

AN ABSTRACT OF THE THESIS OF

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The purpose of this descriptive case study analysis was to provide portraits of the heuristics students used and difficulties they encountered solving conditional probability problems prior to and after two-week instruction on sample space, probability, and conditional probability. Further analysis consisted of evaluating the data in relation to a previously designed Conditional Probability Framework for assessing students levels of thinking developed by Tarr and Jones (1997). Five volunteer participants from a contemporary college mathematics course participated in pre- and post-interviews of a Probability Knowledge Inventory. The Inventory consisted of seven tasks on sample space, probability, and conditional probability. The semi-structured interviews provided participants' explanations on the development of their solutions to the seven tasks.

Among the five participants, rationalizing, finding the odds, computing the percentages, and stating the ratio of a problem were the preferred heuristics used to solve the problems on the Probability Knowledge Inventory. After the two-week instruction, two of the four participants who did not previously use computation of probability to

solve the problem changed their use of heuristics. The difficulties the students encountered prior to instruction included understanding the problem; recognizing the original sample space and when it changes; lacking probability vocabulary knowledge; comparing probability after the sample space changed; understanding the difference between probability and odds; and interchanging ratio, odds, and percentages - sometimes incorrectly - to justify their solution. After the two-week instruction, the students' difficulties diminished. Some improvements included a greater ability to understand the question of interest, to recognize the change in the sample space after a conditioning event, to use probability terminology consistently, and to compare probability after the sample space has changed.

Comparisons to the Probability Framework revealed that four of the five participants exemplified Level 3 thinking - being aware of the role that quantities play in forming conditional probability judgements. One participant exemplified a Level 4 thinking - being aware of the composition of the sample space, recognizing its importance in determining conditional probability and assigning numerical probabilities spontaneously and with explanation.

The Heuristics College Students Use and the Difficulties they Encounter
Solving Conditional Probability Problems: A Case Study Analysis

by

Mary E. Bamberger

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

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TABLE OF CONTENTS

	Page
Chapter I: A Need for Understanding Probability	
Introduction.....	1
Statement of the Problem.....	4
Chapter II: Review of the Literature	
Introduction.....	6
Research on Probabilistic Heuristics.....	6
Research on Conditional Probability Heuristics and Difficulties.....	10
Research on the Teaching and Learning of Probability.....	20
Research on the Framework for Students' Probabilistic Thinking in Instruction	45
Conclusion.....	61
Chapter III: Design and Method	
Introduction.....	65
Subjects.....	66
Overview of Two-Week Probability and Conditional Probability Instruction.....	68
Data Collecting Procedures.....	75
Data Analysis.....	77
Summary.....	79
Chapter IV: Results	
Introduction.....	80
Case Studies: Portraits of the Five Participants.....	81
Response to Research Questions.....	111

TABLE OF CONTENTS (Continued)

	Page
Chapter V: Discussion	
Interpretation and Discussion.....	121
Limitations of the Study.....	123
Implications and Recommendations for Future Studies.....	125
Bibliography.....	129
Appendices.....	132
APPENDIX A: Probability Knowledge Inventory.....	133
APPENDIX B: Consent Forms.....	136
APPENDIX C: Classroom Experiments.....	139
APPENDIX D: Monty Hall's Dilemma.....	144
APPENDIX E: Monty Hall's Dilemma Experiment.....	145

LIST OF TABLES

<u>Table</u>		Page
1	Refined Probabilistic Thinking Framework for Assessing Probabilistic Thinking.....	47
2	Refined Framework for Assessing Middle School Students' Thinking in Conditional Probability and Independence.....	51
3	Refined Framework for Assessing Middle School Students' Thinking in Conditional Probability.....	118

DEDICATION

This thesis is dedicated to the loving memory of my Grandmother, Stacia Prekel. She once told me, "If you enjoy what you do, you never have to work a day in your life". She knew how much I enjoy teaching, and her thoughts and words of wisdom will always help guide me.

The Heuristics College Students Use and the Difficulties they Encounter Solving Conditional Probability Problems: A Case Study Analysis

Chapter I

The Need for Understanding Probability

Introduction

Random events play increasingly common roles in our daily lives. From shuffling cards for a game of cribbage, to selecting an appropriate retirement fund, to relying on a new treatment for cancer, most people encounter chance daily. Random events are a part of the natural world. They can be found in random noise, arrangements of the petals of a flower, and the rolling of a fair die. Fortunately, some "random events" do obey laws of some kind. If we knew these laws, it would simplify some of our lives.

The use of chance mechanisms and the recognition of random events dates back to ancient Egypt, prior to 3500 BC. Archeological digs have uncovered board games from 3500 BC, perfectly balanced fired-pottery die from 3000 BC, and evidence of Egyptians playing the game odd-or-even in 2000 BC (Lightner, 1991; Bennett, 1998; Borovcnik, Bentz, & Kapadia, 1991). Random events also played a vital role for ancient people in their daily lives: settling disputes among neighbors, selecting a course of military strategy, dividing property, and delegating civic responsibilities or privileges (Bennett, 1998). Despite the everyday presence of chance and random events, the development of the laws regarding chance and certain random events is quite young. This lack of early knowledge of pattern recognition in rolling dice or playing cards has

puzzled present day mathematical historians and philosophers. Some theories have been developed in an attempt to explain the lack of recognition of the mathematics behind random events. Among the many theories, two theories appear to be the most acceptable. First, it was possible that the cultures and beliefs of the past may have had an influence on the inability to recognize this link. Evidence of a belief that God or gods directed earthly events in a predetermined plan, in which randomness was not considered, is demonstrated in the early use of lotteries and dice for consulting gods (Lightner, 1991; Hacking, 1975). A second theory asserts a lack of appropriate mathematical notations, symbols, and numerate people, which was evident by the origin of the pips (or dots) on dice. Earlier people recognized the relationships of greater than and less than, but did not know the concept of numbers and numeracy. It was not until the Renaissance Period (14th - 17th centuries) and the development of algebra that the ability to write and calculate with Hindu-Arabic numerals was developed by scholars (Lightner, 1991). Once mathematical notations and symbols were invented, and the church was more open to scientific inquiry, mathematicians started to recognize number sense, number patterns, and empirical frequencies associated with certain random events. By the late 15th century a true mathematical treatment of random events, and the study of chance eventually turned into the branch of mathematics called probability.

Although probability had its origins in the games of chance, probability has become a branch of mathematics with wide ramifications in scientific research, business and industry, politics, and practical daily activity. As these examples illustrate, probability permeates day-to-day life:

- *There is a 20% chance of rain tomorrow.*
- *A screening test for a certain virus is 95% accurate for both infected and uninfected persons.*
- *The new reading program implemented in the local school system increased the students' reading scores ($p < .05$).*

In recent years, organizations have recognized the need for teaching probability.

The National Council of Teachers of Mathematics (1989), in its publication of *Curriculum and Evaluation Standards*, called for introducing a number of probability concepts throughout the K-12 school curricula. The National Council of Teachers of Mathematics (2000) continues its support of teaching probability concepts in its most recent publication of *Principles and Standards for School Mathematics*. Similarly, at the post-secondary level, the American Mathematical Association of Two-Year Colleges (1995), in its publication of *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus*, also recognizes the need to introduce non-mathematics and science major students to the basic probability laws. Finally, the American Association for the Advancement of Science (1993), in its publication of *Benchmarks for Science Literacy*, discusses the need for probability in the K-12 curriculum to enhance students' understanding of real-world events. These proposals call for an increased emphasis on probability in the mathematics and science curriculum.

Since the emphasis of introducing and expanding on the concept of probability in the mathematics and science curriculum is quite recent in the U. S. curriculum, there has been little impetus to carry out research regarding the effects teaching has on student understanding of probability. In fact, most of the contributions on probability research have concentrated on student misconceptions and intuitions of probability. However, these studies conducted by cognitive psychologists mainly focused on the misconceptions

and intuitions people have, not on how the influence of instruction may change student thinking (Cohen, 1957, 1960; Falk, 1986, 1988, 1989b; Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973, 1974, 1980, 1982). The majority of the studies focusing on the effects teaching has on student understanding of probability derive from research on non-North American mathematics and statistics researchers, thus, looking at a different curriculum (Castro, 1998; Fischbein & Gazit, 1984; Fischbein & Schnarch, 1998). By combining the efforts of the cognitive psychologists and the mathematics and statistics educators, the proposals of various mathematics and science organizations can become an effective tool in educating the population on the concept of probability.

Statement of the Problem

Most mathematics educators and many teachers have accepted the increased attention given to probability and statistics in national curriculum reform statements. This emphasis on broader explorations of probability concepts in the curriculum has established a need for further research into the probabilistic thinking of students of all ages. Although there has been substantial research on students' probabilistic thinking (Cohen, 1957, 1960; Falk, 1986, 1988, 1989b; Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973, 1974, 1980, 1982), little of that research has focused on student's probabilistic thinking in the classroom (Fischbein & Schnarch, 1997; Pollatsek, Well, Konold, Hardiman, & Cobb, 1987), and even fewer studies focus on the teaching and learning of probability at the collegiate level (Austin, 1974; Shaughnessy, 1977). More research is needed on the teaching and learning of probability.

The teaching and learning of probability is a complex process; therefore, conducting research on the teaching and learning of probability is a multi-faceted enterprise. By focusing on one concept in the probability classroom and trying to understand how teaching affects the learning of a particular concept, the researcher can solve the puzzle piece by piece. The intent of this study is to look at the teaching and learning of conditional probability in a college level math course. In particular, the questions of interest for this study were:

1. What are some of the heuristics college students use, and what are some of the difficulties they encounter solving conditional probability problems prior to receiving instruction on sample space, probability of an event, and conditional probability?
2. After attending a two-week class on sample space, probability of an event, and conditional probability, in what ways did the students' heuristics change, and in what ways were they able to overcome difficulties they had previously encountered when solving conditional probability problems?
3. How does each student's understanding of conditional probability compare to the Conditional Probability Framework developed by Tarr and Jones (1997)?

Chapter II

Review of the Literature

Introduction

The purpose of this study is to explore the interrelationships between the teaching and learning of conditional probability at the post-secondary level. Four bodies of research inform this investigation and provide the foundation for the theoretical framework: research on

- probabilistic heuristics,
- conditional probability heuristics and difficulties,
- teaching and learning of probability, and
- framework for student probabilistic thinking in instruction.

Within these four bodies of research, this review will include more elaborate investigations on the specific studies pertaining to the teaching and learning of conditional probability at the collegiate level.

Research on Probabilistic Heuristics

Suppose one is faced with determining the outcome of an election, the guilt of a defendant, or the chance of winning at the roulette table. If this person has no exposure to knowledge of chance, or the statistical theory of prediction, they will try to reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations. The reduction of complex tasks of assessing probabilities and predicting

values to simpler judgmental operations has been defined in research on the understanding of probability as probabilistic heuristics (Tversky & Kahneman, 1974). These heuristics, which sometimes yield reasonable judgements, can also lead to severe and systematic errors (Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973). Defining a naïve subject as a person who has not had a formal class in probability, a series of studies with both naïve and educated subjects has supported this hypothesis. This section gives an overview of four of the probabilistic heuristics researchers found that are used to assess the likelihood of an event:

- Representativeness
- Availability
- Positive and Negative Recency Effects (Gambler's Fallacy)
- Conjunction Fallacy

Representativeness

One heuristic Kahneman and Tversky (1972) associate with subjective probability is representativeness. A person who follows this heuristic estimates that the probability of an uncertain event is based on how well an outcome represents some aspect of its parent population, or how the event reflects the prominent features of the process by which it is generated. Tversky and Kahneman (1982) pursued their interest in representativeness to define six subcategories: insensitivity to prior probability of outcomes, insensitivity to sample space, misconceptions of chance, insensitivity to predictability, illusion of validity, and misconceptions of regression. For example, the representativeness of misconceptions of chance state that people expect that a sequence

of events generated by a random process will represent the essential characteristics of that process even when the sequence is short. People believing that the sequence of flipping a coin five times and obtaining H-T-H-T-H is more likely than H-H-H-T-T, or even the sequence H-H-H-T-H, easily illustrate misconception of chance.

Availability

Availability is another heuristic associated with subjective probability. Tversky and Kahneman (1973) described a person who uses availability as one who evaluates the probability of an event by the ease with which relevant instances come to mind. Tversky and Kahneman (1982) continued their interest in availability to find three subcategories: bias due to retrievability of instances, bias of imaginability, and illusory correlation. For example, suppose a word is randomly picked from an English Dictionary. Is it more likely that the word begins with the letter K, or that K is its third letter? Availability tells the naïve person, one who has not had formal education in probability, that since it is much easier to think of words starting with K than of words in which K is the third letter, they would believe the word is more likely to start with K. Unfortunately, in the English language, there are about twice as many words with K in the third position than in the first (Tversky & Kahneman, 1973).

Positive and Negative Recency Effects (Gambler's Fallacy)

Some researchers of probabilistic heuristics classify positive and negative recency effects as subcategories of representativeness. However, the research conducted on

recency effects took place before the recognition of the representativeness heuristic (Cohen, 1957, 1960). Recency occurs when a person is uncertain how to calculate the outcome of the next event, given the results of the previous independent trials. For example, a person using the positive recency heuristic when predicting a head or tail on a flip of a coin tends to believe that after a run of heads, a head is more likely to occur in the next toss. Thus, the positive recency effect causes the person to assume incorrectly that the conditions were not fair. A person using a negative recency heuristic when predicting a head or tail on a flip of a coin tends to believe that after a run of heads, a tail is more likely to occur in the next toss. Thus, the negative recency effect causes the person to believe intuitively that the alternating outcomes seem to better represent a random sequence. The idea of negative recency effect has also been known as "Gambler's Fallacy", in which the gambler believes the events will balance at the end.

Conjunction Fallacy

The conjunction fallacy stems from the extension rule of the Law of Probability: If $A \supset B$, then $P(A) \geq P(B)$. Since the set of possibilities associated with the conjunction (A and B) is included in the set of possibilities associated with B, the same principle can also be expressed by the conjunction rule: $P(A \text{ and } B) \leq P(B)$. However, Tversky and Kahneman (1983) found in their study that when a person is given an uncertain event involving conjunctions, people tend to use the representativeness and availability heuristics to make a conjunction appear more probable. One of their studies showed that 85-90% of their subjects violated the conjunction rule of probability. This was illustrated when after people were given a description of a fictitious female character,

who is "bright, single, 31 years old, outspoken, and concerned with issues of social justice", the subjects were more likely to believe that the person was a bank teller and was active in the feminist movement, than that the person was just a bank teller.

Conclusion

By investigating these primitive conceptions, intuitions of probability, misconceptions, fallacies in thinking, and judgmental biases, researchers have been able to construct a framework of how and when people use these heuristics. Based on these descriptive results of heuristics, teachers can become familiar with student's preexisting probabilistic conceptions before they try to teach the mathematical concepts of probability. A few of the following studies reviewed in this chapter use some of the findings of these psychologists when conducting research on the teaching and learning of conditional probability.

Research on Conditional Probability Heuristics and Difficulties

The probability of an event will vary depending upon the occurrences or nonoccurrence of one or more related events. For example, Oregon sport fishermen are vitally interested in the probability of rain. The probability of rain on a given day, ignoring the daily atmospheric conditions or any other events, is the fraction of days in which rain occurs over a long period of time. This would be called "unconditional probability". Consider the chance of it raining tomorrow. It has rained almost continuously for two days and a storm is heading up the coast. This probability is

conditional on the occurrence of several events, and Oregonian would tell you that it is much larger than the unconditional probability of rain. Thus, the "conditional probability" of an event is the probability of the event given the fact that one or more events have already occurred. Problem I, II and III illustrate other conditional probability problems found in textbooks, life experiences, and professional decisions:

Problem I (Falk 1988, p. 292):

An urn contains two white balls and two black balls. We blindly draw two balls, one after the other *without* replacement from that urn.

- a. What is the probability that the second ball was white, given the first was also white?
- b. What is the probability that the first ball was white, given that the second ball was also white?

Problem II (Tversky & Kahneman, 1980, p. 51):

Which of the following is more probable:

- a. That a girl has blue eyes if her mother has blue eyes
- b. That the mother has blue eyes, if her daughter has blue eyes
- c. The two events are equally probable

Problem III (Falk, 1986):

Which statement is the definition of a Type I error, α , in hypothesis testing:

- a. The probability that one will reject the null hypothesis, given that the null hypothesis is true.
- b. The probability of the null hypothesis is true, given that we rejected the null hypothesis

However, taking into consideration people's experiences with conditional probability, researchers identified common heuristics and difficulties people have in solving these types of problems. Time-axis (Falk, 1983, 1988) and causal bias (Tversky & Kahneman, 1980) are two conditional heuristics used by people who are unsure how to solve the complex tasks of assessing conditional probabilities. Calculating the inverse of the

condition, identifying the conditional event, and confusion due to the wording or framing of the conditional probability are three difficulties people encounter when trying to solve a conditional statement (Falk, 1989a).

Time-Axis Fallacy Heuristic (Falk Phenomenon)

The time-axis fallacy is the most prominent heuristic of conditional probability. Falk (1983, 1988) recognized that when a person is given a conditional probability situation, and asked about the probability of the first event happening, given the second has occurred, they have a difficult time going "back in time" to comprehend the question correctly. This heuristic is illustrated in Problem I, from above. Falk (1983, 1989b) found that the subjects of her study were able to answer part (a) correctly - one third - however, some subjects did not believe part (b) had an answer. The subjects argued that the probability of an outcome of a draw on an event that occurs later is not permissible. Others argued that since the first ball does not care whether the second is black or white, the answer will be one half. Hence, those who use the time-axis heuristic want to compute the probability of an event occurring at the immediate point of time at which the event takes place.

Causal Bias Heuristic

A causal scheme follows a course of cause to consequence. However, when people are faced with finding the probability of an uncertain causal event, they may find it easier to invert this sequence and reason from consequence to cause. Research

conducted by Tversky and Kahneman (1980) tested this hypothesis by asking people which is more probable: $P(X|Y)$ or $P(Y|X)$ when X is the natural cause of Y and $P(X) = P(Y)$. In this study, the majority of the subjects answered $P(Y|X) > P(X|Y)$. An example used in their study was Problem II, from above. Since the distribution of eye color is essentially the same in successive generations, more subjects regarded answer (a), the causal answer, as the correct answer, over (b), the diagnostic answer.

