THE USE OF PROCESS FRACTION DEFECTIVE AS A BASIS FOR THE DESIGN OF $\bar{X}$ CHART SAMPLING PROCEDURES

by

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THE USE OF PROCESS FRACTION DEFECTIVE AS A BASIS FOR THE DESIGN OF $\bar{X}$ CHART SAMPLING PROCEDURES

INTRODUCTION

In 1931 W. A. Shewhart published his book Economic Control of Quality of Manufactured Product wherein he set forth the basic principles of control charts for maintaining control of a process. (8, p. 301-422) From that time to the present, although control charts have been widely used, there has been little change in the design of control chart procedures from those proposed by Shewhart. The sample sizes used are nearly always four or five and the control limits are almost without exception set at three standard deviations. The interval between samples is determined from "experience."

This standardization of control chart procedures is not entirely wrong because they do work well in a majority of cases. In addition, the simplicity gained from a standardized procedure has surely helped to promote the rapid growth in the number of firms using control charts. The standardized procedures do not always work well, however, and modifications are necessary to make the sampling procedure fit the needs of the process.

Two rather complete analyses have been made which attempt to determine the sample size, the location of control limits, and the frequency of sampling. Dudley J. Cowden (3) presents a method for evaluating alternate procedures. Acheson Duncan (4) has developed a technique to calculate the optimum design. Both use the total cost of the system as the basic design criteria. The cost is assumed to be a continuous function of the fraction defective which are produced.

This paper presents a method for determining the sampling procedures when the cost is treated as a discontinuous function of the fraction defective. Since the function is not continuous, the basic criteria for design is the fraction defective produced rather than the cost. A method is developed whereby the sampling interval is determined when the "standard" sample size and limits are used. The method is then modified to fit different situations. The methods are presented in the four parts:

Part IA - with the sample size and the location of the control limits fixed, the sampling interval is determined so that the average fraction defective is controlled.

Part IB - the same as IA, except that the maximum fraction defective is controlled.

Part IIA - with the sampling interval fixed, the economic combination of the sample size and the location of the limits are determined to control the average fraction defective.

Part IIB - the same as IIA, except that the maximum fraction defective is controlled.

The application of these methods requires that the frequency and the distribution of the shifts in the process average be known. Since in practice these must be estimated from the history of the process, an analysis is included to test the sensitivity of the methods to error in estimation. The results indicate that the sampling interval is not greatly influenced so long as some continuous function is used.

An example application of the methods for determining the correct sampling interval is included to show how these techniques would be used in an industrial situation. The sampling interval is derived such that the average fraction defective is .001 or the maximum fraction defective is .003.

THE TRADITIONAL CONCEPTS OF PROCESS CONTROL

The original function of the control chart technique as proposed by Shewhart (8) in 1931 was to provide a system whereby assignable cause changes in a process could be distinguished from the chance variations. The procedure is basically a systematic test of hypothesis about the mean of a process statistic. The hypothesis is that the mean of the statistic has remained unchanged. If the hypothesis is rejected by a point falling out of the control limits, an assignable cause change is assumed to have taken place and a search is instituted to locate the source of the change. Some other indicators are used in practice to indicate an assignable cause change. The appearance of trend lines, runs of points on one side of the average, and groupings of points near but within the control limits are the more common. We will consider here the use of points beyond the control limits only. The effect of this will be to make any design of the sampling procedure a bit on the "safe" side.

Control charts operating on this principle may be applied to the process mean, the process variation, the fraction defective, or the number of defects. We shall consider only the $\bar{X}$ chart as used for controlling the process average. In most applications of the $\bar{X}$ chart a range chart is used concurrently. There are surely cases where there is interaction between the two; however we shall consider the $\bar{X}$ chart as being independent.

DESIGN OF THE SAMPLING PROCEDURE

In order to determine the procedure for the maintenance of the control chart, the following parameters must be fixed:

(a) the size of the sample, hereafter denoted by n

(b) the location of the control limits. The limits are fixed at some number of standard deviations of the sample means. The number of standard deviations will be denoted by K.

(c) the time interval between successive samples, denoted here by h.

The sample size is normally set at four or five. This size is not determined by any consideration of the particular application, but is a figure found by experience to work well in most situations. Juran (6, p. 394) estimates that over 90 per cent of existing $\bar{X}$ control charts are based on subgroups of four or five.

The location of the control limits is also traditional. Shewhart proposed the use of three sigma limits and these are generally used in the United States. Accepted practice in Great Britain is to use probability limits set so that the probability of accepting the process when no change has occurred is at a given percentage, usually .99 or .998 (3, p. 235). In either case, three sigma or probability limits, the location of the control limits is a customary procedure rather than a decision based on the particular application. The justification for this procedure is that it is extremely simple and works well in the majority of cases.

The frequency of sampling is not determined by tradition as explicitly as are the sample size and control limits. Practical guides such as Juran's Quality-Control Handbook give only vague suggestions as to sampling frequency. A typical comment is:

> "Although the problem of the proper sample size and frequency of sampling has not been completely solved, solutions of one sort or another have been worked out in practice." (5, p. 371)

IMPROVEMENTS TO THE TRADITIONAL CONCEPTS

Although a great deal has been written about the deficiencies of the traditional concepts of the $\bar{X}$ chart for process control, not many proposals for improvement have been advanced. Dudley Cowden (3) and Acheson Duncan (4) have each presented rather complete proposals for determining the most economic procedures. Since they both pertain directly to this investigation a short critique of each follows.

Cowden's Proposals (3)

In Statistical Methods in Quality Control Cowden presents a method for determining the total cost expected from a process controlled by an $\bar{X}$ chart. The method used does not determine the sampling procedure but is a method for evaluating various arbitrarily selected procedures. Some of his assumptions which are of interest here are:

(a) the process may go out of control at any time, but once corrected will remain in control for the remainder of the day.

(b) the cost of producing defective items is proportional to the fraction defective produced. The relationship may be either linear or exponential. No limits are set on the defectives which will be accepted.

(c) specification limits are set at three standard deviations of the population.

Some of Cowden's findings gained from evaluation of various sampling schemes are:

(a) the frequency of sampling is affected only moderately by the distribution of the expected deviations from the desired average.

(b) the optimum location of the control limits was found to be about 2.5 sigma. Considerable latitude can be tolerated.

(c) without exception a sample size between 3 and 6 was found to be best.

Duncan's Proposals (4)

In an article titled "The Economic Design of $\bar{X}$ Charts Used to Maintain Current Control of a Process," Duncan presents an analytical technique for determining the economic sample size, sampling frequency, and location of limits. Some of the assumptions are:

(a) the average number of times that the process goes out of control per hour is taken to be a point quantity.

(b) the cost of producing defective units is a direct linear function of the fraction defective produced.

(c) the cost of looking for trouble is independent of the amount of shift.

(d) when the process goes out of control, it goes out by a fixed amount. The shift, δ , is regarded as a point quantity.

The method used is to equate the first derivative of the total cost function to zero, and solve for the sample size, sampling frequency, and location of control limits. The resulting equations cannot be solved directly but approximation techniques are given. Significant findings from application of this method are:

(a) the sample size is determined largely by the expected shift in the mean (δ) for:

$\delta = 2$, $n = 2$ to 6

$\delta = 1$, $n = 8$ to 20

$\delta = .05$, $n = 40$ or more

(b) variations in the loss associated with producing defective units has its dominant effect on the sampling interval. If the loss rate is large, the interval should be small. The loss rate has little effect on sample size or location of limits.

(c) the cost of looking for trouble has its greatest effect on the location of the limits. Optimum location of the limits ranges from 2.5 to 3.5 standard deviations.

EXTENSION OF THE IMPROVEMENTS

The basic purpose of the techniques of Duncan and Cowden has been to find the most economic sampling scheme when no limits are imposed on the fraction defective which may be produced. It is proposed that a more useful criteria would be to fix the sampling scheme such that the fraction defective were set at a given amount. This would result in a discontinuous function of the cost with respect to the fraction defective. The cost relationships are shown in Figure 1.

In the proposed method the cost would be set at zero or at some small amount so long as the fraction defective was at or below a fixed amount, P'. If the fraction defective goes above P', the cost jumps to a very high value. Such a system would provide protection for the producer when the product is to be subjected to some form of acceptance sampling plan by the customer.

