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The extension and convergence of positive operators is investigated by means of a monotone approximation technique. Some generalizations and extensions of Korovkin's monotone operator theorem on C[0, 1] are given.

The concept of a regular set is introduced and it is shown that pointwise convergence is uniform on regular sets. Regular sets are investigated in various spaces and some characterizations are obtained.

These concepts are applied to the approximate solution of a large class of integral equations.

Convergence of Positive Operators

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CONVERGENCE OF POSITIVE OPERATORS

I. INTRODUCTION

§1. Historical Remarks

The ordinary Riemann integral can be regarded as an extension of the integral of a continuous function to a larger space in the following way. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ denote respectively the linear spaces of all, bounded, and continuous real valued functions on $[0, 1]$. For x in $\mathcal{C}$ define

$$P_0 x = \int_0^1 x(t)dt \quad ,$$

then P_0 is a linear functional defined on $\mathcal{C}$. Moreover P_0 is positive, i. e., $x(t) \geq 0$, x in $\mathcal{C}$ implies $P_0 x \geq 0$.

The usual definition of the Riemann integral in terms of the upper and lower sums yields the following.

THEOREM 1. 1. Suppose $x \in \mathcal{B}$. Then x is Riemann integrable if and only if for each $n = 1, 2, \cdots$, there exist $x_n \in \mathcal{C}$ and $x^n \in \mathcal{C}$ such that

(i) $$x_n(t) \leq x(t) \leq x^n(t), \quad 0 \leq t \leq 1 \; ,$$

(ii) $$P_0(x^n - x_n) \to 0 \text{ as } n \to \infty .$$

If (i) and (ii) are satisfied, then from the positivity of P_0 it follows that the numerical sequences $\{P_0 x_n\}$ and $\{P_0 x^n\}$ both converge to

$$\int_0^1 x(t)dt \quad .$$

If we set

$$Px = \lim_{n\to\infty} P_0 x^n = \lim_{n\to\infty} P_0 x_n$$

then Theorem 1.1 yields a subspace $\mathcal{R}$ (the Riemann integrable functions) such that $\mathcal{C} \subset \mathcal{R} \subset \mathcal{B}$ and P is an extension of P_0 to $\mathcal{R}$.

This extension procedure has been generalized to other positive linear functionals [4] and it yields some useful characterizations of sets of uniform convergence for such functionals. There are applications to the approximate solution of integral equations by numerical integration [1].

We will extend this procedure to more general positive operators and investigate some applications.

§ 2. Ordered Vector Spaces

In this section we establish our notation and conventions and note some standard results which will be used subsequently.

Because of the order theoretic nature of the considerations all vector spaces are real. Usually the results can be extended to complex spaces by considering real and imaginary parts.

An ordered vector space X is a vector space equipped with a transitive, reflexive, antisymmetric relation $\leq$ satisfying the following conditions.

(i) If x, y, z are elements of X and $x \leq y$, then

$$x + z \leq y + z .$$

(ii) If x, y are elements of X and $\alpha > 0$, then

$$x \leq y \text{ implies } \alpha x \leq \alpha y .$$

The positive cone K in an ordered vector space X is defined by $K = \{x \in X : x \geq 0\}$.

Some examples of ordered vector spaces with which we shall be working are:

(1.1) The set of all real sequences, $x = \{x_n\}_1^\infty$, where $x \leq y$ means $x_n \leq y_n$, $n = 1, 2, \cdots$.

(1.2) The set $B(\Omega)$ of all bounded real valued functions f defined on a nonempty set Ω. Here $f \leq g$ means $f(x) \leq g(x)$ for all x in Ω.

(1.3) The set of all measurable functions f defined on some measure space $(\Omega, \mathcal{A}, \mu)$. In this case $f \leq g$ means $f(x) \leq g(x)$ a. e. $[\mu]$.

Whenever we deal with one of these examples, or a subspace, we use the order relation given above.

If A is a subset of an ordered vector space X and if x in X has the following properties:

(i) $x \geq a$ for all a in A,

(ii) $z \geq x$ whenever $z \geq a$ for all a in A,

then x is called the supremum of A and we write $x = \sup A$. The infimum of A, denoted by $\inf A$, is defined dually. If the supremum and infimum of $\{x, y\}$ exist for all x, y in X, then X is called a vector lattice and we write $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$. If X is a vector lattice, then for each x in X the positive part x^+, the negative part x^- and the absolute value $|x|$ are defined by

$$x^+ = x \vee 0, \qquad x^- = x \wedge 0, \qquad |x| = x \vee (-x).$$

If X is a vector lattice the following relations hold for all x, y, z in X:

(1.3) $\quad x = x^+ - x^-, \quad |x| = x^+ + x^-,$

(1.4) $\quad x \leq y$ if and only if $x^+ \leq y^+$ and $y^- \leq x^-$,

(1.5) $\quad |x| \leq y$ if and only if $-y \leq x \leq y$,

(1.6) $\quad |x + y| \leq |x| + |y|, \quad \||x| - |y|\| \leq |x - y|,$

(1.7) $\quad |x^+ - y^+| \leq |x - y|, \quad |x^- - y^-| \leq |x - y|,$

(1.8) $\quad (x + y)^+ \leq x^+ + y^+, \quad (x + y)^- \leq x^- + y^-,$

(1.9) $\quad |(x \vee z) - (y \vee z)| \leq |x - y|.$

The above definitions and identities can be found in any standard reference on ordered vector spaces, for example [13, 15, 17].

DEFINITION 2.1. An ordered vector space X is an ordered topological vector space if X is a topological vector space.

Since this definition does not require any relationship to exist between the order and topological structure, many authors require some additional restrictions on the space X. However there does not seem to be any standard definition, and when additional conditions are needed we shall specifically say so.

If A is a subset of an ordered vector space X, the full hull of A is defined by

$$[A] = \{z \in X: x \leq z \leq y, x, y \in A\} .$$

If $A = [A]$, then A is said to be full.

Suppose X is an ordered topological vector space. The positive cone K is said to be normal if there if there is a neighborhood basis of 0 consisting of full sets.

The proof of the following useful result can be found in [15, p. 62].

PROPOSITION 2.1. If X is an ordered topological vector space with positive cone K, then the following are equivalent.

(i) K is normal.

(ii) There is a neighborhood basis of 0 consisting of sets V for which $0 \leq x \leq y$ and y in V imply x in V.

(iii) For any two nets $\{x_\beta: \beta \in I\}$ and $\{y_\beta: \beta \in I\}$ in X, if

$0 \leq x_\beta \leq y_\beta$ for all β in I and if $\{y_\beta: \beta \in I\}$ converges to 0, then $\{x_\beta: \beta \in I\}$ converges to 0.

An ordered locally convex space is an ordered topological vector space equipped with a Hausdorff locally convex topology. An ordered normed (Banach) space is an ordered topological vector space which is a normed (Banach) space.

The next proposition is proved in [15, p. 63].

PROPOSITION 2. 2. If X is an ordered locally convex space with positive cone K, then the following assertions are equivalent:

(i) K is normal.

(ii) There is a family $\{p_i: i \in I\}$ of seminorms generating the topology such that $0 \leq x \leq y$ implies $p_i(x) \leq p_i(y)$ for all $i \in I$.

A subset B of a vector lattice X is solid if x in B and $|y| \leq |x|$ imply y in B. An ordered topological vector space which is a vector lattice is called a topological vector lattice if there is a basis of neighborhoods of 0 consisting of solid sets. A vector lattice equipped with a norm $\|\cdot\|$ is a normed vector lattice if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$; if X is complete for this norm, X is called a Banach lattice.

Examples of Banach lattices are:

(1.10) $B(\Omega)$ with the supremum norm,

(1.11) $C(\Omega)$ with the supremum norm, Ω a compact Hausdorff topological space, (1.12) $R[0, 1]$, the space of Riemann integrable functions on $[0, 1]$ equipped with the supremum norm,

(1.1) $L_p(\Omega)$, $1 \leq p < \infty$, where $(\Omega, \mathcal{A}, \mu)$ is any measure space.

A map from an ordered vector space X into a vector space Y is called positive if its maps the positive cone in X into the positive cone in Y.

Next we give a result [15, p. 86] that guarantees the continuity of positive linear maps.

PROPOSITION 2.3. If X and Y are ordered topological vector spaces and if the positive cone in Y is normal and the positive cone in X has nonempty interior, then every positive linear map from X into Y is continuous.

COROLLARY 2.1. If X is a toplogical vector space ordered by a cone with nonempty interior, then every positive linear functional on X is continuous.

If X is a Hausdorff topological vector space then X can be completed and the completion of X is essentially unique (cf. [18, pp. 41-48] for the details). A subset A of X is said to be relatively compact if the closure of A is compact. A subset A of a Hausdorff topological vector space is said to be precompact if A is relatively compact when viewed as a subset of the completion of X.

Some authors use the term totally bounded for precompact, but we will reserve the following meaning for this term.

DEFINITION 2.2. Let X be a topological vector space, S a subset. We say S is totally bounded if for each neighborhood U of 0 in X, there exists a finite subset F of S such that $S \subset F + U$.

If X is Hausdorff then S is totally bounded if and only if S is precompact.

PROPOSITION 2.4. A set $S \subset X$ is totally bounded if (and obviously only if) for each neighborhood U of 0 in X there exists a finite set $F \subset X$ (not necessarily contained in S) such that $S \subset F + U$.

PROOF. Let U be any neighborhood of 0 and V a balanced neighborhood of 0 such that $V + V \subset U$. Then there exist points $x_1, \cdots, x_r$ in X with $S \subset (x_1 + V) \cup \cdots \cup (x_r + V)$. We may assume $(x_i + V) \cap S \neq \emptyset$, $i = 1, \cdots, r$. Select y_i in $(x_i + V) \cap S$, $i = 1, \cdots, r$. Then $x_i + V \subset y_i + V + V \subset y_i + U$, which implies S is totally bounded.

§3. Summary of Results

In Chapter II the pointwise convergence of positive operators is examined. A procedure is given that extends a positive operator P_0 mapping a subspace of an ordered vector space into an ordered Banach space. It is shown that the extension P inherits some important properties, involving convergence of operators, of P_0. A generalization of Korovkin's theorem on $C[0, 1]$ to an ordered Banach space setting is given. Some results involving weak convergence of positive operators are established and a Korovkin type theorem on $L_p[0, 1]$ is presented.

Uniform convergence of positive operators is examined in Chapter III. If P, P_k, $k = 1, 2, \cdots$, are positive operators mapping an ordered topological vector space X into an ordered topological vector space Y and $P_k x \to Px$ for each x in X, then on what sets is the convergence uniform? The concept of a regular (or P - regular) set is formulated and it is shown that the convergence is uniform on each P-regular set. Every totally bounded set is regular and every regular set is bounded, but the converses are false in general. Other properties of regular sets are established. For example, regular sets may be combined in various ways to produce other regular sets. The relationship between regular sets and sets which are totally bounded in some locally convex topology is examined. It is shown that every regular set is totally bounded in some locally convex topology, but the regular sets in general do not coincide with the totally bounded sets in any locally convex topology.

In Chapter IV an attempt is made to characterize the regular sets in specific spaces. For the spaces ℓ_p, $0 < p < \infty$, and c_0, the regular sets are the totally bounded sets. If φ is a positive linear functional on c, then the φ-regular sets are either the totally bounded sets or the bounded sets, depending on φ. For $L_p(\Omega)$, $1 \leq p < \infty$, where $(\Omega, \mathcal{A}, \mu)$ is a totally finite measure space, the characterization is more involved. For φ in $L_p^* = L_q$, $1/p + 1/q = 1$, let $\chi(\varphi) = \chi_{[\varphi > 0]}$. If S is a dominated family in $L_p(\Omega)$ such that $\chi(\varphi)S$ is totally bounded in $L_p(\Omega)$, then S is a φ-regular set. Conversely if S is a φ-regular set, then $\chi(\varphi)S$ is totally bounded in $L_p(\Omega)$. The P - regular sets in $R[0, 1]$, where P is the Riemann integral, are investigated. No characterization is obtained, but several results giving sufficient conditions for a set to be regular are given. For example, any bounded set of monotone functions is regular. These results have important applications to the approximate solution of integral equations.

Finally, in Chapter V the previous theory is applied to the integral equation

$$(3.1) \qquad x(t) - \int_\Omega x(s)k(s, t)d\mu(s) = y(t)$$

where Ω is a compact Hausdorff topological space, μ a finite Baire measure, k a bounded kernel and y a continuous function. We assume there is a sequence $\{\varphi_n\}$ of continuous linear functionals converging pointwise to φ, the functional associated with μ. Let μ_n be the Baire measure associated with φ_n and x_n the solution to

$$(3.2) \qquad x(t) - \int_\Omega x(s)k(s, t)d\mu_n(s) = y(t) \quad .$$

If k is continuous the concept of a regular set in conjunction with the theory of collectively compact operator sets yield

$$(3.3) \qquad x_n(t) \to x(t), \quad \text{as } n \to \infty, \text{ uniformly for } t \in \Omega,$$

where x is the solution of (3.1). If k is not continuous, the extension procedure of Chapter II is used to extend φ to a space $R(\Omega)$ containing $C(\Omega)$. It is shown that a function x is in $R(\Omega)$ if and only if x is continuous a. e. $[\mu]$. Again we obtain (3.3) by means of certain convergence theorems established in Chapter II, the concept of a regular set, and the theory of collectively compact operator sets.

II. MONOTONE APPROXIMATION AND POSITIVE OPERATORS

§1. Extensions of Positive Operators

As we have seen, one can consider the Riemann integral as the extension of the integral of continuous or step functions to a larger space of functions. We will show that this same procedure can be applied to a positive operator defined on an ordered vector space.

THEOREM 1.1. Let X_1 be an ordered vector space, X_0 a subspace and Y an ordered Banach space with a normal closed positive cone. Assume P_0 is a positive linear operator mapping X_0 into Y. We define a set X as follows. Given x in X_1, x is in X if and only if, for $n = 1, 2, \cdots$, there exist x_n, x^n in X_0 with

(i) $$x_n \leq x \leq x^n \; ,$$

(ii) $$P_0 x^n - P_0 x_n \to 0 \text{ as } n \to \infty \; .$$

Then X is a subspace and $X_0 \subset X \subset X_1$. The sequences $\{P_0 x^n\}$, $\{P_0 x_n\}$ converge to the same limit. Define

(iii) $$Px = \lim_{n\to\infty} P_0 x^n = \lim_{n\to\infty} P_0 x_n \, , \quad x \in X \; .$$

Then Px is independent of the choices of $\{x^n\}$ and $\{x_n\}$ and P is a positive linear operator mapping X into Y.

PROOF. Let $\{x^n\}$, $\{x_n\}$ be sequences in X_0 such that (i) and (ii) hold. Then for any integers $n, m \geq 1$, $x_n \leq x \leq x^m$ so $P_0x_n \leq P_0x^m$. Let p be any positive integer, then

$$-(P_0x^n - P_0x_n) \leq P_0x^{n+p} - P_0x^n \leq P_0x^{n+p} - P_0x_{n+p}$$

so $\{P_0x^n\}$ is a Cauchy sequence by Proposition 2.1 of Chapter I, whence $P_0x^n \to y$ for some y in Y. Then (ii) implies $P_0x_n \to y$.