Difficulties with Calculating Conditional Probability

It is possible that a person is proficient in the computing of conditional probability. Due to confusion between a conditional and its inverse, difficulties identifying the conditional event, and uncertainty of the question of interest due to the wording or framing, they may have approached the problem incorrectly (Falk, 1989a). Problem III, from above, is an example of confusion between a conditional and its inverse. The probability that one will reject the null hypothesis, given that the null hypothesis is true is the definition of a Type I error. However, due to some linguistic ambiguities, a person familiar with hypothesis testing may have a tendency to interpret the inverse as the definition (Falk, 1986, 1989a).

Conditional Probability Heuristics and Difficulties in College Students

College instructors generally agree that people have a great deal of difficulty with conditional probability. Through two experiments, Pollatsek, et al. (1987) tried to examine three areas concerning the heuristics and difficulties college students have in

their study of college students' understanding of conditional probabilities. Defining "naïve student" as an undergraduate who has not taken a college level statistics course, Pollatsek, et al. (1987) first investigated naïve students of conditional probability. They wanted to explore the possibility that naïve students have a fundamental inability to deal with conditional probabilities. Their second interest consisted of the misconception of the notation $P(B|A)$. Pollatsek, et al. (1987) wanted to determine if the problem laid in the confusion of the conceptual understanding between $P(B|A)$ and $P(A \text{ and } B)$. Their third issue concerned casual bias. Tversky and Kahneman (1980) have argued that a causal bias exists in judging conditional probabilities. Pollatsek, et al. (1987) wanted to test if this causal bias is truly powerful and persuasive as claimed.

The first experiment explored naïve students' fundamental inability to deal with conditional probability and the effect of causal bias. On the first day of class, the 86 undergraduate students enrolled in a lower division psychology course completed a six-question, forced answered questionnaire. Each question consisted of an event in which the students were asked which of the following three options stated the correct relationship between $P(A|B)$ and $P(B|A)$: $P(A|B) < P(B|A)$, $P(A|B) > P(B|A)$, or $P(A|B) = P(B|A)$. Of the six questions, three questions were events chosen to contradict the idea of causality. These three questions consisted of scenarios in which Event A could be thought as necessary but not an adequate cause of Event B, and Event A is strongly implied by Event B; however, the two events are related. For example, if Event A was "being sick" and Event B was "having a fever", Event A both causes and is implied by Event B; however, $P(A|B) > P(B|A)$. The remaining three questions were chosen to be sensitive to any causal bias that might have existed. The first question considered a

scenario in which Event A does not cause Event B, and the two events are not related. The last two questions did not have correct answers for the information given. Thus, students were forced to pick a relationship based on their bias.

The unit of analysis consisted of the percentage of correct responses to each of the questions from the entire population. The data analysis consisted of visually comparing the percentages, without statistical support. The results of the study indicate that most of the subjects (72% - 87%) gave the correct answer to the three problems testing causality. In all three cases, the students strongly preferred the alternative consistent with $P(A|B) > P(B|A)$. The three questions referring to the sensitivity of any causal bias, indicate the students had a tendency to choose the alternative consistent with $P(\text{effect}|\text{cause}) > P(\text{cause}|\text{effect})$. This result indicates the students were not sensitive to causal bias. Also, 50% of the students were able to choose $P(A|B) = P(B|A)$, as the correct answer being sought. Hence, the results from the first experiment illustrate that naïve students do have a fundamental ability to deal with conditional probabilities; however, certain factors did interfere with their judgement: wording of the problem and unfamiliarity with the context of the problem.

A second experiment explored the misconception of the notation $P(B|A)$ and possible hindrances naïve students may have calculating these probabilities. The two main hindrances explored in this experiment were the possibility of the wording of the problem affecting student's judgement and the confusion between conjunction and conditional probability. For this experiment, 120 students were recruited from various sections of an introductory psychology course designed for psychology majors and

received course credit for participation. These students were also considered naïve in probability since they did not take a college level probability course.

The data collected for the second experiment consisted of the results from two different questionnaires. The first questionnaire consisted of seven forced-choice questions judging whether $P(A|B)$ was greater, less than, or equal to $P(B|A)$. The first questionnaire was presented to the student in two different formats, but with the same seven events. Half the students answered questions posed in a probability format, while the other half answered questions posed in a percentage format. The distinction between questions posed in the probability and percentage questions was used to see if the wording caused the difficulty in student understanding of the problem. The second questionnaire consisted of the same seven questions; however, the students were asked to estimate the conditional probability in percentages, with justifications of their answers. Students were given fixed amounts of time to complete each section of the questionnaire, to discourage the students from answering hastily. The estimation questions were used to judge the students' confusion between conjunction and conditional probability questions. The study does not discuss the validity or reliability of either questionnaire.

The unit of analysis consisted of the percentage of correct responses to each of the seven questions from each of the two groups. The data analysis consisted of visually comparing the percentages, without statistical support. The results from the forced-choice questionnaire indicated that the performance of judging conditional probability varied widely across the problems - from 30% to 80% correct. When comparing the probability form of the test against the percentage form, student performance was similar, with an average of 57.0% and 56.7% correct respectively. These results indicate that

there was no difference between student responses with respect to the wording of the questions.

The results from the estimation questionnaire indicated there was almost an 80% agreement between the forced choice responses and the estimation responses, with slightly better performance on the estimation section. Finally, the estimation data were analyzed for their "reasonableness". For this study, reasonableness was defined as answers that met certain criteria for each of the seven questions. The patterns found within the estimations of reasonableness suggest that some subjects may have confused conditional and joint probabilities.

Fischbein and Schnarch (1997) investigated the evolution, with age of the student, on probabilistic intuitions and misconceptions, in which, one of the misconceptions considered is conditional probability. Intuition, as defined in this study, is "a cognition that appears *subjectively* as self-evident, directly acceptable, holistic, coercive, and extrapolative" (p. 96). Intuitive cognition is differentiated from an analytically and logically based cognition by intuitive cognition producing a feeling of obviousness and of intrinsic certainty when the student solves a problem.

The sample for the study consisted of five groups of Israeli students without previous instruction on probability: 20 fifth grade students, 20 seventh grade students, 20 ninth grade students, 20 eleventh grade students, and 19 college students. The students were administered a questionnaire with seven probability misconceptions identified in past papers on probability heuristics: representativeness, negative and positive recency effects, simple and compound events, conjunction fallacy, effects of sample size, availability and the time-axis fallacy. Each question consisted of a description of an

event with three forced-choice possible answers: the correct response, the common incorrect misconception response, and a distracter. The questionnaire was administered to the students during a regularly scheduled class, allowing them one hour to complete the questionnaire.

The unit of analysis consisted of the average score of all five groups on each question. An average score for each question was compiled by computing the percentage of students in each group who answered one of the three responses. The analysis of the results consisted of comparing the average percentages for each misconception reply across all age levels. By comparing the average percentages of students answering the questions with the main misconception, the study indicates that the misconceptions of representativeness (from 75% to 22%), negative recency effect (from 35% to 0%), and the conjunction fallacy (from 85% to 44%) decrease with age. The results also show that availability (from 10% to 72%) and the effect of the time-axis (from 5% to 44%) increased with age; and positive recency effects (from 0% to 6%), and compound and simple events (from 70% to 78%) remained stable with age. However, the question pertaining to the misconception of the effect of sample space remained a strong misconception, with only one student from the entire sample answering the question correctly.

A closer look at the effect of the time-axis indicates the lack of understanding of conditional probability. As indicated earlier, the misconception of the time-axis increased with the age of the student (from 5% to 44%). The initial question on the questionnaire consisted of two parts (p. 99):

Yoav and Galit each receive a box containing two white marbles and two black marbles.

- a. Yoav extracts a marble from his box and finds out that it is a white one. Without replacing the first marble, he extracts a second marble. Is the likelihood that this second marble is also white smaller than, equal to, or greater than the likelihood that it is a black marble?
- b. Galit extracts a marble from her box and puts it aside without looking at it. She then extracts a second marble and sees that it is white. Is the likelihood that the first marble she extracted is smaller than, equal to, or greater than the likelihood that it is black?

The responses to this question were divided into three categories. In Category I, both responses were correct; in Category II, the first response was correct and the second incorrect; in Category III, both responses are incorrect. The category of interest is Category II, since this category illustrates that the student understands probability; however, the student had the main misconception of time-axis. The apparently causal order of the story, as it is told in a sequence of events, hides the genuine probabilistic structure of the problem: the two questions actually express the same problem. Due to the method of collection of the data and the nature of the analysis, it is unclear as to why the misconception of time-axis increased with age.

The results from this study suggest there is an instability of probabilistic intuitions as the student ages. The researchers hoped to justify that the intuitions tend to stabilize and become resistant to the influence of age and instruction. However, this study indicates that some misconceptions diminish with age, one was stable, and some gained greater influence. The results from this study had impressed the researchers. The question of interest was to see if the probabilistic misconceptions - combined - increased

or decreased with age. They did not expect the results to be scattered among the various misconceptions.

The results from the two previous studies on the conditional probability heuristics and difficulties in College Students found two main observations: conditional probability intuition decreases with age (Fischbein & Schnarch, 1997), and the major source of error in computing conditional probability was the confusion between conditional and joint probabilities (Pollatsek, et al., 1987). However, as indicated by Pollatsek, et al. (1987), college students misunderstanding of the problem was not due to the word choice of probability or percentage, and college students were not influenced by causal bias. These results may indicate that college students do have a fundamental ability to deal with conditional probability.

Research on the Teaching and Learning of Probability

The previous two sections looked at the studies investigating the heuristics people might use, and the difficulties they might encounter when trying to solve probability and conditional probability problems. This section investigates the studies that look at the effects of instruction on students at all levels and how the intervention may influence their use of heuristics and difficulties when solving probability and conditional probability problems.

Collegiate Teaching Programs

Austin (1974) conducted an experimental investigation of the effectiveness of manipulatives in the teaching of probability and statistics to university-level students. The study consisted of 80 non-math and science students at Purdue University enrolled in two different sections of a sophomore level probability course. Before the experiment began, the students were ranked on the basis of their previous mathematics grades for each section. Each of the two sections was divided into three treatment groups: Manipulative Pictorial (MP), Pictorial (P), and Symbolic (S). In each section, the first three students were randomly assigned to one of three treatment groups. This procedure continued until all students were assigned to a treatment group.

The three treatment groups for each section met in separate tape laboratories during the regular class hours. The students did not have any contact with the instructor, and the laboratory assistants had no knowledge of the purpose of the study. Neither the students nor the assistants were told they were involved in an experiment. They were told they were part of a study on the feasibility of video taped instruction. During each class meeting, every student received a written lesson and a tape of the lecture. Thus, students could listen to the lecture at their own pace. Daily homework was assigned and returned, graded by the instructor.

Each of the three groups had the same written lessons with the same objectives; however, the treatment differed in their lecture portion. The MP groups performed experiments on random processes found in discrete probability. The students conducted random experiments using coins, dice, random-number tables, and marble selection devices. The students also used graphs, diagrams, and figures to motivate their learning.

The P groups were similar to the MP groups; however, the P groups did not perform the experiments. Instead, the P groups used data generated by the instructor. The students did not see the experiments or the use of the physical objects being used. The pictorial aspect of the instruction was not changed. The S groups were similar to the P groups; however, the S groups did not have any pictorial aids. The written material was altered and only words and mathematical symbols were used. Overall, in the MP, P, and S groups, the behavioral objective and problems were identical.

The experiment took place the first four weeks of the term, with the class meeting three times a week. Each lecture was approximately 30 minutes long. At the end of the experiment, an examination covering the twelve lessons was given during an evening meeting, so that all students took the same examination at the same time.

The data for the experiment were collected through the students' previous math grades and the final exam. The final exam consisted of 40 questions, stratified into cognitive levels, based on the taxonomy used in the National Longitudinal Study of Mathematical Abilities: comprehension, computation, application, and analysis. The percentages of items on the exam approximated the percentages of the behavioral objectives used in the lessons. The results of the exam were broken into five dependent variables: comprehension score, computation score, application score, analysis score and total score.

The analysis and results for the test were stated in two categories: students' previous math grades and the five dependent variables of the final exam. Using the student's previous mathematics grades, a two-way analysis was conducted that compared the two factors of class (2 levels) and treatment (3 levels). With $\alpha = .05$, analysis of

variance indicates there was no difference between the two sections and the three treatment groups in their previous mathematics grades ($p > .24$ for the three comparisons). The results from the final exam were also computed using the two-way analysis on each of the five dependent variables. The two factors were class (2 levels) and treatment (3 levels). Homogeneity of cell variance for each variable was tested using Bartlett's χ^2 method. With $\alpha = .05$, only the computational sub-score test rejected the homogeneity hypothesis ($p < .05$ for computational, $p > .05$ for the other four variables). The rejection of the homogeneity hypothesis indicates the assumption of equal population variance was not met; thus the data did not meet the assumptions of a two-way ANOVA (Huck & Cormier, 1996). When the hypothesis of no difference in the examination score means among the three treatment groups was rejected for a particular variable, Scheffé's method was used to make the pairwise comparisons of treatment means. Scheffé's method allows multiple comparison based on the F -test when the assumption of equal population variance is not met (Huck & Cormier, 1996). In the analysis of variance, pooling was done when possible. Factors were pooled if the F -test was not significant at the .25 level. For the total examination scores, equality of treatment means was rejected. The class and interaction effects were pooled, and the Scheffé's test conducted. The results from the Scheffé's test indicated the symbolic treatment mean (μ_S) was less than the manipulative pictorial (μ_{MP}) and the pictorial means (μ_P). However, the hypothesis of equality of the manipulative pictorial mean and the pictorial mean was not rejected. Thus the ordering indicated by Scheffé's method indicated $\mu_S < \mu_P$; $\mu_S < \mu_{MP}$; and $\mu_P = \mu_{MP}$. The Scheffé's method also found the same ordering for the application, analysis, and examination scores. However, for the comprehension sub-score, the two-way analysis

permitted the pooling of class and interaction effects. The resulting analysis rejected the equality of treatment means. Scheffé's test indicated that the symbolic treatment mean was less than the pictorial treatment mean. However, neither the hypothesis that the manipulative-pictorial means is equal to the symbolic mean nor the hypothesis that the manipulative-pictorial means is equal to the pictorial mean was rejected. Thus the order was $\mu_S < \mu_P$; $\mu_S < \mu_{MP}$; and $\mu_P = \mu_{MP}$. For the computation sub-score, homogeneity of cell variance was rejected with $\chi^2_{\text{obs}}(5) = 16.5$. The two-way analysis was not made. Rather, two one-way analyses were made on the scores from each of the two classes. With the separate classes, homogeneity of variance was not rejected in either. Neither of the two one-way analyses rejected the equality of treatment means ($p > .50$, pairwise analysis of treatment averages).

The results from the analysis show four major trends. First, there was no difference in student computational achievement among the three instructional methods. Second, the use of graphs, figures, and diagrams significantly improved the student's application and analysis, and total examination scores. Third, if graphs, figures, and diagrams were used, then students' application, analysis and total scores do not indicate any significant difference between the manipulative-pictorial students and the pictorial students. Finally, if students did use graphs, figures and diagrams, the comprehension score may indicate that students who performed the manipulation did not perform as well as those who were only told the outcome of the experiments.

Shaughnessy (1977) conducted a study on an experimental model of mathematics instruction in elementary probability and statistics and how the instruction maximized the student's chances of overcoming their misconceptions of probability and statistics. The

population for this study consisted of 80 college undergraduate students enrolled in four sections of a finite mathematics course. The four sections were randomly selected for this experiment from a total of seven sections being offered that term. The remaining three sections were defined to be the control group. The experimental groups participated in activity based courses constructed as an alternative to the lecture method. The experimental group participated in a series of nine researcher-designed activities covering five probability concepts: probability, combinatorics, game theory, expected value, and elementary statistics.

The nine activities were carried out in small groups of four or five students, with the students rotating groups for each activity. Each activity required the groups to perform experiments, gather data, organize and analyze the data, and reach a conclusion that could be stated in the form of a mathematical principle or model. The role of the instructor during these activities was similar to a facilitator - clarifying students' questions, assisting groups stalled on a particular problem, or answering a question with a question. However, the instructor did not provide the students with answers. The control group continued the traditional method of lecture.

The data for this study were collected in three areas: researcher observations, pre-test, and post-test. The researcher observations were mainly collected to record student-student interactions, instructor-student interactions, and unique occurrences noted by the researcher. The notes taken by the researcher were not used for data analysis, but instead, to augment the study and to give an insight to the reader the activities and interactions that occurred in the experimental classroom. The 20-question forced-choice pre- and post-tests used for the study were developed based on the questions used by Kahneman

and Tversky in their research. The tests measured the students' knowledge of some probability concepts and for their reliance upon representativeness and availability in estimating the likelihood of events. Besides being a forced-choice test, the test also asked the student to supply a reason for each response. The two tests were identical and were given to both the experimental and control groups.

The data analysis consisted of compiling the pre- and post-test results of the experimental and control groups. To check for possible change in students' use of representativeness and availability, some specific questions on the test were analyzed separately. Oddly, the study does not indicate which statistical analysis was conducted to reach the conclusions. Hence, the results of the analysis were stated with their p-values, with no indication of an alpha-value for comparison. The analysis indicated that the experimental group was more successful at overcoming reliance upon representativeness ($p < .05$), and tend to be more successful at overcoming reliance upon availability ($p < .19$).

The results of this study supported the hypothesis of Kahneman and Tversky, which claimed that combinatorially, naïve college students relay upon availability and representativeness to estimate the likelihood of events. The results on the post-test suggest that the manner in which college student learn probability may make a difference in their ability to overcome misconceptions that arise from availability and representativeness. This experiment suggests that the course methodology and teaching model used in an elementary probability course can help develop student's intuition for probabilistic thinking.

Secondary-Level Teaching Programs

Fischbein and Gazit (1984) investigated the possibility of influencing junior high students' intuitive probabilistic judgement through classroom instruction. The term "intuition", defined by the researchers, is the cognitive belief based on a "global, synthetic, non-explicitly justified evaluation or prediction" (p. 2). For their study, Fischbein and Gazit believed that intuitive attitudes could be developed only through personal involvement of the learner in a practical activity. These intuitions cannot be modified by only verbal explanation. Therefore, the primary purpose of this exploratory study was to evaluate probabilistic intuitions and their development under the influence of systematic instruction. A secondary purpose for this study was to explore the possibility that the student's age may have some impact on the learning capacity of probability.

