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The problem of a general incompressible viscous fluid flow past a flat plate with heat transfer due to forced convection is considered in this thesis. The synthetic method developed by B. R. Seth is applied to the Navier-Stokes equations and the equation of energy governing the flow to obtain the dynamic and thermal boundary layer solutions as asymptotic limits of an extended field. As a result, new formulas are derived for both the dynamic and thermal boundary layer thicknesses. Also, algorithms for estimating all the parameters involved in the analysis are provided and boundary layer functions based on the new solutions are determined.

Synthetic Method in Thermal Boundary Layer Transition

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Synthetic Method in Thermal Boundary Layer Transition

I. INTRODUCTION

1.1 Preliminary Remarks

One of the many long-standing problems in fluid mechanics today is to provide global analytic solutions to the Navier-Stokes equations, which is a coupled system of highly nonlinear partial differential equations, for the motion of viscous fluids. This problem is as yet unresolved, except in a few cases, owing to enormous mathematical difficulties encountered. A variety of assumptions have therefore been made to facilitate analysis of the motions in particular cases. Stokes, for example, linearized the equations by neglecting the inertia terms, thus obtaining the slow motion solution of a viscous fluid past a sphere and a circular cylinder. Oseen improved the approximation by taking the inertia terms partially into account. Prandtl proposed, in his effort to resolve the well-known d'Alembert's paradox, to split up an entire flow field into subfields; he derived, through an order-of-magnitude analysis, the boundary layer equations as approximate forms of the Navier-Stokes equations holding only within a narrow layer near the boundary. Yet, no exact solution is known even for a sphere or a cylinder moving through a viscous fluid. The Prandtl system of boundary layer equations is still nonlinear and is further approximated in most solution processes, leading to results having serious defects of their own. See reference [1] for a complete discussion.

The Navier-Stokes system is elliptic in character with respect to the coordinates, whereas the Prandtl system has features of a parabolic equation. The types of problems which are well-posed for the two models are consequently drastically different. The truncated boundary layer equations of Prandtl and the inviscid fluid equations of Euler are also fundamentally different in character. Hence, it may not be possible to obtain a single solution from patching up local solutions, as is often the case in current literature, which holds at all points near and away from the boundary. Nevertheless, one of the conventional ways of obtaining a picture of the flow field of a viscous fluid outside the wake goes as follows: the entire flow field is divided into two separate regions, namely a boundary layer region where the flow field is obtained from boundary layer equations, and an outer region where the Euler equations are used to obtain an external flow. As a result, there has been a considerable discussion about where and how to match the two parts of the flow field. We shall briefly review the Prandtl's boundary layer theory in Sections 1.4 and 2.2.

The present thesis was motivated by a series of papers by Seth [1,2,3,4] in which the main thrust is on the need of what he terms a synthetic method and its importance as a fresh mathematical method to solve a large class of nonlinear partial differential equations often arising out of physical problems. The main crux of this method, which will be discussed in detail in Sections 1.5 and 2.3, is that it is possible that, instead of narrowing down a field, its extension might give exact solutions of nonlinear problems. In general, the physical problem will become a limiting case, in a certain sense, of the problem investigated because of the extension of field of investigation. The synthetic method has indeed met with great success in treating the Navier-Stokes equations in its complete form for several viscous fluid flow problems, including the motion of a sphere or a circular cylinder through a viscous fluid; various local solutions were obtained as special cases from an exact solution to the Navier-Stokes equations.

As we shall see in the next two sections, the boundary layer formation is an asymptotic phenomenon in the sense that it involves a break-down of uniformity near the boundary. The mathematical structure of the boundary layer phenomena may accordingly be founded on asymptotic analyses, as will be discussed in Sections 1.2 and 1.3. The synthetic method is essentially an attempt in this direction. In the present thesis, the problem of a general incompressible viscous fluid flow past a flat plate with heat transfer due to forced convection is considered based on the synthetic method. In Chapter IV the synthetic method is applied to the Navier-Stokes equations and the equation of energy, derived in Chapter III for linear thermoviscous fluids, governing the flow to obtain the dynamic and thermal boundary layer solutions as asymptotic limits of an extended field.

1.2 Asymptotic Phenomena in Mathematical Physics

It is still rather an overlooked fact that asymptotic description is not merely a convenient way of obtaining approximate solutions to physical problems, but it has a more fundamental significance in the mathematical treatment of nature. One of the most important classical cases is the asymptotic solution of the Schrödinger equation. Birkhoff [5] discovered that the Schrödinger equation is merely the "principal equation" which has the usual Hamilton-Jacobi partial differential equation as its "multiplier equation." In other words, the classical theory of mechanics results from the leading term in the asymptotic expansion of quantum mechanics. Indeed, if we take the Schrödinger equation in the time-independent form:

$$\nabla^2\psi + \lambda^2[E - V(\underline{x})]\psi = 0 \quad ,$$

where $\lambda = \frac{\sqrt{2m}}{\hbar}$ is a large parameter and the function $V(\underline{x})$ is defined over a certain set in which $E - V(\underline{x})$ changes sign, and if we attempt to solve it with the formal asymptotic process:

$$\psi = e^{i\lambda S} \sum_{n=o}^{\infty} v_n(\underline{x})\lambda^{-n} \quad ,$$

we obtain

$$(\nabla S)^2 + V(\underline{x}) = E \quad ,$$

which is the well-known Hamilton-Jacobi partial differential equation in the time-independent form.

Friedrichs [6] describes phenomena such as discontinuities, quick transitions, nonuniformities, or other incongruities resulting from approximate descriptions as asymptotic. Elastic-plastic deformations, visco-elastic deformations, boundary layers, gas dynamic shocks, creep, fatigue, relaxation oscillations, stability, and adiabatic theorem in quantum mechanics are well-known examples. A great deal of research has been done on these transition phenomena by such workers as Stokes, Poincare, Horn, Birkhoff, Noaillon, Tamarkin, Perron, Trijitzinski, Turitten, Fukuhara, Jeffreys, Langer, Wasow, Levinson, Bellman, Stoker, Reissner, Lin, Friedrichs [6] and Seth [7,8].

The occurrence of asymptotic phenomena in nature is often associated with transition processes. Two cases in point are the boundary layer phenomenon and the Stokes phenomenon. The boundary layer transition, touched very briefly upon in the Introduction and to be discussed more thoroughly in the next two sections, is not the only form of break-down of uniformities. Such a break-down may also happen in the interior of the domain of interest. The well-known Stokes phenomenon is such an occurrence. The

Stokes phenomenon obtains at a line if the asymptotic expansion of the analytic continuation of a solution of a differential equation across this line is not given by the analytic continuation of the terms of the asymptotic expansion. The leading term of the asymptotic series of the actual solution representing a physical quantity would therefore suffer a discontinuity upon crossing the Stokes line. There would then exist the possibility of describing continuous quantities as discontinuous ones by describing them asymptotically, namely a discontinuity resulting from an approximate description. Thus, a large class of discontinuity phenomena in mathematical physics, including, e.g., the formation of a shadow cast by an obstacle due to an incident light beam, may be interpreted as Stokes or Boundary layer phenomena.

Such asymptotic phenomena may be treated by various asymptotic analyses. Friedrichs and others successfully employed the method of coordinate-stretching for a number of cases; especially effective for problems of boundary layer type. Lighthill's important extension of Poincare's method, to render approximate solutions to physical problems uniformly valid, has proven to be very general in concept, at the same time very effective upon its application, and yields useful results for a large class of problems. His method [9,10], based on the ingenious idea of expanding not only the dependent variable but also the independent variable, depending on a parameter, is designed to eliminate possible singularities of a power series expansion in the very small parameter, and to render the expansion uniformly valid over the whole domain of interest. Also, in connection with the Stokes phenomena the turning point theory, originated by Jeffreys [11] and Birkhoff [5] and rigorously advanced by Langer [12,13, 14], has played an important role in the analysis of the behaviors of solutions about turning points, and consequently in the neighborhoods of Stokes lines. It is also noteworthy in the context of the present thesis that Meksyn [15], taking advantage of the fact that there occurs an abrupt change of velocity along the normal direction inside the boundary layer, demonstrated the applicability of saddle point method to boundary layer problems in general.

In the next two sections we shall attempt to characterize boundary layer formation as an asymptotic phenomenon and shall also review the conventional (i.e., Prandtl's) treatment of the boundary layer phenomenon, thus indicating in which way we shall approach our problem in this thesis.

1.3 Boundary Layer As An Asymptotic Phenomenon

The Prandtl's boundary layer theory, speculative and approximate as it may be, is essentially valid in the real world because of the vast applicability of the theory and its close agreement with observed phenomena. The boundary layer theory has, however, not been completely understood on a solid mathematical basis. One firm step forward in understanding the mathematical structure of boundary layer effects was taken by Friedrichs [6] who characterized it as an asymptotic phenomenon. The main feature of the asymptotic phenomenon he speaks of is manifested in the drop of the order of the associated differential equation accompanied by a corresponding sacrifice of the boundary conditions to be satisfied. In order to illustrate the process of recognizing the boundary layer as an asymptotic phenomenon, we shall consider, for simplicity, a second order ordinary differential equation

$$\varepsilon u''(z) + a(z)u'(z) + b(z)u(z) = 0 \quad , \qquad (1.3.1)$$

where ε is a small parameter, coupled with the boundary conditions at $z = 0$ and $z = \infty$. We now pose the question how its solution behaves as $\varepsilon \to 0$. To this end, we suppose that the first order differential equation, as obtained in the limit $\varepsilon \to 0$, satisfies the boundary condition at infinity. It is then clear that for small ε the solution of the second order differential equation fairly agrees with the limit solution near infinity. On the other hand, we cannot expect the uniform convergence of the solution to the limit solution near the other boundary. If the parameter ε is small, the solution will run near the limit solution down to $z = \delta << 1$. A quick transition then follows towards the point $z = 0$. The quick transition must occur since a boundary condition is about to be lost and this loss in turn is necessary since the order of the differential equation is about to drop.

To investigate this transition we resort to the method of coordinate-stretching. Specifically, we take the ratio

$$\zeta = \frac{z}{\varepsilon} \tag{1.3.2}$$

as a new independent variable instead of z. Then the original equation with u as a function of ζ becomes

$$u'' + au' + \varepsilon bu = 0 \quad . \tag{1.3.3}$$

This equation remains of the second order when ε is put to zero and therefore both boundary conditions can be satisfied. It is then not difficult to see that the new limit process is uniform even near $z = 0$. Hence, the new limit solution may serve as an approximate description of the quick change of u in the transition layer.

