$

where

$$\lambda^2 = \frac{R_g(R + 2R_k)}{R_k(R + R_k)} = \frac{h^2\kappa(2\mu + \kappa)}{\gamma(\mu + \kappa)}. \tag{2.3.9}$$

Of importance is the fact that $\lambda^2 \geqq 0$ via the inequalities (1.2.9). The general solution of (2.3.8) is

$$\nu(z) = B_2 e^{\lambda z} + B_3 e^{-\lambda z} - \frac{RR_k}{R + 2R_k}(Az + B_1), \tag{2.3.10}$$

where B_2 and B_3 are additional constants of integration. With this expression for $\nu(z)$, equation (2.3.7) reveals

$$u(z) = \frac{M}{\lambda R_k}\left(B_2 e^{\lambda z} - B_3 e^{-\lambda z} \right) - \frac{M(R + R_k)}{R + 2R_k} (Az^2 + 2B_1 z) + B_4 , \tag{2.3.11}$$

where B_4 is another integration constant.

Assuming the experimentally achievable assumption that $A = -dp/dx = 0$, and with the boundary conditions (2.3.5), the integration constants B_1, B_2, B_3, B_4 for equations (2.3.10-11) are found to be:

$$B_1 = \lambda(R + 2R_k)\xi, \quad B_2 = B_3 = \lambda RR_k \operatorname{sech}(\lambda)\xi/2, \text{ and } B_4 = 1/2, \tag{2.3.12}$$

where

$$1/\xi = 2RM \tanh(\lambda) - 4\lambda RR_k .$$

Therefore, the microgyration is

$$\nu(z) = \lambda RR_k \xi\{\cosh(\lambda z)/\cosh(\lambda) - 1\}. \tag{2.3.13}$$

And the velocity is

$$u(z) = RM\xi \sinh(\lambda z)/\cosh(\lambda) - 2\lambda RR_k \xi z + 1/2 . \tag{2.3.14}$$

Equations (2.3.13) and (2.3.14) completely determine the flow profiles for steady, laminar plane Couette flow of a micropolar fluid for the given boundary conditions (2.3.5) and $A = 0$.

For the corresponding unsteady, laminar plane Couette flow problem, the velocity and microgyration fields $u(z,t)$ and $\nu(z,t)$, respectively, reduce to $u(z)$ and $\nu(z)$. That is,

$$u(z,t) = u(z) \quad \text{and} \quad \nu(z,t) = \nu(z). \tag{2.3.15}$$

Especially note that solving equation (2.3.8) when $\lambda^2 = 0$ (i.e. $R_k \to \infty$), we derive $\nu(z) = 0$ and $u(z) = (z + 1)/2$. As expected, the fields revert back to those derived for classical incompressible viscous flows.

III. BASIC ROTATIONAL COUETTE FLOW

III.1 Geometry of Rotational Couette Flow

The rotational Couette flow is defined to mean any flow, occurring in the annulus between two coaxial cylinders, rotating relative to each other about their common axis.

We will use cylindrical coordinates (r,θ,z). These cylinders have the z-axis as the center axle. r denotes the radial direction, and θ denotes the azimuthal direction. The inner cylinder is of radius R_1, and the outer cylinder has radius R_2, with $0 < R_1 < R_2$ always.

Each cylinder is assumed to be of infinite length. The relative motion of the two cylinders will be chosen so that the inner cylinder is rotating at the constant angular velocity Ω_1, and the outer cylinder is rotating at the constant angular velocity Ω_2. Also, the fluid flows to be considered are assumed to be under no external pressure gradients.

The velocity field will be $\underset{\sim}{v} = (u, v, w)$.

The microgyration field will be $\underset{\sim}{\nu} = (\zeta, \eta, \nu)$.

We will be non-dimensionalizing all equations, choosing as reference length, $d = R_2 - R_1$, which represents the gap-width between the cylinders, and as reference velocity[5], $\Omega_1 R_1$, which represents the constant velocity of the inner cylinder. These reference parameters, along with the constant density ρ, are used to provide a reference time, $t = \bar{t}d/\Omega_1 R_1$; a reference pressure gradient, $\nabla p = \rho\Omega_1^2 R_1^2\, \bar{\nabla}\bar{p}/d$; a reference microinertia, $j = \bar{j}d^2$; and a reference microgyration, $\nu = \Omega_1 R_1 \bar{\nu}/d$. The over-bar denotes a dimensionless variable.

Four non-dimensional numbers, for the micropolar theory, are defined as follows:

$$R = \frac{\rho d\Omega_1 R_1}{\mu}; \quad R_k = \frac{\rho d\Omega_1 R_1}{\kappa}; \quad R_g = \frac{\rho d^3\Omega_1 R_1}{\gamma}; \text{ and } R_b = \frac{\rho d^3\Omega_1 R_1}{\alpha+\beta}. \tag{3.1.1}$$

As before, we define

$$1/M = 1/R + 1/R_k .$$

For reference purposes, the surmised field equations are next presented in cylindrical coordinates.

III.2 Field Equations — Cylindrical Coordinates

Listed below are the seven field equations (1.4.1) - (1.4.3) that were deduced in section I.4. These non-dimensionalized field equations, in cylindrical coordinates, are:

$$\frac{1}{r}(ru)_{,r} + \frac{1}{r} v_{,\theta} + w_{,z} = 0 \tag{3.2.1}$$

$$- p_{,r} + \frac{1}{rM}(ru_{,rr} + \frac{1}{r} u_{,\theta\theta} + ru_{,zz} + u_{,r} - \frac{u}{r} - \frac{2}{r} v_{,\theta}) + \frac{1}{R_k}(\frac{1}{r} \nu_{,\theta} - \eta_{,z}) =$$

$$= u_{,t} + uu_{,r} + \frac{v}{r} u_{,\theta} - v/r^2 + wu_{,z} \tag{3.2.2}$$

$$- \frac{1}{r} p_{,\theta} + \frac{1}{rM}(rv_{,rr} + \frac{1}{r} v_{,\theta\theta} + rv_{,zz} + v_{,r} - \frac{v}{r} + \frac{2}{r} v_{,\theta}) + \frac{1}{R_k}(\zeta_{,z} - \nu_{,r}) =$$

$$= v_{,t} + uv_{,r} + \frac{v}{r} v_{,\theta} + \frac{uv}{r} + wv_{,z} \tag{3.2.3}$$

$$- p_{,z} + \frac{1}{rM}(rw_{,rr} + \frac{1}{r} w_{,\theta\theta} + rw_{,zz} + w_{,r}) + \frac{1}{R_k}(\eta_{,r} + \frac{\eta}{r} - \frac{1}{r} \zeta_{,\theta}) =$$

$$= w_{,t} + uw_{,r} + \frac{v}{r} w_{,\theta} + ww_{,z} \qquad (3.2.4)$$

$$\frac{1}{rR_b}(r\zeta_{,rr} + \zeta_{,r} - \frac{\zeta}{r} - \frac{1}{r} \eta_{,\theta} + \eta_{,r\theta} + r\nu_{,rz}) + \frac{1}{R_k}(\frac{1}{r} w_{,\theta} - v_{,z}) +$$

$$+ \frac{1}{rR_g}(r\zeta_{,rr} + \frac{1}{r} \zeta_{,\theta\theta} + r\zeta_{,zz} + \zeta_{,r} - \frac{2}{r} \eta_{,\theta} - \frac{\zeta}{r}) = \frac{2\zeta}{R_k} +$$

$$+ j(\zeta_{,t} + u\zeta_{,r} + \frac{v}{r} \zeta_{,\theta} - \frac{v\eta}{r} + w\zeta_{,z}) \qquad (3.2.5)$$

$$\frac{1}{rR_b}(\zeta_{,\theta r} + \frac{1}{r} \zeta_{,\theta} + \frac{1}{r} \eta_{,\theta\theta} + \nu_{,\theta z}) + \frac{1}{R_k}(u_{,z} - w_{,r}) +$$

$$+ \frac{1}{rR_g}(r\eta_{,rr} + \frac{1}{r} \eta_{,\theta\theta} + r\eta_{,zz} + \eta_{,r} + \frac{2}{r} \zeta_{,\theta} - \frac{\eta}{r}) = \frac{2\eta}{rR_k} +$$

$$+ j(\eta_{,t} + u\eta_{,r} + \frac{v}{r} \eta_{,\theta} + \frac{v\zeta}{r} + w\eta_{,z}) \qquad (3.2.6)$$

$$\frac{1}{R_b}(\zeta_{,zr} + \frac{1}{r} \zeta_{,z} + \frac{1}{r} \eta_{,z\theta} + \nu_{,zz}) + \frac{1}{R_k}(v_{,r} + \frac{v}{r} - \frac{1}{r} u_{,\theta}) +$$

$$+ \frac{1}{rR_g}(r\nu_{,rr} + \frac{1}{r} \nu_{,\theta\theta} + r\nu_{,zz} + \nu_{,r}) = \frac{2\nu}{R_k} +$$

$$+ j(\nu_{,t} + u\nu_{,r} + \frac{v}{r} \nu_{,\theta} + w\nu_{,z}) \qquad (3.2.7)$$

The over-bars on nondimensional quantities have been omitted for convenience, and a concise notation for partial derivatives was employed.

III.3 Laminar Rotational Couette Flow

The rotational Couette flow problem for steady, laminar micropolar fluid flows in the $r\theta$-plane is now presented and solved.

The laminar flow is assumed to be in the θ-direction. For steady, laminar rotational Couette flow, we prescribe the velocity and microgyration fields, respectively, to be:

$$\underset{\sim}{v} = \{0,\ v(r),\ 0\} \quad \text{and} \quad \underset{\sim}{\nu} = \{0,\ 0,\ \nu(r)\}. \tag{3.3.1}$$

The field equations (3.2.1) - (3.2.7), in accord with the prescription (3.3.1), reduce to:

$$\frac{1}{M}\left(d^2v/dr^2 + \frac{d}{dr}(v/r)\right) = \frac{1}{R_k}\frac{d\nu}{dr}\ , \tag{3.3.2}$$

$$\frac{1}{R_g}\left(d^2\nu/dr^2 + \frac{1}{r}\frac{d\nu}{dr}\right) + \frac{1}{R_k}\left(\frac{dv}{dr} + \frac{v}{r}\right) = \frac{2\nu}{R_k}\ , \tag{3.3.3}$$

$$\frac{dp}{dr} = \frac{v}{r^2}\ . \tag{3.3.4}$$

Note that $\partial p/\partial\theta = 0$ because of axial symmetry, and $\partial p/\partial z = 0$ since there is no axial motion.

Enforcing the strict adherence boundary conditions[4] gives, non-dimensionally,

$$\nu(R_1) = 0 = \nu(R_2),\quad v(R_1) = 1,\ \text{and}\quad v(R_2) = \Omega_2 R_2/\Omega_1 R_1, \tag{3.3.5}$$

where $\bar{R}_1 = R_1/d$ and $\bar{R}_2 = R_2/d$. The over-bars on these nondimensional quantities are omitted for convenience.

We now solve equations (3.3.2-3) for $\nu(r)$ and $v(r)$. Integrating equation (3.3.2) with respect to r yields

$$\frac{1}{M}\left(\frac{dv}{dr} + \frac{v}{r}\right) = \frac{\nu}{R_k} - B_1 , \tag{3.3.6}$$

where B_1 is a constant of integration. Substituting for $dv/dr + v/r$ from (3.3.6) into equation (3.3.3) yields

$$d^2\nu/dr^2 + \frac{1}{r}\frac{d\nu}{dr} - \lambda^2\nu = MR_g B_1/R_k , \tag{3.3.7}$$

again (see expressions (2.3.9)) defining

$$\lambda^2 = \frac{R_g(R + 2R_k)}{R_k(R + R_k)} .$$

The general solution of (3.3.7) is well-known, in terms of the modified zero-order Bessel functions I_0 and K_0 of the first and second kind respectively, to be:

$$\nu(r) = B_2 I_0(\lambda r) + B_3 K_0(\lambda r) - \frac{RR_k B_1}{R + 2R_k} , \tag{3.3.8}$$

where B_2 and B_3 are additional integration constants. With this expression for $\nu(r)$, equation (3.3.6) reveals

$$v(r) = \frac{M}{\lambda R_k}\{B_2 I_1(\lambda r) - B_3 K_1(\lambda r)\} - \frac{RR_k B_1 r}{R + 2R_k} + B_4/r , \tag{3.3.9}$$

where I_1 and K_1 are modified first-order Bessel functions of the first and second kind respectively, and B_4 is another integration constant.

With boundary conditions (3.3.5), the integration constants B_1, B_2, B_3, and B_4 for equations (3.3.8-9) are found to be:

$$B_1 = -(R + 2R_k)gB_3/(ARR_k), \quad B_2 = -CB_3/A, \quad B_3 = (\Omega R_2 - R_1)A/G,$$

$$B_4 = R_1R_2\{S' + gR_2 - \Omega(S + gR_2)\}/G. \tag{3.3.10}$$

Here

$$A = I_0(\lambda R_1) - I_0(\lambda R_2), \quad C = K_0(\lambda R_1) - K_0(\lambda R_2),$$

$$S = M\{K_0(\lambda R_2)I_1(\lambda R_1) + K_1(\lambda R_1)I_0(\lambda R_2) - 1/(\lambda R_1)\}/(\lambda R_k),$$

$$S' = -M\{K_0(\lambda R_1)I_1(\lambda R_2) + K_1(\lambda R_2)I_0(\lambda R_1) - 1/(\lambda R_2)\}/(\lambda R_k),$$

$$g = I_0(\lambda R_2)K_0(\lambda R_1) - I_0(\lambda R_1)K_0(\lambda R_2), \quad \Omega = \Omega_2R_2/(\Omega_1R_1),$$

$$G = g(R_2^2 - R_1^2) + R_2S' - R_1S.$$

Therefore, the microgyration is

$$\nu(r) = \frac{(\Omega R_2 - R_1)}{G}\{AK_0(\lambda r) - CI_0(\lambda r) + g\}. \tag{3.3.11}$$

And the velocity is

$$v(r) = \frac{M(R_1 - \Omega R_2)}{\lambda R_k G}\{AK_1(\lambda r) + CI_1(\lambda r)\} + \frac{g(\Omega R_2 - R_1)}{G}r + B_4/r. \tag{3.3.12}$$

Equations (3.3.11) and (3.3.12) completely determine the flow profiles for steady, laminar rotational Couette flow of a micropolar fluid for the given boundary conditions (3.3.5).

For the corresponding unsteady, laminar rotational Couette flow problem, the velocity and microgyration fields $v(r,t)$ and $\nu(r,t)$, reduce to $v(r)$ and $\nu(r)$. That is,

$$v(r,t) = v(r) \quad \text{and} \quad \nu(r,t) = \nu(r). \tag{3.3.13}$$

Especially note that when $\lambda^2 = 0$ (i.e. $R_k \to \infty$), one derives $\nu(r) = 0$ and

$$v(r) = \frac{1}{R_2^2 - R_1^2}\left(r(\Omega_2 R_2^2 - \Omega_1 R_1^2) - \frac{R_1^2 R_2^2}{r}(\Omega_2 - \Omega_1)\right).$$

As expected, the fields revert back to those derived for classical incompressible viscous flows.

IV. STABILITY OF A BASIC PLANE COUETTE FLOW

The laminar plane Couette flow elucidated in chapter II, will now be disturbed by the imposition of a disturbance wave. The stability analysis of this chapter follows the procedures of the Stuart energy method (Stuart,1958). Note that this procedure will use the solutions of the linearized theory, which are derived in section IV.1. The solutions sought, in this (linear) case, satisfy the micropolar analog to the Orr-Sommerfeld (MOS-) energy equations, which are also derived.

To study its stability, the basic flow is superimposed with a two-dimensional, finite disturbance. The imposed disturbances, having zero mean, provide homogeneous boundary conditions for the nonlinear equations of motion governing the disturbance flow. (See section IV.2.)

The disturbance energy equations are derived from the disturbance equations in section IV.3. The energy equations invite a physical interpretation of the possible mechanisms involved in the transition from stable to unstable flow. (See section IV.4.) These nonlinear energy equations (hence, the nomenclature of energy method) are then assumed to be solved by wave forms of the same spatial form as the 'marginal' disturbances of the linearized theory, but with unknown amplitude. In fact, the solution to the nonlinear disturbance energy equations are assumed to be separable into a spatial part, which is known from the linearized theory, and a temporal (time) part, which defines the amplitude of the imposed disturbances (at least, near marginal or critical stability).

Since the spatial part of the disturbance is known, ordinary differential equations, describing the disturbance amplitudes, are found from the disturbance energy equations. Such equations are called amplitude equations when micropolar theory is involved, or else Landau equations when classical theory is used. The possible growth, decay, or equilibrium states of these disturbance amplitudes can then provide the stability criteria. For instance, in sections IV.6 and IV.7, we derive the marginal stability surface, and extract a theoretical prediction for the critical non-dimensional numbers, R_c, R_{gc}, and R_{kc}, involved in the stability of plane Couette flows.

In essence, we are re-working the stability problem for plane Couette (parallel) flows, with the enhanced insight permitted by the micropolar theory of fluid dynamics.

IV.1 Linear Stability Analysis

Employment of the Stuart energy method will require the shape of the marginal disturbances of the linearized theory, so that numerical calculations for theoretical predictions can be performed. Thus, the main goal of this section is to derive the MOS-energy equations. The solution to these coupled equations yields the 'shape' of the disturbances that we will be utilizing in later calculations. Enroute, we will also prove, why only considering two-dimensional infinitesimal disturbances, is sufficient to obtain the minimum critical non-dimensional numbers R_c, R_{kc}, and R_{gc}. This proof suggests the feasibility of the two-dimensional nonlinear stability analysis that we will be undertaking in later sections.

From section I.4, the surmised field equations (1.4.1) - (1.4.3), in non-dimensional form, are:

$$\nabla\cdot\underset{\sim}{v} = 0, \tag{4.1.1}$$

$$-\nabla p + \frac{1}{M}\nabla^2\underset{\sim}{v} + \frac{1}{R_k}\nabla\times\underset{\sim}{\nu} = \frac{\partial\underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v}, \tag{4.1.2}$$

$$\frac{1}{R_b}\nabla(\nabla\cdot\underset{\sim}{\nu}) + \frac{1}{R_g}\nabla^2\underset{\sim}{\nu} + \frac{1}{R_k}\nabla\times\underset{\sim}{v} = \frac{2\underset{\sim}{\nu}}{R_k} + j\frac{\partial\underset{\sim}{\nu}}{\partial t} + j\underset{\sim}{v}\cdot\nabla\underset{\sim}{\nu}, \tag{4.1.3}$$

where $\underset{\sim}{v} = (u, v, w)$ and $\underset{\sim}{\nu} = (\zeta, \nu, \eta)$. We are still using a rectangular Cartesian coordinate system, in that, $\underset{\sim}{x} = (x,y,z)$.

In section II.4, the basic flow was derived to be of the form $\underset{\sim}{v} = \bar{u}(z)\hat{i}$ and $\underset{\sim}{\nu} = \bar{\nu}(z)\hat{j}$, where $\hat{i}$ and $\hat{j}$ are respectively, unit vectors along the x- and y-axes of the rectangular Cartesian coordinate system. To study the stability of this flow, we now superimpose a disturbance on the basic flow as follows:

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \bar{u}(z)\hat{i} + \underset{\sim}{v}'(\underset{\sim}{x},t); \quad \underset{\sim}{\nu}(\underset{\sim}{x},t) = \bar{\nu}(z)\hat{j} + \underset{\sim}{\nu}'(\underset{\sim}{x},t);$$

$$p(\underset{\sim}{x},t) = \text{constant} + p'(\underset{\sim}{x},t). \tag{4.1.4}$$

where $\underset{\sim}{v}'$ is the disturbance velocity, $\underset{\sim}{\nu}'$ is the disturbance microgyration, and p' is the disturbance pressure. On substituting these expressions into equations (4.1.1) -(4.1.3), we obtain the equations of motion governing the disturbed flow.

By utilizing the fact that the basic flow already satisfies the equations of motion, we have

$$\nabla \cdot \underset{\sim}{v}' = 0, \tag{4.1.5}$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\,\frac{\partial}{\partial x}\right)\underset{\sim}{v}' + w'\frac{d\bar{u}}{dz}\,\hat{i} + \underset{\sim}{v}'\cdot\nabla\underset{\sim}{v}' = -\,\nabla p' + \frac{1}{M}\,\nabla^2\underset{\sim}{v}' + \frac{1}{R_k}\,\nabla\times\underset{\sim}{\nu}', \tag{4.1.6}$$

and

$$\left(j\frac{\partial}{\partial t} + j\bar{u}\,\frac{\partial}{\partial x} + \frac{2}{R_k}\right)\underset{\sim}{\nu}' + jw'\frac{d\bar{\nu}}{dz}\,\hat{j} + j\underset{\sim}{v}'\cdot\nabla\underset{\sim}{\nu}' = \frac{1}{R_b}\,\nabla(\nabla\cdot\underset{\sim}{\nu}') + $$

$$+ \frac{1}{R_g}\,\nabla^2\underset{\sim}{\nu}' + \frac{1}{R_k}\,\nabla\times\underset{\sim}{v}'. \tag{4.1.7}$$

By neglecting the quadratic terms $\underset{\sim}{v}'\cdot\nabla\underset{\sim}{v}'$ and $\underset{\sim}{v}'\cdot\nabla\underset{\sim}{\nu}'$ (or equivalently, assuming the disturbances are infinitesimal), we obtain the <u>linearized</u> equations of motion governing the disturbed flow. Since the coefficients of $\underset{\sim}{\nu}'$ and $\underset{\sim}{v}'$, in the linearized equations, depend only on z, the equations admit solutions which depend on x, y, and t exponentially. Consider therefore solutions of the form

$$\begin{aligned} \underset{\sim}{v}'(\underset{\sim}{x},t) &= \hat{\underset{\sim}{v}}(z)\ \exp\Big(i(bx + ay - bct)\Big)\ ; \\ \underset{\sim}{\nu}'(\underset{\sim}{x},t) &= \hat{\underset{\sim}{\nu}}(z)\ \exp\Big(i(bx + ay - bct)\Big)\ ; \\ p'(\underset{\sim}{x},t) &= \hat{p}(z)\ \exp\Big(i(bx + ay - bct)\Big)\ . \end{aligned} \tag{4.1.8}$$

The real parts of the expressions are to be taken to obtain physical quantities. Requiring that the solutions remain bounded as $x,y \to \pm\infty$ implies that the wavenumbers b and a must be real. The wave speed c may be complex, i.e. $c = c_r + ic_i$. The expressions thus represent waves which travel in the direction (b,a,0), with wave speed $bc_r/(b^2 + a^2)^{\frac{1}{2}}$, and which grow or decay like $\exp(bc_i t)$. Note that a wave is said to be (asymptotically) stable if $bc_i \leqq 0$, unstable if $bc_i > 0$, and neutrally stable if $bc_i = 0$. Marginal stability occurs if $bc_i = 0$ for critical values of the parameters (e.g. R, R_k, and R_g) on which the 'eigenvalue' c depends, but $bc_i > 0$ for some neighboring values of the parameters.