Let $\{y_n\}$, $\{y^n\}$ be two other sequences in X_0 satisfying (i) and (ii). Then $x_n \leq x \leq y^m$ implies $P_0x_n \leq P_0y^m$ and, since the positive cone in Y is closed,

$$\lim_{n\to\infty} P_0x^n = \lim_{n\to\infty} P_0x_n \leq \lim_{n\to\infty} P_0y^n = \lim_{n\to\infty} P_0y_n \ .$$

By symmetry, $\lim\limits_{n\to\infty} P_0x^n = \lim\limits_{n\to\infty} P_0y^n$ and so P is well defined by (iii). If $x \in X_0$ take $x_n = x^n = x$, $n = 1, 2, \cdots$, whence $X_0 \subset X$. If $x \in X$ and $x \geq 0$, then $0 \leq x \leq x^n$, $n = 1, 2, \cdots$, so $Px = \lim\limits_{n\to\infty} P_0x^n \geq 0$, i.e., P is a positive operator. The fact that X is a subspace and P is linear follow from these respective properties of X_0 and P_0. This completes the proof.

It is clear from the proof that the extension procedure may be used on an operator P_0 which is positive, but not necessarily linear. For example if X_0 is a subset of X_1 then X would be a subset containing X_0 and P a positive operator defined on X extending P_0.

It should be noted that the norm on Y was never used in the proof, in fact the theorem is still true if Y is a sequentially complete, Hausdorff topological vector space ordered by a normal closed positive cone. However, if the positive cone has nonempty interior, then Y is normable [15, p. 67], so little would be gained by this apparent generality.

If X_1 is a topological vector space, then X need not be closed. To see this let $X_1 = L_1[0,1]$, X_0 the subspace consisting of the continuous functions and P_0 the Riemann integral. Then X is the set of Riemann integrable functions which is not closed in $L_1[0,1]$.

However we do have the following result.

PROPOSITION 1.1. Assume the hypotheses of Theorem 1.1 hold and in addition suppose X_0, X_1 are vector lattices. Then X is a vector lattice.

PROOF. Suppose $x, y \in X$, $x_n \leq x \leq x^n$, $y_n \leq y \leq y^n$, $n = 1, 2, \cdots$, $(P_0 x^n - P_0 x_n) \to 0$, $(P_0 y^n - P_0 y_n) \to 0$. Then
$x_n \vee y_n \leq x \vee y \leq x^n \vee y^n$ and, by (1.9) of Chapter I,
$(x^n \vee y^n) - (x_n \vee y_n) \leq (x^n - x_n) + (y^n - y_n)$. This proves $x \vee y \in X$ and so X is a lattice.

The next result shows that, in Theorem 1.1, X is complete in the sense that another application of the extension procedure yields no further extension.

PROPOSITION 1.2. Let X_0, X, X_1, Y, P_0, P be as in Theorem 1.1. Define a set X_2 by $x \in X_2$ if and only if for each $n = 1, 2, \cdots$, there exist x^n, $x_n \in X_1$ such that $x_n \leq x \leq x^n$, $Px^n - Px_n \to 0$. Then $X_1 = X_2$.

PROOF. The proof is obvious.

A special case of the extension procedure described in Theorem 1.1 has been used in the approximate solutions of certain integral equations by numerical integration [2]. It will be applied by us in Chapter V to a larger class of integral equations.

If, in Theorem 1.1, Y is the real line with the usual topology, then we have the following special case.

COROLLARY 1.1. Let X_1 be an ordered vector space, X_0 a subspace and φ_0 a positive linear functional defined on X_0. We define a set X as follows. Given x in X_1, x is in X if and only if, for $n = 1, 2, \cdots$, there exist x_n, x^n in X_0 such that

(i) $$x_n \leq x \leq x^n$$

(ii) $$\varphi_0 x^n - \varphi_0 x_n \to 0 .$$

Then X is a subspace and $X_0 \subset X \subset X_1$. The sequences $\{\varphi_0 x^n\}$ and $\{\varphi_0 x_n\}$ converge to the same number and if we define

(iii) $$\varphi x = \lim_{n\to\infty} \varphi_0 x^n = \lim_{n\to\infty} \varphi_0 x_n$$

then $\varphi(x)$ is independent of the choices of $\{x^n\}$ and $\{x_n\}$ and φ is a positive linear functional defined on X.

§2. Pointwise Convergence of Positive Operators

We wish to investigate the pointwise convergence of positive operators defined on the spaces X_0, X of Theorem 1.1 .

Theorem 2.1. Let X be an ordered vector space, Y an ordered Banach space with a normal positive cone, and X_0 a subset of X. Assume P is a positive operator mapping X into Y such that for $n = 1, 2, \cdots$, there exist x^n, x_n in X_0 with

(i) $$x_n \leq x \leq x^n ,$$

(ii) $$Px^n - Px_n \to 0 .$$

Let $\{P_i: i \in I\}$ be a net of positive operators mapping X into Y such that $P_i x$ converges to Px for each x in X_0. Then $P_i x$ converges to Px for each x in X.

PROOF. From the inequality $x_n \leq x \leq x^n$ there follows $Px_n \leq Px \leq Px^n$ and $P_i x_n \leq P_i x \leq P_i x^n$ for $i \in I$, $n = 1, 2, \cdots$. Hence

$$P_i x - Px \leq (P_i x^n - Px^n) + (Px^n - Px_n) , \text{ and}$$

$$P_i x - Px \geq (P_i x_n - Px_n) + (Px_n - Px^n) .$$

The theorem follows by Proposition 2.1, Chapter I.

Note that neither P, nor P_i, $i \in I$, was required to be linear. The positivity was the crucial property.

EXAMPLE 2.1. For x in $R[0, 1]$ let

$$Px = \int_0^1 x(t)\,dt ,$$

$$P_n x = \sum_{k=1}^{n} w_{nk} x(t_{nk}), \quad w_{nk} \geq 0, \quad 0 \leq t_{nk} \leq 1, \quad n = 1, 2, \cdots .$$

If $P_n x \to Px$ for all x in $C[0, 1]$, then $P_n x \to Px$ for all x in $R[0, 1]$. Most of the usual quarature formulas, Newton-Cotes excepted, have the above properties.

The next result replaces the operator P with a set of operators.

PROPOSITION 2.1. Let X be an ordered vector space, Y an ordered Banach space with a normal positive cone. Let $\{P_t\colon t \in \Omega\}$ be a set of positive maps from X into Y such that for each x in X, $n = 1, 2, \cdots$, t in Ω, there exist x_{nt}, x^{nt} in X satisfying

(i) $$x_{nt} \leq x \leq x^{nt},$$

(ii) $$P_t x^{nt} - P_t x_{nt} \to 0, \text{ uniformly for } t \text{ in } \Omega .$$

For each t in Ω, suppose $\{P_{it}\colon i \in I\}$ is a net of positive operators mapping X into Y such that for each x in X and each n,

(iii) $$P_{it} x_{nt} \to P_t x_{nt}, \text{ uniformly in } t,$$

(iv) $$P_{it} x^{nt} \to P_t x^{nt}, \text{ uniformly in } t .$$

Then for each x in X

(v) $$P_{it} x \to P_t x, \text{ uniformly for } t \text{ in } \Omega .$$

PROOF. The proof is the same as that for Theorem 2.1 except for the added dependence on t.

DEFINITION 2.1. Let X be a normed linear space and X^* its normed dual. We say that a subset A of X^* is norm-determining if for each x in X we have

$$||x|| = \sup\{|f(x)| : f \in A, ||f|| = 1\}.$$

For example X^* is a norm-determining subset of itself. If Ω is a compact metric space, then the point evaluation functionals comprise a norm-determining subset of $C(\Omega)^*$. If X is a normed linear space ordered by a normal positive cone then each continuous linear functional can be written as the difference of two positive linear functionals [15, p. 72], so there exists at least one subset A of Y^* consisting of positive functionals such that $A - A$ is a norm-determining set.

PROPOSITION 2.2. Let X be an ordered normed linear space and Ω a set of positive linear functionals such that Ω-Ω is a norm determining subset of X^*. Suppose P is a continuous positive linear map from X into X. For x in X, $n = 1, 2, \cdots, t \in \Omega$, assume there exist x_{nt}, x^{nt} such that

(i) $$x_{nt} \leq x \leq x^{nt},$$

(ii) $$t(Px^{nt} - Px_{nt}) \to 0 \text{ as } n \to \infty, \text{ uniformly for } t \text{ in } \Omega.$$

Let $\{P_i : i \in I\}$ be a net of continuous positive linear operators mapping X into X such that for each x in X and each n we have

(iii) $$t(P_i x_{nt}) \to t(Px_{nt}), \text{ uniformly for } t \text{ in } \Omega,$$

(iv) $$t(P_i x^{nt}) \to t(Px^{nt}), \text{ uniformly for } t \text{ in } \Omega.$$

Then $P_i x \to Px$ for each x in X.

PROOF. We define $\varphi_t = P^* t$, $\varphi_{it} = P_i^* t$, t in Ω, i in I. Observe that φ_t, φ_{it} are positive continuous linear functionals on X and we can apply Proposition 2.1 to obtain

$$t(P_i x - Px) \to 0, \text{ uniformly for } t \text{ in } \Omega\text{-}\Omega.$$

Hence $||P_i x - Px|| = \sup_{\substack{||t||=1 \\ t\in\Omega-\Omega}} t(P_i x - Px) \to 0.$

As a special case of this theorem we have the following result.

COROLLARY 2.1. Let X be a normed linear space of bounded real functions $x(t)$, $t \in S$, with the sup norm. Let P be a positive linear operator on X into X. For each x in X, $n = 1, 2, \cdots$, and t in S, assume there exist x_{nt}, $x^{nt} \in X$ such that

(i) $$x_{nt} \leq x \leq x^{nt},$$

(ii) $$(Px^{nt})(t) - (Px_{nt})(t) \to 0 \text{ as } n \to \infty, \text{ uniformly for } t \text{ in } S.$$

Let $\{P_i : i \in I\}$ be a net of positive linear operators mapping X into X such that for each x in X and each n,

(iii) $(P_i x_{nt})(t) \to (Px_{nt})(t)$, uniformly for t in S,

(iv) $(P_i x^{nt})(t) \to (Px^{nt})(t)$, uniformly for t in S.

Then $||P_i x - Px|| \to 0$ for each x in X.

PROOF. For t in S let f_t in X^* be point evaluation at t and set $\Omega = \{f_t: t \in S\} \cup \{0\}$. Then apply Proposition 2.2.

§3. Generalizations of Korovkin's Theorem

We now prove a result which generalizes Korovkin's theorem [12, p. 14] to an arbitrary ordered Banach space.

THEOREM 3.1. Let X be an ordered Banach space, X_0 a subspace and Ω a set of positive continuous linear functionals in X^* such that $\Omega - \Omega$ is a norm-determining subset of X^*. Let P be a continuous, positive linear map from X into X. For each x in X, $n = 1, 2, \cdots$, t in Ω, assume there exists x_{nt}, x^{nt} in X_0 such that

(i) $x_{nt} \leq x \leq x^{nt}$,

(ii) $t(Px^{nt}) - t(Px_{nt}) \to 0$ as $n \to \infty$, uniformly for t in Ω, and, for each x in X and each n,

(iii) $\{x_{nt}: t \in \Omega\}$, $\{x^{nt}: t \in \Omega\}$ are totally bounded.

Let $\{P_k\}$ be a sequence of continuous, positive linear operators on X into X such that

(iv) $||P_k x - Px|| \to 0$ as $k \to \infty$, for each x in X_0.

Then $||P_k x - Px|| \to 0$ as $k \to \infty$, for each x in X.

PROOF. Since $P_k x \to Px$ for each x in X_0 and X is a Banach space, the uniform boundedness principle implies that the convergence is uniform on totally bounded subsets of X_0; in particular, as $k \to \infty$,

$$||P_k x_{nt} - Px_{nt}|| \to 0 \text{ for each } n, \text{ uniformly in } t,$$

$$||P_k x^{nt} - Px^{nt}|| \to 0 \text{ for each } n, \text{ uniformly in } t.$$

Therefore the hypotheses of Proposition 2. 2 are satisfied and so $||P_k x - Px|| \to 0$ for each x in X.

COROLLARY 3. 1. Let X be a Banach space of bounded real valued functions $x(t)$, t in S, with the sup norm and P a positive linear operator on X into X. Suppose X_0 is a subspace such that for each x in X, $n = 1, 2, \cdots$, t in S, there exist x_{nt}, x^{nt} in X_0 with

(i) $\quad x_{nt} \leq x \leq x^{nt}$,

(ii) $\quad (Px^{nt})(t) - (Px_{nt})(t) \to 0$ as $n \to \infty$, uniformly for t in S,

and for each x in X and each n

(iii) $\quad \{x_{nt}:\ t \in S\}$ and $\{x^{nt}:\ t \in S\}$ are totally bounded.

Let $\{P_k\}$ be a sequence of positive linear operators on X into X such that

(iv) $||P_k x - Px|| \to 0$ for each x in X_0.

Then $||P_k x - Px|| \to 0$ for each x in X.

This Corollary is due to Anselone [4]. Korovkin's monotone operator theorem is a special case.

COROLLARY 3.2. (Korovkin) Let P_k, $k = 1, 2, \cdots$, be positive linear operators mapping $C[0, 1]$ into itself. If $P_k x \to x$, as $k \to \infty$, for the three functions $x(t) = 1, t, t^2$, then $P_k x \to x$, as $k \to \infty$, for every x in $C[0, 1]$.

PROOF. Let $X = C[0, 1]$, X_0 the subspace spanned by the three functions $x(t) = 1, t, t^2$. For each x in X and $n = 1, 2, \cdots$, there exist x_{nt}, x^{nt} in X_0 of the forms

(3.1) $$x_{nt}(s) = x(t) - 1/n - a_n(s - t)^2$$

(3.2) $$x^{nt}(s) = x(t) + 1/n + a_n(s - t)^2$$

such that

(3.3) $$x_{nt} \leq x \leq x^{nt}.$$

This follows from the uniform continuity of each x in X. Note that

$$x^{nt}(t) - x_{nt}(t) = 2/n$$

and that, for each x and each n, the sets

$$\{x_{nt}: 0 \leq t \leq 1\}, \{x^{nt}: 0 \leq t \leq 1\},$$

are bounded and equicontinuous, hence totally bounded. Therefore the result follows from Corollary 3.1 with $P = I$.

We now give some results concerning weak convergence of positive operators. If X is a Banach space and x_n, $x \in X$, $n = 1, 2, \cdots$, we shall denote weak convergence of the sequence $\{x_n\}$ to x by $x_n \xrightarrow{w} x$. The proof of the first result is similar to that of Theorem 2.1.

PROPOSITION 3.1. Let X be an ordered vector space, Y an ordered Banach space, and $X_0 \subset X$. Assume P is a positive operator mapping X into Y such that for $n = 1, 2, \cdots$, there exist x^n, $x_n \in X_0$ with

(i) $$x_n \leq x \leq x^n ,$$

(ii) $$P(x^n - x_n) \xrightarrow{w} 0 .$$

Let $\{P_i: i \in I\}$ be a net of positive operators mapping X into Y such that $P_i x \xrightarrow{w} Px$ for each x in X_0. Then $P_i x \xrightarrow{w} Px$ for each x in X.

THEOREM 3.2. Let X be an ordered vector space, Y an ordered Banach space, Ω a set of positive continuous linear functionals on Y and $X_0 \subset X$. Let P be a positive operator on X into Y. For each x in X, $n = 1, 2, \cdots$, and t in Ω, assume there exist x_{nt}, x^{nt} in X_0 such that

(i) $$x_{nt} \leq x \leq x^{nt} ,$$

(ii) $$t[Px^{nt} - Px_{nt}] \to 0 \text{ as } n \to \infty.$$

Let $\{P_i \colon i \in I\}$ be a net of positive operators on X into Y such that

(iii) $$t[P_i x - Px] \to 0 \text{ for each } x \text{ in } X_0, \ t \text{ in } \Omega.$$

Then $t[P_i x - Px] \to 0$ for each x in X, t in Ω.