The population for this study consisted of 18 junior high classes divided into two groups: experimental group and control group. The experimental group participated in 12 lessons covering the concepts of certain, possible, and impossible events; outcomes and events in a chance experiment; the concepts of chance and of quantifying chances; the concepts of probability and relative frequency and the relation between them; counting outcomes; and simple and compound events and their probabilities.

The data for this study were collected by two open-ended questionnaires: Questionnaire A and Questionnaire B. Questionnaire A was devised to assess the teaching effects of the students attending the experimental class. Since the questionnaire was designed to test the extent to which the concepts taught in the experimental groups were assimilated correctly and efficiently, it was not administered to the control group.

Questionnaire B was devised to assess the indirect effect of instruction on the student's intuitively based misconceptions. Questionnaire B was administered to both the experimental and control classes since it did not require special knowledge of probability.

The unit of analysis for this study was the three grade levels of the classes participating for the study: fifth, sixth, and seventh. The results were collected according to if the student could correctly or incorrectly answer the questions and percentages of the correct responses were calculated for each grade level.

The results for this study were reported according to percentages. The responses to the first three questions indicated that distinguishing between certain, chance, and impossible events increased with age: with an average of 70% of the fifth graders able to give at least one example of each event compared to the average of 85% of the sixth graders and 95% of the seventh graders.

The fourth question considered to what extent the students were able to use the procedure they were taught for calculating the probability of independent and dependent events (p. 4):

In a box, there are three red marbles and four black ones. One extracts a marble by chance (without looking).

- a. What is the probability of drawing a red marble?
- b. What is the probability of drawing a black marble?
- c. Let us consider that the drawn marble was a black one. After extracting it you put it back in the box and you make a second extraction. What is the probability that this time too the extracted marble will be black?
- d. An extraction has been performed and the extracted marble was black. This time, the marble is not returned to the box. A second marble is drawn by chance. What is the probability that this marble too, will be black? Or red?

Responses of students differed as the age of the students increased. The majority of the fifth graders - 76% - were not able to solve independent event problems, while 78% were unable to solve dependent event problems. As the age of the student increased, there was a dramatic improvement for the solving of independent event problems: 70% of the sixth graders and 92% of the seventh graders were able to solve the independent event problems. However, the ability to use the procedures taught in class to calculate the dependent events did not grow as fast: 42% of the sixth graders and 72% of the seventh graders were able to solve dependent event problems. Through categorizing the responses to the question, three main categories of misconceptions emerged, possibly explaining the student errors. The first misconception was the tendency to relate the two sets of marbles involved one to the other, instead of relating the set of expected outcomes to the whole set of possible outcomes. The second misconception was that of expressing the probability as a $1/n$ ratio. A third misconception was that the student forgot that the box has one less marble and does not adjust the number of possible outcomes accordingly. The researchers defined the misconception of the student forgetting to subtract one from the number representing the whole set "coordinating capacity".

The remaining questions on Questionnaire A for the experimental group also found that as the age of the student increased, the number of probability questions pertaining to compound events; proportional reasoning; and describing examples of certain, chance and impossible events, simple and compound events improved with age. From the results of Questionnaire A, it was apparent that conceptual and procedural acquisitions vary across the ages. The students in the fifth grade did not benefit as much from the experimental class as the seventh graders. However, this statement can only be

said when comparing the percentages. The researchers did not provide statistical evidence to this claim.

Similar findings of student's age were noticed on Questionnaire B. The older the student, the more likely they were to understand the concept. Questionnaire B determined the students understanding of the conception of luck, representativeness, negative recency effect, outcomes in a stochastic experiment, and proportional reasoning. Questionnaire B does not require special knowledge of probability; hence, Questionnaire B was administered to both the experimental and control groups. The first question referring to the concept of certain events, the results show that the factor of age was more effective than the factor of instruction. When asked about luck and chance, the results were the same for the control and experimental group. The majority of the questions comparing the control group to the experimental group indicated there was a clear and steady increase with age of the students than of the affect of the instruction with 60-70% of the sixth graders and 80-90% of the seventh graders were able to understand and use correctly most of the concepts covered in class. Unfortunately, it appeared that the majority of the concepts were too difficult for the fifth graders.

The researchers acknowledged that they did not conduct a pre / post-test analysis of the experimental group to statistically justify that the experimental lessons did or did not influence students' probabilistic intuitions. The researchers also acknowledged that the questions posed to the students might have been too difficult for the students to understand, and were not appropriate for this age level. This was apparent in the analysis of the test results from the fifth grade classes. The fifth grade classes scored low in all categories - below 50% - on both questionnaires. This may be an indication that the

intuitive attitudes were not properly assessed for this age level. Similarly, the fifth grade students had a difficult time understanding the true meaning of the questions and either did not answer the question or answered at random. However, when comparing the sixth and seventh grade classes, the results indicated that in both the experimental and control groups, age was a larger factor in answering the questions correctly, than the experimental lesson factor.

Observing that most mathematics teachers present probability in a traditional tautological manner, Castro (1998) noticed the need to develop an innovative teaching approach. Traditionally, math is taught through logic with statements such as "If P, then Q"; "Not P implies not Q", etc. making the learning of mathematics more of an empirical and a priori model. However, Castro believed the learning of probability is more quasi-empirical. The quasi-empirical perceptive found in teaching probability highlights counter-examples, refutations, and critiques. Since the teaching of probability cannot be compared to teaching mathematics, the learning of probability cannot be similar to the learning of mathematics. Knowing that most mathematics teachers do want to use the hypothetical-deductive nature of mathematical theories in teaching probability, teachers will try new strategies to form a series of stimulating classroom activities to explicitly show students the mathematical relationships. However, whenever teachers get a sense of imbalance of their new teaching strategies, and at the first sign of difficulties in the educational process, they tend to return to traditional activities with which they feel more comfortable. Castro (1998) was interested in constructing an instructional model based upon conceptual change to improve student's performance in elementary probability. Hence, in this study, Castro looked at four main improvements in student's conceptual

change with the use of a new instructional model: performance in elementary calculations, performance in reasoning in probability while diminishing the biases and mistaken conceptions, improve students' attitudes toward mathematics compared with the traditional method, and a higher conceptual change in the students than that produced by the traditional method.

The population for this study consisted of six first-year high school classes in Madrid, Spain from three different schools. In Spain, the first year students are typically 14-15 years old. The six classes were divided into two groups: experimental group and control group.

The experiment consisted of 15 instructional hours on a probability theory course at the introductory level. Both the experimental and control groups used the same textbook, course syllabus, lesson objectives, and time constraints. The control group presented the content using lessons explained by the teacher, and the students solving the problems from the book. The experimental group consisted of a quasi-empirical teaching method. The quasi-empirical method required the teachers to diagnose the thinking of the students, clarify student's ideas, carry out random experiments to encourage cognitive conflict, apply new ideas to new contexts, and revise the previous ideas with the knowledge gained through the process.

The data for this study were collected in four categories: student performance in probability calculations, student performance in probability reasoning, student attitudes towards mathematics, and level of student conceptual change. The student performance in probability calculations was measured using a pre-test / post-test questionnaire. The questionnaire asked the students about concepts in probability, principles and procedures,

and problem solving tasks. The questions had been selected from a common mathematics text used in Spain during the first year-high school. The student performance in probability reasoning was measured using an open-ended, qualitative questionnaire in a pre-test / post-test format. The questionnaire included items that dealt with everyday random phenomena and with deceptive situations in which the student's intuitions enter into conflict with the formal laws of chance. The items on the qualitative probability-reasoning questionnaire had been selected from the abundance of literature on probability thinking. Student attitudes towards mathematics was also measured using a pre-test / post-test format using the Gairin's 22-item Likert Scale attitude test. The study did not give further information about the attitude scale, nor did it indicate if it is specifically for attitude towards math. Finally, the level of student conceptual change was measured by counting the number of previous mistaken conceptions that were positively overcome after the experimental period. Also, the number of previous conception that were correct intuitions and that the teaching did not change into incorrect ones. To measure this variable, the study used the results from the probability reasoning and calculation tests. For the four main categories of data collection, the study does not discuss the validity or reliability of the questionnaires, attitude test, or computing the conceptual change of the students using scores from the questionnaires.

The unit of analysis was the combined scores of the experimental groups against the combined score of the control group. The study does not state an alpha level for a significant range. The data for this study were analyzed using an analysis of variance of one factor -- the treatment -- against three independent variables: performance in probability calculations, performance in probability reasoning, and attitudes towards

mathematics. The results indicate that comparing the dependent factor -- the treatment -- against the probability reasoning pre-test, the probability calculation pre-test, and the initial attitude test, there was no statistically significant difference between the experimental and control groups ($p > .05$, no F-score reported). This result indicated that both groups started with the same attitude towards mathematics and from the same level of skills and knowledge in probability. Given the design of the experiment -- pre-test, intervention, post-test -- a variance-covariance analysis was conducted. The factor was the two teaching models, and the covariant variables were the results of the intuitive reasoning post-test, the probability calculation post-test, and the attitude towards mathematics after the intervention. The results of the influence of intuition on performance in probability and attitude towards mathematics showed a significant difference existing in the mean scores of the probability reasoning post-test ($F_{1,116} = 46.18$, $p < .01$) and the probability calculation post-test ($F_{1,116} = 26.30$, $p < .01$) in favor of the experimental group. There were no significant differences with regard to the teaching method and attitude towards mathematics after the intervention ($p > .05$, no F-scores reported).

To analyze the conceptual change of the students over time, the study used the results from the probability reasoning and calculation tests. First, items from the two tests were selected and categorized as either intuitive or counter-intuitive. Next, each item was treated as a variable, and assigned a value of 0, 1, or -1 based on if there was no change, a change to incorrect to correct, or a change from correct to incorrect in the responses from the pre-test to post-test results. To compare the levels of conceptual change between the experimental and control group, a chi-square test was used to

contrast the percentage of values of each item value. The results indicate a statistical significant difference in seven of the 12 items used for this analysis, in which the percentages of 1 values are significantly higher in the experimental groups and the percentages of -1 are significantly higher in the control group (no test scores reported).

The results from this study do indicate that the instruction model based on a quasi-empirical and conceptual change perspective do significantly improve students' skills in elementary probability calculations, compared to the traditional instruction method. Also the model of instruction through conceptual change significantly improves student's intuitive probability reasoning, compared to the traditional instruction model. However, the models of instruction through conceptual change did not modify significantly students' attitudes towards mathematics. Finally, a higher level of conceptual change was produced in the group that followed conceptual change methodology than in the traditional teaching group.

Elementary-Level Teaching Programs

Ojemann, Maxey, and Snider (1965a) devised a guided learning program for helping a child learn the elementary aspects of probability. In their study, they wanted to test the effectiveness of this program at the third-grade level.

The guided learning program for this study was based upon Piaget's learning theory, and a learning theory constructed by the researchers. The theoretical framework used for the guided learning program was (p. 321):

When a child is confronted with a situation in which he has to make a choice or decision, the problem is one of selecting the response that has the best chance of producing the result he desires. He thus has to make an estimate or a prediction. If the information he has is some way incomplete, the logical procedure is to put the prediction in probability terms.

To make a decision in probability terms, the subject has to abstract "completeness of information available" from other aspects of the situation, such as the size, shape, color of the objects with which he is dealing; the appearance of the people involved in the situation; the particular place where the event takes place; what others are doing, and so on. When the request is made to "choose one" or "make a guess", these stimuli should arouse the responses, "What information about this do I have or can I get?"

When a child is placed initially in a situation in which he has to make a decision, a variety of responses may be aroused depending on previous interaction of organism and environment. The responses may be related to various aspects of the situation - the people involved, the nature of the objects, if any, that he must manipulate, the familiarity or strangeness of the situation.

The problem in designing the sequence of experiences is one of using stimuli to intensify the responses represented by "What information do I have?" so that when the child is asked to "make a guess", or "pick one", there will arise the responses represented by "What do I know that will help me?" and responses to other aspect of the situation will be minimized.

The guided learning program consisted of five consecutive days of 30-minute instruction, discussions, and hands-on activities taught by one instructor. The program consisted of discussions on various situations that occur in everyday life in which one has to make a best guess or prediction; the possibility of gathering more information to help make a prediction; the effect of additional variables added to the events; and the collection of data and a discussion on frequency.

The population for this study consisted of 48 students within two third grade classes of a mid-west elementary school. The school was located in a section of the community consisting of slightly above middle-class families. One class was chosen to

be the experimental group in which the students participated in the five-day guided learning program. The second class was the control group. To avoid possible Hawthorne Effect, the instructor also visited the control group and told them they were the experimental class. However, it is unclear as to what the instructor did while he visited the control class. The mortality rate consisted of the loss of seven students due to incomplete data and absences

In lieu of a pre-test, the researchers compared the groups according to their scores of the Verbal portion of the Lorge-Thorndike IQ test and the Composite and Arithmetic scores on the Iowa Test of Basic Skills (ITBS). Using the t-test and not stating an alpha-value, the study indicates that there was no statistical significant differences between the control and experimental groups on the Verbal IQ scores ($t = .994$, no p-value reported); the Arithmetic ITBS scores ($t = .880$, no p-value reported); or the Composite ITBS scores ($t = .754$, no p-value reported).

The data for this study were collected through the administering of four post-tests. The instructor administered two of the tests, and a person unknown administered the other two tests to both groups. The tests were administered the week following the conclusion of the learning program, and each test was given on a separate day. The first test consisted of 25 questions based on a situation involving the selection of various proportions of colored objects from a bag. The questions were presented orally, and the students wrote their responses. This test yielded a reliability coefficient (KR-formula 21) of .85 using all experiments and control subjects. The second test consisted of two 12-question sections. The first section consisted of predicting an image on the top card of a deck based on various proportions of shapes written on each card. The second section

consisted of showing the students the top card of a deck and the students had to predict the image on the second card. This test yielded a reliability coefficient (KR-formula 21) of .72 using all experimental and control students. The third test was based upon the Decision Location Test developed for another study. The test consisted of presenting a series of 15 slides, beginning with a slide that displayed a small segment of an object, and continuing with each successive slide adding something so that the object was practically complete on the final slide. The students were asked what they thought the object was with each slide. The final test consisted of placing a constant proportion of black and white objects in a box and not allowing the students to know the proportion. The students were asked to predict the color that would be drawn in the final eight blocks of 12 predictions, knowing the first four draws.

The unit of analysis was the average scores earned on the post-test for each the control group and the experimental group. The results from the four post-tests were analyzed using a t-test comparing the experimental against the control group. An alpha-value was not reported for statistical significance. The first test on predicting the color of the object drawn from a bag when the proportion was known showed the experimental group answered statistically significantly more questions correctly than the control group ($t = 6.46, p < .01$). The second test on predicting shapes drawn on cards indicated the experimental group answered statistically significantly more questions correctly than the control group ($t = 3.49, p < .01$). The results from the Decision Location Test were reported in two different formats: comparison of the mean number of guesses other than "don't know" prior to point of correct perception, and comparison of slide number at which "earliest guess" other than "don't know" occurs. Comparing the mean number of

guesses other than "don't know" prior to point of correct perception indicates the experimental group answered all four slide presentations statistically significantly sooner than the control group ($t = 3.29$ to 4.21 , $p < .01$ for all four examples). Comparing the slide number at which "earliest guess" other than "don't know" occurs indicates the experimental group was able to predict three of the four slide presentations statistically significantly sooner than the control group ($t = 2.56$ to 4.76 , $p < .05$ for three slide presentations; $t = 1.70$, p -value not reported for fourth slide presentation).

On the second test, the students had to predict the color of the object drawn from a bag with an unknown constant proportion of two different colors. The scores were compared in two categories: trials 5-8 and trials 9-12, and knowing the outcomes of trials 1-4. There was no statistically significant difference in predictions between the control and experimental groups in either category ($t = .94$ and 1.27 , no p -values reported). However, using descriptive statistics, the researchers plotted the mean scores per trial for the experimental group and control group on a line-graph. The line-graph indicated that the experimental group consistently answered more correct responses than the control group.

The results from this study indicate that the students in the experimental group were acquiring a considerable ability to relate their predictions to the events presented to them. When information was given to them, they were able to correctly predict the outcome more often than the control group. Also, they tended to wait before making a prediction when only a small amount of information was available and more would be supplied.

In a second study, Ojemann, Maxey, and Snider (1965b) investigated the effect of their guided learning program on a fifth grade class. The population for this study consisted of six fifth grade classes from three different mid-west elementary schools in similar neighborhoods. Two classes from School A and one class from School B were assigned as the experiment group, while the three classes in School C were assigned as the control group. This report does not indicate how the three elementary schools were chosen, how each class was assigned to an experimental or control group, or if the schools had similar curriculums.

The study consisted of subjecting the experimental group to a series of five 30-minute guided learning programs, and both the control and experimental groups watched two movies: "The Scientific Method" and "Weather, Why It Changes". To avoid the Hawthorne Effect, both the control and experimental groups were told that they were the experimental group. The five guided learning programs consisted of short lessons on the discussion of risk, chance, prediction, and maximizing; experiments to illustrate difference chances of objects of equal probability to be drawn; discussions on predictions based on a proportion of objects with similar characteristics; the need of information for predictions; discussion and experiments on conditional probability; and discussions have to revise predictions as additional information becomes available. One outside instructor taught the five 30-minute guided learning programs to the experimental group. The study does not mention if this instructor also visited the control groups, as in the previous study. However, the movies were shown by a research assistant who did not teach the classes.

The data were collected using one pre-test and four post-tests. Both the pre-test and post-test were administered to the control and experimental groups. Also, for comparison, the scores of the Otis Group Intelligence Scale and the composite and arithmetic scores of the Iowa Test of Basic skills were collected for each student. The pre-test consisted of 15 questions asking the students to predict the color of an object drawn from a pool of known proportions of two colored objects. The first test consisted of 25 questions based on a situation involving the selection of various proportions of colored objects from a bag, knowing the proportion of each color represented. The questions were presented orally, and the students wrote their responses. The second test consisted of two 12-question sections. The first section consisted of predicting a shape on the top card of a deck based on various proportions of shapes written on each card. The second section consisted of showing the students the top card of a deck and the students had to predict the shape of the second card. The third test was based from the Decision Location Test developed for another study. The test consisted of presenting a series of 15 slides, beginning with a slide that presented a small segment of an object, and continuing with each successive slide adding something so that the object was practically complete on the final slide. The students were asked what they thought the object was with each slide. The final test consisted of placing a constant proportion of black and white objects in a box and not allowing the students to know the proportion. The students were asked to predict the color that would be drawn in the final eight blocks of 12 predictions, knowing the first four draws. The study does not discuss the validity or reliability for the pre- and post-tests.