The ratios of the parameters R, R_g, and R_k for marginal stability are found in section IV.6 and IV.7, which can thus give some criteria for stability. The <u>critical</u> relationship between the parameters, when discovered, yields the marginal stability surface. Note that neutral stability is not necessarily marginal stability. For comparison, note that on a neutral atability surface, $bc_i = 0$, but bc_i is not necessarily positive for any neighboring values of the parameters. The minimum values of R, R_k, and R_g on the marginal stability surface are called the critical numbers R_c, R_{kc}, and R_{gc}; hence, there is flow instability for any $R > R_c$, $R_k > R_{kc}$, or $R_g > R_{gc}$.

For completeness, we mention that if $bc_r \neq 0$ as bc_i approaches zero from above for a disturbance, oscillatory instability sets in. This is sometimes called overstability. Also, if $bc = 0$ at marginal stability, i.e. $bc_i = 0 = bc_r$, then there is said to be an 'exchange of stabilities', whereby instability sets in as a steady secondary flow, such as in the case of the convection cells that arise when a fluid is heated from below (Perez-Garcia & Rubi,1982).

If we now let $D = d/dz$, then on substituting the expressions (4.1.8) into the <u>linearized</u> equations (4.1.5) - (4.1.7), we obtain the following (coupled) system of ordinary differential equations:

$$i(b\hat{u} + a\hat{v}) + D\hat{w} = 0, \tag{4.1.9}$$

$$\left(D^2 - (b^2 + a^2) - ibM(\bar{u} - c)\right) \hat{u} = M\bar{u}'\hat{w} + (D\hat{\nu} - ia\hat{\eta})M/R_k + ibM\hat{p}, \tag{4.1.10}$$

$$\left(D^2 - (b^2 + a^2) - ibM(\bar{u} - c)\right) \hat{v} = iaM\hat{p} - (D\hat{\zeta} - ib\hat{\eta})M/R_k, \tag{4.1.11}$$

$$\left(D^2 - (b^2 + a^2) - ibM(\bar{u} - c)\right) \hat{w} = MD\hat{p} - (ib\hat{\nu} - ia\hat{\zeta})M/R_k, \tag{4.1.12}$$

$$\left(D^2 - (b^2 + a^2) - ibjR_g(\bar{u} - c) - 2R_g/R_k\right)\hat{\zeta} = (b^2\hat{\zeta} + ba\hat{\nu} - ibD\hat{\eta})R_g/R_b + + (D\hat{v} - ia\hat{w})R_g/R_k, \tag{4.1.13}$$

$$\left(D^2 - (b^2 + a^2) - ibjR_g(\bar{u} - c) - 2R_g/R_k\right)\hat{\nu} = (ba\hat{\zeta} + a^2\hat{\nu} - iaD\hat{\eta})R_g/R_b + + jR_g\bar{\nu}'\hat{w} - (D\hat{u} - ib\hat{w})R_g/R_k, \tag{4.1.14}$$

$$\left(D^2 - (b^2 + a^2) - ibjR_g(\bar{u} - c) - 2R_g/R_k\right)\hat{\eta} = - i\left(D(b\hat{\zeta} + a\hat{\nu} + D\hat{\eta})\right) R_g/R_b + (ia\hat{u} - ib\hat{v})R_g/R_k. \tag{4.1.15}$$

Here primes denote differentiation with respect to z.

The strict adherence boundary conditions, applied to the disturbance flow, imply

$$\hat{u} = \hat{v} = \hat{w} = 0 = \hat{\zeta} = \hat{\nu} = \hat{\eta} \quad \text{at} \quad z = \pm 1. \tag{4.1.16}$$

The three-dimensional problem defined by equations (4.1.9) - (4.1.16) can be reduced to an (almost) equivalent two-dimensional problem by the use of the micropolar analog of the Squire transformation.

Let

$$B = (b^2 + a^2)^{\frac{1}{2}}, \quad BV = b\hat{u} + a\hat{v}, \quad P/B = \hat{p}/b, \quad W = \hat{w},$$

$$C = c, \text{ and } B\tilde{M} = bM \quad \text{(as used in classical viscous theory).} \tag{4.1.17}$$

Also, let

$b\hat{\nu} - a\hat{\zeta} = BN$, $B\tilde{R}_k = bR_k$, and $B\tilde{R}_g = bR_g$ (as is now needed for micropolar theory).

By introducing the relations (4.1.17) into equations (4.1.9) - (4.1.16), and after some simple manipulations, we obtain

$$iBV + DW = 0, \tag{4.1.18}$$

$$\left(D^2 - B^2 - iB\tilde{M}(\bar{u} - C)\right)V = \tilde{M}\bar{u}'W + \tilde{M}DN/\tilde{R}_k + iB\tilde{M}P, \tag{4.1.19}$$

$$\left(D^2 - B^2 - iB\tilde{M}(\bar{u} - C)\right)W = \tilde{M}DP - (iBN)\tilde{M}/\tilde{R}_k, \tag{4.1.20}$$

$$\left(D^2 - B^2 - iB\tilde{R}_g(\bar{u} - C) - 2\tilde{R}_g/\tilde{R}_k\right)N = -\tilde{R}_g(DV - iBW)/\tilde{R}_k + j\tilde{R}_g\bar{\nu}'W, \tag{4.1.21}$$

with boundary conditions

$$V = W = 0 = N \quad \text{at} \quad z = \pm 1. \tag{4.1.22}$$

Equations (4.1.18) - (4.1.22) have the same mathematical structure as equations (4.1.9) - (4.1.16) with $a = \hat{v} = 0 = \hat{\zeta} = \hat{\eta}$, and they thus define the equivalent two-dimensional linear problem. Note, however, that transforms for R_b and $\hat{\eta}$ were not required in the derivation, and hence are still arbitrary. (Moreover, equation (4.1.15) implies the trivial equality $0 = 0$.)

Since $B \geqq b$, it immediately follows that $\tilde{M} \leqq M$, $\tilde{R}_k \leqq R_k$, and $\tilde{R}_g \leqq R_g$. Since by definition $1/M = 1/R + 1/R_k$, $\tilde{R} \leqq R$ also follows. (Recall that the tilda $\sim$ terms stem from the two-dimensional problem.)

We have thus proved, for infinitesimal disturbances, the following

Theorem. To obtain minimum critical nondimensional numbers R_c, R_{kc}, and R_{gc}, it is sufficient to consider only two-dimensional disturbances.

(For a rendition of Squire's theorem for classical viscous theory, refer to Drazin and Reid, Hydrodynamic Stability, p. 155.)

Restricting now to two-dimensional disturbances in the xz-plane, we introduce the disturbance stream function

$$\psi'(x,z,t) = \phi(z)\,\exp\bigl(ib(x - ct)\bigr)\,.$$

Thus, the two components of the disturbance velocity will be

$$u' = \partial\psi'/\partial z \quad \text{and} \quad w' = -\,\partial\psi'/\partial x.$$

Furthermore,

$$\hat{u} = d\phi/dz \quad \text{and} \quad \hat{w} = -\,ib\phi. \tag{4.1.23}$$

Notice that $\phi(z)$ is a complex-valued function, in that, $\phi(z) = \phi_r(z) + i\phi_i(z)$ where ϕ_r is the real part of $\phi(z)$ and ϕ_i is the imaginary part of $\phi(z)$.

From equations (4.1.10) - (4.1.16) with $a = \hat{v} = 0 = \hat{\zeta} = \hat{\eta}$, we now have

$$\left(D^2 - b^2 - ibM(\bar{u} - c)\right)\hat{u} = M\bar{u}'\hat{w} + MD\hat{\nu}/R_k + ibM\hat{p}, \tag{4.1.24}$$

$$\left(D^2 - b^2 - ibM(\bar{u} - c)\right)\hat{w} = MD\hat{p} - M(ib\hat{\nu})/R_k, \tag{4.1.25}$$

$$\left(D^2 - b^2 - ibjR_g(\bar{u} - c) - 2R_g/R_k\right)\hat{\nu} = -\,R_g(D\hat{u} - ib\hat{w})/R_k + jR_g\bar{\nu}'\hat{w}, \tag{4.1.26}$$

with boundary conditions

$$\hat{u} = \hat{w} = 0 = \hat{\nu} \quad \text{at} \quad z = \pm 1. \tag{4.1.27}$$

Remember, in these equations, primes denote differentiation with respect to z, $i = \sqrt{-1}$, and the scalar j is the microinertia.

By inserting (4.1.23) into the three equations (4.1.24-26), we reduce the unknowns to three: ϕ, $\hat{\nu}$, and $\hat{p}$. The resulting equations are:

$$\{D^2 - b^2 - ibjR_g(\bar{u} - c) - 2R_g/R_k\}\hat{\nu} = ibjR_g\bar{\nu}'\phi - R_g(\phi'' - b^2\phi)/R_k , \tag{4.1.28}$$

$$\{D^2 - b^2 - ibM(\bar{u} - c)\}(-ib\phi) = MD\hat{p} - M(ib\hat{\nu})/R_k , \tag{4.1.29}$$

$$\{D^2 - b^2 - ibM(\bar{u} - c)\}\phi' = M\bar{u}'(-ib\phi) + MD\hat{\nu}/R_k + ibM\hat{p}. \tag{4.1.30}$$

Now, eliminating $\hat{p}$ from the above equations, and effecting some rearrangement, gives the micropolar analog to the Orr-Sommerfeld (MOS-) energy equations. The result is

$$\frac{(D^2 - b^2)^2}{ibM}\phi = (\bar{u} - c)(D^2 - b^2)\phi - \bar{u}''\phi + \frac{(D^2 - b^2)\hat{\nu}}{ibR_k} \tag{4.1.31}$$

and

$$\frac{(D^2 - b^2)\hat{\nu}}{ibR_g} + \frac{(D^2 - b^2)\phi}{ibR_k} = j(\bar{u} - c)\hat{\nu} + \frac{2\hat{\nu}}{ibR_k} + j\bar{\nu}'\phi \tag{4.1.32}$$

with boundary conditions

$$D\phi = b\phi = 0 = \hat{\nu} \quad \text{at } z = \pm 1. \tag{4.1.33}$$

The unknowns ϕ and $\hat{\nu}$ governed by (4.1.31-32) will be required in later calculations, as disturbance shapes. Also, in the calculations, $\bar{u}$ and $\bar{\nu}$ are taken as (2.3.14) and (2.3.13), respectively. Note, however, that in the nonlinear theory, the mean velocity $\bar{u}$ and the mean micogyration $\bar{\nu}$ are different from those of laminar flow (as taken above) beacause of the interactions between the mean flow and the disturbances.

Next, we study, with the nonlinear theory, the stability of a basic flow, which is disturbed by a two-dimensional, finite disturbance.

IV.2 Imposition of Finite Disturbances on a Basic Flow

Two-dimensional plane Couette flow refers to a plane Couette flow that is theoretically restricted to two spatial variables. We will be analyzing the stability of a basic plane Couette flow, that is disturbed by a two-dimensional, finite disturbance in the xz-plane. Moreover, since the channel is assumed to extend to infinity in both the positive and negative y-directions, the velocity and microgyration fields will be independent of the y-coordinate.

Restricting to two-dimensional flows, the continuity equation $\partial u/\partial x + \partial w/\partial z = 0$ defines a stream function for the fluid flow. This stream function is decomposed into the sum of two functions, one representing the mean flow, the other representing the finite disturbance flow. The analysis of the nonlinear field equations governing the finite disturbance flow employs procedures embodied in the Stuart energy method, with which we begin.

The energy method to be used here, was established by J. T. Stuart in his fundamental paper published in 1958 (Stuart,1958). The energy method of Stuart is an approximate method which assumes that the spatial form (shape) of the nonlinear disturbances is the same as the shape of marginal disturbances of the linearized theory, but with unknown amplitude. (For additional references and criticism, refer to D. D. Joseph's book, Stability of Fluid Motions I (Joseph, 1976).) This energy method was further developed and applied by Stuart (Stuart,1960) and by J. Watson (Watson,1960). Also, of special note are the elucidations of the Stuart energy method made by A. Davey (Davey,1962).

For flow under no pressure gradient between two parallel plates in constant relative motion (plane Couette flow), we impose a disturbance travelling in the direction of the basic flow, which has the form[6]

$$\psi(x,z,t) = K\psi_1(z)\,\exp\Big(ib(x - ct)\Big) + \tilde{K}\tilde{\psi}_1(z)\,\exp\Big(-ib(x - \tilde{c}t)\Big) \tag{4.2.1}$$

where K is an arbitrary complex scalar, wave speed $c = c_r + ic_i$ $(c_r \geq 0)$, and the tilda $\sim$ denotes a complex conjugate.

Now, let the stream function for the flow be represented by the Fourier series expansion in x (Watson,1960), as

$$\psi(x,z,t) = \bar{\phi} + \phi' =$$

$$= \phi_0(z,t) + \sum_{n=1}^{\infty} \{\phi_n(z,t)\, e^{nibx} + \tilde{\phi}_n(z,t)\, e^{-nibx}\}. \tag{4.2.2}$$

In the linearized theory, $\bar{\phi}$ represents the steady stream function, and ϕ' reduces to (4.2.1) (or equivalently, the function $\psi'(x,z,t)$ used in section IV.1). For the nonlinear theory, the sum on the right represents the finite disturbance ϕ', while $\bar{\phi}$ is the mean stream function, where the mean (average) is taken with respect to x over a disturbance wavelength of $2\pi/b$. b is the period[7] of the disturbance wave.

With the stream function ψ, the Fourier series expansion gives

$$u = \partial\psi/\partial z = \bar{u} + u' =$$

$$= \bar{u}(z,t) + \sum_{n=1}^{\infty} \{u_n'(z,t)\, e^{nibx} + \tilde{u}_n'(z,t)\, e^{-nibx}\};$$

$$w = -\partial\psi/\partial x = w' =$$

$$= \sum_{n=1}^{\infty} \{w_n'(z,t)\, e^{nibx} + \tilde{w}_n'(z,t)\, e^{-nibx}\};$$

$$\nu = \bar{\nu} + \nu' =$$

$$= \bar{\nu}(z,t) + \sum_{n=1}^{\infty} \{\nu_n'(z,t)\, e^{nibx} + \tilde{\nu}_n'(z,t)\, e^{-nibx}\}. \quad (4.2.3)$$

The over-bar now denotes a mean value, and the primed terms denote disturbance flow variables.

Defined for equations (4.2.3) are

$$\bar{u}(z,t) = \partial\phi_0/\partial z; \quad u_n'(z,t) = \partial\phi_n/\partial z; \text{ and } \quad w_n'(z,t) = -nib\phi_n. \quad (n \geq 1) \quad (4.2.4)$$

For the resulting two-dimensional flow, mathematically, we prescribe that the velocity and microgyration vector fields are, respectively,

$$\underset{\sim}{v} = \{u(x,z,t),\ 0,\ w(x,z,t)\} \quad \text{and} \quad \underset{\sim}{\nu} = \{0,\ \nu(x,z,t),\ 0\}. \qquad (4.2.5)$$

Applying the prescription (4.2.5) to the field equations (2.2.1) - (2.2.7) yields:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad (4.2.6)$$

$$-\frac{\partial p}{\partial x} + \frac{1}{M}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{1}{R_k}\frac{\partial \nu}{\partial z} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}, \qquad (4.2.7)$$

$$-\frac{\partial p}{\partial z} + \frac{1}{M}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{1}{R_k}\frac{\partial \nu}{\partial x} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}, \qquad (4.2.8)$$

$$\frac{1}{R_g}\left(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial z^2}\right) + \frac{1}{R_k}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{2\nu}{R_k} + j\left(\frac{\partial \nu}{\partial t} + u\frac{\partial \nu}{\partial x} + w\frac{\partial \nu}{\partial z}\right). \qquad (4.2.9)$$

The conditions to be applied throughout this procedure are:

(C1) that the mean velocity $\bar{u}$ and the mean microgyration $\bar{\nu}$ assume the same values on the plates as do the undisturbed velocity and the undisturbed microgyration;

(C2) that the disturbance velocities u' and w', and the disturbance microgyration ν' vanish on the plates; and

(C3) that there is a suitable condition on the mean pressure gradient in the flow direction or, equivalently on the mean velocity and the mean microgyration.

Consequently, for disturbed plane Couette flow, acceptable non-dimensional boundary conditions are:

$$\bar{u} = 0 \quad \text{at} \quad z = -1, \qquad \bar{u} = 1 \quad \text{at} \quad z = 1, \qquad \bar{\nu} = 0 \quad \text{at} \quad z = \pm 1;$$

$$\partial\phi_n/\partial z = 0 = \phi_n = \nu' \quad \text{at} \quad z = \pm 1. \quad (n = 1,2,3,\ldots) \tag{4.2.10}$$

In order that the pressure gradients shall balance the remaining terms, the pressure must be of the form

$$p = xp^*(t) + p^{**}(z,t) + p'(x,z,t) =$$

$$= xp^*(t) + p^{**}(z,t) + \sum_{n=1}^{\infty} \{p_n'(z,t)e^{nibx} + \tilde{p}_n'(z,t)e^{-nibx}\} \tag{4.2.11}$$

where the part $p^*(t)$ is a purely time-dependent mean pressure term, $p^{**}(z,t)$ is the mean pressure independent of x, and $p'(x,z,t)$ is the disturbance pressure.

Substitute equations (4.2.3) and (4.2.11) into equations (4.2.7) - (4.2.9), and equate the Fourier components. The equations arising from equating the terms independent of x are equivalently found by taking the mean of equations (4.2.7) - (4.2.9). The result is:

$$\bar{u}_{,t} + \overline{u'u'_{,x}} + \overline{w'u'_{,z}} + \bar{\nu}_{,z}/R_k = -p^* + \bar{u}_{,zz}/M \tag{4.2.12}$$

$$\overline{u'w'}_{,x} + \overline{w'w'_{,z}} = -p^{**}_{,z} \tag{4.2.13}$$

$$\bar{u}_{,z}/R_k + \bar{\nu}_{,zz}/R_g = 2\bar{\nu}/R_k + j(\bar{\nu}_{,t} + \overline{u'\nu'_{,x}} + \overline{w'\nu'_{,z}}) \tag{4.2.14}$$

Note that the quantities $\overline{u'^2}$, $\overline{w'^2}$, $\overline{u'w'}$, $\overline{\nu'^2}$, $\overline{u'\nu'}$, and $\overline{w'\nu'}$ are independent of x. Also, note that the continuity equation (4.2.6) implies that

$$u'_{,x} = -w'_{,z}. \tag{4.2.15}$$

The equations governing the disturbance field quantities are now found on subtracting (4.2.12) from (4.2.7), (4.2.13) from (4.2.8), and

(4.2.14) from (4.2.9), to be:

$$\frac{\partial u'}{\partial t} + \bar{u}\,\frac{\partial u'}{\partial x} + w'\,\frac{\partial \bar{u}}{\partial z} + \chi_1 = -\,\frac{\partial p'}{\partial x} + \frac{1}{M}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial z^2}\right) - \frac{1}{R_k}\,\frac{\partial \nu'}{\partial z} \qquad (4.2.16)$$

$$\frac{\partial w'}{\partial t} + \bar{u}\,\frac{\partial w'}{\partial x} + \chi_2 = -\,\frac{\partial p'}{\partial z} + \frac{1}{M}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \frac{1}{R_k}\,\frac{\partial \nu'}{\partial x} \qquad (4.2.17)$$

$$\frac{1}{R_k}\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) + j\left(\frac{\partial \nu'}{\partial t} + \bar{u}\,\frac{\partial \nu'}{\partial x} + w'\,\frac{\partial \bar{\nu}}{\partial z} + \chi_3\right) = \frac{1}{R_g}\left(\frac{\partial^2 \nu'}{\partial x^2} + \frac{\partial^2 \nu'}{\partial z^2}\right) - \frac{2\nu'}{R_k} \qquad (4.2.18)$$

where

$$\chi_1 = u'\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z} - \frac{\partial}{\partial z}(\overline{u'w'}),$$

$$\chi_2 = u'\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z} - \frac{\partial}{\partial z}(\overline{w'^2}),$$

$$\chi_3 = u'\frac{\partial \nu'}{\partial x} + w'\frac{\partial \nu'}{\partial z} - \frac{\partial}{\partial z}(\overline{w'\nu'}).$$

These nonlinear equations will be referred to as the disturbance equations.

As a preview, the terms $\frac{\partial}{\partial z}(\overline{u'w'})$ and $\frac{\partial}{\partial z}(\overline{w'^2})$ are the familiar Reynolds stress terms. The term $j\frac{\partial}{\partial z}(\overline{w'\nu'})$ is a mean couple stress term. These concepts are discussed in section IV.4.

IV.3 Disturbance Energy Equations

In this section we will derive the disturbance energy balance equations for two-dimensional, finite disturbances. By properly preparing the disturbance equations (4.2.16) - (4.2.18), we derive the two-dimensional form of the micropolar analog to the Reynolds-Orr (MRO-) energy equations. Note that since the channel is unbounded in the x-direction, we have assumed the disturbances $\underset{\sim}{u}'$ and $\underset{\sim}{\nu}'$ to be (spatially) periodic in x, and thus, the following integrations with respect to x can be taken over exactly one wavelength.

Begin by multiplying (4.2.16) by u' and (4.2.17) by w'. Add the equations; then utilize simplifications similar to those demonstrated in Appendix A, while integrating dxdz. Thus, the first Disturbance Energy Equation is derived without approximation to be:

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}(u'^2 + w'^2)\,dxdz = \iint (-\overline{u'w'})\frac{\partial \bar{u}}{\partial z}\,dxdz - \frac{1}{M}\iint\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right)^2 dxdz - \frac{1}{R_k}\iint\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right)\nu'\,dxdz \tag{4.3.1}$$

where the integrals are evaluated over a volume bounded by the plates $z = \pm 1$, and by one wavelength $x = 0,\ 2\pi/b$.

Note that in the first integral on the right-hand side of (4.3.1), $u'w'$ was replaced by $\overline{u'w'}$ because only the mean part contributes to the integral (Stuart,1956).

The mean velocity occurring in (4.3.1), is derived from (4.2.12), and is given by

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial z}(\overline{u'w'}) + \frac{1}{R_k}\frac{\partial \bar{\nu}}{\partial z} = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{M}\frac{\partial^2 \bar{u}}{\partial z^2} . \tag{4.3.2}$$

Next, multiply the remaining equation (4.2.18) by ν', simplify, then integrate; thus deriving the companion Disturbance Energy Equation without approximation to be:

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2} j \nu'^2 \, dxdz = \iint (-j\overline{w'\nu'}) \frac{\partial \bar{\nu}}{\partial z} dxdz - \frac{1}{R_g}\iint \left(\left(\frac{\partial \nu'}{\partial x}\right)^2 + \left(\frac{\partial \nu'}{\partial z}\right)^2\right) dxdz -$$

$$- \frac{1}{R_k}\iint \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right)\nu' \, dxdz - \frac{2}{R_k}\iint \nu'^2 \, dxdz \tag{4.3.3}$$

where again the integrals are evaluated over a volume bounded by the plates $z = \pm 1$, and by one wavelength $x = 0, 2\pi/b$. Note that $\nu'w'$ was replaced by $\overline{\nu'w'}$, as was similarly done in (4.3.1).