<u>PROOF.</u> From (i) we obtain $Px_{nt} \leq Px \leq Px^{nt}$, $P_i x_{nt} \leq P_i x \leq P_i x^{nt}$, $i \in I$, $t \in \Omega$, $n = 1, 2, \cdots$. From this we obtain

$$t[P_i x - Px] \leq t[P_i x^{nt} - Px^{nt}] + t[Px^{nt} - Px_{nt}],$$

$$t[P_i x - Px] \geq t[P_i x_{nt} - Px_{nt}] + t[Px_{nt} - Px^{nt}],$$

and the theorem follows from (i) and (ii).

<u>COROLLARY 3.3.</u> Let $\{P_k\}$ be a sequence of positive linear operators mapping $C[0, 1]$ into itself. If the norms $||P_k||$, $k = 1, 2, \cdots$, are uniformly bounded and $(P_k x)(t) \to x(t)$, $0 \leq t \leq 1$, for the three functions $x(t) = 1, t, t^2$, then $(P_k x)(t) \to x(t)$, $0 \leq t \leq 1$, for all x in $C[0, 1]$.

<u>PROOF.</u> This follows from Theorem 3.2, where x_{nt}, x^{nt} are defined by (3.1), (3.2) respectively and the fact that for bounded sequences in $C[0, 1]$, weak convergence is equivalent to pointwise convergence.

<u>COROLLARY 3.4.</u> Let $\{P_k\}$ be a sequence of positive linear operators mapping $C(-\infty, \infty)$ (the space of continuous functions defined on the real line) into itself. Suppose for each compact interval I of

$(-\infty, \infty)$ there exists a constant M_I such that $|P_k(1)(t)| \leq M_I$, for $t \in I$, $k = 1, 2, \cdots$. If $(P_k x)(t) \to x(t)$, $-\infty < t < \infty$, for the three functions $x(t) = 1, t, t^2$, then $(P_k x)(t) \to x(t)$, $-\infty < t < \infty$, for every x in $C(-\infty, \infty)$.

Now we shall consider the convergence of operators on the space $L_p[0, 1]$, $1 \leq p < \infty$.

<u>THEOREM 3. 3.</u> Let $\{P_k\}$ be a sequence of positive linear operators mapping $L_p[0, 1]$ into itself. Suppose $P_k x \overset{w}{\to} x$ for the three functions $x(t) = 1, t, t^2$. Then $P_k x \overset{w}{\to} x$ for every x in $L_p[0, 1]$ if and only if the norms $\|P_k\|$, $k = 1, 2, \cdots$, are uniformly bounded.

<u>PROOF.</u> If $P_k x \overset{w}{\to} x$ for each x in $L_p[0, 1]$, the the norms $\|P_k\|$, $k = 1, 2, \cdots$, would be bounded by the uniform boundedness principle. Suppose there is a constant M such that $\|P_k\| \leq M$, $k = 1, 2, \cdots$. Let $x \in L_p[0, 1]$, assume x is continuous, and let $f \in L_p^*[0, 1] = L_q[0, 1]$, $1/p + 1/q = 1$, $f \geq 0$. Define $\varphi_t(s) = (t - s)^2$, $0 \leq s, t \leq 1$. For each $n = 1, 2, \cdots$, and each t in $[0, 1]$ we have x_{nt}, x^{nt} as defined in (3. 1), (3. 2) and satisfying (3. 3). From the positivity of f and P_k we obtain for each $k, n \geq 1$

$$(3.4) \qquad \int_0^1 f(t)[P_k x)(t) - x(t)]\, dt \leq \int_0^1 x(t)f(t)[P_k 1)(t) - 1]\, dt$$

$$+ n^{-1} \int_0^1 f(t)(P_k 1)(t)dt + a_n \int_0^1 f(t)[P_k \varphi_t)(t)]\, dt,$$

$$(3.5) \qquad \int_0^1 f(t)[(P_k x)(t) - x(t)]\,dt \geq \int_0^1 x(t)f(t)[(P_k 1)(t) - 1]\,dt$$

$$- n^{-1} \int_0^1 f(t)(P_k 1)(t)dt - a_n \int_0^1 f(t)[P_k \varphi_t)(t)]\,dt .$$

The fact that $\int_0^1 f(t)[(P_k\varphi_t)(t)]\,dt \to 0$ as $k \to \infty$ yields $f(P_k x) \to f(x)$. Now let x in $L_p[0, 1]$ be arbitrary. There exists a sequence $\{x_n\}$ of continuous functions such that $x_n \to x$ in $L_p[0, 1]$. Then $|f(P_k x - x)| \leq |f(P_k x - P_k x_n)| + |f(P_k x_n - x_n)| + |f(x_n - x)| \leq$ $||f||\,(M + 1)\,||x_n - x|| + |f(P_k x_n - x_n)|$ so $f(P_k x) \to f(x)$. Finally for arbitrary f in $L_p^*[0, 1]$ write $f = f^+ - f^-$ to obtain $f(P_k x) \to f(x)$. The proof is complete.

Now we can give a Korovkin type theorem for $L_p[0, 1]$.

THEOREM 3.4. Let $\{P_k\}$ be a sequence of positive linear operators mapping $L_p[0, 1]$ into itself. If

(i) the norms $||P_k||$, $k = 1, 2, \cdots$, are uniformly bounded,

(ii) $P_k 1 \to 1$,

(iii) $P_k x \overset{w}{\to} x$ for the two functions $x(t) = t, t^2$, then $P_k x \to x$ for all x in $L_p[0, 1]$.

PROOF. Let G be the set of all $g \in L_p[0, 1]$ such that g is the characteristic function of subinterval of $[0, 1]$ or the characteristic function of the complement of such a subinterval. Since the closed linear span of G is $L_p[0, 1]$ and the norms $||P_k||$ are uniformly bounded, it suffices to show $P_k g \to g$ for each g in G. For g in G let $Z_g = \{t \in [0, 1]: g(t) = 0\}$. By Theorem 3.3 we have $P_k g \xrightarrow{w} g$ which implies

(3.6) $$\int_{Z_g} (P_k g)(t)dt \to 0 \text{ as } k \to \infty, \; g \in G.$$

Let g in G be fixed and f = 1 - g. Then f is in G and by (3.6) we have

$$\int_{Z_f} (P_k f)(t)dt \to 0 \text{ as } k \to \infty.$$

Now

$$\int_{Z_f} (P_k f)(t)dt = \int_{Z_f} [(P_k 1)(t) - (P_k g)(t)]\, dt$$

$$= \int_{Z_f} [(P_k 1)(t) - 1]\, dt + \int_{Z_f} [1 - (P_k g)(t)]\, dt$$

which implies that

$$\int_{Z_f} |1 - (P_k g)(t)|\, dt \to 0 \text{ as } k \to \infty.$$

Then (3.6) yields

$$\int_0^1 |(P_k g)(t) - g(t)|\, dt \to 0 \text{ as } k \to \infty.$$

Let $\{P_{k'} g\}$ be an arbitrary subsequence of $\{P_k g\}$. Then there is a further subsequence, say $\{P_{k''}\}$ such that $(P_{k''} g)(t) \to g(t)$ a.e., as $k'' \to \infty$. Let $\varepsilon > 0$ be given, then by Egoroff's theorem there exists as set A with $m(A) < \frac{1}{4}\varepsilon[2^p + 2^{2p}]^{-1}$ and on $[0, 1] - A = A^c$ such that $(P_{k''}g)(t) \to g(t)$ uniformly. Hence for k'' sufficiently large

$$\int_{A^c} |(P_{k''}g)(t) - g(t)|^p dt < \frac{1}{2}\varepsilon .$$

On the other hand

$$\int_A |(P_{k''}g)(t) - g(t)|^p dt \leq m(A)[2^p + 2^{2p}] + 2^{2p}(||P_{k''}1 - 1||^p < \frac{1}{2}\varepsilon$$

for k'' sufficiently large. Therefore $P_{k''}g \to g$ in $L_p[0, 1]$ and thus $P_k g \to g$ in $L_p]0, 1]$. This completes the proof.

A similar result was established in[10] using the three functions $x(t) = 1$, $\sin t$, $\cos t$, except it was assumed that $P_k x$ converged strongly to x for all three functions.

III. UNIFORM CONVERGENCE OF POSITIVE OPERATORS AND REGULAR SETS

§1. Uniform Convergence of Positive Operators

Let X and Y be topological vector spaces and P, P_k, k = 1, 2, ⋯, continuous linear maps from X into Y. Assume $P_k x \to Px$ for each x in X. We wish to determine subsets of X on which the convergence is uniform. If the spaces X and Y have an order structure and P_k, P are positive operators then the sets on which convergence is uniform will be larger than otherwise.

We state a generalization of the Banach-Steinhaus theorem, a proof of which can be found in [18, p. 347].

PROPOSITION 1.1. Let X be a barreled topological vector space, and Y a locally convex space. The following properties of a subset H of the space L(X, Y) of continuous linear maps of X into Y are equivalent.

(i) H is bounded for the topology of pointwise convergence;

(ii) H is bounded for the topology of bounded convergence;

(iii) H is equicontinuous.

PROPOSITION 1.2. Let X be a barreled topological vector space, Y a locally convex topological vector space. Let P, P_k, k = 1, 2, ⋯, be linear maps from X into Y, P_k continuous, such that $P_k x \to Px$ for each x in X. Then P is continuous and the convergence is uniform on totally bounded subsets of X.

PROOF. It follows from the pointwise convergence that the set $H = \{P_k: k = 1, 2, \cdots\}$ is bounded for the topology of pointwise convergence. Thus H is equicontinuous by Proposition 1.1. If V is any closed neighborhood of 0 in Y, then there exists a neighborhood U of 0 in X such that $P_k U \subset V$, $k = 1, 2, \cdots$, and this implies $PU \subset V$, i. e., P is continuous.

Now we prove that the convergence is uniform on totally bounded subsets. Let S be a totally bounded subset of X, V an arbitrary neighborhood of 0 in Y and W a balanced neighborhood with $W + W + W \subset V$. There exists a neighborhood U of 0 in X such that $PU \subset W$ and $P_k U \subset W$, $k = 1, 2, \cdots$. There exist points $x_1, \cdots, x_r$ in X such that

$$S \subset (x_1 + U) \cup \cdots \cup (x_r + U)$$

and so there is an N such that $k > N$ implies $P_k x_j - Px_j \in W$, $j = 1, \cdots, r$. Then for x in S and $k > N$ we have $x \in (x_j + U)$ for some j, whence $P_k x - Px = P_k(x - x_j) + (P_k x_j - Px_j) + P(x_j - x) \in W + W + W \subset V$, and the proof is complete.

DEFINITION 1.1. Let X, Y be ordered topological vector spaces and P a positive linear map from X into Y. A set $S \subset X$ is said to be regular if for each x in X and each neighborhood U of 0 in Y, there exist x_U, x^U in X such that

(i) $x_U \leq x \leq x^U$,

(ii) $P(x^U - x_U) \in U$,

(iii) $S_U = \{x_U: x \in S\}$, $S^U = \{x^U: x \in S\}$ are totally bounded.

When we wish to emphasize the dependence on P we shall write P-regular.

If P is a positive linear functional defined on X the definition of a regular set S becomes the following. Given x in S, $\varepsilon > 0$, there exist x_ε, x^ε in X such that

(i) $x_\varepsilon \leq x \leq x^\varepsilon$,

(ii) $P(x^\varepsilon - x_\varepsilon) < \varepsilon$,

(iii) $S^\varepsilon = \{x^\varepsilon: x \in S\}$, $S_\varepsilon = \{x_\varepsilon: x \in S\}$ are totally bounded.

The significance of regular sets is indicated by the following theorem.

<u>THEOREM 1.1.</u> Let X be an ordered, barreled topological vector space and Y a locally convex space ordered by a normal cone. Suppose P, P_k, $k = 1, 2, \cdots$, are positive linear operators mapping X into Y, P_k continuous, such that $P_k x \to Px$ for each x in X. Then P is continuous and the convergence is uniform on P-regular sets.

<u>PROOF.</u> The continuity of P follows from the Proposition 1.2. Let S be a P-regular subset of X and U a neighborhood of 0 in Y. Then there exist subsets S_U, S^U of X satisfying (i), (ii), (iii) of

Definition 1.2. The equalities

$$Px_U \leq Px \leq Px^U, \quad P_k x_U \leq P_k x \leq P_k x^U, \quad k = 1, 2, \cdots,$$

yield

$$P_k x - Px \leq (P_k x^U - Px^U) + (Px^U - Px_U),$$

$$P_k x - Px \geq (P_k x_U - Px_U) + (Px_U - Px^U), \quad k = 1, 2, \cdots,$$

and the conclusion follows by Proposition 1.2 and Proposition 2.1 of Chapter I.

The concept of a regular set was introduced in [1] by P. M. Anselone to deal with integral equations having discontinuous kernels. In that paper $X = R[0, 1]$, P was the Riemann integral and P_k numerical quarature. Also see [2, 3, 4].

§2. Elementary Properties of Regular Sets

We shall show that regular sets behave much like totally bounded sets, although there are some differences. The first proposition is clear.

PROPOSITION 2.1. Let X, Y be ordered topological vector spaces and P a positive linear map from X into Y. Then every totally bounded set is a regular set.

The converse is not true in general. To see this let $X = C[0, 1]$ and P the Riemann integral. Set $S = \{x_n : n = 1, 2, \cdots\}$, where

$x_n(t) = 1$ for $1/n \leq t \leq 1$, $x_n(0) = 0$, and x_n is linear between 0 and $1/n$. Then S is not totally bounded since it is not equicontinuous. On the other hand it is easy to show that S is regular.

LEMMA 2.1. Let X be a topological vector space ordered by a normal cone, S a subset of X. Assume there are bounded subsets S_1, S_2 of X such that for each x in S there exist $x_1 \in S_1$, $x_2 \in S_2$ with $x_1 \leq x \leq x_2$. Then S is bounded.

PROOF. First we observe that if S_1 and S_2 are bounded then $S_2 - S_1$ is bounded. Let U be any neighborhood of 0 and let V be a full neighborhood of 0 such that $V + V \subset U$. There exists $\lambda > 0$ such that $S_2 - S_1 \subset \lambda V$ and $S_1 \subset \lambda V$. Now $x_1 \leq x \leq x_2$ implies $0 \leq x - x_1 \leq x_2 - x_1$ so that $1/\lambda (x - x_1) \in V$. Hence $(1/\lambda)x = (1/\lambda)(x - x_1) + (1/\lambda)(x_i) \in V + V \subset U$, i.e., $S \subset \lambda U$.

PROPOSITION 2.2. Let X, Y be ordered topological spaces with X possessing a normal positive cone. Suppose P is a positive linear map from X into Y. Then every regular set is bounded.

PROOF. Observe that totally bounded sets are bounded and use the preceding lemma.

The converse is again false in general. Let $X = C[0, 1]$, P the Riemann integral. Set $S = \{x_n: n = 1, 2, \cdots\}$, where $x_n(t) = \cos(2n\pi t)$. Define

$$P_n x = \frac{1}{n} \sum_{1}^{n} x(\frac{k}{n}) .$$

We know that $P_n x \to Px$ for each x in $C[0, 1]$, but $P_n x_n = 1$ and $Px_n = 0$ for $n = 1, 2, \cdots$. Therefore, by Theorem 1.1 S cannot be a regular set.

From Proposition 2.1 and 2.2, we see that the regular sets falls somewhere between the bounded sets and the totally bounded sets.