The unit of analysis was the average scores earned on the pre- and post-test for each the control group and experimental group. The results from tests were analyzed using a T-test comparing the experimental to the control group. An alpha-value was not reported for statistical significance. The pre-test results indicate there is no statistical difference between the control group and experimental group on the 15-item pre-test ($t = -1.57$, no p-value reported), the Otis Intelligence Scale ($t = -.27$, no p-value reported), ITBS arithmetic component ($t = -.62$), or the ITBS composite ($t = -.62$, no p-value reported). The first post-test on prediction the color of the object drawn from a bag when the proportion was know showed the experimental group answered statistically significantly more questions correctly than did the control group ($t = 7.07$, $p < .01$). The second test on predicting shapes drawn on cards indicated the experimental group answered statistically significantly more questions correctly than the control group ($t = 3.97$, $p < .01$). The results from the Decision Location Test were reported in two different formats: comparison of the mean number of guesses other than "don't know" prior to point of correct perception, and comparison of slide number at which "earliest guess" other than "don't know" occurs. Comparing the mean number of guesses other than "don't know" prior to point of correct perception indicates the experimental group answered three of the four slide presentations statistically significantly sooner than the control group ($t = -1.91$, no p-value reported; and -2.47 to -3.18 , $p < .01$ for the other three examples). Comparing the slide number at which "earliest guess" other than "don't know" occurs indicates the experimental group was able to predict two of the four slide presentations statistically significantly sooner than the control group ($t = 2.04$ to 2.44 , $p < .05$ for two slide presentations; $t = 1.11$ to 1.33 , p-value not reported for the remaining

two slide presentations). Analyzing the results for the final test, when the students had to predict the color of the object drawn from a bag with an unknown constant proportion of two different colors, the scores were compared for each of the 15 trials knowing the outcomes of trials beforehand. There was no statistically significant difference in predictions between the control and experimental groups for the first trial ($t = .27$, no p -values reported). However, for the remaining 14 trials, there was a statistically significant difference between the control and experimental group ($t = 2.25$ to 5.35 , $p < .5$ for all 14 trials). The results of the final post-test were also reported using descriptive statistics. The researchers plotted the mean scores per trial for the experimental group and control group on a line-graph. The line-graph indicated that the experimental group consistently answered more correct responses than the control group.

The results of the first three post-tests indicated that the fifth grade students had developed a considerable ability to relate their predictions from the information available. The results of the fourth post-test, concerning the ability to predict the outcome without prior knowledge of the possible chances of success, indicate that the fifth grade experimental class was successful in using the information that they gathered from the previous events to predict the future events.

Conclusion

The studies investigating the effects of probabilistic instruction on students illustrate some trends and patterns. A major trend that is noticed was the recognition that the teaching and learning of probability changed with age. Ojemann, et al.(1965a, 1965b) observed in their experimental studies on third and fifth graders that those

students exposed to probabilistic instruction did improve their understanding of basic probabilistic understanding, and the experimental students waited for more information on an event before claiming a prediction of the outcome. When conducting experiments with middle school level students, Fischbein and Gazit (1984) concluded that age was more important in student understanding of probability than the experiments conducted in class. Finally, collegiate level studies conducted by Austin (1974) and Shaughnessy (1977) found that instruction improved students' probabilistic understanding and reasoning. This trend indicates that age is a factor in student understanding of probability, and understanding of instruction. This result is also supported by the previous findings of Fischbein and Schnarch (1997) on the investigation of evolution, with age, of probabilistic intuitively based misconceptions.

A pattern resulting from these studies also indicates that the method of teaching may have an impact on student learning of probability. Castro (1998) indicates that the instructional format - deductive as opposed to inductive - improves students' probabilistic understanding and reasoning, while changing student's misconceptions. Collegiate level studies conducted by Austin (1974) showed that the use of manipulatives in the classroom did not make a significant contribution to student's learning. However, the use of graphs, figures, diagram, and results of experiments significantly improved college student's understanding of probability.

Teachers of probability generally agree that people have a great deal of difficulty with conditional probability. The research conducted by cognitive psychologists and mathematics and statistics educators had an effect on the type of questions that mathematics educators explored in the teaching and learning of probability. The studies

reviewed in this section indicate the possibility that age is a large factor in the understanding of probability and the method of instruction used in the classroom may affect how students learn. The results of these studies allow mathematics and statistics educators to gain a better understanding of how to conduct their lessons and create a better curriculum.

Research on the Framework for Students' Probabilistic Thinking in Instruction

Jones, et al. (1997, 1999) and Tarr and Jones (1997) claim that in order to create appropriate curriculum, a theoretical framework of how students understand a concept needs to be developed. Based on this theoretical framework, teachers are given a coherent picture of student understanding that is needed to guide classroom instruction and assessment. The next three studies included in this review illustrate their current work on creating a research based framework for assessing elementary and middle school students' probabilistic thinking.

Previously, researchers looked at young children's probabilistic thinking, defining students' heuristics used to solve probabilistic problems, and conducting experiments to see if instruction can improve a student's choice of heuristics for solving the problem. However, Jones, et al. (1997) noticed a gap in the research on development of a framework for systematically describing and predicting young children's thinking in probability as they learn the theory behind probability. In their first study, Jones, et al. (1997) conducted a longitudinal study to develop and evaluate a possible Probabilistic Theoretical Framework for assessing probabilistic thinking in children. The framework was developed based on four main probability constructs: sample space, probability of an

event, probability comparisons, and conditional probability. Each of the four constructs was divided into four levels: Subjective Thinking, Transitional, Informal Quantitative, and Numerical Reasoning.

Observing two third grade classes during the course of one school year, the researchers collected data through three structured interviews on eight case studies concerning 20 probabilistic tasks. The interviews took place during three assessment points during the school year: fall, winter, and spring. Each class participated in an eight week Probability Problem Task Program implemented at different times throughout the school year. The first class took part in the program immediately following the first interview, and prior to the second. The second class participated in the program after the second interview, and prior to the third.

The results of the study were described in three areas: refinements made to the Probabilistic Thinking Framework, profiles and stability of the students' thinking across the four constructs, and analysis of the probabilistic thinking at each level. Figure 1 shows the final Probabilistic Thinking Framework after the refinements were made from the interviews.

CONSTRUCT	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Qualitative	Level 4 Numerical
SAMPLE SPACE	<ul style="list-style-type: none"> • Lists an incomplete set of outcomes for a one stage experiment 	<ul style="list-style-type: none"> • Lists a complete set of outcomes for a one-stage experiment • Sometimes lists a complete set of outcomes for a two-stage experiment using limited and unsystematic strategies 	<ul style="list-style-type: none"> • Consistently lists the outcomes of a two-stage experiment using a partially generative strategy 	<ul style="list-style-type: none"> • Adopts and applies a generative strategy which enables a complete listing of the outcomes for a two and three stage case
PROBABILITY OF AN EVENT	<ul style="list-style-type: none"> • Predicts most/least likely events based on subjective judgements • Recognizes certain and impossible events 	<ul style="list-style-type: none"> • Predicts most/least likely event based on quantitative judgements but may revert to subjective judgements 	<ul style="list-style-type: none"> • Predicts most/least likely events based on quantitative judgements including situations involving non-contiguous outcomes • Uses numbers informally to compare probabilities • Distinguishes "certain", "impossible", and "possible" events, and justifies choice quantitatively 	<ul style="list-style-type: none"> • Predicts most/least likely events for single stage experiments • Assigns a numerical probability to an event (it may be a real probability or a form of odds.)
PROBABILITY COMPARISONS	<ul style="list-style-type: none"> • Compares the probability of an event in two different sample spaces, usually base on various subjective or numeric judgements. • Cannot distinguish "fair" probability situations from "unfair" ones 	<ul style="list-style-type: none"> • Makes probability comparisons based on quantitative judgements (may not quantify correctly and may have limitations where non-contiguous events are involved) • Begins to distinguish "fair" probability situations from "unfair" ones 	<ul style="list-style-type: none"> • Makes probability comparisons based on consistent quantitative judgements • Justifies with valid quantitative reasoning, but may have limitations where non-contiguous events are involved • Distinguishes "fair" and "unfair" probability generators based on valid numerical reasoning 	<ul style="list-style-type: none"> • Assigns a numerical probability measure and compares • Incorporates non-contiguous and contiguous outcomes in determining probabilities • Assigns equal numerical probabilities to equal likely events
CONDITIONAL PROBABILITY	<ul style="list-style-type: none"> • Following one trial of a one-stage experiment, does not give a complete list of outcomes even though a complete list was given prior to the first trial • Recognizes when certain and impossible events arise in non-replacement situations 	<ul style="list-style-type: none"> • Recognizes that the probability of some events changes in a non-replacement situation; however, recognition is incomplete and is usually restricted only to events that have previously occurred 	<ul style="list-style-type: none"> • Can determine changing probability measures in a non-replacement situation • Recognizes that the probability of all event change in a non-replacement situation 	<ul style="list-style-type: none"> • Assigns numerical probabilities in replacement and non-replacement situations • Distinguishes dependent and independent events

Table 1: Refined Probabilistic Thinking Framework for Assessing Probabilistic Thinking (Jones, et al., 1997, p. 111)

The second result consists of the profile and stability of thinking across the four constructs. The profiles of the case studies indicate that the internal consistency of the child's probabilistic thinking across the constructs were consistent before instruction.

However, after receiving instruction, the data indicated a lack of internal consistency in the students, with a tendency for sample space and conditional probability to lag behind the others.

Third, the researchers took a closer look at the probabilistic thinking at each level, hoping to explain the inconsistencies among the analysis from the previous part. From the recorded interviews, student responses and thinking strategies were analyzed. Due to the mixture of responses by the students, and the various levels of understanding recorded, the study then picked exemplary students to analyze for each of the four levels. The students who represented Level 1 of thinking appeared to have a narrow perspective in relation to probability situations. Level 1 students did not recognize random phenomena. These students also based their responses on subjective beliefs. Their responses to sample space knowledge reflected that they could not provide a complete listing, but they tend to focus subjectively on what is more likely to happen than on all the possibilities. Finally, in response to the other three constructs, the students used a subjective reasoning to answer the question, rather than rely on quantitative logic. The students who represented Level 2 thinking appeared to be in a transition between subjective and informal quantitative judgements. When responding to the questions pertaining to sample space, Level 2 students were able to identify the complete set of outcomes for a one-stage experiment, and partial lists for a two-stage experiment. Despite the greater understanding of sample space, the students still tended to overlook all outcomes when asked to determine probabilities. Still, in Level 2, the idea of conditional probability is still incomprehensible. Level 2 students recognize the existence of conditional probability, but they tend not to recognize the effect on the

resultant probability. The students who represented Level 3 thinking appeared to start using quantitative judgements in determining the probabilities. The Level 3 students were not always able to use correct probabilities; however, they were able to assign quantitative values and make comparisons. Level 3 students were also able to understand conditional probability in the sense that they recognize that the probabilities of all events change in a non-replacement situation. Finally, evidence that the students used strategies when listing their outcomes defined the students moving to a more quantitative reasoning. The students who represented Level 4 thinking were consistently able to adopt the strategies and use them systematically to generate the outcomes of an experiment. Level 4 students were also able to assign and use numerical probabilities in both equally and non-equally likely situations. They were able to use a generative strategy to list the outcomes of both two- and three-strategy experiments, and they appear to use sample space as the basis for finding and comparing numerical probabilities. Their ability to compute probabilities effectively was also evident in their recognition and computation of conditional probabilities.

The results of this longitudinal study have shown that the internal consistencies of the students across the constructs decreased during the school year and exposure to the program. However, it may not be reasonable to suggest that the level of the probabilistic thinking of all students will follow an ordered progression through the levels of the Probabilistic Thinking Framework, or even that their thinking should be consistent. It was evident that if the student was weak in the understanding of sample space, the level of the student's thinking continued to be low, until they were able to gain the understanding of sample space.

Tarr and Jones (1997) further formulated, refined, and validated a framework for assessing middle school students thinking in conditional probability and independence, based on the previous work conducted by Jones, et al. (1997). The population for this study consisted of 15 students who had not been exposed to probability instruction. The 15 students were randomly selected from fourth and fifth grade classes in an elementary school and sixth, seventh, and eighth grade classes in a middle school. The randomization process consisted of selecting a student from the top third, middle third, and lowest third in each grade based on scores of a mathematics achievement test.

The 15 students participated in a structured interview consisting of eight tasks assessing the students on conditional probability, and six tasks assessing independence. From the data collected in the interviews, the researchers were able to refine their results and validate a new framework. The results of the validation process was presented in three parts: refinements made to the framework following data collection, profiles and stability of students' thinking across the two construction, and summaries and exemplars to illustrate the four levels of probabilistic thinking in the revised framework. The refined framework was developed following the analysis of the case-study data. Table 2 illustrates the final framework.

CONSTRUCT	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Qualitative	Level 4 Numerical
CONDITIONAL PROBABILITY	<ul style="list-style-type: none"> • Recognizes when "certain" and "impossible" events arise in replacement and non-replacement situations • Generally uses subjective reasoning in considering the conditional probability of any even in a "with" or "without" replacement situation • Ignores given numerical information in formulating predictions 	<ul style="list-style-type: none"> • Recognizes that the probabilities of some events change in a "without replacement" situation. Recognition is incomplete, however, and is usually confined to events that have previously occurred • Inappropriate use of numbers in determining conditional probabilities. For example, when the sample space contains two outcomes, always assumes that the two outcomes are equally likely • Representativeness acts as a confounding effect when making decisions about conditional probability. • May revert to subjective judgements 	<ul style="list-style-type: none"> • Recognizes that the probabilities of all events change in a "without replacement" situation, and that none change in a "with replacement" situation • Keeps track of the complete composition of the sample space in judging the relatedness of two events in both "with" and "without" replacement situations • Can quantify, albeit imprecisely, changing probabilities in a "without replacement" situation 	<ul style="list-style-type: none"> • Assigns numerical probabilities in "with" and "without" replacement situations • Uses numerical reasoning to compare the probabilities of events before and after each trial in "with" and "without" replacement situations • States the necessary conditions under which two events are related
INDEPENDENCE	<ul style="list-style-type: none"> • Predisposition to consider that consecutive events are always related • Pervasive belief that they can control the outcome of an event • Uses subjective reasoning which precludes any meaningful focus on the independence • Exhibits unwarranted confidence in predicting successive outcomes 	<ul style="list-style-type: none"> • Shows some recognition as to whether consecutive events are related or unrelated • Frequently uses a "representativeness" strategy, either a positive or negative recency orientation • May also revert to subjective reasoning 	<ul style="list-style-type: none"> • Recognizes when the outcome of the first event does or does not influence the outcome of the second event. In "with replacement" situations, sees the sample space as restored • Can differentiate, albeit imprecisely, independent and depended events in "with" and "without" replacement situations • May revert to the use of a representativeness strategy 	<ul style="list-style-type: none"> • Distinguishes dependent and independent events in "with" and "without" replacement situations, using numerical probabilities to justify their reasoning. • Observes outcomes of successive trials but rejects a representativeness strategy • Reluctance or refusal to predict outcomes when events are equally likely

Table 2: Refined Framework for Assessing Middle School Students' Thinking in Conditional Probability and Independence (Tarr & Jones, 1997, p. 48)

The second data analysis consisted of looking at the profiles and stability of the students' thinking across the two constructs. The results indicate that 11 of the 15 students had consistency in their thinking patterns across the two constructs. In the remaining

four cases, the difference in the thinking levels were only one level apart. These observations support the stability hypothesis for the framework.

The last analysis consisted of a closer look at the probabilistic thinking at each level, combining the results from each grade. The responses and thinking of each of the students who served as case studies were analyzed, and summaries and exemplars were produced to illustrate the thinking patterns outlined in the refined framework. The students who represented Level 1 thinking appeared to rely on subjective judgements, ignore relevant quantitative information, and generally believe that they can control the outcome of an event. Since they reason without quantifiable justifications, Level 1 students form conditional probability judgements by constructing their own reality by searching for patterns which do not exist or by imposing their own system of regularity. By using their own recent experience, they tend to predict or estimate the chance of an event based on that experience. In judgement of independence, Level 1 students tend to believe those previous outcomes always influence future outcomes, basically denying the existence of independence. The students who represent Level 2 are in transition between subjective and informal quantitative thinking. When computing conditional probability, Level 2 students can sometimes make appropriate use of quantitative information, but are easily distracted by irrelevant features. When outcomes do not occur as expected, Level 2 students revert to subjective outcomes. In addition, they are still prone to assuming those two outcomes are equally likely when a situation has two outcomes. Level 2 students are able to recognize that the probabilities of some events change in "without replacement" situations, but recognition is incomplete and is usually confined to events that have previously occurred. The students who represent the third level of thinking are

aware of the role that quantities play in forming conditional probability judgements. Without using precise numerical probabilities, Level 3 students are able to solve conditional probabilities through the use of relative frequencies, ratios, or some form of odds. Level 3 students can keep track of the sample space, especially in conditional probability events. By the ability to keep track of sample space, they are also able to recognize independent events. However, Level 3 students sometimes revert to representativeness strategies in dealing with a series of independent trials. Finally, the students who represent the fourth level of thinking do use numerical reasoning to interpret probability situations. Level 4 students recognize sample space and its importance in determining conditional probability. Level 4 students are able to calculate the conditional probability, while also giving a thorough explanation of the computations involved. Using numerical thinking, Level 4 students consistently distinguish between independent and dependent events and can identify the conditions under which the two events are related. Their reliance on numerical reasoning enables them to hold strong convictions when making conditional probability judgements, and these students are reluctant to make predictions when all outcomes are equally likely. Finally, Level 4 students recognize that certain outcomes do arise, even if it is against all odds.

Unlike the validation of the Probabilistic Thinking Framework conducted by Jones, et al. (1997), the validation of the Conditional and Independent Probabilistic Framework by Tarr and Jones (1997) was conducted through interviews of students who have not had instruction on probability. This was not a longitudinal study, nor did it check for student understanding after instruction. However, the information gathered

from this study can provide insight and direction for further validation of such a framework on student understanding of conditional probability and independence.

Continuing with their interest in student's probabilistic thinking, Jones, et al. (1999) used the Probabilistic Thinking Framework developed in their previous study (Jones, et al. 1997) to create and evaluate an instructional program in probability on student learning. The main goal of the study was to evaluate the thinking of third grade students according to the Framework and observe possible growth due to the program. This longitudinal experimental study intended to evaluate the effects of the instructional program on student learning on two third grade classes during the course of a school year.

