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Title: Stability Analysis of Homogeneous Shear Flow: Linear and Nonlinear Theories and a Hamiltonian Formulation

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Abstract approved:

The stability of steady-state solutions of the equations governing two-dimensional, homogeneous, incompressible fluid flow are analyzed in the context of shear-flow in a channel. Both the linear and nonlinear theories are reviewed and compared. In proving nonlinear stability of an equilibrium, emphasis is placed on using the stability algorithm developed in Holm *et al.* (1985). It is shown that for certain types of equilibria the linear theory is inconclusive, although nonlinear stability can be proven.

Establishing nonlinear stability is dependent on the definition of a norm on the space of perturbations. McIntyre and Shepherd (1987) specifically define five norms, two for corresponding to one flow state and three to a different flow state, and suggest that still others are possible. Here, the norms given by McIntyre and Shepherd (1987) are shown to induce the same topology (for the corresponding flow states), establishing their equivalence as norms, and hence their equivalence as measures of stability. Summaries of the different types of stability and their mathematical definitions are presented.

Additionally, a summary of conditions on shear-flow equilibria under which the various types of stability have been proven is presented.

The Hamiltonian structure of the two-dimensional Euler equations is outlined following Olver (1986). A coordinate-free approach is adopted emphasizing the role of the Poisson bracket structure. Direct calculations are given to show that the Casimir invariants, or distinguished functionals, are time-independent and therefore are conserved quantities in the usual sense.

Stability Analysis of Homogeneous Shear Flow: The Linear and Nonlinear Theories and a Hamiltonian Formulation

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1. INTRODUCTION

A. Motivation for the present study

The transition from laminar to turbulent motion in fluids has been studied using various approaches since the middle of the last century. A few comprehensive reviews of the literature have emerged over the past several years (Drazin and Reid, 1981; Swinney and Gollub (editors), 1985; Chandrasekhar, 1961; Joseph, 1976 (vols. 1 and 2)). One of the continuing tasks is to combine the results of various theoretical approaches with observations to improve our intuition and direct research for the future. To accomplish this a unified theory is essential. This study is an effort to link the relatively new ideas and results of fluid dynamics in a fully nonlinear, Hamiltonian framework with the classical results based on linear theory. While this has been done in the literature in various contexts the mathematical tools necessary have become increasingly sophisticated and there is a continuing need to make these accessible to a wider audience. The present work is an attempt to make a unified overview of these ideas in the specific context of homogeneous shear flow.

There have been recent contributions in the general theory of Hamiltonian fluid dynamics which are useful both qualitatively and computationally in atmospheric dynamics (see for example Holm *et al.*, 1983, 1985; Holm and Long, 1988, Abarbanel *et al.*, 1984, 1986; Olver 1982, 1986; Salmon, 1983). Yet there are still various aspects which remain as interesting problems. For example, it is not always clear how to extend rather specialized results to geophysical flows and is particularly difficult in the case of the atmosphere where its particular thermodynamics, compressibility, and boundary conditions tend to complicate the modeling.

The motivation for studying stability of homogeneous laminar flow is more than just historical precedent. Laminar, steady (time-independent) solutions to the equations of motion governing homogeneous flow are possible and correspond to observed characteristics of real fluids. Since in some cases there is an observed transition to turbulence and in other cases the flow remains laminar, there is a desire to classify these flows according to whether or not such a transition will occur. This has theoretical importance in understanding the mechanisms of generating turbulence and may eventually have practical value in prediction of atmospheric events. Beyond any of this is the pedagogic appeal of dealing with a relatively simpler case to learn the mathematical framework necessary. The Hamiltonian formulation of fluid dynamics is fairly involved but it unifies the theories of stability analysis for unstratified and stratified flows. Of course the stability results can be rather different, but the mathematical techniques are much the same. This study is an attempt to present the theory in the homogeneous case with as few digressions as possible. The ideas extend to the stratified case and references will be given.

The original emphasis in the study of stability of (homogeneous) shear flow was on the analysis of linearized equations of motion. This approach is reviewed here in order to compare its results to more recent results based on nonlinear theories. One important goal is to define carefully what is meant by stability in these two cases (linear and nonlinear) as a means of comparison. Stability of shear flow is the primary concern of this study, but there may be applications to convective stability by analyzing an appropriate model with the same mathematical techniques. The nonlinear techniques and Hamiltonian formulation have been used in analyzing the stability of stratified shear flow (Abarbanel *et al.*, 1984, 1986; Abarbanel and Holm, 1987; Holm and Long, 1989; Holm *et al.*, 1983, 1985), modeling quasigeostrophic dynamics (Blumen, 1968, 1971, 1978; Dikii, 1965a,b; Holm, 1986, 1988; Swaters, 1986; Weinstein 1983; Andrews, 1984) and implemented in numerical schemes (eg. Salmon, 1983). The main advantage of the nonlinear approach is that it has the promise of giving more general stability criteria, and consequently will be more realistic. However, we will see that since the nonlinear theory only provides sufficient conditions for stability that there is always the need to refine the estimates to include a wider class of flows to be analyzed. The linear theory can essentially only establish criteria for instability while the nonlinear theory can establish criteria for stability (Drazin and Reid, 1986)—examples which follow illustrate this point. Together these theories approach a complete description of stability.

B. An overview of dynamic stability

A standard theory regarding stability of equilibrium points in dynamical systems is now briefly reviewed. The terminology is appropriate for evolution equations such as the Navier-Stokes equations or any of its approximations, in particular the equations describing two-dimensional homogeneous shear flow. Consider an evolution equation

$$\frac{\partial u}{\partial t} = F(u)$$

where u is in some appropriate class of functions comprising a normed linear space, and F is an operator, possibly nonlinear, defined on this space.

Equilibrium, or *stationary* solutions u_e satisfy $F(u_e) = 0$. This equilibrium is said to be *Lyapunov stable* if any solution $u(t)$ beginning near u_e at $t = 0$ stays near u_e for all time. That is,

$$\forall\,\epsilon > 0,\ \exists\,\delta > 0 \quad \text{such that} \quad \|u_e - u(0)\| < \delta \quad \Rightarrow \quad \|u_e - u(t)\| < \epsilon\ ,\forall t\ .$$

The goal of an analysis of dynamic stability is to establish both necessary and sufficient conditions under which an equilibrium is stable. Usually the evolution equations for the motion of a perturbation linearized about the equilibrium solution,

$$\frac{\partial \tilde{u}}{\partial t} = DF(u_e) \cdot \tilde{u}\ ,$$

are studied by looking at the spectrum[1] of the linear operator $DF(u_e)$. This essentially now describes the dynamics of a "new" linear system which approximates the nonlinear system for $\tilde{u}$ near u_e. Qualitatively, if the solution, $\tilde{u}$, to this equation decays in amplitude with time, this is an indication of the stability of the original equilibrium. Formally, if the solution to the linearized equation is Lyapunov stable then the equilibrium is said to be *linearly stable.* A special case of linear stability is when the spectrum of $DF(u_e)$ consists only of imaginary values [2], which is called *neutral-(spectral)* stability of the equilibrium. In the broadest sense, spectral stability means that the spectrum of the linearized operator has no positive real part. This is the same as neutral stability in the case of Hamiltonian systems since without dissipation the spectrum can not have a negative real part. For detailed discussion of

[1] The *spectrum* of a linear operator $DF(u_e)$ is defined to be the set of values, λ, for which $(DF(u_e) - \lambda)\tilde{u} = 0$ has nontrivial solutions. This arises from assuming a separation of variables form for $\tilde{u}$. Most geophysical fluid dynamics references *specify* a specific type of solution such as $\tilde{u} = \exp ik(x - ct)$, where k is real and c is complex, which additionally specifies that $-ikc$ is in the spectrum.

[2] This would mean showing that the imaginary part of c is zero.

linear techniques in various contexts of fluid dynamics see Drazin and Reid (1981) and Chandrasekhar (1961). These points will be made again in the specific context of homogeneous shear flow in what follows.

Another approach introduced by Arnold (1965, 1969) uses the idea that if an equilibrium solution can be shown to be the minimum (or maximum) of a conserved functional, say H, then it is stable in the sense of Lyapunov. If for the moment we assume that the evolution equation is in Hamiltonian form (which requires special characteristics of F and is defined in Chapter 4, section B), then the method uses the second variation, $\delta^2 H$, of the conserved functional H, and allows *finite* perturbations to the equilibrium. That is, $\tilde{u}$ need not be close to u_e as in the linearized case. Definiteness of the second variation is sufficient for linear stability, but not sufficient to prove nonlinear stability (Arnold, 1969; Abarbanel *et al.*, 1986; Holm *et al.*, 1985). However, it is an indication of nonlinear stability and following Holm *et al.* (1985) we will call this *formal stability*. Formal stability implies linear stability but the converse is not true. See Holm *et al.* (1985) and Abarbanel *et al.* (1986) for discussion and proofs of these points. To prove nonlinear stability requires further convexity estimates described below.

C. The Energy-Casimir Convexity Method

The approach taken here is one that can be classified as an energy method and was first developed by Arnold (1965, 1969). The outline which follows is a generalization of Arnold's approach due to Holm *et al.* (1985) who refers to it as the "stability algorithm". (It appears in various papers since then (Abarbanel *et al.*, 1986; Holm and Long, 1988). The outline given below is left in a somewhat general form with comments related to the application to follow.

1.) *Equations of motion and Hamiltonian*

We begin with an appropriate space $\mathcal{F}$ of functions u and the equations governing the time evolution of u given as

$$\frac{\partial u}{\partial t} = F(u) \ . \tag{1.1}$$

The function u will turn out to be the vorticity of a two-dimensional flow and $\mathcal{F}$ will be the class of possible solutions in a certain domain satisfying given boundary conditions. The evolution equation will be the vorticity equation.

We associate a Poisson bracket structure with $\mathcal{F}$ on the space of real-valued functionals on $\mathcal{F}$. The Poisson bracket of two functionals must be a functional which depends bilinearly on the respective functional derivatives. Using this structure we can write the evolution equation (1.1) in Hamiltonian form as

$$\frac{\partial u}{\partial t} = \mathcal{D} \cdot \delta H[u] \ , \tag{1.2}$$

where $\mathcal{D}$ is a differential operator with corresponding Poisson bracket given by

$$\{\mathcal{P}, \mathcal{Q}\} = \int_D \delta\mathcal{P} \cdot \mathcal{D}\delta\mathcal{Q} \, d\vec{x} \ ,$$

for functionals $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{F}$. H is a Hamiltonian of the equation(s) (1.1). δH, $\delta\mathcal{P}$ and $\delta\mathcal{Q}$ are the variational or functional derivatives of H, $\mathcal{P}$, and $\mathcal{Q}$ respectively. The square bracket containing u is used to indicate that δH may depend on u and derivatives of u. Refer to Chapter IV for definitions and further details regarding the notation and construction of these relations. Appendix B discusses the definition of functional derivatives.

2.) *Constants of the motion*

There will be essentially two kinds of conserved quantities. One is the Hamiltonian, H, which is necessarily given as part of the structure in the forgoing formulation. The other type of conserved quantities are *distinguished functionals* or *Casimirs.*[3] These are functionals C such that $\{C, G\} = 0$ for all functionals G. As Holm *et al.* (1985) points out there may also be other conserved functionals associated with symmetries of the Hamiltonian. In the application considered here, these symmetries give rise to conservation of angular momentum and energy. (For further discussion see Olver (1982, 1986).)

3.) *Equilibria as critical points of a conserved functional*

The equilibrium solutions, u_e, of the evolution equations (1.1) can now be associated with extremals of conserved functionals. This is done by requiring that the first variation of $H_C = H + C$ have a critical point at u_e. (That is, the first variation of H_C at u_e is zero.) We are trying to establish that the equilibrium states will occur where the energy functional has a minimum (or maximum) and so its first variation is required to vanish there. This places restrictions on C but there may remain a certain degree of flexibility in C.

4.) *Convexity estimates*

If the second variation of H_C is definite (i.e. either strictly positive or strictly negative) at the equilibrium solutions then the system is called

[3] Olver (1986) uses the term "distinguished functions" which is in the tradition of S. Lie. Sudarshan and Mukunda (1974) use "Casimir"—this is the earliest reference using this terminology of which I am aware.

formally stable. This is not sufficient to prove nonlinear stability—see Holm *et al.* (1985), and references mentioned there. This is related to the fact that only the space consisting of smooth solutions is being considered, and it is not complete. (That is, a sequence of smooth solutions may converge to a discontinuous function which is no longer an allowed solution.)

To show nonlinear stability, convexity of H_C is used to estimate H_C for finite perturbations (away from the equilibrium). This is accomplished by finding quadratic forms on the solution space so that for *finite* perturbations $\delta u = u - u_e$,

$$Q_1(\delta u) \leq H(u_e + \delta u) - H(u_e) - DH(u_e) \cdot (\delta u) \tag{1.3}$$

$$Q_2(\delta u) \leq C(u_e + \delta u) - C(u_e) - DC(u_e) \cdot (\delta u) \ . \tag{1.4}$$

It is required that $Q_1 + Q_2$ be positive for all nonzero δu in $\mathcal{F}$. Otherwise, the same argument applies to $-H_C$ (see comments following (1.9)).

5.) *A priori estimates*

We can then show the following estimate on (δu)

$$Q_1(\delta u(t)) + Q_2(\delta u(t)) \leq H_C(u(0)) - H_C(u_e) \ . \tag{1.5}$$

The proof is the following argument. By adding the forms Q_1 and Q_2, noting that $DH_C(u_e) \cdot (\delta u) = 0$ by step (3), and noting that H_C is a constant of motion (being the sum of two constants of motion) so that

$$H_C(u(t)) - H_C(u_e) = H_C(u(0)) - H_C(u_e) \ ,$$

the result is immediate (Holm *et al.*, 1985).

6.) *Nonlinear stability*

If the quadratic form given by $Q_1 + Q_2$ defines a norm[4] on the space of perturbations, we may write

$$\|(\delta u)\|^2 = Q_1(\delta u) + Q_2(\delta u) \ . \tag{1.6}$$

If H_C is continuous in this norm at u_e then u_e is (Lyapunov) stable, *in this norm*. This can be seen as follows. Continuity of H_C means that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|\delta u(t)\| < \delta \qquad \Rightarrow \qquad |H_C(u_e + \delta u) - H_C(u_e)| < \epsilon$$

for all t. Note that H_C at any t is the same as H_C at $t = 0$. Then the a priori estimate (1.5) together with the above implication gives

$$\|\delta u(t)\| \leq |H_C(u_e + \delta u(0)) - H_C(u_e)| < \epsilon$$

for all t.

As Holm *et al.* (1985) shows, a sufficient condition for continuity of H_C is that there are constants C_1 and C_2 so that

$$H(u_e + \delta u) - H(u_e) - DH(u_e) \cdot (\delta u) \leq C_1 \|\delta u\|^2 \tag{1.7}$$

$$C(u_e + \delta u) - C(u_e) - DC(u_e) \cdot (\delta u) \leq C_2 \|\delta u\|^2 \ . \tag{1.8}$$

In this case stability is given by the estimate in the sense that

$$\|\delta u(t)\|^2 = (Q_1 + Q_2)(\delta u(t)) \leq (C_1 + C_2)\|\delta u(0)\|^2 \ . \tag{1.9}$$

It may turn out that Q_1 and Q_2 are not both be positive yet their sum is positive, or that their sum is always negative (Holm *et al.*, 1985). In the

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[4] Recall that a norm on a linear space is a real-valued function denoted by $\| \cdot \|$ with the properties (i) $\|u\| \geq 0$, (ii) $\|u\| \neq 0$ if $u \neq 0$, (iii) $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$, and (iv) $\|cu\| = |c| \, \|u\|$ for c a constant.

latter case stability is proved using minus the sum of Q_1 and Q_2 as the norm and showing that $-H_C$ is continuous in this norm. That is, u_e becomes a local minimum for $-H_C$ by showing $-H_C$ is convex. An example of the former (Q_1 and Q_2 not both positive, but $Q_1 + Q_2$ is positive), first presented by Arnold (1965), will be given where it is shown that $-H_C$ is continuous at the equilibrium in this case also.