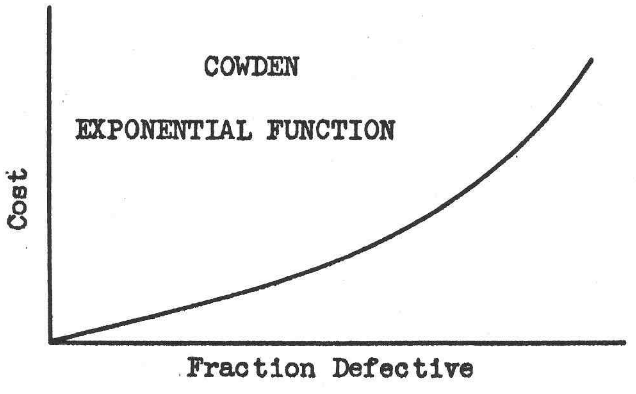
In the following sections a method for determining a sampling plan based on the fraction defective criteria will be developed and illustrated. Four cases will be considered. They are:

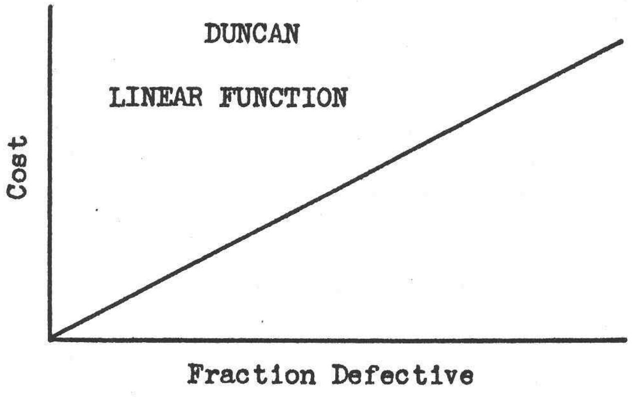
Part IA - The sample size and location of control limits are fixed in advance. The sampling interval will be derived to hold the average fraction defective produced by the process at or below a given amount.

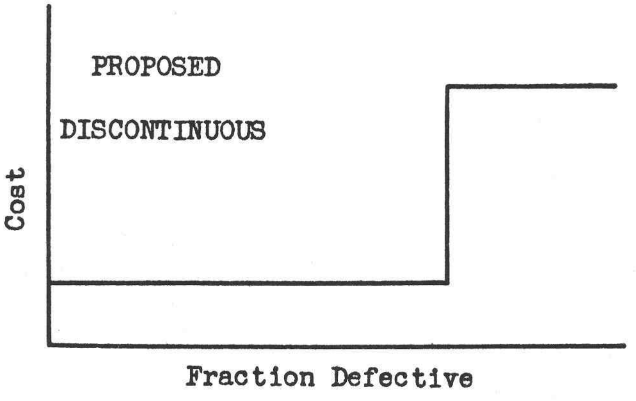
Part IB - The sample size and location of control limits are fixed. The sampling interval will be derived to hold the maximum fraction defective produced by the process at or below a given amount.

FIGURE 1

POSSIBLE RELATIONSHIPS OF COST TO THE
FRACTION DEFECTIVE PRODUCED BY A PROCESS







Part IIA - The sampling interval is fixed in advance. The sample size and location of control limits are derived to fix the average fraction defective.

Part IIB - The sampling interval is fixed in advance. The sample size and location of control limits are derived to fix the maximum fraction defective.

DETERMINATION OF THE SAMPLING INTERVAL TO CONTROL
THE AVERAGE FRACTION DEFECTIVE

PART IA

Statement of Problem: With n and K fixed, find h such that

$\bar{p} \leq \bar{p}'$

THE FRACTION DEFECTIVE

A defective item is defined as any item which falls beyond the specification limits. Since there will be variation between items even when the process is in control, we are forced to consider not individual items but the fraction of the items which will be defective under any particular conditions. If we assume that the distribution of items is normal with constant and known standard deviation, the fraction defective will be a function of the process average and the location of the specification limits.

If the process average is at $\bar{X}$, and the specification limits are designated as USL and LSL, the fraction defective may be found by evaluating the normal function between the limits $-\infty$ to Z_1, and Z_2 to $+\infty$, where $Z_1 = \frac{USL - \bar{X}}{\sigma'}$ and $Z_2 = \frac{\bar{X} - LSL}{\sigma'}$

The fraction defective which will be produced at any value of the process average is easily found by use of normal tables.

For computation it is much easier to express the specification limits in terms of their distance from the desired average as measured in standard deviations. Let:

$$\delta = \frac{\bar{X}' - \bar{X}}{\sigma'} \quad \text{and} \quad S = \frac{USL - \bar{X}'}{\sigma'}$$

If the upper and lower specification limits are equidistant from the desired mean, we obtain

$$Z_1 = S - \delta \quad \text{and} \quad Z_2 = S + \delta$$

The corresponding values of P are found from the normal tables.

The average fraction defective produced by the process over an hours time will be the percent defective produced while the process is "out of control" (P) times the amount of time that the process remains "out of control" ($\bar{T}$) times the frequency of occurrence of the "out of control" condition (λ). Thus:

$$\bar{P} = P\,\bar{T}\lambda \qquad \text{No. 1}$$

Of these variables, λ and P are determined by the process capability. They may of course be changed by changing the process or the location of the specification limits. This is not to imply that λ or P are point quantities, they are random variables and may assume various values within a distribution. $\bar{T}$ is dependent on the sampling scheme.

TIME "OUT OF CONTROL"

$\bar{T}$, the amount of time that the process remains out of control may be varied by changes in the sample size (n), the frequency of sampling, (h), and the number of standard deviations at which we set the control limits (K). We find from Cowden (3), p. 288) that

$$\bar{N} = \frac{1+\beta}{2\alpha} \qquad \text{No. 2}$$

where $\bar{N}$ is the average number of samples which will be taken before the control chart shows the process as "out of control." β is defined as the probability of accepting the process on the basis of a single sample and α is the probability of rejecting. $\alpha(1-\beta)$ For convenience let us substitute R for α and (1-R) for β. If the control limits are set at K standard deviations of the sample averages, R will be the probability that a sample mean lies outside the interval (-K to + K). From the relationship $\sigma_{\bar{x}} = \sigma'/\sqrt{n}$ we note that if the mean shifts to $\delta\sigma'$ the shift is equivalent to $\delta\sqrt{n}\,\sigma_{\bar{x}}$. We may find R by entering the normal tables at the point

$$Z = K - \delta\sqrt{n}$$

The average amount of time that the process remains out of control is then

$$\bar{T} = \bar{N}h = \frac{(2-R)}{(2R)}h \qquad \text{No. 3}$$

where h is the time interval between samples and R is the probability of rejection. The variable R will be dependent upon the sample size and the location of the control limits.

DETERMINATION OF THE SAMPLING INTERVAL

Combining equations No. 1 and No. 3, we obtain the average per-cent defective produced by the process over a long period of time:

$$\bar{P} = P\,\lambda\,\frac{(2-R)}{(2R)}h$$

Since the criteria for solution is that $\bar{P} \leq \bar{P}'$, we substitute $\bar{P}'$ for $\bar{P}$ and solve for h.

$$h = \frac{\bar{P}'}{\lambda P\,\frac{(2-R)}{(2R)}} \qquad \text{No. 4}$$

Of the variables in equation No. 4, only λ, P, and R need be calculated. P' is the desired value of the process fraction defective. The number of occurrences per hour of "out of control" conditions, λ, is a random variable and will assume various values. However, if we assume that the control chart is to operate a long time with respect to λ, then we may use the expected value of λ in our calculations.

The variables P and R are both greatly influenced by the variations in the process mean. R is also affected by the sample size and the location of the control limits. Because of this dependence the proposed method of calculation uses an average value of the term $P\,\frac{(2-R)}{(2R)}$, weighted to correspond to the expected distribution of δ.

THE DISTRIBUTION OF δ

Before any calculations can be made to design the sampling procedure some information must be available regarding the distribution of shifts in the process and the frequency of occurrence of the shifts.

For most operations there will be a large number of different things which may cause a shift. Tool wear, set-up error, equipment failure, and variations in raw material are a few of the more common. If the shifts can result from any of a number of independent and randomly distributed causes, the central limit theorem would indicate that the distribution of δ would be approximately normal.