The researchers did acknowledge that the population for this study was the same as their previous study (Jones, et al., 1997). The population for this study consisted of two third grade classes. One class was chosen to be the early-instruction group, and the second class was the delayed-instruction group. Two students from each class were selected to be the target students for the study. Jana and Kerry were selected from the early-instruction group, and Cory and Deidra were selected from the delayed class.

The instructional program used for this study was based on the Probabilistic Thinking Framework created and validated in their previous study (Jones, et al., 1997). Although the framework was being validated while conducting this study, the researchers claim "it is true that the instructional program was implemented using the unrefined framework. However, reference to our validation study will reveal that minimal changes were made to the framework during the refinement process" (p. 490). Hence, the Probabilistic Thinking Framework upon which the instructional program was based was

considered to be similar enough to the refined framework, thus the instructional program was claimed to be based upon the refined framework as well.

The eight-week instructional model implemented in the instructional program consisted of two main designs. First, the instructors planned and implemented their instruction based on the Probabilistic Thinking Framework. The second design was consistent with a socio-constructivist orientation to learning. These two key designs were used with the Probabilistic Thinking Framework to create the instructional program used in this study.

The duration of the experiment was one school year. During the year, each of the two classes participated in three probability knowledge assessments: Fall, Winter, and Spring. In addition, each of the classes participated in the instructional program three different times in the school year. The early-instruction class participated in the instructional program after the Fall assessment and before the Winter assessment. The delayed-instruction class participated in the instructional program after the Winter assessment and before the Spring assessment. By comparing the delayed-instruction group with the early-instruction group, it was possible to measure possible benefits of the students being exposed to additional mathematical experiences before participating in the instructional program. It was noted that the delayed-instructional group only received more instruction on whole numbers.

The data for this study were collected using three observational tools: research generated probability knowledge assessments, Mentor Summary Evaluations (MSE), and researcher narratives. The research generated probability knowledge assessments were interviews conducted three times during the school year: Fall, Winter, and Spring. The

interviews consisted of observing each student in the class complete 20 tasks on continuous and discrete events: five sample space tasks, four probability of an event tasks, seven probability comparison tasks, and four conditional probability tasks. The Mentor Summary Evaluations (MSE) consisted of three to five questions designed to help the mentors evaluate each student on the basis of their probability thinking each day. The MSE allowed the study to create an ongoing profile of each student and their probabilistic thinking. The final data collection instrument consisted of researcher narratives of each class session. However, the researcher narratives were only conducted on the four target students - two from each class - each student being observed by a different researcher. On basis of field notes, each researcher constructed a Researcher Narrative Summary (RNS) for their target student each session. To augment the RNS, each target student was videotaped four times over the 16 sessions. The RNS and videotapes provided a description of the target students' thinking on the mathematical tasks presented in the sessions. In addition, the ongoing annotations in the notes help identify trends in the student's thinking.

The data analysis consisted of repeated measures of analysis of variance test and a triangulation of the MSE, RNS, and videotapes. First, the data collected from the probability knowledge assessment tests provided one dependent variable - the students - and two independent variable - the early-instruction group and the delayed-instruction group. From these categories, repeated measures analysis of variance were conducted to check for significant differences between the three assessment points: Fall, Winter, and Spring. The data gathered by the MSE, RNS, and videotapes allowed a triangulation of data analysis for each of the four target students. Each target student was assigned

multiple codes to their 16 MSEs and 16 RNSs. These codes were based on the four constructs and four thinking levels associated with the framework. From the codes, it was possible to devise a time-ordered matrix allowing the display of changes and trends in their thinking levels across the four probability constructs. The triangulation allowed the data to be scrutinized for consistency, commonality, and alternative interpretation of the data. Data from mentors and researchers were further triangulated with assessment data to generate a clear perspective on each target student.

The results of the data analysis were reported in three areas: descriptive analysis of the two classes, analysis of the two instructional groups, and analysis of the four target students. The descriptive analysis consisted of presenting the frequencies of the modal probabilistic thinking levels for each student at the three assessment points. From the data gathered on the MSE, the frequency table indicated that immediately prior to instruction, the delayed-instruction group showed fewer students ($n = 5$) at Level 1 and more students at Levels 2 and 3 (total of 14) than the early-instruction group ($n = 9$ for Level 1 and $n = 9$ for Levels 2 and 3). However, at the beginning of the school year, the delayed-instruction group had a similar class make-up to the early-instruction group (delayed group had 7 students at Level 1 and 12 students at Level 2 and 3, while early group had 9 students at Level 1 and 9 students at Level 2 and 3). This discrepancy, as noted by the researchers, may be due to the delayed-instruction class had an increase in their probabilistic thinking levels during the school year without the benefits of the program. Immediately after instruction, seven students in the early-instruction and twelve in the delayed instruction exhibited a modal of Level 3 thinking. Finally, at the end of

the school year, a total of three students from the two classes still were at the modal of Level 1 thinking.

The second set of results consisted of a statistical analysis comparing the two classes. First, using the students probability performances measured on the assessment tests, a repeated measures analysis of variance revealed significant difference for the three assessment points ($F_{2,70} = 12.88, p < .001$) and significant groups by assessment points interaction ($F_{2,70} = 6.38, p < .01$). Both classes had similar mean scores (early = 12.4 and delay = 13.11), while at the second assessment point, the early instruction group had a higher mean than the delayed (early = 15.93, delayed = 13.95). However, at the final assessment point, the roles reversed (early = 15.00, delayed = 16.63). This disordinal analysis was considered statistically significant (Tukey-HSD test, $p < .05$ in each case).

The final set of results consisted of an analysis of the four target students. From the triangulation of the RNS, MSE, and videotaped data, a number of learning patterns emerged, with four main patterns recognized. First, there was an indication of misconceptions of sample space. The misconceptions appeared to be influenced by subjective judgements and were deep-seated. The second learning pattern concerned the application of part-part reasoning. Part-part reasoning was defined to be thinking that involves the relationship of two parts to each other. This misconception of part-part reasoning influenced students quantifying probability situations in a meaningful way. The third learning pattern concerned the application of both part-part and part-whole relationships in probability situations. Part-whole reasoning was defined to be thinking that represents the relationship of a part to a whole. The ability to recognize the different

concepts in various situations is the key to producing growth in probabilistic thinking. Finally, the last learning pattern consisted of the use of invented or conventional language. As the students used their own language to describe the situation, it was beneficial for them when developing further probabilistic thinking.

After recognizing possible emerging learning patterns, the researchers summarized the learning of each of the four target students: Jana, Kerry, Corey, and Diedra. Jana and Kerry were selected from the early-instruction group, and Cory and Deidra were selected from the delayed-instruction group. Jana reflected a student who was not able to understand the concept of sample space. Due to this, her thinking levels across the four constructs were unstable following instruction than prior to it. Also, the continual misconception of sample space was a hindrance in her continued understanding of the other three concepts. Despite effort during instruction to make the connection of sample space to the concept of probability, subjective judgements dominated her thinking in situations related to all constructs. As mentioned earlier, this basic misconception was not unique. At the beginning of the school year, 15 students displayed this misconception, and by the end of the school year, five students still continued with this misconception. Kerry reflected a student who experienced difficulty in quantifying probabilities. Kerry also had difficulty with the concept of sample space at the beginning of the school year; and her scores reflected an unstable comprehension of the four constructs after instruction. However, once Kerry understood the concept of sample space, she started to use quantitative reasoning for the other three constructs. However, her difficulty was her part-part reasoning. Kerry grouped all situations as "equally likely". Kerry typifies three other students in the study who also had difficulty with part-

part reasoning. Corey reflected a student who showed a growth in probabilistic thinking during the intervention. Corey's initial scores indicated an unstable understanding of the four concepts at the beginning of the school year; however, the instruction helped him build his probabilistic thinking and received more uniform scores at the end. Corey also had no evidence of subjective judgements throughout the study. The instruction was able to clarify his misconceptions and by the end of the study, he was able to obtain Level 3 knowledge of probability. This growth in probabilistic knowledge was also evident in 19 other students. Finally, Deidra reflected a student who recognized the use of fractions in probability, despite her misunderstanding of sample space. Deidra's sample space reasoning still consisted of subjective reasoning. After instructional intervention and a better understanding of sample space, Deidra was able to obtain a Level 4 understanding of the four concepts. However, Deidra still had conflicts understanding sample space and tried not to let her subjective reasoning influence her decisions. Deidra was able to integrate her new knowledge and apply it to probability comparisons in a more quantitative format. No other student in the study was represented by Deidra's growth in probabilistic knowledge.

Jones, et al. (1999) believe the results of this study are able to help create a more effective instructional program for young students. The apparent learning patterns that evolved from the data analysis may have ramifications for children's understanding of probability. The statistical analysis indicates both the early and delayed instruction groups showed significant probability knowledge growth in their performance after instruction. The researchers acknowledge that the students in this study benefited from

working in pairs with adult mentors, receiving individualized instruction that is not possible in a regular classroom.

The Probabilistic Thinking Framework generated by these previous studies enables children's probabilistic thinking to be described and predicted in a coherent and systematic manner. Jones, et al. (1997, 1999) and Tarr and Jones (1997) used case studies to validate the Framework. Jones, et al. (1997, 1999) looked at longitudinal student profiles and the effect of a Probability Problem Task Program, while Tarr and Jones (1997) interviewed students without previous instruction on probability. During the case study analysis, each study made the same assumption: at any given time, a student's probabilistic thinking is stable across the constructs. Although the results from these studies might not apply to the probabilistic thinking of college students, the Framework does present a starting point for similar studies with college students.

Conclusion

This study investigates college students' understanding of conditional probability and how instruction influences their thinking. The studies examined in this literature review took several forms and employed many research techniques. The samples ranged from third grade students (Jones, et al., 1999; Jones, et al., 1997; Ojemann, et al., 1965a) to college level students (Pollatsek, et al., 1987; Fischbein & Schnarch, 1997; Shaughnessy, 1977; Austin, 1974). The majority of the studies utilized a quantitative methodology to answer their research question, Five of the seven quantitative studies used the experimental model (Castro, 1998; Fischbein & Gazit, 1984; Austin, 1974; Ojemann, et al., 1965a; Ojemann, et al., 1965b) and two studies using the descriptive

model (Pollatsek, et al., 1987; Fischbein & Schnarch, 1997). Two studies used a combination of quantitative and qualitative methods (Jones, et al., 1999, Shaughnessy, 1977). The final two studies used a qualitative method exclusively (Tarr & Jones, 1997; Jones, et. al., 1997).

Many studies used the results of Tversky and Kahneman's heuristic studies on probabilistic intuitions as their theoretical framework for their study (Fischbein & Schnarch, 1997; Fischbein & Gazit, 1984; Shaughnessy, 1977). Other studies based their framework on teaching methodological research and intellectual development (Jones, et al., 1999; Jones, et al., 1997; Austin, 1974, Castro, 1998).

The probability topics covered in all eleven of the studies could be found in a typical introductory probability course. However, some studies found the age of the student to be a limitation (Fischbein & Schnarch, 1997; Fischbein & Gazit, 1984). Among the studies measuring the effect of a teaching program to student learning, the time frame of the instructional experiment ranged from one week (Ojemann, et al., 1965a; Ojemann, et al., 1965b) to one school year or term (Jones, et al., 1997, Jones, et al., 1999).

The research highlighted several aspects of the influence of instruction on student understanding of probability. Specifically, college students do have an intuitive understanding of conditional probability (Fischbein & Schnarch, 1997, Pollatsek, et al., 1987; Austin, 1974); the major source of error in computing conditional probability was the confusion between conditional and joint probability (Pollatsek, et al., 1987); the type of course teaching methodology and teaching model has an influence on the development of student's intuition of probabilistic thinking (Shaughnessy, 1974; Castro, 1998); and the

use of graphs, figures, diagrams, and results from experiments significantly improve students understanding of probability, while manipulations and experiments do not affect student understanding (Austin, 1974).

Questions remain on how instruction influences college student's understanding of probability. One issue considered by many studies was the cognitive development of a student's understanding of probability. Many researchers questioned the appropriate age for students to begin studying probability theory (Fischbein & Schnarch, 1997; Fischbein & Gazit, 1984; Ojemann, et al., 1965a; Ojemann, et al., 1965b). A second issue noted by one study consisted of the effects of manipulatives and experiments in the college level mathematics classroom (Austin, 1974). A final issue consists of a Probabilistic Thinking Framework for college level students. The Probabilistic Thinking Framework validated by Jones, et al. (1997) considered the thinking of an elementary student. Further investigation for the possibility of its applications to college level instruction could provide valuable insight. The evidence addressing the question about the impact of instruction on college students use of heuristics and overcoming their difficulties of solving conditional probability problems is obviously very limited. What research exists seems to suggest that these incorrect heuristics and difficulties may be difficult to overcome, even with systematic instruction. Therefore, the purpose of this study is to address the following questions:

1. What are some of the heuristics college students use, and what are some of the difficulties they encounter solving conditional probability problems prior to receiving instruction on sample space, probability of an event, and conditional probability?

2. After attending a two-week class on sample space, probability of an event, and conditional probability, in what ways did the students' heuristics change, and in what ways were they able to overcome difficulties they had previously encountered when solving conditional probability problems?
3. How does each student's understanding of conditional probability compare to the Conditional Probability Framework developed by Tarr and Jones (1997)?

Chapter III

Design and Method

Introduction

Although there has been substantial research on students' probabilistic thinking (Cohen, 1957, 1960; Falk, 1986, 1988, 1989; Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973, 1974, 1980, 1982), little of that research has focused on student's probabilistic thinking in the classroom (Fischbein & Schnarch, 1997; Pollatsek, et al., 1987), and even fewer studies focus on the teaching and learning of probability at the collegiate level (Austin, 1974; Shaughnessy, 1977). This study investigates college students' use of various heuristics, the difficulties they encounter when solving conditional probability problems, and how instruction influences their understanding. This is accomplished through a Probability Knowledge Inventory (see Appendix A) and semi-structured interviews with students enrolled in a mathematics course at a medium-sized state university. The interviews explore the student's knowledge of sample space, probability, and conditional probability and demonstrate in what ways instruction may influence student's use of heuristics and assist them in overcoming their difficulties in solving conditional probability problems.

The research questions are:

1. What are some of the heuristics college students use, and what are some of the difficulties they encounter solving conditional probability problems prior to

receiving instruction on sample space, probability of an event, and conditional probability?

2. After attending a two-week class on sample space, probability of an event, and conditional probability, in what ways did the students' heuristics change, and in what ways were they able to overcome difficulties they had previously encountered when solving conditional probability problems?
3. How does each student's understanding of conditional probability compare to the Conditional Probability Framework developed by Tarr and Jones (1997)?

Subjects

The population for this study consisted of 20 students enrolled in a contemporary mathematics course at a comprehensive university in the Pacific Northwest. This institution was chosen for three reasons. First, the location made it convenient for the study. Second, the majority of the students enrolled in the mathematics course had had no formal instruction in probability. Third, the researcher was the instructor for the class.

The 10-week contemporary mathematics course was designed as a terminal mathematics course to satisfy the baccalaureate core requirement for students not majoring in math, science, or engineering. The text adopted for the course was *Mathematics in Life, Society, and the World* by Parks, Musser, Burton, and Siebler (2000). The authors' intent for this book was to provide students with an enjoyable mathematics class while illustrating the necessity of mathematics in their lives. The material covered in this course was designed to give the students a fundamental

background in statistics, computing interest rates, probability, management mathematics, and game theory.

The students involved in this study were participants in a program designed to assist in the retention of minority, disadvantaged, and disabled students who have traditionally been denied equal access to higher education. This program serves those who may or may not meet the current university admission requirements, but are recognized as having the potential to complete a college degree program.

The class consisted of a wide variety of students with various backgrounds. Students ranged from 18 to 42 years old, eight were freshmen, three were sophomores, seven were juniors, and two were seniors. The subjects in this study did not have strong backgrounds in college level mathematics. Only nine students indicated they had had other college level mathematics courses prior to enrolling in the course, and none of the students had previously enrolled in this course. Only four students in the group indicated that they had had any previous work in probability. Two of the students had previously taken courses taught by the researcher.

At the beginning of the term, the students were presented a short description of the study and they were asked for their cooperation. At that time, the entire class was given Class Participation Informed Consent Forms (see Appendix B), explaining their involvement in the experiment as a class, and Interview Participation Consent Forms, for those who were interested in participating in interviews. Five female students volunteered to be interviewed, one participant was a prior student of the instructor.

Overview of Two-Week Probability and Conditional Probability Instruction

The entire class participated in a two-week probability and conditional probability unit during the fourth and fifth week of the course. Instruction occurred in three, 50-minute classes per week and the instruction over the six days covered the entire chapter on probability and conditional probability. The purpose of this section is to describe the two-week instruction, including a description of the teaching objectives, questions posed to the class, and classroom activities in which the students participated.

Day 1: Defining Probability Terminology

The goal for the first day was to familiarize students with the terminology associated with probability and to conduct an experiment using the new terminology.

The class began with a discussion of the following problem written on the board:

How would you interpret:

1. Your roommate contracted the measles. You had a measles immunization shot; however the doctor told you, you still have a 10% chance of contracting the measles.
2. The probability of a clear sky today is 80%.
3. Your morning cereal box claims there is a game ticket inside. The chance of winning the grand prize is 1 in 150,000.

The class discussed the possible interpretation of each statement, and developed a consensus on each of the interpretations. The students were then given a simple experiment: toss a coin twice and record the results. Prior to conducting the experiment, the students were asked what type of questions could be asked, what are the answers to the questions, and how would you interpret these answers. As a class, the students

developed a working definition for probability. Using the simple experiment as a guideline, the class generated definitions for the experiment, sample space, and possible outcomes. Allowing further exploration of their definitions, the class generated data for this experiment by reproducing the experiment 20 times. Once the data was generated, the students calculated the percentages of each outcome. Then the students were asked if the results were reasonable, and if they could state a theoretical solution.

For the remainder of the class, the students were placed into five groups. Each group was asked to generate data for five different experiments, compute the experimental probability, and discuss what the group believed to be the theoretical probability for each outcome (see Appendix C). The results from the experiments were written on the board for presentation and discussion the next day.

Day 2: Computing Simple Probability

The main goal for the second day was that students would define probability and compute simple probabilities for various experiments and events within the experiment. At the beginning of class, each of the five groups presented their experiments, from the day before, and explained how they calculated the experimental and theoretical probability for their experiments. After the entire class agreed upon the solutions, the class was asked if the working definition constructed the day before could be refined. At this time, the class agreed on a final definition of probability: the relative frequency at which we can expect an outcome to occur. As reinforcement of the concept of probability and some of the properties the students noticed, the students were shown the

Properties of Probability, and asked to summarize each property, in their own words, for homework.