The derivatives which appear here in "generic" form are the usual linear maps in finite dimensions, but in the application to follow these become *variational derivatives* of functionals. A brief overview is given in Appendix B. For more details see Guenther and Lee (1988) and Olver (1986).

2. LINEAR THEORY OF HOMOGENEOUS SHEAR FLOW

This review mainly consists of expressing known results without developing them from first principles. The linear theory has been covered extensively in other literature (Drazin and Reid, 1981; Chandrasekhar, 1961) and is reviewed here for completeness and to establish notational consistency for comparison throughout this study.

A. Governing Equations

Consider the case of two-dimensional, unstratified (homogeneous) incompressible parallel shear flow (without viscosity). The approach has been to decompose the flow into mean and perturbation parts and linearize the governing equations using the Reynolds assumptions of averaging—that is, products of perturbation quantities are neglected (cf. Stull, 1988).

Let $U(y)$ be the equilibrium (mean) velocity of the fluid moving in the x-direction, where y is the vertical coordinate. (Since gravitational effects are hydrostatic in this setting, the actual orientation of the (x, y)-plane does not affect the dynamics.) The perturbation velocities are denoted by u (in the x-direction) and v (in the y-direction). The momentum equations are then

$$u_t + Uu_x + vU' = -p_x \tag{2.1}$$

$$v_t + Uv_x = -p_y \tag{2.2}$$

where subscripts denote partial derivatives and $' := \partial/\partial y$ on the mean variables. The equation of mass continuity is

$$u_x + v_y = 0 \tag{2.3}$$

which allows the definition of a stream function ψ so that

$$u = \psi_y \qquad v = -\psi_x \ . \tag{2.4}$$

The problem of stability is governed by equations (2.1)–(2.3) and appropriate boundary conditions. In principle we can pose appropriate initial value problems and solve these equations to observe the evolution of an initial perturbation. In this sense we can loosely define instability to be conditions under which the perturbation quantities grow with time, and stability otherwise.

The scalar vorticity of this system is defined to be $\omega = \hat{k} \cdot \nabla \times (u, v) = v_x - u_y$, and is referred to simply as vorticity. The equation governing the evolution of the vorticity is obtained by taking $\partial/\partial x$ of equation (2.2) and subtracting $\partial/\partial y$ of equation (2.1). The result is

$$\omega_t + U\omega_x - U'(u_x + v_y) - U''v = 0 \ . \tag{2.5}$$

The first term is the time evolution of vorticity, next is the change in ω due to horizontal advection of (perturbation) vorticity by the mean flow. The third term represents vortex stretching (of the mean flow) due to convergence (of the perturbation flow—in this case zero) and the last term is "vertical" advection (i.e. in the y-direction) of mean vorticity by the perturbation flow. As mentioned, the third term is zero using the equation of continuity (2.3), leaving

$$\omega_t + U\omega_x - U''v = 0 \ . \tag{2.6}$$

So any change in perturbation vorticity is due to the interaction of advection by the mean flow (of perturbation vorticity) and advection by the perturbation flow (of mean vorticity). Note that in terms of the stream function,

$\omega = v_x - u_y = (-\psi_x)_x - (\psi_y)_y = -\Delta\psi$. So equation (2.6) can be written

$$\Delta\psi_t + U(\Delta\psi)_x - U''\psi_x = 0 \ . \tag{2.7}$$

This equation, (2.7), can also be derived by linearizing the nonlinear vorticity equation directly. The relationship between certain conventions followed in two-dimensional fluid dynamics and Arnold's papers is discussed in Appendix A.

We restrict our attention to a channel flow where there are rigid boundaries (parallel to the x-axis) at $y = y_o$ and $y = y_1$. The boundary conditions will be no flow through the channel wall which may be characterized in any one of the following ways.

(bc.1.) $v(y_0) = v(y_1) = 0$

(bc.2.) $\psi(y_0) =$ constant, $\psi(y_1) =$ (different) constant

(bc.3.) $\psi_x(y_0) = \psi_x(y_1) = 0$.

Additionally, in order to consider a finite domain and maintain conservation of mass, the total flow across any two points x_0 and x_1 along the channel must be the same

$$\int_{y_0}^{y_1} [U(y) + u(x_0, y; t)]\, dy = \int_{y_0}^{y_1} [U(y) + u(x_1, y; t)]\, dy \ .$$

For this last condition it suffices to assume the flow variables, either u or ψ, are periodic in x, say of period L. In which case if we choose $x_1 = x_0 + L$ then $u(x_1, y; t) = u(x_0 + L, y; t) = u(x_0, y; t)$ and the total flow must be the same at the endpoints of the channel. Though this assumption of periodicity is made by Arnold (1965, 1969) for his development, it is somewhat restrictive from a physical point of view. Unfortunately, it is rather closely tied to the success of the approach (Holm *et al.*, 1986). This is covered in some detail in Appendix A, part (iv).

At this point note that the system (2.1)–(2.4), (2.6) or (2.7) (with boundary conditions) are all equivalent statements of the problem.

B. Normal Modes and the Rayleigh Equation

Consider solutions where all perturbation quantities are decomposable into periodic modes which have the common factor $e^{ik(x-ct)}$ where k is real and c is complex. This amounts to assuming a separable form of solution (for (2.7)) which has a prescribed periodic structure. Specifically, let the perturbation stream function ψ have a normal mode decompostion as

$$\psi = \phi(y)e^{ik(x-ct)} \ ,$$

and substitute into (2.7). The result is the Rayleigh equation

$$(U-c)(\phi'' - k^2\phi) - U''\phi = 0 \qquad (2.8)$$

with zero boundary conditions

$$k\phi = 0 \qquad \text{on} \qquad y = y_0 \ , \quad y = y_1 \ .$$

(The resulting equation is the same having had taken (2.1)–(2.3) instead, and appropriate normal mode expansions of u, v, and p.)

An equivalent nondimensional version of this equation may be obtained by use of the new variables

$$\tilde{y} = \frac{y-y_0}{d} \ , \quad \tilde{\phi}(\tilde{y}) = \frac{\phi(y)}{Vd} \ , \quad \tilde{U}(\tilde{y}) = \frac{U(y)}{V} \ , \quad \tilde{c} = \frac{c}{V} \quad \text{and} \quad \alpha = dk \ .$$

Here V denotes a characteristic velocity scale in the x-direction, usually thought of as the cross-channel average of the mean profile U. The characteristic length scale is the cross channel width $d = y_1 - y_0$. The (dimensional) Rayleigh equation (2.8) can then be written

$$(V\tilde{U}(\tilde{y}) - V\tilde{c})(\frac{V}{d}\tilde{\phi}''(\tilde{y}) - \alpha^2\frac{V}{d}\tilde{\phi}(\tilde{y})) - V\tilde{U}''(\tilde{y})\frac{V}{d}\tilde{\phi}(\tilde{y}) = 0$$

where the prime $'$ is now $\partial/\partial\tilde{y}$. Factoring out the constant scale values

$$\frac{V^2}{d}\left\{(\tilde{U}-\tilde{c})(\tilde{\phi}''-\alpha^2\tilde{\phi})-\tilde{U}''\tilde{\phi}\right\}=0\ .$$

After dropping the tildes and dividing by the constant this leaves us with the nondimensional Rayleigh equation

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi=0 \tag{2.9}$$

with corresponding zero boundary conditions

$$\alpha\phi=0 \qquad \text{on} \qquad y=0\ , \quad y=1\ .$$

Note that the boundary conditions are unchanged when α is replaced by $-\alpha$ so the solutions will be independent of such a choice. Therefore assume $\alpha\geq 0$. Also note that if (ϕ, c) is a solution of (2.9) (and boundary conditions) then so also is the conjugate pair (ϕ^*, c^*). That is, to each unstable mode there is a corresponding stable mode and vice versa (Drazin and Reid, 1981). Adopting *convention* we say that if there is a solution ϕ for some $c=c_r+ic_i$ where $c_i>0$, then the equilibrium is unstable. What this really means is that in this case the notion of asymptotic stability (perturbations tending toward an equilibrium with time) does not exist and the strongest notion of stability possible is to say the equilibrium is stable if there are only modes with $c_i=0$. The convention is to say an unstable mode is one for which $c_i>0$ and ignore the corresponding conjugate solution.

Arnold (1978) makes the more general observation that

> ...for hamiltonian[5] systems asymptotic stability is impossible, so stability in a linear approximation is always neutral and insufficient for a conclusion about the stability of an equilibrium position of the nonlinear problem.

As Drazin and Reid (1981) point out, the above formulation of the problem would be entirely equivalent to solving the appropriate initial value problem for a perturbation if the spectrum of modes were complete (cf. comments following (2.4)). That is, an arbitrary disturbance could be written as a superposition of modes and the linearity of the system would allow analysis of each mode separately. The spectrum can be shown to be finite in non-singular modes (modes where $c \neq U$ anywhere in the domain) and continuous in singular modes (where $U = c$ at *some* point in the domain)—together these constitute a complete spectrum. The continuous spectrum is made up of modes which are stable (c is real) and can be disregarded if seeking conditions for instability. To show these properties of the spectrum is in itself somewhat involved and will not be considered here. For more details see Drazin and Reid (1981), section 24.

By assuming $c_i > 0$ (instability), dividing equation (2.9) by $(U-c)$, multiplying by ϕ^*, and integrating by parts we get Rayleigh's integral condition for instability

$$\int_0^1 (|\phi'|^2 + \alpha^2|\phi|^2)\, dy + \int_0^1 \frac{U''}{U-c}|\phi|^2\, dy = 0\ .$$

Since the first term is real, the imaginary part is the imaginary part of the second term and must be equal to zero. This yields Rayleigh's necessary

[5] Arnold uses a lower-case "h" throughout his book (1978).

condition for instability,

Instability implies $U''(y_c) = 0$ *for some* $y_c \in (0,1)$.

The contrapositive yields

Theorem (2–1) (Rayleigh's Theorem). $U''(y) \neq 0$ *for all* $y \in (0,1)$ *implies stability.*

Of course this is meant in the sense that the equilibrium profile, U, is neutrally stable in that the spectrum of the operator linearized about U is purely imaginary. (That is, stability *means* there are no eigenfunction-eigenvalue pairs (ϕ, c) with $c_i > 0$ which satisfy the Rayleigh equation for any α with the given U.)

C. Fjørtoft's Extension of Rayleigh's Theorem

By taking the real and imaginary parts of Rayleigh's integral condition, adding an appropriate constant times the imaginary part to the real part the Rayleigh-Fjørtoft necessary condition for instability is obtained (Drazin and Reid, 1981),

Instability implies $U''(U) < 0$ *for some* $y \in (0,1)$.

Let $y_c \in (0,1)$ be the point where $U'' = 0$, then by an appropriate Galilean shift of U by a constant we may take $U(y_c) = 0$. Again using the contrapositive a sufficient condition for spectral-linear stability of the mean state is obtained which will be refered to as the Rayleigh-Fjørtoft theorem.

Theorem (2–2) (Rayleigh-Fjørtoft). *Given that* $U''(y_c) = 0$ *and* $U(y_c) = 0$ *then* $U''(U) \geq 0$ *for all* $y \in (0,1)$ *implies stability.*

3. NONLINEAR THEORY

Arnold (1965, 1969) addressed the question of Lyapunov stability of fluid equilibria as outlined in the introduction. In this section his stability results are derived and their relation to the linear results through certain examples is shown. The development is done here to illustrate the stability algorithm of Holm *et al.* (1985) and put Arnold's results explicitly into this framework. This was in fact done by Holm *et al.* (1985) but is restated here to make certain aspects of this process explicit. It will be seen that this brings out some of the subtle points in an application of the algorithm. This topic is also developed in McIntyre and Shepherd (1987), without reference to Holm's technique, where they point out some of the complications Arnold did not address in detail. Comment on some of their results will also be made.

A. Basic method; U/U'' positive

1.) *Equations of motion and a Hamiltonian*

The flow is still partitioned into a mean (stationary) part and a perturbation part but the nonlinear equations must now be considered. If variables with an asterisk represent the total flow in dimensional form let

$$u^* = U(y) + u(x, y; t) \quad \text{and} \quad v^* = v(x, y; t) \ .$$

The equations of motion can then be written

$$u_t^* + u^* u_x^* + v^* u_y^* = -p_x^* \tag{3.1a}$$

$$v_t^* + u^* v_x^* + v^* v_y^* = -p_y^* \tag{3.1b}$$

with mass continuity being

$$u_x^* + v_y^* = 0 \ .$$

The vorticity equation is derived by taking $\partial/\partial x$ of equation (3.1b) and subtracting $\partial/\partial y$ of equation (3.1a). Using the continuity equation the vorticity equation for the total flow is

$$\omega_t^* + u^*\omega_x^* + v^*\omega_y^* = 0 \ . \tag{3.2}$$

The continuity equation implies the existence of a stream function for the total flow which we denote by ψ. Then equation (3.2) takes the form

$$(\Delta\psi)_t - \psi_y(\Delta\psi)_x + \psi_x(\Delta\psi)_y = 0 \ , \tag{3.3}$$

where $\Delta\psi = \psi_{xx} + \psi_{yy}$ is the Laplacian of the stream function. Since the perturbations must also satisfy the continuity equation, define a perturbation stream function, ϕ, and a stream function for the equilibrium state, $\Psi = \Psi(y)$ giving $\psi = \Psi(y) + \phi(x, y; t)$. As previously mentioned if we now linearized (3.2) we would obtain (2.6) and linearizing (3.3) produces (2.7).

Following Arnold's examples we restrict our attention to flows which are periodic in x, say of period L. That is, assume *all* flow variables, including ψ, derivatives of ψ and similarly for p^*, are periodic in x. The domain of the flow may then be restricted to a section of channel $D = \{(x, y) : x_0 \le x \le x_1, y_0 \le y \le y_1\}$. For convenience we will label the boundaries as follows

$$\Gamma_{y_0} = \{(x, y_0) : x_0 \le x \le x_1\}$$

$$\Gamma_{y_1} = \{(x, y_1) : x_0 \le x \le x_1\}$$

$$\Gamma_{x_0} = \{(x_0, y) : y_0 \le y \le y_1\}$$

$$\Gamma_{x_1} = \{(x_1, y) : y_0 \le y \le y_1\} \ ,$$

and denote the total boundary, $\cup\Gamma_i$, by ∂D.

The total energy of the system

$$E(\psi) = \frac{1}{2}\iint_D \nabla\psi\cdot\nabla\psi\, dx\, dy \tag{3.4}$$

is conserved which follows readily from the assumed periodicity (see the following section). It turns out that E is the Hamiltonian and (3.3) is in Hamiltonian form (Olver, 1982, 1986). To show this requires further theoretical development and is discussed in Chapter IV.