In addition to the distribution of δ, information is also necessary on the frequency of the shifts. No such handy assumptions can be made concerning $\bar{\lambda}$; the values used must be based on observation of the process. Since data gathered to determine the frequency of deviations must fit the distribution of δ, the assumption of normality does not appear to be reasonable. It is not reasonable because the small deviations, which comprise the bulk of occurrences in the normal distribution, would simply not be discovered. By way of explanation, let us examine how $\bar{\lambda}$ and δ might be determined in practice.

The most convenient source of data would be past records of control charts, if such exist. Each time that the charts showed an "out of control" condition would be recorded as one occurrence, and the amount of the deviation would be plotted to form a frequency histogram. Since a control chart is normally a very weak tool for detecting small changes in the process, the small deviations simply would not show on the records even if they did occur. It would be too likely that another deviation would occur, a large one, which would throw the chart out of control before the small deviation was ever detected. Since only those deviations which are observed can serve as data for the determination of $\bar{\lambda}$ the assumption of a normal distribution appears inadvisable.

Rather than make any assumptions regarding the distribution of δ, it is suggested that the distribution be derived empirically. Schlaifer (7, p. 109) states:

> "Basically, the problem is one to be decided by judgement, and judgement must be based more on a

general understanding of the real phenomena under study than on statistical theory."

Schlaifer also suggests the following steps in smoothing the historical frequency distribution. (7, p. 108)

1. Fit by eye a smooth curve which has the right general shape.
2. Adjust the curve so that the probabilities will add to one.

APPLICATION OF THE GENERAL EQUATION

In order to solve equation No. 4 to find the sampling interval the following information must be obtained.

(a) n, the sample size

(b) K, the location of the control limits

(c) $\bar{P}'$, the desired average percent defective

(d) the probability distribution of δ

(e) $\bar{\lambda}$, the frequency of occurrence of shifts

(f) the location of the specification limits

(g) $\bar{X}'$, the population mean

(h) σ', the population standard deviation

The procedure for solution is best shown by illustration.

Example Calculation

Assume the following information:

(a) $n = 4$ (b) $K = 3$

(c) $\bar{P}' = .01$ (d) $\bar{\lambda} = .1$

(e) $S = 3$ (f) $\bar{X}' = 0$

(g) $\sigma' = 1$

(h) The distribution of δ is triangular, with mean 0 and variance 1. The range of δ is then from $-\sqrt{6}$ to $+\sqrt{6}$

For calculation purposes, the probability function of δ is divided into intervals and treated as a discrete function. In this example the function has been divided into 25 intervals. Since the function is symmetrical, the probabilities are doubled and only one side of the distribution is used. If greater accuracy is desired the distribution can be divided into a greater number of intervals.

The calculations required to evaluate the weighted average of the term $P \frac{(2-R)}{(2R)}$ are shown in Table 1.

From Table 1, $P \frac{(2-R)}{(2R)} = 0.32424$

and, $h = \frac{.01}{(.1)(.32424)} = .308$ hours

or approximately one sample every 20 minutes.

TABLE I

SAMPLE CALCULATION OF $P \frac{(2-R)}{(2R)} W$

δ Class Interval	δ Mid-Point	Weight	$K-\delta\sqrt{N}$	R	$\frac{2-R}{2R}$	P	$P \frac{(2-R)}{(2R)} W$
-.1 to .1	0	.08002	3	.00270	370	.00270	.07980
.1 to .3	.2	.14992	2.6	.00466	214	.00325	.10430
.3 to .5	.4	.13660	2.2	.01390	71.5	.00500	.04880
.5 to .7	.6	.12328	1.8	.03590	27.4	.00820	.02770
.7 to .9	.8	.10996	1.4	.08080	11.9	.01390	.01820
.9 to 1.1	1.0	.09664	1.0	.15870	5.8	.0228	.01280
1.1 to 1.3	1.2	.08332	0.6	.2743	3.14	.0359	.00965
1.3 to 1.5	1.4	.07000	0.2	.4207	1.88	.0548	.00722
1.5 to 1.7	1.6	.05668	-0.2	.5793	1.21	.0808	.00554
1.7 to 1.9	1.8	.04336	-0.6	.7257	.875	.1151	.00437
1.9 to 2.1	2.0	.03004	-1.0	.8413	.690	.1587	.00328
2.1 to 2.3	2.2	.01676	-1.4	.9192	.587	.2119	.00208
2.3 to $\sqrt{6}$	2.4	.00340	-1.8	.9641	.537	.2743	.00050
	Σ	1.0000				Σ	.32424

DETERMINATION OF THE SAMPLING INTERVAL TO CONTROL THE MAXIMUM FRACTION DEFECTIVE

PART IB

Statement of Problem: With n and K fixed, find h such that

$P \leq P'$.

In part IA an expression was derived which would determine the sampling interval such that the average fraction defective would be controlled. The expression can be converted so that the maximum fraction defective is controlled. This is done by replacing the term for the average number of periods before a shift is detected by the maximum number.

MAXIMUM TIME "OUT OF CONTROL"

If the probability of rejecting a process on the basis of any one sample is given as R, the probability of accepting it will be (1-R). Let Q = (1-R). If the probability of the process being accepted at the first sample after a shift is Q, the probability of acceptance of both the first and the second is Q^2, the first, second and third is Q^3, etc. The probability of the shift going undetected for h samples would be Q^h.

Since there is a finite though small probability that the shift will never be detected we cannot say with certainty what the maximum number of sampling periods will be. We can, however, state the probability that the shift will not be detected by the h^{th} sample. Let this probability be designated as ϵ . Then:

$$\epsilon = Q^h$$

and $$h = \frac{\log \epsilon}{\log Q} \qquad \text{No. 5}$$

We can say with a confidence of $(1 - \epsilon)$ that h will be the maximum number of sampling periods before the shift is detected.

Substituting No. 5 into equation No. 4, we obtain

$$h = \frac{P' \log Q}{\lambda P \log \epsilon}$$

or $$h = \frac{P' \log (1-R)}{\lambda P \log \epsilon} \qquad \text{No. 6}$$

Example Calculation

For comparison assume the same information as used in example IA. In addition, let $\epsilon = .10$. The calculations for computing the weighted average of the term $\frac{\ln (1-R)}{P'}$ are shown in Table 2.

From Table 2, $\frac{\ln (1-R)}{P'} = -5.7832$

and, $$h = \frac{(.01)\ (-5.7832)}{(.1)\ (-2.303)}$$

$$h = .251 \text{ hours}$$

or approximately one sample every 15 minutes.

TABLE 2

SAMPLE CALCULATIONS OF $\frac{\ln(1-R)}{P}$

δ Class Interval	δ Mid-Point	Weight	R	(1-R)	ln(1-R)	P	$\frac{\ln(1-R)W}{P}$
-.1 to .1	0	.08002	.00270	.99730	-.0030	.00270	-.0890
.1 to .3	.2	.14992	.00466	.99534	.0050	.00325	-.2310
.3 to .5	.4	.13660	.01390	.98610	.0141	.00500	-.3850
.5 to .7	.6	.12328	.03590	.96410	.0366	.00820	-.5500
.7 to .9	.8	.10996	.08080	.91920	.0845	.01390	-.6680
.9 to 1.1	1.0	.09664	.15870	.84130	.1720	.0228	-.7330
1.1 to 1.3	1.2	.08332	.27430	.72570	.3216	.0359	-.7450
1.3 to 1.5	1.4	.07000	.42070	.57930	.5465	.0548	-.6970
1.5 to 1.7	1.6	.05668	.57930	.42070	.8675	.0808	-.6100
1.7 to 1.9	1.8	.04336	.72570	.27430	-1.2946	.1151	-.4870
1.9 to 2.1	2.0	.03004	.84130	.15870	1.8389	.1587	-.3480
2.1 to 2.3	2.2	.01676	.91920	.08080	2.5133	.2119	-.1990
2.3 to $\sqrt{6}$	2.4	.00340	.96410	.03590	3.3242	.2743	-.0412
						Σ	-5.7832

DETERMINATION OF THE SAMPLE SIZE AND LOCATION OF CONTROL LIMITS TO CONTROL THE AVERAGE FRACTION DEFECTIVE

PART IIA

Statement of Problem: With h fixed, determine the most economic n and K such that $\bar{\bar{P}} \leq \bar{\bar{P}}'$.