The mean microgyration occurring in (4.3.3), is derived from (4.2.14), and is given by

$$\frac{1}{R_k}\frac{\partial \bar{u}}{\partial z} + \frac{1}{R_g}\frac{\partial^2 \bar{\nu}}{\partial z^2} = \frac{2\bar{\nu}}{R_k} + j\left(\frac{\partial \bar{\nu}}{\partial t} + \frac{\partial}{\partial z}(\overline{w'\nu'})\right) . \tag{4.3.4}$$

Equations (4.3.1) and (4.3.3) comprise the two-dimensional MRO-energy equations. It can be shown, so far as two-dimensional disturbances are concerned, that the energy integrals of the MOS-energy equations are equivalent to the above two-dimensional form of the MRO-energy equations. (To see this, let $u' = \mathrm{Re}\{(D\phi)\, e^{ib(x-ct)}\}$, $w' = -\mathrm{Re}\{ib\phi e^{ib(x-ct)}\}$, and $\nu' = \mathrm{Re}\{\hat{\nu}(z)\, e^{ib(x-ct)}\}$. Then integrating over one wavelength and dividing by the wavelength (i.e. averaging with respect to x), one derives the imaginary part of the energy integral of the MOS-energy equations.) This is to be expected, since in the derivation of equations (4.3.1) and (4.3.3), the nonlinear terms, χ_1, χ_2, and χ_3, in the disturbance equations disappear in the process of integration.

Following a physical interpretation of the flow mechanisms suggested by equations (4.3.1) and (4.3.3), we will derive the amplitude equations with the Stuart energy method.

IV.4 Physical Interpretation of the MOS-Energy Equations

The goal of this section is to identify and physically interpret the terms appearing in the disturbance energy equations (4.3.1) and (4.3.3). To enrich the physical interpretation, equations (4.3.1) and (4.3.3) are, respectively, re-written in dimensional form and a briefer notation is introduced.

$$\frac{\partial}{\partial t}\iint \frac{\rho}{2}(u'^2 + w'^2)\, dxdz = \iint (-\rho\overline{u'w'})\, \frac{\partial \bar{u}}{\partial z}\, dxdz -$$

$$- (\mu+\kappa)\iint \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right)^2 dxdz - \kappa\iint \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right)\nu'\, dxdz.$$

Briefly,

$$\frac{\partial E}{\partial t} = I_1 - (\mu+\kappa)I_2 + \kappa I_3. \qquad (4.4.1)$$

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}\rho j\nu'^2 \, dxdz = \iint (-\rho j\overline{w'\nu'}) \frac{\partial\bar{\nu}}{\partial z} dxdz - \gamma \iint \left(\left(\frac{\partial\nu'}{\partial x}\right)^2 + \left(\frac{\partial\nu'}{\partial z}\right)^2\right) dxdz -$$

$$- \kappa \iint \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right)\nu' \, dxdz - 2\kappa \iint \nu'^2 \, dxdz.$$

Briefly,

$$\frac{\partial e}{\partial t} = H_1 - \gamma H_2 + \kappa I_3 - 2\kappa H_3. \tag{4.4.2}$$

In equation (4.4.1), the term on the left-hand side gives the rate of growth of the disturbance kinetic energy within the volume considered. On the right-hand side of (4.4.1), the term I_1 is the integral of the product of the Reynolds stress and the mean velocity gradient, and represents the "translational" rate of transfer of kinetic energy from the mean flow to the disturbance. The term $(\mu+\kappa)I_2$ is always positive; so $-(\mu+\kappa)I_2$ represents the rate of $(\mu+\kappa)$-viscous dissipation of the kinetic energy of the disturbance due to translational and rotational effects of the macro-volume elements in the volume considered.

The term κI_3 is the common link between equations (4.4.1) and (4.4.2). The term I_3 is the integral of the dot product of the curl $\underset{\sim}{v}$ and the microgyration, and physically represents the Swirl created by the disturbance. Mathematically,

$$I_3 = \underset{\sim}{\nu}'\cdot(\nabla\times\underset{\sim}{v}') = \begin{vmatrix} 0 & & 0 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u' & 0 & w' \end{vmatrix} = -\nu'\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right).$$

Notice that the scalar triple product $\underset{\sim}{\nu}'\cdot(\nabla\times\underset{\sim}{v}')$ can also be written in two other forms. In that,

$$\underset{\sim}{\nu}'\cdot(\nabla\times\underset{\sim}{v}') = \underset{\sim}{v}'\cdot(\nabla\times\underset{\sim}{\nu}') = \nabla\cdot(\underset{\sim}{v}'\times\underset{\sim}{\nu}') .$$

The third variation is the divergence of the Coriolis acceleration.

The Coriolis acceleration, common to the mechanics of moving coordinate systems, equals $2\underset{\sim}{\nu}'\times\underset{\sim}{v}'$. Note that microgyration is angular velocity. For a fluid motion described by micropolar theory, the Coriolis acceleration, $2\underset{\sim}{\nu}'\times\underset{\sim}{v}'$, represents the resultant from the interaction of the rotation of moving micro-volume elements and the present motion of the ambient macro-volume elements for the existing flow in the volume considered. Thus, a nonzero swirl, i.e. $\underset{\sim}{\nu}'\cdot(\nabla\times\underset{\sim}{v}') = \nabla\cdot(\underset{\sim}{v}'\times\underset{\sim}{\nu}') \neq 0$, acts as a source, if $I_3 > 0$ (or a sink, if $I_3 < 0$), for spreading (or gathering) the energy necessary to create a turbulent flow. Most of all, the swirl is the coupling mechanism between the micro- and macro-continuum volume elements.

In equation (4.4.2), the term on the left-hand side gives the rate of growth of the disturbance Microenergy of Rotation[8] within the volume considered. On the right-hand side of (4.4.2), the term H_1 is the integral of the product of the mean couple stress and the mean microgyration gradient, and represents the "rotational" rate of transfer of microenergy of rotation from the mean (micro)flow to the disturbance. The term γH_2 is always positive; so $-\gamma H_2$ represents the rate of γ-viscous dissipation of the microenergy of rotation of the disturbance due to the translational and rotational effects of the micro-volume elements in the volume considered. The term $2\kappa H_3$ is always positive; so $-2\kappa H_3$ represents the κ-viscous dissipation of the microenergy of rotation of the disturbance due to the rotational effects of the micro-volume elements in the volume considered.

In equation (4.4.1), the nonlinear Reynolds stress term $\rho\overline{u'w'}$ has the units of force per area. In equation (4.4.2), the nonlinear mean couple stress term $\rho j\overline{w'\nu'}$ has the units of force times distance per area. With microspin, $\sigma = j\nu$, the mean couple stress term can also be written as $\rho\overline{w'\sigma'}$. One of the main advantages that microcontinuum mechanics has over classical continuum mechanics, is its recognition of couple stresses (and body couples).

In summary, fluid flow stabilizers are the terms $(\mu+\kappa)I_2$, γH_2, and $2\kappa H_3$ which represent viscous dissipation mechanisms. Fluid flow destabilizers are the terms E, e, I_1, and H_1. Note that listing I_1 and H_1 as destabilizers, presumes these terms to be positive (which may not always be true). The intermediary between stability and instability is the swirl term κI_3.

Combining equations (4.4.1) and (4.4.2), reveals

$$\frac{\partial(E + e)}{\partial t} = I_1 + H_1 - (\mu+\kappa)I_2 - \gamma H_2 - 2\kappa H_3 + 2\kappa I_3. \qquad (4.4.3)$$

Suppose

$$I_1 + H_1 + 2\kappa I_3 > 0.$$

If

$$I_1 + H_1 + 2\kappa I_3 > (\mu+\kappa)I_2 + \gamma H_2 + 2\kappa H_3$$

then

$$\partial(E + e)/\partial t > 0.$$

This means the disturbance energies are growing, and the disturbances are increasing in amplitude (i.e. the flow is becoming unstable).

Conversely, if

$$I_1 + H_1 + 2\kappa I_3 < (\mu+\kappa)I_2 + \gamma H_2 + 2\kappa H_3$$

then

$$\partial(E + e)/\partial t < 0.$$

This means the disturbance energies are decaying, and the disturbances are decreasing in amplitude (i.e. the flow is becoming stable).

Ideally, if

$$I_1 + H_1 + 2\kappa I_3 < 0$$

then

$$\partial(E + e)/\partial t < 0,$$

in that, the flow is becoming stable. Finally, equations (4.3.2) and (4.3.3) show how the distribution of mean velocity and mean microgyration are affected by the viscous stresses[9], pressure gradients, Reynolds stress, and mean couple stress, due to the disturbance. An equilibrium flow is possible if $\bar{u}$ and $\bar{\nu}$ can be so distorted, by the Reynolds stress and the mean couple stress, that

$$I_1 + H_1 + 2\kappa I_3 = (\mu+\kappa) I_2 + \gamma H_2 + 2\kappa H_3,$$

which implies

$$\partial(E + e)/\partial t = 0,$$

in that, equilibrium. The equilibrium state will play a crucial role in the analysis presented in the next sections.

IV.5 Amplitude Equations

Recall, in the Stuart energy method, the stream function for the disturbed flow, ψ, represents a mean flow together with a periodic disturbance consisting of the fundamental harmonic, ϕ_0, with wavelength $2\pi/b$, and higher harmonics, ϕ_1, ϕ_2, ..., having wavenumbers nb ($n \geq 1$), but the same (real) wave velocity, c_r, which is assumed to be independent of time. The amplification or damping of the finite disturbance, and the consequent changes in the mean velocity $\bar{u}$ and the mean microgyration $\bar{\nu}$, are accounted for by the dependence of all the ϕ-functions on time t.

We assume that the higher harmonics ϕ_2, ϕ_3, ... are zero. Furthermore, we assume that the disturbances are under 'supercritical' conditions meaning that the non-dimensional numbers R, R_k, R_g, and R_b are <u>above</u> the value which is critical for the linearized <u>instability</u> theory. (For motivation, see the proof of the micropolar analog to Squire's theorem, given in section IV.1.) Moreover, a disturbance under supercritical conditions amplifies for small amplitudes. A suitable initial condition, therefore, is that the function $\phi_1(z,t)$ shall be an exponentially increasing function of time in the limit as $t \to -\infty$; in fact, ϕ_1 has to be the appropriate function, $\phi(z)\exp(bc_i t)$, where $c_i > 0$, of the linearized instability theory (Stuart, 1960).[12]

Assume disturbances u', w', and ν' are similar in 'shape' to the solution given by an amplitude factor, a(t) or, A(t) in the case of ν'. That is

$$\phi_1(z,t) = a(t)\phi(z) \quad \text{and} \quad \nu'(z,t) = A(t)\hat{\nu}(z).$$

For an equilibrium state, we presume that

$$\partial\bar{\nu}/\partial t = 0 = \partial\bar{u}/\partial t.$$

With this presumption, and assuming constant mean pressure $\bar{p}$, we have equations (4.3.2) and (4.3.4) yielding

$$\frac{d\bar{\nu}}{dz} = \frac{R_k}{M} d^2\bar{u}/dz^2 - R_k \frac{d}{dz}(\overline{u'w'}) \tag{4.5.1}$$

and

$$\frac{d\bar{u}}{dz} = 2\bar{\nu} + R_k j \frac{d}{dz}(\overline{w'\nu'}) - \frac{R_k}{R_g} d^2\bar{\nu}/dz^2. \tag{4.5.2}$$

Integrating (4.5.1), and using (4.5.2), reveals

$$d^2\bar{\nu}/dz^2 - \lambda^2\bar{\nu} = f(z) = R_g j\frac{d}{dz}(\overline{w'\nu'}) - MR_g(\overline{u'w'})/R_k - MR_g K_1/R_k{}^2 \tag{4.5.3}$$

where $f(z)$ represents the non-homogeneous part of (4.5.3), K_1 is an integration constant, and λ^2 is as given in $(2.3.9)_1$.

The homogeneous equation from (4.5.3) is solved by

$$\bar{\nu}_h(z) = K_2 e^{\lambda z} + K_3 e^{-\lambda z} \tag{4.5.4}$$

where K_2 and K_3 are additional integration constants.

Remember, we are seeking $d\bar{\nu}/dz$ and $d\bar{u}/dz$ for the disturbance energy equations (4.3.1) and (4.3.3), so that the amplitude equations can be derived.

Using variation of parameters, we find a particular solution for equation (4.5.3) of the form

$$\bar{\nu}_p(z) = \frac{1}{\lambda}\int_1^z \sinh\big(\lambda(z - s)\big)\ f(s)\ ds. \tag{4.5.5}$$

So, in integral form, the general solution to equation (4.5.3) for the mean microgyration is found to be

$$\bar{\nu}(z) = K_2 e^{\lambda z} + K_3 e^{-\lambda z} + \frac{1}{\lambda}\int_1^z \sinh\big(\lambda(z - s)\big)\ f(s)\ ds. \tag{4.5.6}$$

The mean microgyration gradient is

$$\frac{d\bar{\nu}}{dz} = \lambda(K_2 e^{\lambda z} - K_3 e^{-\lambda z}) + \int_1^z \cosh\big(\lambda(z-s)\big)\, f(s)\ ds. \tag{4.5.7}$$

Next, integrating equation (4.5.2) with respect to z, after having incorporated (4.5.6), gives the mean velocity to be:

$$\bar{u}(z) = \frac{R}{\lambda(R + R_k)} (K_2 e^{\lambda z} - K_3 e^{-\lambda z}) + \frac{2}{\lambda} \int_1^z \int_1^r \sinh(\lambda(z-s)) f(s)\, ds dr +$$

$$+ R_k j(\overline{w'\nu'}) - \frac{R_k}{R_g} \int_1^z \cosh(\lambda(z-s)) f(s)\, ds + K_4 \qquad (4.5.8)$$

where K_4 is another integration constant.

The mean velocity is

$$\frac{d\bar{u}}{dz} = \frac{R}{R + R_k} (K_2 e^{\lambda z} + K_3 e^{-\lambda z}) + \frac{R}{\lambda(R + R_k)} \int_1^z \sinh(\lambda(z-s)) f(s)\, ds +$$

$$+ 2 \int_1^z \int_1^r \cosh(\lambda(z-s)) f(s)\, ds dr + M\overline{u'w'} + MK_1/R_g. \qquad (4.5.9)$$

The strict adherence boundary conditions (4.2.10) are:

$$\bar{\nu}(\pm 1) = 0 = \nu'(\pm 1) = u'(\pm 1) = w'(\pm 1) = \bar{u}(-1) \quad \text{and} \quad \bar{u}(1) = 1. \qquad (4.5.10)$$

As an integrating aid for $f(z)$, it is reasonable to suppose that <u>in the mean</u> $\bar{u}'$ is an odd function, and that $\bar{w}'$ and $\bar{\nu}'$ are even functions. Hence, $\overline{u'w'}$ is an odd function, and $\overline{w'\nu'}$ is an even function. Also, facts like integrating an even function gives an odd function are used when determining the integration constants.

The integration constants K_1, K_2, K_3, and K_4 for equations (4.5.6) and (4.5.8) are found to be:

$$K_1 = \frac{R + 2R_k}{R(\cosh(2\lambda) - 1)} \left(\frac{C}{B} (e^{\lambda} - e^{-3\lambda}) + \frac{I^*}{\lambda} \right), \qquad K_2 = -\frac{C}{B} e^{-2\lambda},$$

$$K_3 = \frac{C}{B}, \quad \text{and} \quad K_4 = 1 + \frac{2RCe^{-\lambda}}{\lambda B(R + R_k)}, \qquad (4.5.11)$$

where

$$C = \frac{(R + R_k)}{(R + 2R_k)} \left(\frac{\cosh(2\lambda) - 1}{\sinh(2\lambda)}\right)\left(1 - \frac{R_k}{R_g} \tanh(\lambda)\ I^*\right) - \frac{I^*}{\lambda} ,$$

$$B = \frac{2\left(\cosh(2\lambda) - 1\right)}{\lambda(R+2R_k)\sinh(2\lambda)} \{\lambda(R+2R_k)e^{-\lambda}\left(\cosh(2\lambda) + 1\right) + R\left(\cosh(2\lambda) - 1\right)\},$$

$$I^* = \cosh(\lambda) \int_1^{-1} \{R_g j\lambda\cosh(\lambda s)\ \overline{w'v'} + \frac{MR_g}{R_k} \sinh(\lambda s)\ \overline{u'w'}\ \}\ ds.$$

Incorporating the expressions (4.5.11), equations (4.5.7) and (4.5.9) become

$$\frac{d\bar{v}}{dz} = - \frac{\lambda C}{B} \{e^{\lambda(z-2)} + e^{-\lambda z}\} - \frac{MR_g}{\lambda R_k^2} K_1 \{\cosh\left(\lambda(z-1)\right) - 1\} + \\ + \int_1^z \cosh\left(\lambda(z-s)\right) \{R_g j \frac{d}{ds}(\overline{w'v'}) - \frac{MR_g}{R_k} (\overline{u'w'})\}\ ds, \qquad (4.5.12)$$

and

$$\frac{d\bar{u}}{dz} = \frac{RC}{B(R+R_k)} \{e^{-\lambda z} - e^{\lambda(z-2)}\} + \frac{MK_1}{R+2R_k} \{\cosh\left(\lambda(z-1)\right) - 1\} + \\ + \frac{R}{\lambda(R+R_k)} \int_1^z \cosh\left(\lambda(z-s)\right) \{R_g j \frac{d}{ds}(\overline{w'v'}) - \frac{MR_g}{R_k} (\overline{u'w'})\}\ ds + \\ + M\overline{u'w'} - \frac{MR_g K_1}{\lambda R_k^2} (z - 1) \cosh\left(\lambda(z-1)\right) + \frac{MK_1}{R_g} + \\ + 2\int_1^z \int_1^r \cosh\left(\lambda(z-s)\right) \{R_g j \frac{d}{ds}(\overline{w'v'}) - \frac{MR_g}{R_k}(\overline{u'w'})\}\ dsdr. \qquad (4.5.13)$$

If this were classical viscous theory, the amplitude equations, that we derive in this section, would be referred to as a Landau equation. We would then comment on the Landau equation as an appropriate description of the nonlinear self-interaction of the most unstable mode (stemming from normal mode analysis) when slightly supercritical. We assume that this single, weakly unstable mode and its lower harmonics (e.g. ϕ_0 and ϕ_1) dominate the flow.

The derivation takes the MOS-energy equations, and substitutes a solution with the same spatial form as the solution of the linearized problem. Thus, we make what is called the 'shape assumption', namely that the finite disturbances (e.g. u', w', and ν') have the same spatial structure as the linear ones, although their amplitudes (e.g. a(t) and A(t)) may differ. This approximation serves to give a simplified derivation of the amplitude equations by neglect of the harmonics and neglect of the distortion of the fundamental ϕ_0. It is a good approximation only if the total nonlinear effect is nearly the same as that due only to the distortion of the mean flow.

Similar computations, to those needed in deriving equations (4.5.15) and (4.5.16), are presented in Appendix B.

The amplitude equation for equation (4.3.1), incorporating (4.5.13), is found to be

$$\gamma_1 \frac{da^2}{dt} = -\gamma_2 a^2 - \gamma_3 a^3 A - \gamma_4 a^4 - \gamma_5 a^3 A - \gamma_6 a^4 - \frac{\gamma_7}{M} a^2 - \frac{\gamma_8}{R_k} aA, \qquad (4.5.14)$$

where

$$\gamma_1 = \int_{-1}^{1} \{ |\phi'|^2 + b^2|\phi|^2 \} dz; \qquad \gamma_2 = 2b\int_{-1}^{1} F_1(z)\{\phi_r'\phi_i - \phi_r\phi_i'\} dz,$$

for
$$F_1(z) = \frac{RC}{B(R+R_k)} \{e^{-\lambda z} - e^{\lambda(z-2)}\} + \frac{MK_1}{R+2R_k} \{\cosh(\lambda(z-1)) - 1\} - \frac{MR_g K_1}{\lambda R_k^2} (z-1) \cosh(\lambda(z-1)) + \frac{MK_1}{R_g};$$

$$\gamma_3 = 4b^2 \int_{-1}^{1} (\phi_r'\phi_i - \phi_i'\phi_r) \int_{1}^{z} G_1 \frac{d}{ds}(\phi_r\Phi_i - \Phi_r\phi_i) \, ds dz,$$

for $G_1 = \frac{RR_g j}{\lambda(R+R_k)} \cosh(\lambda(z-s))$;

$$\gamma_4 = 4b^2 \int_{-1}^{1} (\phi_r'\phi_i - \phi_i'\phi_r) \int_1^z G_2 (\phi_r'\phi_i - \phi_i'\phi_r)\, dsdz,$$

for $G_2 = -\frac{RMR_g}{\lambda R_k(R+R_k)} \cosh(\lambda(z-s))$;

$$\gamma_5 = 4b^2 \int_{-1}^{1} (\phi_r'\phi_i - \phi_i'\phi_r) \int_1^z\int_1^r G_3 \frac{d}{ds}(\phi_r\Phi_i - \Phi_r\phi_i)\, dsdrdz,$$

for $G_3 = 2R_g j \cosh(\lambda(z-s))$;

$$\gamma_6 = 4b^2 \int_{-1}^{1} (\phi_r'\phi_i - \phi_i'\phi_r) \int_1^z\int_1^r G_4 (\phi_r'\phi_i - \phi_i'\phi_r)\, dsdrdz,$$

for $G_4 = -2\frac{MR_g}{R_k} \cosh(\lambda(z-s))$;

$$\gamma_7 = 2\int_{-1}^{1} \{|\phi''|^2 + 2b^2|\phi'|^2 + b^4|\phi|^2\}\, dz;$$

$$\gamma_8 = 2\int_{-1}^{1} \{b^2(\phi_r\Phi_r + \phi_i\Phi_i) - (\phi_r''\Phi_r + \phi_i''\Phi_i)\}\, dz.$$

Primes indicate differentiation with respect to z.