2.) *Constants of the motion*

Conservation of energy E is most easily shown by deriving the energy equation directly. Begin by taking u^* times (3.1a) and adding v^* times (3.1b), yielding

$$\frac{1}{2}((u^*)^2+(v^*)^2)_t = -\frac{1}{2}(u^*,v^*)\cdot\nabla((u^*)^2+(v^*)^2) - (u^*,v^*)\cdot\nabla p^* \ .$$

Now integrating over the domain, the left hand side becomes the time rate of change of energy,

$$\frac{\partial E}{\partial t} = -\frac{1}{2}\iint_D (u^*,v^*)\cdot\nabla((u^*)^2+(v^*)^2)\, dx\, dy - \iint_D (u^*,v^*)\cdot\nabla p^*\, dx\, dy \ .$$

The right hand side can be changed into an integral of a divergence by recalling that $\nabla\cdot(u^*,v^*)=0$ so that

$$\begin{aligned}
-\frac{1}{2}&\iint_D (u^*,v^*)\cdot\nabla((u^*)^2+(v^*)^2)\, dx\, dy - \iint_D (u^*,v^*)\cdot\nabla p^*\, dx\, dy \\
&= -\frac{1}{2}\iint_D \nabla\cdot[((u^*)^2+(v^*)^2)(u^*,v^*)]\, dx\, dy \\
&\quad + \frac{1}{2}\iint_D [((u^*)^2+(v^*)^2)\nabla\cdot(u^*,v^*)]\, dx\, dy \\
&\quad - \iint_D \nabla\cdot[p^*(u^*,v^*)]\, dx\, dy + \iint_D [p^*\nabla\cdot(u^*,v^*)]\, dx\, dy \\
&= -\iint_D \nabla\cdot\left\{[\frac{1}{2}((u^*)^2+(v^*)^2)+p^*](u^*,v^*)]\right\}\, dx\, dy \ .
\end{aligned}$$

Applying the divergence theorem this can be turned into a boundary integral,

$$-\iint_D \nabla \cdot \left\{ [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*](u^*, v^*)] \right\} \, dx \, dy$$
$$= -\int_{\partial D} [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*](u^*, v^*)] \cdot \hat{n} \, ds$$

(where $\hat{n}$ is an outward unit normal)

$$= -\left\{ \int_{x=x_0}^{x_1} [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*][-v^*]_{(x,y_0)} \, dx + \right.$$
$$\int_{x=x_0}^{x_1} [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*][v^*]_{(x,y_1)} \, dx +$$
$$\int_{y=y_0}^{y_1} [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*][-u^*]_{(x_0,y)} \, dy +$$
$$\left. \int_{y=y_0}^{y_1} [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*][u^*]_{(x_1,y)} \, dy \right\} .$$

The first and second terms are zero since $v^*(x, y_i) = 0$. Since $u^*(x_0, y) = u^*(x_1, y)$ the last two terms may be combined so that

$$\frac{\partial E}{\partial t} = -\int_{y=y_0}^{y_1} \left\{ [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*]_{(x_1,y)} - \right.$$
$$\left. [\frac{1}{2}((u^*)^2 + (v^*)^2) + p^*]_{(x_0,y)} \right\} u^*(x_0, y) \, dy .$$

The terms in the braces cancel by the assumed periodicity of the flow variables. Consequently we have conservation of energy.

Equation (3.2) (equivalently (3.3)) is the statement that the total vorticity is conserved following the fluid motion. From this and boundary conditions we get that for any smooth function $\Phi(\xi)$,

$$C(\psi) = \iint_D \Phi(\omega^*) \, dx \, dy = \iint_D \Phi(\Delta\psi) \, dx \, dy \tag{3.5}$$

is also conserved. That is,

$$\frac{\partial C}{\partial t} = \iint_D \Phi'(\omega^*) \frac{\partial \omega^*}{\partial t} \, dx \, dy = \iint_D \Phi'(\omega^*) \left[-u^* \omega_x^* - v^* \omega_y^* \right] \, dx \, dy$$
$$= \iint_D \Phi'(\Delta\psi) \left[\psi_y \Delta\psi_x - \psi_x \Delta\psi_y \right] \, dx \, dy \; ,$$

and integrating by parts this turns into

$$\int_{y_0}^{y_1} \left[(\Phi\psi_y \Big|_{x=x_0}^{x=x_1} - \int_{x_0}^{x_1} \Phi\psi_{yx}\, dx \right] dy - \int_{x_0}^{x_1} \left[(\Phi\psi_x \Big|_{y=y_0}^{y=y_1} + \int_{y_0}^{y_1} \Phi\psi_{xy}\, dy \right] dx = 0\,.$$

C is a distinguished functional (Casimir) of the Hamiltonian system as described in the introduction (section C, part 2). The proof of this is described in Chapter IV and shown in Olver (1982, 1986).

Since the equilibrium state (described by $\Psi(y)$) is time-independent, equation (3.3) gives us

$$\Psi_x(\Delta\Psi)_y - \Psi_y(\Delta\Psi)_x = 0\ ,$$

or restated this is

$$\frac{\partial(\Psi, \Delta\Psi)}{\partial(x, y)} = 0\ .$$

This means that stationary solutions are those for which $\nabla\Psi$ and $\nabla\Delta\Psi$ are collinear (i.e. streamlines are parallel to lines of constant vorticity). In other words, stationary solutions are those for which vorticity advection is zero. A sufficient condition for this to occur is the existence of a function Λ so that

$$\Psi = \Lambda(\Delta\Psi)$$

which in this particular case means that

$$\Psi = \Lambda(-U')\ . \tag{3.6}$$

A sufficient condition for Λ to be single-valued is that $U'' \neq 0$ for all $y \in (y_0, y_1)$ so that U' is monotonic in y. However, this functional relation still holds in some cases where inflection points ($U'' = 0$) are present. An example of this is considered later in this chapter. McIntyre and Shepherd (1987) pursue the case where Λ is multi-valued which is not discussed further here.

3.) *Equilibria as critical points of a conserved functional*

Now define a new functional made by adding (3.4) and (3.5)

$$H_C(\psi) = E(\psi) + C(\psi) = \frac{1}{2}\iint_D \nabla\psi \cdot \nabla\psi \, dx\, dy + \iint_D \Phi(\Delta\psi)\, dx\, dy \ . \quad (3.7)$$

The arbitrary function Φ will be determined so as to make the stationary flows extremals of this functional. That is, consider the first variation of H_C evaluated at the stationary flow Ψ (see Appendix B),

$$\begin{aligned} \delta H_C &= \iint_D [\nabla\Psi \cdot \nabla\phi + \Phi'(\Delta\Psi)]\ dx\, dy \\ &= \iint_D [(\Phi'(\Delta\Psi) - \Psi)\Delta\phi]\ dx\, dy \ . \end{aligned}$$

(Recall Φ is a function of a single variable and Φ' is just its derivative with respect to that dependence.)

Extremals of H_C are flows for which $\delta H_C = 0$. If we choose $\Phi'(\Delta\Psi) = \Psi = \Lambda$ then the bracket in the preceding equation is zero and $\delta H_C = 0$ at Ψ. That is, by specifying Φ as that function for which

$$\Phi'(\xi) = \Lambda(\xi) \ , \quad (3.8)$$

the stationary flow Ψ then corresponds to a critical point of H_C. This step of the algorithm determines Φ to the extent that Λ is determined by the stationary flow.

The second variation is found to be (see Appendix B)

$$\delta^2 H_C = \iint_D \left[\Phi''(\Delta\Psi)(\Delta\phi)^2 + (\nabla\phi)^2\right]\ dx\, dy \ .$$

We can calculate Φ'' (in terms of the mean velocity) using the relation (3.6)

$$\Psi = \Lambda(-U') \quad \Rightarrow \quad \frac{d\Psi}{dy} = \Lambda'(-U')(-U'')$$

$$\text{i.e.} \quad -U = \Lambda'(-U')(-U'') \quad \Rightarrow \quad \Lambda'(-U') = \frac{U}{U''} \ .$$

This gives us that

$$\Phi''(\Delta\Psi) = \Phi''(-U') = \Lambda'(-U') = \frac{U}{U''} \ .$$

We arrive at the second variation in the form

$$\delta^2 H_C = \iint_D \left[\frac{U}{U''} (\Delta\phi)^2 + (\nabla\phi)^2 \right] \ . \tag{3.9}$$

From (3.9) it appears that with appropriate restrictions on U/U'' we could make the second variation positive (or possibly negative) definite. This would prove formal stability and we could conclude that the equilibrium is linearly stable. This suggests looking for convexity estimates is worthwhile in an effort to show nonlinear stability.

Note that the Rayleigh-Fjørtoft theorem from the linear theory predicts stability if U/U'' is nonnegative, which would correspond to (3.9) being positive definite. In which case formal stability and spectral stability coincide. But, as noted in the introduction, Holm *et al.* (1985) discusses the fact that this does not suffice to establish nonlinear stability.

Also worth noting at this point is that there is the potential of showing (3.9) to be *negative* definite, for cases where U/U'' is negative, admitting the possibility of showing stability in such cases. There is *no* result from the linear theory in this situation (the linear theory does not apply to the case U/U'' negative). Further discussion and examples of this occur in material to follow.

4.) *Convexity estimates*

Following the stability algorithm, consider step (4) and look at the right hand sides of the inequalities (1.3) and (1.4).

$$
\begin{aligned}
&E(\Psi+\phi)-E(\Psi)-\delta E(\Psi)\cdot(\phi)\\
&\quad=\frac{1}{2}\iint_D(\nabla\Psi+\nabla\phi)^2\,dx\,dy\\
&\qquad\qquad-\frac{1}{2}\iint_D(\nabla\Psi)^2\,dx\,dy-\left\{-\iint_D\Psi\Delta\phi\,dx\,dy\right\}\\
&\quad=\frac{1}{2}\iint_D(\nabla\phi)^2\,dx\,dy+\iint_D\nabla\Psi\cdot\nabla\phi\,dx\,dy+\iint_D\nabla\phi\Delta\phi\,dx\,dy\\
&\quad=\frac{1}{2}\iint_D(\nabla\phi)^2\,dx\,dy+\oint_{\partial D}\Psi\frac{\partial\phi}{\partial\nu}\,d\sigma\\
&\qquad\qquad\text{by Green's identity---see Appendix A(i)}\\
&\quad=\frac{1}{2}\iint_D(\nabla\phi)^2\,dx\,dy\ .
\end{aligned}
$$

So it is clear that if we choose

$$
Q_1(\phi):=\frac{1}{2}\iint_D(\nabla\phi)^2\,dx\,dy \tag{3.10}
$$

then inequality (1.3) is satisfied (we just happen to have equality).

Similarly consider

$$
\begin{aligned}
&C(\Psi+\phi)-C(\Psi)-\delta C(\Psi)\cdot(\phi)\\
&\quad=\iint_D[\Phi(\Delta\Psi+\Delta\phi)-\Phi(\Delta\Psi)-\Phi'(\Delta\Phi)\Delta\phi]\,dx\,dy\ .
\end{aligned}
$$

Now a condition on the convexity of Φ must be used to find Q_2. Assume there is a constant c so that

$$
0<c\leq\Phi''(\xi)\quad\text{for all}\quad\xi\in[\min\Delta\Psi,\max\Delta\Psi]
$$

and extend the definition of $\Phi(\xi)$ to all of the real line subject to this inequality. Then it follows (see Appendix C) that for any ϕ in the class of perturbations

$$
\frac{c}{2}(\Delta\phi)^2\leq\Phi(\Delta\Psi+\Delta\phi)-\Phi(\Delta\Psi)-\Phi'(\Delta\Psi)\Delta\phi\ .
$$

Consequently we may choose

$$Q_2(\phi) := \iint_D \frac{c}{2}(\Delta\phi)^2 \, dx \, dy \tag{3.11}$$

and satisfy the inequality (1.4).

5.) *A priori estimates*

It can now be shown that the a priori estimate (1.5) given by part (5) of the stability algorithm is satisfied. That is, we will show that (corresponding to (1.5))

$$Q_1(\phi) + Q_2(\phi) \leq H_C(\Psi + \phi_0) - H_C(\Psi) \tag{3.12}$$

where ϕ_0 means $\phi(x, y; t = 0)$. The following lemma is used.

lemma (3–1). *The functional given by*

$$\widetilde{H}(\phi) := \iint_D \frac{1}{2}(\nabla\phi)^2 + [\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Psi)\Delta\phi] \, dx \, dy$$

is constant with respect to t. That is,

$$\widetilde{H}(\phi(x, y; t)) = \widetilde{H}(\phi(x, y; 0)) \; .$$

Proof: Consider that $H_C(\phi)$ is independent of t. Then $\widehat{H}_C(\phi) = H_C(\Psi + \phi) - H_C(\Psi)$ is also independent of t. But

$$\begin{aligned}
\widehat{H}_C(\phi) &= \iint_D \left[\frac{1}{2}(\nabla\Psi + \nabla\phi)^2 + \Phi(\Delta\Psi + \Delta\phi) - \frac{1}{2}(\Delta\Psi)^2 - \Phi(\Delta\Psi)\right] \, dx \, dy \\
&= \iint_D [\nabla\Psi \cdot \nabla\phi + \Phi'(\Delta\Psi)\Delta\phi] \, dx \, dy \\
&\quad + \iint_D \left[\frac{1}{2}(\nabla\phi)^2 + \Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Psi)\Delta\phi\right] \, dx \, dy \\
&= H_1(\phi) + \widetilde{H}_C(\phi) \; ,
\end{aligned}$$

by defining

$$H_1(\phi) := \iint_D [\nabla\Psi \cdot \nabla\phi + \Phi'(\Delta\Psi)\Delta\phi] \, dx \, dy \; .$$

Note that by Green's identity (cf. Appendix A.)

$$
\begin{aligned}
H_1(\phi) &= \iint_D \left[\nabla\Psi \cdot \nabla\phi + \Phi'(\Delta\Psi)\Delta\phi\right]\, dx\, dy \\
&= -\iint_D \left[\Psi\Delta\phi + \Phi'(\Delta\Psi)\Delta\phi\right]\, dx\, dy + \oint_{\partial D} \Psi \frac{\partial\phi}{\partial\nu}\, d\sigma \;.
\end{aligned}
$$

But we have chosen $\Phi'(\Delta\Psi) = \Psi$ (by requiring the first variation of H_C to be zero) so the first term on the right hand side is zero. Since Ψ is constant on (connected components of) the boundary, the boundary integral will be zero if we restrict perturbations to those for which the normal derivative of their stream function is zero. This amounts to requiring that the perturbations preserve total circulation of the mean flow (see Appendix A). Consequently we have that $\widetilde{H}_C(\phi) = \widehat{H}_C(\phi)$ and since $\widehat{H}_C(\phi)$ is independent of t, $\widetilde{H}_C(\phi)$ must be as well. ▮

This is used to show the a priori estimate (3.12) as follows.

$$
\begin{aligned}
Q_1(\phi) + Q_2(\phi) &= \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2\right]\, dx\, dy \\
&\le \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + \Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Psi)\Delta\phi\right]\, dx\, dy \\
&= \widetilde{H}(\phi) = \widetilde{H}(\phi_0) = \widehat{H}_C(\phi_0) \\
&= H_C(\Psi + \phi_0) - H_C(\Psi) \;.
\end{aligned}
$$

6.) *Nonlinear stability*

With some further restriction on U/U'' nonlinear stability is proven if it can be shown that H_C is continuous in the norm defined by

$$
\|\phi\|^2 := Q_1(\phi) + Q_2(\phi) = \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2\right]\, dx\, dy \;. \tag{3.13}
$$

(This is equation (1.6) in the stability algorithm.) We are able to achieve the sufficient conditions corresponding to (1.7) and (1.8) if there is a constant C

so that

$$\frac{U}{U''} = \Phi''(\xi) \le C < \infty \, .$$

First find C_1 so that (1.7) holds by using

$$\begin{aligned} &E(\Psi+\phi) - E(\Psi) - \delta E(\Psi)\cdot(\phi) \\ &= \frac{1}{2}\iint_D (\nabla\phi)^2\,dx\,dy \le \frac{1}{2}\iint_D \left[(\nabla\phi)^2\,dx\,dy + c(\Delta\phi)^2\right]\,dx\,dy \\ &\qquad = Q_1(\phi) + Q_2(\phi) = \|\phi\|^2 \, . \end{aligned}$$

(In this case we may simply choose $C_1 = 1$.) Next find C_2 so that (1.8) holds by using

$$\begin{aligned} &C(\Psi+\phi) - C(\Psi) - \delta C(\Psi)\cdot(\phi) \\ &= \iint_D [\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Phi)\Delta\phi]\,dx\,dy \\ &\le \iint_D \frac{C}{2}(\Delta\phi)^2\,dx\,dy = \frac{C}{2c}\iint_D c(\Delta\phi)^2\,dx\,dy \\ &\le \frac{C}{c}\frac{1}{2}\iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2\right]\,dx\,dy \\ &= \frac{C}{c}\|\phi\|^2 \, , \end{aligned}$$

(and choosing $C_2 = C/c$).