This break-down of uniformity is precisely what happens with a viscous fluid flow past an obstacle as the viscosity approaches zero. The Navier-Stokes equations for a two-dimensional viscous fluid flow is of the fourth order in terms of the stream function and has four boundary conditions to be satisfied. Prandtl's boundary layer equations derived from the Navier-Stokes equations by an order-of-magnitude analysis is of third order. On the other hand, the inviscid fluid flow equation for the stream function is of second order. If one considers the orders of the above governing differential equations, one would notice that in passing from the Prandtl's equation to the inviscid fluid flow equation there occurs a drop of order one, while in passing from the general viscous fluid flow equation to the inviscid fluid flow equation the drop in order would actually be two. However, in the former case, the derivations are rather hypothetical since the pressure term within the boundary layer is replaced by the external pressure and one of the equations of motion is dropped altogether. Thus the actual drop in order in passing from the general viscous flow to the inviscid flow should be two instead of one. Consequently, as remarked earlier in connection with Equations (1.3.1) through (1.3.3), there occurs a breakdown of the uniformity of the solution within a narrow layer of fluid near the boundary known as the boundary layer. It may be footnoted that problems of boundary layer type have been treated from the point of view of singular perturbations using matched asymptotic expansions valid over different subdomains in an infinite domain. But, the present treatment given in this thesis is a departure from the traditional singular perturbation methods and is based on transition concepts and synthetic method, as will be discussed in the following chapters.

1.4 Prandtl's Boundary Layer Concept—Dynamic and Thermal

Prandtl's ingenious idea of a thin layer of quick transition near the solid boundary emerged out of an attempt to resolve the well-known d'Alembert's paradox of late 19th century. The paradox was that for fluids with low viscosity the assumption of absence of viscosity led to a very satisfactory description of the flow around the body, although it did not offer an explanation for the resistance experienced by the body. Prandtl, in 1904, advanced the hypothesis that for sufficiently small viscosities the viscous effects are confined to a very thin layer near the surface of the body, the velocity gradient normal to which is very large. Based on this hypothesis, he was able to analyze the fundamental differences in the behavior of inviscid and viscous fluids and suggested to split the entire flow field into two separate regions: one a very narrow region near the surface of the body and the other away from the body. While the latter flow is quite accurately approximated by the inviscid fluid theory of Euler, equations governing the flow in the immediate neighborhood of the body were derived by appraising the order of magnitude of the various terms of the Navier-Stokes equations and rejecting the relatively insignificant terms.

The Navier-Stokes equations governing the motion of an incompressible viscous fluid along a flat plate are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (1.4.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (1.4.2)$$

with the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \qquad (1.4.3)$$

where u and v are the tangential and normal velocity components, respectively, p the fluid pressure, and ν the kinematic viscosity. In deriving Equations (1.4.1) - (1.4.3) it is assumed that the plate coincides with y = 0, on the positive side of x; the main flow is in the direction of x positive. The assessment of the order of magnitude of the terms in Equations (1.4.1) - (1.4.3) are based on the following assumptions:

$$\begin{aligned} x &\sim 1 \ , \quad u \sim 1 \ , \quad \frac{\partial}{\partial x} \sim 1 \ , \\ y &\sim \delta_s \ , \quad v \sim \delta_s \ , \quad \frac{\partial}{\partial y} \sim \delta_s^{-1} \ , \end{aligned} \qquad (1.4.4)$$

where $\delta_s << 1$ is the thickness of the boundary layer and the notation $\sim$ stands for " is of the order of magnitude of ". Since very near the boundary the viscous effects are predominant and away from the boundary inertia terms are more important, Prandtl made the further assumption that inside the boundary layer the viscous and inertia terms are of the same order of magnitude. This assumption led to the semi-empirical formula for the thickness of the boundary layer as proportional to $R^{-1/2}$ R being the Reynolds number. Substituting the order estimates (1.4.4) into Equations (1.4.1) - (1.4.3), we can deduce that in Equation (1.4.1) the term $\frac{\partial^2 u}{\partial x^2}$ can be disregarded compared with the term $\frac{\partial^2 u}{\partial y^2}$. The equations of motion (1.4.1) - (1.4.3) accordingly become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (1.4.5)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \qquad (1.4.6)$$

where (1.4.2) altogether drops out in view of the order of magnitude analysis carried out according to Prandtl. Equations (1.4.5) and (1.4.6) are called Prandtl's boundary layer equations. Since in the main flow the action of viscosity can be neglected, we obtain from the inviscid fluid theory of Euler that

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \qquad (1.4.7)$$

where U is the given streaming speed. Prandtl's boundary layer system as given by (1.4.5) and (1.4.6) are two equations in three unknowns u, v, and p. Thus, mathematically speaking, an indeterminancy enters into the problem. There has been as yet no method of theoretically deriving the pressure at the outer edge of the boundary layer. In the case of a flat plate, this difficulty is overcome by supposing the pressure distribution inside the boundary layer the same as at its outer edge, i.e., by using the pressure from Equation (1.4.7) or by borrowing the pressure from experiments. We shall present a critical review of the Prandtl's boundary layer theory in Section 2.2.

Boussinesq, in 1903, introduced the concept of thermal boundary layer before Prandtl proposed an analogous idea for the transfer of momentum. The transfer of heat between a solid body and a fluid flow is governed by an equation of energy besides the usual Navier-Stokes equations. The heat transfer may be due to natural or forced convection. Flows in which buoyancy forces brought about by large temperature differences through variations in density are dominant are called natural convection flows. For natural convections, there occurs a mutual interaction between the velocity and temperature fields. On the other hand, when buoyancy forces are negligible, and when the properties of the fluid may be independent of temperature, mutual interaction ceases, that is to say, the velocity field no longer depdnds on the temperature field but the converse does not hold. This happens at large Reynolds numbers and small temperature differences. Such flows are termed forced convection flows. In the case of small conductivity as for gases and liquids, there is also a thin temperature boundary layer where heat exchange takes place between the solid boundary and the fluid, there is a very steep temperature gradient in the normal direction to the solid boundary, and the heat flux due to conduction is of the same order of magnitude as that due to convection.

The system of equations of motion and energy for an incompressible two-dimensional steady flow with constant properties is given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1.4.8)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x\beta(\phi - \phi_\infty) \qquad (1.4.9)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} \quad \frac{\partial^2 v}{\partial y^2}\right) + g_y\beta(\phi - \phi_\infty) \qquad (1.4.10)$$

$$u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = a\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + \frac{\nu}{c}\Phi \ , \qquad (1.4.11)$$

where a is the thermal diffusivity, c the specific heat, β the coefficient of expansion, g_x and g_y the x- and y-components of the vector of gravitational acceleration, ϕ_∞ the fluid temperature at infinity, and the function $\Phi = \Phi(x,y)$ the viscous dissipation function given by

$$\Phi = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 . \qquad (1.4.12)$$

The boundary layer equations for heat transfer are derived in the same way as in the case of a momentum boundary layer. Thus, similar order considerations can be carried out and the boundary layer equations of motion and energy become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1.4.13)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + g_x\,\beta(\phi - \phi_\infty) \qquad (1.4.14)$$

$$u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = a\,\frac{\partial^2 \phi}{\partial y} + \frac{\nu}{c}\left(\frac{\partial u}{\partial y}\right)^2 . \qquad (1.4.15)$$

It is to be noted here that the partial differential equations for the dynamic and thermal boundary layer, (1.4.14) and (1.4.15), are very similar in structure. It is assumed in deriving Equations (1.4.13) - (1.4.15) that inside the thermal boundary layer the conduction and convection terms are of the same order of magnitude. This assumption has led to the semi-empirical formula for the ratio δ_t/δ_s, where δ_t is the thickness of the thermal boundary layer, as proportional to $P^{-1/2}$, P being the Prandtl number.

For forced convection flows with large Reynolds numbers and small temperature differences, the buoyancy forces will drop out of Equations (1.4.9) and (1.4.10), thus leading to the independence of velocity from temperature while retaining the dependence of the latter on the former.

1.5 Synthetic Method for Boundary Layer Analysis

The usual mathematical treatment of fluid flow problems is based on such assumptions that restrict the types of solutions to rather special cases and consequently limit the scope of such solutions to an extent where no amount of refinements, such as higher approximations, can produce a complete solution. Indeed, the basic assumptions, on which the various reductions of the Navier-Stokes equations have been carried out, limit the range of investigation and the associated equations fail to give satisfactory results when the ranges are extended. For example, the truncated Prandtl's boundary layer equations and the linear differential equation of the perfect fluid flow restrict the range of investigation to certain regions near and away from the boundary, respectively. The celebrated Blasius solution is defective in that the smooth transition into the perfect fluid flow is not accomplished. We shall give a critical review of the Prandtl's boundary layer theory in Section 2.2 of Chapter II.

It may be observed that all analytical solutions represent idealized cases of the actual state of a physical phenomenon. The ever-growing mathematical modeling is a case in point. Quite frequently, many phenomena of physical interest can be identified with the limiting behaviors of the solutions of the underlying mathematical models. The limiting state can be approached in many ways. It has been rather a common trend in dealing with nonlinear differential equations to employ reducing methods such as linearization and semilinearization. By reducing the field of investigation to a particular aspect of an event, an overall description is not obtained and a number of important effects remain uncharted. It does, therefore, seem necessary that the field of investigation should be enlarged rather than restricted further. In general, the physical problem will become a limiting case, in a certain sense, of the problem investigated because of the extension of field of investigation. In particular, in the case of flow past an obstacle, if we seek a solution holding near the boundary and merging smoothly into the perfect fluid solution away from the boundary, we shall have to employ a more correct form of the governing field equations than currently used in fluid mechanics.

The synthetic method consists first in an extension of a field which is achieved by introducing a constraining term or terms into the field equations of a problem. Plausible solutions compatible with the requirements of the problem are chosen at the outset. These solutions should contain a parameter or parameters to be controlled in the process of relaxing the constraint(s) and then the desired solutions will be obtained over the required domain from the field equations as limiting cases of the extended field. In particular, by the asymptotic vanishing of the constraint(s) is meant that their dominant terms should become vanishingly small, let alone the other trailing terms. An abrupt change in the magnitude of a function associated with a certain physical quantity, such as an apparent discontinuity in the boundary of the shadow cast by an object due to an incident light beam and similarly, in the case of flow past an obstacle, a quick transition of the flow velocity near the boundary, is manifested in the dominant term of its asymptotic expansion. In the boundary layer setting, the constraining terms are a force and an energy. They are concentrated in the immediate neighborhood of the boundary and at other points they are vanishingly small, thus implying abrupt changes in their magnitudes near the boundary. Therefore, the asymptotic vanishing of the constraining force and energy near the boundary indeed corresponds to the boundary layer formation, which is asymptotic in the sense of Friedrichs, as described in Sections 1.2 and 1.3. We shall show in Chapter IV that our solutions correspond to the dynamic and thermal boundary layer solutions with the desired property of smooth transition into the main stream flow. A detailed discussion of the synthetic method is given in Section 2.3 of Chapter II.