We recall that, by solving equations (4.1.31-32), the unknowns ϕ and $\hat{\nu}$ are determined. In the above, $\phi_1 = \phi_r + i\phi_i = \phi$ and $\hat{\nu} = \Phi = \Phi_r + i\Phi_i$. Also, we applied the shape assumption when we utilized the expressions

$$u'(z,t) = a(t)\frac{d\phi}{dz}, \quad w'(z,t) = -ib\phi(z)\, a(t), \quad \nu'(z,t) = A(t)\,\Phi(z).$$

Continuing, the amplitude equation corresponding to equation (4.5.3), incorporating (4.5.12), is found to be

$$\delta_1 \frac{dA^2}{dt} = - \delta_2 aA - \delta_3 a^2A^2 - \delta_4 a^3A - \frac{\delta_5}{R_g} A^2 - \frac{\gamma_8}{R_k} aA - \frac{2}{R_k} \delta_6 A^2, \qquad (4.5.15)$$

where

$$\delta_1 = j \int_{-1}^{1} |\Phi|^2 \, dz; \qquad \delta_2 = 2bj \int_{-1}^{1} F_2(z)\{\Phi_r\phi_i - \phi_r\Phi_i\} \, dz,$$

for

$$F_2(z) = - \frac{\lambda C}{B} \{e^{\lambda(z-2)} + e^{-\lambda z}\} - \frac{MR_gK_1}{\lambda R_k{}^2} \{\cosh(\lambda(z-1)) - 1\};$$

$$\delta_3 = 4b^2 \int_{-1}^{1} j(\phi_r\Phi_i - \Phi_r\phi_i) \int_{1}^{z} G_5 \frac{d}{ds} (\phi_r\Phi_i - \Phi_r\phi_i) \, dsdz,$$

for

$$G_5 = R_g j \cosh(\lambda(z-s));$$

$$\delta_4 = 4b^2 \int_{-1}^{1} j(\phi_r\Phi_i - \Phi_r\phi_i) \int_{1}^{z} G_6 (\phi'_r\phi_i - \phi'_i\phi_r) \, dsdz,$$

for

$$G_6 = - \frac{MR_g}{R_k} \cosh(\lambda(z-s));$$

$$\delta_5 = 2 \int_{-1}^{1} \{ |\Phi'|^2 + b^2|\Phi|^2 \} \, dz; \text{ and } \quad \delta_6 = 2 \int_{-1}^{1} |\Phi|^2 \, dz = 2\delta_1/j.$$

We have thus derived the amplitude equations (4.5.14-15) for finite disturbances imposed on a basic plane Couette flow between two parallel plates.

Next, we examine the disturbance amplitudes at the threshold between stability and instability.

IV.6 Criticality

The threshold between stability and instability is criticality. For the amplitude equations, criticality implies that the magnitude of all the disturbance amplitudes are not changing as time changes. Mathematically, such a state of equilibrium implies that

$$\frac{da}{dt} = 0 = \frac{dA}{dt} \, . \tag{4.6.1}$$

Additionally, we assume that $a^4 = 0 = a^3A = a^2A^2$; in that, a^2 and A^2 are much greater than a^4, a^3A, and a^2A^2.

Hence, at critical stability (criticality), the amplitude equations (4.5.14) and (4.5.15) yield

$$0 = \gamma_{2c}a^2 + \frac{\gamma_7}{M_c} a^2 + \frac{\gamma_8}{R_{kc}} aA, \tag{4.6.2}$$

and

$$0 = \delta_{2c}aA + \frac{\delta_5}{R_{gc}} A^2 + 2\frac{\delta_6}{R_{kc}} A^2 + \frac{\gamma_8}{R_{kc}} aA. \tag{4.6.3}$$

The 'c' affixed to nondimensional numbers indicates a 'critical value'. Note that γ_{2c} and δ_{2c} contain critical numbers.[13]

Integrating relation (4.6.1) suggests that $a = mA$, in that, these two disturbance amplitudes are multiples of each other at criticality. For instance, $m = a(0)/A(0)$. Remember that initial conditions are plausible since disturbances are under supercritical conditions. In particular, if we select $m = 1$, then the two disturbance amplitudes are initially of equal magnitude. Then equations (4.6.2) and (4.6.3) become, respectively,

$$0 = \gamma_{2c} + \gamma_7/M_c + \gamma_8/R_{kc}, \tag{4.6.4}$$

and

$$0 = \delta_{2c} + \delta_5/R_{gc} + \gamma_8/R_{kc} + 2\delta_6/R_{kc}. \tag{4.6.5}$$

We have discovered, with some approximation, the critical relationship between the parameters R, R_g, and R_k ! This critical relationship is defined by equations (4.6.4-5), and thus yields the marginal stability surface, $S_m = S_m(b,R,R_g,R_k,c)$ where the 'eigenvalue' c is the wavespeed with restrictions imposed on it by the assumed supercritical conditions[10].

The marginal stability surface is

$$S_m = \delta_{2c} - \gamma_{2c} + \delta_5/R_{gc} + 2\delta_6/R_{kc} - \gamma_7/M_c = 0. \tag{4.6.6}$$

The graph of S_m would indicate, at a glance, the combination of parameters R, R_g, and R_k that lead to a stable flow, an unstable flow, or a flow in equilibrium. Since the graph of S_m is a hypersurface, only traces of S_m can be plotted.

Before graphing S_m, we should decompose γ_{2c} and δ_{2c}, with the intention of recovering the critical numbers that these relations contain. The major difficulty is liberating λ from the exponential and hyperbolic functions, while maintaining the existing integrity of the integrations.

We have three options. First, we could empirically estimate the probable expressions for γ_{2c} and δ_{2c}, and then try to write a costly algorithm to generate the traces of S_m. Second, we could linearize

$F_1(z)$ and $F_2(z)$, thereby extracting λ from the exponential and hyperbolic functions. The third option is to find another option, like the one we will pursue in the next section.

The second option is known as the narrow gap approximation. Mathematically, this approximation means that z (for us, λz) is assumed small. Employing this approximation at this stage of the analysis is burdened by the difficulty of knowing how to express such constants, as $\cosh(\lambda)$, linearly. To ease this burden, the narrow gap approximation should first be utilized when equation (4.5.4) is invoked into the "nonlinear" analysis; that is, linearize $\bar{\nu}$ and $\bar{u}$.

IV.7 Determination of the Constant, λ

In section II.3, equation (2.3.14) describing the velocity field for steady, laminar plane Couette flow was derived, in accordance with the assumptions of section I.4 and boundary conditions (2.3.5), to be:

$$u(z) = \frac{\sinh(\lambda z)/\cosh\lambda - 2\lambda\Lambda z}{2\tanh\lambda - 4\lambda\Lambda} + \frac{1}{2} \qquad (4.7.1)$$

where

$$\Lambda = R_k/M = 1 + R_k/R > 1. \qquad (4.7.2)$$

Similarly, equation (2.3.13) describing the microgyration field for steady, laminar plane Couette flow was derived to be:

$$\nu(z) = \frac{\lambda\Lambda\{\cosh(\lambda z)/\cosh\lambda - 1\}}{2\tanh\lambda - 4\lambda\Lambda}. \qquad (4.7.3)$$

We notice that

$$\lambda^2 = \frac{R_g(R + 2R_k)}{R_k(R + R_k)} = \frac{\Gamma(2\Lambda - 1)}{\Lambda} \qquad (4.7.4)$$

where

$$\Gamma = R_g/R_k > 0. \qquad (4.7.5)$$

So, $\lambda = \lambda(\Gamma,\Lambda)$ is a function of the two ratios, Γ and Λ. If any two of the triple, Γ, Λ, and λ, is known, the other can be determined.

We rely, as one ultimately must, on experimental data to dictate the value of λ for the fluid flowing between the experimenter's parallel plates.

To illustrate the selection of λ, and to demonstrate the range of velocity and microgyration, as λ varies, for fixed Λ and z, we present Table 4.1. Input values for the calculations are z, Λ, and λ. Output values for u(z) and $\nu(z)$ are calculated from equations (4.7.1) and (4.7.3), respectively. Also, $(4.7.4)_2$ allows us to determine Γ from the input values.

As the tabulations in Table 4.1 indicates, the values of u(z) and $\nu(z)$ vary slightly for $\lambda > 10$ (at a fixed z). Furthermore, increasing Λ tends to promote a more rapid convergence to velocity (and microgyration) values that we would expect from classical viscous theory. Notice, also, the lower values, at $z = 0.99$, for the microgyration as it complies with the boundary condition of $\nu(1) = 0$.

Once λ is determined, useful ratios of the nondimensional numbers can be calculated. From the expression

$$\lambda^2 = R_g(2R_k - M)/R_k^2 \qquad (4.7.6)$$

we get

$$\Gamma = R_g/R_k = \lambda^2/(2 - M/R_k). \qquad (4.7.7)$$

Table 4.1. Velocity and microgyration for various λ.

Γ	λ	u(.25)	ν(.25)	u(.50)	ν(.50)	u(.75)	ν(.75)	ν(.99)
*** Λ = 2 ***								
7 E-5	0.01	0.62500	0.00002	0.75000	0.00001	0.87500	0.00001	3 E-7
7 E-3	0.10	0.62506	0.00155	0.75010	0.00124	0.87509	0.00073	0.00003
0.667	1	0.62912	0.10239	0.75665	0.08314	0.88091	0.04971	0.00234
16.67	5	0.63101	0.25646	0.76101	0.24141	0.88720	0.18772	0.01283
66.67	10	0.62820	0.25627	0.75632	0.25468	0.88356	0.23536	0.02440
267	20	0.62658	0.25316	0.75316	0.25315	0.87970	0.25146	0.04589
600	30	0.62605	0.25210	0.75210	0.25210	0.87815	0.25196	0.06534
1667	50	0.62563	0.25126	0.75126	0.25126	0.87688	0.25126	0.09886
4267	80	0.62539	0.25078	0.75078	0.25078	0.87618	0.25078	0.13810
8067	110	0.62528	0.25057	0.75057	0.25057	0.87585	0.25057	0.16716
15000	150	0.62521	0.25042	0.75042	0.25042	0.87563	0.25042	0.19454
*** Λ = 10 ***								
5 E-5	0.01	0.62500	0.00001	0.75000	0.00001	0.87500	0.00001	3 E-7
5 E-3	0.10	0.62501	0.00123	0.75002	0.00098	0.87501	0.00057	0.00003
0.526	1	0.62569	0.08618	0.75112	0.06997	0.87600	0.04184	0.00197
13.2	5	0.62615	0.24610	0.75211	0.23166	0.87837	0.18014	0.01231
52.6	10	0.62563	0.25112	0.75124	0.24956	0.87668	0.23063	0.02391
211	20	0.62531	0.25063	0.75063	0.25062	0.87584	0.24894	0.04543
474	30	0.62521	0.25042	0.75041	0.25042	0.87563	0.25028	0.06490
1316	50	0.62513	0.25025	0.75025	0.25025	0.87538	0.25025	0.09847
3368	80	0.62508	0.25016	0.75016	0.25016	0.87523	0.25016	0.13775
6368	110	0.62506	0.25011	0.75011	0.25011	0.87517	0.25011	0.16686
11842	150	0.62504	0.25008	0.75008	0.25008	0.87513	0.25008	0.19428
*** Λ = 100 ***								
5 E-5	0.01	0.62500	0.00001	0.75000	0.00001	0.87500	0.00001	3 E-7
0.005	0.10	0.62500	0.00117	0.75000	0.00094	0.87500	0.00055	0.00002
0.503	1	0.62507	0.08321	0.75011	0.06757	0.87510	0.04040	0.00190
12.56	5	0.62511	0.24388	0.75021	0.22957	0.87523	0.17852	0.01220
50.25	10	0.62506	0.24999	0.75012	0.24844	0.87517	0.22959	0.02380
201.0	20	0.62503	0.25006	0.75006	0.25005	0.87509	0.24838	0.04533
452.2	30	0.62502	0.25004	0.75004	0.25004	0.87506	0.24990	0.06481
1256	50	0.62501	0.25003	0.75003	0.25003	0.87504	0.25002	0.09838
3216	80	0.62501	0.25002	0.75002	0.25002	0.87502	0.25002	0.13768
6080	110	0.62501	0.25001	0.75001	0.25001	0.87502	0.25001	0.16679
11307	150	0.62500	0.25001	0.75001	0.25001	0.87501	0.25001	0.19422
Γ	λ	u(.25)	ν(.25)	u(.50)	ν(.50)	u(.75)	ν(.75)	ν(.99)

Then

$$R_g/R = (R_g/R_k)(R_k/M) - R_g/R_k \tag{4.7.8}$$

and

$$R/R_k = (R_g/R_k)(R/R_g). \tag{4.7.9}$$

The numbers, that are the ratios (4.7.7-9), still apply at criticality.

The values of the critical non-dimensional numbers R_c, R_{gc}, and R_{kc}, involved in the stability of plane Couette flows, can now be theoretically predicted. From the marginal stability surface (4.6.6), the relation $R_{kc}S_m = 0$ yields

$$R_{kc} = \{-(R_{kc}/R_{gc})\delta_5 - 2\delta_6 + (R_{kc}/M_c)\gamma_7\}/(\delta_{2c} - \gamma_{2c}). \tag{4.7.10}$$

Similarly,

$$R_{gc} = \{-\delta_5 - 2(R_{gc}/R_{kc})\delta_6 + (R_{gc}/M_c)\gamma_7\}/(\delta_{2c} - \gamma_{2c}), \tag{4.7.11}$$

$$M_c = \{-(M_c/R_{gc})\delta_5 - 2(M_c/R_{kc})\delta_6 + \gamma_7\}/(\delta_{2c} - \gamma_{2c}), \tag{4.7.12}$$

$$R_c = M_cR_{kc}/(R_{kc} - M_c). \tag{4.7.13}$$

Recall that the difference, $\delta_{2c} - \gamma_{2c}$, contains additional critical numbers. Consequently, the relations (4.7.10-13) only implicitly establish values for R_c, R_{gc}, and R_{kc}. Or so it seems. With the ratios of the nondimensional numbers (4.7.7-9), $\delta_{2c} - \gamma_{2c}$, can indeed be shown to be a constant. (Refer to section VI.5.) This result, though, requires that λ is known.

Numerical procedures for this plane Couette flow problem are postponed until chapter VI.

V. STABILITY OF A BASIC ROTATIONAL COUETTE FLOW

The laminar rotational Couette flow elucidated in chapter III, will now be disturbed by the imposition of a disturbance wave. The stability analysis of this chapter follows the procedures of the Stuart energy method (Stuart,1958). Note that this procedure will use the solutions of the linearized theory, which are pursued in section V.1. The solutions sought, in this (linear) case, satisfy the micropolar analog to the Orr-Sommerfeld (MOS-) energy equations, which are also derived.

To study its stability, the basic flow is superimposed with an axisymmetric, finite disturbance. (See section V.2.) The imposed disturbance, having zero mean, provides homogeneous boundary conditions for the nonlinear equations of motion governing the disturbance flow.

The disturbance energy equations are derived from the disturbance equations in section V.3. The energy equations suggest a physical interpretation of the possible mechanisms involved in the transition from stable to unstable flow. (See section V.4.) These nonlinear energy equations (hence, the nomenclature of energy method) are then assumed to be solved by wave forms of the same spatial form as the 'marginal' disturbances of the linearized theory, but with unknown amplitude. In fact, the solution to the nonlinear disturbance energy equations are assumed to be separable into a spatial part, which is known from the linearized theory, and a temporal (time) part, which defines the amplitude of the imposed disturbances (at least, near marginal or critical stability).

Since the spatial part of the disturbance is known, ordinary differential equations, describing the disturbance amplitudes, are found from the disturbance energy equations. Such equations are called amplitude equations when micropolar theory is involved, or else Landau equations when classical theory is used. The possible growth, decay, or equilibrium states of these disturbance amplitudes can then provide the stability criteria. For instance, in section V.6, we derive marginal stability surfaces, and extract a theoretical prediction for the critical nondimensional numbers, R_c, R_{gc}, and R_{kc}, involved in the stability of rotational Couette flows.

In essence, we are re-working the stability problem for Couette flows between coaxial, rotating cylinders, with the enhanced insight permitted by the micropolar theory of fluid dynamics.

V.1 Linear Stability Analysis

Employment of the Stuart energy method, will require the shape of the marginal disturbances of the linearized theory, so that numerical calculations for theoretical predictions can be performed. Thus, the goal of this section is to derive the MOS-energy equations pertaining to the linearized rotational Couette disturbed flow problem. The solution to these coupled equations is the shape of the marginal disturbances that we will be utilizing in later calculations.

From section I.4, the surmised field equations (1.4.1) - (1.4.3), in nondimensional form, are:

$$\nabla \cdot \underset{\sim}{v} = 0, \tag{5.1.1}$$

$$- \nabla p + \frac{1}{M} \nabla^2 \underset{\sim}{v} + \frac{1}{R_k} \nabla \times \underset{\sim}{\nu} = \partial \underset{\sim}{v}/\partial t + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v}, \tag{5.1.2}$$

$$\frac{1}{R_k} \nabla(\nabla \cdot \underset{\sim}{\nu}) + \frac{1}{R_g} \nabla^2 \underset{\sim}{\nu} + \frac{1}{R_k} \nabla \times \underset{\sim}{v} = 2\underset{\sim}{\nu}/R_k + j\partial \underset{\sim}{\nu}/\partial t + j\underset{\sim}{v} \cdot \nabla \underset{\sim}{\nu}, \tag{5.1.3}$$

where now $\underset{\sim}{v} = (u, v, w)$ and $\underset{\sim}{\nu} = (\zeta, \eta, \nu)$. We will maintain a cylindrical coordinate system, in that, the point $\underset{\sim}{x} = (r,\theta,z)$.

In section III.3, the basic flow was derived to be of the form $\underset{\sim}{v} = \bar{v}(r)\, \hat{e}_\theta$, $\underset{\sim}{\nu} = \bar{\nu}(r)\, \hat{e}_z$, and $p = \bar{p}(r)$, where $\hat{e}_\theta$ and $\hat{e}_z$ are respectively unit vectors along the θ- and z-axes of the cylindrical coordinate system. To study the stability of this flow, let

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \bar{v}(r)\, \hat{e}_\theta + \underset{\sim}{v}'(\underset{\sim}{x},t), \quad \underset{\sim}{\nu}(\underset{\sim}{x},t) = \bar{\nu}(r)\, \hat{e}_z + \underset{\sim}{\nu}'(\underset{\sim}{x},t),$$
$$p(\underset{\sim}{x},t) = \bar{p}(r) + p'(\underset{\sim}{x},t), \tag{5.1.4}$$

where $\underset{\sim}{v}'$ is the disturbance velocity, $\underset{\sim}{\nu}'$ is the disturbance microgyration, and p' is the disturbance pressure. On substituting these expressions into equations (5.1.1) - (5.1.3), we obtain the equations of motion governing the disturbed flow.

By utilizing the fact that the basic flow already satisfies the equations of motion, we have

$$\nabla \cdot \underset{\sim}{v}' = 0, \tag{5.1.5}$$

$$- \nabla p' + \frac{1}{M} \nabla^2 \underset{\sim}{v}' + \frac{1}{R_k} \nabla \times \underset{\sim}{\nu}' = \left(\frac{\partial}{\partial t} + \frac{\bar{v}}{r}\frac{\partial}{\partial \theta}\right)\underset{\sim}{v}' + u'\frac{d\bar{v}}{dr}\, \hat{e}_\theta + \underset{\sim}{v}' \cdot \nabla \underset{\sim}{v}', \tag{5.1.6}$$

$$\frac{1}{R_b} \nabla(\nabla \cdot \underset{\sim}{\nu}') + \frac{1}{R_g} \nabla^2 \underset{\sim}{\nu}' + \frac{1}{R_k} \nabla \times \underset{\sim}{v}' = \left(j\frac{\partial}{\partial t} + j\frac{\bar{v}}{r}\frac{\partial}{\partial \theta} + \frac{2}{R_k}\right)\underset{\sim}{\nu}' + ju'\frac{d\bar{\nu}}{dr}\, \hat{e}_z +$$
$$+ j\underset{\sim}{v}' \cdot \nabla \underset{\sim}{\nu}'. \tag{5.1.7}$$

By neglecting the quadratic terms $\underset{\sim}{v}' \cdot \nabla \underset{\sim}{v}'$ and $\underset{\sim}{v}' \cdot \nabla \underset{\sim}{\nu}'$ (or equivalently, assuming the disturbances are infinitesimal), we obtain the linearized equations of motion governing the disturbed flow. Since the coefficients of $\underset{\sim}{\nu}'$ and $\underset{\sim}{v}'$, in the linearized equations, depend only on r, the equations admit solutions which depend on z and t exponentially. To maintain a physically realistic wave, the effects of axisymmetry (i.e. solutions independent of θ) are induced.

Consider therefore solutions of the form

$$\underset{\sim}{v}'(\underset{\sim}{x},t) = \hat{\underset{\sim}{v}}(r) \exp\{ib(z - ct)\};$$

$$\underset{\sim}{\nu}'(\underset{\sim}{x},t) = \hat{\underset{\sim}{\nu}}(r) \exp\{ib(z - ct)\};$$

$$p'(\underset{\sim}{x},t) = \hat{p}(r) \exp\{ib(z - ct)\}. \qquad (5.1.8)$$

The real parts of the expressions must be taken to obtain physical quantities. Requiring that the solutions remain bounded as $z \to \pm\infty$ implies that the wavenumber b must be real. The wave speed c may be complex, in that, $c = c_r + ic_i$. The expressions thus represent waves which travel in the direction (0,0,b), with wave speed c_r, and which grow or decay in time like $\exp(bc_i t)$. Note that a wave is said to be (asymptotically) stable if $bc_i \leq 0$, unstable if $bc_i > 0$, and neutrally stable if $bc_i = 0$. Marginal stability occurs if $bc_i = 0$ for critical values of the parameters (e.g. R, R_k, R_g, and R_b) on which the 'eigenvalue' c depends, but $bc_i > 0$ for some neighboring values of the parameters.

The ratios of the parameters R, R_k, R_g, and R_b for marginal stability are found in section V.7, which can thus give some criteria for stability. The critical relationship between the parameters, when discovered, yields the marginal stability surface. Note that neutral stability is not necesarily marginal stability. For comparison, note that on a neutral stablility surface, $bc_i = 0$, but bc_i is not necessarily positive for any neighboring values of the parameters. The minimum values of R, R_k, R_g, and R_b on all the marginal stability surfaces are called the critical numbers R_c, R_{kc}, R_{gc}, and R_{bc}; hence, there is flow instability for any $R > R_c$, $R_k > R_{kc}$, $R_g > R_{gc}$, and $R_b > R_{bc}$.

For completeness, we mention that if $bc_r \neq 0$ as bc_i approaches zero from above for a disturbance, oscillatory instability sets in. This is sometimes called overstability. Also, if $bc = 0$ at marginal stability (i.e. $bc_i = 0 = bc_r$), then there is said to be an 'exchange of stabilities', whereby instability sets in as a steady secondary flow, such as in the case of the convection cells that arise when a fluid is heated from below (Perez-Garcia & Rubi, 1892).