The terms “equally likely” and “events” were written on the board, and the groups were asked to use previous experiments, as examples to develop definitions for these two terms. The discussion on the definitions involved students questioning the sample space of certain events. From this dilemma, the students developed a general equation for the Fundamental Counting Principle of an experiment with two events: if an event A can occur in x ways, and for each of these x ways, and event B , can occur in y ways, then the number of ways events A and B can occur, is x times y . Using their prior experiments as examples, each group was able to list the entire sample and confirm its size using the Fundamental Counting Principle.

Allowing each group to reevaluate their experiments with the event listed on their worksheet, each group derived the probability of the event. Each group also defined two more events that could stem from their experiment, and calculated the probability of each of those events. These results were reported in a whole class discussion.

Day 3: Computing Probability in Complex Experiments

The goal of the third day was for the students to recognize a complex experiment, to list all the possible outcomes using a tree diagram, and, using the tree diagram, to derive various probabilities for each defined event. The class began with a discussion of the following questions written on the board:

A fair coin is flipped four times. Which result is most likely to occur:

- a. HTHT
- b. TTHH
- c. HTTT
- d. HHHH
- e. None of these - explain why not.

After the discussion of the problem, the class was asked to define the term experiment. Referring to the previous experiments conducted in class, the students were shown which experiments were simple experiments and which experiments were complex experiments. Based on this comparison, the students developed a definition for experiment and complex experiment. Those groups who did have a complex experiment were then asked how they derived all the outcomes for the sample space. The explanations led in a discussion on various methods for listing the elements in a sample space of a complex experiment, and further use of the Fundamental Counting Principle developed the day before.

Given the experiment of flipping a coin, then rolling a die, the students were shown a method called "Probability Tree Diagram". Pros and cons of a Probability Tree were discussed against the other methods of generating sample space and probability of various events used by the other groups. The students practiced the concept of developing a Probability Tree Diagram on other experiments. Combining ideas together, the students tried to apply the Fundamental Counting Principle to the Probability Tree.

The class concluded with the students constructing a Probability Tree to list all the possible outcomes of flipping a coin four times, stating the probability of each event, and reaching a class consensus of the solution posed at the beginning of class.

Day 4: Introduction to Conditional Probability

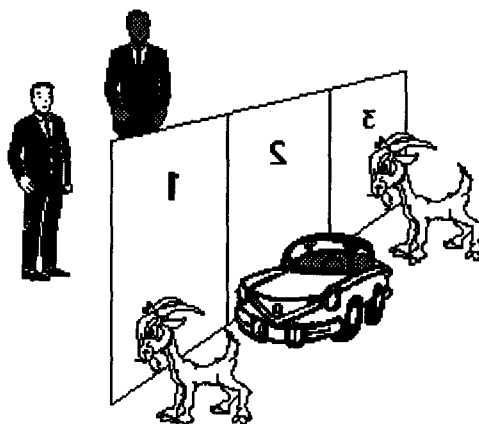
The goal for the fourth day was to introduce the students to the concept of conditional probability and to compare it to simple probability. The class began with a discussion of the following questions written on the board:

A jar contains two white balls and two orange balls. Two balls are drawn, in order, without replacement. Find the following:

1. What is the probability that the second ball is white, given that the first ball was white?
2. What is the probability that the second ball was white?
3. What is the probability that the first ball was white, given that the second ball was also white?

This problem generated an extensive discussion on logic, syntax, and language of probability. As a class, the students determined the probability for each situation, using past knowledge. After discussing the problems, the instructor wrote the definition of conditional probability on the board. From this definition, the students re-calculated the probabilities. After the students were satisfied with their answers, the remaining class time was devoted to discussing solutions to conditional probability problems. The conditional probability problems discussed were similar to the type of problems found on the Probability Knowledge Inventory, in which the condition event in the problem is the change in the sample space.

Before the end of the class, the students were assigned the Monty's Dilemma Problem (See Appendix D for the actual assignment). Students were asked to prepare their answers before the sixth class session.



Suppose you're on a game show, and you are given the choice of three doors. Behind one door is a car, behind the other two doors are goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?". Is it to your advantage to switch your choice of doors? If you were the contestant, which of the following strategies would you choose, and why?

- a. Strategy 1 (stick): Stick with the original door.
- b. Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- c. Strategy 3 (switch): Switch to the other door.

Day 5: Conditional Probability and Independence

The goal of the fifth day was to review the concept of conditional probability, observe the probability of independent events. The class started with a review of conditional probability, and students computed conditional probability using Venn diagrams and data tables. After a review of conditional probability, its definition, and applications, the students were asked the following question:

Roll a 4 sided, fair die, with the numbers 1, 2, 3, and 4 written on the faces.

- a) What is the probability of rolling a 3 on the first roll?
- b) Roll the die again. What is the probability of rolling a 3, given that you rolled a 2 on the first roll?

After a long discussion, the class concluded that the outcome of the second roll was independent of the first roll. This example led to a discussion of independent events. After developing a definition of independent events, the students listed independent experiments and computed their probability.

To review the concepts covered during the past two weeks, the class was divided into the same five groups, and asked to develop problems and solutions for a practice mid-term review sheet.

Day 6: Monte's Dilemma

Each chapter of the text used for this course concluded with a “real world” problem. This problem in the chapter on probability was the Monte's Dilemma Problem. On the fourth day, the students were given background information on Monte's Dilemma, and were asked to prepare a solution to the problem. The various solutions were discussed, and the class did not come to a consensus on the result. Since the class was only 50 minutes, the students were given worksheets to guide them through the problem (see Appendix E). Also to save time, the class was divided into three groups and each group was assigned a particular strategy:

- Strategy 1 (stick): Stick with the original door.
- Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- Strategy 3 (switch): Switch to the other door.

At the end of class, each group reported the results of their strategy. For homework, the students were asked to analyze the data presented by each group, derive a conclusion to the question, and justify their solution to the following question:

If you were the contestant, which of the following strategies would you choose?

- a. Strategy 1 (stick): Stick with the original door.
- b. Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- c. Strategy 3 (switch): Switch to the other door.

Conclusion

Data for this study was collected prior to and after this two-week instruction on sample space, probability, and conditional probability. The two-week instruction on probability gave the class a general overview of conditional probability, mainly exploring conditional problems in which the removal of an outcome from a sample space affects the probability of an event. This course was not designed to be an in-depth study of the concept of conditional probability.

Data Collecting Procedures

Data collection occurred in two stages, with two parts in each stage. The first stage consisted of administering the Probability Knowledge Inventory (see Appendix A) to the entire class prior to the two-week instruction on sample space, probability, and conditional probability. After the administration of the Probability Knowledge Inventory, the researcher interviewed the five participants, assessing the participant's solutions to the

various tasks on the Inventory. The second stage occurred after the two-week instruction on sample space, probability, and conditional probability. The second stage was similar to the first. The Probability Knowledge Inventory was administered to the entire class, followed by an interview assessing the participants thinking on the various tasks in the Inventory.

The researcher constructed the Probability Knowledge Inventory, an open-ended questionnaire based on the course curriculum and the literature on conditional probability (Jones, et al, 1997, 1999; Tarr & Jones, 1997; Parks, et al, 2000). The Probability Knowledge Inventory comprised seven tasks in which three assessed thinking of sample space; three assessed thinking of probability, conditional probability, and comparing probability; and one assessed thinking in independence. The sample space task focused on the student's ability to identify the complete set of outcomes in a one-stage or a two-stage experiment. The conditional probability task focused on the probability situations involving "with" and "without" replacement conditions. The final task in independence assessed whether the student could recognize independent and dependent trials, thus recognizing when conditional probability is the appropriate method of computation. The broad array of probability tasks allowed the researcher to assess the probabilistic background of the students, prior to assessing problems on conditional probability. Face, content, and construct validity was verified by mathematics educators, probability theorists, and course curriculum developers. A pilot study of the Probability Knowledge Inventory was conducted with students enrolled in an intermediate algebra course the previous term. The material covered in the intermediate algebra course was considered the prerequisite for students enrolled in the course involved for this study.

The Probability Knowledge Inventory provided the design of the semi-structured interview protocol conducted prior to, and after, the instruction on probability. A semi-structured interview consists of asking a series of structured questions and then probing more deeply using open-form questions to obtain additional information (Gall, Borg, and Gall, 1996). The researcher asked participants to explain their solutions to the seven tasks on the Inventory, and to explain their choice of solution strategies. Pilot interviews conducted with two intermediate algebra students were valuable in providing information on communication problems, evidence of inadequate motivation on the part of the interviewees, and other clues that suggested the need for rephrasing questions or revising the procedure. Interviews were audiotaped and transcribed for subsequent analysis.

Data Analysis

Due to the nature of data collection, qualitative analysis was used to address the questions of interest. One of the main characteristics of qualitative research is its focus on the intensive study of specific instances of a phenomenon. More specifically, this study focused on case study analysis, a particular approach in qualitative analysis (Gall, Borg, & Gall, 1996). One advantage of a case study is its possibility of an in-depth attempt to understand an individual, allowing the researcher to seek and to explain, not merely to record, an individual's behavior (Ary, Jacobs, & Razavieh, 1990). Researchers generally do case studies for one of three purposes: to produce detailed description of a phenomenon, to develop possible explanations of it, or to evaluate the phenomenon (Gall, et al., 1996). For this study, since the questions of interests are descriptive in nature, the purpose of this qualitative case study analysis was to provide a detailed description of

students' strategies and the difficulties they encountered solving conditional probability problems prior to and after receiving instruction on sample space, probability, and conditional probability.

The analysis of the data occurred in three stages: pre-instructional interviews, post-instructional interviews, and comparisons of pre- and post-instructional interviews. The process of analyzing the pre- and post-instructional interviews was identical. During these interviews, the researcher noted difficulties students experienced while solving the problems as well as interesting statements made by the student. Once the interviews were transcribed, the researcher analyzed them by reading the transcriptions, searching for difficulties that may have been overlooked during the interview, and examining statements made by the student. The collection of student's problems and statements formed preliminary categories of problem solving strategies and difficulties encountered by the students while solving the problems. The researcher then reread the transcripts searching for disconfirming evidence of these categories. In order to stay focused on each case study, the researcher analyzed all the data for each individual at one time. Once sufficient evidence and patterns were developed for the individual case study, the first draft of the case study was written.

At the conclusion of the two-week instruction, the five participants returned for a second interview. A similar process of checking for patterns, statements, and drafts describing the participants continued for the second interview. However, during the analysis of the second interview, the notes and drafts of the previous interview were not referred to in order to lessen the possibility of bias towards observations of certain patterns, while unintentionally ignoring others.

The final data analysis stage consisted of comparing the pre- and post-instructional interviews and noting similarities and differences in the pre- and post-instructional interviews for each participant and among the group of the five participants. Based on the results of the three stages of data analysis, the results from the study could be compared to previous research on the teaching and learning of conditional probability.

Summary

Ultimately, the purpose of descriptive case study analysis is to provide rich portraits of the heuristics the students used and the difficulties they encountered solving conditional probability problems prior to and after instruction on sample space, probability, and conditional probability. Data collection consisted of the administration of the Probability Knowledge Inventory and semi-structured interviews assessing the five participants. From this data, case study analysis allowed for a description of each participant's use of heuristics and the difficulties they encountered solving conditional probability problems. Together, the analysis and results from previous studies create a profile of the heuristics students used and difficulties they encountered solving conditional probability problems.

Chapter IV

Results

Introduction

Through the Probability Knowledge Inventory and semi-structured interviews, this investigation attempted to serve two main purposes. The first purpose was to gain a better understanding of college students' use of heuristics and the difficulties they encounter solving conditional probability problems prior to and after attending a two-week class on sample space, probability of an event, and conditional probability. The second purpose of the investigation was to compare the results found in this study with prior research on the teaching and learning of conditional probability. In order to achieve these two purposes, this chapter has two objectives. The first objective is to portray each of the five participants prior to and after instruction. The portraits include a description of the heuristics the participants used and the difficulties they encountered solving probability and conditional probability problems. The second objective is to analyze the data in response to the three research questions:

1. What are some of the heuristics college students use, and what are some of the difficulties they encounter solving conditional probability problems prior to receiving instruction on sample space, probability of an event, and conditional probability?
2. After attending a two-week class on sample space, probability of an event, and conditional probability, in what ways did the students' heuristics probability

change, and in what ways were they able to overcome difficulties they had previously encountered when solving conditional probability problems?

3. How does each student's understanding of conditional probability compare to the Conditional Probability Framework developed by Tarr and Jones (1997)?

Case Studies: Portraits of the Five Participants

The presentation of the data consists of a portrait of each of the five participants. The portraits include information about the student's mathematics and probability background, their attitudes towards mathematics, the heuristics they used to solve the problems, and difficulties they encountered during the first and second administration of the Probability Knowledge Inventory. The five students who volunteered to participate in this study are called Angela, Beth, Cathy, Debra, and Emily. Pseudonyms are used to assure the anonymity of the study participants.

The descriptions of the heuristics students used to solve the probability problems include five categories: probability, percentage, ratio, odds, and rationalization. The term "probability" indicates the participant used the definition and properties of probability to solve the problem. "Percentage" indicates that the participant used either properties of probability or properties of percentages to solve the problem; however, the participant used percentages when stating the results. "Ratio" indicates that the participant compared one group to another group to solve the problem. "Odds", though similar to ratio, indicates that the participant stated that they solved the problem using odds. If neither the concept of ratio or odds was implied, it is assumed the student was using ratio to solve the problem. Finally, when the participant did not seem to use one of the previously

stated heuristics, but used prior mathematical or practical knowledge such as reasoning skills or intuition, the student was defined to be "rationalizing" the solution.

Angela

Angela was a 19-year-old freshman who said she would rather play golf than attend a mathematics class. Despite her strong background in mathematics, including successful completion of college level algebra, Angela was unsure of her future academic goals. She knew she did not want to pursue a career in mathematics or science, so she was currently pursuing a Liberal Arts degree. Angela was briefly exposed to probability in two of her high school classes, Geometry and Algebra II. When asked which probability concepts she recalled from these previous classes, Angela could not remember what topics were covered and did not remember studying conditional probability.

Angela appeared to approach the problems on the Probability Knowledge Inventory as word problems with specific algorithms. Angela claimed on Problem 1: "Then I was thinking, OK, this is probability, so I added them all up". From there, when starting to explain each problem, she would start with "I add them (the number of possible outcomes) all up...", and proceed with solving the problem. Although the first two problems only asked for the outcomes of the sample space of the experiment, Angela used her algorithm to find the total of all the numbers.

- I: Problem 2: Spin both spinners. If you were to sum up the total of the numbers selected on the two spinners, what are all the possibilities you could get?
- S: Um, I just, um, the first little spinner has a 1 and a 2 and the next has a 3 and a 4, so you can either get a 1 and a 3; or a 1 and a 4; or a 2 and a 4; and a 2 and a 3 for the spinners. Then I added

them all up and I got 20 for that one. That's what I got on that one.

Angela acknowledged her dislike for word problems in the first interview and she appeared to have difficulty approaching the problems on the Probability Knowledge Inventory. During the administration of the Inventory, she asked for clarification of Problem 2, and during the interview, she said that she did not understand Problem 3 or 7. Angela seemed to feel more comfortable asking about the problems, and she was able to discuss her solutions in more depth. A second indication that she struggled with the intent of the problems was during the administration of the test. Angela gave more elaborate solutions to Problems 1 and 2 than was required. Problems 1 and 2 asked her to list the sample space for the experiments. For Problem 1, Angela calculated the probability for each outcome, and for Problem 2, she stated various combinations of possible sums.

- I: Problem 1: In a bag, there are 4 green marbles, 3 red marbles, and 2 yellow marbles. If you close your eyes and draw a marble from the bag, what possible colors could your marble be?
- S: I thought, first, I was like OK, there could be green, red, and yellow marbles. Then I was thinking, OK this is probability, so I added up all, um, 4, 3 and 2...And got 9. Right? Ok, so I thought there was a 4 to 9 chance I could get green because there are four marbles, and then I thought there is a 3 to 9 chance I could get red and then a 2 to 9 chance I could get yellow. So I thought the best probability would be 4 out of 9, with green, and that's what I thought on that one.

Despite her confusion with the statement of the problem, during the interview, Angela was proficient at finding the entire original sample space and computing simple probabilities. When first starting to solve the problems, her solutions were stated as

probabilities, but by Problem 6, she started to use the concepts of ratios and odds, while interchanging the terminology, sometimes incorrectly. Angela was unsure why she switched from "there is a 4 to 9 chance I would get it (a Snickers bar)", to "there is a 3 to 2 ratio that boys will be picked". However, her inconsistency between ratios, odds, and probability did confuse her for Problem 6d when she responded "he (Rick) still has a 1 to 3 possibilities (of winning), so it has changed. Before he had 1 to 4, now he has 1 to 3". Where 1 : 4 and 1 : 3 are the correct ratios for this problem, and 1 out of 5 and 1 out of 4 are the correct probabilities for this problem.

In the problems containing a condition, Angela had difficulty computing the new sample space. First, she was able to recognize the change in the sample space of the number of candy bars in Problem 4b, and the change in the sample space of the number of candidates in Problem 6b; however, Angela was frustrated by the calculations needed to answer Problem 5c. Angela noted a difference in sample space after the blue marble was replaced and was able to apply it to her probability, but was unable to notice the change in the sample space for her friend's probability.

- I: Problem 5c: Your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day changed, or is it the same as the day before? Has your friend's chance of winning changed, or is it the same as before?
- S: Um, I thought to myself that, mine became equal, like...It says, OK, and then my friend's has not changed, there is still only one marble in there, but there is only, there's three blue ones and now three green ones, and two red, and one yellow. So now there is about an equal chance that blue and green, there is a same probability that the blue and green could be picked, so I said mine changed too, so, mine changed.
- I: And your friend's has not changed?
- S: Yes, my friend's stayed the same. Yea!

Problem 6 seemed to confuse Angela. It was intended to check if Angela understood the concept of independent events. This problem followed three conditional probability problems in which the sample space and probability changed among the solutions. Angela seemed to have difficulty knowing where to begin solving Problem 6:

- I: Problem 6: You rolled the die and got a 2. Now you are going to roll the die again. Does the outcome of the first roll affect the possibility of rolling a 2 the second time?
- S: I said no. I said no because I thought ... OK, I think the probability does change, but I do not know how. I really don't. I think it does change, but I don't know how to explain it, how to do it. So, I just said no.