In Parts IA and IB the sampling procedure was developed by fixing n and K and determining the sampling interval. Such a procedure would be appropriate for situations where the charts were maintained by operating personnel. By using "standard" values of n and K the simplicity of the control chart is preserved.

If the sampling and the maintenance of the charts is to be done by non-operating personnel it may be more desirable to fix the sampling interval and derive n and K. This would be particularly appropriate where the sampling was done by a roving inspector and the interval was fixed by the time required for him to make his rounds.

Examination of equation No. 4 shows that the only term affected by n and K is the probability of rejecting the process. Solving equation No. 4 for R, we obtain:

$$R = \frac{2\bar{\lambda}Ph}{2P' + \lambda Ph} \qquad \text{No. 7}$$

Since P is dependent on δ, we must again use a weighted average for P according to the distribution of δ. The value of R obtained in this manner is taken to be the required probability of rejection when the process has shifted to the average value of δ.

Since the control chart is a two-sided test of hypothesis regarding the mean with known standard deviation, we may use the following expression for finding n or K.

$$n = \frac{(K+K_R)^2 \sigma^2}{(X - \bar{X})^2} \qquad (2, \text{p. } 123)$$

or

$$n = \frac{(K + K_R)^2}{\delta^2} \qquad \text{No. } 8$$

For application of equation No. 8 we use the mean deviation of δ and obtain K_R from tables of the normal curve from the relationship

$$R = \int_{-\infty}^{-K_R} f(x)dx + \int_{+K_R}^{+\infty} f(x)dx$$

For most δ , one or the other of the terms will be negligible, so only one need be evaluated. When using the normal tables to find K_R care must be taken to select the correct sign. If R is less than 0.50, K_R will be negative.

Evaluation of equation No. 8 requires that a value of n or K be assumed. Since there are infinite combinations of n and K which will satisfy the requirements, the economics of the situation must be considered to determine which combination is best.

ECONOMIC CONSIDERATIONS

The costs of having a control chart which are influenced by n or K are:

(a) the cost of looking for trouble when none exists.

Let this cost be C_1. The probability of this happening is dependent on K. Let α be the probability of looking for trouble when none exists.

(b) the cost of taking and plotting one observation. Let this cost be C_2.

The cost per sample will be

$$C = \alpha C_1 + nC_2 \qquad \text{No. 9}$$

Since we are interested in the least total cost, equating the first derivative of equation No. 9 to zero and solving for n and K would appear to be the appropriate method. However, the resulting expression must be solved by enumeration, so enumeration of the original function appears to be the easiest method.

Since n must be an integer, enumeration is best done by assuming values of n and solving for K and the total cost. Equation No. 8 is converted to:

$$K = \delta\sqrt{n} \; - K_R \qquad \text{No. 10}$$

That combination of n and K which results in the least total cost is taken to be the optimum design of the sampling procedure when the interval is fixed at h.

Example Calculations

To illustrate the calculations required to determine R and K_R, assume the following information:

(a)	$\bar{P}' = .01$	(d)	$\bar{\lambda} = .10$
(b)	$S = 3$	(e)	$\bar{X}' = 0$
(c)	$\sigma' = 1$	(f)	$h = .3$ hours

(g) the distribution of δ is triangular, with mean 0 and variance 1.

The calculations required to determine R and n are shown in Table 3. From Table 3, the weighted average of P is .031768. Then

$$R = \frac{(2)\ (.10)\ (.031768)\ (.3)}{(2)\ (.01) + (.1)\ (.031768)\ (.3)}$$

$$= \frac{.001908}{.02 + .000954}$$

$$= \frac{.001908}{.020954}$$

$$R = .0913$$

From the normal tables, $K_R = -1.33$

In order to determine the least cost combination of n and K, assume:

$$C_1 = \$20$$

$$C_2 = \$.10$$

The sample calculations are shown in Table 4. The minimum cost occurs at $n = 3$ and $K = 2.74$.

TABLE 3
SAMPLE CALCULATIONS OF R AND md

δ Mid-Point	Weight	P	PW	δ W
0	.08002	.00270	.000216	0
.2	.14992	.00325	.000487	.0300
.4	.13660	.00500	.000682	.0547
.6	.12328	.00820	.001010	.0740
.8	.10996	.0139	.001520	.0877
1.0	.09664	.0228	.002200	.0966
1.2	.08332	.0359	.002990	.1000
1.4	.07000	.0548	.003840	.0988
1.6	.05668	.0808	.004570	.0907
1.8	.04336	.1151	.005000	.0780
2.0	.03004	.1587	.004770	.0602
2.2	.01676	.2119	.003550	.0368
2.4	.00340	.2743	.000933	.0082
			Σ .031768	Σ .8157

TABLE 4
SAMPLE CALCULATIONS OF n AND K

n	$n(\delta)$	$\left[n(\delta) - K_R\right]^K$	α	$C_1\alpha$	C_2	C
1	.82	2.15	.0216	.432	.10	.532
2	1.15	2.48	.0131	.262	.20	.462
3	1.41	2.74	.0061	.122	.30	.422
4	1.63	2.96	.0031	.062	.40	.462
5	1.82	3.15	.0019	.038	.50	.538
6	2.00	3.33	.0010	.010	.60	.610
8	2.31	3.64	.0003	.006	.80	.806
10	2.58	3.91	.0001	.002	1.00	1.002
12	2.73	4.06	.00006	.0012	1.20	1.2012
14	3.05	4.38	.00001	.0002	1.40	1.4002

DETERMINATION OF THE SAMPLE SIZE AND LOCATION OF THE CONTROL LIMITS TO CONTROL THE MAXIMUM FRACTION DEFECTIVE

PART IIB

Statement of Problem: With h fixed, determine the most economic n and K such that $P \leq P'$

The procedures developed in Part IIA for controlling the average fraction defective may be used with slight modification. In order to use maximum fraction defective as a criteria, we need only substitute the expression for maximum time before process rejection in place of the average time. The expression for maximum fraction defective becomes:

$$P' = h\bar{\lambda}P \frac{(\log \epsilon \quad)}{(\log (1-R))}$$

solving for log (1-R)

$$\log (1-R) = h\bar{\lambda}P \frac{(\log \epsilon \quad)}{P'} \qquad \text{No. 11}$$

The methods for solution are similar to those used in Part IIA. Weighted averages are used for P and n in order to find R and K_R. The economic combination of n and K is found by enumeration.

Example Calculations

To illustrate the calculations required to determine R and K_R, assume the following information:

(a) $P' = .01$	(d) $\bar{\lambda} = .10$
(b) $S = 3$	(e) $\bar{X}' = 0$
(c) $\sigma' = 1$	(f) $h = .25$ hours
(g) $\epsilon = .10$	

(h) The distribution of δ is triangular, with mean 0 and variance 1.

The calculations for P and n are the same as for Part IIA and are shown in Table 3.

$$\ln(1-R) = \frac{(.25)(.10)(.031768)(\ln.10)}{.01}$$

$$= -.183$$

$$1-R = .833$$

$$R = .167$$

From the normal tables, $K_R = -0.97$

Calculations for the minimum cost combination of n and K are shown in Table 5. The minimum cost occurs at $n = 4$ and $K = 2.60$.

TABLE 5
SAMPLE CALCULATIONS OF n AND K

n	$\sqrt{n}(\delta)$	K $(\sqrt{n}\,\delta - K_R)$	α	$C_1\alpha$	$C_2 n$	C
1	.82	1.79	.0734	1.468	.10	1.568
2	1.15	2.12	.0348	.696	.20	.896
3	1.41	2.38	.0173	.346	.30	.646
4	1.63	2.60	.0093	.181	.40	.581
5	1.82	2.79	.0053	.106	.50	.606
6	2.00	2.97	.0030	.060	.60	.660
8	2.31	3.28	.0014	.028	.80	.828
10	2.58	3.55	.0005	.010	1.00	1.010
12	2.73	3.70	.0002	.004	1.20	1.204

SENSITIVITY ANALYSIS

In the development of the procedures for determining a sampling scheme a basic assumption was made that all relevant data could be accurately obtained. The assumption is necessary from an academic point of view but is not practical for an industrial application of the procedures. The cost would be prohibitive in many cases. Before attempting to apply these procedures we should first determine their sensitivity to error in the data.

We will consider only the procedure for determining the sampling interval as developed in Part IA. Since the other procedures are simply modifications or reversals of procedure IA, the same arguments should apply to them all.