If we now let $D = d/dr$ and $D_* = d/dr + 1/r$, then on substituting the expressions (5.1.8) into the linearized equations (5.1.5-7), we obtain, after some rearrangement, the following (coupled) system of ordinary differential equations:

$$D_*\hat{u} + b\hat{w} = 0, \tag{5.1.9}$$

$$(DD_* - b^2 + ibMc)\hat{u} = MD\hat{p} + ibM\hat{\eta}/R_k, \tag{5.1.10}$$

$$(D_*D - b^2 - 1/r + ibMc)\hat{v} = M(ib\hat{\zeta} - D\hat{\nu})/R_k + M\bar{v}'\hat{u}, \tag{5.1.11}$$

$$(D_*D - b^2 + ibMc)\hat{w} = ibM\hat{p} - MD_*\hat{\eta}/R_k, \tag{5.1.12}$$

$$(DD_* - b^2 + ibcjR_g - 2R_g/R_k)\hat{\zeta} = ibR_g\hat{v}/R_k - R_g(DD_* + ibD)\hat{\nu}/R_b, \tag{5.1.13}$$

$$(D_*D - b^2 - 1/r^2 + ibcjR_g - 2R_g/R_k)\hat{\eta} = R_g(ib\hat{u} - D\hat{w})/R_k, \tag{5.1.14}$$

$$(D_* D - b^2 + ibcjR_g - 2R_g/R_k)\hat{\nu} = ibR_g(D_*\hat{\zeta} - ib\hat{\nu})/R_b - R_g D_*\hat{v}/R_k +$$
$$+ jR_g\bar{\nu}'u. \tag{5.1.15}$$

Here primes denote differentiation with respect to r. Notice that $DD_* \neq D_*D$.

The strict adherence boundary conditions, applied to the disturbance flow, imply

$$\hat{u} = \hat{v} = \hat{w} = 0 = \hat{\zeta} = \hat{\eta} = \hat{\nu} \quad \text{at} \quad r = R_1, R_2. \tag{5.1.16}$$

Having restricted to axisymmetric disturbances, we introduce the disturbance stream function

$$\Psi'(r,z,t) = \Phi(r)\exp\{ib(z - ct)\}.$$

Thus, two of the disturbance velocity components will be

$$u' = -\frac{1}{r}\frac{\partial\Psi}{\partial z} \quad \text{and} \quad w' = \frac{1}{r}\frac{\partial\Psi}{\partial r}.$$

Furthermore,

$$\hat{u} = -ib\Phi/r \quad \text{and} \quad \hat{w} = \frac{1}{r}\frac{d\Phi}{dr} = D_*(\Phi/r). \tag{5.1.17}$$

Notice that $\Phi(r) = \Phi_r + i\Phi_i$ is a complex-valued function.

By inserting (5.1.17) into the six equations (5.1.10-15), and eliminating $\hat{p}$ from these equations, we reduce the unknowns to five: Φ, $\hat{v}$, $\hat{\zeta}$, $\hat{\eta}$, and $\hat{\nu}$. Finally, effecting some rearrangement gives the micropolar analog to the Orr-Sommerfeld (MOS-) energy equations. The result is

$$\frac{(DD_* - b^2)^2}{ibM}\left(\frac{\Phi}{r}\right) + \frac{(DD_* - b^2)}{ibR_k}\hat{\eta} = b^2\left(\frac{DD_*}{ibM} + c\right)\left(\frac{\Phi}{r}\right) \quad (5.1.18)$$

$$\frac{(D_*D - b^2 - 1/r^2)}{ibM}\hat{v} + c\hat{v} + \bar{v}'\Phi/r = \frac{ib\hat{\zeta} - D\hat{\nu}}{ibR_k} \quad (5.1.19)$$

$$\frac{(DD_* - b^2)}{ibR_g}\hat{\zeta} + \frac{DD_*\hat{\nu}}{ibR_b} + \frac{D\hat{\nu}}{R_b} + cj\hat{\zeta} = \frac{\hat{v}}{R_k} + \frac{2\hat{\zeta}}{ibR_k} \quad (5.1.20)$$

$$\frac{(DD_* - b^2 - 1/r^2)}{ibR_g}\hat{\eta} + \frac{(DD_* - b^2)}{ibR_k}\left(\frac{\Phi}{r}\right) + cj\hat{\eta} = \frac{2\hat{\eta}}{ibR_k} \quad (5.1.21)$$

$$\frac{(DD_* - b^2)}{ibR_g}\hat{\nu} + cj\hat{\nu} + \frac{D_*\hat{v}}{ibR_k} + j\bar{\nu}'\Phi/r = \frac{2\hat{\nu}}{ibR_k} + \frac{D_*\hat{\zeta}}{R_b} + \frac{b^2\hat{\nu}}{ibR_b} \quad (5.1.22)$$

The boundary conditions are

$$b\Phi = \hat{v} = D\Phi = 0 = \hat{\zeta} = \hat{\eta} = \hat{\nu} \quad \text{at} \quad r = R_1, R_2. \quad (5.1.23)$$

The unknowns Φ, $\hat{v}$, $\hat{\zeta}$, $\hat{\eta}$, and $\hat{\nu}$ governed by (5.1.18-22) will be required in later calculations, as disturbance shapes. Also, in the calculations, $\bar{v}$ and $\bar{\nu}$ are taken as in (3.3.12) and (3.3.11), respectively. Note, however, that in the nonlinear theory, the mean velocity $\bar{v}$ and the mean microgyration $\bar{\nu}$ are different from those of laminar flow (as taken above) because of the interactions between the mean flow and the disturbances.

Next, we study, with the nonlinear theory, the stability of the basic flow, which is disturbed by an axisymmetric, finite disturbance.

V.2 Imposition of Finite Disturbances on a Basic Flow

Axisymmetric Couette flow refers to a Couette flow regime that is (theoretically) independent of the azimuthal coordinate θ. We will be analyzing the stability of a basic rotational Couette flow, that is disturbed by an axisymmetric, finite disturbance in the rz-plane. Moreover, since the cylinders are assumed to extend to infinity in both the positive and negative z-directions, the velocity and microgyration fields will be independent of the θ-coordinate.

Restricting to axisymmetric flows, the continuity equation $\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$ defines a Stokes stream function for the fluid flow. This stream function is decomposed into the sum of two functions, one representing the mean flow, the other representing the finite disturbance flow. The analysis of the nonlinear field equations governing the finite disturbance flow employs procedures embodied in the Stuart energy method, with which we begin.

For flow between two coaxial cylinders in constant relative rotation (rotational Couette flow), we impose a disturbance travelling in the direction of the basic flow which has the form

$$\Psi(r,z,t) = K\Psi_1(r)\ \exp\{ib(z - ct)\} + \tilde{K}\ \tilde{\Psi}_1(r)\ \exp\{-ib(z - \tilde{c}t)\} \qquad (5.2.1)$$

where K is an arbitrary complex constant, wave speed $c = c_r + ic_i$ $(c_r \geqq 0)$, and the tilda $\sim$ denotes a complex conjugate.

Now, let the stream function for the flow be represented by the Fourier series expansion in z, as

$$\Psi(r,z,t) = \bar{\phi} + \phi' =$$

$$= \bar{\phi}_0(r,t) + \sum_{m=1}^{\infty} \{\phi_m(r,t)\, e^{mibz} + \tilde{\phi}_m(r,t)\, e^{-mibz}\}. \qquad (5.2.2)$$

In the linearized theory, $\bar{\phi}$ represents the steady stream function, and ϕ' reduces to (5.2.1) (or equivalently, the function $\Psi'(r,z,t)$ used in section V.1). For the nonlinear theory, the sum on the right represents the finite disturbance ϕ' , while $\bar{\phi}$ is the mean stream function, where the mean (average) is taken with respect to z over a disturbance wavelength of $2\pi/b$, b being the period[7] of the disturbance wave.

With the stream function Ψ, the Fourier series expansion gives

$$u = -\frac{1}{r}\frac{\partial\Psi}{\partial z} = u' = \sum_{m=1}^{\infty} \{u_m'(r,t)\, e^{mibz} + \tilde{u}_m'(r,t)\, e^{-mibz}\},$$

$$v = \bar{v} + v' = \bar{v}(r,t) + \sum_{m=1}^{\infty} \{v_m'(r,t)\, e^{mibz} + \tilde{v}_m'(r,t)\, e^{-mibz}\},$$

$$w = \frac{1}{r}\frac{\partial\Psi}{\partial r} = w' = \sum_{m=1}^{\infty} \{w_m'(r,t)\, e^{mibz} + \tilde{w}_m'(r,t)\, e^{-mibz}\},$$

$$\nu = \bar{\nu} + \nu' = \bar{\nu}(r,t) + \sum_{m=1}^{\infty} \{\nu_m'(r,t)\, e^{mibz} + \tilde{\nu}_m'(r,t)\, e^{-mibz}\},$$

$$\zeta = \zeta' = \sum_{m=1}^{\infty} \{\zeta_m'(r,t)\, e^{mibz} + \tilde{\zeta}_m'(r,t)\, e^{-mibz}\},$$

$$\eta = \eta' = \sum_{m=1}^{\infty} \{\eta_m'(r,t)\, e^{mibz} + \tilde{\eta}_m'(r,t)\, e^{-mibz}\}. \qquad (5.2.3)$$

The over-bars now denote a mean value, and the primed terms denote disturbance flow variables.

Defined for equations (5.2.3) are

$$u'_m(r,t) = -\, mib\phi_m/r, \qquad w'_m(r,t) = \frac{1}{r}\,\partial\phi_m/\partial r \quad (m \geqq 1). \tag{5.2.4}$$

For the resulting axisymmetric flow, mathematically, we prescribe that the velocity and microgyration vector fields are, respectively,

$$\underset{\sim}{v} = \{u(r,z,t),\ v(r,z,t),\ w(r,z,t)\},\ \underset{\sim}{\nu} = \{\zeta(r,z,t),\ \eta(r,z,t),\ \nu(r,z,t)\}. \tag{5.2.5}$$

Applying the prescription (5.2.5) to the field equations (3.2.1) - (3.2.7) yields:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{5.2.6}$$

$$-\frac{\partial p}{\partial r} + \frac{1}{M}\{\partial^2 u/\partial r^2 + \frac{\partial}{\partial r}(u/r) + \partial^2 u/\partial z^2\} - \frac{1}{R_k}\frac{\partial \eta}{\partial z} =$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}, \tag{5.2.7}$$

$$\frac{1}{M}\{\partial^2 v/\partial r^2 + \frac{\partial}{\partial r}(v/r) + \partial^2 v/\partial z^2\} + \frac{1}{R_k}\left(\frac{\partial \zeta}{\partial z} - \frac{\partial \nu}{\partial r}\right) =$$

$$= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z}, \tag{5.2.8}$$

$$-\frac{\partial p}{\partial z} + \frac{1}{M}\{\partial^2 w/\partial r^2 + \frac{1}{r}\frac{\partial w}{\partial r} + \partial^2 w/\partial z^2\} + \frac{1}{R_k}\left(\frac{\partial \eta}{\partial r} + \frac{\eta}{r}\right) =$$

$$= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}, \tag{5.2.9}$$

$$\frac{1}{R_b}\ \{\partial^2\zeta/\partial r^2 + \frac{\partial}{\partial r}(\zeta/r) + \frac{\partial^2\nu}{\partial r\partial z}\} + \frac{1}{R_g}\ \{\partial^2\zeta/\partial r^2 + \frac{\partial}{\partial r}(\zeta/r) + \partial^2\zeta/\partial z^2\} =$$

$$= \frac{1}{R_k}\frac{\partial v}{\partial z} + \frac{2\zeta}{R_k} + j\left(\frac{\partial\zeta}{\partial t} + u\,\frac{\partial\zeta}{\partial r} - \frac{v\eta}{r} + w\,\frac{\partial\zeta}{\partial z}\right), \qquad (5.2.16)$$

$$\frac{1}{R_g}\ \{\partial^2\eta/\partial r^2 + \frac{\partial}{\partial r}(\eta/r) + \partial^2\eta/\partial z^2\} = \frac{1}{R_k}\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) + \frac{2}{R_k}\ \frac{\eta}{r} +$$

$$+ j\{\partial\eta/\partial t + u\ \partial\eta/\partial r + v\zeta/r + w\ \partial\eta/\partial z\}, \qquad (5.2.11)$$

$$\frac{1}{R_b}\left(\frac{\partial^2\zeta}{\partial r\partial z} + \frac{1}{r}\frac{\partial\zeta}{\partial z} + \partial^2\nu/\partial z^2\right) + \frac{1}{R_g}\ \{\partial^2\nu/\partial r^2 + \frac{1}{r}\frac{\partial\nu}{\partial r} + \partial^2\nu/\partial z^2\} =$$

$$= -\frac{1}{R_k}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + \frac{2\nu}{R_k} + j\left(\frac{\partial\nu}{\partial t} + u\,\frac{\partial\nu}{\partial r} + w\,\frac{\partial\nu}{\partial z}\right). \qquad (5.2.12)$$

The conditions to be applied throughout this procedure are:

(C1) that the mean velocity $\bar{v}$ and the mean microgyration $\bar{\nu}$ assume the same values on the cylinders as do the undisturbed velocity and microgyration;

(C2) that the disturbance velocities u', v', w' and the disturbance microgyrations ζ', η', ν' vanish on the cylinders; and

(C3) that just enough external power is supplied to maintain the angular speeds of the cylinders at constant values, in accordance with the variation with time of the mean skin friction on the cylinders.

Consequently, for disturbed rotational Couette flow, acceptable non-dimensional boundary conditions are:

$$\bar{v} = 1 \text{ at } r = R_1, \quad \bar{v} = \Omega_2 R_2/\Omega_1 R_1 \text{ at } r = R_2, \quad \bar{\nu} = 0 \text{ at } r = R_1, R_2,$$

and

$$u' = v' = w' = 0 = \zeta' = \eta' = \nu' \quad \text{at } r = R_1, R_2. \qquad (5.2.13)$$

In order that the pressure gradients will also balance the remaining terms, the pressure must be of the form

$$p = rp^*(t) + p^{**}(r,t) + p'(r,z,t) =$$

$$= rp^*(t) + p^{**}(r,t) + \sum_{m=1}^{\infty} \{p_m(r,t)\, e^{mibz} + \tilde{p}_m(r,t)\, e^{-mibz}\} \tag{5.2.14}$$

where asterisks denote labels for the terms in the decomposition of the mean pressure, as used in equation (4.2.11). Substitute equations (5.2.3) and (5.2.14) into equations (5.2.7) - (5.2.12), and equate the Fourier components. The equations arising from equating the terms independent of z are equivalently found by taking the mean of equations (5.2.7-12). The result is:

$$- p^* = \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'^2}) - \frac{1}{r}(\bar{v}^2 + \overline{v'^2}), \tag{5.2.15}$$

$$\frac{1}{M}\{\partial^2/\partial r^2 + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\}\bar{v} - \frac{1}{R_k}\frac{\partial\bar{\nu}}{\partial r} = \frac{\partial\bar{v}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{u'v'}), \tag{5.2.16}$$

$$- \frac{\partial}{\partial z} p^{**}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'w'}) = 0, \tag{5.2.17}$$

$$0 = j\left(\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\zeta'}) - \overline{v'\eta'}/r\right), \tag{5.2.18}$$

$$0 = j\left(\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\eta'}) + \overline{v'\zeta'}/r\right), \tag{5.2.19}$$

$$\frac{1}{R_g}\{\partial^2/\partial r^2 + \frac{1}{r}\frac{\partial}{\partial r}\}\bar{\nu} = -\frac{1}{R_k}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\bar{v} + \frac{2\bar{\nu}}{R_k} + j\left(\frac{\partial\bar{\nu}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\nu'})\right). \tag{5.2.20}$$

Note that quanities involving the mean disturbance variables, like $\overline{u'v'}$ and $\overline{u'\nu'}$, are independent of z. Also, note that the continuity equation (5.2.6) implies that $\frac{1}{r}\frac{\partial}{\partial r}(ru') = -\frac{\partial w'}{\partial z}$. (5.2.21)

The equations governing the disturbance field quantities are now found on subtracting (5.2.15) from (5.2.7), (5.2.16) from (5.2.8), ... , and (5.2.20) from (5.2.12), to be:

$$-\frac{\partial p'}{\partial r} + \frac{1}{M}\left(\frac{\partial^2 u'}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u'}{r}\right) + \frac{\partial^2 u'}{\partial z^2}\right) - \frac{1}{R_k}\frac{\partial \eta'}{\partial z} = \frac{\partial u'}{\partial t} - 2\frac{v'\bar{v}}{r} + \chi_1, \quad (5.2.22)$$

$$\frac{1}{M}\left(\frac{\partial^2 v'}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v'}{r}\right) + \frac{\partial^2 v'}{\partial z^2}\right) + \frac{1}{R_k}\left(\frac{\partial \zeta'}{\partial z} - \frac{\partial \nu'}{\partial r}\right) = \frac{\partial v'}{\partial t} + u'\left(\frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\right) + \chi_2, \quad (5.2.23)$$

$$-\frac{\partial p'}{\partial z} + \frac{1}{M}\left(\frac{\partial^2 w'}{\partial r^2} + \frac{1}{r}\frac{\partial w'}{\partial r} + \frac{\partial^2 w'}{\partial z^2}\right) + \frac{1}{R_k}\left(\frac{\partial \eta'}{\partial r} + \frac{\eta'}{r}\right) = \frac{\partial w'}{\partial t} + \chi_3, \quad (5.2.24)$$

$$\frac{1}{R_b}\left(\frac{\partial^2 \zeta'}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{\zeta'}{r}\right) + \frac{\partial^2 \nu'}{\partial r \partial z}\right) + \frac{1}{R_g}\left(\frac{\partial^2 \zeta'}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{\zeta'}{r}\right) + \frac{\partial^2 \zeta'}{\partial z^2}\right) = \frac{1}{R_k}\frac{\partial v'}{\partial z} + \frac{2\zeta'}{R_k} + j\left(\frac{\partial \zeta'}{\partial t} - \frac{\bar{v}\eta'}{r} + \chi_4\right), \quad (5.2.25)$$

$$\frac{1}{R_g}\left(\frac{\partial^2 \eta'}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{\eta'}{r}\right) + \frac{\partial^2 \eta'}{\partial z^2}\right) = \frac{1}{R_k}\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) + \frac{2}{R_k}\frac{\eta'}{r} + j\left(\frac{\partial \eta'}{\partial t} + \frac{\bar{v}\zeta'}{r} + \chi_5\right), \quad (5.2.26)$$

$$\frac{1}{R_b}\left(\frac{\partial^2 \zeta'}{\partial r \partial z} + \frac{1}{r}\frac{\partial \zeta'}{\partial z} + \frac{\partial^2 \nu'}{\partial z^2}\right) + \frac{1}{R_g}\left(\frac{\partial^2 \nu'}{\partial r^2} + \frac{1}{r}\frac{\partial \nu'}{\partial r} + \frac{\partial^2 \nu'}{\partial z^2}\right) = -\frac{1}{R_k}\left(\frac{\partial v'}{\partial r} + \frac{v'}{r}\right) + \frac{2\nu'}{R_k} + j\left(\frac{\partial \nu'}{\partial t} + u'\frac{\partial \bar{\nu}}{\partial r} + \chi_6\right), \quad (5.2.27)$$

where

$$\chi_1 = u'\frac{\partial u'}{\partial r} - \frac{v'^2}{r} + w'\frac{\partial u'}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'^2}) + \frac{1}{r}\overline{v'^2} ,$$

$$\chi_2 = u'\frac{\partial v'}{\partial r} + \frac{u'v'}{r} + w'\frac{\partial v'}{\partial z} - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{u'v'}) ,$$

$$\chi_3 = u'\frac{\partial w'}{\partial r} + w'\frac{\partial w'}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'w'}) ,$$

$$\chi_4 = u'\frac{\partial \zeta'}{\partial r} + w'\frac{\partial \zeta'}{\partial z} - \frac{v'\eta'}{r} + \frac{\overline{v'\eta'}}{r} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\zeta'}) ,$$

$$\chi_5 = u'\frac{\partial \eta'}{\partial r} + \frac{v'\zeta'}{r} + w'\frac{\partial \eta'}{\partial z} - \frac{\overline{v'\zeta'}}{r} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\eta'}) ,$$

$$\chi_6 = w'\frac{\partial \nu'}{\partial z} + u'\frac{\partial \nu'}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\nu'}) .$$

These nonlinear equations will be referred to as the disturbance equations.

V.3 Disturbance Energy Equations

In this section we will derive the disturbance energy balance equations for axisymmetric, finite disturbances. By properly preparing the disturbance equations (5.2.22) - (5.2.27), we derive the integral form of the MOS-energy equations. Note that since the cylinders are unbounded (in the z-direction), we have assumed the disturbances $\underset{\sim}{u}'$ and $\underset{\sim}{\nu}'$ to be (spatially) periodic in z, and thus, the following integrations with respect to z can be taken over exactly one wavelength.

Begin by multiplying (5.2.22) by u', (5.2.23) by v', and (5.2.24) by w'. Add these equations; then utilize the simplifications and integrations demonstrated in Appendix A. Thus, the first Disturbance Energy Equation is derived without approximation to be:

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}(u'^2 + v'^2 + w'^2)\ rdrdz = \iint (-\overline{u'v'})\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) rdrdz \ -$$

$$-\frac{1}{M}\iint\left(\left(\frac{\partial v'}{\partial z}\right)^2 + \left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv')\right)^2\right) rdrdz \ -$$

$$-\frac{1}{R_k}\iint\left(\zeta'\frac{\partial v'}{\partial z} + \eta'\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) - \frac{\nu'}{r}\frac{\partial}{\partial r}(rv')\right) rdrdz \qquad (5.3.1)$$

where the integrals are evaluated over a volume bounded by the cylinders $r = R_1, R_2$ and by one wavelength $z = 0, 2\pi/b$.

The mean velocity $\bar{v}$ occurring in equation (5.3.1), is derived from (5.2.16), and is given by

$$\frac{1}{M}\{\partial^2/\partial r^2 + \frac{1}{r}\frac{\partial}{\partial r} - 1/r^2\}\bar{v} - \frac{1}{R_k}\frac{\partial \bar{\nu}}{\partial r} = \frac{\partial \bar{v}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{u'v'}). \qquad (5.3.2)$$

Next, multiply (5.2.25) by ζ', (5.2.26) by η', and (5.2.27) by ν'; simplify; then integrate; thus deriving the companion Disturbance Energy Equation without approximation to be:

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2} j\,(\zeta'^2 + \eta'^2 + \nu'^2)\ rdrdz \;=\; \iint(-j\overline{u'\nu'})\ \frac{\partial\bar{\nu}}{\partial r}\ rdrdz \;-$$

$$- \frac{2}{R_k}\iint\left(\zeta'^2 + \frac{\eta'^2}{r} + \nu'^2\right) rdrdz \;-$$

$$- \frac{1}{R_k}\iint\left(\zeta'\frac{\partial v'}{\partial z} + \eta'\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) - \frac{\nu'}{r}\frac{\partial}{\partial r}(rv')\right) rdrdz \;-$$

$$- \frac{1}{R_b}\iint\left(\left(\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\right)^2 + 2\,\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z} + \left(\frac{\partial\nu'}{\partial z}\right)^2\right) rdrdz \;-$$

$$- \frac{1}{R_g}\iint\left(\left(\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\eta')\right)^2 + \left(\frac{\partial\nu'}{\partial r}\right)^2 + \left(\frac{\partial\zeta'}{\partial z}\right)^2 + \left(\frac{\partial\eta'}{\partial z}\right)^2 + \left(\frac{\partial\nu'}{\partial z}\right)^2\right) rdrdz. \qquad (5.3.3)$$

where again the integrals are evaluated over a volume bounded by the cylinders $r = R_1, R_2$ and by one wavelength $z = 0, 2\pi/b$.