On the first administration of the Probability Knowledge Inventory, Angela used two main heuristics: the use of probability to reason a solution, and "adding them all up". Angela tended to have the most difficulty in understanding what the problem was asking, distinguishing between reporting a solution using ratios or proportions, and comparing probabilities after a conditioning event has occurred. With these various heuristics and difficulties, Angela was not confident in her solutions.

Angela claimed that after the two weeks of instruction, she felt more confident in her solutions to the second administration of the Probability Knowledge Inventory. Angela started the interview stating "The first one, I was totally guessing my answer, because it has been a long time. But I know this one (referring to this test), I did it way different than I did the first time. I felt a little more confident. I did feel better about it".

For Problems 1, 2, and 3, Angela was able to grasp the idea that the sample space consisted of combinations of possible outcomes, but she was not able to compute the sample space for Problem 3. When asked in Problems 2 and 3 how she knew when she

had all of the possible outcomes, despite Angela listing the entire sample space for Problem 2, she could not justify how she knew she had the entire sample space. On Problem 3, Angela was not able to list the entire sample space, nor state the number of outcomes in the sample space. These second responses to Problems 1, 2, and 3 were unlike her first attempt. During the first administration of the Inventory, Angela had difficulties understanding the question of interest in Problems 2 and 3. Also, Angela was not able to state the sample space for Problems 2 and 3.

The frustration of not being able to describe the sample spaces did not hinder Angela's thinking for the remaining problems. For Problems 4 and 5, Angela solved the problems, and justified her solutions using probability. However, in Problem 6, Angela switched her solutions between ratios and probability. On the previous administration of the Inventory, Angela interchanged the use of ratio and odds to state her solutions. When asked why she used the two different methods on the second test, and if they were the same process, Angela could not give a reason. It was not evident that Angela recognized there was a difference between the two methods of solving the problem:

- I: So, my question is, in Problem 6a, "who is more likely to become class president, a boy or a girl", your answer was "most likely a boy with a ratio of 3 to 2". On Problem 6b, "is it more likely Rick will be chosen", your answer was "Rick has a 1 out of 5 chance to be chosen". Is there a difference between reporting a solution as 3 to 2 compared to 1 out of 5?
- S: Is there a difference... Well, this is there is 2 girls and 3 boys, and here it says Rick, so that is one person, so I said Rick is one person out of 5.
- I: Do you know why you reported some answers using ratios, and others using probability?
- S: Do I know why... Um, well there is a probability that if this is boys, I don't know... the probability 1 to 5... hmm, I just looked at it as a ratio. I know there is a probability, probability that there will be 3. Hmm...I don't know.

- I: Where are you getting the 3 and the 2 (written on her Inventory)?
- S: I don't know why. Maybe I shouldn't have written those down. If you want a probability, like there is 1 out of 5, like with Rick, which I got. And then with boys and girls, there is 3 boys out of 2 girls. There is 2 girls to 3 boys. I am thinking there is a 3 to 2 ratio that...I don't know...
- I: I was wondering why you are reporting in ratio for those two problems and probability for the other two.
- S: I don't know.

Angela did have some difficulty with solving the conditional probability problems. At first, she did not recognize that the sample space changed, hence the probability changed. However, while explaining her reasoning behind her solutions, she was able to correct herself:

- I: Problem 4c: Has your chance of drawing a Snicker's bar in part (b) changed, or is it the same chance as in part (a)?
- S: Uhm, well, it is the same.
- I: Why is it the same?
- S: Because you still have a 4 to 9 probability that you will pick a Snicker, no you don't. Oh, no... now there is only 8 possibilities, and now there is only 8 Snickers, so this outcome is right. There is only 8.
- I: What does the 8 represent?
- S: Because you took one of the Hershey bars away, so now there is only 8 possible candy bars. And so, yea... there is half and half... but there is still 4 Snickers.
- I: Returning to the question, has the chance of selecting a Snickers bar changed?
- S: No.
- I: It has not changed.
- S: No, yea! Ok. It's changed from...yea those are different. Yes, they are different.
- I: How do you know they are different?
- S: Because the outcomes have changed. Instead of 9, there is now 8 possible outcomes.
- I: Where are the 9 and the 8 coming from?
- S: How many candy bars you have...

Angela was able to use the reasoning from Problem 4 and apply it to Problem 5 and report the correct solution. On the Inventory, Angela wrote the response that her friend's chance of winning has not changed, but after looking at Problem 4 again, Angela decided to write out the sample space and solve the problem on a piece of paper:

- I: Problem 5c: And then has your friend's chance of winning changed, or is it the same as before?
 S: What did my friend have again?
 I: Yellow.
 S: Yellow... he had 1 out of 9. Yea, his has changed because he now has ... it was 1 to 10, now it is 1 to 9 that he could get picked.
 I: What do the 9 and the 10 represent?
 S: All the marbles. All of the possible outcomes.

Problem 6 allowed Angela to demonstrate that she recognized when the sample space changed and when it did not change.

- I: Problem 6b: You rolled the die and got a 2. Roll the die again. Does the outcome of the first roll affect the possibility of rolling a 2 the second time?
 S: This one, no. There is a 1 to 6 probability that um, that you roll a 2. And there is still 1 to 6 that you are going to roll the 2 again.
 I: So, they are the same?
 S: Yea.
 I: Then, how come on Problem 4, when we are choosing Snicker, Hershey and Butterfinger bars, when you first reach into the bag, you said the probability of getting a Snickers is 4 to 9. But when you reached in for the second time, you said the probability of getting a Snickers bar is 4 to 8. How come the probability changed for that problem, but not for this one?
 S: Because there is still, they did not take away the 2 on the die. So, it is equal...Because I know they are still the same. The probability of getting a 2 again.

Angela was more satisfied with her responses on the second administration of the Inventory. Angela recognized the question of interest, a difficulty she overcame from the

first interview, and solved the problem accordingly. Despite her continued difficulty identifying the sample space in Problems 2 and 3, the remaining problems indicated that Angela recognized when they did and did not change. By recognizing that the sample space did change, Angela identified in the second interview that the probability also changed; a weakness Angela encountered during the first interview. In solving her problems during the second interview, Angela approached the solution with two different methods: probability and ratios, but was not able to explain why she would use one method over the other. Angela's approach changed from the first interview when she chose to state her solutions using ratios or proportions.

Beth

After serving in the National Guard for 16 years, Beth returned to college to complete her degree in sociology and criminal justice. Beth, a senior, had completed two junior level statistics courses at the university prior to enrolling in this course. She claims her mathematical background was not strong, and that she had only completed pre-college algebra courses. When asked which probability concepts she recalled from her prior classes, Beth said that her first formal introduction to probability was in her previous statistics course, and she was hesitant to try to list some of the terminology and concepts covered in the class.

Beth's solutions and explanations to the problems on the Probability Knowledge Inventory indicated that Beth had difficulties explaining her solutions using probability terminology. For example, when answering Problem 1, Beth believed that the possible color that could be chosen from the bag was green, because "there is more probability in

green". Her main confusion in probability terminology also stemmed from her confusion between probability and odds. For Problem 4b Beth would interchange terminology associated with odds and probability:

- I: Problem 4b: Suppose you chose a Hershey Bar and ate it. If you reached into the bag again, what kind of candy bar do you have the most chance of drawing?
- S: The Snickers bar, because there is still more Snickers. There is four Snickers. Well, I guess it will be an even chance of drawing, because the chance of drawing of the other two, there is four Snickers and four of the other two kind. There are two Hershey and two Butterfingers. Four out of four. The chance of still drawing, I would say is a Snickers bar, because, there are more of them. Your odds are better.

Beth's continued confusion between the terminology associated with odds and probability was evident in the methods she used to approach solving three of the problems, thus indicating a confusion between the concepts of probability and odds. For Problems 4 and 5, Beth used odds to solve the problems and state the solution. However, in Problem 6, Beth converted to solving the problems using probability, with the concept of odds used sparingly. When trying to clarify why she used the two different methods for solving the problems, Beth was confused:

- I: Problem 6a: Is it more likely that the class president will be a boy or a girl?
- S: A boy, because there are more of them.
- I: Is it more likely that Rick will be chosen or Rick will not be chosen?
- S: Well, he has a 1 to 5 chance over everyone else, but he is 1 in 3 with the boys, because there are 3 boys.
- I: You wrote that "he has a 1 / 5 chance but his odds are better than the girls". What do you mean that the odds are better than the girls?
- S: There are 3 boys and 2 girls.
- I: So Rick's odds are...

- S: 1 in 3
- I: And what does the 3 represent?
- S: The 3 boys.
- I: In your answer you wrote, what does the 5 represent?
- S: Well, that will be everyone. Be he really had... Well, he has the same chance as any, realistically, but it should be a 1 in 5 chance. But his odds are better than the girls.
- I: So, Rick was 1 and 5 was the total of everyone. Let's look at the previous problems. In Problem 4, we had 4 Snickers Bars, and there were 5 remaining Hershey and Butterfinger bars. Whereas in this problem...
- S: So, 1 in 4 would probably be better than...(hesitation of completing sentence)
- I: I was just wondering which one would sound more reasonable to you: the 1 student out of the total number of students, or the 4 Hershey bars to the remaining 5 candy bars.
- S: Um... I see what I was doing here, I was counting up these and separating them. I was not doing that over here. Because those are different items and those were just people. Granted, they are boys and girls, and I did separate them by gender, but they are not Snickers.

A second difficulty Beth had when solving the problems was the ability to recognize the sample space for most of the problems, especially when a condition was on the original event. For Problem 3, when Beth had to list all the possible combinations of coloring three sections with three different colored pencils, Beth first felt comfortable suggesting there are only three possible combinations. When asked about a combination she did not list, she saw that she was missing some combinations. Frustrated at the problem, she concluded there were only nine possible combinations for a three by three matrix, but did not want to list them all. Beth's frustration with recognizing the sample space also carried into her solutions of the conditional probability questions. Beth recognized that the sample space did change when a candy bar was eaten, or a marble was taken away, but had difficulties assigning quantifiable statements to her solutions once the sample space changed. For the final problem, Problem 6, Beth did recognize

that after rolling a 2, the probability of rolling another 2 did not change, and the sample space did not change from one event to the next.

With the difficulty in recognizing the complete sample space and the confusion between probability and odds, Beth was still able to rationalize the correct solutions to the probability questions. She felt confident of her solutions, appearing to use her intuition more than quantifiable methods. Unfortunately, when Beth tried conditional problems, her main struggles were computing the size of the new sample space, and not being confused with the problem. Overall, Beth used a variety of problem solving techniques to find solutions - intuition, odds, probability - without using one method more frequently than the other.

Beth was a bit hesitant during her second interview. She was behind on some homework and was very tired. Her current state of mind may have influenced her responses to the questions, but Beth seemed to try hard to answer the questions.

Beth approached most of the problems on the second administration of the Probability Knowledge Inventory through rationalizing the problem and hoping her solution was correct. Her attempt to rationalize the solutions was more apparent in the second interview than the first. Her main problem solving approaches in the second interview consisted of rationalizing and using ratios sporadically. For instance, she recognized in Problem 4b that there were 4 Snicker bars to 4 other candy bars, and in Problem 6c there were "1 girl to 3 boys", but for Problem 5a, Beth selected blue because "there are more blue marbles in the bag than the other marbles". Beth appeared to use the ratios to derive her solutions, but did not use the ratios to answer the questions.

However, her limited use of ratios and rationalizing the solution did confuse her on responding to one problem:

- I: Problem 5b: Suppose you are assigned a green and your friend is assigned a yellow marble. Your teacher reached in and drew a marble. Which of the four colors do you predict will be drawn?
- S: I am betting on blue, but it would be nice if it was green.
- I: Why are you saying blue?
- S: Because there are more of them.
- S: Why would you like it to be green?
- I: Because I don't want to take the test.
- S: On the Inventory you wrote, "Really this is a 50/50 chance, but I don't want to take the test". What do you mean by 50/50 chance?
- I: Well, both me and my friend have an even chance of having a marble picked.
- S: So you and your friend have an even chance of getting your marble picked?
- I: Well, no, not really. The yellow one has less of a chance than I do.
- S: How do you know that?
- I: Because there is only one marble.

This interaction also indicates that Beth had difficulty recognizing the sample space of the problem and the outcomes of the event. She tried to use humor to explain her answer, but appeared to confuse herself more. When asked why she preferred to solve some problems with numerical values, and others through reasoning, Beth replied "I don't know". In addition, when asked if she preferred to reason out the solution or use numerical values, Beth replied, "It doesn't make any difference".

Beth's previous difficulty with recognizing the sample space was also evident in another problem when she had to compare simple probabilities. For Problem 6b, Beth forgot to include the girls in her sample:

- I: Problem 6b: Is it more likely that Rick will be chosen or Rick will not be chosen?
- S: Rick has just as much of a chance as the other 3 boys. But yea, he could be chosen.

Her difficulty with recognizing the sample space continued with the problems concerning conditional probability. Beth recognized that the original sample space changed, but is unsure how to explain the change:

- I: Problem 5c: Your teacher chose a blue marble and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day changed or is it the same as the day before?
- S: It's changed.
- I: How do you know it has changed?
- S: Because there is one less marble.
- I: Has your friend's chance changed, or is it the same as the day before?
- S: It has changed. There is one less marble.
- I: What do you mean by "there is one less marble"?
- S: The blue marble is gone, so that is one less marble.

During the second administration of the Inventory, Beth seemed to not have a preferred method for approaching the solutions. When trying to solve the problems, Beth used a combination of both rationalization and ratios. In the previous administration of the Inventory, Beth used a variety of problem solving techniques to find the solution - intuition, odds, probability- without using one method more frequently than the other. Beth's main difficulty continued to be recognizing the correct sample space, thus making her feel unsure with her explanations of the solutions. Without a solid understanding of sample space, Beth was not able to clearly describe the conditional events and what had occurred in these problems. The apparent difficulties in recognizing the complete sample

space and computing the size of the new sample space Beth had on the first administration of the Inventory continued with her second attempt of solving problems on the Inventory.

Cathy

Cathy enrolled in the course with a strong background in mathematics and probability. In high school, Cathy had taken a functions, statistics, and trigonometry course introducing her to the basic concepts of probability. Prior to enrolling in this course, Cathy completed both a sophomore level statistics course and college algebra course. Cathy was a 19-year-old sophomore, majoring in Housing Studies, and enrolled in the course because she needed three more credits during the spring term.

After a review of Cathy's solutions and the interview, it was evident that Cathy was proficient in solving problems requiring probability and conditional probability. Without prompting, Cathy was able to systematically explain how she reached her conclusions, and to compare her solutions. Her scratchwork on the Inventory indicated that Cathy had also devised her own system for attacking probability questions. Cathy was proficient at providing quantifiable solutions, and she used correct terminology in answering the questions. Cathy's response to Problem 5c illustrates her thought processes:

- I: Problem 5c: Your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day changed, or is it the same as the day before? Has your friend's chance of winning changed, or is it the same as before?
- S: Well, since she only drew a blue marble, and I was assigned a

green marble, my chances have changed from 3 in 10; to 3 in 9. And then my friend, since his marble wasn't chosen either, which was yellow, which was 1 in 10, since he did not replace the marble, it is 1 in 9 chance now. And it changed because the entire number of marbles in the bag decreased.

Cathy expressed the solutions to all the problems using probability, and was able to use the correct probability terminology in solving her problems. She did not have difficulties understanding the problem, and seemed to enjoy the challenge.

Cathy enjoyed the class on probability, and stated that it helped clarify some of the questions she had about probability. Cathy underlined key words on each of the problems on the Probability Knowledge Inventory and explained that the underlining of key words helped her approach the problems. Cathy was confident in her solutions and was able to explain her reasoning behind her solutions. On the conditional probability problems, Cathy recognized the change in sample space and adjusted her solutions accordingly:

- I: Problem 5c: Your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day changed, or is it the same as the day before?
- S: Well, since I have the blue marble, there was 3 of them. And originally it was 3 out of 10, because there was 10 marbles. Since she drew one blue marble and did not replace it, my chance is now 3 out of 9.
- I: Did your friend's chance of winning change?
- S: Yes it did. It went from 1 out of 10 to 1 out of 9.

Cathy also recognized when the sample space and the probability did not change:

- I: You rolled the die and got a 2. Roll the die again. Does the outcome of the first roll affect the possibility of rolling a 2 the second time?

- S: No, because the die doesn't care what you rolled before. It will still come up randomly, whatever it comes up.
- I: In Problem 4, when you had candy bars, 4 Snickers, 3 Hershey, and 2 Butterfingers, at the first drawing you said Snickers had the best chance of being selected with a 4 to 9 chance. When you had the second drawing, you said the Snickers bar had 4 to 8. How come when you rolled the die the second time ...
- S: There is always going to be 6 numbers that can come up. And there is only one of each number.
- I: So, why does the probability change with the Snickers bar.
- S: It changed because it originated with 9 bars in the bag, and after one was eaten, there was only 8 left.

Cathy did not appear to have difficulties in listing the sample space, determining the sample space, and justifying her solutions using probability. Cathy recognized the change in sample space and the change in probability of the problems containing conditional probability. For all of the problems, Cathy chose to use probability to justify her solutions. It was evident that Cathy was proficient solving problems requiring probability and conditional probability at level of the course.

Debra

Debra was an 18-year-old freshman majoring in communications. Debra was apprehensive about taking the math class because she thought her mathematics skills were not strong enough. The year before, she had completed an intermediate algebra course in high school. Hoping to gain more mathematics skills to give her the necessary background to the required for her degree, she enrolled the previous summer in a basic mathematics course. Debra continued building up her mathematics knowledge her freshman year by enrolling in another basic mathematics course and a pre-college algebra course. She recalls learning probability in high school, but could not recall any of the

topics covered. Debra had had the instructor for this course for her basic mathematics course in the fall term.