Examination of equation No. 4 shows that the parameters which may cause difficulty are $\bar{P}'$, the desired average percent defective; σ', the population standard deviation; $\bar{\lambda}$, the frequency of "out of control", and the distribution of δ.

$$h = \frac{\bar{P}'}{\bar{\lambda} P \frac{(2-R)}{(2R)}} \qquad \text{No. 4}$$

Since h is linear with respect to $\bar{P}'$ and $\bar{\lambda}$, any error in these parameters will be reflected in the same magnitude by h. The value of $\bar{P}'$ to be used must be chosen to fit the actual requirements of the product. If $\bar{P}'$ is set at .001 to be conservative when a value of .01 would be acceptable, the sampling frequency and the cost of maintaining control will be increased tenfold. The cost of sampling and the importance of maintaining the desired percent defective must be carefully considered.

It should also be pointed out that a control chart cannot maintain the fraction defective at a point below the process capabilities. If the process variation and the location of the specification limits is such that P percent defective are produced even when the process is in control, the average percent defective cannot be lower than P no matter how small the sampling interval. One of the assumptions made in the development of equation No. 4 was that the percent defective produced was virtually zero until the process went "out of control" by some amount δ. Defective items were then produced until the condition was signalled by the control chart. If the specification limits are such that the percent defective produced when the process has not shifted is not much less than $\bar{\bar{P}}'$ the assumption is not valid and equation No. 4 cannot be used.

The percent defective produced at any given process average is dependent on the population standard deviation and will be influenced by error in the measurement of sigma. The relationship is not linear but is a function of the area under the normal curve. Calculations of the sampling interval are quite sensitive to σ'. However if a sufficient history of the process is available the standard deviation can be quite adequately determined and should cause no difficulty in the application of the procedures.

In Part IA the suggested method for application of equation No. 4 was to use an average weighted to correspond to the expected distribution of δ. In order to test the sensitivity of the sampling interval to the distribution of δ, sample calculations were made for normal, triangular, rectangular, and point distributions. For comparison, all distributions have a mean of zero and a standard deviation of one. The test distributions are shown in Figure 2. Tolerance limits were set at 2.5, 3.0, 3.5, and 4.0 standard deviations. These may be considered as changes in the tolerance limits or as changes in the standard deviation. Similar calculations were made of K = 2.5 and K = 3.0. Tightening the control limits makes the control chart a more powerful test and hence lengthens the sampling interval. Increasing the sample size would have the same effect. The results of these calculations are shown in Tables 6 and 7.

Examination of Tables 6 and 7 shows that the normal, triangular, and rectangular distributions yield essentially the same results. This is perhaps to be expected since they all yield a normal distribution of the sample averages. The point distribution gives a sampling interval which is significantly greater than the others.

This may be explained by noting what happens at the center and extreme points of the distributions. When a small shift in the process mean occurs a rather small fraction defective is produced. Although the amount is small, the number of sampling periods which elapse before the shift is detected is very large. In the point distribution these shifts do not occur.

At the extreme points of the distributions the opposite occurs. At values of $\delta = 2.5$ or $\delta = 3.0$ the fraction defective becomes quite large. Although the control chart detects the shift very quickly the result is still a shortening of the sampling interval to protect against these infrequent occurrences.

Both of these arguments against the assumption of a point distribution diminish as the tolerance limits become further from the mean. This is shown in Tables 6 and 7 in that the difference in the sampling interval between the point distribution and the others is much less pronounced when the tolerance limits are at four standard deviations.

FIGURE 2

TEST DISTRIBUTIONS USED IN THE ANALYSIS OF SENSITIVITY TO δ

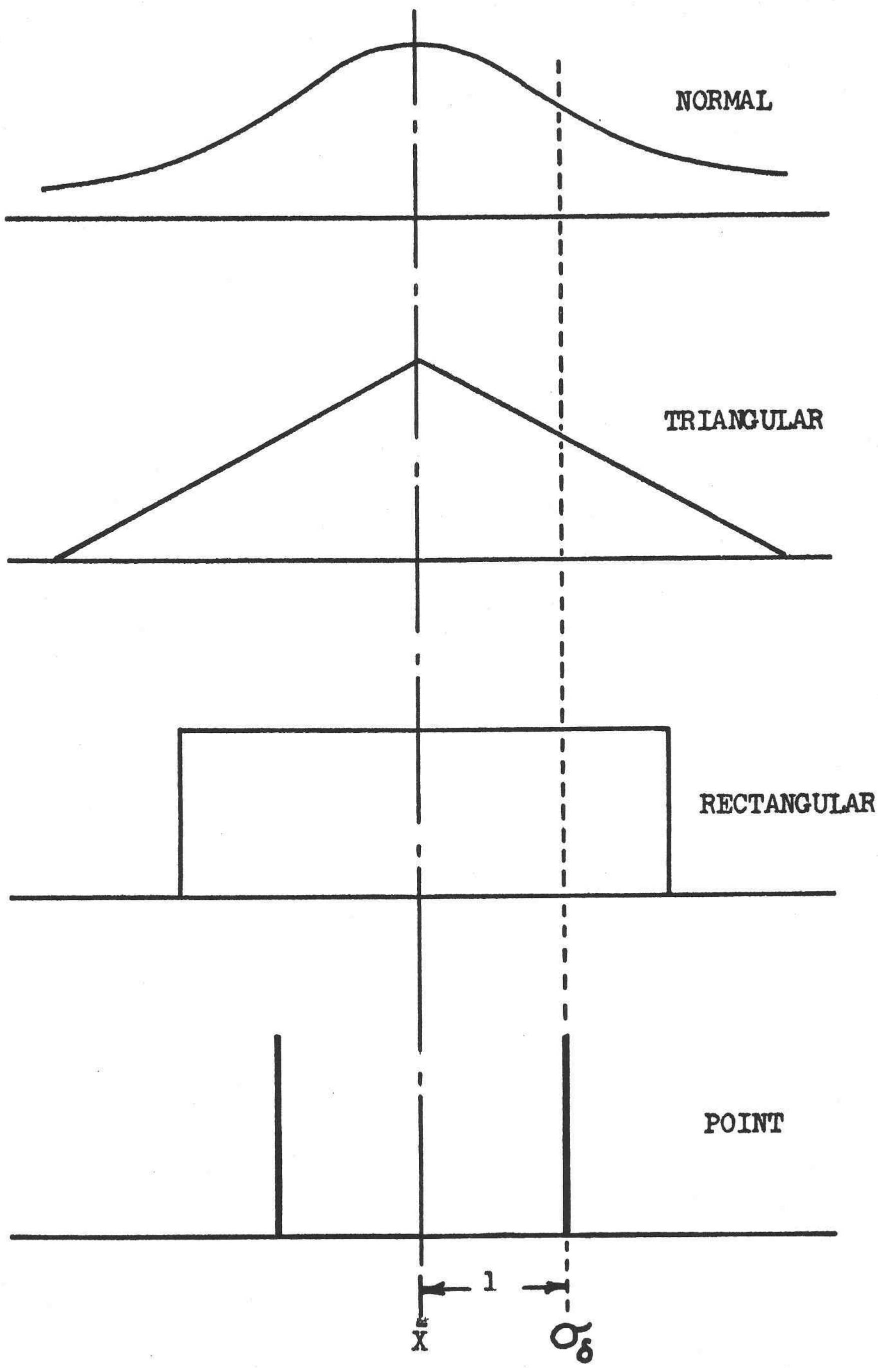


TABLE 6
SAMPLING INTERVALS FOR VARIOUS DISTRIBUTIONS AND TOLERANCE LIMITS

$n = 4, K = 3$

Distribution \ S	2.5	3.0	3.5	4.0
Normal	.072	.301	1.44	10.6
Triangular	.035	.308	1.46	10.0
Rectangular	.121	.360	1.69	10.2
Point	.258	.760	2.78	12.8

TABLE 7
SAMPLING INTERVALS FOR VARIOUS DISTRIBUTIONS AND TOLERANCE LIMITS

$n = 4, K = 2.5$

Distribution \ S	2.5	3.0	3.5	4.0
Normal	.256	.95	3.96	20.3
Triangular	.288	.96	4.13	22.2
Rectangular	.300	1.08	4.50	25.0
Point	.550	1.61	5.80	27.0

EXAMPLE APPLICATION

In order to further illustrate the use of the techniques developed in the previous sections a step-by-step example follows. Historical data were obtained on a heat-treating process for small steel parts. This process was selected because a rather long history in the form of $\bar{X}$ and R charts was available. In order to preserve the confidential nature of these data, the company will be designated as the ABC company.