As shown in stability algorithm part (6), this suffices to show continuity of H_C and therefore Lyapunov (nonlinear) stability of Ψ in the given norm. That is, given any $\epsilon > 0$, by the continuity of H_C at Ψ we may find a $\delta > 0$ so that $\|\phi\| < \delta$ implies $|H_C(\Psi+\phi) - H_C(\Psi)| < \epsilon$ for all time. Then from the a priori estimate (3.12) we see that $\|\phi\|^2 < |H_C(\Psi+\phi_0) - H_C(\Psi)| < \epsilon$ for all time.

In summary, if there are constants c and C so that

$$0 < c \le \Phi'' = \frac{U}{U''} \le C < \infty \, , \tag{3.14}$$

it is possible to show Lyapunov stability of the equilibrium (considered as the flow profile U or its stream function Ψ) in the normed space of perturbations

ϕ, which satisfy the basic boundary conditions and additionally leave the total circulation unchanged, with norm defined by

$$\|\phi\|^2 := \frac{1}{2} \iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2 \right] \, dx \, dy \ .$$

Note that if U'' is not zero (so it is of one sign throughout the channel) we may add a constant to the mean flow (a Galilean transformation) so that U and U'' are of the same sign (the sign of U'') and bounded away from zero. Then (3.13) defines a norm and stability follows (provided C exists). This is in the same sense as Rayleigh's theorem, ($U'' \neq 0$ implies spectral stability) but stronger in that this shows Lyapunov stability.

If there is one point of inflection and U is antisymmetric with respect to that point, we locate the x-axis at the inflection point (so $y \in [a, b]$ with $U''(0) = 0$) and again through a Galilean transformation (location of the y-axis) find a reference frame for which $U(0) = 0$. In this reference frame U/U'' is positive and the result still holds.

Of course, if the stability condition (3.14) does not hold then the flow is not necessarily unstable. This is part of the weakness in finding sufficient conditions for stability. However, in example (2) below, it will be seen that in certain instances both necessary and sufficient conditions for stability have been found.

Example 1.)

It is interesting to note a few subtleties regarding the above results. Arnold (1965) gave as an example (a version of) the flow with $U(y) = \beta y + \gamma y^3$ which is stable for $\beta\gamma > 0$ but points out that Tollmien (see Drazin and Reid, 1981) showed that the flow $U(y) = \gamma y^3$ in the case where $y_0 = -y_1$ is unstable (also see Meshalkin and Sinai, 1962). Tollmien's example is the limit of Arnold's as $\beta \to 0^+$. The definition of the norm is dependent on β and as β becomes small the norm of a given perturbation becomes smaller. In effect this only changes the "shape" of open neighborhoods (see part C to follow). The stability argument breaks down for vanishing β in step (6)—C_2 fails to exist.

B. The case U/U'' negative

One possibility we have not yet considered is that $U(y)$ has one inflection point but U/U'' is negative. To do this we need to reconsider the stability algorithm. If we assume there is a c so that

$$0 < c \leq -\Phi''(\xi) = -\frac{U}{U''} \quad \text{for all} \quad \xi \in [\min\Delta\Psi, \max\Delta\Psi]$$

(extended to $\xi \in R$) then it follows that

$$\frac{c}{2}(\Delta\phi)^2 \leq -\{\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Phi)\Delta\phi\}$$

(as in Appendix C, but include the sign change). If we define

$$\|\phi\|^2 := \frac{1}{2}\iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right]\, dx\, dy \tag{3.15}$$

this will in fact be a norm on the space of perturbations provided that it is positive definite. It turns out that a necessary and sufficient condition for (3.15) to be positive definite is related to the size of the domain (McIntyre and Shepherd, 1987). That is, the scale of the disturbance (defined as the ratio of enstrophy to energy) must be restricted by the scale of the domain. As shown below, this in essence means that for a given domain the choices for c are limited.

To obtain as estimate for the scale of the disturbance, this scale and the scale of the domain are characterized in terms of properties of the perturbations. This is done in the following way. The least eigenvalue for the domain is characterized by considering the eigenvalue problem described by

$$\Delta\phi + k\phi = 0 \quad \text{in} \quad D \tag{3.16}$$
$$\phi = 0 \quad \text{on} \quad \partial D \ ,$$

ϕ being in the class of perturbation stream functions restricted to those that vanish on the boundary. If we multiply by ϕ and integrate over the domain we have

$$\iint_D [\phi\Delta\phi + k^2\phi^2]\, dx\, dy = 0 \ .$$

Integrating by parts (Green's identity, Appendix A) with the given boundary conditions yields

$$k^2 \iint_D \phi^2\, dx\, dy = \iint_D (\nabla\phi)^2,\, dx\, dy \ .$$

We define the least eigenvalue k_0 of the eigenvalue problem by the variational formulation

$$k_0^2 := \inf \left[\frac{\iint_D (\nabla\phi)^2\, dx\, dy}{\iint_D \phi^2\, dx\, dy} \right] \tag{3.17}$$

where the infimum is taken over a suitable class of functions (we eventually will need this to contain functions whose "second derivatives", $\Delta\phi$, are square integrable).

A rough scaling argument shows that the least eigenvalue is indeed related to the domain size. Let L be a characteristic scale of the domain. Then $\nabla\phi \sim \bar{\phi}/L$, where $\bar{\phi}$ is an appropriate scale for ϕ (e.g. the spatial average of ϕ). Then $k_0^2 \sim (\bar{\phi}/L)^2/(\bar{\phi})^2 = 1/L^2$. This is mainly to note that a decrease in the domain size results in a corresponding increase in the least eigenvalue.

We define the scale of the perturbation, $1/\kappa$, as the ratio of its enstrophy to energy,

$$\kappa^2(\phi) := \frac{\iint_D (\Delta\phi)^2\, dx\, dy}{\iint_D (\nabla\phi)^2\, dx\, dy} \ . \tag{3.18}$$

By expanding ϕ in terms of the eigenfunctions defined by (3.16) the following inequality, analogous to the *Poincaré inequality* (cf. Guenther and Lee, 1988 pg.461) is established (Holm *et al.*, 1986; McIntyre and Shepherd, 1987)

lemma (3–2).

$$\kappa(\phi) \geq k_0 \ . \tag{3.19}$$

Proof: Let $\phi = \sum a_j \phi_j$ where $\{\phi_j\}$ is the (orthogonal) eigenbasis defined by (3.15). Then

$$\Delta\phi = \Delta\left(\sum a_j\phi_j\right) = \sum a_j \Delta\phi_j = -\sum a_j k_j^2 \phi_j \ ,$$

where the basis is assumed to be C^∞ with convergence of its derivatives. Now consider that

$$\begin{aligned}
\iint_D (\Delta\phi)^2 \, dx\, dy &= \langle \Delta\phi, \Delta\phi\rangle = \langle \sum a_j k_j^2 \phi_j, \sum a_i k_i^2 \phi_i\rangle \\
&= \sum_j \sum_i a_j a_i k_j^2 k_i^2 \langle \phi_j, \phi_i\rangle = \sum a_n^2 k_n^4 \\
&\geq \sum a_n^2 k_0^4 \qquad \text{since } k_0 \text{ is the least eigenvalue} \\
&= k_0^4 \langle \sum a_j\phi_j, \sum a_i \phi_i\rangle = k_0^4 \langle \phi, \phi\rangle = k_0^4 \iint_D \phi^2 \, dx\, dy \ .
\end{aligned}$$

That is,

$$\iint_D \phi^2 \, dx\, dy \leq \frac{1}{k_0^4} \iint_D (\Delta\phi)^2 \ . \tag{3.20}$$

Also, using the Schwarz inequality, $(\int fg)^2 \leq \int f^2 \int g^2$, we have

$$\begin{aligned}
\iint_D (\nabla\phi)^2 \, dx\, dy &= -\iint_D \phi\Delta\phi \, dx\, dy \quad \text{(Green's identity)} \\
&\leq \left\{\iint_D \phi^2 \, dx\, dy\right\}^{1/2} \left\{\iint_D (\Delta\phi)^2 \, dx\, dy\right\}^{1/2} \\
&\qquad\qquad \text{(by Schwarz inequality)} \\
&\leq \frac{1}{k_0^2} \left\{\iint_D (\Delta\phi)^2 \, dx\, dy\right\}^{1/2} \left\{\iint_D (\Delta\phi)^2 \, dx\, dy\right\}^{1/2} \\
&\qquad\qquad \text{(by (3.20))} \\
&= \frac{1}{k_0^2} \iint_D (\Delta\phi)^2 \, dx\, dy \ .
\end{aligned}$$

That is,

$$\iint_D (\Delta\phi)^2 \, dx \, dy \geq k_0^2 \iint_D (\nabla\phi)^2 \, dx \, dy \ . \tag{3.21}$$

From this we get the desired result

$$\kappa^2(\phi) := \frac{\iint_D (\Delta\phi)^2 \, dx \, dy}{\iint_D (\nabla\phi)^2 \, dx \, dy} \geq k_0^2 \qquad \blacksquare$$

It is now apparent that if $ck_0^2 > 1$, then

$$\iint_D c(\nabla\phi)^2 \, dx \, dy \geq ck_0^2 \iint_D (\nabla\phi)^2 \, dx \, dy \geq \iint_D (\nabla\phi)^2 \, dx \, dy \ .$$

This condition then suffices to make (3.15) positive, and zero only in the case that $\phi = 0$ identically. That is, if $ck_0^2 > 1$ then (3.15) is positive definite. However, this condition is also necessary, as shown in the following lemma.

lemma (3–3).

$$\frac{1}{2} \iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2 \right] \, dx \, dy > 0 \quad \Rightarrow \qquad ck_0^2 > 1 \ .$$

Proof: If we suppose that equality holds in (3.21) for *some* ϕ,

$$\iint_D (\Delta\phi)^2 \, dx \, dy = k_0^2 \iint_D (\nabla\phi)^2 \, dx \, dy \ ,$$

and use Green's identity (Appendix A) on the right hand side, we have

$$\begin{aligned}
\iint_D (\Delta\phi)^2 \, dx \, dy &= k_0^2 \iint_D (\nabla\phi)^2 \, dx \, dy \\
&= -k_0^2 \iint_D \phi(\Delta\phi) \, dx \, dy \\
&= -k_0^2 \iint_D (-\frac{1}{k^2})(\Delta\phi)^2 \, dx \, dy \\
&\qquad\qquad \text{(using (3.16))} \\
&= \frac{k_0^2}{k^2} \iint_D (\Delta\phi)^2 \, dx \, dy \ ,
\end{aligned}$$

which is true if and only if

$$k^2 = k_0^2 \ .$$

That is, (3.15) is zero if and only if ϕ is proportional to the eigenfunction ϕ_0 corresponding to k_0. So we have that k_0^2 is an infimum of the set of possible κ's. Now suppose (3.15) is positive definite so that

$$\frac{\iint_D (\Delta\phi)^2\,dx\,dy}{\iint_D (\nabla\phi)^2\,dx\,dy} > \frac{1}{c}\ .$$

Then $1/c$ is a lower bound for the set of κ's, in which case it must be smaller than the infimum. That is,

$$\frac{1}{c} < k_0^2 \qquad \text{or} \qquad ck_0^2 > 1 \qquad\qquad \blacksquare$$

With these lemmas the following theorem is established.

Theorem (3–4).

$$\|\phi\|^2 := \frac{1}{2}\iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right]\,dx\,dy$$

is positive definite if and only if

$$ck_0^2 > 1\ .$$

This result can be viewed in two ways. If c is given, then by restricting the domain size appropriately the least eigenvalue, k_0, will be large and satisfy the above inequality. Then the norm is legitimate and may seek conditions for stability. Alternatively, if the domain is specified (as is usual in most situations) then (3.15) can not be used as a norm unless there is some large enough lower bound on the ratio of velocity and vorticity (that is, c must be large enough to make (3.15) positive while maintaining $c \leq -U/U''$). This begins to show that flows for which we may be able to establish stability can be characterized by such bounds.

We now proceed to the a priori estimate corresponding to (1.5) for $-H_C$, assuming we have $ck_0^2 > 1$ and thus a norm given by (3.15). The following inequalities hold from previous computations and lemma (3–1) with a change of sign. Also recall the bound on Φ'',

$$0 < c \leq -\Phi''(\xi) \quad \text{for all} \quad \xi \in R .$$

$$\begin{aligned}
0 < \|\phi\|^2 &= \frac{1}{2}\iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right] dx\,dy \\
&\leq \iint_D \left[-(\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Psi)\Delta\phi) - \frac{1}{2}(\nabla\phi)^2\right] dx\,dy \\
&= -\widetilde{H}_C(\phi(x,y;t)) = -\widetilde{H}_C(\phi(x,y;0)) \\
&= -\widehat{H}_C(\phi_0) = -[H_C(\Psi + \phi_0) - H_C(\Psi)] .
\end{aligned}$$

(See proof of lemma (3–1) for notation.) This is the a priori estimate corresponding to (1.5) in general and (3.12) of the previous case ($U/U'' > 0$). To show stability we need to prove that $-H_C$ is continuous in this norm at Ψ. To do this we assume there is a constant C so that

$$0 < c \leq -\Phi''(\xi) = -\frac{U}{U''} \leq C < \infty ,$$

for all real ξ. From this bound we have that

$$\begin{aligned}
|-H_C(\Psi + \phi) + H_C(\Psi)| &= -\widehat{H}_C(\phi) = -\widetilde{H}_C(\phi) &\quad (3.22a)\\
&= -\iint_D \left\{\frac{1}{2}(\nabla\phi)^2 \right.\\
&\qquad \left. + [\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Psi)\Delta\phi]\right\} dx\,dy \\
&= \iint_D \left\{-[\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Phi'(\Delta\Psi)\Delta\phi] \right.\\
&\qquad \left. - \frac{1}{2}(\nabla\phi)^2\right\} dx\,dy \\
&\leq \iint_D \{\frac{C}{2}(\Delta\phi)^2 - \frac{1}{2}(\nabla\phi)^2\}\, dx\,dy . &\quad (3.22b)
\end{aligned}$$

Employing the facts contained in theorem (3–4) and lemma (3–2), which are

(i) $\qquad ck_0^2 > 1$

and

$$(ii) \qquad \iint_D (\Delta\phi)^2\, dx\, dy \geq k_0^2 \iint_D (\nabla\phi)^2\, dx\, dy\ ,$$

the last term, (3.22*b*), can be bounded as follows.