II. SYNTHETIC METHOD VERSUS PRANDTL'S APPROACH TO BOUNDARY LAYER PHENOMENA

2.1 Preliminary Remarks

We have already indicated in the last chapter that the classical boundary layer theory based on Prandtl's order of magnitude analysis is hypothetical in nature and as such leads to certain serious questions from both mathematical and physical standpoints. It also has been pointed out there that only when the boundary layer theory is fitted into the framework of asymptotic analysis does its mathematical structure and physical ramifications become transparent. A critical review of Prandtl's theory will be made subsequently in Section 2.2. Although the Friedrich's approach of characterizing the boundary layer as an asymptotic phenomenon leads to a definite clarification of the state of boundary layer theory, yet it does not yield a rigorous justification of the theory itself, the main difficulty being the nonlinearity involved in the analytical treatment. The synthetic method, devised as an alternative to the boundary layer treatment due to Prandtl, enables us to obtain a single flow pattern which not only provides the boundary layer effects near the solid boundary but also accomplishes smooth merger of the flow into the main stream. The main features of this method are succinctly laid down in Section 2.3.

2.2 Limitations of Prandtl's Boundary Layer Theory

1. The order of the Prandtl's boundary layer equations governing the flow past a flat plate is one less than the order of the full Navier-Stokes equations and hence one boundary condition has to be relaxed. This boundary condition corresponds to the one imposed on the velocity in the normal direction to the plate at infinity. As a consequence of this situation, this normal velocity component $v_\infty = 0.8604U\sqrt{\nu/xU}$ and thus the smooth transition into the Eulerian flow away from the boundary is not accomplished.

2. The number of the truncated boundary layer equations becomes less than the number of the unknowns--the normal and tangential velocity components and the pressure distribution. This mathematical indeterminacy has been overcome hypothetically by using the pressure either from the perfect fluid theory or from experiments.

3. A key assumption made in Prandtl's boundary layer theory is that the ratio of the viscous to inertia forces is uniformly of the order of unity within the boundary layer. This assumption is not quite reasonable since the viscous effects as compared to inertia effects are predominant near the boundary but negligibly small not very far from it. In fact, there exists a sudden drop from an infinite value on the boundary to a sufficiently small quantity at the edge of the boundary layer.

4. The above-mentioned assumption that viscous and inertia forces are of the same order in the boundary layer fixes the order of the boundary layer thickness once and for all as $R^{-1/2}$, whereas the edge of the boundary layer is an arbitrary line in the fluid such that the viscous effects dominant near the boundary die out rapidly as we proceed way from it, and that the vorticity is very large near the boundary and vanishes asymptotically away from it. Interestingly enough, Proudman [16] has taken the order of boundary layer thickness to lie between $R^{-1/3}$ and $R^{-1/4}$. Hence, we may say that there is a definite need to reexamine the concept of boundary layer thickness.

5. The equation of motion governing the normal velocity component in the Navier-Stokes equations is totally dropped, but the order of the normal velocity v is not prescribed with respect to the normal length y.

6. The boundary layer solutions based on Prandtl's theory fail to show, for example, the formation of Cotes' spirals in a flow past a flat plate, which is a commonly occurring transition phenomenon observed in experiments.

7. Boundary layer equations of Prandtl are valid only from a region away from the leading edge up to the point of separation of the boundary layer. No solutions are possible outside this region.

8. In spite of the reduction of order of the field equations

and reduced complexity of the boundary layer equations no exact solution is available for Prandtl's equations. Series solutions are obtained but their convergence is not proved satisfactorily.

2.3 Advantages of the Synthetic Method for Boundary Layer Analysis

1. The field equations of a general viscous flow—the Navier-Stokes equations and the equation of energy—are retained in full and they are further extended by introducing some arbitrary external constraining force and energy supply.

2. Since no exact solutions of the above system are available and no approximate solutions are sought, plausible flow patterns are to be constructed and fitted into the field equations and boundary conditions.

3. The vanishing of the constraining force and energy supply everywhere ensures the correctness of the solutions.

4. The force and energy supply thus introduced vanish exactly only in the case of inviscid or vortex motion and in the case of uniform temperature distribution, while for a general viscous fluid theory they can be made to vanish only asymptotically. Thus the actual boundary layer solutions can be identified with asymptotic limiting cases of an extended field. This is in full agreement with the observation that the boundary layer formation is an asymptotic phenomenon.

5. The synthetic method not only provides a flow field capable of smooth transition into the main stream flow in both velocity and temperature, but also indicates the existence of a boundary layer analytically rather than hypothetically.

III. NAVIER-STOKES FLUIDS AS A SUBCLASS OF THERMOVISCOUS FLUIDS

3.1 Preliminary Remarks

The present chapter provides the mechanics, thermodynamics and constitutive equations of nonpolar (no couple stress or body couples exist) thermoviscous fluids, of which the Navier-Stokes fluids form a subclass. The approach is that of continuum theory. In Section 3.2 we present a quick review of balance laws for mass, momenta and energy, and axiomatize a principle of entropy. We then define temperature-rate-independent thermoviscous fluids and formulate the constitutive relations accordingly in Section 3.3. Finally, in Section 3.4, the linear thermoviscous constitutive theory is formulated and subsequently the celebrated Navier-Stokes equations and the equation of energy are derived based on the linear constitutive theory in anticipation of the problem to be treated in the next chapter.

3.2 Basic Principles of Continuum Thermo-Mechanics

All materials treated as continua are assumed to obey a certain set of fundamental balance laws. The basic axioms of thermo-mechanics consist of conservation of mass, balance of momentum, balance of moment of momentum, conservation of energy and law of entropy.

The intergral formulations of the mechanical balance laws may be written in cartesian tensor notation as [17]:

Conservation of Mass:

$$\frac{d}{dt}\int_{V-\sigma} \rho dv = 0 \tag{3.2.1}$$

Balance of Momentum:

$$\frac{d}{dt}\int_{V-\sigma} \rho v_\ell dv = \int_{S-\sigma} t_{k\ell} da_k + \int_{V-\sigma} \rho \; f_\ell dv \tag{3.2.2}$$

Balance of Moment of Momentum:

$$\frac{d}{dt}\int_{V-\sigma} \rho e_{k\ell m} X_\ell v_m dv = \int_{S-\sigma} e_{k\ell m} X_\ell t_{nm} da_n$$

$$+ \int_{V-\sigma} \rho e_{k\ell m} X_\ell f_m dv \quad , \tag{3.2.3}$$

where $\rho \equiv$ mass density,

$t_{k\ell} \equiv$ stress tensor,

$f_\ell \equiv$ body force density,

$v_k \equiv$ velocity vector,

$X_\ell \equiv$ position vector, and

$e_{k\ell m} \equiv$ alternating tensor.

The balance laws (3.2.1) - (3.2.3) are formulated over a material volume V, enclosed by a surface S, which is being swept by a discontinuity surface σ with a velocity $\underset{\sim}{\nu}$. In continuum mechanics it is axiomatized that the balance laws (3.2.1) - (3.2.3) are valid for every part of the material volume V. Hence, the local balance laws are thus obtained:

Conservation of Mass:

$$\dot{\rho} + (\rho v_k)_{,k} = 0 \quad \text{in} \quad V - \sigma \tag{3.2.4}$$

$$\underset{\sim}{[}\rho\,(v_k - \nu_k)\,\underset{\sim}{]}\, n_k = 0 \quad \text{on } \sigma \tag{3.2.5}$$

Balance of Momentum:

$$t_{k\ell,k} + \rho(f_\ell - \dot{v}_\ell) = 0 \text{ in } V - \sigma \tag{3.2.6}$$

$$\underset{\sim}{[}t_{k\ell} - \rho v_\ell\,(v_k - \nu_k)\,\underset{\sim}{]}\, n_k = 0 \text{ on } \sigma \tag{3.2.7}$$

Balance of Moment of Momentum:

$$t_{k\ell} = t_{\ell k} \quad \text{in } V - \sigma \quad , \tag{3.2.8}$$

where n_k is the exterior unit normal vector to σ. The brackets $\underset{\sim}{[} \quad \underset{\sim}{]}$ indicate the jump across σ of the appropriate material properties as the discontinuity surface is approached from positive and negative sides of $\underset{\sim}{n}$ of σ. In the above equations a superposed dot indicates material differentiation and a subscript following a comma denotes partial differentiation, e.g.

$$\dot{v}_\ell \equiv \left.\frac{\partial v_\ell}{\partial t}\right|_{\underset{\sim}{X}} = \frac{\partial v_\ell}{\partial t} + v_{\ell,m}v_m$$

$$v_{k,\ell} \equiv \frac{\partial v_k}{\partial x_\ell} \quad .$$

The jump conditions (3.2.5) and (3.2.7) give the boundary conditions on the surface S of the body when σ is taken to coincide with S. In this case, (3.2.5) becomes an identity and (3.2.7) gives

$$t_{k\ell}n_k = t_{(\underset{\sim}{n})\ell} \quad \text{on } S \; , \tag{3.2.9}$$

where $t_{(\underset{\sim}{n})\ell}$ is the surface traction on S.

We shall now state the two fundamental laws of thermodynamics. The first law of thermodynamics is the law of conservation of energy. The integral form of this law is expressed as [17]:

Conservation of Energy:

$$\frac{d}{dt}\int_{V-\sigma} \rho\,(\varepsilon + \frac{1}{2}\, v_k v_k)dv = \int_{S-\sigma} (t_{k\ell}v_\ell + q_k)da_k$$

$$+ \int_{V-\sigma} \rho(f_k v_k + h)dv \quad , \tag{3.2.10}$$

where $\varepsilon \equiv$ total internal energy density,

$q_k \equiv$ total energy flux, and

$h \equiv$ energy supply density.

When Equation (3.2.10) is postulated to be valid for every part of the body, we obtain the localized form of conservation of energy:

Conservation of Energy:

$$\rho\dot{\varepsilon} = t_{k\ell}v_{\ell,k} + q_{k,k} + \rho h, \text{ in } V-\sigma \tag{3.2.11}$$

$$\underset{\sim}{[}\ \rho(\varepsilon + \frac{1}{2} v_k v_k)(v_k - \nu_k) - t_{k\ell}v_\ell - q_k\ \underset{\sim}{]}\ n_k = 0 \text{ on } \sigma. \tag{3.2.12}$$

The heat flux $q_{(\underset{\sim}{n})}$ on the surface S may be prescribed from (3.2.12) by taking σ as S:

$$\underset{\sim}{q} \cdot \underset{\sim}{n} = q_{(\underset{\sim}{n})} \quad \text{on} \quad S \quad , \tag{3.2.13}$$

where $q_{(\underset{\sim}{n})}$ is the normal component of the surface heat vector.