Although a number of different parts are treated in the process, most of them are about 1/8 by 1/2 by 1 inches. The parts are carried in a continuous flow through a furnace and are then quenched to harden them. The parts are tempered in a salt bath to the desired hardness, which is controlled by adjusting the temperature of the salt bath. Since the temperatures cannot be changed rapidly, about twenty minutes are required for an adjustment to have any effect.

This process has been controlled by $\bar{X}$ and R charts for some time and had achieved a good state of statistical control before any data were taken for this study. The present system uses samples of five taken at half-hour intervals. The control limits are set at three standard deviations of the sample averages. The operating personnel have accepted the control chart procedures and are able to interpret the results. It is felt that any change in the sample size or the location of the control limits would cause considerable confusion and would not be worthwhile. The sampling interval can be changed without much difficulty.

The procedures developed in Parts IA and IB will be applied to this example to determine the sampling interval necessary to achieve an average fraction defective of .001 or a maximum fraction defective of .003.

THE COLLECTION OF NECESSARY DATA

Since the present control charts are operated on the basis of three standard deviation limits and samples of five, these will be retained. The specification limits for this product are set at 49 and 52 Rockwell C. The desired average is 50.5. In order to apply procedures IA and IB, the process standard deviation, the frequency of shifts, and the distribution of shifts must be determined from the history of the process. For this application, the data were taken from control charts covering a period of 575 hours, or 1,150 samples.

The Standard Deviation

The most preferable method of determining the standard deviation would be to compute it from a large number of observations by the relationship

$$\sigma' = \sqrt{\frac{\Sigma(X-\bar{X})^2}{N-1}}$$

In this application the standard deviation was calculated from the control limits. From the relationship

$$\sigma' = \frac{\sqrt{N}\,(UCL-\bar{X})}{K}$$

$$\sigma' = \frac{\sqrt{5}\ (51.0 - 50.5)}{3}$$

$$= .373$$

Figure 3 shows the relationship of the process average, the control limits, and the specification limits.

The Distribution of Shifts

In order to determine the distribution of the shifts in the process mean the control charts were examined for out of control points. The process operated in a steady state only so long as the type of part or the material remained the same. When these were changed, the charts often showed out of control conditions until the temperature could be adjusted sufficiently. Points falling out of control immediately after a material change were not counted. After control had been achieved, an occurrence of a shift was recorded whenever the chart again went out of the limits. Successive points beyond the limits were counted as one occurrence unless notes on the chart indicated that the process had been adjusted between the samples.

These data were then grouped and plotted to form the frequency histogram shown in Figure 4. The histogram appears to be the two tails from an approximately normal distribution. If the sample were sufficiently large, one could use the actual frequencies as the weights for the computation of the sampling interval. Since this sample is rather small, a smoothed distribution is used.

Before drawing the smoothed distribution one must first consider

FIGURE 3

THE CONTROL CHART

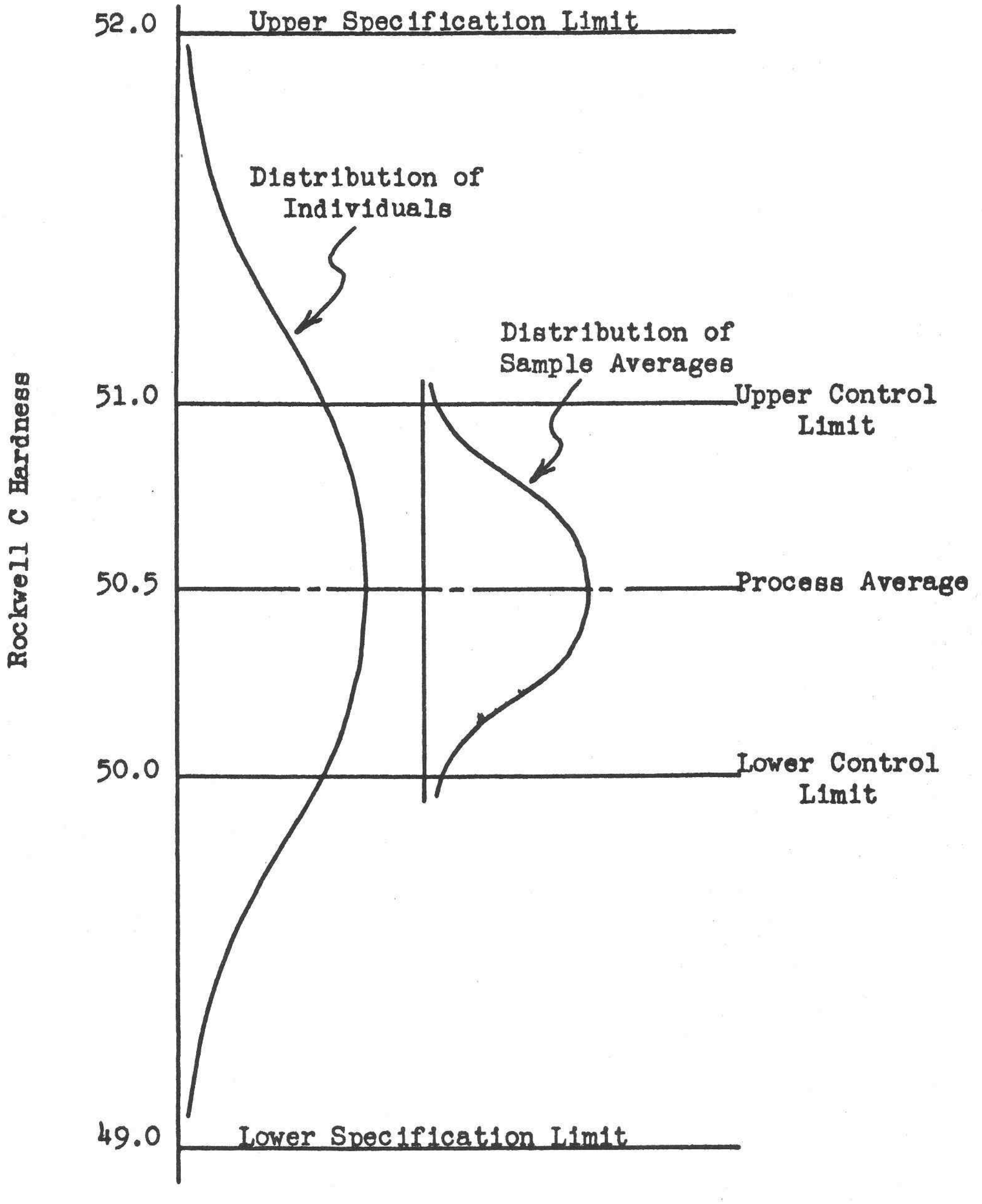


FIGURE 4

FREQUENCY DISTRIBUTION OF SHIFTS

Smooth Frequency	Actual Frequency
.5	
1.5	
2.5	
3.5	2
10.0	11
30.0	
38.0	
40.0	
44.0	
42.0	
39.0	
35.0	34
10.0	9
5.5	1
3.0	3
2.0	1
1.0	
.5	