PROPOSITION 2.3. Let X be an ordered topological vector space, Y a topological vector space ordered by a normal cone, P a positive linear map from X into Y and S a subset of X. Assume that for each x in S and each neighborhood U of 0 in Y there exist x^U, x_U in X such that

(i) $x_U \leq x \leq x^U$,

(ii) $P(x^U - x_U) \in U$,

(iii) $S_U = \{x_U: x \in S\}$, $S^U = \{x^U: x \in S\}$ are P-regular.

Then S is a P-regular subset of X.

PROOF. Let U be a full neighborhood of 0 in Y, V a neighborhood of 0 such that $V + V + V \subset U$. If $x \in S$, then there exist $x^V \in S^V$, $x_V \in S_V$ satisfying (i), (ii) and (iii). Since S^V, S_V are regular, there exist a_V, a^V, b_V, b^V in X such that

$$a_V \leq x_V \leq a^V, \; b_V \leq x^V \leq b^V ,$$

$$P(a^V - a_V) \in V, \; P(b^V - b_V) \in V$$

and these sets of approximations are all totally bounded. Observe that

$a_V \le x \le b^V$ and $0 \le b^V - a_V \le (b^V - b_V) + (x^V - x_V) + (a^V - a_V)$, whence $P(b^V - a_V) \in U$ and S is a regular set.

This result will be used later in forming new regular sets from other regular sets.

In some cases totally bounded sets may be replaced by finite sets in the definition of a regular set.

PROPOSITION 2.4. Let X be a subspace of $B(\Omega)$ which contains the constants and let φ be a positive linear functional defined on X. A subset S of X is φ-regular if and only if the following holds. Given $\varepsilon > 0$ and x in S, there exist x^ε, x_ε in X with $x_\varepsilon \le x \le x^\varepsilon$, $\varphi(x^\varepsilon - x_\varepsilon) < \varepsilon$ and the sets $S^\varepsilon = \{x^\varepsilon\colon\ x \in S\}$, $S_\varepsilon = \{x_\varepsilon\colon\ x \in S\}$ are finite.

PROOF. Let S be a φ-regular subset of X and $\varepsilon > 0$. For x in S, there exist x^ε, x_ε in X with $x_\varepsilon \le x \le x^\varepsilon$, $\varphi(x^\varepsilon - x_\varepsilon) < \frac{1}{2}\varepsilon$ and $S^\varepsilon = \{x^\varepsilon\colon\ x \in S\}$, $S_\varepsilon = \{x_\varepsilon\colon\ x \in S\}$ are totally bounded. Let $\{x_1^\varepsilon, \cdots, x_n^\varepsilon\}$, $\{x_{1\varepsilon}, \cdots, x_{m\varepsilon}\}$ be finite $\varepsilon[8\varphi(1)]^{-1}$-nets for S^ε, S_ε respectively. Then for some i, j and all $t \in \Omega$ we have

$$x_i^\varepsilon(t) - \varepsilon[8\varphi(1)]^{-1} \le x^\varepsilon(t) \le x_i^\varepsilon(t) + \varepsilon[8\varphi(1)]^{-1} ,$$

$$x_{j\varepsilon}(t) - \varepsilon[8\varphi(1)]^{-1} \le x_\varepsilon(t) \le x_{j\varepsilon}(t) + \varepsilon[8\varphi(1)]^{-1} .$$

Since

$$\varphi((x_i^\varepsilon + \varepsilon[8\varphi(1)]^{-1}) - (x_{j\varepsilon} - \varepsilon[8\varphi(1)^{-1})) < \varepsilon ,$$

we can replace S^{ε} by $\{x_i^{\varepsilon} + \varepsilon[8\varphi(1)]^{-1}: i = 1, \cdots, n\}$ and S_{ε} by $\{x_{i\varepsilon} - \varepsilon[8\varphi(1)]^{-1}: i = 1, \cdots, m\}$.

This result applies in particular to $X = R[0, 1]$ and φ the Riemann integral.

PROPOSITION 2.5. Let X be a complete topological vector space ordered by a closed positive cone, Y an ordered topological vector space, and P a positive continuous linear map X into Y. If $S \subset X$ is a regular set, then the closure $\overline{S}$ of S is a regular set.

PROOF. Let $\{x_i: i \in I\}$ be a net in S which converges to an element x_0 in X, and U a closed neighborhood of 0 in Y. Then there exist totally bounded subsets S_U, S^U of X such that for each x in S we have x^U in S^U and x_U in S_U such that $x_U \leq x \leq x^U$ and $P(x^U - x_U) \in U$. In particular, for each $i \in I$ we have $x_U(i) \in S_U$ and $x^U(i) \in S^U$ with $x_U(i) \leq x_i \leq x^U(i)$ and $P(x^U(i) - x_U(i)) \in U$. There exist subnets $x_U(i)$ and $x^U(i)$ with $i \in I' \subset I$, such that $x_U(i) \to x_U \in \overline{S}_U$ and $x^U(i) \to x^U \in \overline{S}^U$ with $i \in I'$. Since the positive cone in X is closed we have $x_U \leq x_0 \leq x^U$ and $P(x^U - x_U) \in U$ since U is closed. Thus $\overline{S}$ is a regular set.

The next proposition is similar to the result that a continuous image of a compact set is compact.

PROPOSITION 2.6. Let X be an ordered topological vector space and Y an ordered normed space with a normal cone. If P is a continuous positive linear map from X into Y and $S \subset X$ is a regular set, then $P(S)$ is a totally bounded subset of Y.

PROOF. Let $\{P(x_n)\}$ be a sequence in $P(S)$. For each m, $n = 1, 2, \cdots$, there exist $x_m(n) \in S_m$ and $x^m(n) \in S^m$ with S_m and S^m totally bounded, such that

$$x_m(n) \leq x_n \leq x^m(n), \quad ||x^m(n) - x_m(n)|| < 1/m .$$

Using the total boundedness of S_m, S^m and the Cantor diagonlization process we can find a subsequence $\{n_k\}$ of the natural numbers such, for each $m = 1, 2, \cdots$, the sequences $\{x_m(n_k)\}_{k=1}^{\infty}$ and $\{x^m(nk)\}_{k=1}^{\infty}$ are Cauchy. Then

$$Px_m(n_k) \leq Px_{n_k} \leq Px^m(n_k) ,$$

$$Px_{n_k} - Px_{n_{k'}} \leq [Px^m(n_k) - Px^m(n_{k'})] + [Px^m(n_{k'}) - Px_m(n_{k'})],$$

$$Px_{n_k} - Px_{n_{k'}} \geq [Px_m(n_k) - Px_m(n_{k'})] + [Px_m(n_{k'}) - Px^m(n_{k'})],$$

and the proposition follows by the normality of the positive cone in Y.

Now we consider how one may combine two or more regular sets to produce another regular set. The following is clear.

PROPOSITION 2.7. Suppose X, Y are ordered topological spaces, P a positive linear map from X into Y. If S, S_1, S_2 are regular subsets of X, then the following are also regular sets:

(i) $S_1 + S_2 = \{x_1 + x_2 \colon\ x_1 \in S_1\}$,

(ii) $rS = \{rx\colon\ x \in S\}$ for each real number r,

(iii) $S_1 \cup S_2$.

We now consider the case when X is a topological vector lattice. We need the following lemma.

LEMMA 2.1. Let X be a topological vector lattice and S, S_1, S_2 totally bounded subsets of X. Then the following sets are totally bounded:

(i) $|S| = \{|x| : x \in S\}$,

(ii) $S^+ = \{x^+ : x \in S\}$,

(iii) $S^- = \{x^- : x \in S\}$,

(iv) $S_1 \wedge S_2 = \{x_1 \wedge x_2 : x_1 \in S_1,\ x_2 \in S_2\}$,

(v) $S_1 \vee S_2 = \{x_1 \vee x_2 : x_1 \in S_1,\ x_2 \in S_2\}$.

PROOF. Let U be any solid neighborhood of 0. Then there exist points $x_1, \cdots, x_r$ in X such that $S \subset (x_1 + U) \cup \cdots \cup (x_r + U)$. Using the fact that U is solid and the inequality $||x| - |y|| \leq |x - y|$, we obtain $|S| \subset (|x_1| + U) \cup \cdots \cup (|x_r| + U)$. So (i) is proved. (ii) follows from the inequality $|x^+ - y^+| \leq |x - y|$ and (iii) from (i) and (ii). Since $x \wedge y = -1/2[x - y - (x - y)] + y$ we have $S_1 \wedge S_2 \subset -1/2[S_1 - S_2 - (S_1 - S_2)] + S_2$ and so (iv) holds. Finally (v) follows from (iv) and the identity $x \vee y = -[-(-x) \wedge (-y)]$.

PROPOSITION 2.8. Let X be a topological vector lattice, Y a topological vector space ordered by a normal cone, P a positive linear map from X into Y and S, S_1, S_2 regular subsets of X.

Then the following are regular sets:

(i) $S_1 \vee S_2$,

(ii) $S_1 \wedge S_2$,

(iii) S^+,

(iv) S^-,

(v) $|S|$.

PROOF. We first prove (i). Let U be any full neighborhood of 0 in Y and let V be a neighborhood of 0 such that $V + V \subset U$. Then there exist totally bounded sets S_1^V, S_{1V}, S_2^V, S_{2V} such that for x_1 in S_1 and x_2 in S_2 we have

$$x_{1V} \leq x_1 \leq x_1^V, \qquad x_{2V} \leq x_2 \leq x_2^V$$

and

$$P(x_1^V - x_{1V}) \in V_1, \qquad P(x_2^V - x_{2V}) \in V$$

for some

$$x_1^V \in S_1^V,\ x_{1V} \in S_{1V},\ x_2^V \in S_2^V,\ x_{2V} \in S_{2V} .$$

From this we obtain

$$x_{1V} \vee x_{2V} \leq x_1 \vee x_2 \leq x_1^V \vee x_2^V$$

and, by (1.9) of Chapter I,

$$P(x_1^V \vee x_2^V) - (x_{1V} \vee x_{2V})] \leq P(x_1^V - x_{1V}) + P(x_2^V - x_{2V}) \in V + V \subset U.$$

Then $S_1 \vee S_2$ is regular by Lemma 2.1 (v). Similarly for (ii); (iii) follows from (i) by setting $S_2 = \{0\}$, and (iv) follows from (ii). Finally $|S| \subset S^+ + S^-$ and so (v) follows by (iii), (iv) and Proposition 2.7 (i).

PROPOSITION 2.9. Let $X \subset B(\Omega)$ be a subspace which is a vector lattice with $1 \in X$, Y an ordered topological vector space, and P a positive linear map from X into Y. If S_1, S_2 are regular subsets of X, then the set

$$S_1 S_2 = \{x_1 x_2 : x_1 \in S_1, x_2 \in S_2\}$$

is regular.

PROOF. Since $S_1 S_2 \subset S_1^+ S_2^+ - S_1^+ S_2^- - S_1^- S_2^+ + S_1^- S_2^+$, we may assume that S_1, S_2 consist entirely of nonnegative functions. Let M be a bound for S_1 and S_2. Let U be a neighborhood of 0 in Y, V a neighborhood of 0 with $V + V \subset U$. There exist totally bounded sets S_1^V, S_{1V}, S_2^V, S_{2V} such that for x_1 in S_1 and x_2 in S_2 we have

$$0 \leq x_{1V} \leq x_1 \leq x_1^V \leq M ,$$

$$0 \leq x_{2V} \leq x_2 \leq x_2^V \leq M ,$$

$$P(x_i^V - x_{iv}) \in (1/M)V ,$$

for some

$$x_{iV} \in S_{iV}, \; x_i^V \in S_i^V, \; i = 1, 2.$$

Hence $x_{1V}x_{2V} \le x_1x_2 \le x_1^Vx_2^V$ and $P(x_1^Vx_2^V - x_{1V}x_{2V}) \in U$. Since it is easy to verify that the product of two totally bounded subsets of $B(\Omega)$ is again totally bounded, the proof is complete.

§3. Regular Sets and Locally Convex Topologies

In this section we examine more closely the relationship between regular sets and those sets which are totally bounded with respect to some locally convex topology. While not every regular set is totally bounded in the original topology the following result shows that it is totally bounded in some locally convex topology.

If Y is an ordered locally convex space with a normal positive cone then by Proposition 2.2 there is a basis of monotone seminorms q, i.e., $0 \le x \le y$ implies $q(x) \le q(y)$.

THEOREM 3.1. Let X be an ordered topological vector space, Y an ordered locally convex space with a normal positive cone, P a positive continuous linear map from X into Y and q a monotone continuous seminorm on Y. If S is a regular subset of X, then S is totally bounded in the locally convex topology $\mathcal{T}_c$ induced on X by the seminorm $p(x) = q(Px)$.

PROOF. We shall let $\mathcal{T}$ denote the original topology on X. Since P is continuous in the $\mathcal{T}$ topology we have $\mathcal{T}_c \subset \mathcal{T}$. Observe that the sets $U_\varepsilon = \{x \in X: \ p(x) < \varepsilon\}$ form a basis of neighborhoods for $\mathcal{T}_c$. For each $\varepsilon > 0$, there exist sets S_ε and S^ε which are totally bounded in $\mathcal{T}$ and hence in $\mathcal{T}_c$, such that, for each x in S, there are $x_\varepsilon \in S_\varepsilon$ and $x^\varepsilon \in S^\varepsilon$ with $x_\varepsilon \leq x \leq x^\varepsilon$ and $p(x^\varepsilon - x_\varepsilon) < \frac{1}{2}\varepsilon$. Since S_ε is totally bounded in $\mathcal{T}_c$ there exist points $x_1, \cdots, x_r$ in X such that

$$S_\varepsilon \subset (x_1 + U_{\frac{1}{2}\varepsilon}) \cup \cdots \cup (x_r + U_{\frac{1}{2}\varepsilon}).$$

Let x, x_ε be fixed. Then we have $p(x_\varepsilon - x_j) \leq \frac{1}{2}\varepsilon$ for some j, $1 \leq j \leq r$. Since $0 \leq x - x_\varepsilon \leq x^\varepsilon - x_\varepsilon$ and q is monotone it follows that $p(x - x_j) < \varepsilon$. Therefore

$$S \subset (x_1 + U_\varepsilon) \cup \cdots \cup (x_r + U_\varepsilon)$$

and S is totally bounded in $\mathcal{T}_c$.

There exist totally bounded sets in $\mathcal{T}_c$ which are not regular. To demonstrate this let $X = C[0, 1]$ and let φ in $C[0, 1]^*$ be point evaluation at 0. If $x_n(t) = nt$, $0 \leq t \leq 1$, and $S = \{x_n\colon n = 1, 2, \cdots\}$, then S is totally bounded in the locally convex topology induced on $C[0, 1]$ by $p(x) = |\varphi(x)|$, but S is not bounded in $C[0, 1]$ and so cannot be a φ-regular set.

<u>COROLLARY 3.1.</u> Let X be a topological vector lattice and φ a positive continuous linear functional on X. If S is a φ-regular subset of X, then X is totally bounded in the locally convex topology induced by the seminorm $q(x) = \varphi(|x|)$.

<u>PROOF.</u> This follows from $|\varphi(x)| \leq \varphi(|x|)$ and Theorem 3.1.

COROLLARY 3.2. Let $X = R[0, 1]$ and φ be the Riemann integral. Then every φ-regular set is totally bounded in $L_1[0, 1]$.

Now we turn to a question posed by Anselone [4]. Is it possible to find a locally convex topology $\mathcal{T}_c$ such that a set is regular if and only if S is totally bounded in $\mathcal{T}_c$? We shall show that the answer is in general no.

For ℓ_p, $0 < p < 1$,

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$$

defines a metric and this metric induces a topology $\mathcal{T}$ on ℓ_p which is not locally convex [9]. With this metric ℓ_p is a complete metric space.