The second law of thermodynamics is stated in the form of a Clausius-Duhem inequality. It is postulated that the time rate of change of the total eutropy H is never less than the sum of entropy influx $\underset{\sim}{S}$ through the surface S of the body and the volume entropy supply B in the body. According to this postulate, we get

$$\Gamma \equiv \frac{dH}{dt} - B - \int_{S-\sigma} \underset{\sim}{S} \cdot d\underset{\sim}{a} \geq 0, \tag{3.2.14}$$

where Γ so defined is called total entropy production. For a continuous mass medium this can be expressed as

$$\frac{d}{dt}\int_{V-\sigma}\rho\eta\, dv - \int_{V-\sigma}\frac{\rho h}{\theta}\, dv - \int_{S-\sigma}\frac{\tilde{q}}{\theta}\cdot da \geq 0, \qquad (3.2.15)$$

where η is the entropy density and $\theta > 0$ is the absolute temperature; $\frac{\tilde{q}}{\theta}$ and $\frac{h}{\theta}$ are, respectively, the entropy influx and entropy sources due to heat flux. We mention in passing that it is always possible to modify these relations by taking into consideration all other effects [17]. Localizing (3.2.15) in the same way as before, we obtain

$$\rho\gamma \equiv \rho\dot{\eta} - \frac{\rho h}{\theta} - \left(\frac{q_k}{\theta}\right)_{,k} \geq 0 \quad \text{in } V - \sigma \qquad (3.2.16)$$

$$\underset{\sim}{[}\rho\eta(v_k - \nu_k) - \frac{q_k}{\theta}\underset{\sim}{]}\, n_k \geq 0 \quad \text{on } \sigma\ , \qquad (3.2.17)$$

where γ is the entropy production density. Eliminating the heat source term h between (3.2.11) and (3.2.16), we get the Clausius-Duhem inequality for a simple thermodynamical process:

$$\rho\gamma \equiv \rho(\dot{\eta} - \frac{\dot{\varepsilon}}{\theta}) + \frac{1}{\theta}\, t_{k\ell}v_{\ell,k} + \frac{1}{\theta^2}q_k\theta_{,k} \geq 0 \text{ in } V\text{-}\sigma \qquad (3.2.18)$$

$$\underset{\sim}{[}\rho\eta(v_k - \nu_k) - \frac{q_k}{\theta}\underset{\sim}{]}\, n_k \geq 0 \text{ on } \sigma\ . \qquad (3.2.19)$$

Introducing the free energy function ψ by

$$\psi \equiv \varepsilon - \theta\eta ,$$

we can write (3.2.18) in terms of the free energy as

$$-\frac{\rho}{\theta}(\dot{\psi}+\eta\dot{\theta}) + \frac{1}{\theta} t_{k\ell}v_{\ell,k} + \frac{1}{\theta^2} q_k \theta_{,k} \geq 0 \qquad (3.2.20)$$

On the surface S of the body, (3.2.19) reduces to

$$\underset{\sim}{[} \frac{q_k}{\theta} \underset{\sim}{]} n_k \geq 0 \text{ on S.} \qquad (3.2.21)$$

The balance laws, including the Clausius-Duhem inequality, must be satisfied by all materials regardless of their nature, i.e., fluids, solids, etc.

3.3 Constitutive Relations of a Temperature-Rate-Independent Thermoviscous Fluid

In order to describe the nature and response of a material, we need not only the balance laws discussed in the last section but also the constitutive make-up of the material. Thus there arises a need to incorporate material characterization in order to bring out the various types of responses experienced by different types of materials subjected to the same external stimuli. Such a process necessarily involves the cause and effect relationships, leading to the constitutive theory for the materials.

In this thesis we consider a constitutive theory of temperature-rate-independent thermoviscous fluids in which constitutive functionals reduce to ordinary functions of the following arguments [18]:

$$\rho,\ v_{k,\ell},\ \theta,\ \theta_{,k}\ . \tag{3.3.1}$$

Thus we assume that the constitutive functions for the free energy, entropy, heat flux vector, and stress tensor are given by

$$\psi = \hat{\psi}\ (\rho,\ v_{k,\ell}\ ,\ \theta,\ \theta_{,k}) \tag{3.3.2}$$

$$\eta = \hat{\eta}\ (\rho,\ v_{k,\ell}\ ,\ \theta,\ \theta_{,k}) \tag{3.3.3}$$

$$q_k = \hat{q}_k\ (\rho,\ v_{k,\ell}\ ,\ \theta,\ \theta_{,k}) \tag{3.3.4}$$

$$t_{k\ell} = \hat{t}_{k\ell}(\rho,\ v_{k,\ell}\ ,\ \theta,\ \theta_{,k}) \tag{3.3.5}$$

These constitutive functions are subject to the axiom of objectivity, namely, they should remain invariant under a change of spatial frame of the form

$$\underset{\sim}{x}' = \underset{\sim}{Q}(t)\underset{\sim}{x} + \underset{\sim}{b}(t)\ , \tag{3.3.6}$$

where $\underset{\sim}{b}(t)$ is an arbitary time-dependent vector and $\underset{\sim}{Q}(t)$ is an arbitrary time-dependent orthogonal tensor, i.e., $\det(\underset{\sim}{Q}(t)) = \pm 1$. Therefore, we have as a consequence of the axiom of objectivity:

$$\psi = \tilde{\psi}\ (\rho,\ d_{k\ell},\ \theta,\ \theta_{,k}) \tag{3.3.7}$$

$$\eta = \tilde{\eta}(\rho, d_{k\ell}, \theta, \theta_{,k}) \tag{3.3.8}$$

$$q_k = \tilde{q}_k(\rho, d_{k\ell}, \theta, \theta_{,k}) \tag{3.3.9}$$

$$t_{k\ell} = \tilde{t}_{k\ell}(\rho, d_{k\ell}, \theta, \theta_{,k}) \quad , \tag{3.3.10}$$

where

$$d_{k\ell} = \frac{1}{2}(v_{k,\ell} + v_{\ell,k}) \tag{3.3.11}$$

is the well-known deformation-rate tensor.

A further consequence of the above invariance requirement is that the constitutive functions must be isotropic functions relative to the full group of all orthogonal transformations. Therefore, we require:

$$\tilde{\psi}(\rho, \underset{\sim}{Q}\, \underset{\sim}{d}\, \underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\nabla\theta) = \tilde{\psi}(\rho, \underset{\sim}{d}, \theta, \nabla\theta) \tag{3.3.12}$$

$$\tilde{\eta}(\rho, \underset{\sim}{Q}\, \underset{\sim}{d}\, \underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\nabla\theta) = \tilde{\eta}(\rho, \underset{\sim}{d}, \theta, \nabla\theta) \tag{3.3.13}$$

$$\tilde{\underset{\sim}{q}}(\rho, \underset{\sim}{Q}\, \underset{\sim}{d}\, \underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\nabla\theta) = \underset{\sim}{Q}\, \tilde{\underset{\sim}{q}}\, (\rho, \underset{\sim}{d}, \theta, \nabla\theta) \tag{3.3.14}$$

$$\tilde{\underset{\sim}{t}}(\rho, \underset{\sim}{Q}\, \underset{\sim}{d}\, \underset{\sim}{Q}^T, \theta, \underset{\sim}{Q}\nabla\theta) = \underset{\sim}{Q}\, \tilde{\underset{\sim}{t}}\, (\rho, \underset{\sim}{d}, \theta, \nabla\theta)\, \underset{\sim}{Q}^T, \tag{3.3.15}$$

where $\underset{\sim}{Q}^T$ represents the transpose of the matrix $\underset{\sim}{Q}$ and ∇ is the gradient operator.

Assuming that these functions are polynomials, they may be expressed more explicitly using appropriate representation theorems [19] as follows:

$$q_k = (\kappa_0 \delta_{k\ell} + \kappa_1 d_{k\ell} + \kappa_2 d_{km} d_{\ell m})\theta_{,\ell} \tag{3.3.16}$$

$$t_{k\ell} = \alpha_0\delta_{k\ell} + \alpha_1 d_{k\ell} + \alpha_2 d_{km}d_{m\ell} + \alpha_3\theta_{,k}\,\theta_{,\ell}$$

$$+ \frac{1}{2}\alpha_4(\theta_{,k}d_{m\ell} + \theta_{,\ell}\, d_{mk})\,\theta_{,m} \tag{3.3.17}$$

$$+ \frac{1}{2}\alpha_5\, d_{mn}\theta_{,m}\,(\theta_{,k}d_{n\ell} + \theta_{,\ell}d_{nk}) ,$$

where κ_r and α_r are polynomials in the invariants:

$$I_1 \equiv d_{kk} ,\ I_2 \equiv d_{k\ell}d_{\ell k},\ I_3 \equiv d_{k\ell}d_{\ell m}d_{mk}$$

$$I_4 \equiv \theta_{,k}\,\theta_{,k} ,\ I_5 \equiv d_{k\ell}\theta_{,k}\,\theta_{,\ell},$$

$$I_6 \equiv d_{k\ell}d_{\ell m}\,\theta_{,m}\,\theta_{,k} , \tag{3.3.18}$$

i.e.,

$$\kappa_r = \kappa_r(\rho,\theta;\ I_1,I_2,\ldots,I_6),\ r = 0,1,2 \tag{3.3.19}$$

$$\alpha_r = \alpha_r(\rho,\theta;\ I_1,I_2,\ldots,I_6),\ r = 0,1,2,3,4,5 \tag{3.3.20}$$

Similarly, we have

$$\psi = \psi\ (\rho,\theta:I_1,I_2,\ldots,I_6) \tag{3.3.21}$$

$$\eta = \eta\ (\rho,\theta:I_1,I_2,\ldots,I_6). \tag{3.3.22}$$

We now make the rather strong assumption that (3.2.18) is to be postulated for all admissible thermodynamical processes characterized by the constitutive Equations (3.3.7) - (3.3.10). This requirement places further restrictions on the forms of the constitutive functions. In order to enforce this restriction we proceed as follows:

The local form of the entropy inequality (3.2.20) may be written in a slightly different form as

$$- \rho(\dot{\psi} + \eta\dot{\theta}) + t_{k\ell}d_{k\ell} + \frac{q_k}{\theta}\,\theta_{,k} \geq 0 \quad . \tag{3.3.23}$$

Here we have utilized the symmetry property of $t_{k\ell}$ and $\theta > 0$. In view of (3.3.7), we may write (3.3.23) explicitly in the form:

$$-\rho\left(\frac{\partial\psi}{\partial\rho}\dot{\rho} + \frac{\partial\psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,k}}\dot{\overline{\theta_{,k}}} + \eta\,\dot{\theta}\right)$$
$$+ t_{k\ell}d_{k\ell} + \frac{q_k}{\theta}\,\theta_{,k} \geq 0. \tag{3.3.24}$$