The mean microgyration $\bar{\nu}$ occurring in equation (5.3.3), is derived from (5.2.20), and is given by

$$\frac{1}{R_g}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{\nu} = \frac{2\bar{\nu}}{R_k} - \frac{1}{R_k}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\bar{v} + j\left(\frac{\partial\bar{\nu}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'\nu'})\right). \qquad (5.3.4)$$

Equations (5.3.1) and (5.3.3) comprise the integral form of the micropolar analog to the Orr-Sommerfeld (MOS-) energy equations.

Following a physical interpretation of the flow mechanisms suggested by equations (5.3.1) and (5.3.3), we will derive the amplitude equations, with the Stuart energy method.

V.4 Physical Interpretation of the MOS-Energy Equations

The goal of this section is to identify and physically interpret the terms appearing in the disturbance energy equations (5.3.1) and (5.3.3). (Interpretation is similar to that given in section IV.4.) To enrich the physical interpretation, equations (5.3.1) and (5.3.3) are, respectively, re-written in dimensional form and a briefer notation introduced.

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}\rho(u'^2 + v'^2 + w'^2)\ rdrdz = \iint(-\rho\overline{u'v'})\left(\frac{\partial\bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) rdrdz -$$

$$- (\mu+\kappa)\iint\left(\left(-\frac{\partial v'}{\partial z}\right)^2 + \left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv')\right)^2\right) rdrdz -$$

$$- \kappa\iint\left(\zeta'\frac{\partial v'}{\partial z} + \eta'\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) - \frac{\nu'}{r}\frac{\partial}{\partial r}(rv')\right) rdrdz.$$

Briefly, $\frac{\partial E}{\partial t} = I_1 - (\mu+\kappa)I_2 + \kappa I_3.$ (5.4.1)

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}\rho j(\zeta'^2 + \nu'^2 + \eta'^2)\ rdrdz = \iint(-\rho j\overline{u'\nu'})\ \frac{\partial\bar{\nu}}{\partial r}\ rdrdz -$$

$$- \gamma\iint\left(\left(\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\eta')\right)^2 + \left(\frac{\partial\nu'}{\partial r}\right)^2 + \left(\frac{\partial\zeta'}{\partial z}\right)^2 + \right.$$

$$\left. + \left(\frac{\partial\eta'}{\partial z}\right)^2 + \left(\frac{\partial\nu'}{\partial z}\right)^2\right) rdrdz -$$

$$- \kappa\iint\left(\zeta'\frac{\partial v'}{\partial z} + \eta'\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) - \frac{\nu'}{r}\frac{\partial}{\partial r}(rv')\right) rdrdz -$$

$$- 2\kappa \iint (\zeta'^2 + \frac{\eta'^2}{r} + \nu'^2)\, rdrdz -$$

$$- (\alpha+\beta) \iint \left(\left(\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\right)^2 + 2\,\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z} + \left(\frac{\partial\nu'}{\partial z}\right)^2 \right) rdrdz.$$

Briefly, $\frac{\partial e}{\partial t} = H_1 - \gamma H_2 + \kappa I_3 - 2\kappa H_3 - (\alpha+\beta)H_4.$ (5.4.2)

In equation (5.4.1), the term on the left-hand side gives the rate of growth of the disturbance kinetic energy within the volume considered. On the right-hand side of (5.4.1), the term I_1 is the integral of the product of the Reynolds stress and the flow shear $(\partial\bar{v}/\partial r - \bar{v}/r)$, and represents the "translational" rate of transfer of kinetic energy from the mean flow to the disturbance. The term $(\mu+\kappa)I_2$ is always positive; so $-(\mu+\kappa)I_2$ represents the rate of $(\mu+\kappa)$-viscous dissipation of the kinetic energy of the disturbance due to translational and rotational effects of the macro-volume elements in the volume considered.

The term κI_3 is the common link between equations (5.4.1) and (5.4.2). The term I_3 is the integral of the dot product of the micro-gyration with the curl $\underset{\sim}{v}$, and geometrically represents the Swirl created by the disturbance. Mathematically,

$$I_3 = \underset{\sim}{\nu}'\cdot(\nabla\times\underset{\sim}{v}') = \frac{1}{r}\begin{vmatrix} \zeta' & r\eta' & \nu' \\ \partial/\partial r & \frac{1}{r}\partial/\partial\theta & \partial/\partial z \\ u' & rv' & w' \end{vmatrix}.$$

Notice that the scalar triple product $\underset{\sim}{\nu}'\cdot(\nabla\times\underset{\sim}{v}')$ can also be written in two other forms. In that,

$$\underset{\sim}{\nu}' \cdot (\nabla \times \underset{\sim}{v}') = \underset{\sim}{v}' \cdot (\nabla \times \underset{\sim}{\nu}') = \nabla \cdot (\underset{\sim}{v}' \times \underset{\sim}{\nu}') .$$

The third variation is the divergence of the Coriolis acceleration.

The Coriolis acceleration, common to the mechanics of moving coordinate systems, equals $2\underset{\sim}{\nu}' \times \underset{\sim}{v}'$. Note that microgyration is angular velocity. For a fluid motion described by micropolar theory, the Coriolis acceleration, $2\underset{\sim}{\nu}' \times \underset{\sim}{v}'$, represents the resultant from the interaction of the rotation of moving micro-volume elements and the present motion of the ambient macro-volume elements for the existing flow in the volume considered. Thus, a nonzero swirl, i.e. $\underset{\sim}{\nu}' \cdot (\nabla \times \underset{\sim}{v}') = \nabla \cdot (\underset{\sim}{v}' \times \underset{\sim}{\nu}') \neq 0$, acts as a source, if $I_3 > 0$ (or a sink, if $I_3 < 0$), for spreading (or gathering) energy necessary to create a turbulent flow. Most of all, the swirl is the coupling mechanism between the micro- and macro-continuum volume elements.

In equation (5.4.2), the term on the left-hand side gives the rate of growth of the disturbance Microenergy of Rotation[8] within the volume considered. On the right-hand side of (5.4.2), the term H_1 is the integral of the product of the mean couple stress and the mean microgyration gradient, and represents the "rotational" rate of transfer of microenergy of rotation from the mean (micro)flow to the disturbance. The term γH_2 is always positive; so $-\gamma H_2$ represents the rate of γ-viscous dissipation of the microenergy of rotation of the disturbance due to the translational and rotational effects of the micro-volume elements in the volume considered. The term $2\kappa H_3$ is always positive; so $-2\kappa H_3$ represents the κ-viscous dissipation of the microenergy of rotation of the disturbance due to the rotational effects of the micro-volume elements in the volume considered. The term $(\alpha+\beta)H_4$ may always be positive depending on the sign and magnitude of $2\frac{\partial v'}{\partial r}\frac{\partial \zeta'}{\partial z}$; if $H_4 > 0$ then $-(\alpha+\beta)H_4$ represents the rate of $(\alpha+\beta)$-viscous dissipation of the microenergy of rotation of the disturbance due to the rotational, translational, and dilational effects of the microvolume elements in the volume considered.

In summary, fluid flow stabilizers are the terms $(\mu+\kappa)I_2$, γH_2, $2\kappa H_3$, and $(\alpha+\beta)H_4$ which represent viscous dissipation mechanisms. Fluid flow destabilizers are the terms E, e, I_1, H_1. Note that listing I_1 and H_1 as destabilizers, presumes that these terms are positive. The intermediary between stability and instability is the swirl term κI_3.

Combining equations (5.4.1) and (5.4.2), reveals

$$\frac{\partial}{\partial t}(E + e) = I_1 + H_1 - (\mu+\kappa)I_2 - \gamma H_2 - 2\kappa H_3 - (\alpha+\beta)H_4 + 2\kappa I_3. \tag{5.4.3}$$

(Note that $2\kappa I_3$ now possesses the factor of 2, in agreement with the accepted definition of the Coriolis acceleration.) Suppose that $I_1 + H_1 + 2\kappa I_3 > 0$. If $I_1 + H_1 + 2\kappa I_3 > (\mu+\kappa)I_2 + \gamma H_2 + 2\kappa H_3 + (\alpha+\beta)H_4$ then $\partial(E + e)/\partial t > 0$. This means the disturbance energies are growing, and the disturbances are increasing in amplitude (i.e. the flow is becoming unstable). Conversely, if $I_1 + H_1 + 2\kappa I_3 < (\mu+\kappa)I_2 + \gamma H_2 + 2\kappa H_3 + (\alpha+\beta)H_4$ then $\partial(E + e)/\partial t < 0$. This means the disturbance energies are decaying, and the disturbances are decreasing in amplitude (i.e. the flow is becoming stable). Ideally, if $I_1 + H_1 + 2\kappa I_3 < 0$ then $\partial(E + e)/\partial t < 0$, i.e. the flow is becoming stable.

Finally, equations (5.3.2) and (5.3.4) show how the distribution of mean velocity and mean microgyration are affected by the viscous stresses, pressure gradients, Reynolds stress, and mean couple stress, due to the disturbance. An equilibrium flow is possible if $\bar{v}$ and $\bar{\nu}$ can be so distorted, by the Reynolds stress and the mean couple stress, that $I_1 + H_1 + 2\kappa I_3 = (\mu+\kappa)I_2 + \gamma H_2 + 2\kappa H_3 + (\alpha+\beta)H_4$, which implies then that $\partial(E + e)/\partial t = 0$; in that, equilibrium. The equilibrium state will play a crucial role in the analysis presented in the next sections.

V.5 Amplitude Equations

Recall, in the Stuart energy method, the stream function for the disturbed flow, Ψ, represents a mean flow together with a periodic disturbance consisting of the fundamental harmonic, ϕ_0, with wave-length $2\pi/b$, and higher harmonic components, ϕ_1, ϕ_2, ... , having wavenumbers nb ($n \geq 1$), but the same (real) wave velocity, c_r, which is assumed to be independent of time. The amplification or damping of the finite disturbance, and the consequent changes in the mean velocity $\bar{v}$ and the mean microgyration $\bar{\nu}$ are accounted for by the dependence of all the ϕ-functions on time t.

We assume that the higher harmonics ϕ_2, ϕ_3, ... are zero. Furthermore, we assume that the disturbances are under 'supercritical' conditions meaning that the nondimensional numbers R, R_k, R_g, and R_b are above the value which is critical for the linearized instability theory.

Moreover, a disturbance under supercritical conditions amplifies for small amplitudes. A suitable initial condition, therefore, is that the function $\phi_1(r,t)$ shall be an exponentially increasing function of time t in the limit as $t \to -\infty$; in fact, ϕ_1 has to be the appropriate function $\Phi(r)\exp(bc_i t)$, where $c_i > 0$, of the linearized instability theory (Stuart, 1960).[12]

Assume disturbances u', v', w', ζ', η', and ν' are similar in 'shape' to the solution given by the linear theory, but that the solution is multiplied by an amplitude factor: a(t) in the case of $\underset{\sim}{v}'$, and A(t) in the case of $\underset{\sim}{\nu}'$. I.e. $\phi_1'(r,t) = a(t)\Phi(r)$, $v_1'(r,t) = a(t)\hat{v}(r)$, and $\underset{\sim}{\nu}_1'(r,t) = A(t)\hat{\underset{\sim}{\nu}}(r)$.

For an equilibrium state, we presume that $\partial\bar{\nu}/\partial t = 0 = \partial\bar{v}/\partial t$. With this presumption, we have equations (5.3.2) and (5.3.4) yielding

$$\frac{1}{M}\left(\frac{d^2\bar{v}}{dr^2} + \frac{d}{dr}\left(\frac{\bar{v}}{r}\right)\right) - \frac{1}{R_k}\frac{d\bar{\nu}}{dr} = \frac{1}{r^2}\frac{d}{dr}(r^2\overline{u'v'}) \tag{5.5.1}$$

and

$$\frac{d\bar{v}}{dr} + \frac{\bar{v}}{r} = 2\bar{\nu} + R_k j\,\frac{1}{r}\frac{d}{dr}(r\overline{u'\nu'}) - \frac{R_k}{R_g}\left(\frac{d^2\bar{\nu}}{dr^2} + \frac{1}{r}\frac{d\bar{\nu}}{dr}\right). \tag{5.5.2}$$

Integrating (5.5.1) and using (5.5.2) leads to

$$\frac{d^2\bar{\nu}}{dr^2} + \frac{1}{r}\frac{d\bar{\nu}}{dr} - \lambda^2\bar{\nu} = f(r) =$$

$$= -\frac{MR_g}{R_k}(\overline{u'v'}) - 2\frac{MR_g}{R_k}\int_{R_1}^{r}\frac{1}{s}\,\overline{u'v'}\,ds + jR_g\,\frac{1}{r}\frac{d}{dr}(r\overline{u'\nu'}) + \frac{MR_g}{R_k}C_1 \tag{5.5.3}$$

where f(r) represents the non-homogeneous part of (5.5.3), C_1 is an integration constant, and again λ^2 is as given in $(2.3.9)_1$.

The homogeneous equation from (5.5.3) is solved by

$$\bar{\nu}_h(r) = C_2 I_0(\lambda r) + C_3 K_0(\lambda r) \tag{5.5.4}$$

where I_0 and K_0 are modified zero-order Bessel functions of the first and second kind, respectively; and C_2 and C_3 are additional integration constants.

Our immediate purpose is to find $d\bar{\nu}/dr$ and $d\bar{\nu}/dr - \bar{\nu}/r$, for the disturbance energy equations (5.3.1) and (5.3.3), so that the amplitude equations can be derived.

Using variation of parameters, we find a particular solution for equation (5.5.3) of the form

$$\bar{\nu}_p(r) = I_0(\lambda r) \int_{R_1}^{r} sK_0(\lambda s)\ f(s)\ ds - K_0(\lambda r) \int_{R_1}^{r} sI_0(\lambda s)\ f(s)\ ds. \tag{5.5.5}$$

So, in integral form, the general solution to equation (5.5.3) is the mean microgyration, given by

$$\bar{\nu}(r) = \bar{\nu}_h(r) + \bar{\nu}_p(r). \tag{5.5.6}$$

The mean microgyration gradient, in the radial direction, is

$$\frac{d\bar{\nu}}{dr} = \lambda\Big(C_2 I_1(\lambda r) - C_3 K_1(\lambda r)\Big) + \lambda I_1(\lambda r)\int_{R_1}^{r} sK_0(\lambda s) f(s)\ ds + $$
$$+ \lambda K_1(\lambda r) \int_{R_1}^{r} sI_0(\lambda s) f(s)\ ds. \tag{5.5.7}$$

Next, integrating equation (5.5.2) with respect to r, after having incorporated (5.5.6), leads to the mean velocity as follows:

Let $g(r)$ represent the right-hand side of (5.5.2). Then $d\bar{v}/dr + \bar{v}/r = g(r)$ has solution

$$\bar{v}(r) = \frac{1}{r}\left(\int sg(s)\, ds + C_4 \right) \tag{5.5.8}$$

where C_4 is another integration constant.

The mean velocity is

$$\frac{d\bar{v}}{dr} = -\frac{1}{r^2}\left(\int sg(s)\, ds + C_4 \right) + g(r). \tag{5.5.9}$$

Therefore, the flow shear, required in (5.3.1), is given by

$$\frac{d\bar{v}}{dr} - \frac{\bar{v}}{r} = -\frac{2}{r^2}\left(\int sg(s)\, ds + C_4 \right) + g(r). \tag{5.5.10}$$

Let

$$F(s) = f(s) - \frac{MR_g C_1}{R_k}.$$

Then

$$\int sg(s)\, ds = \frac{R\, r}{\lambda(R+R_k)}\Big(C_2 I_1(\lambda r) - C_3 K_1(\lambda r) \Big) - \frac{MR_g}{\lambda R_k} C_1\, r^2 +$$

$$+ 2\int \xi I_0(\lambda\xi) \int_{R_1}^{\xi} sK_0(\lambda s)F(s)\, ds d\xi - 2\int \xi K_0(\lambda\xi) \int_{R_1}^{\xi} sI_0(\lambda s)F(s)\, ds d\xi +$$

$$+ R_k j(\overline{ru'v'}) - \frac{R_k}{R_g}\lambda r\left(I_1(\lambda r)\int_{R_1}^{r} sK_0(\lambda s)F(s)\, ds + K_1(\lambda r)\int_{R_1}^{r} sI_o(\lambda s)F(s)\, ds \right). \tag{5.5.11}$$

For brevity in the calculations that follow, we write

$$\int sg(s)\,ds = \frac{R\,r}{\lambda(R+R_k)}\left(C_2 I_1(\lambda r) - C_3 K_1(\lambda r)\right) - \frac{MR_g}{\lambda R_k} C_1 r^2 + G(r), \tag{5.5.12}$$

where $G(r)$ represents the last four terms on the right-hand side of (5.5.11). $G(r)$ contains all the nonlinear mean stress and mean couple stress terms for this integral of $sg(s)$.

Utilizing the boundary conditions (5.2.13), the integration constants C_1, C_2, C_3, and C_4 for equations (5.5.6) and (5.5.8) are:

$$C_1 = \frac{R_k}{MR_g H_2}(H_1 + g_0 H'/H), \quad C_2 = -H' K_0(\lambda R_1)/H, \quad C_3 = H' I_0(\lambda R_1)/H,$$

$$C_4 = 1 + \frac{H'}{H}\left(\frac{R}{\lambda^2(R+R_k)} + \frac{g_0 R_1^2}{\lambda H_2}\right) + \frac{H_1 R_1^2}{\lambda H_2}\,. \tag{5.5.13}$$

Here

$$H_1 = I_0(\lambda R_2)\int_{R_1}^{R_2} sK_0(\lambda s)F(s)\,ds - K_0(\lambda R_2)\int_{R_1}^{R_2} sI_0(\lambda s)F(s)\,ds,$$

$$H_2 = \frac{(R_2-R_1)}{\lambda R_2}\left(1 - \lambda R_2\{I_0(\lambda R_2)K_1(\lambda R_1) - K_0(\lambda R_2)I_1(\lambda R_1)\}\right),$$

$$H' = 1 - \Omega R_2 + \frac{(R_1^2 - R_2^2)}{\lambda H_2} H_1 + G(R_2)/R_2\,, \quad \Omega = \frac{\Omega_2 R_2}{\Omega_1 R_1}\,,$$

$$g_0 = I_0(\lambda R_1)K_0(\lambda R_2) - I_0(\lambda R_2)K_0(\lambda R_1)\,,$$

$$H = \frac{R}{\lambda^2(R+R_k)}\left(\lambda R_2\{K_0(\lambda R_1)I_1(\lambda R_2) + I_0(\lambda R_1)K_1(\lambda R_2)\} - 1\right) + \frac{g_0}{\lambda H_2}(R_2^2 - R_1^2).$$

Incorporating these integration constants (5.5.13), equations (5.5.7) and (5.5.10) become:

$$\frac{d\bar{v}}{dr} = -\frac{\lambda H'}{H}\{K_0(\lambda R_1)I_1(\lambda r) + I_0(\lambda R_1)K_1(\lambda r)\} + \frac{\lambda(r - R_1)}{HH_2}\{I_1(\lambda r)K_1(\lambda R_1) -$$

$$- K_1(\lambda r)I_1(\lambda R_1)\}\{HH_1 + g_0H'\} + \lambda I_1(\lambda r)\int_{R_2}^{r} sK_0(\lambda s)F(s)\,ds +$$

$$+ K_1(\lambda r)\int_{R_2}^{r} sI_0(\lambda s)F(s)\,ds, \tag{5.5.14}$$

and

$$\frac{d\bar{v}}{dr} - \frac{\bar{v}}{r} = \frac{(\lambda-2)}{\lambda}\,\frac{HH_1 + g_0H'}{HH_2} - \frac{2}{r^2}C_4 + \frac{2RH'}{\lambda(R+R_k)H\,r}\{K_0(\lambda R_1)I_1(\lambda r) +$$

$$+ I_0(\lambda R_1)K_1(\lambda r)\} + \frac{RH'}{(R+R_k)H}\{I_0(\lambda R_1)K_0(\lambda r) - K_0(\lambda R_1)I_0(\lambda r)\} +$$

$$+ \frac{2(r - R_1)}{\lambda H\,r}\{HH_1 + g_0H'\}\Big(\lambda r\{I_0(\lambda r)K_1(\lambda R_1) + K_0(\lambda r)I_1(\lambda R_1)\} - 1\Big) +$$

$$+ \frac{\lambda M}{r}C_1(r-R_1)\{K_1(\lambda r)I_1(\lambda R_1) - I_1(\lambda r)K_1(\lambda R_1)\} + \chi(r), \tag{5.5.15}$$

where

$$\chi(r) = -\frac{4}{r^2}\int \xi I_0(\lambda\xi)\int_{R_1}^{\xi} sK_0(\lambda s)F(s)\,ds\,d\xi + R_k\frac{j}{r}\frac{d}{dr}(r\overline{u'v'}) +$$

$$+ \frac{4}{r^2}\int \xi K_0(\lambda\xi)\int_{R_1}^{\xi} sI_0(\lambda s)F(s)\,ds\,d\xi - \frac{2}{r}R_k j(\overline{u'v'}) + F(r) +$$

$$+ \frac{R}{R+R_k}\{I_0(\lambda r)\int_{R_1}^{r} sK_0(\lambda s)F(s)\,ds - K_0(\lambda r)\int_{R_1}^{r} sI_0(\lambda s)F(s)\,ds\} -$$

$$- \frac{\lambda}{r}\frac{R_k}{R_g}\{I_1(\lambda r)\int_{R_1}^{r} sK_0(\lambda s)F(s)\,ds + K_1(\lambda r)\int_{R_1}^{r} sI_0(\lambda s)F(s)\,ds\}.$$

The symbol $O(\text{amp})^n$ will be used to denote an <u>n:th order amplitude</u> that is formed by any product of $a(t)$ and/or $A(t)$. The expression

$\chi(r)$, as expresses above, leads to $O(amp)^4$ terms in the amplitude equations.

The derivation takes the MOS-energy equations, and substitutes a solution with the same spatial form as the solution of the linearized problem. Thus, we make what is called the 'shape assumption', namely that the finite disturbances (e.g. $\underset{\sim}{v}'$ and $\underset{\sim}{\nu}'$) have the same spatial structure as the linear ones, although their amplitudes (e.g. a(t) and A(t)) may differ. This approximation serves to give a simplified derivation of the amplitude equations by neglect of the harmonics, as well as neglect of the distortion of the fundamental ϕ_0. It is a good approximation only if the total nonlinear effect is nearly the same as that due only to the distortion of the mean flow.