LEMMA 3.1. A bounded set $S \subset \ell_p$, $0 < p < 1$, is totally bounded (with respect to $\mathcal{T}$) if and only if

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} |x_n|^p = 0 \text{ uniformly for } x \text{ in } S.$$

PROOF. Suppose S is totally bounded. Fix $\varepsilon > 0$. Then there exist a finite $\frac{1}{2}\varepsilon$ - net $\{x^1, \cdots, x^M\}$ for S and $N = N(\varepsilon)$ such that

$$\sum_{n=m}^{\infty} |x_n^j|^p < \frac{1}{2}\varepsilon, \; j = 1, \cdots, M.$$

For x in S, we have $d(x, x^j) < \frac{1}{2}\varepsilon$ for some j, $1 \leq j \leq M$, which implies

$$\sum_{n=N}^{\infty} |x_n|^p < \varepsilon .$$

Conversely let S be a bounded set with the property that for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\sum_{n=N+1}^{\infty} |x_n|^p < \frac{1}{2}\varepsilon, \quad x \text{ in } S.$$

For x in S, let $x^N = (x_1, x_2, \cdots, x_n)$ in Euclidean R^N. The set $S^N = \{x^N: x \in S\}$ is bounded and hence totally bounded. Since all finite dimensional topological vector spaces are homeomorphic, S^N is totally bounded in R^N with the p-metric. Hence S_N has a finite $\frac{1}{2}\varepsilon$-net, say $x^{Nj} = (x_1^j, \cdots, x_N^j)$, $j = 1, \cdots, L$. Then the vectors $x^j = (x_1^j, \cdots, x_N^j, 0, \cdots)$ provide an ε-net for S and so S is totally bounded in ℓ_p.

<u>LEMMA 3. 2.</u> Let $S \subset \ell_p$, $0 < p < 1$, and assume there exist totally bounded sets $S_1, S_2 \subset \ell_p$ such that for each x in S, there are x^1 in S_1 and x^2 in S_2 with $x^1 \leq x \leq x^2$. Then S is totally bounded.

<u>PROOF.</u> For $\varepsilon > 0$. Then by Lemma 3. 1 there is a positive integer $N = N(\varepsilon)$ such that

$$\sum_{n=N}^{\infty} |x_n|^p < \frac{1}{4}\varepsilon \quad \text{for } x \text{ in } S_1 \cup S_2 .$$

Let $x \in S$, $x^1 \leq x \leq x^2$, and $x^1 \in S_1$, $x^2 \in S_2$. Then

$$\sum_{n=N}^{\infty} |x_n|^p \leq \sum_{n=N}^{\infty} |x_n - x_n^1|^p + \sum_{n=N}^{\infty} |x_n^1|^p$$

$$\leq \sum_{n=N}^{\infty} |x_n^2 - x_n^1|^p + \sum_{n=N}^{\infty} |x_n^1|^p < \varepsilon \ .$$

By Lemma 3.1, S is totally bounded.

<u>COROLLARY 3.3.</u> A set $S \subset \ell_p$, $0 < p < 1$, is regular if and only if S is totally bounded.

<u>LEMMA 3.3.</u> Let X be a locally convex topological vector space and $S \subset X$ totally bounded. Then the convex hull $H(S)$ is totally bounded.

<u>PROOF.</u> First, if S is a finite set, $\{x_1, \cdots x_n\}$, then $H(S)$ is compact because it is the image of the compact simplex

$$\{(\lambda_1, \cdots, \lambda_n) \colon \lambda_i \geq 0, \ \sum_1^n \lambda_i = 1\} \subset R^n$$

under the continuous map

$$(\lambda_1, \cdots, \lambda_n) \rightarrow \sum_1^n \lambda_i x_i \ .$$

Now let S be any totally bounded set in X, U a neighborhood of 0, and V a convex balanced neighborhood of 0 such that $V + V \subset U$. Then there exists a finite set $A \subset S$ with $S \subset A + V$, $H(A)$ is compact, and $H(S) \subset H(A) + V$. Since $H(A)$ is compact, there is a finite set B such that $H(S) \subset B + V + V \subset B + U$. Therefore $H(S)$ is totally bounded.

We now show that there cannot exist a locally convex topology $\mathcal{T}_c$ whose totally bounded sets are the same as the totally bounded sets of $\mathcal{T}$.

At this point we set $p = 1/2$.

Define $x^k \in \ell_{1/2}$, $k = 1, 2, \cdots$, by $x^k_n = 0$ for $n \neq k$ and $x^k_k = 1/k$. Let $S = \{x^k: \ k = 1, 2, \cdots\}$. Then S is totally bounded in $\mathcal{T}$ (in fact, $x^k \to 0$). We now shows that the convex hull of X is not totally bounded in $\mathcal{T}$. Let K be an arbitrary positive integer and observe that

$$\sum_{K+1}^{2K} 1/K \geq K(1/2K) = 1/2.$$

Then

$$1/2 \leq \sum_{K+1}^{2K} 1/k = \sum_{K+1}^{2K} [(1/k)(1/k)]^{1/2} \leq \sum_{K+1}^{2K} [(1/2K)(1/k)]^{1/2}$$

$$= 2^{-1/2} \sum_{K+1}^{2K} [(1/K)(1/k)]^{1/2} .$$

Let $y^K = (1/K)(x^{K+1} + x^{K+2} + \cdots + x^{2K})$. Then

$$\sum_{K+1}^{2K} |y_n^K|^{1/2} = \sum_{K+1}^{\infty} |y_n^K|^{1/2} \geq 1/2 \;\; 2^{1/2}.$$

Since K was arbitrary, the convex hull of S does not satisfy Lemma 3.1 and so S is not totally bounded. On the other hand, if the totally bounded sets in $\mathcal{T}$ coincided with those in some locally convex topology $\mathcal{T}_c$, then the convex hull of S would also be totally in $\mathcal{T}$ by Lemma 3.3. Therefore the regular sets in $\ell_{1/2}$ cannot coincide with the totally bounded sets in any locally convex topology on $\ell_{1/2}$.

IV. REGULAR SETS IN SPECIAL SPACES

§1. Sequential Spaces

We shall investigate the concept of a regular set in the various sequence spaces and obtain several characterizations. Actually it turns out that most of the time the regular sets are precisely the totally bounded sets. Recall that this is the case for ℓ_p, $0 < p < 1$. First we investigate the spaces ℓ_p, $1 \leq p < \infty$. The order relation that we shall be using in this section is given by (1.1) in Chapter I.

We will use the following well known characterization of totally bounded sets in ℓ_p, $1 \leq p < \infty$, which is given in [11, p. 338].

LEMMA 1.1. A subset S of ℓ_p, $1 \leq p < \infty$, is totally bounded if and only if it is bounded and

$$\lim_{m\to n} \sum_{n=m}^{\infty} |x_n|^p = 0 ,$$

uniformly for $x = (x_1, x_2, \cdots)$ in S.

LEMMA 1.2. Let $S \subset \ell_p$, $1 \leq p < \infty$, and suppose there exist totally bounded sets S_1, S_2 in ℓ_p such that for each x in S, there are elements x^1 in S_1 and x^2 in S_2 with $x^1 \leq x \leq x^2$. Then S is totally bounded.

PROOF. Since the proof is similar to the proof of Lemma 3. 2 Chapter II, it is omitted.

From this and Corollary 3.3, Chapter II, we have the following result.

THEOREM 1.1. A set $S \subset \ell_p$, $0 < p < \infty$, is regular if and only if S is totally bounded.

We consider next the space c of real convergent sequences with the norm $\|x\| = \sup_k |x_k|$ for $x = (x_1, x_2, \cdots)$. The discussion includes the subspace c_0 of sequences converging to zero.

The next result is given in [11, p. 339].

LEMMA 1.3. A set S in c or c_0 is totally bounded if and only if it is bounded and the limit $\lim_{k \to \infty} x_k$ exists uniformly for $x = (x_1, x_2, \cdots)$ in S.

LEMMA 1.4. Let $S \subset c_0$ and assume there exist totally bounded sets S_1, S_2 in c_0 such that for each x in S, there are x^1 in S_1 and x^2 in S_2 with $x^1 \leq x \leq x^2$. Then S is totally bounded.

PROOF. If $x^1 \leq x \leq x^2$, then $x_n^1 \leq x_n \leq x_n^2$, $n = 1, 2, \cdots$, and the proof follows by Lemma 1.3.

From Lemma 1.4 we obtain the following.

THEOREM 1.2. A subset S of c_0 is regular if and only if S is totally bounded.

The dual c^* of c is isometrically isomorphic to ℓ_1 and the representation is given by (cf. [19, p. 115])

$$\varphi(x) = a_0 \lim x_n + \sum_1^{\infty} a_n x_n \quad ,$$

$$\varphi = (a_0, a_1, \cdots) \epsilon\, c^* \,,$$

$$x = (x_1, x_2, \cdots) \epsilon\, c \,.$$

We will investigate the φ-regular subsets of c with φ in c^*.

<u>LEMMA 1.5.</u> A functional $\varphi = (a_0, a_1, \cdots)$ in c^* is positive if and only if $a_n \geq 0$, $n = 0, 1, \cdots$.

<u>PROOF.</u> It is clear that $a_n \geq 0$, $n = 0, 1, \cdots$, implies that φ is a positive functional. Conversely, assume φ is positive and for each $k \geq 1$ let x^k be an element in c with $x_k^k = 1$ and $x_n^k = 0$ for $n \neq k$. Then $\varphi x^k = a_k \geq 0$, $k = 1, 2, \cdots$. Now we show $a_0 \geq 0$. If $a_0 < 0$, then there exists N such that $\sum_{N+1}^{\infty} a_n < -\frac{1}{2} a_0$. Choose x in c with $x_n = 0$, $1 \leq n \leq N$ and $x_n = 1$, $n > N$. Then x is positive and $\varphi x < \frac{1}{2} a_0 < 0$ and so φ is not positive.

<u>LEMMA 1.6.</u> Let $\varphi = (0, a_1, a_2, \cdots) \epsilon\, c^*$, where $a_n \geq 0$, $n = 1, 2, \cdots$. If $S \subset c$ is a bounded set, then S is a φ-regular set.

<u>PROOF.</u> Suppose S is bounded by M and let $\varepsilon > 0$ be given. Let $N(\varepsilon)$ be such that

$$\sum_{N+1}^{\infty} a_n < \varepsilon [2M]^{-1} \,.$$

If we set

$$S^{\varepsilon} = \{x \in c: \ |x_n| \leq M, \ n = 1, 2, \cdots, N; \ x_n = M, \ n > N\},$$

$$S_{\varepsilon} = \{x \in c: \ |x_n| \leq M, \ n = 1, 2, \cdots, N; \ x_n = -M, \ n > N\},$$

then S^{ε}, S_{ε} are totally bounded sets by Lemma 1.3. For each $x \in S$ pick $x_{\varepsilon} \in S_{\varepsilon}$ and $x^{\varepsilon} \in S^{\varepsilon}$ such that $x_{\varepsilon n} = x_n = x^{\varepsilon}_n$, $n = 1, 2, \cdots, N$. Then $x_{\varepsilon} \leq x \leq x^{\varepsilon}$ and $\varphi(x^{\varepsilon} - x_{\varepsilon}) < \varepsilon$ and so S is φ-regular.

<u>LEMMA 1.7.</u> Let $\varphi \in c^*$, $\varphi = (a_0, 0, 0, \cdots)$, $a_0 > 0$. If S is a φ-regular subset of c, then S is totally bounded.

<u>PROOF.</u> Let $\varepsilon > 0$ be given. Then, by Proposition 2.4, Chapter II, there exist finite sets S^{ε}, S_{ε} such that, for each x in S, there are x_{ε} in S_{ε} and x^{ε} in S^{ε} with $x_{\varepsilon} \leq x \leq x^{\varepsilon}$ and $(a_0)\lim_{n\to\infty}(x^{\varepsilon}_n - x_{\varepsilon n}) = \varphi(x^{\varepsilon} - x_{\varepsilon}) < (\frac{1}{4}\varepsilon)\, a_0$. Let $L(x) = \lim_{n\to\infty} x_n$ for $x \in c$. Then there exists $N(\varepsilon)$ such that $n > N$ implies $|x^{\varepsilon}_n - L(x^{\varepsilon})| < \frac{1}{2}\varepsilon$, $|x_{\varepsilon n} - L(x_{\varepsilon})| < \frac{1}{2}\varepsilon$ and $|x^{\varepsilon}_n - x_{\varepsilon n}| < \frac{1}{2}\varepsilon$ for all x in S. From $x_{\varepsilon n} \leq x_n \leq x^{\varepsilon}_n$ we obtain $L(x_{\varepsilon}) \leq L(x) \leq L(x^{\varepsilon})$, which yields $-\varepsilon < x_n - L(x) < \varepsilon$ for $n > N$. Hence, by Lemma 1.3, S is totally bounded.

<u>THEOREM 1.3.</u> Let $\varphi = (a_0, a_1, \cdots)$ be a positive linear functional in c^* and S a subset of c.

(i) Assume $a_0 = 0$. Then S is φ-regular if and only if S is bounded.

(ii) Assume $a_0 > 0$. Then S is φ-regular if and only if S is totally bounded.

PROOF. In view of Lemma 1.6 we need prove only (ii). Let $\varphi_0 = (a_0, 0, 0, \cdots)$ and note that $\varphi_0 x \leq \varphi x$ if $x \geq 0$. From this it follows that if S is a φ-regular set, then S is φ_0-regular. By Lemma 1.7, S is totally bounded.

§2. Regular Sets in L_p, $1 \leq p < \infty$

Let $(\Omega, \mathcal{A}, \mu)$ be a totally finite measure space. We shall attempt to characterize the φ-regular sets in $L_p(\Omega)$, $1 \leq p < \infty$, when φ is an element of $L_p^*(\Omega)$. By the Riesz representation theorem, for $1 \leq p < \infty$ we may identify $L_p^*(\Omega)$, with $L_q(\Omega)$ where $1/p + 1/q = 1$.

Recall that for measurable functions f and g, $f \leq g$ means $f(t) \leq g(t)$ a. e. For f and g in L_p, $1 \leq p < \infty$, $f \leq g$ means this inequaltiy is satisfied by all functions in the respective equivalence classes.

Suppose φ is a positive linear functional on $L_p(\Omega)$. Define a function $\chi(\varphi)$ in $L_\infty(\Omega)$ by letting $\chi(\varphi)$ be the characteristic function of the set $[\varphi > 0] = \{t \in \Omega : \varphi(t) > 0\}$.

THEOREM 2.1. Let $S \subset L_p(\Omega)$, $1 \leq p < \infty$, and φ a positive linear functional on $L_p(\Omega)$. Assume that the set $\chi(\varphi)S$ is a φ-regular (or totally bounded). Assume that $[1 - \chi(\varphi)]S$ (or S itself) is dominated, i. e., there exists $g \in L_p(\Omega)$ such that

$$-g \leq [1 - \chi(\varphi)]x \leq g, \qquad x \in S.$$

Then S is φ-regular.

PROOF. For each $x \in S$,

$$x = \chi(\varphi)x + [1 - \chi(\varphi)]x, \quad -[1 - \chi(\varphi)]g \leq [1 - \chi(\varphi)]x \leq [1 - \chi(\varphi)]g .$$

Since $\varphi([1 - \chi(\varphi)]g) = 0$, the set $[1 - \chi(\varphi)]S$ is a φ-regular. Since

$$S \subset \chi(\varphi)S + [1 - \chi(\varphi)]S ,$$

S is φ-regular by Proposition 2.7, Chapter III.