Through the continuity Equation (3.2.4), the above inequality (3.3.24) becomes

$$-\rho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} - \rho\left(\frac{\partial\psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{\partial\psi}{\partial\theta_{,k}}\dot{\overline{\theta_{,k}}}\right)$$
$$+ \left(t_{k\ell} + \rho^2\frac{\partial\psi}{\partial\rho}\delta_{k\ell}\right)d_{k\ell} + \frac{q_k}{\theta}\,\theta_{,k} \geq 0. \tag{3.3.25}$$

Since the inequality (3.3.25) is linear in the quantities $\dot{d}_{k\ell}$, $\dot{\overline{\theta_{,k}}}$, and $\dot{\theta}$, they can be chosen arbitrarily and independently of the other terms in the inequality. For (3.3.25) to be maintained for all such independent variations of the processes involved, we should have

$$\frac{\partial\psi}{\partial d_{k\ell}} = 0 \quad , \quad \frac{\partial\psi}{\partial\theta_{,k}} = 0 \tag{3.3.26}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} \quad . \tag{3.3.27}$$

Equations (3.3.26) lead to the conclusion that

$$\psi = \psi(\rho,\theta) \quad . \tag{3.3.28}$$

(3.3.25) now reduces to

$$(t_{k\ell} + \rho^2 \frac{\partial\psi}{\partial\rho}\delta_{k\ell})d_{k\ell} + \frac{q_k}{\theta}\theta_{,k} \geq 0. \tag{3.3.29}$$

Writing

$$t_{k\ell} = -\pi\rho_{k\ell} + {}_D f_{k\ell} \tag{3.3.30}$$

so as to introduce the dissipative part of the stress tensor

$${}_d f_{k\ell} = {}_D \tilde{f}_{k\ell}\,(\rho,\ \underset{\sim}{d},\ \theta,\ \nabla\theta) \tag{3.3.31}$$

and the thermodynamic pressure

$$\pi(\rho,\theta) \equiv \rho^2 \frac{\partial\psi}{\partial\rho} \quad , \tag{3.3.32}$$

the inequality (3.3.29) becomes

$$\rho\gamma \equiv {}_D f_{k\ell} d_{k\ell} + \frac{q_k}{\theta} \theta_{,k} \geq 0 \qquad (3.3.33)$$

for all independent variations of $\underset{\sim}{d}$ and $\nabla\theta$. In terms of the polynomial constitutive fucntions (3.3.16) and (3.3.17) the dissipation inequality (3.3.33) places severe restrictions on the coefficients κ_r and α_r :

$$\bar{\alpha}_o I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \frac{\kappa_o}{\theta} I_4 + (\alpha_3 + \frac{\kappa_1}{\theta}) I_5$$

$$+ (\alpha_4 + \frac{\kappa_2}{\theta}) I_6 + \alpha_5 (I_1 I_6 - \frac{1}{2} I_1{}^2 I_5 + \frac{1}{2} I_2 I_5$$

$$+ \frac{1}{6} I_1^3 I_4 - \frac{1}{2} I_1 I_2 I_4 + \frac{1}{3} I_3 I_4) \geq 0 \qquad (3.3.34)$$

where $\bar{\alpha}_o \equiv \alpha_o + \pi$.

In obtaining (3.3.34) we used the Cayley-Hamilton theorem in three dimensions. In addition, ${}_D\underset{\sim}{f} = \underset{\sim}{0}$ and $\underset{\sim}{q} = \underset{\sim}{0}$ when $\underset{\sim}{d} = \underset{\sim}{0}$ and $\nabla\theta = \underset{\sim}{0}$.

3.4 Navier-Stokes Equations and Energy Equation

The master Equations (3.3.16) and (3.3.17) can be employed to derive various approximate constitutive theories. Here we are only concerned with the linear theory in that Equations (3.3.12) - (3.3.15) are linear in the deformation-rate tensor and temperature gradient. In this case, we have

$$\psi = \psi(\rho,\theta) \qquad (3.4.1)$$

$$\eta = \eta(\rho,\theta) \tag{3.4.2}$$

$$q_k = \kappa(\rho,\theta)\theta_{,k} \tag{3.4.3}$$

$$t_{k\ell} = [-\pi(\rho,\theta) + \lambda(\rho,\theta)I_1]\delta_{k\ell} + 2\mu(\rho,\theta)d_{k\ell} \; . \tag{3.4.4}$$

Equations (3.4.1) and (3.4.2) already satisfy the requirements of the entropy inequality (3.3.25). The entropy η is derivable from (3.4.1) by direct differentiation with respect to θ. Equation (3.4.3) is of the form of Fourier law of heat conduction, where $\kappa(\rho,\theta)$ plays the role of heat conductivity. Equation (3.4.4) has the form of the generalized Newtonian law of viscosity with $\lambda(\rho,\theta)$ and $\mu(\rho,\theta)$ playing the roles of viscosity coefficients. $\pi(\rho,\theta)$ appearing in (3.4.4) and defined earlier in (3.3.32) plays the role of the thermodynamic pressure. Thus, ${}_Df_{k\ell}$ can now be identified, in view of (3.3.30), by

$${}_df_{k\ell} = \lambda(\rho,\theta)I_1\delta_{k\ell} + 2\mu(\rho,\theta)d_{k\ell}, \tag{3.4.5}$$

which plays the role of the viscous stress tensor. The dissipation inequality (3.3.33) now becomes

$$\rho\gamma = (\lambda + \tfrac{2}{3}\mu)I_1^2 + 2\mu\bar{d}_{k\ell}\bar{d}_{\ell k} + \frac{\kappa}{\theta}\,\theta_{,k}\theta_{,k} \geq 0 \; , \tag{3.4.6}$$

where $\bar{d}_{k\ell} \equiv d_{k\ell} - \frac{1}{3}I_1\delta_{k\ell}$ is the deviatoric part of the deformation-rate tensor $d_{k\ell}$.

In order that the entropy production γ may be a positive semi-definite function of I_1, $\overline{d}_{k\ell}$, and $\theta_{,k}$, the entropy inequality requires

$$3\lambda + 2\mu \geq 0 \ , \ \mu \geq 0 \ , \ \kappa \geq 0 \ . \tag{3.4.7}$$

The celebrated Navier-Stokes equations may now be obtained by replacing the term in the equation of motion (3.2.6) involving the stress tensor $t_{k\ell}$ by the constitutive relation (3.4.4):

$$- \pi_{,k} + (\lambda+\mu)\, v_{\ell,\ell k} + \mu v_{k,\ell\ell} + \rho f_k$$

$$= \rho\frac{\partial v_k}{\partial t} + \rho v_{k,\ell} v_\ell = \rho \dot{v}_k \ . \tag{3.4.8}$$

The equation of energy is obtained by substituting $\varepsilon = \psi + \theta\eta$ into (3.2.11) and using (3.3.32), (3.4.1), and (3.4.2):

$$\rho\theta\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta} - \rho^2\theta\frac{\partial^2\psi}{\partial\theta\partial\rho}d_{kk} + \lambda(d_{kk})^2 + 2\mu d_{k\ell}d_{\ell k}$$

$$+ (\kappa\theta_{,k})_{,k} + \rho h = 0 \ . \tag{3.4.9}$$

For the incompressible fluids we have the internal constraints that ρ = constant = ρ_o . Hence $\frac{\partial\psi}{\partial\rho}$ in (3.3.32) is undefined. The thermodynamic pressure π is replaced by an unknown pressure $p(\underset{\sim}{x},t)$, which is to be determined through solutions of the field equations under a given set of boundary conditions. In this case, λ and μ are functions of θ only. The incompressibility condition is equivalent to setting $I_1 = 0$. Incompressible thermoviscous fluid dynamics, linear in stress constitutive equations, is thus based on the field equations (3.2.4), (3.4.8), and (3.4.9) for the five unknowns: p, v_k, and θ.

IV. THERMAL BOUNDARY LAYER TRANSITION

4.1 Preliminary Remarks

In the present chapter we turn our attention to the problem of thermal boundary layer transition for a viscous, incompressible, steady laminar flow past a flat plate with heat transfer due to forced convection.

Before going any further we find it worthwhile here to highlight the main influence of Prandtl's discovery of the boundary layer theory on various aspects of theoretical development of fluid dynamics and flow measurements. Following Prandtl's development of boundary layer theory, both its mathematical and physical implications have become the subject of intense investigation by many theorists and experimentalists. The reason for such a sustaining interest is two-fold—one lies in its outstanding success in opening out an immense possibility of treating the still indomitable Navier-Stokes equations for their solution along with the prospects of encountering many interesting mathematical challenges and the other lies in harvesting the technological and experimental outcome of this solution in design work in a variety of fields including aeronautics, locomotion and other self-propelled systems.

Prandtl's derivation of the boundary layer equations from the full Navier-Stokes equations governing a two-dimensional incompressible steady laminar flow of a viscous fluid past a flat plate is based on a consideration of relative orders of magnitude of various terms near and away from the solid boundary. This procedure resulted not only in the elimination of certain terms from the Navier-Stokes equations but also in dropping the equation governing the velocity component normal to the wall altogether. The resulting number of equations fell short of the number of unknowns thus causing a mathematical indeterminacy. This indeterminacy led Prandtl to borrow for the unknown pressure inside the boundary layer the pressure from the perfect fluid theory or from experiments. It was pointed out by Seth [1] that the reduced boundary layer equations of Prandtl provide a solution only in the thin region surrounding the body; however, the smooth transition into the free stream flow region away from the boundary is not accomplished. Moreover, the boundary layer solutions obtained on this basis resulted in serious error in so far as it makes the velocity component normal to the wall infinite at infinity. As observed by Seth [1], a careful examination of Prandtl's order of magnitude analysis reveals that it assumes that the ratio of the viscous to inertia terms is of order unity inside the boundary layer while actually this ratio undergoes rather a drastic variation from infinite value near the boundary to almost zero at the outer edge of the boundary layer. Also, this order approximation procedure results in presupposing the order of the thickness of the boundary layer. When a fluid flows past a solid body the nonviscous motion away from the body is irrotational. This means that all the vorticity is confined within the thin boundary layer region. Yet the vorticity does not vanish abruptly, it dies out asymptotically. Hence, the formula for the thickness of the boundary layer should be dependent on what order of vorticity one is prepared to regard as negligible at the edge of the boundary layer. Furthermore, in spite of the reduction and approximation procedures employed, the classical boundary layer equations are still nonlinear and are further approximated in most solution processes leading to results having serious defects of their own (see reference [1] for complete discussion).