$$\begin{aligned}
&\iint_D \{\frac{C}{2}(\Delta\phi)^2 - \frac{1}{2}(\nabla\phi)^2\}\, dx\, dy \\
&= \frac{ck_0^2 - 1}{2(ck_0^2 - 1)} \iint_D \left\{C(\Delta\phi)^2 - (\nabla\phi)^2\right\} dx\, dy && (3.23a) \\
&= \frac{1}{2(ck_0^2 - 1)} \iint_D \left\{cCk_0^2(\Delta\phi)^2 - ck_0^2(\nabla\phi)^2 - C(\Delta\phi)^2 + (\nabla\phi)^2\right\} dx\, dy \\
&= \frac{1}{2(ck_0^2 - 1)} \iint_D \left\{Ck_0^2[c(\Delta\phi)^2 - \frac{1}{k_0^2}(\Delta\phi)^2] + [1 - ck_0^2](\nabla\phi)^2\right\} dx\, dy \\
&\leq \frac{1}{2(ck_0^2 - 1)} \iint_D \left\{Ck_0^2[c(\Delta\phi)^2 - \frac{1}{k_0^2}(\Delta\phi)^2]\right\} dx\, dy \quad (\text{by } (i)) \\
&\leq \frac{1}{2(ck_0^2 - 1)} \iint_D \left\{Ck_0^2[c(\Delta\phi)^2 - (\nabla\phi)^2]\right\} dx\, dy \quad (\text{by } (ii)) \\
&= \frac{Ck_0^2}{2(ck_0^2 - 1)} \iint_D \left\{c(\Delta\phi)^2 - (\nabla\phi)^2\right\} dx\, dy\ . && (3.23b)
\end{aligned}$$

Writing (3.22*a*)-(3.23*b*) together to consolidate the result,

$$\begin{aligned}
|-H_C(\Psi + \phi) + H_C(\Psi)| &= -H_C(\Psi + \phi) + H_C(\Psi) \\
&\leq \iint_D \{\frac{C}{2}(\Delta\phi)^2 - \frac{1}{2}(\nabla\phi)^2\}\, dx\, dy \\
&= \frac{Ck_0^2}{2(ck_0^2 - 1)} \iint_D \left\{c(\Delta\phi)^2 - (\nabla\phi)^2\right\} dx\, dy \\
&= \frac{Ck_0^2}{2(ck_0^2 - 1)} \|\phi\|^2\ .
\end{aligned}$$

That is, we have

$$|-H_C(\Psi + \phi) + H_C(\Psi)| \leq \frac{Ck_0^2}{2(ck_0^2 - 1)} \|\phi\|^2\ ,$$

expressing the continuity of $-H_C$ at the equilibrium Ψ.

Summarizing (the case $U/U'' \leq 0$), it was shown that if there are constants c and C so that

$$0 < c \leq -\Phi''(\xi) = -\frac{U}{U''} \leq C < \infty$$

and the domain is small enough so that the least eigenvalue of (3.16) satisfies $ck_0^2 > 1$, then the equilibrium Ψ is Lyapunov stable in the norm given by

$$\|\phi\| := \frac{1}{2} \iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right] \, dx \, dy \; .$$

This represents a substantial departure from the linear theory in that it shows stability of certain equilibrium flows where the linear theory is indeterminate. Recall that Rayleigh's theorem only shows stability in the absence of an inflection point ($U'' \neq 0$), and the Rayleigh–Fjørtoft theorem only covers the additional case where U/U'' is positive—*neither* of these are met in this case, where $U/U'' \leq 0$. Furthermore, the structure of the perturbations are completely arbitrary (up to satisfying boundary conditions).

Example 2.)

The example which follows might be considered as a "standard example" since it has been shown in several papers (Arnold, 1965; Holm *et al.*, (1985); Drazin and Reid, 1981). We present it again here since it illustrates so well the foregoing ideas, and additionally some observations not dealt with in the literature.

We still consider a channel flow confined between walls at y_0 and y_1 with $y_0 < 0 < y_1$ (the width of the channel being $d = y_1 - y_0$) and periodic in the x-coordinate of period L. The domain D is again confined to the rectangle between $x = 0$ and $x = L$.

Let $U(y) = \sin y$ be the equilibrium velocity profile. Note that $U/U'' = -1$ for all y in $[y_0, y_1]$. This clearly falls into the category considered in section B above and as noted the Rayleigh and the Rayleigh-Fjørtoft theorems do not apply. Then to choose c and C so that

$$0 < c \leq -U/U'' \leq C < \infty \; ,$$

that is, $0 < c \leq 1 \leq C < \infty$, the best estimates will be given by $c = C = 1$. With appropriate restrictions on the size of the domain (channel width) we can establish stability with respect to arbitrary perturbations within an appropriate class.

It is shown in Appendix A, part (iv), that the least eigenvalue determined by (3.16) is π/d. Consequently, if $c(\pi/d)^2 = (\pi/d)^2 > 1$ then $ck_0^2 = k_0^2 > 1$ and we have Lyapunov stability of U subject to perturbations which preserve the total flow rate, have period L in the x-coordinate, and preserve circulations on Γ_{y_i}. Viewing this inequality in a slightly different way, if $d < \pi$ then we have stability.

Drazin and Reid (1981), using the linear theory, demonstrate that the flow is unstable for $d > \pi$ (see their references) which complements the above Lyapunov stability result. This raises a few important questions. It first appears that the problem is completely specified and the problem of stability of U is resolved. That is, $d < \pi$ is both a necessary and sufficient condition for stability of the equilibrium U. Consider the configuration shown in figure 3.1 (next page). The left profile is (Lyapunov) stable since $d < \pi$. Notice in particular that the inflection point is included and also the critical point of U (where $U'(y) = 0$). Compare this to the configuration shown on the right. No characteristics of the flow have changed, that is, the inflection point and the critical point are still present, but now $d > \pi$. This raises the question as to *why* one configuration is stable and the other is not.

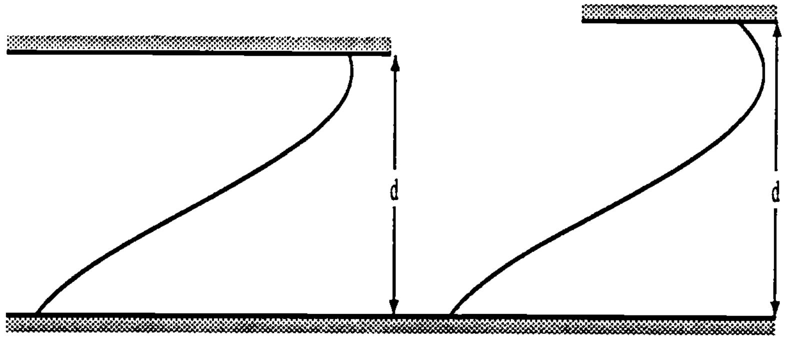

Figure 3.1 Lyapunov stable configuration, $d < \pi$ (*left*).
Linearly unstable configuration, $d > \pi$ (*right*).

Based on these two figures, one might hope to show the second profile (in fig. 3.1) is also stable. Yet, the linear theory has established that this profile is unstable. It appears here (and in other results such as Chandrasekhar (1961)) that the scale of the domain is a determining factor of stability.

C. Stability in other norms

Since in general there may be more than one way to find quadratic forms Q_1 and Q_2 in step 4 of the stability algorithm, we may end up with various possible norms on the space of perturbations. Furthermore, it is not entirely evident that we have a uniquely defined problem from the point of view of choices of distinguished functions (step 2) So the question arises as to how much difference this choice can make, and where the differences are important.

The norm plays the role of defining a topology on the space of perturbations—that is, the nature of open sets. We need not go into the details of definitions in topology. Instead, it is easy to use a simple finite dimensional analogy to see how using different norms changes the "shape" of open sets. If we construct a neighborhood of a "point", x_0 (in our case a function in the space of perturbations) by defining it to be all points x which are within ρ of x_0, i.e. $\{x : \|x - x_0\| < \rho\}$, and do this for different norms, the *shape* of the neighborhood changes. A simple illustration is to consider the (x, y)-plane with two norms given by

$$\|(x,y)\|_a = \sqrt{x^2 + y^2}$$

$$\|(x,y)\|_b = \max(|x|, |y|) \ .$$

The neighborhoods are given by circles and squares, respectively. The same is true in function spaces, but we can no longer draw the results and compare.

Recall that to show stability in step 6 we must show continuity of the conserved functional H_C. Continuity is essentially a property described in terms of open sets (a function is continuous if the inverse image of open sets are open) and so we might expect that showing continuity in one norm may be different than in another norm. There is a situation where it does not make a difference—if we can show, roughly speaking, for each point x_0 that for any neighborhood of x_0 in one norm we can find a neighborhood in the other which is entirely inside the first, then do the same vice versa, the topologies are *equivalent*. This means that showing continuity of H_C in the one norm is *entirely equivalent* to showing continuity in the other norm.

In what follows the equivalence of several specific topologies on the space of perturbations is shown. The method of proof will be the following. Given two norms, $\|\cdot\|_a$ and $\|\cdot\|_b$, show there are constants K_1 and K_2 so that

$$\|\cdot\|_a \leq K_1 \|\cdot\|_b \quad \text{and} \quad \|\cdot\|_a \geq K_2 \|\cdot\|_b \ .$$

It can then be concluded that, since continuity is equivalent using these norms, that stability is independent of the choice between those specific norms.

Consider the case where U/U'' is positive. McIntyre and Shepherd (1987) give the following as two possible norms which can be used to show stability in this case.

$$\|\phi\|_1^2 := \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2\right]\, dx\, dy \ ,$$

and

$$\|\phi\|_2^2 := \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + C(\Delta\phi)^2\right]\, dx\, dy \ .$$

Since $c \leq C$, it is immediate that $\|\cdot\|_1 \leq \|\cdot\|_2$. Furthermore, since $C/c \geq 1$,

$$\begin{aligned}
\|\cdot\|_2^2 &= \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + C(\Delta\phi)^2\right]\, dx\, dy \\
&= \frac{1}{2}\iint_D \left[(\nabla\phi)^2 + \frac{C}{c}c(\Delta\phi)^2\right]\, dx\, dy \\
&\leq \frac{1}{2}\iint_D \left[\frac{C}{c}(\nabla\phi)^2 + \frac{C}{c}c(\Delta\phi)^2\right]\, dx\, dy \\
&= \frac{C}{2c}\iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2\right]\, dx\, dy = \frac{C}{c}\|\phi\|_1^2 \ .
\end{aligned}$$

Therefore, these two norms generate equivalent topologies and showing stability in one is equivalent to the other.

In the case U/U'' negative, Arnold (1969) suggests that if we assume that (3.14) is positive definite, then we can define a norm by

$$\|\phi\|^2 = \frac{c}{2}\iint_D (\Delta\phi)^2\, dx\, dy \ ,$$

(see also McIntyre and Shepherd (1987)). The a priori estimate (step 5) follows easily from the above calculations and the continuity of $-H_C$ in this norm follows from

$$\begin{aligned}
-H_C(\Psi+\phi) + H_C(\Psi) &= \\
&- \left[E(\Psi+\phi) - E(\Psi) - DE(\Psi)\phi\right] \\
&- \left[C(\Psi+\phi) - C(\Psi) - DC(\Psi)\phi\right] \\
&\leq -\frac{1}{2}\iint_D (\nabla\phi)^2\, dx\, dy + \frac{C}{c}\iint_D (\Delta\phi)^2\, dx\, dy \\
&\leq \frac{2C}{c}\|\phi\|^2 \ .
\end{aligned}$$

What remains unclear is whether or not this norm generates a topology equivalent to that of (3.15) and another norm given by McIntyre and Shepherd (1987) (listed below). To investigate this let

$$\|\phi\|_a^2 := \frac{1}{2}\iint_D (\Delta\phi)^2\,dx\,dy\ ,$$

$$\|\phi\|_b^2 := \frac{1}{2}\iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right]\,dx\,dy$$

and

$$\|\phi\|_c^2 := \frac{1}{2}\iint_D \left[C(\Delta\phi)^2 - (\nabla\phi)^2\right]\,dx\,dy.$$

It is immediate that $\|\phi\|_b^2 \leq c\|\phi\|_a^2$, so continuity in the norm $\|\cdot\|_a$ implies continuity in the norm $\|\cdot\|_b$. We need to show the other direction.

$\|\cdot\|_b$ is a norm only in the case that its defining integral is positive definite which depends on the least eigenvalue k_0 (Theorem 3–4). Then we anticipate that the equivalence of these norms, a and b, is also dependent on the least eigenvalue. Recall that in the setting we consider, a lower bound on the least eigenvalue is π/d and we have the following string of inequalities

$$k_0 \geq \frac{\pi}{d} > 1\ .$$

Then

$$\begin{aligned}
\|\phi\|_b^2 &:= \frac{1}{2}\iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right]\,dx\,dy \\
&\geq \frac{1}{2}\iint_D \left[c(\Delta\phi)^2 - \frac{1}{k_0^2}(\Delta\phi)^2\right]\,dx\,dy \quad \text{by (3.21)} \\
&= \frac{ck_0^2 - 1}{2k_0^2}\iint_D (\Delta\phi)^2\,dx\,dy \\
&= \frac{ck_0^2 - 1}{k_0^2}\|\phi\|_b^2\ .
\end{aligned}$$

Thus, if we show continuity of H_C in norm b, we can show it in norm a. So the two norms are equivalent, at least in this context, and showing stability in one is equivalent to showing stability in the other.

The fact that $\|\cdot\|_c$ is equivalent to $\|\cdot\|_a$ is precisely the same calculation using C in place of c. We can conclude that stability determined by any one of these is equivalent to stability determined by any of the others.

Arnold (1965, 1969) originally showed stability by bounding the perturbation by its initial condition. This is the most generic notion of stability of an equilibrium. McIntyre and Shepherd (1987) restate Arnold's results in this same way. Holm *et al.* (1985) showed that what this really amounts to is showing that the conserved functional H_C (or $-H_C$) is continuous at the equilibrium, with the definition of continuity being in terms of a norm determined by the problem. McIntyre and Shepherd (1987) point out that the perturbation can be shown to be bounded in more than one norm. It turns out that the norms they give generate equivalent topologies, and therefore showing continuity (or boundedness) in one is just as good as another. Their point, however, is that we can also try to show that *any* finite perturbation— no matter how large— is bounded for all time. Consider the definition of Lyapunov stability given in the introduction.

$$\forall \epsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad \|u_e - u(0)\| < \delta \quad \Rightarrow \quad \|u_e - u(t)\| < \epsilon \,, \forall t \,.$$

We may instead define another type of stability as follows.

$$\forall \delta \quad \text{and} \quad \forall u(0) \quad \text{such that} \quad \|u_e - u(0)\| < \delta$$

$$\exists \epsilon > 0 \quad \text{such that} \quad \|u_e - u(t)\| < \epsilon \,, \forall t \,.$$

That is, any initial perturbation is bounded for all time.

A particle in static equilibrium in an infinite potential well, or any situations analogous to this, will be stable in this sense. An important question is whether or not there are *any* configurations where a steady fluid flow represents an equilibrium "in an *infinitely deep* potential well" (quote from McIntyre and Shepherd, 1987). We do not address this question here but merely point out that this seems physically implausible—large enough perturbations should change the equilibrium in the inviscid setting, and likely most other settings concerning fluids.