The assumption that S is dominated cannot be dropped. Let $\Omega = [0, 1]$ and μ Lebesgue measure. If we let $\varphi = \chi_{[0, 1/2]}$, $\varphi_n = \chi_{[0, 1/2 + n^{-1})}$, $n = 2, 3, \cdots$, then by Lebesgue's dominated convergence theorem we have $\varphi_n x \to \varphi x$ for all x in $L_p[0, 1]$, $1 \leq p < \infty$. Define $x_n = n\chi_{(1/2, 1/2 + n^{-1})}$, $n = 2, 3, \cdots$, and set $S = \{x_n: n = 2, 3, \cdots\} \subset L_p[0, 1]$. Now $\chi(\varphi)S$ is totally bounded in $L_p[0, 1]$, but φ_n does not converge uniformly to φ on S since $\varphi_n x_n = 1$ and $\varphi x_n = 0$, $n = 2, 3, \cdots$. Therefore by Theorem 1.1, Chapter III, S cannot be a regular set.

We now set about establishing a converse of Theorem 2.1.

LEMMA 2.1. Let $(\Omega, \mathcal{A}, \mu)$ be as before and suppose $\varphi \in L_1^*(\Omega)$, $\varphi \geq \delta > 0$. Then a subset S of $L_1(\Omega)$ is φ-regular if and only if S is totally bounded.

PROOF. Suppose S is φ-regular and let $\varepsilon > 0$ be given. For x in S, there exist x_ε, x^ε in $L_1(\Omega)$ such that $x_\varepsilon \leq x \leq x^\varepsilon$, $\varphi(x^\varepsilon - x_\varepsilon) < \frac{1}{2}(\delta\varepsilon)$ and S^ε, S_ε are totally bounded in $L_1(\Omega)$. Then

$$||x^{\varepsilon} - x_{\varepsilon}||_1 = \int_{\Omega} (x^{\varepsilon} - x_{\varepsilon})\, d\mu \leq \delta^{-1} \int_{\Omega} (x^{\varepsilon} - x_{\varepsilon})\varphi\, d\mu < \frac{1}{2}\, \varepsilon$$

which implies $||x^{\varepsilon} - x|| < \frac{1}{2}\, \varepsilon$. So any $\frac{1}{2}\, \varepsilon$-net for S^{ε} will be an ε-net for S.

LEMMA 2.2. Let $E \in \mathcal{Q}$, $\Psi \in L_{\infty}(\Omega)$, $\varphi = \chi_E \Psi$. If $\varphi \geq \delta \chi_E$, $\delta > 0$, and S is a φ-regular subset of $L_1(\Omega)$, then $\chi(\varphi)S$ is totally bounded in $L_1(\Omega)$.

PROOF. The proof is similar to that of Lemma 2.1.

LEMMA 2.3. Suppose S is a bounded set in $L_{\infty}(\Omega)$ and φ a positive functional in $L_1^*(\Omega)$. If S is a φ-regular subset of $L_1(\Omega)$, then $\chi(\varphi)S$ is a totally bounded subset of $L_1(\Omega)$.

PROOF. By a typical construction there exists an increasing sequence $\{\varphi_n\}$ of simple functions such that $||\varphi_n - \varphi||_{\infty} \to 0$. Let M be the $L_{\infty}(\Omega)$ supremum of S and fix $\varepsilon > 0$. Then there exists $N(\varepsilon)$ such that $||\varphi_N - \varphi||_{\infty} < \varepsilon[4M]^{-1}$. Since $\varphi_N \leq \varphi$, S is a φ_N-regular set and by Lemma 2.2, $\chi(\varphi_N)S$ is a totally bounded subset of $L_1(\Omega)$. Let $\{\chi(\varphi_N)x_1, \cdots, \chi(\varphi_N)x_r\}$ be a finite $\frac{1}{2}\, \varepsilon$-net for $\chi(\varphi_N)S$ and let $x \in S$. Then for some j, $1 \leq j \leq r$, we have $||\chi(\varphi_N)x_j - \chi(\varphi_N)x||_1 < \frac{1}{2}\, \varepsilon$. Therefore

$$||\chi(\varphi)x - \chi(\varphi)x_j||_1 = \int_{\Omega} [\chi(\varphi) - \chi(\varphi_N)]\, |x - x_j|\, d\mu + \int_{\Omega} \chi(\varphi_N)\, |x - x_j|\, d\mu < \varepsilon$$

and so $\{\chi(\varphi)x_1, \cdots, \chi(\varphi)x_r\}$ is a finite ε-net for $\chi(\varphi)S$ in $L_1(\Omega)$.

It is obvious that if S is a totally bounded subset of $L_p(\Omega)$ for some $p > 1$, then S is also totally bounded in $L_1(\Omega)$. Under certain conditions the converse is true.

LEMMA 2.4. Let S be a bounded set in $L_\infty(\Omega)$. If S is totally bounded in $L_1(\Omega)$, then S is also totally bounded in $L_p(\Omega)$, $1 < p < \infty$.

PROOF. Let M be the bound on S and let $x, y, \epsilon (1/2M)S$. Then $|x - y| \leq 1$, whence

$$\int_\Omega |x - y|^p d\mu \leq \int_\Omega |x - y| d\mu$$

and this implies $(1/2M)S$ is totally bounded in $L_p(\Omega)$ and so S itself is totally bounded in $L_p(\Omega)$.

Now we can extend Lemma 2.3 to the $L_p(\Omega)$, $1 < p < \infty$, case.

LEMMA 2.5. Let S be a bounded set in $L_\infty(\Omega)$, $\varphi \epsilon L_\infty(\Omega)$, $\varphi \geq 0$. If S is a φ-regular subset of $L_p(\Omega)$ where $1 \leq p < \infty$, then $X(\varphi)S$ is a totally bounded subset of $L_p(\Omega)$.

PROOF. If S is a φ-regular subset of $L_p(\Omega)$ then it is certainly a φ-regular subset of $L_1(\Omega)$ and, by Lemma 2.3, $X(\varphi)S$ is a totally bounded subset of $L_1(\Omega)$. Whence, by Lemma 2.4, $X(\varphi)S$ is totally bounded in $L_p(\Omega)$.

Lemma 2.6. Let S be a bounded set in $L_\infty(\Omega)$, φ a positive linear functional on $L_p(\Omega)$, $1 \leq p < \infty$. If S is a φ-regular subset of $L_p(\Omega)$, then $X(\varphi)S$ is totally bounded in $L_p(\Omega)$.

PROOF. Let Ψ be the characteristic function of $[\varphi \leq 1]$ and set $\hat{\varphi} = \Psi\varphi + (1 - \Psi)$. Then $\hat{\varphi} \in L_\infty(\Omega)$, $0 \leq \hat{\varphi} \leq \varphi$ and $\chi(\hat{\varphi}) = \chi(\varphi)$. Hence S is a $\hat{\varphi}$-regular set in $L_p(\Omega)$ and, by Lemma 2.5, $\chi(\varphi)S$ is totally bounded in $L_p(\Omega)$.

Now we want to weaken the hypotheses on S. The next result generalizes the classical result on the continuity of the integral.

LEMMA 2.7. Suppose S is a totally bounded subset of $L_p(\Omega)$, $1 \leq p < \infty$, and let $\varepsilon > 0$ be given. Then there exists $\delta(\varepsilon) > 0$ such that $A \in \mathcal{Q}$, $\mu(A) < \delta$ implies $||\chi_A x|| < \varepsilon$ for all $x \in S$.

PROOF. Let $\{x_1, \cdots, x_r\}$ be a finite $\frac{1}{2}\varepsilon$-net for S and let $\delta(\varepsilon) > 0$ be such that $A \in \mathcal{Q}$, $\mu(A) < \delta$ implies

$$||\chi_A x_j|| < \varepsilon, \; j = 1, \cdots, r .$$

Then for each x in S we have $||x - x_j|| < \frac{1}{2}\varepsilon$ for some j, $1 \leq j \leq r$, and

$$||\chi_A x|| \leq ||x - x_j|| + ||\chi_A x_j|| < \varepsilon .$$

Now suppose S is any subset of $L_p(\Omega)$. For x in S and n a positive integer let $x_n^+(x)$ be the characteristic function of $[x > n]$, $\chi_n^-(x)$ the characteristic function of $[x < -n]$ and $\chi_n(x) = \chi_n^+(x) + \chi_n^-(x)$. If S is bounded in $L_p(\Omega)$ by say M, then $M \geq ||x|| \geq ||x\chi_n(x)|| \geq ||n\chi_n(x)||$, whence

$$||\chi_n(x)|| \to 0 \text{ as } n \to \infty, \text{ uniformly for } x \text{ in } S. \tag{2.1}$$

LEMMA 2.8. Let S be a totally bounded subset of $L_p(\Omega)$, $1 \leq p < \infty$ and fix $\varepsilon > 0$. Then there exists $N(\varepsilon)$ such that $n > N$ implies

$$\|x \chi_n(x')\| < \varepsilon \quad \text{for all } x, \ x' \text{ in } S.$$

PROOF. This follows from (2.1) and Lemma 2.7.

LEMMA 2.9. Let S be a subset of $L_p(\Omega)$, $1 \leq p < \infty$, and suppose there are totally bounded sets S_1 and S_2 in $L_p(\Omega)$ such that for each x in S, there are x_1 in S_1 and x_2 in S_2 with $x_1 \leq x \leq x_2$. Then

$$\lim_{n\to\infty} \|x\chi_n(x')\| = 0 \text{ uniformly for } x, \ x' \text{ in } S.$$

PROOF. From $x_1 \leq x \leq x_2$ and $x_1' \leq x' \leq x_2'$ we have

$$|x| \leq |x_1| + |x_2| , \qquad \chi_n(x') \leq \chi_n(x_1') + \chi_n(x_2') \ .$$

The assertion follows by Lemma 2.8.

Suppose S is any subset of $L_p(\Omega)$, $1 \leq p < \infty$ and $x \in S$. For $n = 1, 2, \cdots$, we define

$$x_n = n\chi_n^+(x) + (1 - \chi_n(x))x - n\chi_n^-(x) ,$$

and

$$S_n = \{x_n : \ x \in S\} .$$

Note that $S_n \subset L_\infty(\Omega)$. If $x, \ y \in S$, then

$$|x_n - y_n| \leq |x - y| .$$

From this it follows that S_n is totally bounded in $L_p(\Omega)$ if S is.

LEMMA 2.10. Let φ be a positive linear functional on $L_p(\Omega)$, $1 \leq p < \infty$ and S a φ-regular subset of $L_p(\Omega)$. Then S_n is also φ-regular.

PROOF. If $x, y \in L_p(\Omega)$ and $x \leq y$, then $x_n \leq y_n$ and $\varphi(y_n - x_n) \leq \varphi(y - x)$. The lemma now follows from the preceding paragraph.

We are now ready to state and prove the theorem we have been working towards.

THEOREM 2.2. Let φ be a positive linear functional on $L_p(\Omega)$, $1 \leq p < \infty$, and S a φ-regular subset of $L_p(\Omega)$. Then $\chi(\varphi) S$ is a totally bounded subset of $L_p(\Omega)$.

PROOF. Fix $\varepsilon > 0$. By Lemma 2.9 there exists $N(\varepsilon)$ such that

$$||x \chi_N(x)|| < \varepsilon \quad \text{for all} \quad x \in S.$$

By Lemma 2.10, S_N is φ-regular. So $\chi(\varphi)S_N$ is totally bounded in $L_p(\Omega)$ from Lemma 2.6. Now

$$||x \chi(\varphi) - x_N \chi_N(\varphi)|| \leq ||x \chi_N(x)|| < \varepsilon \quad \text{for all} \quad x \in S.$$

Hence $\chi(\varphi)S_N$ is a totally bounded ε-net for $\chi(\varphi)S$ which implies $\chi(\varphi)S$ is totally bounded.

COROLLARY 2.1. Suppose $\varphi \in L_p^*(\Omega)$, $\varphi > 0$. Then S is a φ-regular subset of $L_p(\Omega)$ if and only if S is a totally bounded subset of $L_p(\Omega)$.

It is interesting that the property of φ-regularity depends only on $X(\varphi)$ and not on the values of φ.

§3. Regular Sets in $C[0, 1]$ and $R[0, 1]$

We want to consider the concept of a φ-regular set in $R[0, 1]$ and also in the subspace $C[0, 1]$. In contrast to the previous sections we will not obtain characterizations of regular sets in these spaces but will give some sufficient conditions for a set to be regular. In terms of applications, perhaps the most important case is $R[0, 1]$ with φ being the Riemann integral.

DEFINITION 3.1. Let Ω be a topological space and K a subset of $C(\Omega)$. Then K is said to be equicontinuous if for each $\varepsilon > 0$ and each t in Ω there is a corresponding neighborhood $N = N(t, \varepsilon)$ of t with

$$\sup_{f \in K} \sup_{s \in N} |f(t) - f(s)| < \varepsilon .$$

The following classical result [11, p. 226] gives a nice characterization of the totally bounded subsets of $C(\Omega)$.

THEOREM 3.1. (Arzela-Ascoli) If Ω is compact then a set in $C(\Omega)$ is totally bounded if and only if it is bounded and equicontinuous.

We will let φ denote the Riemann integral defined on $R[0, 1]$ until otherwise stated.

LEMMA 3.1. Let S be a bounded set of step functions such that the number of discontinuities of each element of S are uniformly bounded. Then S is a φ-regular subset of $R[0, 1]$.

PROOF. Since the number of discontinuities are uniformly bounded we can approximate each step function above and below by trapezoidal functions with fixed slopes. Such a set of functions is clearly equicontinuous and bounded, hence totally bounded.

We can use this lemma to build new classes of φ-regular sets.

If $x \in R[0, 1]$ and A is a subset of $[0, 1]$ we define $\omega(x, A) = \sup_{s, t \in A} |x(s) - x(t)|$. If I is a subinterval of $[0, 1]$ let $\ell(I)$ denote its length.

THEOREM 3.2. Let S be a bounded subset of $R[0, 1]$. Assume that, for each $\varepsilon > 0$ and $x \in S$ there exist positive integers $N(\varepsilon)$ and $K(\varepsilon)$ such that

(i) there are disjoint intervals $J_x^1, \cdots, J_x^n$, where $n \leq N$ and

$$\sum_{m=1}^{n} \ell(J_x^m) < \varepsilon ,$$

(ii) the complement of $\bigcup_{m=1}^{n} J_x^m$ in $[0, 1]$ is the disjoint union of intervals $I_x^1, \cdots, I_x^k$ where $k \leq K$ and $\omega(x, I_x^m) \leq \varepsilon$, $m = 1, \cdots, k$.

Then S is a φ-regular set.

PROOF. Since a scalar multiple of a regular set is again regular we may assume that S is bounded by $\frac{1}{2}$. Assuming (i) and (ii) hold we define

$$x^{\varepsilon}(t) = \begin{cases} \frac{1}{2} & \text{if } t \in J_x^m,\ m = 1, \cdots, n \\ \sup_{t \in I_x^m} x(t) & \text{if } t \in I_x^m,\ m = 1, \cdots, k. \end{cases}$$

Similarly

$$x_{\varepsilon}(t) = \begin{cases} -\frac{1}{2} & \text{if } t \in J_x^m,\ m = 1, \cdots, n \\ \inf_{t \in I_x^m} x(t) & \text{if } t \in I_x^m,\ m = 1, \cdots, k. \end{cases}$$

Then $x_{\varepsilon} \leq x \leq x^{\varepsilon}$ and $\varphi(x^{\varepsilon} - x_{\varepsilon}) \leq \varepsilon$. By Lemma 3.1 the sets $S^{\varepsilon} = \{x^{\varepsilon}:\ x \in S\}$ and $S = \{x_{\varepsilon}:\ x \in S\}$ are φ-regular. So S is a φ-regular set by Proposition 2.4 of Chapter III.