The above considerations led Seth [2] to trace the source of the inherent problems with the classical boundary layer theory to the procedure of narrowing down or reducing the field equations on one or more assumptions. By reducing the field of investigation to a particular aspect of an event, a complete description is not obtained and a number of important effects remain uncharted. Consequently, he initiated the concept of the synthetic method which consists in extending the field by means of a few arbitrary forces introduced into the full Navier-Stokes equations. It is to be noted that the introduction of such body forces will make the already complicated Navier-Stokes equations even more formidable unless the choice of the arbitrary terms added is made in such a manner that the integration of the equations of motion is rendered possible. This will then permit plausible solutions to be selected and fitted so as to satisfy exactly the boundary conditions and the equations of motion. The vanishing of the arbitrary forces at all points ensures the correctness of the solution. But this is found to vanish exactly only in the case of nonviscous and vortex motion, while for a general viscous fluid it can be made to vanish asymptotically. This indicates the existence of a thin region near the boundary where viscous effects are predominant and this region is identified as the boundary layer region. Thus the actual boundary layer solution is obtained as the limiting case of an extended field. This viewpoint is in full agreement with the observation made by Friedrichs [6] that the boundary layer formation is an asymptotic phenomenon. The synthetic method developed by Seth [1,2,3,4] has indeed met with great success in treating the Navier-Stokes equations in their complete form for several fluid flow problems [20,21,22].

In this chapter, we investigate the problem of a general steady laminar incompressible viscous fluid flow past a flat plate with heat transfer due to forced convection, our main objective being the study of the thermal boundary layer formation on the flat plate. Our approach consists of an extension of the synthetic method,described earlier, to the problem on hand by introducing in addition to anexternal constraining force an external constraining energy supply into the appropriate field equations governing the flow. We then relax the external constraints thus introduced by requiring them to vanish asymptotically everywhere in the field which, as we find in the course of our analysis, leads to the actual thermal boundary layer solution arising as a limiting case of an extended field.

4.2 Formulation of the Problem

We consider a two-dimensional steady laminar flow of a viscous incompressible fluid with constant properties past a flat plate with heat transfer due to forced convection. We choose the x-axis along the length of the plate with the origin coinciding with the leading edge. The y-axis is chosen normal to the plate and the undisturbed flow is along the x-direction. At large Reynolds numbers and small temperature differences, the buoyancy forces are negligible in comparison with the inertia and viscous forces, the velocity field no longer depends on the temperature field but the latter does on the former; such flows are termed forced flows. The governing field equations of motion, continuity and energy for such flows are,(Landau [23]), respectively, in the absence of body forces and external energy supply:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u, \qquad (4.2.1)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v, \qquad (4.2.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (4.2.3)$$

$$u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = \frac{\nu}{c} \Phi + a\nabla^2 \phi, \tag{4.2.4}$$

where $u = u(x,y)$, $v = v(x,y)$, $\phi = \phi(x,y)$ are, respectively, the x- and y-components of the velocity, and the temperature field. p is the pressure, ν = kinematic viscosity coefficient; $a = \kappa/\rho c$ is the thermometric conductivity of the fluid, κ, ρ and c being, respectively, the thermal conductivity, the mass density, and specific heat of the fluid. The function $\Phi = \Phi(x,y)$ is the viscous dissipation function given by

$$\Phi = 2 \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right\} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 .$$

The boundary conditions on the velocity field for our problem are

$$u = 0 = v \text{ on } y = 0, \tag{4.2.5}$$

$$u = U,\ v = 0 \text{ at } y = \infty , \tag{4.2.6}$$

where U is the constant velocity of the undisturbed flow at infinity in the x-direction.

The variety of possible sets of boundary conditions is much greater for the temperature field than for the velocity field. In the present thesis, we shall treat the so-called adiabatic wall problem, namely, the heated fluid flows past a thermally insulated plate. This problem thus corresponds to the boundary condition

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0 , \tag{4.2.7}$$

that is, the condition that there is no heat flux through the surface of the plate. Finally, we assume the boundary condition on the temperature field at infinity to be

$$\phi = \phi_{\infty} \text{ (= constant) at } y = \infty . \qquad (4.2.8)$$

As remarked before, we note from Equations (4.2.1) - (4.2.3) that the velocity field no longer depends on the temperature field but the latter does on the former. This observation will permit us to take advantage of the already available Seth's synthetic method of solution [1] for treating the Navier-Stokes Equations (4.2.1) - (4.2.3) to obtain the viscous boundary layer solution for the flat plate case. Our main concern in this thesis is the investigation of the thermal boundary layer formation on the flat plate. For this purpose we need to extend the synthetic method further for treating the energy Equation (4.2.4). Before doing so, we consider it worthwhile presenting an improved version of Seth's synthetic method of solution for the viscous boundary layer for the sake of providing continuity of ideas for the convenience of our readers.

4.3 Synthetic Method for Viscous Flow Along a Flat Plate

Seth's extended field equations [1] can be obtained by introducing constraining force components X and Y, respectively, into (4.2.1) and (4.2.2) to obtain

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u ,$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v , \qquad (4.3.1)$$

and (4.2.3) can be satisfied by a stream function $\psi = \psi(x,y)$ such that

$$u = \frac{\partial\psi}{\partial y} , \quad v = -\frac{\partial\psi}{\partial x} . \qquad (4.3.2)$$

Substituting (4.3.2) into (4.3.1) and eliminating p by cross-differentiation we obtain

$$\nu\nabla^4\psi + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} . \qquad (4.3.3)$$

Choosing the similarity transformations defined by

$$\eta = \frac{y}{\sqrt{Lx}} , \quad \psi = U\sqrt{Lx}f(\eta) \qquad (4.3.4)$$

where L is some length measured along the plate, we obtain

$$u = Uf' , \quad v = \frac{1}{2} U\sqrt{\frac{L}{x}}(\eta f' - f) \qquad (4.3.5)$$

the primes denoting differentiation with respect to η. Substituting (4.3.5) into (4.3.3) we obtain

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = \frac{\nu U}{(Lx)^{3/2}} \Big[f^{iv} + \left(\frac{L}{2x}\right)(\eta^2 f^{iv} + 5\eta f''' + 3f'')$$

$$+ \left(\frac{L}{4x}\right)^2 (\eta^2 f^{iv} + 14\eta^3 f''' + 45\eta^2 f'' + 15\eta f' - 15f) \Big]$$

$$+ \frac{U^2}{2\sqrt{Lx}^3} \Big[ff''' + f'f'' + \left(\frac{L}{4x}\right)(\eta^2 ff''' + 3\eta^2 f'f''$$

$$+ 3\,ff'' + 3\eta f'^2 - 3ff') \Big] . \qquad (4.3.6)$$

Equation (4.3.6) can be satisfied if we choose

$$Y = 0 \text{ for all } x \text{ and } \eta$$

and

$$X = -\frac{U^2}{2x} \Big[\left(\frac{2}{R} f''' + ff''\right) + \left(\frac{L}{4x}\right) \left\{ \frac{4}{R}(\eta S'' + 2S')\right.$$

$$\left. - \eta (fS)' + 2fS \right\} + \left(\frac{L}{4x}\right)^2 \frac{2}{R} \{\eta^3 S'' + 9\eta^2 S'$$

$$+ 15\eta S\} \Big] , \qquad (4.3.7)$$

where

$$S = S(\eta) = \eta f' - f, \; R = UL/\nu = \text{Reynolds number}, \qquad (4.3.8)$$

and, as before, the primes in (4.3.7) and (4.3.8) denote differentiation with respect to η.

The force X also vanishes if

$$S = \eta f' - f = 0, \text{ or } f = A\eta, \qquad (4.3.9)$$

which gives the non-viscous solution, where A is an arbitrary constant of integration.

It can readily be seen that the first bracketed expression on the right-hand side in (4.3.7) set equal to zero, namely,

$$\frac{2}{R} f''' + ff'' = 0,$$

yields the well-known Blasius equation whose solution is defective in that the velocity in the direction normal to the plate does not vanish at infinity. Using (4.3.5) the boundary conditions (4.2.5) and (4.2.6) can be written as

$$f(0) = 0 \ , \ \lim_{\eta \to 0} [\eta f' - f] = 0 \ ,$$

$$f'(\infty) = 1 \ , \ \lim_{\eta \to \infty} [\eta f' - f] = 0. \qquad (4.3.10)$$

For a general viscous solution holding near and away from the boundary, in view of the boundary condition at infinity, $f'(\infty) = 1$, and the retarded-motion consideration near the boundary we choose the following form for the similarity velocity profile $f'(\eta)$:

$$f'(\eta) = 1 + F'(\eta)e^{-k\eta} \qquad (4.3.11)$$

where the function $F'(\eta)$ in (4.3.11) will be determined by the use of the boundary conditions of the problem as well as the property of smooth transition of the flow from the boundary layer region to that of the free stream; and k is as yet an undetermined parameter. On careful examination of (4.3.11), we find that in order that there may exist a boundary layer on the plate which is capable of smooth transition into the free stream flow, k is to be chosen sufficiently large so as to permit such a transition, i.e., $k \gg 1$.

In order to determine the nature of the conditions to be satisfied by the function $F(\eta)$ we integrate (4.3.11) to obtain

$$f(\eta) = \eta + \int_0^{\eta} e^{-k\tau} F'(\tau) d\tau. \tag{4.3.12}$$

Now, the consequences of the boundary conditions (4.3.10) lead to the following conditions on the functions f and F:

$$F'(0) = -1, \ f(0) = 0, \ \lim_{\eta \to \infty} F'(\eta) e^{-k\eta} = 0,$$

$$\int_0^{\infty} e^{-k\tau} F'(\tau) d\tau = 0. \tag{4.3.13}$$

For a general viscous solution holding near and away from the boundary one can satisfy all of the boundary conditions (4.3.13) by choosing the plausible form for $F'(\eta)$:

$$F'(\eta) = k\eta - 1 \tag{4.3.14}$$

and thus obtain from (4.3.11), the similarity velocity distribution $f'(\eta)$ in the form

$$f'(\eta) = 1 - (1 - k\eta)e^{-k\eta} \quad . \tag{4.3.15}$$

An integration of (4.3.15) and the use of the boundary conditions (4.3.13) yields the expression obtained by Seth [1]:

$$f(\eta) = \eta(1 - e^{-k\eta}). \tag{4.3.16}$$

As for requiring the solution (4.3.16) to satisfy the governing differential equation, it will suffice, equivalently, to consider the satisfaction of Equation (4.3.7) examining the circumstances under which X can be made to vanish asymptotically everywhere in the flow regime. We observe from (4.3.15) and (4.3.16) that since $k >> 1$, it follows that $f(\eta) \sim \eta$, $f'(\eta) \sim 1$, $f''(\eta) \sim 0$, and $f'''(\eta) \sim 0$, where the notation ~ stands for " is of the order of magnitude of ". Therefore the solution (4.3.16) approaches the irrotational value at all points except near the plate. Also, it follows from (4.3.7) that when $k >> 1$, X is small not very far from the boundary. On the plate $\eta = 0$,

$$X = - \frac{U^2}{x}\frac{f'''(0)}{R} = \frac{U^2}{x}\frac{3k^2}{R} \sim \frac{k^2}{R} \,. \tag{4.3.17}$$

This can also be made small provided we choose

$$R \sim k^{2+\lambda}, \quad \lambda > 0. \tag{4.3.18}$$

Using $f(\eta)$ given by (4.3.16) and its derivatives in the expression (4.3.7), we obtain

$$X = \frac{U^2}{x}\frac{k^2}{R}e^{-k\eta}\{[(3-k\eta)+\frac{1}{2}\frac{R}{k^2}(k^2\eta^2-2k\eta)(1-e^{-k\eta})]$$

$$+(\frac{L}{4x})^2\eta^2[2(6-k\eta-\frac{6}{k\eta})+\frac{R}{2k^2}(k^2\eta^2-5k\eta+2)$$

$$-\frac{R}{2k^2}(k\eta-2)(2k\eta-1)]+(\frac{L}{4x})^2\eta^4(13-k\eta-\frac{35}{k\eta})\}. \tag{4.3.19}$$

The first square-bracketed expression on the right-hand side of (4.3.19) is the dominant term in X for small η. Since our solution approaches the perfect fluid solution except in a thin layer close to the boundary, we only need be concerned with small values of η for which the behavior of X remains to be examined. Under the condition (4.3.18) the dominant term in the expression for X can be readily seen to vanish asymptotically and so does X at all points near the boundary. Since we already have shown that X asymptotically vanishes away from the boundary approaching the perfect fluid solution, it follows that X asymptotically vanishes at all points of the flow regime.