Representative computations and important relations are presented in Appendix B.

The amplitude equation for equation (5.3.1), incorporating (5.5.15), is

$$\gamma_1 \frac{da^2}{dt} = -\gamma_2 a^2 - \gamma_3 O(amp)^4 - \gamma_4 \frac{a^2}{M} - \gamma_5 \frac{aA}{R_k}, \tag{5.5.16}$$

where

$$\gamma_1 = \int_{R_1}^{R_2} \left\{\frac{b^2}{r^2} |\Phi|^2 + |v|^2 + \frac{1}{r^2} |\Phi'|^2\right\} r dr;$$

$$\gamma_2 = 2b \int_{R_1}^{R_2} Q(r)\{\Phi_i v_r - \Phi_r v_i\}\, dr, \qquad Q(r) = \frac{d\bar{v}}{dr} - \frac{\bar{v}}{r} - \chi(r)$$

(see (5.5.15));

$$\gamma_3 = 2b \int_{R_1}^{R_2} \chi(r)\{\Phi_i v_r - \Phi_r v_i\}\, dr;$$

$$\gamma_4 = \int_{R_1}^{R_2} \{2b|v|^2 + \frac{2}{r^2}|\Phi''|^2 + \frac{2}{r^4}|\Phi'|^2 + \frac{b^4}{r^2}|\Phi|^2 - \frac{4b^2}{r^2}(\Phi_r''\Phi_r + \Phi_i''\Phi_i) -$$

$$- \frac{4}{r^3}(\Phi_r''\Phi_r' + \Phi_i''\Phi_i') + \frac{2b^2}{r^3}(\Phi_r'\Phi_r + \Phi_i'\Phi_i) + \frac{2}{r^2}|v|^2 + 2|v'|^2 +$$

$$+ \frac{2}{r}(v_r v_r' + v_i v_i')\} \, rdr;$$

$$\gamma_5 = \int_{R_1}^{R_2} \{2b(\zeta_i v_r - \zeta_r v_i) + \frac{1}{r}(\eta_r \Phi_r'' + \eta_i \Phi_i'') - \frac{1}{r^2}(\eta_r \Phi_r' + \eta_i \Phi_i') -$$

$$- \frac{b^2}{r}(\eta_r \Phi_r + \eta_i \Phi_i) - \frac{1}{r}(\nu_r v_r + \nu_i v_i) + (\nu_r v_r' + \nu_i v_i')\} \, rdr.$$

Primes indicate differentiation with respect to r.

We recall that, by solving equations (5.1.24-28), the unknowns Φ, $\hat{v}$, $\hat{\zeta}$, $\hat{\eta}$, and $\hat{\nu}$ are determined. In the above, $\phi_1 = \Phi = \Phi_r + i\Phi_i$, $\hat{v} = v = v_r + iv_i$, $\hat{\zeta} = \zeta = \zeta_r + i\zeta_i$, $\hat{\eta} = \eta = \eta_r + i\eta_i$, and $\hat{\nu} = \nu = \nu_r + i\nu_i$. Also, we applied the shape assumption when we utilized the expressions: $u'(r,t) = -ib\,\frac{\Phi(r)}{r}\,a(t)$, $v'(r,t) = a(t)v(r)$, $w'(r,t) = a(t)\,d\Phi/dr$, $\zeta'(r,t) = A(t)\zeta(r)$, $\eta'(r,t) = A(t)\eta(r)$, and $\nu'(r,t) = A(t)\nu(r)$. All disturbances are approximated up to the first harmonic, so that, for example, $\zeta' = \zeta_1' = \zeta$.

The amplitude equation for equation (5.3.3), incorporating (5.5.14), is

$$\delta_1 \frac{dA^2}{dt} = - \delta_2 aA - \delta_3 O(\text{amp})^4 - \delta_4 \frac{A^2}{R_g} - \gamma_5 \frac{aA}{R_k} - \delta_5 \frac{2A^2}{R_k} - \delta_6 \frac{A^2}{R_b}, \quad (5.5.17)$$

where

$$\delta_1 = j \int_{R_1}^{R_2} \{|\zeta|^2 + |\eta|^2 + |\nu|^2\} \, rdr;$$

$$\delta_2 = 2bj \int_{R_1}^{R_2} Q_1(r)\{\Phi_i \nu_r - \Phi_r \nu_i\}\, dr,$$

$$Q_1(r) = -\frac{\lambda H'}{H}\{K_0(\lambda R_1)I_1(\lambda r) + I_0(\lambda R_1)K_1(\lambda r)\} +$$

$$+ \frac{\lambda(r-R_1)}{HH_2}1 \{I_1(\lambda r)K_1(\lambda R_1) - K_1(\lambda r)I_1(\lambda R_1)\}\{HH_1 + g_o H'\};$$

$$\delta_3 = 2bj \int_{R_1}^{R_2} Q_2(r)\{\Phi_i \nu_r - \Phi_r \nu_i\}\, dr, \qquad Q_2(r) = \frac{d\bar{\nu}}{dr} - Q_1(r)$$

(see (5.5.14));

$$\delta_4 = 2 \int_{R_1}^{R_2} \{(1 + rb^2)(|\zeta|^2 + |\eta|^2 + |\nu|^2) + r|\zeta'|^2 + r|\eta'|^2 + r|\nu'|^2$$

$$+ \zeta_r\zeta_r' + \zeta_i\zeta_i' + \eta_r\eta_r' + \eta_i\eta_i'\}\, dr;$$

$$\delta_5 = 2 \int_{R_1}^{R_2} \{r|\zeta|^2 + |\eta|^2 + r|\nu|^2\}\, dr;$$

$$\delta_6 = 2 \int_{R_1}^{R_2} \{\frac{2}{r^2}|\zeta|^2 + |\zeta'|^2 + |\nu'|^2 + \frac{1}{r}(\zeta_r\zeta_r' + \zeta_i\zeta_i') + 2b(\nu_i'\zeta_r - \nu_r'\zeta_i\}\, rdr.$$

We have thus derived the amplitude equations (5.5.16) and (5.5.17) for finite disturbances imposed on a basic Couette flow between rotating, coaxial cylinders.

Next, we examine the disturbance amplitudes at the threshold between stability and instability.

V.6 Criticality

The threshold between stability and instability is criticality. For the amplitude equations, criticality implies that the magnitude of all the disturbance amplitudes are not changing as time changes. Mathematically, such a state of equilibrium proposes that

$$\frac{da}{dt} = 0 = \frac{dA}{dt} . \tag{5.6.1}$$

Additionally, we assume that $O(amp)^n = 0$ for $n > 3$; specifically, $O(amp)^2$ is much greater than $O(amp)^4$.

Hence, at critical stability (criticality), the amplitude equations (5.5.16) and (5.5.17) produce

$$0 = \gamma_{2c} a^2 + \gamma_4 \frac{a^2}{M_c} + \gamma_5 \frac{aA}{R_{kc}} , \tag{5.6.2}$$

$$0 = \delta_{2c} aA + \delta_4 \frac{A^2}{R_{gc}} + \gamma_5 \frac{aA}{R_{kc}} + \delta_5 \frac{2A^2}{R_{kc}} + \delta_6 \frac{A^2}{R_{bc}} . \tag{5.6.3}$$

The 'c' affixed to nondimensional numbers indicates a 'critical value'. Note that γ_{2c} and δ_{2c} contain critical numbers.

Integrating relation (5.6.1) suggests that $a = mA$, in that, these two disturbance amplitudes are multiples of each other at criticality. For instance, $m = a(0)/A(0)$. Remember that initial conditions are plausible since disturbances are under supercritical conditions. In particular, if we select $m = 1$, then the two disturbance amplitudes are initially of equal magnitude. Then equations (5.6.2) and (5.6.3) become, respectively,

$$0 = \gamma_{2c} + \gamma_4/M_c + \gamma_5/R_{kc} \,, \qquad (5.6.4)$$

$$0 = \delta_{2c} + \delta_4/R_{gc} + \gamma_5/R_{kc} + 2\delta_5/R_{kc} + \delta_6/R_{bc}. \qquad (5.6.5)$$

We have discovered, with some approximation, the critical relationship between the parameters R, R_k, R_g, and R_b ! This critical relationship is defined by equations (5.6.4-5), and thus yields the marginal stability surface, $S_m = S_m(b,R,R_k,R_g,R_b,c)$ where the 'eigenvalue' c is the wave speed with restrictions imposed on it by the assumed supercritical conditions[10].

The marginal stability surface is

$$S_m = \delta_{2c} - \gamma_{2c} + \delta_4/R_{gc} + 2\delta_5/R_{kc} + \delta_6/R_{bc} - \gamma_4/M_c = 0. \qquad (5.6.6)$$

The graph of S_m would indicate, at a glance, the combination of parameters R, R_g, R_b, and R_k that lead to a stable flow, an unstable flow, or a flow in equilibrium. Since the graph of S_m is a hypersurface, only (two- or three-dimensional) traces of S_m can be plotted.

Before graphing S_m, we should decompose γ_{2c} and δ_{2c}, with the intention of recovering the critical numbers that these relations contain. The major difficulty is liberating λ from the Bessel functions, while maintaining the existing integrity of the integrations.

One option is to make the narrow gap approximation. Mathematically, this approximation means that the gap-width $d = R_2 - R_1 \ll 1$. Employing this approximation at this stage of the analysis is burdened by the difficulty of knowing how to express such constants, as $K_0(\lambda R_1)$ and $I_0(\lambda R_2)$, linearly[11]. To ease this burden, the narrow gap approximation should first be utilized when equation (5.5.4) is invoked into the "nonlinear" analysis; that is, linearize $\bar{\nu}$ and $\bar{v}$.

The option we would pursue, involves the utilization of known experimental data, and is similar to the procedure of section VI.1. The ratios (4.7.7) - (4.7.9) are still applicable here, once λ is determined (for fixed Λ). The λ determined for plane Couette flow could be used here, if the same fluid is involved, and vice versa. As in section IV.7, the values of the critical numbers, R_c, R_{gc}, and R_{kc}, involved in the stability of rotational Couette flows, can now be theoretically predicted. Note, however, that we must assume $R_b = \infty$ (i.e. $\alpha = -\beta$). From the adjusted marginal stability surface (5.6.6), the relation $R_{kc}S_m = 0$ yields

$$R_{kc} = \{-(R_{kc}/R_{gc})\delta_4 - 2\delta_5 + (R_{kc}/M_c)\gamma_4\}/(\delta_{2c} - \gamma_{2c}). \qquad (5.6.7)$$

Similarly,

$$R_{gc} = \{-\delta_4 - 2(R_{gc}/R_{kc})\delta_5 + (R_{gc}/M_c)\gamma_4\}/(\delta_{2c} - \gamma_{2c}), \qquad (5.6.8)$$

$$M_c = \{-(M_c/R_{gc})\delta_4 - 2(M_c/R_{kc})\delta_5 + \gamma_4\}/(\delta_{2c} - \gamma_{2c}), \qquad (5.6.9)$$

$$R_c = M_cR_{kc}/(R_{kc} - M_c). \qquad (5.6.10)$$

Utilizing the ratios of the nondimensional numbers (4.7.7-9), the constant, $\delta_{2c} - \gamma_{2c}$, can be evaluated, once λ is known. Finally, to determine R_b, also, requires more detailed analysis of S_m (for instance, its graph).

Numerical procedures, for the rotational Couette flow problem, would be similar to the numerical procedures outlined for the plane Couette flow problem in chapter VI.

VI. NUMERICAL PROCEDURES — PLANE COUETTE FLOW

In chapter IV, a qualitative analysis of the stability of a basic plane Couette flow was presented. In this chapter, we will outline the numerical procedures that will quantitatively substantiate the nonlinear analysis for a basic plane Couette flow.

We begin by listing the sequence of steps necessary to graph the marginal stability surface S_m, and to calculate the theoretically predicted critical numbers R_c, R_{kc}, and R_{gc}.

Step 1: Algorithm to determine λ;

Step 2: Determine the ratios of the nondimensional numbers;

Step 3 (optional): Plot $\bar{u}$ and $\bar{\nu}$ (equations (4.7.1) and (4.7.3)) for $-1 \leqq z \leqq 1$;

Step 4: (a). Determine the unknowns ϕ_r, ϕ_i, ν_r, and ν_i;

(b)(optional). Plot these functions for $-1 \leqq z \leqq 1$;

Step 5: Determine the coefficients of the amplitude equations;

Step 6: Calculate the values of the critical numbers; and

Step 7: Plot traces of $S_m = S_m(b,R,R_k,R_g,c)$ for fixed b and c.

To demonstrate these steps, we choose $\Lambda = 2$ and $\lambda = 5$. This choice was made because $\lambda = 5$ seems to represent fairly, a typical fluid with moderate micropolar properties.

VI.1 Algorithm to Determine λ

If experimental velocity data is available for steady, laminar plane Couette flow, we can then determine λ, for any predetermined Λ, by solving equation (4.7.1) for λ. From (4.7.1), we obtain a function of λ only, such that

$$F(\lambda) = G(z)\sinh\lambda - \sinh(\lambda z) + 2\lambda\Lambda(z - G(z))\cosh\lambda = 0 \qquad (6.1.1)$$

where

$$G(z) = 2u(z) - 1. \qquad (6.1.2)$$

Since $\lambda = 0$ (which represents the classical case) is a root of $F(\lambda)$, we divide through by λ (now requiring $\lambda \neq 0$), to get

$$P(\lambda) = \frac{F(\lambda)}{\lambda} = \frac{G(z)\sinh\lambda - \sinh(\lambda z)}{\lambda} + 2\Lambda(z - G(z))\cosh\lambda = 0. \qquad (6.1.3)$$

Using Newton's method, we obtain the following iterative formula:

$$\lambda_{n+1} = \lambda_n - P(\lambda_n)/P'(\lambda_n) \qquad (n = 0,1,2,3,\ldots) \qquad (6.1.4)$$

where

$$P'(\lambda) = \{G(z)\cosh\lambda - z\cosh(\lambda z)\}(\lambda - 1)/\lambda^2 + 2\Lambda(z - G(z))\sinh\lambda. \qquad (6.1.5)$$

We can predict an initial approximation, λ_0, from Table 4.1; or we can use a method such as linear interpolation on $P(\lambda)$.

Warning: If $G(z) = z$, at least to within the accuracy of the data, then no other roots will be found for $P(\lambda)$. Note that we will always expect $G(z) \simeq z$, because classically $u(z) = (z + 1)/2$, which implies that $G(z) = z$.

To reduce the choice of Λ, from being predetermined, to being 'experimentally' determined, in conjunction with the above algorithm (6.1.4), requires an additional constraint, such as that provided by equation (4.7.3). Of course, we would then require accurate experimental microgyration data for steady, laminar plane Couette flow.

A source of experimental data for velocity (although given graphically) is Reichardt (1956). Other papers of interest, that present numerical insight into classical plane Couette flow are Ellingsen, Gjevik, and Palm (1970), and Orszag and Kells (1980). (Refer to the bibliography for the journal reference.)

VI.2 Ratios of Nondimensional Numbers

Formulas for useful ratios between the parameters, R, R_g, and R_k, were derived in section IV.7. With our demonstration values of $\lambda = 5$ and $\Lambda = 2$, we compute, from equations (4.7.7-9), that

$$\Gamma = R_g/R_k = 50/3 \simeq 16.\bar{6}, \tag{6.2.1}$$

$$R_g/R = 50/3, \tag{6.2.2}$$

$$R/R_k = 1. \tag{6.2.3}$$

These ratios, (6.2.1-3), are required in the execution of the remaining steps. Also, according to our foregoing choices of λ and Λ, we note from (6.2.3) that R coincides with R_k (i.e. $\mu = \kappa$).

VI.3 Graphs of the Laminar Velocity and Microgyration Fields

Figure 6.1 illustrates the velocity given by (2.3.14), with $\lambda = 5$ and $\Lambda = 2$. Recall that all quantities (e.g. u and z) are nondimensional. (Refer to section II.1.) The dashed line in figure 6.1 represents the classical velocity field, $u(z) = (z + 1)/2$.

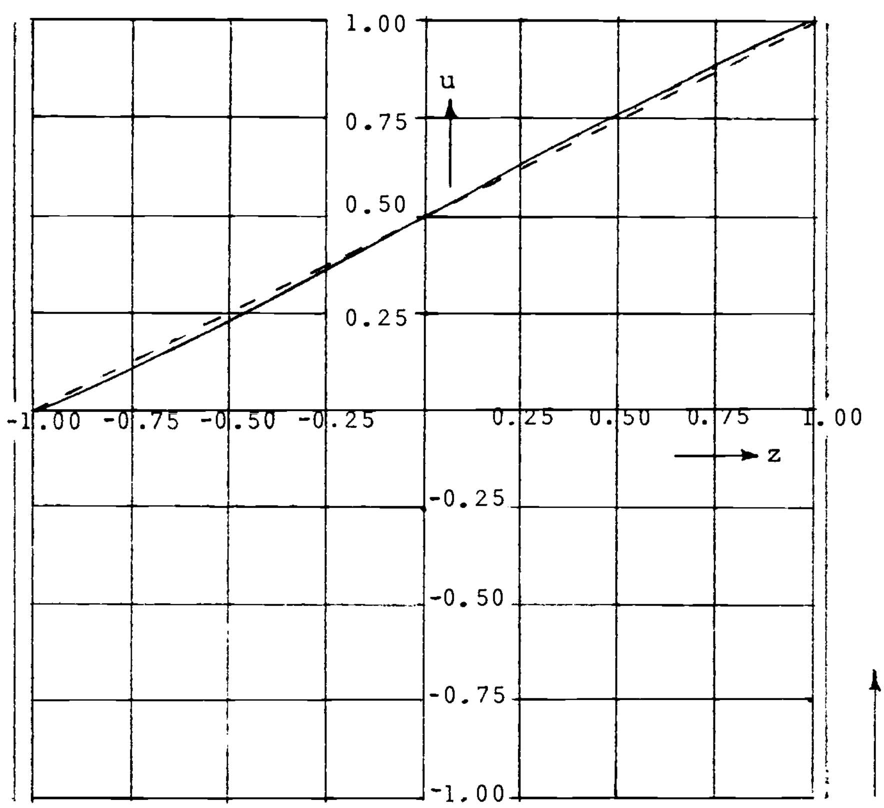

Figure 6.1. Steady, laminar velocity for a micropolar fluid.

In comparing micropolar with classical velocity profiles, we notice the subtle effects of the hyperbolic functions (largest deviation from classical theory is 0.0126 at $z = \pm 0.7$), present in the micropolar solutions of steady, laminar plane Couette flow. Such subtle effects should not lead one to expect them in the case of transition to turbulence also.

Figure 6.2 illustrates the microgyration profile given by (2.3.13), with $\lambda = 5$ and $\Lambda = 2$.

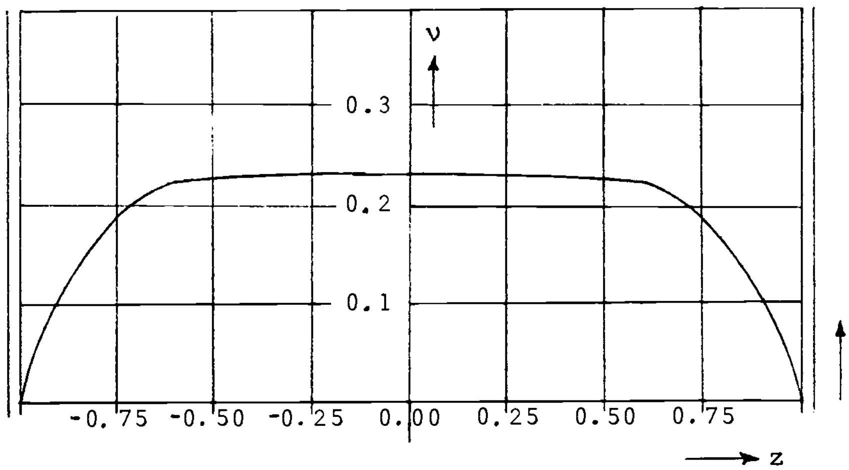

Figure 6.2. Steady, laminar microgyration for a micropolar fluid.

VI.4 Determining the Functions ϕ_r, ϕ_i, ν_r, and ν_i

The MOS-energy equations (4.1.31) and (4.1.32), with (4.1.33) as the boundary conditions, are decomposed into their real and imaginary parts, thereby resulting in the following system of coupled differential equations for the unknown functions ϕ_r, ϕ_i, ν_r, and ν_i:

$$\Lambda_1 D\phi_r = \bar{u}''\phi_i - (\bar{u} - c_r)D\phi_i + \Lambda_2 D\nu_r , \quad (6.4.1)$$

$$\Lambda_1 D\phi_i = (\bar{u} - c_r)D\phi_r - \bar{u}''\phi_r + \Lambda_2 D\nu_i , \quad (6.4.2)$$

$$\Lambda_3 D\nu_r + \Lambda_2 D\phi_r = \Lambda_4 \nu_i - j(\bar{u} - c_r)\nu_i - j\Lambda\bar{u}''\phi_i , \quad (6.4.3)$$

$$\Lambda_3 D\nu_i + \Lambda_2 D\phi_i = \Lambda_4 \nu_i + j(\bar{u} - c_r)\nu_r + j\Lambda\bar{u}''\phi_r , \quad (6.4.4)$$

where

$$D = d^2/dz^2 - b^2, \quad \Lambda_1 = 1/(bM) + c_i, \quad \Lambda_2 = 1/(bR_k),$$

$$\Lambda_3 = 1/(bR_g), \quad \Lambda_4 = 2/(bR_k) - jc_i, \quad \Lambda = R_k/M,$$

$$\bar{u} - c_r = \frac{\sinh(\lambda z)/\cosh\lambda - 2\lambda\Lambda z}{2\tanh\lambda - 4\lambda\Lambda} + \frac{1}{2} - c_r ,$$

$$\bar{u}'' = \frac{\lambda^2\sinh(\lambda z)/\cosh\lambda}{2\tanh\lambda - 4\lambda\Lambda} .$$

Coincidentally, $1/M = 2/R_k$, when $\lambda = 5$ and $\Lambda = 2$.

The boundary conditions (4.1.33) are:

$$\frac{d}{dz}(\phi_r + i\phi_i) = b(\phi_r + i\phi_i) = 0 = \nu_r + i\nu_i \quad \text{at} \quad z = \pm 1. \quad (6.4.5)$$

The fact that $\bar{\nu}' = \Lambda\bar{u}''$ was utilized in equations (6.4.3) and (6.4.4). (Refer to (2.3.2).)