THEOREM 3.3. Let S be a bounded subset of $R[0, 1]$. Assume that for each $\varepsilon > 0$, there exist positive integers $N(\varepsilon)$ and $K(\varepsilon)$ such that: for x in S there exist disjoint intervals $J_k^1, \cdots, J_x^k$, $k \leq K$ and $\Sigma \ell(J_x^m) \leq \varepsilon$; and on each complementary subinterval I_x^m, x is a piecewise monotone function with at most N distinct oscillations. Then S is a φ-regular subset of $R[0, 1]$.

PROOF. We assume that S is bounded by 1 and then partition the interval $[-1, 1]$ into L subintervals each of length less than ε. Each subinterval can be further partitioned into at most N subintervals on which x is monotone. Then a further subdivision of these into at most L intervals I_x^m yields $\omega(x, I_x^m) < \varepsilon$. There are at most $(K + 1)(N)(L)$ of the intervals I_x^m and the assertion follows by Theorem 3.2.

This theorem has some important corollaries.

COROLLARY 3.1. Any bounded set of monotone functions in $R[0, 1]$ is a φ-regular set.

COROLLARY 3.2. Any bounded set S of functions on $[0, 1]$ of bounded variation with uniformly bounded total variations is a φ-regular set.

PROOF. Any function x in S can be written as the difference of two increasing functions x_1 and x_2 such that the sets $S_1 = \{x_1: x \in S\}$ and $S_2 = \{x_2: x \in S\}$ are bounded. Then S_1, S_2 are φ-regular and since $S \subset S_1 - S_2$, S is φ-regular.

Theorem 3.2 and Corollaries 3.1, 3.2 are important in the numerical approximation of integral operators with discontinuous kernels [3].

We now mention some results that hold for $C[0, 1]$. If g is a continuous, nondecreasing function defined on $[0, 1]$ then

$$\varphi_g = \int_0^1 x(t)\, dg(t)$$

defines a continuous positive linear functional on $C[0, 1]$.

It is not difficult to show that Theorems 3.2, 3.3 and Corollaries 3.1, 3.2 remain valid for the linear functional φ_g. Here trapezoidal functions are used in place of step functions in S^{ε}, S_{ε}. Since the proofs are similar to the ones given they will be omitted.

V. APPLICATIONS TO INTEGRAL EQUATIONS

§1. Integral Equations with Continuous Kernels

In this chapter we shall use the idea of the extension of a positive operator and of a regular set to obtain approximate solutions for a large class of integral equations.

In §1 and §2 of this chapter, unless otherwise stated, Ω will denote a compact Hausdorff (and therefore uniform) topological space.

The class of Baire sets is defined to be the smallest σ-algebra $\mathcal{A}$ of subsets of Ω such that each function x in $C(\Omega)$ is measurable with respect to $\mathcal{A}$. If φ is a continuous linear functional on $C(\Omega)$, then there exists by the Riesz representation theorem [16, p. 310], a unique, finite signed Baire measure μ on Ω such that

$$\varphi(x) = \int_\Omega x d\mu, \text{ for each } x \text{ in } C(\Omega)$$

and $||\varphi|| = |\mu|(\Omega)$. Moreover if φ is positive then μ is a measure.

Let k be a continuous real-valued function defined on $\Omega \times \Omega$. Since $\Omega \times \Omega$ is compact k is uniformly continuous.

Assume φ is a linear functional on $C(\Omega)$ and μ the associated signed Baire measure. We want to consider integral equations of the form

$$(1.1) \qquad \lambda x(s) - \int_\Omega k(s, t) x(t) d\mu(t) = y(s), \; s \in \Omega,$$

where $y \in C(\Omega)$, λ is a non-zero real number and x is the unknown function.

One method of solution is to replace the integral by some approximation, solve this approximate equation and show that the approximate solutions that are obtained converge in $C(\Omega)$ to a solution of (1. 1). An example of this is $\Omega = [0, 1]$ and φ the Riemann integral. The functionals discussed in Example 2. 1 of Chapter II can be used to approximate φ. For a discussion of this classical case and other references see [2].

Getting back to the general case, we shall assume that there exists a sequence $\{\varphi_n\}$ of continuous linear functionals on $C(\Omega)$ such that

$$\varphi_n x \to \varphi x \text{ for each } x \text{ in } C(\Omega). \tag{1. 2}$$

Let μ_n be the signed Baire measure associated with φ_n, $n = 1, 2, \cdots$. Define linear operators K and K_n, $n = 1, 2, \cdots$, on $C(\Omega)$ into $C(\Omega)$ by

$$(Kx)(s) = \int_\Omega k(s, t)x(t)d\mu(t), \quad x \in C(\Omega), \quad s \in \Omega, \tag{1. 3}$$

$$(K_n x)(s) = \int_\Omega k(s, t)x(t)d\mu_n(t), \quad x \in C(\Omega), \quad s \in \Omega. \tag{1. 4}$$

Note that K, K_n are compact operators.

Let $k_s(t) = k(s, t)$ for $s, t \in \Omega$. Then $k_s \in C(\Omega)$ and

(1.5) $$(Kx)(s) = \varphi(k_s x), \; s \in \Omega,$$

(1.6) $$(K_n x)(s) = \varphi_n(k_s x), \; s \in \Omega.$$

By (1. 2) we have

(1.7) $$(K_n x)(s) \to (Kx)(s) \text{ for each } s \in \Omega, \; x \in C(\Omega).$$

It follows from (1. 2) and the uniform boundedness principle that the norms $||\varphi_n||$, $n = 1, 2, \cdots$, are uniformly bounded. Then by the uniform continuity of k we obtain for each x in $C(\Omega)$.

(1.8) $\{\varphi_n(k_s x)\colon \; n = 1, 2, \cdots\}$ is an equicontinuous family.

From this and (1. 6) it follows [16, p. 178] that $\varphi_n(k_s x) \to \varphi(k_s x)$ uniformly for s in Ω. Thus,

(1.9) $$||K_n x - Kx|| \to 0 \text{ for each } x \text{ in } C(\Omega).$$

At this point we need to review some of the theory of collectively compact operator approximations. This theory has been developed within the last six or seven years. Some references are [2, 3, 5, 6, 14].

Let X be a Banach space and $[X]$ the set of all continuous linear operators mapping X into X. A set of operators $\mathcal{K} \subset [X]$ is collectively compact if the set $\{Tx\colon \; T \in \mathcal{K}, \; ||x|| \leq 1\}$ is totally bounded in X.

Recalling the definition of K_n in (1. 4) it follows from the continuity of k and the uniform boundedness of $|\mu_n|(\Omega) = ||\varphi_n||$, $n =$

1, 2, $\cdots$, that

(1.10) $\{K_n : n = 1, 2, \cdots\}$ is collectively compact.

We need the following result.

PROPOSITION 1.1. Let T_n, $T \in [X]$, $\lambda \neq 0$. Assume

(i) $\|T_n y - Ty\| \to 0$ for each y in X,

(ii) $\{T_n : n = 1, 2, \cdots\}$ is collectively compact,

(iii) T is compact.

Then $(\lambda - T)^{-1}$ exists if and only if $(\lambda - T_n)^{-1}$ exists and is uniformly bounded for n sufficiently large. In either case

$$(\lambda - T_n)^{-1} \to (\lambda - T)^{-1}.$$

Actually (iii) is a consequence of (i) and (ii). In particular, it follows from the contraction mapping theorem that $(\lambda - K)^{-1}$ exists for λ sufficiently large, i. e., (1.1) has a unique solution for each y in $C(\Omega)$.

Fix λ so that $(\lambda - K)^{-1}$ exists and let $y \in C(\Omega)$. Then, by Proposition 1.1, $(\lambda - K_n)^{-1}$ exists for n sufficiently large, say $n \geq N$, so the equation

$$(1.11) \qquad \lambda x(s) - \int_\Omega k(s, t)x(t)d\mu_n(t) = y(s), \quad s \in \Omega, \quad n \geq N,$$

has a unique solution x_n.

Then by (1.9) and Proposition 1.1 we have the following.

THEOREM 1.1. Suppose x is the unique solution of (1.1) and x_n the unique solution of (1.11). Then $x_n(t)$ converges uniformly to $x(t)$ as $n \to \infty$.

§ 2. The Extension of Positive Linear Functionals on $C(\Omega)$

We henceforth assume that φ is a positive linear functional on $C(\Omega)$ and μ the associated Baire measure. Note that $C(\Omega) \subset B(\Omega)$, where $B(\Omega)$ is the set of all real valued bounded functions defined on Ω with the sup norm. Now by Corollary 1.1, Chapter II, φ may be extended to a subspace of $B(\Omega)$ which we shall denote by $R(\Omega)$. We shall denote the extension also by φ.

LEMMA 2.1. Each x in $R(\Omega)$ is a Baire measurable function.

PROOF. Let $x \in R(\Omega)$, then for each positive integer n there are x_n, x^n in $C(\Omega)$ such that $x_n \leq x \leq x^n$ and $\varphi(x^n - x_n) < 1/n$. Moreover by redefining the functions x_n and x^n if necessary we may assume that $x_n \uparrow$, $x^n \downarrow$. Hence there exist Baire measurable functions x_1, x_2 such that $x_n(t) \to x_1(t)$, $x^n(t) \to x_2(t)$ for each t in Ω. By Lebesgue's dominated convergence theorem

$$\int_\Omega (x^n - x_n)d\mu \to \int_\Omega (x_2 - x_1)d\mu \,.$$

But since

$$\varphi(x^n - x_n) = \int_\Omega (x^n - x_n)d\mu \to 0$$

we have

$$\int_{\Omega}(x_2 - x_1)d\mu = 0,$$

whence $x_2(t) = x_1(t)$ a. e. $[\mu]$. Since $x_1(t) \leq x(t) \leq x_2(t)$ for all t in Ω we have $x_1(t) = x(t) = x_2(t)$ a. e. $[\mu]$. Thus x is Baire measurable.

LEMMA 2. 2. $R(\Omega)$ is a closed subspace of $B(\Omega)$ and consequently $R(\Omega)$ is a Banach space.

PROOF. Let $x_n \to x$, and $x_n \in R(\Omega)$. For each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$x_N - \varepsilon[4\varphi(1)]^{-1} \leq x \leq x_N + \varepsilon[4\varphi(1)]^{-1}$$

and there exist $x_{N\varepsilon}$, x_N^{ε} in $C(\Omega)$ such that

$$x_{N\varepsilon} \leq x_N \leq x_N^{\varepsilon} \text{ and } \varphi(x_N^{\varepsilon} - x_{N\varepsilon}) < \frac{1}{2}\varepsilon .$$

If we set $x^{\varepsilon}(t) = x_N^{\varepsilon}(t) + \varepsilon[4\varphi(1)]^{-1}$, $t \in \Omega$,

$$x_{\varepsilon}(t) = x_{N\varepsilon}(t) - \varepsilon[4\varphi(1)]^{-1}, \quad t \in \Omega,$$

then x^{ε}, $x_{\varepsilon} \in C(\Omega)$, $x_{\varepsilon} \leq x \leq x^{\varepsilon}$ and $\varphi(x^{\varepsilon} - x_{\varepsilon}) < \varepsilon$. Thus $x \in R(\Omega)$.

This lemma follows from a later result, but we wanted to give it an independent proof.

A different characterization of $R(\Omega)$ will be given, but first we need some results concerning semi-continuous functions.

DEFINITION 2. 1. Let Ω be a topological space. We say that $x \in B(\Omega)$ is upper (lower) semi-continuous at t_0 if given $\varepsilon > 0$ there exists a neighborhood U of t_0 such that $t \in U$ implies $x(t_0) > x(t) - \varepsilon$ $(x(t_0) < x(t) + \varepsilon)$.

If Ω is a uniform topological space, then $x \in B(\Omega)$ is upper (lower) semi-continuous on Ω if and only if x is the inf (sup) of a family of continuous functions on Ω [8, p. 146]. If Ω is metrizable then $x \in B(\Omega)$ is upper (lower) semi-continuous if and only if x is the pointwise limit of a decreasing (increasing) sequence of continuous functions [8, p. 155].

If Ω is a topological space and $t \in \Omega$ let $\mathcal{N}_t$ denote the collection of all neighborhoods of t.

<u>DEFINITION 2. 2.</u> If Ω is a topological space and $x \in B(\Omega)$, the upper envelope of x is

$$\overline{x}(t_0) = \inf_{U \in \mathcal{N}_{t_0}} \sup_{t \in U} x(t), \quad t_0 \in \Omega,$$

and the lower envelope of x is

$$\underline{x}(t_0) = \sup_{U \in \mathcal{N}_{t_0}} \inf_{t \in U} x(t), \quad t_0 \in \Omega.$$

<u>LEMMA 2. 3.</u> Let Ω be a topological space and $x \in B(\Omega)$. Then

(i) $\overline{x}, \underline{x} \in B(\Omega)$,

(ii) $\underline{x} \leq x \leq \overline{x}$,

(iii) $\overline{x}$ is upper semi-continuous,

(iv) $\underline{x}$ is lower semi-continuous,

(v) $x(t) = \overline{x}(t)$ if and only if x is upper semi-continuous at t,

(vi) $x(t) = \underline{x}(t)$ if and only if x is lower semi-continuous at t.

<u>PROOF.</u> (i) is clear since $x \in B(\Omega)$.

(ii) Let $U \in \mathcal{N}_{t_0}$. Then $t_0 \in U$ and so $x(t_0) \leq \sup_{t \in U} x(t)$ which implies $x(t_0) \leq \overline{x}(t_0)$. Similarly, $\underline{x} \leq x$.

(iii) Let $t_0 \in \Omega$ and $\varepsilon > 0$ be given. There exists an open neighborhood U of t_0 such that $\overline{x}(t_0) > \sup_{t \in U} x(t) - \varepsilon$. Let $s \in U$. Then U is a neighborhood of s, whence $\inf_{U \in \mathcal{N}_s} \sup x(t) \leq \sup_{t \in U} x(t)$. Therefore $\overline{x}(t_0) > \overline{x}(s) - \varepsilon$ for all s in U, so $\overline{x}$ is upper semi-continuous.

(iv) is proved in a similar fashion.

(v) Suppose x is upper semi-continuous at t_0 and let $\varepsilon > 0$ be given. Then there exists a neighborhood U of t_0 such that $t \in U$ implies $x(t_0) > x(t) - \varepsilon$. Then $x(t_0) \geq \sup_{t \in U} x(t) - \varepsilon \geq \inf_{U \in \mathcal{N}_{t_0}} \sup_{t \in U} x(t) - \varepsilon = \overline{x}(t_0) - \varepsilon$. Hence $x(t_0) \geq \overline{x}(t_0)$ and by (ii) $x(t_0) = \overline{x}(t_0)$. Conversely suppose $x(t_0) = \overline{x}(t_0)$ and let $\varepsilon > 0$ be given, then by (iii) there exists a neighborhood U of t_0 such that $x(t_0) = \overline{x}(t_0) > \overline{x}(t) - \varepsilon$ for t in U and so from (ii) we have $x(t_0) > x(t) - \varepsilon$ for t in U so x is upper semi-continuous at t_0.

(vi) is proved in a similar fashion.

<u>LEMMA 2.4.</u> Suppose Ω is a topological space and $x \in B(\Omega)$. Then x is continuous at t_0 if and only $\underline{x}(t_0) = x(t_0) = \overline{x}(t_0)$.