Next, we evaluate the vorticity from $(4.3.4)_2$ and (4.3.16) obtaining

$$\zeta^* = \frac{\zeta\sqrt{Lx}}{U} = \frac{\sqrt{Lx}}{U}\nabla^2\psi = ke^{-k\eta}[(2-k\eta)$$

$$+ \quad (\frac{L}{4x})\eta^2(3 - k\eta)]. \qquad (4.3.20)$$

We find that the vorticity $\nabla^2\psi$ is of order k near the boundary (on the boundary as well) and k is large, so that the vorticity is large near the boundary but dies out rapidly as we proceed away from the boundary, thus confirming the existence of a thin region surrounding the boundary which we define as the boundary layer region. The thickness of this layer will depend on the order of vorticity allowable outside this region. If $\eta = \delta_s$ represents the edge of the boundary layer, the vorticity outside this layer must then be vanishingly small. This is possible if $k\delta_s$ is large while δ_s is small. Or, equivalently, we have

$$\delta_s \sim k^{-m}, \; 0 < m < 1 \qquad (4.3.21)$$

which can be established as follows. Let $\delta_s \sim k^{\alpha}$ for some α. If $\alpha > 0$, then $\delta_s \sim 1$ or larger, which is impossible. Hence, $\alpha < 0$. Then we may let $\alpha = -\alpha_0$, with $\alpha_0 > 0$. If $\alpha_0 > 1$, then put $\alpha_0 = 1 + \beta$, with $\beta > 0$. Accordingly, $\delta_s \sim k^{-(1+\beta)}$, i.e., $k\delta_s \sim k^{-\beta} << 1$, since $k >> 1$. This is impossible also since $k\delta_s$ is to be large. Thus, α_0 cannot be greater than 1. We must therefore have $0 < \alpha_0 < 1$, thus establishing (4.3.21).

It now follows from (4.3.18) and (4.3.21) that the generalized formula for the boundary layer thickness is:

$$\delta_s \sim R^{-m/(2+\lambda)}, \ \lambda > 0, \ 0 < m < 1. \tag{4.3.22}$$

Hence, the thickness of the boundary layer is not of the order of $R^{-1/2}$ customarily used in literature, but is indeed of the order of $R^{-m/(2+\lambda)}$, where $\lambda > 0$ and $0 < m < 1$, so that the ratio $m/(2+\lambda) < \frac{1}{2}$. This result confirms the view of Proudman [16] that the order of thickness of the boundary layer ought to be greater than $R^{-1/2}$. Thus in this analysis the field equations and the boundary conditions are all satisfied. A single solution holding near and away from the boundary is obtained. For a given value of R, k should be taken from the order of large vorticity present near the boundary. λ is then given by (4.3.18). From the order of small vorticity allowable outside the layer we obtain m. For example, let $R \sim 10^5$ and let k, which represents the order of vorticity on the plate be 10^2. Then $\lambda = 1/2$. Further if the vorticity outside the layer be of order 0.01, m is found to be equal to 1/2 which is obtained by using the order of magnitude of ζ^* as given by (4.3.20) for large $k\eta = k\delta_s = k^{1-m}$. Therefore, the thickness of the boundary layer now becomes of order $R^{-1/5}$ from (4.3.22).

The ratio of viscous to inertia forces becomes

$$\frac{V}{I} = \frac{\nu\nabla^2 u}{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}} = -\frac{2}{R}\ \frac{1}{ff''}\left[f''' + \left(\frac{L}{4x}\right)(\eta^2 f'''' + 3\eta f'')\right]$$

$$= \frac{2}{R}\ \frac{k^2(3 - k\eta) - (L/4x)k\eta(k^2\eta^2 - 6k + 6)}{k\eta(k\eta - 2)(1 - e^{-k\eta})} \quad . \tag{4.3.23}$$

Near the plate, as $\eta \to 0$, $V/I \to \infty$. At the outer edge of the boundary layer where $k\eta = k\delta_s$, since $k\sigma_s \sim k^{1-m}$ and $R \sim k^{2+\lambda}$, it follows from (4.3.23) that

$$\frac{V}{I} \sim k^{m-1-\lambda} \quad . \tag{4.3.24}$$

Since $\lambda > 0$ and $0 < m < 1$, it follows from (4.3.24) that V/I is small at the outer edge of the boundary layer. Thus, V/I varies from a very large value near the plate to a small quantity of order $k^{m-1-\lambda}$ at the edge of the boundary layer.

We find that the classical boundary layer relation can be obtained as an extreme case by setting in (4.3.22) and (4.3.24)

$$\lambda = 0,\ m = 1 \tag{4.3.25}$$

which yields

$$V/I \sim 1, \quad \delta_s \sim R^{-1/2} \tag{4.3.26}$$

just as derived on the basis of Prandtl's assumptions. The choice of the parameters λ and m made in (4.3.25), however, does not hold since $\lambda = 0$ violates X becoming small throughout the field, and m = 1 makes the vorticity large at the outer edge of the boundary layer.

4.4 Thermal Boundary Layer Analysis

Since the formation of the thermal boundary layer on the plate considered in Section 4.2 is an asymptotic phenomenon, as discussed before, once again it is appropriate to employ the synthetic method in order to solve the equation of energy which governs the flow of heat due to forced convection. We now extend the equation of energy by introducing a constraining energy supply into Equation (4.2.4) obtaining

$$u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = E + a\nabla^2\phi + \frac{\nu}{c}\{ 2[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2]$$

$$+ (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2\} , \quad (4.4.1)$$

where E represents the constraining energy supply. As before, using the similarity variable η and defining certain dimensionless quantities, we may write (4.4.1) in the form

$$[\phi'' + \frac{1}{2} PRf\phi' + \frac{PU^2}{c}(f'')^2] + (\frac{L}{4x})\{\eta^2\phi'' + 3\eta\phi'$$

$$+ \frac{2PU^2}{c}f''(\eta S' - S)\} + (\frac{L}{4x})^2\{\frac{PU^2}{c}(\eta S' + S)^2\} = - \tilde{E} , \quad (4.4.2)$$

where

$\tilde{E} = \tilde{E}(x,\eta) = E(x,y)$, $S = S(\eta) = \eta f' - f$,

$P = \frac{\nu}{a}$ = Prandtl number.

It may be noted that the equation of energy in the classical thermal boundary layer theory is obtained by equating to zero the first square bracket in (4.4.2) and taking $\tilde{E} = 0$.

For analytical convenience, we write the temperature field $\phi(\eta)$ as

$$\phi(\eta;R,P) = \phi_\infty + \frac{U^2}{c}\theta(\eta;R,P). \qquad (4.4.3)$$

The function θ thus defined constitutes the similarity temperature profile. Equation (4.4.2) can then be put in the form

$$\left[\frac{\theta''}{PR} + \frac{1}{2}f\theta' + \frac{(f'')^2}{R}\right] + \left(\frac{L}{4x}\right)\left\{\frac{1}{PR}(\eta^2\theta'' + 3\eta\theta')\right.$$

$$\left. + \frac{2}{R}f''(\eta S' - S)\right\} + \left(\frac{L}{4x}\right)^2 \left\{\frac{1}{R}(\eta S' + S)^2\right\} = -\mathcal{E} \qquad (4.4.4)$$

where

$$\mathcal{E} = c\tilde{E}/U^2PR = c\tilde{E}/U^2Pe,$$

and $Pe = PR$ represents the well-known Peclet number.

We shall now seek the solution of the extended equation of energy, (4.4.4) for the adiabatic wall problem as mentioned in Section 4.2 (see Equations (4.2.7) and (4.2.8)), under the boundary conditions

$$\theta'(0) = 0 \text{ (the adiabatic wall condition)} \qquad (4.4.5)$$

$$\theta(\infty) = 0. \qquad (4.4.6)$$

Clearly, the energy supply E vanishes when θ = constant and $f(\eta) = A$, the perfect fluid solution, which is the case if the flow is uniform in both velocity and temperature fields.