Prior to numerically solving the system (6.4.1-4), we need to be given the wave speed $c = c_r + ic_i$ (obeying the supercritical conditions that we are assuming), the period b, and the constant microinertia j. (For purposes of demonstration, we could arbitrarily select $j = 1$.) Because only values for the ratios of the nondimensional numbers are known, we also equate the classical Reynolds number, Re, to the micropolar parameter, R. Now, the ratios of the nondimesional numbers, as given in (6.2.1-3), can be used to determine the constants Λ_1, Λ_2, Λ_3, and Λ_4.

Remark: An exact solution of the Orr-Sommerfeld equation for plane Couette flows of classical viscous fluids was obtained by W. H. Reid (Reid, 1979).

VI.5 Values for the Coefficients of the Amplitude Equations

Only the coefficients, δ_2, γ_2, δ_5, δ_6, and γ_7, that are required to complete steps 6 and 7, are calculated. Determination of the coefficients is accomplished by numerically integrating the following equations:

$$\gamma_7 = 2\int_{-1}^{1} \{\phi_r''^2 + \phi_i''^2 + 2b^2(\phi_r'^2 + \phi_i'^2) + b^4(\phi_r^2 + \phi_i^2)\}\, dz, \qquad (6.5.1)$$

$$\delta_6 = 2\int_{-1}^{1} \{\nu_r^2 + \nu_i^2\}\, dz, \qquad (6.5.2)$$

$$\delta_5 = 2 \int_{-1}^{1} \{\nu_r'^2 + \nu_i'^2 + b^2(\nu_r^2 + \nu_i^2)\}\, dz \ , \tag{6.5.3}$$

$$\gamma_2 = 2b \int_{-1}^{1} F_1(z)\{\phi_r'\phi_i - \phi_r\phi_i'\}\, dz \ , \tag{6.5.4}$$

where

$$F_1(z) = \frac{C\{e^{-\lambda z} + e^{\lambda(z-2)}\}}{B(1 + R_k/R)} - \frac{\lambda K_1(z - 1)\cosh\{\lambda(z-1)\}}{(1 + 2R_k/R)} +$$

$$+ \frac{C(e^{\lambda} - e^{-3\lambda})\{\cosh\{\lambda(z-1)\} - 1\}}{B(1 + R/R_k)\{\cosh(2\lambda) - 1\}} + \frac{(R_k/R_g)K_1}{1 + R_k/R} \ , \tag{6.5.5}$$

with

$$\frac{C}{B} = \frac{\lambda(1 + R_k/R)/2}{\lambda(1 + R_k/R)e^{-\lambda}\{\cosh(2\lambda) + 1\} + \cosh(2\lambda) - 1} \ , \tag{6.5.6}$$

and

$$K_1 = \frac{C(1 + 2R_k/R)(e^{\lambda} - e^{-3\lambda})}{B\{\cosh(2\lambda) - 1\}} \ , \tag{6.5.6}$$

$$\delta_2 = 2bj \int_{-1}^{1} F_2(z)\{\nu_r\phi_i - \phi_r\Phi_i\}\, dz \ , \tag{6.5.7}$$

where

$$F_2(z) = -\lambda \frac{C}{B}\{e^{\lambda(z-2)} + e^{-\lambda z}\} - \frac{\lambda K_1\{\cosh\{\lambda(z-1)\} - 1\}}{(1 + 2R_k/R)} \ . \tag{6.5.8}$$

Note that primes denote differentiation with respect to z. Also, recall that $\Phi_r + i\Phi_i = \Phi = \nu = \nu_r + i\nu_i$, by our shape assumption.

The remaining steps can now be completed. For step 6, the values of the critical numbers can be calculated from expressions (4.7.10-13). For step 7, traces of the marginal stability surface $S_m = S_m(b,R,R_k,R_g,c)$ (refer to (4.6.6)) can be plotted with the values of b and c, as used in section VI.4.

VII. SUMMARY AND CONCLUSIONS

VII.1 Discussion of the Results

In this thesis we used micropolar fluid dynamics for the problems of flows of micropolar fluids between two parallel plates (plane Couette flow) and between two coaxial, rotating cylinders (rotational Couette flow). Closed-form solutions to these two problems were obtained for steady, laminar flow. A graph of the laminar velocity profile for a plane Couette flow of a micropolar fluid showed only a subtle deviation from the classical plane Couette flow solution predicted from the Navier-Stokes equations. Also, a graph of the laminar microgyration profile for a plane Couette flow of a micropolar fluid was presented, predicting a nearly constant microgyration value of 0.25 throughout the mid-region between the plates.

The two basic flows, laminar plane Couette and laminar rotational Couette, were superimposed by a finite two-dimensional and a finite axisymmetric disturbance, respectively.

The linear theory of micropolar fluid dynamics, for plane Couette and rotational Couette disturbance flows, was briefly pursued, and thus, the micropolar analog of the Orr-Sommerfeld energy equations were derived. Numerical solution of the MOS-energy equations was not performed. The solution to these equations was assumed to be the spatial form (shape assumption) of the superimposed non-linear disturbances.

The nonlinear disturbance equations were derived for the finite disturbance flow. Then, the nonlinear disturbance energy equations were derived. The functions in the disturbance energy equations were assumed to be separable into a spatial part and a temporal part. The spatial part was known from the linearized theory for the disturbance flow. Hence, incorporating these (spatial) solutions into the disturbance energy equations, resulted in the amplitude equations. Then, the equations governing the finite amplitudes of the disturbance flow were known. Finally, the stationary phase of the amplitude functions led to the discovery of the marginal stability surface. Also, using ratios of the nondimensional numbers, the equation for the marginal stability surface directly produced expressions for the critical numbers, R_c, R_{gc}, and R_{kc}. However, for the disturbed rotational Couette flow problem, the values of the critical numbers could only be implicitly implied (due to the presence of R_b) from the marginal stability surface.

Of special importance, was the elucidation of the fluid flow mechanisms induced to deal with the energies of the disturbance flow. A physical interpretation of the disturbance energy equations introduced the concepts of swirl, microenergy of rotation, and mean couple stress, into the repertoire of fluid dynamics. Recall, the 'swirl' created by a disturbance is essentially the divergence of the Coriolis acceleration experienced within the volume elements in the fluid, thereby acting as a source (or sink) for the energy necessary to create turbulent flow. Also, the swirl is the coupling mechanism between the micro- and macrocontinuum volume elements, and thus, provides a tangible link for understanding the transition from laminar to turbulent flow. Recall, the disturbance 'microenergy of rotation' is the kinetic energy of rotation for a micro-volume element about the principal axes (rectangular or cylindrical coordinate axes). A detailed explanation of the fluid flow stabilizers and destabilizers was presented, including inequalities describing flow stability or instability.

All numerical calculations hinge on the establishment of the constant λ. We must emphasize that λ is constant, only because, we have assumed that R, R_k, and R_g remain constant for a given fluid.

The necessary numerical procedures are outlined in chapter VI.

VII.2 Scope of Further Work

Completion of the numerical procedures, as we began to illustrate in chapter VI, is needed to quantitatively substantiate the qualitative nonlinear stability analysis that was presented for disturbed plane Couette and rotational Couette flows. The remaining numerical work is no small task, but will be straightforward from the algorithm of the procedures given.

We anticipate many interesting results from the numerical work for plane Couette and rotational Couette disturbance flows. Also, we expect, that for the first time, theoretically predicted critical numbers will be calculated for the stability of plane Couette flow.

We would like to see a more convincing argument for, or against, the term H_4 (of section V.4) always being positive. (Refer to Appendices A and B.) The implications of H_4 not always being positive are not contradictory, but suggest that the instability (probably caused by vigorous dilating) of the microcontinuum volume elements may, in fact, exist locally; while globally, the flow is stable. Furthermore, a more extensive study of all the flow mechanisms should be made for various flow situations and fluids.

An attempt should be made at finding an exact solution of the MOS-equations for plane Couette flow of micropolar viscous fluids, as was successfully done in the classical case.

Ultimately, we would like to witness, in our lifetime, a presentation of the "closed-form" solutions to the nonlinear equations of micropolar fluid dynamics.

ENDNOTES

1. Hydrodynamic stability applies the abstract concepts of stability for differential equations. The ideas are similar; however, here the physical decay or growth of disturbance waves (solutions) is of paramount importance.

2. An enlightening synopsis of nonlinear stability theory for classical viscous fluids is presented in the book, Hydrodynamic Stability, (Drazin & Reid, 1981, pp. 370-464).

3. A couple in classical continuum mechanics is pictured as a pair of parallel forces having equal magnitude and opposite sense with respect to each other, separated by a moment arm. The moment arm is allowed to tend to zero in a volume element since the latter, regarded as an infinitesimal, approches zero, while the forces are assumed to remain bounded. Thus, the couple vanishes. A similar argument is made for couple stress also, that it vanishes. (Eringen,1967)

4. Strict adherence boundary conditions imply (1) that the microgyration vector vanishes on the plates; and (2) that the fluid comforms to the no-slip condition, meaning that the fluid velocity is equal to the plate velocity when in direct contact with the plate.

5. Selecting this reference velocity precludes the freedom of letting the inner cylinder be stationary.

6. It is found that in turbulent plane Couette flow, the mean flow is antisymmetric, so that although a disturbance of the form (4.2.1) is possible, it would not, in general, lead to an antisymmetrical mean flow; while an infinitesimal disturbance, that is composed of two disturbances travelling in opposite directions, with stream function of the form

$$\psi(x,z,t) = (4.2.1) + \tilde{K}\,\tilde{\psi}_1(-z)\,\exp\{ib(x+\tilde{c}t)\} + K\,\psi_1(-z)\,\exp\{-ib(x+ct)\},$$

does.

7. Period $b = 2\pi f = 2\pi c/\lambda = k$ (wavenumber), where f = frequency, λ = wavelength, and $c = 1$ by the proper choice of units. Hence, the wavenumber and the period are 'equivalent', as shown above.

8. If $\omega_1, \omega_2, \omega_3$ and $\Omega_1, \Omega_2, \Omega_3$ represent the magnitudes of the angular velocities and angular momenta about the principal axes, respectively, then $\Omega_1 = I_1\omega_1$, $\Omega_2 = I_2\omega_2$, and $\Omega_3 = I_3\omega_3$, where I_1, I_2, I_3 are the principal moments of inertia. The kinetic energy of rotation about the principal axes is given by $T = (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)/2$. Thus, we analogously define $e = \frac{j}{2}\nu'^2$ to be called the 'microenergy of rotation' for the disturbance.

9. E.g. $(\mu+\kappa)\partial^2\bar{u}/\partial z^2$, $2\kappa\bar{\nu}$, and $\gamma\partial^2\bar{\nu}/\partial z^2$.

10. Supercritical conditions impose the <u>possibility</u> of flow instabilities prior to the equilibrium state. This instability means $bc_i > 0$ just prior to equilibrium. This situation is marginal stability.

(Definition given in section IV.1.) Yes, it is possible that the flow situation may technically be neutral stability, since we are using non-linear theory.

11. Euler's constant $\Upsilon \simeq 0.5772$

For small values of λr,

$$I_0(\lambda r) \simeq 1 \qquad K_0(\lambda r) \simeq -\Upsilon - \ln(\lambda r/2)$$

$$I_1(\lambda r) \simeq \lambda r/2 \qquad K_1(\lambda r) \simeq 1/(\lambda r) \qquad \text{(Tranter, 1968)}$$

For the small gap-width approximation, assuming $d \ll 1$ implies that $\lambda r = \bar{\lambda}\bar{r}\, d^2 \ll 1$ (since λ is fixed and $R_1 < r < R_2$); λ and r dimensional.

12. The situation is somewhat different when considering a disturbance under 'subcritical' conditions. In this case, a small disturbance does not amplify, but is damped. And now, a suitable 'terminal' condition is applied, namely that the function ϕ_1 shall be an exponentially decreasing function of time in the limit as $t \to +\infty$. By analogy with the supercritical case, ϕ_1 has to be the function $\phi(z)\exp(bc_i t)$, where $c_i < 0$, of the linearized stability theory.

13. The number I^*, found in the constant C of $F_1(z)$ and $F_2(z)$, should be equated to zero (i.e. assumed negligible) since it implies amplitudes of order a^2 and aA. Hence, as seen from equations (4.6.2) and (4.6.3), I^* will lead to amplitudes of order a^4 and a^3A, which we assumed to be zero.

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APPENDICES

APPENDIX A

The derivation of the disturbance energy equations will be, in part, now demonstrated. Multiply equation (5.2.25) by ζ', (5.2.26) by η', and (5.2.27) by ν'; then add the resulting equations to get

$$j\frac{\partial}{\partial t}(\zeta'^2 + \eta'^2 + \nu'^2) + ju'\nu'\frac{\partial\bar{\nu}}{\partial r} + jT_1 + T_2/R_k + 2T_3/R_k = \\ = T_4/R_b + T_5/R_g , \qquad \text{(A.1)}$$

where

$$T_1 = \zeta'\chi_4 + \eta'\chi_5 + \nu'\chi_6;$$

$$T_2 = \zeta'\frac{\partial v'}{\partial z} + \eta'\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) - \nu'\left(\frac{\partial v'}{\partial r} + \frac{v'}{r}\right);$$

$$T_3 = \zeta'^2 + \frac{\eta'^2}{r} + \nu'^2;$$

$$T_4 = \zeta'\{\partial^2\zeta'/\partial r^2 + \frac{\partial}{\partial r}(\zeta'/r) + \partial^2\nu'/\partial r\partial z\} + \\ + \nu'\{\partial^2\zeta'/\partial r\partial z + \frac{\partial}{\partial z}(\zeta'/r) + \partial^2\nu'/\partial z^2\};$$

$$T_5 = \zeta'\{\partial^2\zeta'/\partial r^2 + \frac{\partial}{\partial r}(\zeta'/r) + \partial^2\zeta'/\partial z^2\} + \\ + \eta'\{\partial^2\eta'/\partial r^2 + \frac{\partial}{\partial r}(\eta'/r) + \partial^2\eta'/\partial z^2\} + \\ + \nu'\{\partial^2\nu'/\partial r^2 + \frac{\partial}{\partial r}(\nu'/r) + \partial^2\nu'/\partial z^2\}.$$

Integrate equation (A.1). For $\iint T_1$ rdrdz, the term

$$\nu'\chi_6 = \nu'w'\frac{\partial\nu'}{\partial z} + \nu'u'\frac{\partial\nu'}{\partial r} - \frac{\nu'}{r}\frac{\partial}{\partial r}(r\overline{u'\nu'}).$$

From this, when integrating by parts, a typical term is like

$$\iint \nu' u' \frac{\partial \nu'}{\partial r} rdrdz = \tag{A.2}$$

$$= \int r\nu'^2 u' \Big|_{r=R_1}^{r=R_2} dz \;-\; \int u'\nu'^2\, drdz \;-\; \int \nu' \frac{\partial}{\partial r}(\nu' u')\, drdz \;=$$

$$= S_1 - S_2 - S_3.$$

Due to condition (C2) of section V.2, $S_1 = 0$. Integral S_2 (and, in turn, S_3) equals zero in the mean.

Recall: Mean value of function f on the interval $a \leqq x \leqq b$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Now, since the disturbances are assumed to be of zero mean (with respect to spatial variable z), integral (A.2) equals zero, in the mean.

Specifically,
$$\int_0^{2\pi/b} e^{ibz}\, dz = ib\, e^{ibz}\Big|_0^{2\pi/b} = 0.$$

(E.g. $\nu' = \hat{\nu}\, e^{ib(z-ct)}$)

So, quantites like $\overline{u'\nu'^2}$ are independent of z. Note, however, that $u'\nu'^2$ was replaced by $\overline{u'\nu'^2}$ because only the mean part contributes to the integral (Stuart,1956).

The above result is typical for the term T_1. And hence,

$$\iint T_1\, rdrdz = 0.$$

Let us now look at term T_4. Because

$$\frac{\partial}{\partial z}(\zeta' \partial\nu'/\partial r) = \frac{\partial \zeta'}{\partial z}\frac{\partial \nu'}{\partial r} + \zeta'\, \partial^2\zeta'/\partial r\partial z$$

and

$$\frac{\partial}{\partial r}(\nu' \partial\zeta'/\partial z) = \frac{\partial \nu'}{\partial r}\frac{\partial \zeta'}{\partial z} + \nu'\, \partial^2\zeta'/\partial r\partial z$$

imply that

$$\zeta' \, \partial^2\nu'/\partial r\partial z + \nu' \, \partial^2\zeta'/\partial r\partial z = \frac{\partial}{\partial r}(\nu'\partial\zeta'/\partial z) - 2\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z},$$

we have

$$T_4 = -\left(\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\right)^2 + \frac{\partial}{\partial r}(\nu'\partial\zeta'/\partial z) - 2\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z} + \frac{\nu'}{r}\frac{\partial\zeta'}{\partial z} - \left(\frac{\partial\nu'}{\partial z}\right)^2.$$

Now, since

$$\int \frac{\partial}{\partial r}(\nu'\partial\zeta'/\partial r)\, r dr = -\int \nu'\partial\zeta'/\partial z \, dr,$$

we finally derive

$$\iint T_4 \, rdrdz = -\iint\left(\{\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\}^2 + 2\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z} + \left(\frac{\partial\nu'}{\partial z}\right)^2\right) rdrdz.$$

Since the term $2\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z}$ may <u>not</u> always be positive, poses uncertainty as to the fidelity of T_4 in remaining an energy dissipation mechanism.

We anticipated $\iint 2\frac{\partial\nu'}{\partial r}\frac{\partial\zeta'}{\partial z}$ rdrdz to be equivalent to the integral $\iint\left(\frac{\partial\nu'}{\partial r} + \frac{\partial\zeta'}{\partial z}\right)^2$ rdrdz, but this is false, in general.

If true, then T_4 (i.e. H_4) would always be positive; and hence, $-(\alpha+\beta)H_4$ would always represent the rate of $(\alpha+\beta)$-viscous dissipation of the microenergy of rotation of the disturbance due to the rotational, translational, and dilational effects of the micro-volume elements in the volume considered. The reason this supposition is false, is that the integrals $\int(\partial\nu'/\partial r)^2$ rdr and $\int(\partial\zeta'/\partial z)^2$ dz cannot be shown to be zero, in general. Refer to the presence of these integrals in the term H_2.

<u>Remark</u>: $\zeta' \, \partial^2\zeta'/\partial r^2 + \zeta'\frac{\partial}{\partial r}(\zeta'/r) =$ (A.3)

$$= \frac{\partial}{\partial r}(\zeta'\partial\zeta'/\partial r) + \frac{\zeta'}{r}\frac{\partial\zeta'}{\partial r} - \zeta'^2/r^2 - \partial\zeta'^2/\partial r.$$

Integration by parts yields, with the influence of condition (C2), that

$$\int\frac{\partial}{\partial r}(\zeta'\partial\zeta'/\partial r)\, rdr = r\zeta'\partial\zeta'/\partial r\Big|_{R_1}^{R_2} - \frac{1}{2}\int\partial\zeta'^2/\partial r\, dr =$$

$$= 0 - \zeta'^2/2\Big|_{R_1}^{R_2} = 0.$$

Therefore, integral (A.3) =

$$= -\iint\{\zeta'^2/r^2 + (\partial\zeta'/\partial r)^2\}\, rdrdz = -\iint\left(\frac{1}{r}\frac{\partial}{\partial r}(r\zeta')\right)^2 rdrdz.$$

The sample calculations presented above thus demonstrate how the disturbance energy equations (5.3.3) and (5.3.1) were derived.

APPENDIX B

Representative computations and important relations for the amplitude equations (5.5.16&17) are now presented.

We have assumed the disturbance stream function (as given by (5.2.2)), to be

$$\phi'(r,z,t) \;=\; \phi_1(r,t)e^{ibz} + \tilde{\phi}_1(r,t)e^{-ibz}.$$

With the solutions given by the linearized theory (refer to equations (5.1.18-22)), we make the shape assumption, in that

$$\phi_1 = a(t)\Phi(r), \quad v_1' = a(t)\hat{v}(r), \text{ and } \nu_1' = A(t)\hat{\nu}(r).$$

<u>Notation</u>: $\hat{\underset{\sim}{\nu}}(r) = \underset{\sim}{\nu}_r + i\underset{\sim}{\nu}_i = \underset{\sim}{\nu}$ (E.g. $\hat{\zeta} = \zeta_r + i\zeta_i = \zeta$),

$\hat{v}(r) = v_r + iv_i = v$, and $\Phi(r) = \Phi_r + i\Phi_i$.

Above, primes ´ denoted disturbance quantities. Below, apostrophes ' on Φ, v, ζ, η, and ν denote differentiation with respect to r. The tilda $\sim$ denotes a complex conjugate.

<u>Note</u>: A double arrow $\twoheadrightarrow$ signifies a result after integrating over $z = 0$ to $z = 2\pi/b$. Also, the modulus of a complex number Φ is $|\Phi|^2 = \Phi\tilde{\Phi}$.

$$u'^2 = -\frac{b^2}{r^2}\left(\Phi^2 e^{2ibz} - 2|\Phi|^2 + \tilde{\Phi}^2 e^{-2ibz}\right) \twoheadrightarrow \frac{2b^2}{r^2}|\Phi|^2$$

$$v' = ve^{ibz} + \tilde{v}e^{-ibz}; \quad v'^2 \twoheadrightarrow 2|v|^2$$

$$w'^2 \twoheadrightarrow \frac{2}{r^2}|\Phi'|^2 \quad \text{where} \quad \Phi' \equiv d\Phi/dr.$$

$$\overline{u'v'} = u'v' \twoheadrightarrow \frac{ib}{r}(\tilde{\Phi}v - \Phi\tilde{v}) = (\Phi_r - i\Phi_i)(v_r + iv_i) - (\Phi_r + i\Phi_i)(v_r - iv_i)$$

$$= \frac{2b}{r}(\Phi_i v_r - \Phi_r v_i) \in \text{Reals.}$$

$$2\frac{\partial \nu'}{\partial r}\frac{\partial \zeta'}{\partial z} = 2ib(\nu' e^{ibz} + \tilde{\nu}' e^{-ibz})(\zeta e^{ibz} - \tilde{\zeta}\, e^{-ibz})$$

$$\twoheadrightarrow 2ib(\tilde{\nu}'\zeta - \nu'\tilde{\zeta}) = 4b(\nu_i'\zeta_r - \nu_r'\zeta_i).$$

<u>Important Note</u>: $2\frac{\partial \nu'}{\partial r}\frac{\partial \zeta'}{\partial z} > 0 \Leftrightarrow \nu_i'\zeta_r > \nu_r'\zeta_i$.

Also, of concern in section V.4 is the result that

$$H_4 > 0 \Leftrightarrow \frac{2}{r}(\zeta_r\zeta_r' + \zeta_i\zeta_i') + 4b(\nu_i'\zeta_r - \nu_r'\zeta_i) > 0.$$

This is valid, to the extent of all the assumptions, that we have made up through section V.5, especially the shape assumption.

The sample expressions presented above thus demonstrate how the coefficients for the amplitude equations (5.5.16) and (5.5.17) were derived.