PROOF. This follows from Lemma 2. 3 and the observation that x is continuous at t_0 if and only if x is both upper and lower semi-continuous at t_0.

Now we come back to the case where Ω is a compact Hausdorff topological space. The following theorem gives a complete characterization of $R(\Omega)$.

THEOREM 2. 1. Let $x \in B(\Omega)$. A necessary and sufficient condition that $x \in R(\Omega)$ is that x be continuous a. e. $[\mu]$.

PROOF. The full proof is quite long and complicated and can be found in [7]. We will give here a different proof of the necessity and a simple proof of the sufficiency under the additional assumption that Ω is metrizable.

Assume $x \in R(\Omega)$. Then there exist sequences $\{x^n\}$, $\{x_n\}$ in $C(\Omega)$ such that $x_n \leq x \leq x^n$, $x^n \downarrow$, $x_n \uparrow$ and $\varphi(x^n - x_n) \to 0$. Define $x_2(t) = \lim_{n\to\infty} x^n(t)$, $x_1(t) = \lim_{n\to\infty} x_n(t)$, $t \in \Omega$. Then $x_1 \leq x \leq x_2$, x_2 is upper semi-continuous, x_1 is lower semi-continuous and by Lebesgue's dominated convergence theorem $\varphi(x^n - x_n) \to \varphi(x_2 - x_1)$, whence $\varphi(x_2 - x_1) = 0$. This implies that $x_1(t) = x(t) = x_2(t)$ a. e. $[\mu]$ and so x is continuous a. e. $[\mu]$.

For the converse we assume that Ω is metrizable and suppose x is continuous a. e. $[\mu]$. Let $\overline{x}$, $\underline{x}$ be the upper and lower envelopes respectively of x. By Lemmas 2. 3 and 2. 4 we have $\underline{x} \leq x \leq \overline{x}$ and $\underline{x}(t) = \overline{x}(t)$ a. e. $[\mu]$. Since Ω is metrizable there exist sequences

$\{x^n\}$ and $\{x_n\}$ in $C(\Omega)$ such that $x^n \downarrow \overline{x}$ and $x_n \uparrow \underline{x}$. By Lebesgue's dominated convergence theorem we have $\varphi(x^n - x_n) \to \varphi(\overline{x} - \underline{x}) = 0$, so $x \in R(\Omega)$. The proof is complete.

For $\Omega = [0, 1]$ and φ the Riemann integral, this reduces to the classical result that a function is Riemann integrable if and only if it is continuous except on a set of Lebesgue measure zero.

COROLLARY 2.1. If $x, y \in R(\Omega)$, then the product $xy \in R(\Omega)$, i. e., $R(\Omega)$ is a Banach algebra.

COROLLARY 2.2. If $x \in R(\Omega)$, then the absolute value $|x|$ of x in $R(\Omega)$.

Note that Lemma 2.2 also follows from Theorem 2.1.

§3. Integral Equations with Discontinuous Kernels

We now extend our results in §1 to the case where k may have some points of discontinuity.

Henceforth we assume that Ω is a compact metric space with metric d.

Let us define for x in $R(\Omega)$ a new norm $||x||_1 = \varphi(|x|)$ and let $R_1(\Omega)$ be the space $R(\Omega)$ with this new norm.

We shall assume that k is a bounded kernel such that

(3.1) $$k_s \in R(\Omega) \text{ for each } s \in \Omega .$$

As before we define

(3.2) $$(Kx)(s) = \varphi(k_s x), \qquad s \in \Omega,\ x \in R(\Omega) .$$

PROPOSITION 3.1. Assume that

(3.3) $$\lim_{s' \to s} ||k_s - k_{s'}||_1 = 0 \text{ for each } s \in \Omega.$$

Then

(3.4) $||k_s - k_{s'}||_1 \to 0$ as $d(s, s') \to 0$, uniformly for s, s' in Ω.

(3.5) $K\, R(\Omega) \subset C(\Omega)$.

(3.6) K is a compact linear operator.

(3.7) $\max_{s \in \Omega} \varphi(|k_s|)$ exists and $||K|| \leq \max_{s \in \Omega} \varphi(|k_s|)$.

PROOF. Define $f:\Omega \to R_1(\Omega)$ by $f(s) = k_s$. Then f is continuous by (3.3) and therefore uniformly continuous by the compactness of Ω. This proves (3.4). The function $||k_s||_1$, $s \in \Omega$, is a continuous real valued function defined on a compact set and so it attains its maximum. For each $x \in R(\Omega)$

$$|(Kx)(s)| \leq ||x|| \max_{s \in \Omega} \varphi(|k_s|),$$

$$|(Kx)(s) - (Kx)(s') \leq ||x|| \; ||k_s - k_{s'}||_1 .$$

It follows that $K\, R(\Omega) \subset C(\Omega)$, $||K|| \leq \max_{s \in \Omega} ||k_s||_1$, and K is compact by the Arzela-Ascoli theorem.

PROPOSITION 3.2. Suppose for $m = 1, 2, \cdots$ there are continuous kernels k^m such that

(3.8) $$||k_s^m - k_s||_1 \to 0 \text{ uniformly in } s.$$

Then (3.3) holds and if we define $(K^m x)(s) = \varphi(k_s^m x)$, $m = 1, 2, \cdots$, then

$$||K^m - K|| \to 0 . \tag{3.9}$$

PROOF. Define $f, f^m : \Omega \to R_1(\Omega)$ by $f(s) = k_s$ and $f^m(s) = k_s^m$. Then each f^m is continuous and f^m converges uniformly to f, so f is continuous and (3. 3) holds. Now $||K^m - K|| \leq \max ||k_s^m - k_s||_1 \to 0$ as $m \to \infty$.

DEFINITION 3. 1. The kernel $k(s, t)$ is said to be uniformly t-integrable if for each $\varepsilon > 0$ there exist continuous kernels k^ε, k_ε such that

(3. 10) $\quad k_\varepsilon(s, t) \leq k(s, t) \leq k^\varepsilon(s, t), \; s, t \in \Omega,$

(3. 11) $\quad ||k_s^\varepsilon - k_{\varepsilon s}||_1 < \varepsilon$ for all s in Ω.

THEOREM 3. 1. Suppose k is a uniformly t-integrable kernel. Then (3. 3) holds and $\{k_s : s \in \Omega\}$ is a φ-regular set.

PROOF. It is clear that $\{k_s : s \in \Omega\}$ is a φ-regular set from the definition of a uniformly t-integrable kernel, and (3. 3) follows by Proposition 3. 2.

One may find a discussion of uniformly t-integrable kernels for $\Omega = [0, 1]$, φ the Riemann integral in [1].

We assume there exists a sequence $\{\varphi_n\}$ of positive linear functionals on $C(\Omega)$ such that $\varphi_n x \to \varphi x$ for each x in $C(\Omega)$. By Theorem 3. 1, Chapter II,

$$\varphi_n x \to \varphi x \text{ for each } x \text{ in } R(\Omega) . \tag{3.12}$$

For each n define

$$(K_n x)(s) = \varphi_n(k_s x), \quad x \in R(\Omega), \quad s \in \Omega. \tag{3.13}$$

PROPOSITION 3.3. Each K_n is a continuous linear map from $R(\Omega)$ into $B(\Omega)$ and the norms $||K_n||$, $n = 1, 2, \cdots$, are uniformly bounded.

PROOF. It follows from (3.12) and the principle of uniform boundedness that $||\varphi_n|| \leq M$, $n = 1, 2, \cdots$, for some $M < \infty$. Then

$$|(K_n x)(s)| \leq \varphi_n(|k_s x|) \leq ||x|| \, [M \sup_{s \in \Omega} ||k_s||],$$

which implies

$$||K_n|| \leq M \sup_{s \in \Omega} ||k_s|| .$$

We denote by μ_n the finite Baire measures associated with the φ_n, $n = 1, 2, \cdots$, and let

$$k^t(s) = k(s, t) \quad \text{for} \quad s, t \in \Omega. \tag{3.14}$$

In general one should not expect to have $K_n R(\Omega) \subset R(\Omega)$, $n = 1, 2, \cdots$. The next result gives one condition for which this is true.

PROPOSITION 3.4. Suppose there exist continuous kernels k^m, k_m, $m = 1, 2, \cdots$, satisfying

(i) $\quad k_m(s, t) \leq k(s, t) \leq k^m(s, t)$, $m = 1, 2, \cdots$, $s, t \in \Omega$,

(ii) $\quad \int_\Omega [k^m(s, t) - k_m(s, t) d\mu(s) \to 0$ as $m \to \infty$ for each $t \in \Omega$.

Then $K_n R(\Omega) \subset R(\Omega)$, $n = 1, 2, \cdots$.

PROOF. Without loss of generality assume that $k^m \downarrow$, $k_m \uparrow$. Let $x \in R(\Omega)$, $x \geq 0$ and $n \geq 1$ be fixed. From (i) we obtain

$$\varphi_n(k_{ms}x) \leq \varphi_n(k_s x) \leq \varphi_n(k_s^m x), \quad m = 1, 2, \cdots .$$

As functions of s, $\varphi_n(k_{ms}x)$ and $\varphi_n(k_s^m x)$, $m = 1, 2, \cdots$, are continuous because of the uniform continuity of k_m and k^m. By Fubini's theorem and Lebesgue's dominated convergence theorem

$$\varphi[\varphi_n(k_s^m x) - \varphi_n(k_{ms}x]$$

$$= \int_\Omega x(t)d\mu_n(t) \int_\Omega [k^m(s, t) - k_m(s, t)]\, d\mu(s) \to 0 \text{ as } m \to \infty.$$

Thus, $(K_n x)(s) = \varphi_n(k_s x) \in R(\Omega)$. For an arbitrary x in $R(\Omega)$, write $x = x^+ - x^-$ to obtain $K_n x = K_n x^+ - K_n x^- \in R(\Omega)$.

The next result gives a criterion for each K_n to be compact. Let $n \geq 1$ be fixed.

PROPOSITION 3.5. Suppose there exist continuous kernels k^m, k_m, $m = 1, 2, \cdots$, satisfying

(i) $k_m(s, t) \leq k(s, t) \leq k^m(s, t)$, $s, t \in \Omega$,

(ii) $\varphi_n(k_s^m - k_{ms}) \to 0$ as $m \to \infty$ for each s in Ω.

Then K_n is compact.

PROOF. From the inequalities

$$k_s - k_{s'} \leq (k_s^m - k_{ms}) + (k_{ms} - k_{ms'})$$

$$k_s - k_{s'} \geq (k_{ms} - k_s^m) + (k_s^m - k_{s'}^m),$$

there follows

$$\varphi_n(k_s - k_{s'}) \leq \varphi_n(k_s^m - k_{ms}) + \varphi_n(k_{ms} - k_{ms'}),$$

$$\varphi_n(k_s - k_{s'}) \geq \varphi_n(k_{ms} - k_s^m) + \varphi_n(k_s^m - k_{s'}^m).$$

By the continuity of k^m, k_m and (ii) we have

$$\varphi_n(k_s - k_{s'}) \to 0 \text{ as } s' \to s.$$

$$|(K_n x)(s) - (K_n x)(s')| \leq |\varphi_n(k_s - k_{s'})|, \quad x \in R(\Omega), \quad ||x|| \leq 1.$$

Hence the set of functions $\{K_n x : ||x|| \leq 1\}$ is equicontinuous. These functions are also uniformly bounded and so K_n is compact.

There are other conditions that would guarantee the compactness of each K_n, for example if each K_n had a finite dimensional range. For φ_n of the form

$$\varphi_n x = \sum_{i=1}^{n} w_{ni} x(t_{ni}), \quad w_{ni} \geq 0, \quad t_{ni} \in \Omega, \quad n = 1, 2, \cdots,$$

we need only specify k^t $R(\Omega)$ for each t in Ω to obtain

(3.15) $$K_n R(\Omega) \subset R(\Omega), \qquad n = 1, 2, \cdots,$$

(3.16) $$K_n \text{ is compact}, \qquad n = 1, 2, \cdots.$$

Now we turn to the question of convergence of the approximate solutions of our integral equation to the exact solution.

THEOREM 3.2. Suppose $\{k_s: s \in \Omega\}$ is a φ-regular subset of $R(\Omega)$. Then $||K_n x - Kx|| \to 0$ as $n \to \infty$ for each x in $R(\Omega)$.

PROOF. For x in $R(\Omega)$, (3.12) implies

$$(K_n x)(s) = \varphi_n(k_s x) \to \varphi(k_s x) = (Kx)(s) \text{ for each } s \text{ in } \Omega.$$

But $\{k_s: s \in \Omega\}$ is a φ-regular set and so $(K_n x)(s) \to (Kx)(s)$ uniformly for s in Ω, i.e., $||K_n x - Kx|| \to 0$ as $n \to \infty$.

In particular this result holds if k is uniformly t-integrable.

We need the next result which appears in [3].

PROPOSITION 3.6. Let X be a Banach space and K_n, K_n^m, $m, n = 1, 2, \cdots$, continuous linear maps from X into X. Suppose K_n is compact for each n, $\{K_n^m: n \geq 1\}$ is collectively compact for each m, and

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} ||K_n^m - K_n|| = 0 .$$

Then $\{K_n: n \geq 1\}$ is collectively compact.

THEOREM 3.3. Suppose (3.15), (3.16) hold. If

(i) $\{k_s: s \in \Omega\}$ is a φ-regular set,

(ii) for each $m = 1, 2, \cdots$, there exist continuous kernels k^m such that $||k_s^m - k_s||_1 \to 0$ as $m \to \infty$, uniformly for $s \in \Omega$.

Then $\{K_n: n \geq 1\}$ is collectively compact.

PROOF. For $m, n \geq 1$ define

$$(K_n^m x)(s) = \varphi_n(k_s^m x) , \qquad x \in R(\Omega),\ s \in \Omega .$$

By (1.10) and the continuity of k^m, $\{K_n^m: n \geq 1\}$ is collectively compact for each m. Also by the continuity of k^m, $\{k_s^m: s \in \Omega\}$ is totally bounded, whence $\{|k_s^m - k_s|: s \in \Omega\}$ is a φ-regular set for each m. Since $||K_n^m - K_n|| \leq \sup_{s \in \Omega} \varphi_n(|k_s^m - k_s|)$ we have

$$\overline{\lim_{n\to\infty}} ||K_n^m - K_n|| \leq \sup_{s \in \Omega} \varphi(|k_s^m - k_s|)$$

and so, by (ii), $\lim_{m\to\infty} \overline{\lim_{n\to\infty}} ||K_n^m - K_n|| = 0$. Then by Proposition 3.6, $\{K_n: n \geq 1\}$ is collectively compact.

Let y in $R(\Omega)$ be fixed and consider the equations

$$(3.17) \qquad \lambda x(s) - \int_\Omega k(s, t)x(t)d\mu(t) = y(s),$$

$$(3.18) \qquad \lambda x(s) - \int_\Omega k(s, t)x(t)d\mu_n(t) = y(s), \quad n = 1, 2, \cdots.$$

Suppose (3.17) has a unique solution x in $R(\Omega)$ and suppose the hypotheses of Theorem 3.3 hold. Then (3.18) has a unique solution x_n for n sufficiently large, say $n \geq N$, and by Proposition 1.1 and Theorem 3.2, 3.3 we have $||x_n - x|| \to 0$ as $n \to \infty$. We note in particular that this would hold if k were uniformly t-integrable and (3.15), (3.16) were valid.

Finally we remark that everything in this section is valid if Ω is a compact Hausdorff (not necessarily metrizable) topological space. However in that case some of the notation becomes rather cumbersome. In particular $d(s, s') \to 0$ uniformly in s, s' would have to be replaced by a statement involving the uniformities of Ω.

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