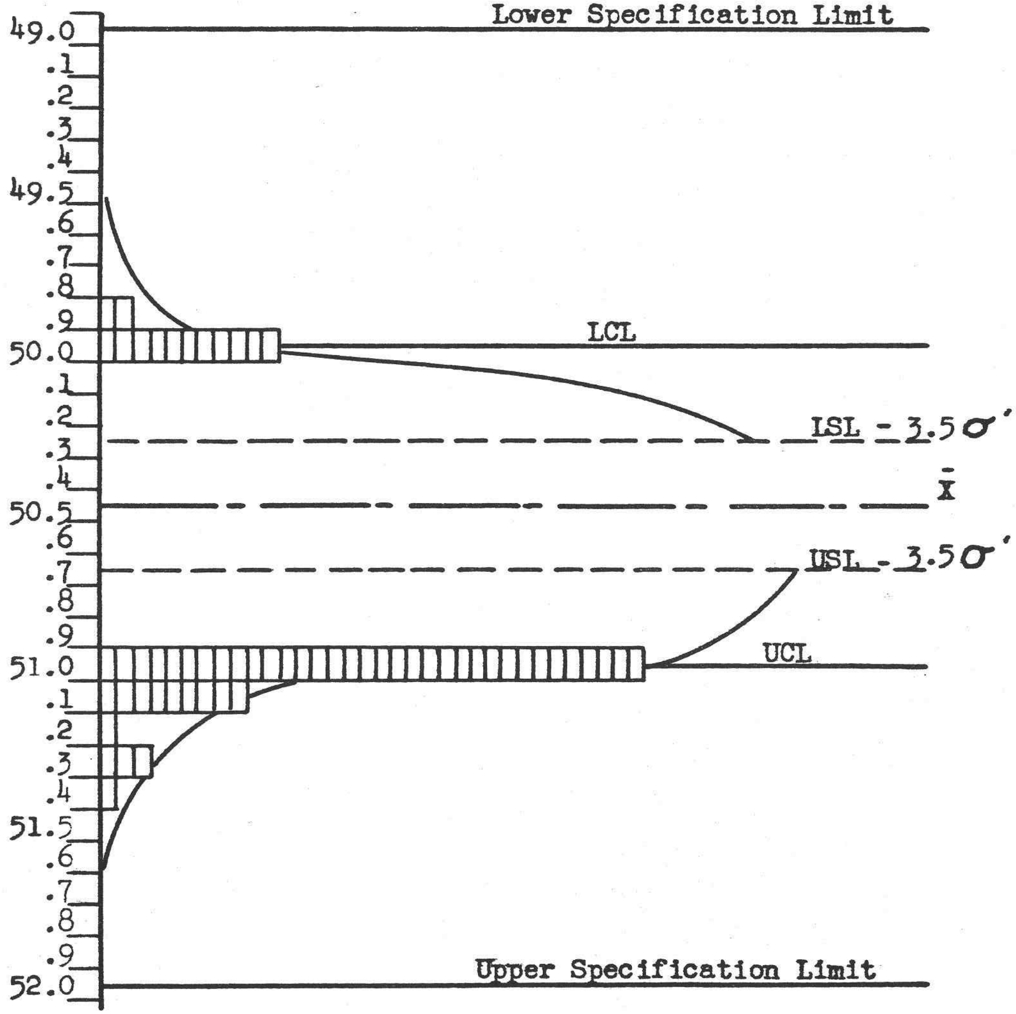


TABLE 8
SMOOTHED PROBABILITY DISTRIBUTION

$\bar{\bar{X}}$	δ		f	W
49.6	-2.41		.5	.00162
49.7	-2.14		1.5	.00487
49.8	-1.88		2.5	.00813
49.9	-1.61		3.5	.01140
50.0	-1.34		10.0	.03230
50.1	-1.07		30.0	.09740
50.2	- .85		38.0	.12300
50.3	- .54		40.0	.13000
- - - -				
50.7	.54		44.0	.14300
50.8	.85		42.0	.13600
50.9	1.07		39.0	.12680
51.0	1.34		35.0	.11300
51.1	1.61		10.0	.03230
51.2	1.88		5.5	.01790
51.3	2.14		3.0	.00974
51.4	2.41		2.0	.00650
51.5	2.68		1.0	.00323
51.6	2.95		.5	.00162
		Σ	308.0	.99881

the process and the source of the data. The nature of the heat treating process is such that small shifts in the mean are more likely than the large ones. Only large shifts appear on the histogram because of the manner of collecting data. Also, many of the smaller shifts will not show because the process was corrected before any points fell beyond the control limits. Shifts, runs, and groups of points near the control limits are used in addition to points beyond the control limits as a basis for adjusting the process. In view of these arguments the curve is extrapolated toward the center. The extrapolation extends to a distance of 3.5 standard deviations from the specification limits. To extend the extrapolation further would make little difference as the fraction defective produced by the smaller shifts is negligible. Such extrapolation is risky business at best, and should not be used unless better data are not available. Even so, the accuracy of the results is surely influenced and should be interpreted accordingly.

The frequency of shifts is read from the smoothed frequency distribution of Figure 4. The shifts are converted to δ by dividing the shift by σ', and the frequencies are converted to probabilities by dividing by the total frequency. This is shown in Table 8.

The Frequency of Shifts

The frequency of shifts in the process mean is calculated from the smoothed frequency distribution by dividing the total frequency of shifts by the total hours of record. Thus:

$$\bar{\lambda} = \frac{308}{575} = .536$$

DETERMINATION OF THE SAMPLING INTERVAL FOR $\bar{P}' = .001$

In order to solve equation No. 4 for the sampling interval, we must first find the weighted average for the term $P\frac{(2-R)}{(2R)}$. The calculations are shown in Table 9. The computational steps are:

1. List the values of δ as computed in Table 8.
2. Calculate $(S - \delta)$. S is found by the relationship

   $$\frac{\text{specification} - \bar{\bar{X}}}{\sigma'}$$

   In this application the specification limits are equidistant from the mean so S is the same for both upper and lower specification limits. The quantity $(S - \delta)$ represents the distance, measured in standard deviations, from the process mean to the specification limit when the mean has shifted to δ.
3. The value P is found by entering the normal tables at the point $(S - \delta)$. P is the area under the normal curve which lies beyond $(S - \delta)$ standard deviations.
4. Calculate $|\delta\sqrt{n}|$. This quantity is the amount of the shift as measured in standard deviations of the sample average.
5. Calculate $K - |\delta\sqrt{n}|$. For this example $K = 3$. This quantity represents the distance from the shifted mean to the control limits, measured in standard deviations of the sample average.
6. Find R, the probability of rejection. This is done

by entering the normal tables at the point $K - |\delta \sqrt{n}|$. Care must be taken in reading R from the tables. Since the tables are not arranged in a standard form no general rule can be given. However, if $K - |\delta \sqrt{n}|$ is negative, R will be greater than .50, and vice versa.

7. Calculate $\frac{1}{R} - \frac{1}{2}$, the average number of periods before the shift is detected. For convenience, this is denoted here by $\bar{N}$.
8. List the weights, or probabilities, of the occurrence of δ. The values shown are as calculated in Table 9.
9. Compute the product $P\bar{N}W$.
10. Sum the term $P\bar{N}W$ for all values of δ to obtain the weighted average.

From Table 9, the weighted average of $P\bar{N}$ is .006501. The sampling interval necessary to maintain an average fraction defective of .001 or less is then

$$h = \frac{\bar{P}'}{\lambda P(\frac{1}{R} - \frac{1}{2})}$$

$$h = \frac{.001}{(.536)(.006501)}$$

$$= .287 \text{ hours}$$

or 17.2 minutes

TABLE 9
EXAMPLE CALCULATIONS

δ	$S-\lvert\delta\rvert$	P	$\lvert\delta\sqrt{n}\rvert$	$3-\lvert\delta\sqrt{n}\rvert$	R	$\frac{1}{R}-\frac{1}{2}$	W	$P\bar{N}W$
-2.41	1.61	.0537	5.5	-2.5	.9938	.51	.00162	.000024
-2.14	1.88	.0301	4.8	-1.8	.9641	.54	.00487	.000079
-1.88	2.14	.0162	4.2	-1.2	.8849	.63	.00813	.000083
-1.61	2.41	.00798	3.6	- .6	.7257	.88	.01140	.000080
-1.34	2.68	.00368	3.0	0	.50	1.50	.03230	.000178
-1.07	2.95	.00159	2.4	.6	.2743	3.15	.09740	.000488
- .85	3.17	.00069	1.9	1.1	.1357	8.88	.12300	.000755
- .54	3.48	.00023	1.2	1.8	.0359	27.30	.13000	.000815
.54	3.48	.00023	1.2	1.8	.0359	27.30	.14300	.000900
.85	3.17	.00069	1.9	1.1	.1357	8.88	.13600	.000855
1.07	2.95	.00159	2.4	.6	.2743	3.15	.12680	.000635
1.34	2.68	.00368	3.0	0	.50	1.50	.11300	.000624
1.61	2.41	.00798	3.6	- .6	.7275	.88	.03230	.000227
1.88	2.14	.0162	4.2	-1.2	.8849	.63	.01790	.000162
2.14	1.88	.0301	4.8	-1.8	.9641	.54	.00974	.000158
2.41	1.61	.0537	5.5	-2.5	.9938	.51	.00650	.000178
2.68	1.34	.0901	6.0	-3.0	.9987	.50	.00323	.000145
2.95	1.07	.1423	6.6	-3.6	.9998	.50	.00162	.000115
							Σ	.006501

DETERMINATION OF THE SAMPLING INTERVAL FOR $P' = .003$

In order to solve equation No. 6 for the sampling interval necessary to maintain the maximum fraction defective at .003 or less, we must first find the weighted average of the term $\frac{\ln(1-R)}{P}$. The calculations required are shown in Table 10, which corresponds to the following steps:

1. List the values of δ as computed in Table 8.
2. List the values of R for each value of δ. The methods for determining R are the same as shown in Table 9.
3. Compute the quantity (1-R).
4. Find the value ln(1-R). Natural logarithms have been used here for convenience. The base of the logarithm is not important so long as it is consistent.
5. List the probability of each value of δ from Table 8.
6. List the values of P to correspond to each value of δ. The values shown in Table 10 were transferred from Table 9. The methods of computation are the same.
7. Compute the product ln(1-R) W/P.
8. Sum the term ln(1-R) W/P to obtain the weighted average.