In the spirit of the synthetic method, we take for the temperature profile θ holding near and away from the boundary the following expression and satisfying the boundary condition (4.4.6):

$$\theta(\eta:R,P) = G(R,P)e^{-n\eta} + H(R,P)\eta e^{-n\eta}, \qquad (4.4.7)$$

where n is, as k, an unknown parameter as yet undetermined. Upon imposing the adiabatic wall condition (4.4.5) on the first derivative of θ with respect to η, i.e.,

$$\theta'(\eta;R,P) = (H - nG)e^{-n\eta} - nH\eta e^{-n\eta}, \qquad (4.4.8)$$

we obtain

$$H = nG. \qquad (4.4.9)$$

Using (4.4.9) we find that Equation (4.4.7) reduces to the form

$$\theta(\eta;R,P) = G(R,P)e^{-n\eta}(1 + n\eta). \qquad (4.4.10)$$

At the boundary where $\eta = 0$, the energy $E(\eta;R,P)$ becomes

$$E(0;R,P) = -\frac{\theta''(0)}{PR} - \frac{(f''(0))^2}{R} = \frac{n^2 G(R,P)}{PR} - 4\frac{k^2}{R} \qquad (4.4.11)$$

where f"(0) is evaluated by using (4.3.16). Now $E\,(0;R,P)$ can be made vanishingly small through the same order as that of the constraining force, namely, $\frac{k^2}{R} = \frac{1}{k^\lambda} \ll 1$, $\lambda > 0$, provided the first term on the extreme right-hand member of Equation (4.4.11) becomes

$$n^2 G \sim k^2 P. \tag{4.4.12}$$

In view of (4.3.18), we can write (4.4.12) as

$$n^2 G \sim R^{\frac{2}{2+\lambda}} P. \tag{4.4.13}$$

We can now rewrite (4.4.13) in a convenient form by factoring P in the form $P = P^{\frac{2}{2+\lambda'}} P^{\frac{\lambda'}{2+\lambda'}}$, for some λ' to be determined, so as to allow, for example, the inclusion of the experimentally well established special case (see Eckert and Weise [24]), namely, G(R,P), which is identically the same as $\theta(0;R,P)$, behaves independently of R and approximately as $P^{1/2}$ for moderate P. Thus the energy $E\,(0;R,P)$ in (4.4.11) can be made vanishingly small by writing (4.4.13) in the form

$$n^2 G \sim R^{\frac{2}{2+\lambda}} P^{\frac{2}{2+\lambda'}} P^{\frac{\lambda'}{2+\lambda'}} \tag{4.4.14}$$

where we clearly identify

$$G \sim P^{\lambda'/(2+\lambda')}, \; \lambda' > 0, \tag{4.4.15}$$

and

$$n \sim R^{\frac{1}{2+\lambda}} P^{\frac{1}{2+\lambda'}}, \; \lambda, \lambda' > 0 \tag{4.4.16}$$

Now, we can inquire about the behavior of E in the general case, namely, $E(\eta;R,P)$ for general values of η. By substituting (4.4.10), (4.4.15), and (4.4.16) into (4.4.4), it is readily verified that the dominant term in the energy expression can be made to vanish asymptotically at all points except in a very thin region near the plate as in the case of the constraining force thus indicating the existence of the thermal boundary layer surrounding the plate.

The thickness of the thermal boundary layer will depend on the order of the temperature difference, θ, allowable outside this layer. If $\eta = \delta_t$ represents the edge of the thermal boundary layer, the temperature difference outside this layer must then be vanishingly small. Then analogous to the determination of the dynamic boundary layer thickness, we shall require that $\eta\delta_t$ be large while δ_t is small. Thus, we obtain

$$\delta_t \sim n^{-m'} \ , \ 0 < m' < 1. \tag{4.4.17}$$

Hence, it follows from (4.4.16) and (4.4.17) that

$$\delta_t \sim R^{-m'/(2+\lambda)} P^{-m'/(2+\lambda')} . \tag{4.4.18}$$

The ratio of δ_t to δ_s is therefore given by

$$\frac{\delta_t}{\delta_s} \sim R^{\frac{m-m'}{2+\lambda}} \ P^{-\frac{m'}{2+\lambda'}} \tag{4.4.19}$$

where $\lambda, \lambda' > 0$ and $0 < m, m' < 1$.

We will now give an algorithm for estimating all the parameters involved in the generalized formula for the thermal boundary layer thickness, (4.4.18), i.e., the parameters, λ, λ' and m' and illustrate it by an example.

For a given R, as mentioned before in the case of viscous boundary layer thickness, k should be taken from the order of large vorticity present near the boundary. λ is then determined from (4.3.18). For example, let $R \sim 10^5$ and $P \sim 0.7$, and G, which is identical with the temperature $\theta(0;R,P)$ is $\sim \sqrt{0.7}$. These choices for P and G are realistic choices for our example since they are involved in the experiments of [24]. Let k which represents the order of vorticity on the plate be 10^2. Then $\lambda = \frac{1}{2}$ which follows from (4.3.18) and λ' is given by (4.4.15), $\lambda' = 2$. Next, n is determined from (4.4.12). Since $P \sim 0.7$, $G \sim (.7)^{1/2}$, and $k = 10^2$, we obtain $n = 10^2(0.7)^{1/4} = 91.47$.

From the order of small temperature difference allowable outside the thermal boundary layer we get m', by using the formula (4.4.10). At the outer edge of this layer represented by $\eta = \delta_t$, for large $n\delta_t$, from (4.4.10), we have

$$\theta \sim Ge^{-n\delta_t}. \tag{4.4.20}$$

Since $\delta_t \sim n^{-m'}$ according to (4.4.17), the formula (4.4.20) becomes

$$\theta \sim G \exp[-n^{(1-m')}] . \tag{4.4.21}$$

If we now specify the order of the small temperature difference, $\theta \sim 10^{-2}$ at the outer edge of the thermal boundary layer, we obtain for the case $G = \sqrt{.7}$ with n as determined above, i.e., n = 91.47, m' = 0.7. Thus the values of $\lambda = \frac{1}{2}$, $\lambda' = 2$, $m' = \frac{7}{10}$. Hence, we obtain the new formula for the thermal boundary layer thickness from (4.4.18), for the example under consideration,

$$\delta_t \sim R^{-7/25} \, P^{-7/40} . \tag{4.4.22}$$

4.5 Boundary Layer Functions

Momentum Thickness:

The loss of momentum in the boundary layer as compared to the potential flow, is given by

$$P = \rho \int_0^\infty u(U - u)dy \tag{4.5.1}$$

so that a new boundary layer thickness can be defined by

$$\rho U^2 \delta_2 = \rho \left| \int_0^\infty u(U - u)dy \right| \tag{4.5.2}$$

where δ_2 is the momentum thickness and is a positive measure. Equation (4.5.2) can be written as

$$\delta_2 = \left| \int_0^\infty \frac{u}{U}(1 - \frac{u}{U})dy \right| . \tag{4.5.3}$$

Using the results $(4.3.5)_1$ and (4.3.15) in (4.5.3), we obtain for δ_2

$$\delta_2 = \frac{5\sqrt{Lx}}{4k} \sim \sqrt{x}R^{-\frac{1}{2+\lambda}} . \tag{4.5.4}$$

Equation (4.5.4) should be compared with the formula due to Blasius:

$$\delta_2 \sim \sqrt{x}R^{-1/2} \tag{4.5.5}$$

<u>Drag</u>:

Since the plate is wetted by the fluid on both sides, the total drag 2D is given by

$$\mathcal{D} = 2D = 2\mu \int_0^L \left(\frac{\partial u}{\partial y}\right)\Bigg|_{y=0} dx , \tag{4.5.6}$$

where μ is the coefficient of viscosity. Hence, we obtain

$$C_{\mathcal{D}} = 2D = 2\mu \int_0^L \frac{2kU}{\sqrt{Lx}} dx = 8k\mu U \sim 8U\mu R^{\frac{1}{2+\lambda}} . \tag{4.5.7}$$

The drag coefficient $C_{\mathcal{D}}$ then becomes

$$C_{\mathcal{D}} = \frac{\mathcal{D}}{\rho U^2 L} = \frac{8k}{R} \sim R^{-(1+\lambda)/(2+\lambda)} . \tag{4.5.8}$$

V. CONCLUSION

5.1 Discussion

Our present analysis has shown that the synthetic method is indeed more effective in achieving global treatment of problems of fluid flows past obstacles than the methods based on Prandtl's order-of-magnitude analysis. The new boundary layer solutions are obtained from the full Navier-Stokes equations and the complete equation of energy as limiting cases of an extended field so that the smooth transition into the Eulerian mass and heat flow away from the boundary is accomplished. The synthetic method has played a fundamental role, in the course of our analysis, of characterizing both dynamic and thermal boundary layer formation as asymptotic phenomena. One of the main features in our analysis is the introduction of an external constraining energy source into the field equations in addition to the force and obtaining solutions of the system by requiring the external constraints to vanish asymptotically in the boundary layer regions. A key assumption proposed in the Prandtl's boundary layer theory is that the ratio of the viscous to inertia forces is uniformly of the order of unity within the dynamic boundary layer. This assumption, shown earlier in the analysis to be rather fallacious, does fix the order of dynamic boundary layer thickness as $R^{-1/2}$. The fact of the matter is that there exists no hard and fast, well-defined, sharp line of demarkation splitting the flow field into subfields. It turns out from our analysis that Reynolds number of the flow by itself is insufficient to determine the order of magnitude of the boundary layer thickness, as is customarily done in current literature. In addition, it becomes important to consider the dependence of the boundary layer thickness on the order of vorticity to be neglected at the outer edge of the boundary layer. Thus both the Reynolds number and the vorticity parameter are essential to characterize the boundary layer thickness. Since the order of vorticity outside the boundary layer is vanishingly small, we have used a parameter m representing the order of vorticity, in addition to the parameter characterizing the Reynolds number, to define a formula for the dynamic boundary layer thickness and have consequently obtained it as $R^{-m/(2+\lambda)}$, where $\lambda > 0$ and $0 < m < 1$. Also, the classical treatment of Prandtl led to the somewhat speculative conclusion that the ratio of the dynamic and thermal boundary layer thicknesses is $P^{-1/2}$. It is, however, found by the synthetic method that the ratio should be $R^{\frac{m-m'}{2+\lambda}} P^{\frac{-m'}{2+\lambda'}}$, where λ', $\lambda > 0$ and $0 < m, m' < 1$. These new formulas seem to confirm the view that the order of the dynamic and thermal boundary layer thicknesses ought to be greater than their classical counterparts. Furthermore, we have obtained algorithms for estimating all the parameters involved in the analysis, thereby making our dynamic and thermal boundary layer solutions complete.

5.2 Scope of Further Work

In the present study it has been successfully demonstrated how the synthetic method renders tractable the nonlinearity involved in the coupled system of the Navier-Stokes equations and the equation of energy, governing the viscous flow past a flat plate with heat transfer due to forced convection. With several important fluid flow problems such as a viscous incompressible flow past a sphere or a cylinder, a nonviscous compressible flow past a sphere or a cylinder, and a non-Newtonian fluid flow, already solved, yielding some illuminating results, it then becomes all the more desirable to extend them to incorporate the corresponding problems of heat transfer by the synthetic method. It is equally worthy of an effort to apply the synthetic method to transition phenomena in general where incongruities such as apparent discontinuities, quick transitions, and nonuniformities are involved. It should be realized and at the same time emphasized that such singularities have resulted from the mathematical methods employed to treat them, while actually Nature would indeed operate in such a way as to accomplish smooth transitions in physical phenomena. A prime candidate for such an interesting investigation would be, for example, the elastic-plastic transition of the stress distribution in a small neighborhood of the tip of a sharp crack in an elastic medium. The classical elasticity theory predicts a stress singularity at the crack tip contrary to the physical reality. It is well recognized that in a small region surrounding the advancing crack tip, the elastic material undergoes a transition to the plastic state while outside this region the material still remains elastic. But this elastic-plastic transition has been treated by splitting the entire stress field into two subfields: the plastic zone and the elastic zone. This treatment involves matching two different types of stress distributions at the common boundary of the subfields. As a consequence the problem of the crack tip stress singularity remains unresolved. We believe that the synthetic method and transition approach will be of greater advantage in dealing with such transition phenomena than the classical matching techniques.

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