4. HAMILTONIAN EVOLUTION EQUATIONS

The point of view taken here follows closely that given by Olver (1986). There are several ways to approach Hamiltonian mechanics and only one is presented in what follows. The first topic to be addressed is in what sense the vorticity equation, (3.2) or (3.3), can be thought of as a Hamiltonian system. Following this, the ways in which this structure can be used to find conserved quantities and study stability are presented.

The key to applying Hamiltonian techniques to evolution equations, and doing so with the least amount of knowledge of differential geometry, is to use a coordinate-free approach. This is done by highlighting the Poisson bracket (defined below) which contains the structural information necessary to determine the useful results of a Hamiltonian formulation. While the counterparts to all of the definitions and theorems in the infinite dimensional setting exist in the finite dimensional setting, there is no need here to consider the details of how these two settings fit together. Mainly this chapter is an exposition of terminology and statements of results using that terminology. (See Olver (1986) for more details on both the finite and infinite dimensional cases in a general setting.)

A. Differential functions

In order to consider in what way a differential operator is Hamiltonian, it must be established what is meant by a differential operator, and on what space these operators act. The terminology will be given primarily in terms of the example of two-dimensional flow governed by the vorticity equation

(3.2) (or (3.3)) that has been considered.

Let D be the domain defined in Chapter 3, that is, $D = \{(x, y) : x_0 \leq x \leq x_1, y_0 \leq y \leq y_1\}$. Let $M = (D \times \mathbb{R}) \times U$ be an open connected subset of the space of independent and dependent variables. A typical element of this space is of the form $(x, y, t; u)$. u represents a place-holder, or coordinate, to be "filled" by any suitable dependent variable such as a stream function fitting the physical setting (*not* a velocity component). A typical partial differential equation (such as the vorticity equation) will involve an element $\phi \in U$ (a stream function) and derivatives of ϕ with respect to x, y, and t. Let U_1 denote the space whose elements consist of (u_x, u_y, u_t) where $u \in U$. That is, U_1 is the space which has the same number of coordinates as there are possible first derivatives of u. Similarly, U_2 will denote the space consisting of elements of the type $(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt})$. That is, U_2 is the space which has the same number of coordinates as there are second derivatives of u.

From these form the space $U^{(2)} = U \times U_1 \times U_2$ whose coordinates represent all derivatives of order up to two (including no derivative) of functions $u = \phi(x, y; t)$. An element of $U^{(2)}$ will be denoted $u^{(2)}$.

Finally, let $M^2 = (D \times \mathbb{R}) \times U^{(2)}$ be the space whose coordinates represent independent and dependent variables and derivatives of the dependent variables up to order two. (This is called the *second order jet space* of the space M. It can be extended easily to include higher order derivatives.)

Let $\mathcal{A}$ denote the space of smooth real-valued functions $P : M^2 \mapsto \mathbb{R}$. The functions in $\mathcal{A}$ are called *differential functions* and are denoted $P(x, u^{(2)})$ or $P[u]$, where the square bracket is used to remind us that the functional may depend on the independent and dependent variables as well as the derivatives of the dependent variable. For example, x, u, and xu are differential functions, as well as uu_y, u_t, and $u^2u_x + u_{yy}$. This space of functions $\mathcal{A}$ is the one on which the differential operator will act.

Finally, let $\mathcal{F}$ denote the space of functionals whose elements are given by

$$\mathcal{P} = \int_D P[u]\, dx\, dy$$

whenever $P \in \mathcal{A}$. [6]

B. Hamiltonian operators and Poisson brackets

Let $\mathcal{D} : \mathcal{A} \mapsto \mathcal{A}$ be a linear differential operator on the space of differential functions. For example, if we consider (3.3) written as

$$\omega_t = \omega_x \psi_y - \omega_y \psi_x \ ,$$

the right hand side can be thought of as $\mathcal{D}(\psi)$ where $\mathcal{D} = \omega_x D_y - \omega_y D_x$ and $\psi \in \mathcal{A}$. Notice that $\mathcal{D}$ includes *physically* nonlinear effects though it is applied in a linear fashion to functions in $\mathcal{A}$. That is, given ϕ_1 and ϕ_2 in $\mathcal{A}$,

$$\mathcal{D}(\phi_1 + \phi_2) = \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2) \ .$$

From Appendix B, if $\mathcal{P} = \int P$ for $P \in \mathcal{A}$, the functional derivative of $\mathcal{P}$ is denoted $\delta\mathcal{P}$ or $\delta\mathcal{P}/\delta u$. Corresponding to the operator $\mathcal{D}$ is a *Poisson bracket* defined on $\mathcal{F}$ given by

$$\{\mathcal{P}, \mathcal{Q}\} = \int_D \delta\mathcal{P} \cdot \mathcal{D}\delta\mathcal{Q}\, dx\, dy \tag{4.1}$$

for any two functionals $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{F}$. Note that $\delta\mathcal{P}$ and $\mathcal{D}\delta\mathcal{Q}$ are functions (since in this case there is only one dependent variable) so that $\delta\mathcal{P} \cdot \mathcal{D}\delta\mathcal{Q}$ is

[6] Notice that two integrands P and Q may lead to the same functional $\mathcal{P}$ if they differ by a total divergence. A careful definition of $\mathcal{F}$ would involve equivalence classes based on this relation.

a differential function and $\{\mathcal{P},\mathcal{Q}\}$ defined by (4.1) is a functional in $\mathcal{F}$. We call $\mathcal{D}$ a *Hamiltonian operator* if its Poisson bracket (4.1) is *skew-symmetric*

$$\{\mathcal{P},\mathcal{Q}\} = -\{\mathcal{Q},\mathcal{P}\} \tag{4.2}$$

and satisfies the *Jacobi identity*

$$\{\{\mathcal{P},\mathcal{Q}\},\mathcal{R}\} + \{\{\mathcal{R},\mathcal{P}\},\mathcal{Q}\} + \{\{\mathcal{Q},\mathcal{R}\},\mathcal{P}\} = 0 \; . \tag{4.3}$$

For a given Hamiltonian operator $\mathcal{D}$, it is shown in Olver (1986) that to each functional $\mathcal{H} = \int H$ in $\mathcal{F}$, there corresponds an evolution equation of the form

$$\frac{\partial u}{\partial t} = \mathcal{D}\left(\frac{\delta \mathcal{H}}{\delta u}\right) \; . \tag{4.4}$$

$\mathcal{H}$ is called the *Hamiltonian* corresponding to equation (4.4). Furthermore, given any other functional $\mathcal{P}$, its rate of change following the motion along solutions u is given by

$$\frac{d\mathcal{P}}{dt} = \{\mathcal{H},\mathcal{P}\} \tag{4.5}$$

The converse is the setting which is of more interest. That is, consider a given evolution equation such as (3.3) written in the form

$$\omega_t = \omega_x \psi_y - \omega_y \psi_x \; . \tag{4.6}$$

We wish to show this is in Hamiltonian form so we must write it in the form (4.4) and show that the candidate operator

$$\mathcal{D} = \omega_x D_y - \omega_y D_x$$

is Hamiltonian (by showing properties (4.2) and (4.3)) and find an appropriate Hamiltonian functional $\mathcal{H}$ so that (4.4) holds. To show that this operator is Hamiltonian by directly confirming (4.2) and (4.3) can be difficult. There are further results regarding the relation between these properties and properties of the operator $\mathcal{D}$ which help. These are formulated in subsequent sections.

C. Skew-adjoint operators and their relation to Poisson brackets

Given an operator $\mathcal{D} : \mathcal{A} \mapsto \mathcal{A}$, $\mathcal{D}$ can be thought of as having the form

$$\mathcal{D} = \sum_J P_J[u] D_J \quad P_J \in \mathcal{A}, \tag{4.7}$$

where the sum is taken over all *unordered* 3-tuples $J = (j_1, j_2, j_3)$, with $0 \leq j_i \leq k$. The order of the derivative is $\#J = j_1 + j_2 + j_3$ and D_J represents a derivative up to $\#J^{\text{th}}$ order. That is,

$$k = 0: \quad J = (0,0,0)\ D_J = D_{(0,0,0)} \quad \text{(no derivative)}$$

$$k = 1: \quad J = (1,0,0),(0,1,0),(0,0,1),\ D_J = D_x, D_y, \text{or } D_t$$

$$k = 2: \quad J = (2,0,0),(1,2,0),\ldots,\ D_J = D_x D_x, D_x D_y^2, \text{ etc.}$$

The following example will clarify the notation.

Let $\mathcal{D} = D_y^2 = D_y D_y$ be a differential operator on the space $\mathcal{A}$. $\mathcal{D}$ is of the form (4.7) where $J = (0,2,0)$, $P_{(0,2,0)} = 1$ and $D_(0,2,0) = D_y D_y$. Similarly, $\mathcal{D} = D_t + u D_x + D_y^2$ is of the form (4.7) where

$$P_{(1,0,0)}[u] = u, \quad P_{(0,2,0)}[u] = 1, \quad P_{(0,0,1)}[u] = 1$$

$$D_{(1,0,0)} = D_x \qquad D_{(0,2,0)} = D_y^2 \qquad D_{(0,0,1)} = D_t$$

The (formal) *adjoint* of $\mathcal{D}$, denoted $\mathcal{D}^*$ is defined by the following relation,

$$\int_\Omega P\mathcal{D}(Q)\,dx\,dy = \int_\Omega Q\mathcal{D}^*(P)\,dx\,dy \tag{4.8}$$

for every pair $P, Q \in \mathcal{A}$ which are zero when $u = 0$, every domain $\Omega \subset D \times \mathbb{R}$ and every function $u = f(x,y)$ of compact support in Ω.

Integrating by parts (in a generalized sense—see Olver (1986)) shows that

$$\mathcal{D}^* = -\sum_J D_J(P_J \cdot) \ .$$

That is, for any function $Q \in \mathcal{A}$

$$\mathcal{D}^*(Q) = -\sum_J D_J(P_J Q) \ .$$

Continuing with the above example, $\mathcal{D} = D_t + uD_x + D_y^2$, the adjoint $\mathcal{D}$ is

$$\mathcal{D}^* = -[D_x u + D_t 1] = -D_t - uD_x - u_x \ .$$

$\mathcal{D}$ is called *skew-adjoint* if $\mathcal{D}^* = -\mathcal{D}$. Note that the operator in the preceding example is *not* skew-adjoint.

Olver (1986) shows that an operator being skew-adjoint and its corresponding Poisson bracket being skew-symmetric (property (4.2)) are equivalent. This is extremely useful in the cases where showing an operator is skew-adjoint is relatively easy.

Consider the vorticity equation in operator form. That is,

$$\omega_t = \mathcal{D}(\psi) \quad \text{where} \quad \mathcal{D} = \omega_x D_y - \omega_y D_x \ . \tag{4.9}$$

This can be thought of as being in form (4.7) where

$$P_{(1,0,0)}[u] = -\omega_y, \quad P_{(0,1,0)}[u] = \omega_x, \quad P_{(0,0,1)}[u] = 0$$

$$D_{(1,0,0)} = D_x \quad D_{(0,1,0)} = D_y \qquad D_{(0,0,1)} = D_t \ .$$

Then the adjoint of $\mathcal{D}$ may be calculated,

$$\begin{aligned}
\mathcal{D}^* &= -[D_{(1,0,0)}P_{(1,0,0)} + D_{(0,1,0)}P_{(0,1,0)}] \\
&= -[D_x(-\omega_y \cdot) + D_y(\omega_x \cdot)] \\
&= \omega_y D_x + \omega_{yx} - \omega_x D_y - \omega_{xy} \\
&= -\omega_x D_y + \omega_y D_x = -\mathcal{D} \ .
\end{aligned}$$

It is therefore shown that $\mathcal{D} = \omega_x D_y - \omega_y D_x$ of (4.9) is a skew-adjoint operator and so its corresponding Poisson bracket will be skew-symmetric.

In order that $\mathcal{D}$ be a Hamiltonian operator it remains to show that (4.3) holds for the Poisson bracket. With further theoretical development (peripheral to the interests of this study) Olver shows that (4.3) does in fact hold. Then to complete the process of showing that (4.9) is in Hamiltonian form we need a functional $\mathcal{H}$ whose functional derivative with respect to ω will be a stream function ψ. Simply note that the kinetic energy of the system given by (3.4),

$$\mathcal{H} = \frac{1}{2} \int_D \nabla\psi \cdot \nabla\psi \, dx \, dy$$

can be written equivalently (using Green's identity) as

$$\mathcal{H} = -\frac{1}{2} \int_D \psi \Delta \psi \, dx \, dy = -\frac{1}{2} \int_D \psi \omega \, dx \, dy \; .$$

So we can say the differential functions $H_1 = \frac{1}{2} \nabla\psi \cdot \nabla\psi$ and $H_2 = -\frac{1}{2}\psi\Delta\psi$ are equivalent, since they give rise to the same functional $\mathcal{H}$. The calculation of $\delta\mathcal{H}/\delta\omega$ follows as

$$\begin{aligned} &\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{H}(\omega + \epsilon\zeta) \quad \text{where} \quad \zeta = -\Delta\phi \\ &= -\frac{1}{2} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_D (\psi + \epsilon\phi)\Delta(\psi + \epsilon\phi) \, dx \, dy \\ &= -\frac{1}{2} \int_D (\phi\Delta\psi + \psi\Delta\phi) \, dx \, dy \\ &= -\int_D \psi\Delta\phi \, dx \, dy \\ &= \int_D \psi\zeta \, dx \, dy \; . \end{aligned}$$

That is,

$$\frac{\delta\mathcal{H}}{\delta\omega} = \psi \; .$$

Finally, we can say that the operator $\mathcal{D}$, shown to be Hamiltonian, and the functional $\mathcal{H}$ give rise to an evolution equation of the form (4.4) which turns out to be the vorticity equation (4.9).

D. Conserved functionals and Casimirs

Olver (1986) defines a *distinguished functional* (or Casimir) for a Hamiltonian operator $\mathcal{D}$ as a functional $\mathcal{C} \in \mathcal{F}$ satisfying $\mathcal{D}(\delta\mathcal{C}) = 0$ for all x, y, t, and u. It then follows that $\mathcal{C}$ is a distinguished functional if and only if $\{\mathcal{C}, \mathcal{G}\} = 0$ for all functionals $\mathcal{G}$, which is how it was defined in the introduction. For the case of the vorticity equation (4.9) the distinguished functionals are those for which

$$\mathcal{D}(\delta\mathcal{C}) = \omega_x D_y(\delta\mathcal{C}) - \omega_y D_x(\delta\mathcal{C}) = 0 \ .$$

Let $P[\phi]$ be a differential function for which $\mathcal{D}(P) = 0$, that is,

$$\omega_x D_y P - \omega_y D_x P = 0 \quad \text{or}$$

$$\omega_x D_y P = \omega_y D_x P \ .$$

This is the statement that the determinant of the Jacobian of the functions P and ω is zero. Consequently, P and ω are functionally related, that is $P = P(\omega)$. Furthermore, this implies that P can not depend on x, y, t, or derivatives of ω except through the direct dependence on ω (Olver, 1986). The complete class of distinguished functionals is therefore given by

$$\mathcal{C}(\omega) = \int_D C(\omega)\, dx\, dy \ ,$$

where C is any smooth function of ω.