From Table 10, $\ln(1-R)/P = -184.632$. Before using equation No. 6 to find the sampling interval we must first select a level of confidence. If we set $\epsilon = .10$, then the level of confidence is $1-\epsilon$ or .90. We would thus be about 90% confident that the fraction defective would not exceed .003. Letting $\epsilon = .10$, we find h as:

$$h = \frac{(.003)(-184.632)}{(.536)(-2.303)}$$

$$= .448 \text{ hours}$$

$$\text{or} \quad 26.9 \text{ minutes}$$

According to this model then, the long term maximum fraction defective from this process should not exceed .003 if samples are taken approximately every thirty minutes. Adequate data are not available as to the fraction defective actually produced by this process. However, estimates made from spot checks are that the average fraction defective is between .001 and .002 and that the maximum fraction defective is seldom above .003.

Although the results of this example appear to agree well with the actual performance of the process, this should not be taken as an adequate test of these procedures. The example is given primarily as an illustration, not as a test.

TABLE 10
CALCULATION OF ln(1-R)P

1	2	3	4	5	6	7
δ	R	(1-R)	ln(1-R)	W	P	$\frac{\ln(1-R)\ W}{P}$
-2.41	.9938	.0062	-5.116	.00162	.0537	- .155
-2.14	.9641	.0359	-3.324	.00487	.0301	- .537
-1.88	.8849	.1151	-2.163	.00813	.0162	- 1.088
-1.61	.7275	.2743	-1.295	.01140	.00798	- 1.850
-1.34	.5000	.5000	- .693	.03230	.00368	- 6.090
-1.07	.2743	.7257	- .322	.09740	.00159	-19.700
- .85	.1357	.8643	- .146	.12300	.00069	-26.000
- .54	.0359	.9641	- .037	.13000	.00023	-20.900
- .54	.0359	.9641	- .037	.14300	.00023	-23.000
.85	.1357	.8643	- .146	.13600	.00069	-28.800
1.07	.2743	.7257	- .322	.12680	.00159	-25.600
1.34	.5000	.5000	- .693	.11300	.00368	-21.250
1.61	.7257	.2743	-1.295	.03230	.00798	- 5.240
1.88	.8849	.1151	-2.163	.01790	.0162	- 2.390
2.14	.9641	.0359	-3.324	.00974	.0301	- 1.080
2.41	.9938	.0062	-5.116	.00650	.0537	- .620
2.68	.9987	.0013	-6.600	.00323	.0901	- .236
2.95	.9998	.0002	-8.500	.00162	.1423	- .096
					Σ	-184.632

SUMMARY

This paper has presented a method for determining the sampling interval for $\bar{X}$ control charts on the basis of the desired fraction defective produced by a process. Although "standard" values for the sample size and the location of the control limits are available and are widely used in industry, no standards are available for the sampling interval. The interval must be determined from experience.

The sampling interval is derived from the process fraction defective by first writing an expression for the fraction defective which will result from any sampling interval and then solving this expression for the appropriate interval. The fraction defective is found to be a function of:

n,	the sample size
K,	the location of the control limits
h,	the sampling interval
δ,	the distribution of shifts in the process average
$\bar{\lambda}$,	the average frequency of shifts in the process average
R,	the probability of a point showing out of control when the process has shifted to δ
P,	the fraction defective produced when the process has shifted to δ

The average fraction defective ($\bar{\bar{P}}$) produced by the process over a long period is

$$\bar{\bar{P}} = \bar{\lambda} p \frac{(2-R)\,h}{(2R)}$$

If $\bar{P}'$ is the desired fraction defective, this expression is solved for the sampling interval required to make $\bar{P} = \bar{P}'$.

$$h = \frac{\bar{P}'}{\bar{\lambda} P \frac{(2-R)}{(2R)}}$$

Since the values of P and R are dependent on the distribution of the shifts in the process average, the expression for h must be weighted to correspond to the expected distribution of δ .

The above expression for the sampling interval can be adjusted so that the maximum rather than the average fraction defective is used as the design criteria. It can also be reversed so that the sample size and the location of the control limits are derived when the sampling interval is fixed.

An analysis of the sensitivity of the sampling interval to the distribution of δ is included. The results indicate that the shape of the distribution is not critical so long as it is a continuous function. Treating δ as a point quantity rather than as a random variable does not appear justified.

In order to more fully illustrate the use of these methods for determining sampling intervals an example application is presented. Although the sampling interval and fraction defective predicted by the example calculations agree well with the actual performance of the process, this example should not be considered an adequate test of these methods.

RECOMMENDATIONS FOR FURTHER STUDY

Before the techniques developed in this paper are applied to industrial situations the following should be considered:

1. The effects of the variance of the distribution of δ

   Although the sampling interval does not appear to be too sensitive to the shape of the distribution, errors in determining the variance of δ may impose limits on the use of these methods.

2. Limits on attainable values of P' or $\bar{P}'$

   As was pointed out in the discussion, the control chart cannot maintain a fraction defective which is below the process capability. Before applying these methods to any situation where the desired fraction defective is close to the process capability an analysis should be made to determine what limits must be imposed on P' or $\bar{P}'$.

3. The effects of treating the frequency of shifts as a point quantity rather than as a random variable

4. Limits on the values of $\bar{\lambda}$

   Derivation of the methods assumed that the frequency of shifts was small with respect to the sampling interval. Just how small it must be should be determined.

5. Emperical testing

   Before these methods are used where the results might be critical, they should be thoroughly tested in industrial situations. Because of the historical data required and the subsequent auditing of actual fraction defective such tests will necessarily cover a rather long time span. They are, however, essential.

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APPENDIX

DEFINITION OF TERMS

Basic Parameters

n - the size of a sample taken for an $\bar{X}$ chart

K - the number of standard deviations of the sample averages at which the limits are set

h - the frequency of sampling, expressed in hours between samples

Variables

P' - the desired maximum percent defective from the system

$\bar{P}'$ - the desired average percent defective from the system

P - the percent defective produced at any given deviation from the process average

S - the amount of the shift in the process average, measured in standard deviations

$$S = \frac{\bar{X} - \bar{\bar{X}}'}{\sigma'}$$

$\bar{\lambda}$ - the average frequency with which the process mean shifts from $\bar{X}'$, measured in occurrences per hour

R - the probability of detecting a shift in the process mean on any one sample

α - the probability of a point showing out of control when the mean has <u>not</u> shifted

ϵ - the level of confidence used when predicting the maximum number of intervals which may elapse before a shift is detected

C_1 - the cost of looking for trouble when none exists

C_2 - the cost of taking and charting one observation

$\bar{N}$ - the average number of periods which will elapse before a shift in $\bar{X}$ is detected

$\bar{T}$ - the average number of hours before a shift in $\bar{X}$ is detected

$\bar{X}$ - the average of a sample

$\bar{X}'$ - the desired process average

σ' - the standard deviation of the individuals

USL - the upper specification limit

LSL - the lower specification limit

UCL - the upper control limit

LCL - the lower control limit

δ - the shift in the mean of the process as measured in standard deviations of the population. If a process shifts to $\bar{X}$ from $\bar{X}'$, then

$$\delta = \frac{\bar{X} - \bar{X}'}{\sigma'}$$

$\bar{P}$ - the average percent defective from the system