The importance of finding distinguished functionals is that they immediately give rise to conservation laws. Recall that the change in $\mathcal{C}$ following a solution is given by $\{\mathcal{H}, \mathcal{C}\}$ which is zero, meaning that $\mathcal{C}$ is a constant of the motion. Furthermore, if we consider the functional $\mathcal{H} + \mathcal{C} = \mathcal{H}_\mathcal{C}$, then for any functional $\mathcal{P}$

$$\begin{aligned}\{\mathcal{H}_\mathcal{C}, \mathcal{P}\} &= \{\mathcal{H} + \mathcal{C}, \mathcal{P}\} \\ &= \{\mathcal{H}, \mathcal{P}\} + \{\mathcal{C}, \mathcal{P}\} \\ &= \{\mathcal{H}, \mathcal{P}\} \ .\end{aligned}$$

Consequently, the system with Hamiltonians given by $\mathcal{H}$ and $\mathcal{H}_\mathcal{C}$ are equivalent. This is another way of viewing the fact that the distinguished functional $\mathcal{C}$ given by (3.5) can be added to the energy (3.4) to give a functional which describes the same dynamics. The addition of the distinguished functional is a way of expanding the number of possible equilibrium solutions corresponding to extremals of a functional.

To summarize, the significance of uncovering a Hamiltonian structure to the vorticity equation is that it facilitates the identification of an appropriate functional for which the equilibrium solutions correspond to extremals of this functional. Furthermore, it sets the stage for the same type of analysis in the three-dimensional case and the stratified cases. As might be anticipated, the Poisson bracket theory for the stratified case turns out to be more complicated and a deeper understanding of what is called the Lie-Poisson structure is needed. For discussions related to this see, for example, Abarbanel *et al.* (1986), Holm *et al.* (1985), Holm *et al.* (1986), and references contained in these.

5. SUMMARY

This chapter contains two sections. The first section contains a sequence of tables intended to summarize the definitions and results of stability analysis as it applies to shear-flow as it has been discussed in the foregoing chapters. The last section is a short discussion of the applicability of these results to geophysical flows.

A. Summary of definitions and results

Table 5.1 summarizes the four basic notions of what is meant by an equilibrium (i.e. steady flow) being stable. The notation corresponds to that used in Chapters 2 and 3 to describe two-dimensional shear flow. "Linearization" means that the equations have been linearized about a mean state, taken to be the stationary velocity profile.

Table 5.2 lists the conditions under which the various types of stability have been proven. The conditions are sufficient in each case, but as indicated in Chapter 3, for certain examples both necessary and sufficient conditions have been found.

Table 5.3 shows the hierarchy of the various types of stability. Note that the equivalence between linear and spectral stability only holds in the case of conservative systems.

Table 5.4 lists the norms considered in this work for which Lyaunov stability has been proven. They are listed according to the case to which they have been applied (either U/U'' positive or U/U'' negative).

Table 5.1

Type of stability	**Definition/Characterization**
Lyapunov Stability	$\forall \epsilon\, \exists \delta$ s.t. $\|\phi(0) - \Psi\| < \delta \Rightarrow \|\phi(t) - \Psi\| < \epsilon\ \forall t$—i.e. a small perturbation of the mean flow will stay "near" the mean flow for all time
Formal Stability	Second variation, δH_C, positive or negative definite—used to prove Linear Stability
Linear (Lyapunov) Stability	Equilibrium is Lyapunov stable as an equilibrium of the *linearized* equations
Spectral Stability	Normal modes of the linearized equations are purely oscillatory—(i.e. $c_i = 0$)

Table 5.2

Type of stability	**Sufficient Conditions**
Lyapunov Stability	(a) $U'' \neq 0$ (b) $\exists\, c,\, C$ s.t. $0 < c \leq U/U'' \leq C < \infty$ (c) $\exists\, c,\, C$ s.t. $0 < c \leq -U/U'' \leq C < \infty$
Formal Stability	(a) $U'' \neq 0$ (b) $\exists\, c$ s.t. $0 < c \leq U/U''$ (c) $\exists\, c$ s.t. $0 < c \leq -U/U''$ and $c < 1/k_0^2$
Linear (Lyapunov) Stability	Shown by proving Formal stability—i.e. holds for the same cases
Spectral Stability	(a) $U'' \neq 0$ (Rayleigh's Theorem) (b) $UU'' \geq 0$ (Rayleigh-Fjørtoft Theorem)

Table 5.3

Lyapunov Stability

$\Downarrow$

Formal Stability

$\Downarrow$

Linear Stability

$\Updownarrow$

Spectral Stability

Table 5.4

Norms on the space of perturbations

Case one: $0 < c \leq \frac{U}{U''} \leq C < \infty$

$$\|\phi\|_1^2 := \frac{1}{2} \iint_D \left[(\nabla\phi)^2 + c(\Delta\phi)^2\right] \, dx \, dy$$
$$\|\phi\|_2^2 := \frac{1}{2} \iint_D \left[(\nabla\phi)^2 + C(\Delta\phi)^2\right] \, dx \, dy$$

Case two: $0 < c \leq -\frac{U}{U''} \leq C < \infty$

$$\|\phi\|_a^2 := \frac{1}{2} \iint_D (\Delta\phi)^2 \, dx \, dy$$
$$\|\phi\|_b^2 := \frac{1}{2} \iint_D \left[c(\Delta\phi)^2 - (\nabla\phi)^2\right] \, dx \, dy$$
$$\|\phi\|_c^2 := \frac{1}{2} \iint_D \left[C(\Delta\phi)^2 - (\nabla\phi)^2\right] \, dx \, dy$$

B. Discussion of assumptions

The assumptions used to formulate the model of channel flow analyzed in Chapters 2 and 3 are that the flow must be

1.) strictly two-dimensional ($\vec{v} = (u(x,y), v(x,y)$, etc.),

2.) incompressible ($d\rho/dt = 0$),

3.) homogeneous ($\rho =$ constant),

4.) with free-slip boundary condition on the channel walls, and

5.) the total flow at the two ends of the channel must be the same.

Additionally, in order that the first variation of H_C vanish at the steady-state velocity profile defined by $\Psi = \Psi(y)$ is it assumed that the perturbations, given by stream functions ϕ, must satisfy all of the above as well as either

6.) requiring the (perturbation) flow variables to be periodic in x, or

7.) $\int_{\Gamma_{x_0}} \Psi(y)(\partial\phi/\partial n)(x_0, y)\, dy = \int_{\Gamma_{x_1}} \Psi(y)(\partial\phi/\partial n)(x_1, y)\, dy$.

Several discussions on the applicability of assuming a flow to be two dimensional can be found in most fluid dynamics texts. The main limitation that this presents here is that vorticity is not tilted or stretched which are identified as mechanisms for instability. This does not rule out the possibility of concentrating vorticity through advection which may be associated with growth of perturbations (See Drazin and Reid (1981) for further discussion of both of these points.) Note that Abarbanel *et al.* (1986) apply the stability algorithm to three-dimensional flows.

The assumption of incompressibility (item 2) leads immediately to the velocity being nondivergent and (with assumption 1) is enough for the existence of a stream function (see Appendix A, part (iii)). There are situations where atmospheric flows may be considered nondivergent (though not necessarily incompressible)—for example, the Boussinesque approximation (see L. Mahrt (1986): On the shallow motion approximation. *Jour. Atm. Sci.* **43 (10)**, 1036–1044).

One of the more restrictive assumptions is that of homogeneity (item 3). Then the body force due to the action of gravity is balanced by the hydrostatic pressure so that these terms are dropped leaving only the remaining "modified pressure" due to the fluid motion. The resulting neglect of bouyancy effects is a fairly severe limitation as far as modeling any atmospheric flows. However, Holm *et al.* (1985) and Abarbanel *et al.* (1986), as well as others have applied the Energy-Casimir Convexity Method to stratified flows. The analysis becomes considerably more complicated.

The free-slip condition at the walls (item 4) is a usual sort of boundary condition since the viscous boundary layer may be considered to be extremely small. (In any true fluid the velocity must be zero on the boundary, at least above the molecular scale, so that the velocities must remain somewhat low to keep the boundary layer negligible.)

Requiring that the flow into the channel be the same as the flow out (item 5) is not so restrictive, and is necessary if the fluid is incompressible and homogeneous.

Restricting the flow to be periodic in x (item 6) means that this is flow "on a cylinder". This requires that any "passive" flow feature which remains intact for a long enough period of time will advect out of the up-stream boundary and simultaneously back in through the down-stream boundary. For example, a nonstationary vortex patch may advect through the domain several times. This seems physically restrictive if what we wish to model is a portion of some "real" channel. One potential relief to this problem is that the periodicity required throughout the analysis is entirely arbitrary.

One way to lessen this restriction somewhat is to require item 7. These are *not* circulations since the paths defining Γ_{x_i} are not circuits (see Appendix A, part (ii)). This might be interpreted as the balance of the flux (i.e. total flux is zero) of perturbation kinetic energy. This seems not to be an improvement over assuming periodicity.

In conclusion, there are several ways of reducing the restrictions that are apparent in the particular situation studied here. Generalizations and applications of the stability algorithm may be found in the references cited. It appears that the biggest limitation in applying the stability algorithm to geophysical flows is in defining boundary conditions which have the desired properties for the theory to apply, while maintaining some realistic model for actual flows. A careful study of how this approach may apply using some kind of radiative boundary conditions may be useful.

The definition of the norm in which stability is proven is determined by the problem, but there is some flexibility. In the case of U/U'' positive, two norms were defined and shown to be equivalent. In the case of U/U'' negative, three different norms were defined and shown to be equivalent. (The norms are listed in Table 5.4.) Recall that if two norms are equivalent, then continuity is independent of the choice between those norms. This leads to the conclusion that stability can be proven in either norm.

The equivalence of the norms in the case where U/U'' is negative is especially interesting. The norm $\|\phi\|_a$ can be interpreted as a measure of the perturbation enstrophy, or rotational energy. Whereas $\|\phi\|_b$ is the residual amount of perturbation enstrophy over perturbation kinetic energy. By showing these induce equivalent topologies suggests that in the case of two-dimensional (homogeneous) flow it is the rotational energy which is the most important factor in determining stability.

In order to establish that the "quadratic forms" listed in Table 5.4 qualify as norms, there must exist a $c > 0$ so that either $c \leq U/U''$ (in the case U/U'' positive), or $c \leq -U/U''$ (in the case U/U'' negative). This essentially says that it suffices that the curvature of the mean velocity profile must be bounded in order to establish a norm on the space of perturbations (using these quadratic forms).

To then establish stability using one of these norms, continuity of H_C at the equilibrium must be shown. It suffices that there is a constant, $0 < C < \infty$, such that either $U/U'' \leq C$ (in the case U/U'' positive), or $-U/U'' \leq C$ (in the case U/U'' negative). This means that for a given curvature, velocity must be bounded, or for a given finite but nonzero velocity, curvature of the velocity is bounded away from zero.

One point of interest is to note that in the case of constant shear, say $U(y) = \beta y$, the stability algorithm breaks down. There is no way to make this steady flow correspond to a minimum of H_C. Recall that Φ is determined such that

$$\delta H_C == \iint_D [(\Phi'(\Delta\Psi) - \Psi)\Delta\phi] \, dx \, dy = 0 \; .$$

But for this example, $\Delta\Psi = U''(y) = 0$ and thus $\Phi'(\Delta\Psi)$ cannot be a function of y, while Ψ is a function of y. There is no way to choose Φ so that $\Phi'(\Delta\Phi) - \Psi = \Phi'(0) - (1/2)\beta y^2 = 0$.

This work has concentrated on finding sufficient conditions for stability of steady-state flows. By contrast, there is still the need to analyze nonlinear evolution of unstable fluids.

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APPENDICES

APPENDIX A

Preliminaries

(i) Green's (first) identity

Let $D \subset R^2$ be a bounded *normal domain* (one in which the Gauss divergence theorem holds) with boundary ∂D. If u and v are functions which have continuous second derivatives on D and continuous first derivatives on the closure of D then,

$$\iint_D [v\Delta u + \nabla v \cdot \nabla u]\, d\vec{x} = \oint_{\partial D} v \frac{\partial u}{\partial n}\, ds ,$$

where n is a unit outward normal and ds a "surface" element (in this dimension simply a line element).

This of course holds in a much broader context, see Guenther and Lee (1988) for further details. The domain considered in the case of a channel flow satisfies the requirements for being a normal domain—a region of R^2 bounded by four piecewise-smooth curves.

(ii) Circulation

A *circuit* is a continuous, piecewise-smooth, oriented curve (in R^2) which has "connected endpoints". This circuit is representable in terms of a parameter,$s \in [0, 1]$ as

$$\gamma(s) = \big(x(s), y(s)\big) ,$$

with $\gamma(0) = \gamma(1)$. The circuit is *simple* if it does not cross itself, that is, $0 < s_1 < s_2 < 1$ implies $\gamma(s_1) \neq \gamma(s_2)$. A circuit is *reducible* if it can be continuously transformed (collapsed) to a point.

If $\vec{v}$ is a vector field defined on some domain D, and γ is a circuit contained in D then the *circulation* of the circuit γ is defined as

$$\Gamma(\gamma) = \int_\gamma \vec{v} \cdot d\vec{x} = \int_0^1 \vec{v} \cdot (d\vec{x}/ds)\, ds \ .$$

(Note that it is only required that γ be a circuit and not necessarily simple.)

Consider the domain of interest—a channel with solid walls at y_0 and y_1, with the flow of the fluid being periodic in x of period L. Then the domain D may be taken to be as given at the beginning of Chapter 3. Let γ_{y_0} be the path defined by

$$\gamma_{y_0}(s) = (x_0 + Ls, y_0) \ .$$

(This is the circuit given by moving from (x_0, y_0) to $(x_0 + L, y_0)$ along the x-axis.) Then γ_{y_0} actually defines a circuit since

$$\gamma_{y_0}(0) = (x_0, y_0) = (x_0 + L, y_0) = \gamma_{y_0}(1) \ ,$$

but this is not a reducible circuit. (In order to reduce this circuit we would need to shrink γ_{y_0} to a point, but to do this would mean shrinking away from at least one of the endpoints and this breaks the circuit.)

Similarly,

$$\gamma_{y_1}(s) = (x_0 + L(1-s), y_1),$$
$$\gamma_{x_0}(s) = (x_0, y_0 s + y_1(1-s)), \qquad \text{and}$$
$$\gamma_{x_1}(s) = (x_1, y_0(1-s) + y_1 s)$$

are all paths with γ_{y_1} being an irreducible circuit. (γ_{x_0} and γ_{x_1} are *not* circuits.) The direction that the path is traversed is built in to the parametrization, and finally note that any circuit completely contained in D is reducible (D is *simply connected*).

(iii) Existence of a stream function

From the equation for incompressibility, which amounts to the vector field of the velocity being nondivergent,

$$\nabla \cdot \vec{v} = \nabla \cdot (u, v) = u_x + v_y = 0 \ ,$$

we see that $u_x = -v_y$. Using results from calculus this means,

$$-v\hat{j} + u\hat{i} = \nabla\psi$$

for some ψ defined on D (provided D is simply connected). That is,

$$u = \psi_x \qquad \text{and} \qquad v = -\psi_y \ .$$

(It is usual to use the orientation described here, though it is entirely possible to have used $-\psi$ instead of ψ for the defining property.) ψ is called the *stream function* of the flow $\vec{v}$ and is unique up to an additive constant.

Let $\gamma(s)$ be any path in D, with unit tangent τ and unit normal n. For such an arbitrary path in R^2 it is possible to choose two different directions for n. If $\gamma(s)$ is a circuit on ∂D then it makes sense to choose n so that it is an outward normal (in the event we wish to apply the Gauss divergence theorem). However, depending on the direction of γ in relation to the domain (clockwise/counterclockwise) it is necessary to specify different directions of n for γ and $-\gamma$ (traversing in the opposite direction). (That is, n points outward in either case, yet $n \times \tau$ changes sign.) It is still possible to derive some desired properties of ψ.

Consider $\gamma(s)$ to be in the interior of D and, without loss of generality, choose n so that $n \times \tau$ is in the $\hat{k}$ direction (the other possible choice being $-\hat{k}$). Denote the transformation which rotates vectors by α by T_α. Then

$$T_{\frac{\pi}{2}}(n) = \tau, \qquad \text{and} \qquad T_{-\frac{\pi}{2}}(\tau) = n \ .$$

Consequently,

$$\nabla\psi \cdot \tau = T_{\frac{\pi}{2}}(\nabla\phi) \cdot T_{\frac{\pi}{2}}(\tau) = T_{\frac{\pi}{2}}(-v,u) \cdot n$$

$$= (u,v) \cdot n = \vec{v} \cdot n \ .$$

Similarly,

$$\nabla\psi \cdot n = T_{-\frac{\pi}{2}}(\nabla\phi) \cdot T_{-\frac{\pi}{2}}(n) = T_{-\frac{\pi}{2}}(-v,u) \cdot \tau$$

$$= (-u,-v) \cdot \tau = -\vec{v} \cdot \tau \ .$$

If the other direction for the normal was chosen it would change only the sign in these calculations. So we have the properties

$$\boxed{\begin{array}{l} \nabla\psi \cdot n = \pm\vec{v} \cdot \tau \\ \nabla\psi \cdot \tau = \mp\vec{v} \cdot n \end{array}}$$

Since $\vec{v}$ is tangent to Γ_{y_i} $(i = 1,2)$, $\vec{v} \cdot n = 0$ on Γ_{y_i}. Then

$$0 = \vec{v} \cdot n = \pm\nabla\phi \cdot \tau$$

implies that $\nabla\psi$ is orthogonal to Γ_{y_i}. Thus, $\psi|_{\Gamma_{y_i}}$ is constant. Moreover, since $\vec{v}(x_0,y) = \vec{v}(x_0 + L, y)$, $\nabla\psi$ is also periodic in x. Then ψ must be periodic up to an additive constant. But since $\psi(x_0,y_i) = \psi(x_0 + L, y_i)$, the constant must be zero which yields $\psi(x_0,y) = \psi(x_0 + L, y)$ for any y. That is, $\psi(x,y)$ is periodic in x of period L.

There is one final property which follows from the above considerations. Suppose that Ψ is a stream function corresponding to the equilibrium flow and ϕ is a stream function for a perturbation flow. Then both satisfy the above properties so that

$$\oint_{\Gamma_{y_i}} \Psi \frac{\partial\phi}{\partial n}\, ds = \Psi|_{y_i} \oint_{\Gamma_{y_i}} \nabla\phi \cdot n\, ds$$

$$= \pm\Psi|_{y_i} \oint_{\Gamma_{y_i}} \vec{v} \cdot \tau\, ds$$

$$= \pm\Psi|_{y_i} \Gamma(\gamma_{y_i}) \ .$$

That is, $\oint_{\Gamma_{y_i}} \Psi \frac{\partial \phi}{\partial n}\, ds$ is a constant times the circulation of the circuit Γ_{y_i}. When the perturbation is required to preserve the circulations on Γ_{y_i} this is equivalent to requiring that

$$\oint_{\Gamma_{y_i}} \Psi \frac{\partial \phi}{\partial n}\, ds = 0 \ .$$

(iv) Least eigenvalue for Δ in D

Let D be given as above. Consider the eigenvalue problem

$$\text{(EVP)} \qquad \begin{cases} \Delta\phi + k^2\phi = 0 & \text{in } D \\ \phi = 0 & \text{on } \Gamma_{y_i}, \text{ and} \\ \phi(x_0, y) = \phi(x_0 + L, y) & \text{for all } y \end{cases}$$

Since the domain is geometrically suitable, use separation of variables to solve the problem. Let

$$\phi(x,y) = X(x)Y(y) \ .$$

The boundary conditions may then be expressed as

$$Y(y_i) = 0 \quad \text{for} \quad i = 1,2 \qquad \text{and} \quad X(x_0 + L) = X(x_0) \ .$$

Substitution of $(A1)$ with these boundary conditions into (EVP) yields two eigenvalue problems.

$$(EVPa) \qquad \begin{cases} Y'' + \mu^2 Y = 0 & \text{for } y \in [y_0, y_1] \\ Y(y_0) = Y(y_1) = 0 \ , \end{cases}$$

and

$$(EVPb) \qquad \begin{cases} X'' + \nu^2 X = 0 & \text{for } x \in [x_0, x_1] \\ X(x_0 + L) = X(x_0) \ , \end{cases}$$

where $\nu^2 = k^2 - \mu^2$.

The general solution of $(EVPa)$ is

$$Y(y) = A \sin \mu y + B \cos \mu y \ .$$

To satisfy the boundary conditions and with some simplification it is apparent that $B = 0$, A is arbitrary, and $\sin \mu(y_1 - y_0) = \sin(\mu d) = 0$. This will be satisfied for the eigenvalues given by

$$\mu_n = \frac{n\pi}{d} .$$

Now consider the second problem $(EVPb)$. The boundary conditions only require that the solution be periodic. Let ϕ be a constant (so it is periodic) and $\nu = 0$. This is a nontrivial solution to $(EVPb)$ and so the least eigenvalue is $\nu = 0$.

Then the eigenvalues of the original problem, (EVP), are given by the relation

$$k_n^2 = \nu^2 + \mu_n^2 = \nu^2 + \left(\frac{\pi n}{d}\right)^2 .$$

The lowest eigenvalue is

$$k_1^2 = \frac{\pi^2}{d^2} .$$

Let $\phi_1 = A_1 \sin(n\pi y/d)$ be the eigenfunction corresponding to k_1. That is, the pair (ϕ_1, k_1) satisfies (EVP),

$$\Delta\phi_1 + k_1^2\phi = 0 .$$

Multiplying by ϕ_1, integrating over D yields

$$\int_D [\phi_1 \Delta\phi_1 + k_1^2 \phi_1^2]\, d\vec{x} = 0$$

or, applying Green's identity,

$$-\int_D (\nabla\phi_1)^2\, d\vec{x} + k_1^2 \int_D \phi_1^2\, d\vec{x} = 0 .$$

Solving for k_1^2,

$$k_1^2 = \frac{\int_D (\nabla\phi_1)^2\, d\vec{x}}{\int_D \phi_1^2\, d\vec{x}} .$$

Since, by construction, k_1^2 is the smallest eigenvalue, its Rayleigh quotient (the above equation) is smaller than any other Rayleigh quotient corresponding to any other eigenvalue. So k_1^2 is the *least* eigenvalue in the same sense as defined by equation (3.17). That is, $k_1 \equiv k_0$.

APPENDIX B

Some calculus of variations

(i) Variational Derivatives

Only the essentials are presented here in order to provide a key to the notation used and to give some idea of the computational aspects of the calculus of variations. The definitions and theorems presented follow closely those given in Olver (1986). For a good introduction and other applications see Guenther and Lee (1988), Chapter 11.

Let Ω be an open connected domain in $\mathbb{R}^p$ with smooth boundary $\partial\Omega$. Let the space of dependent variables $u = (u^1, \ldots, u^q)$ be $\mathbb{R}^q$. (Olver (1986) comments that the following ideas can be extended to variational problems over smooth manifolds—see Olver (1986) for references.)

A *variational problem* consists of trying to maximize or minimize a functional

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})\, d\vec{x} \ ,$$

over some class of functions, $u = f(x)$, defined on Ω. L (called the *Lagrangian for*, or sometimes the *density of* $\mathcal{L}$) is a smooth function of x, u and derivatives, $u^{(n)}$, of u. That is, L is a differential function as defined in Chapter 4. The particular specification of this class depends on boundary conditions and differentiability conditions on the minimizing function. For a given class a minimizing function for a given variational problem need not exist.

Let $\mathcal{L}[u]$ be a variational problem. The *variational derivative* (sometimes referred to as the *functional derivative*) of $\mathcal{L}$ is the unique q-tuple

$$\delta\mathcal{L}[u] = (\delta_1\mathcal{L}[u], \ldots, \delta_q\mathcal{L}[u])$$

for which

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{L}[f+\epsilon\eta] = \int_\Omega \delta\mathcal{L}[f(x)]\cdot\eta(x)\,d\vec{x}\ , \qquad (B.1)$$

whenever $u = f(x)$ is a smooth function on Ω, and $\eta(x) = (\eta^1(x), \ldots, \eta^q(x))$ is smooth with compact support in Ω. $f + \epsilon\eta$ must satisfy any boundary conditions imposed on the class over which $\mathcal{L}$ is to be minimized. The *variational (or functional) derivative of* $\mathcal{L}$ *with respect to* u^α is the α^{th} component, $\delta_\alpha\mathcal{L} = \delta\mathcal{L}/\delta u^\alpha$, of $\delta\mathcal{L}$.

The expression on the left hand side of $(B.1)$ is called the *first variation* of $\mathcal{L}$ (with respect to u) and is denoted by either $\delta\mathcal{L}$, $\delta\mathcal{L}(\eta)$, or $\delta\mathcal{L}(f)\cdot\eta$. It may be thought of as the "directional derivative" of $\mathcal{L}$ in the "direction" of η at the "point" f (see Guenther and Lee, 1988).

In the Introduction, section C, where the stability algorithm first appears, the derivatives were left in a generic form. Once the setting of the problem is specified the notation can be suitably specialized. For example, $DE(u_e)\cdot(u-u_e)$ in equation (1.3) becomes $\delta E(\Psi)\cdot(\phi)$ for calculations leading up to (3.10).

If f is an extremum of $\mathcal{L}[u]$, then for each η with compact support in Ω such that $f+\epsilon\eta$ is in the desired class of functions, $\mathcal{L}[f+\epsilon\eta]$ is a smooth function of ϵ and thus has an extremum at $\epsilon = 0$. Observing this, along with $(B.1)$ and a short argument establishes the following theorem.

Theorem B–1. *If* $u = f(x)$ *is an extremal of* $\mathcal{L}[u]$, *then*

$$\delta\mathcal{L}[f(x)] = 0 \qquad \text{for} \quad x\in\Omega\ . \qquad (B.2)$$

This is the condition required in step 3 of the stability algorithm.

In a similar fashion higher order variational derivatives are defined. We only need the second order variational derivative denoted, $\delta^2\mathcal{L}$, which is easily calculated by extending $(B.2)$ using the second order derivative with respect to ϵ.

(ii) Calculation of δH_C and $\delta^2 H_C$

As given in Chapter 3, equation (3.7), H_C is

$$H_C(\psi) = E(\psi) + C(\psi) = \frac{1}{2}\iint_D \nabla\psi\cdot\nabla\psi\,dx\,dy + \iint_D \Phi(\Delta\psi)\,dx\,dy\ . \quad (B.3)$$

The first variation of H_C at Ψ is found by computing

$$\frac{d}{d\epsilon}\left[H_C(\Psi+\epsilon\phi)\right]_{\epsilon=0}\ .$$

That is,

$$\begin{aligned}\delta H_C(\Psi) &= \frac{d}{d\epsilon}\left[H_C(\Psi+\epsilon\phi)\right]_{\epsilon=0}\\ &= \frac{d}{d\epsilon}\left[\iint_D [\frac{1}{2}\nabla(\Psi+\epsilon\phi)\cdot\nabla(\Psi+\epsilon\phi) + \Phi(\Delta(\Psi+\epsilon\phi)]\,dx\,dy\right]_{\epsilon=0}\\ &= \left[\iint_D [\nabla(\Psi+\epsilon\phi)\cdot\nabla\phi + \Phi'(\Delta(\Psi+\epsilon\phi)\Delta\phi]\,dx\,dy\right]_{\epsilon=0} \qquad (B.4)\\ &= \iint_D [(\Phi)\cdot\nabla\phi + \Phi'(\Delta(\Psi)\Delta\phi]\,dx\,dy\\ &= \iint_D [\Phi'(\Delta\Psi) - \Psi]\Delta\phi\,dx\,dy \qquad \text{(using Green's identity)}\ .\end{aligned}$$

To find the second variation,

$$\frac{d^2}{d\epsilon^2}\left[H_C(\Psi+\epsilon\phi)\right]_{\epsilon=0}\ ,$$

is computed. By picking up the calculation at equation $(B.4)$,

$$\begin{aligned}&\frac{d}{d\epsilon}\left[\iint_D [\nabla(\Psi+\epsilon\phi)\cdot\nabla\phi + \Phi'(\Delta(\Psi+\epsilon\phi)\Delta\phi]\,dx\,dy\right]_{\epsilon=0}\\ &= \left[\iint_D [\nabla\phi\cdot\nabla\phi + \Phi''(\Delta(\Psi+\epsilon\phi)\Delta\phi^2]]\,dx\,dy\right]_{\epsilon=0}\\ &= \iint_D \left[\Phi''(\Delta\Psi)(\Delta\phi)^2 + (\nabla\phi)^2\right]\,dx\,dy\ .\end{aligned}$$

APPENDIX C

An application of Taylor's theorem

One form of Taylor's theorem in one dimension is the following (see most any Calculus book; eg. H. Thurston, *Intermediate Mathematical Analysis*, Oxford (New York), 1988).

Theorem. *If $\Phi^{(k)}(x)$ exists for $x \in [x_0 - h, x_0 + h]$ then there is a θ, $\theta \in [x_0, x_0 + h]$ so that*

$$\Phi(x_0 + h) = \sum_{j=0}^{k-1} \frac{h^j \Phi^{(j)}(x_0)}{j!} + \frac{h^j \Phi^(k)(\theta)}{k!} .$$

Now consider that we have $0 < c < \Phi''(x)$ on $\mathbb{R}$. So this inequality holds on every closed interval, and in particular on $[x_0 - h, x_0 + h]$. Then by Taylor's theorem there is a θ, $\theta \in [x_0, x_0 + h]$, so that

$$\Phi(x_0 + h) = \Phi(x_0) + h\Phi'(x_0) + \frac{h^2}{2}\Phi''(\theta) .$$

But $0 < c < \Phi''(x)$ on $[x_0, h]$ implies $0 < c < \Phi''(\theta)$. So

$$\begin{aligned} \frac{h^2}{2}c &< \frac{h^2}{2}\Phi''(\theta) \\ &= \Phi(x_0 + h) - \Phi(x_0) - h\Phi'(x_0) . \end{aligned}$$

That is,

$$\frac{h^2}{2}c < \Phi(x_0 + h) - \Phi(x_0) - h\Phi'(x_0) . \qquad (C.1)$$

This can be extended to the case where h and x_0 are real-valued, bounded functions by observing that it must hold pointwise. Specifically, if $\Delta\Psi$ and $\Delta\phi$ are real-valued and bounded, then for each (x, y)

$$\Phi(\Delta\Psi + \Delta\phi) = \Phi(\Delta\Psi) + \Delta\phi\Phi'(\Delta\Psi) + \frac{(\Delta\phi)^2}{2}\Phi''(\theta) , \qquad (C.2)$$

where θ is between $\Delta\Psi$ and $\Delta\phi$, and $(C.1)$ follows in the form

$$\frac{(\Delta\phi)^2}{2}c < \Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Delta\phi\Phi'(\Delta\Psi)\ .$$

Similarly, if $\Phi''(x) < C < \infty$ on $\mathbb{R}$, then it holds on each closed interval in $\mathbb{R}$ and $(C.2)$ still holds. Then,

$$\Phi(\Delta\Psi + \Delta\phi) - \Phi(\Delta\Psi) - \Delta\phi\Phi'(\Delta\Psi) < \frac{(\Delta\phi)^2}{2}C\ .$$