Reading the responses on the Probability Knowledge Inventory and during the interview, Debra seemed to prefer to rationalize the solutions to the problems without using numbers to further justify her solutions. Her rationalization did take in consideration the change of the sample space during the conditional probability problems, and she was able to derive the correct solution. In response to Problem 5:

- I: Suppose your teacher is going to have a drawing to see who can miss the final exam without it affecting their grade. If your color is drawn, then you do not have to take the final exam. In a bag, there are 4 blue marbles, 3 green marbles, 2 red marbles, and 1 yellow marble. What color would you like to be?
- S: Blue, because, there is, yes, there's more of the different colors, but they are all broken up into smaller groups and blue you got the most, out of all of them. And so, that is why I chose that one.
- I: Suppose you are assigned green, and your friend is assigned yellow. Your teacher reached in and drew a marble. Which of the four colors will you predict will be drawn?
- S: Blue, isn't it kind of like the same as this question? (pointing to part a)
- I: Yes, it is.
- S: OK, blue, for the same reason.
- I: Your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day changed or is it the same as the day before?
- S: Yea, it's like the Snickers one, because you took out one of the higher, like the one who had the most chance, and now it's like even with the green, and so like you have the same chance of getting blue and green because they both have three in there. So, it's changed now. Because now, I have a better chance of getting it, the marble.

The only time Debra used a quantitative answer to support her original solutions was her response to Problem 4b. For Problem 4b, Debra did notice that the chance of

getting a Snickers bar became 50%, after the Hershey bar was removed from the bag.

After explaining how she derived the 50%, Debra was then asked to return to Problems 5 and 6 and asked if she could also give a numerical solution to each problem. With some hesitation, Debra successfully found corresponding numerical solutions using fractions and percentages, even recognizing the change of the sample space.

Despite her ability to rationalize the solutions to Problems 4 and 5, Debra did not know how to start solving Problem 6:

- I: The ASOSU is electing a president and a vice president. There are five people running: Beth, Jose, Maria, Rick, and Joshua. All five students are considered to have an equal chance of winning. At the end of the day, the results are announced. Is it more likely that the class president will be a boy or a girl, and why?
- S: Since it is all, like, equal, so, I wasn't sure, I mean I did not know any way to pick it or if it was random or I did not know how to work this one.
- I: So, let's look at the second one: Is it more likely that Rick will be chosen or Rick will not be chosen?
- S: Well, again, since it was equal, I did not know how to solve that.
- I: Could you use some of the reasoning that you used on the other problems? For instance, using numbers and setting up fractions.
- S: But, there is like, not a number, like one out of five, or like, I mean, there is no specific number given to them.

Her frustration with equally likely outcomes also appeared in her solution to Problem 7:

- I You have a die. If you rolled the die, can you predict with certainty which number will come up?
- S: No, because it is random, equal, equal chance because the numbers on the die show up once.
- I: You rolled the die and got a 2. Roll the die again. Does the outcome of the first roll affecting the possibility of rolling a 2 the second time?
- S: No, unless you roll it the exact same way, with the exact same ... everything. But now, because the numbers only show up once.

- I: How come on these questions, when you took out a Hershey bar, or taking out a marble...
- S: Because there was not an equal number of, there wasn't like, if there was one Snicker, or one Hershey, or one Butterfinger bar, then it would be like the same (problem). That would be saying like this... the number 4 written on the dice, 4 times and then the 2 written on it twice.
- I: Suppose you wanted to roll a 3. On average, how many rolls should it take to ensure that a 3 would come up?
- S: I have no idea. Because there, I do not know how to solve that. Like I said earlier, they are all equal. You can't like say, there is going to be an average, because there is not. Unless you roll it the exact same way.

During the first administration of the Probability Knowledge Inventory, Debra seemed to prefer providing a reason behind her solutions without numerical justification; however, when asked for a numerical response, she was just as capable and would use fractions and percentages in her responses. Debra was able to recognize when a condition changed the sample space, and her rationalization changed accordingly. Her main difficulty was recognizing the properties of equally likely events and how to compare the events.

When Debra arrived for the second interview, she claimed she was able to recall how she developed the solutions on the initial Probability Knowledge Inventory. Debra believed she solved the problems on the second administration of the Inventory using the same methods. On the second test, Debra wrote out the answers to the problems, without providing elaborate justification. Some of her written responses included "Snickers, because it is in the bag the most", "Blue, there are more", or "Boy, there is more". During the interview, Debra would start explaining her solution using quantities on the simple probability problems, but when a condition was added to the problem, she would start rationalizing her response without using quantitative support. This approach to

solving the problems was similar to her first attempt. Sometimes Debra appeared to not understand how to respond and would use her own terminology to explain the problem:

- I: Problem 4: In a bag you have 4 Snicker bars, 3 Hershey Bars, and 2 Butterfinger Bars. Suppose you closed your eyes and drew a candy bar. What kind of candy bar do you have the most chance of drawing and why?
- S: Snickers bar, because it is the most in there.
- I: Suppose you chose a Hershey Bar and ate it. If you reached into the bag again, what kind of candy bar do you have the most chance of drawing? And why?
- S: Um, now you have an even better chance of getting the Snickers because you are minus 1 of the others, you have less of those, and greater the Snickers.
- I: What do you mean by "greater the Snickers"?
- S: Well, it already.... OK, there are already more Snickers in there, and then, um, if you... and there are less of the other kind, and if you took out one more of the other kind, you have greater the chance of, a more better chance of getting a Snickers, because now it is like, like... you had 3 of the Hershey bars, now you have 2 Hershey bars, basically you have double the Snickers, double the other, hence a greater... do you get what I am saying?
- I: Has your chance of drawing a Snickers Bar in part (b) changed or is it the same chance as in part (a)?
- S: Just like I said before, you have a greater chance because there is one less of the others, so basically, you have... I mean... less of the other kind, and more of the Snickers, you are going to get... like...destiny, or like it's fate that you are going to get the other kind. There is more.

But when Debra was asked to compare two different problems that were affected by a condition, she tried to use probability to justify her responses, but reverted back to rationalization when stating her final response. At the end, Debra used her intuition to solve the problem, but gave the solution she thought was being sought:

- I: Problem 5c: Your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day changed or is it the same as the day before?
- S: It changed. Because now the one that was the most is not... and you basically... it's either, but, I mean, since blue and green are the highest, it is fair game for that one. Most likely, these two will be less chosen (pointing to red and yellow). But now you have a better chance, because blue is not ahead of you. Blue is not more.
- I: Has your friend's chance of winning changed?
- S: No, not really, because it is still 1 out of... I mean, I guess... I know... not really... I don't think so.
- I: So, it has not changed?
- S: I mean... yea, it has changed because there is 1 less of the other color, but still it is the only 1 in there. Majority rules, and it probability will not. Basically your friend has 1 out of how many there are. 1 out of 9 chances, because there are 9 marbles, oh, wait... there is 1 out of 9 marbles. 1 out of the 9, she is 1 out of them. And green is 3 out of the 9.
- I: Returning to the original question has your friend's chance changed?
- S: Um, I guess, I mean, I don't know, I don't think so, personally, but if you want to go the probability of it, yea, because it is not 1 out of 10 anymore. It is 1 out of 9.
- I: Where are you getting 1 out of 10 and 1 out of 9?
- S: Because there was... there was 10 marbles, but your teacher took one away. Now there is 9. So, yes, it has gotten better, I think.
- I: So, going back to the first question has your chance changed?
- S: Yea, because, there's not... there is not more of one color than me. Like green is the highest, more of the color, more of the green in there. First it was more blue, but now there is the same amount of green and blue.

Debra preferred to rationalize here solutions to Problems 4 and 5, whether it was to state the solution to a simple probability or a conditional probability. When the conditioning event occurred to the problem, Debra was able to recognize the change in the sample space, and based her results on the fact there was a smaller sample space:

- I: Suppose you chose a Hershey Bar and ate it. If you reached into the bag again, what kind of candy bar do you have the most chance of drawing? And why?
- S: Um, now you have an even better chance of getting the Snickers because you are minus 1 of the others, you have less of those, and greater the Snickers.
- I: What do you mean by "greater the Snickers"?
- S: Well, it already.... OK, there are already more Snickers in there, and then, um, if you... and there are less of the other kind, and if you took out one more of the other kind, you have greater the chance of, a more better chance of getting a Snickers, because now it is like, like... you had 3 of the Hershey bars, now you have 2 Hershey bars, basically you have double the Snickers, double the other, hence a greater... do you get what I am saying?
- I: Has your chance of drawing a Snickers Bar in part (b) changed or is it the same chance as in part (a)?
- S: Just like I said before, you have a greater chance because there is one less of the others, so basically, you have... I mean... less of the other kind, and more of the Snickers, you are going to get... like... destiny, or like it's fate that you are going to get the other kind. There is more.

This was the same technique used by Debra on the previous administration of the Inventory.

On the first test, Debra did not know how to solve Problem 6. However, on her second test, Debra appeared to understand the problem and used ratios to explain the solution. Since this was her first attempt to use a quantifiable solution to her problems, Debra was asked why she did not use ratios in a similar problem; Debra was surprised at the question, and tried to use ratios. After that, Debra did not try to use ratios to explain a problem:

- I: Problem 6c: Suppose Beth is selected president. A vice president is selected randomly from the remaining candidates. Is it more likely the vice president will be a boy or a girl?
- S: Guy, because if you just drew out of the bag, the ratio is 3 guys to 1 girl. And so, it is more likely you are going to get the guy. Just

like the whole marbles thing. Like if there is more blue than green, you will more likely get the blue because there is more in there.

I: And, how come you did not answer the first one (Problem 6a) with the ratio since we were also looking at if it would be more likely that it was a boy or a girl?

S: Want me to?

I: Sure. What would the ratio be?

S: It would be 3 boys to 2 girls.

It was not apparent that Debra had difficulties understanding the problems or rationalizing the solutions. This was a similar approach to solving the problems during the first administration of the Inventory. Debra admitted that it was difficult for her to justify the correct solution, but believed all her solutions were correct. Debra was able to recognize the change in sample space, but did not use the numerical quantities to help solve a problem. However, Debra was able to recognize when the sample space did change and when it did not.

I: Problem 6b: You rolled the die and got a 2. Roll the die again. Does the outcome of the first roll affect the possibility of rolling a 2 the second time?

S: No, it is the same answer as last time. Unless you roll it the same exact way, everything the same, no.

I: So, how about with the candy bar problem, I asked what is the chance of getting the Snickers bar, you were saying that it changed from the first drawing to the second drawing.

S: Because there is a different amount of numbers on there. This is all on there equally. The Snickers, there was 3 of that, 3 of the Hershey, but 2 of the Butterfingers. So it is not all equal.

I: But you were saying the chance was different for the candy bar problem, and for the die, you are saying it is the same.

S: Because these are all on here (the die) equally. These are all equal.

Debra preferred to rationalize her solutions, taking into account possible changes in sample space. If she appeared to be unable to explain a solution, Debra reverted to

explaining the solution using probability or ratios. However, her final solution was always a rational statement. Debra did not appear to have difficulty understanding the question, or recognizing a conditioning statement. It is evident that Debra understood basic probability and conditional probability questions. Her methods of solving the problems on the second administration of the Inventory did not appear to change from the first administration; however, Debra was able to provide a solution to Problem 6 during the second interview.

Emily

Despite being older than the average student and having difficulties returning to school, Emily, at age 42, was determined to pass all her required mathematics courses for her degree in Housing Studies. Emily was glad this course was her final class after struggling through eight terms of basic arithmetic and pre-algebra at the local community college; however, she was excited to have passed college algebra the previous term. She did not recall having a course covering the concepts of probability, and she was looking forward to learning a new topic that did not involve algebra.

Emily approached the problems on the first administration of the Probability Knowledge Inventory using her intuition and rationalizing the possible solutions. Despite a lack of formal instruction in probability, Emily was able to rationalize the correct solutions to all the problems, using her own terminology to explain her solutions.

- I: Problem 4: In a bag, there are 4 Snickers bars, 3 Hershey bars, and 2 Butterfinger bars. Suppose you closed your eyes and drew a candy bar. What kind of candy bar do you have the most chance of drawing and why?

- S: You have the most chance of drawing a Snickers bar because there are more Snickers than any of the other...bars.
- I: Suppose you chose a Hershey bar and ate it. If you reached into the bag again, what kind of candy bar do you have the most chance of drawing?
- S: You still have the best chance of drawing a Snickers because you, um, eaten one of the others, so you lowered that number, so you raised the chances of Snickers.
- I: Has your chance of drawing a Snickers bar in part (b) changed or is it the same chance as in part (a)?
- S: Chance (b) has increased, because you have, um, you deleted one of the other bars, so you are getting more of chance, one more of a chance to get the Snickers than you are in the first one.
- I: Can you define for me "one more chance"?
- S: Well, here you have 7 options, no, wait, 9 options in (a) and 4 of those are going to be a Snickers. So you have already eaten a Hershey's, so that gives you 4, 5, 7, 8,...8 options, but still, you still have 4 Snickers, so your chances are 50% of getting a Snickers in part (b).

Emily rationalized the solution to Problem 5 in a similar manner using more of her own terminology:

- I: Problem 5: Suppose your teacher is going to have a drawing to see who can miss the final exam without it affecting their grade. If you color is drawn, then you do not have to take the final exam. In a bag, there are 4 blue marbles, 3 green marbles, 2 red marbles, and 1 yellow marble. What color do you want to be? And why
- S: This one confused me a little, but I thought my reasoning was the same as the other one. I thought the blue, because there was more blue marbles than it had of another color.
- I: Suppose you are assigned green and your friend is assigned yellow. Your teacher reached in and drew a marble. Which of the four colors do you predict will be drawn?
- S: I will still predict blue, because there are still more blue marbles than any of the other colors.
- I: And then your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day change, or is it the same as the day before?
- S: Yes, it is better than yesterday, because one of the blue marbles is missing so now there is only three. Three blue and three green. So, green has a 50% chance of ...getting over the blue.

- I: How are you getting 50%?
- S: So, she took a blue one out, now there is 3 blue and 3 green, and 2 red, and 1 yellow. Ok, so it is not a 50% chance... There is going to be the same amount of blue and green marbles now, so if I have a green one, and my friend has a yellow one,
- I: So, has your chance different today than it was yesterday?
- S: Yes, because there is one less blue marble, and I had green.
- I: And then has your friend's chance of winning changed, or is it the same as before?
- S: Her chances are better, one better too, because there is one less blue.
- I: How are you defining "one better"?
- S: One more, one more opportunity to win. One more... what is the word I am looking for... a greater chance, a greater chance.

In her rationalization, Emily did try to assign quantifiable solutions to support her answer. Whenever Emily noticed that there was a 50% chance of either getting the Snickers bar or the blue marble, she reported the solution with the percentage. However, after explaining her rationalization to her solutions, she was asked if she could use percentages to solve the Problem 6. Emily tried, but with some hesitation:

- I: In Problem 4, towards the end, you started assigning percentages to your solutions. For example, in part (b) you said there was a 50% chance of choosing a Snickers bar after the Hershey bar was eaten. For Problem 5, is there a way you can assign percentages to the three questions?
- S: Um, I am not good with percentages... there are more blue, Well, if you add the 3 green, and the 2 red and the 1 yellow, we have 5 chances there, no you have 6 chances, so you have 6 chances with the other colors, and 4 chances with the color blue. So you would not quite have 50% chance, that would be around 30%.
- I: So, if you do not quite have 50% how come you still chose blue?
- S: Um... because there is still more blue marbles, you still have a good chance of getting a blue.

After rationalizing the solutions to Problems 4 and 5, and assigning percentages, whenever she felt comfortable, Emily proceeded to solve Problem 6 using ratios and

probability. Emily noticed Problem 6 was comparing 3 boys to 2 girls, and the chance of Rick being chosen for the first election was that "Rick has a 1 in 5 chances of winning", and "4 chances out of 5, that he will not be chosen". Emily easily recognized in Problem 6 that the sample space did reduce to four after Beth was selected president, and continued solving the problem using proportions with the correct terminology.

During the first administration of the Probability Knowledge Inventory, Emily approached most of the problems by rationalizing the solutions and using her own terminology to explain her solutions. When using quantifiable justifications to her solutions, Emily was not consistent in her use of percentages, ratios, and probability. Overall, Emily was able to solve the problems correctly, recognizing the change in sample space in the conditioning event, and able to recognize independent events.

Emily arrived at the second interview in good spirits, and was excited to answer the questions. She felt that she has learned so much about probability, and was confident she understood the problems better and solved them in "less than half the time it took me to understand them last time". Still nervous about the interview, Emily held a pencil in her hand, which she then realized was a great tool for her to write down her thoughts during the interview.

At the first glance of Emily's written solutions on the Inventory, it appeared that Emily used probability to state her solutions, unlike in her previous attempt when Emily rationalized her solutions. However, as Emily explained her solutions to the problems, she stated her solutions without the probability to support it. Through her explanations, Emily was able to recognize the sample space, the change in sample space, and comparison of probabilities. When questioned about the probabilities stated on the

Inventory, Emily was able to support her rational response with the probabilities she originally wrote on the Inventory.

- I: Problem 4c: Suppose you chose a Hershey bar and ate it. If you reached into the bag again, what kind of candy bar do you have the most chance of winning?
- S: Oh, I chose a Hershey Bar and ate it. Ok, so now there is two Hershey Bars. OK. So, it's Snickers! You still have a chance with Snickers because there is less Hershey bars, now. So you have a better chance of getting a Snickers now. Because you lowered the number of Hershey bars.
- I: On the side of the problem, you wrote $4/8$. What does that mean?
- S: That means 4 possibilities out of 8 total.
- I: Problem 4c: Has your chance of drawing a Snickers bar in part (b) changed or is it the same chance as in part (a)?
- S: Chances increased from part (a) because a Hershey was taken out and not replaced. So the possibility is now 4 out of 8 chances there will be a Snickers.
- I: I have a question. On the margin you wrote 4 out of 9 and 4 out of 8; did you write that when you did (a) and (b) or did you write them when you were doing part (c)?
- S: I did that when I was doing (a) and (b).

Emily explained Problems 4, 5, 6, and 7 using probabilities, first replying her rationale behind the solution, then returning back to her written probability solution for further explanation.

Despite Emily's ability to rationalize the problems and support her solutions with probability, Emily did have problems interpreting the problem and sporadically answer a different problem. It was not clear if Emily did not read the problems correctly, or if she was having difficulty with the problem. While solving Problem 4, Emily thought the Snickers bar was removed, not the Hershey bar. During the interview, she was able to catch her misunderstanding on this problem and corrected her solution accordingly. However, in Problem 5, Emily did not answer the question of interest, nor did she

recognize during the interview that she originally did not answer the question of interest and change it, as in the previous problem.

- I: Problem 5c: Your teacher drew a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has your chance of winning the second day change, or is it the same as the day before?
- S: Yes, my chance has changed because now, there is a blue marble missing. Which brings that down to 3 blue marbles I believe. The chances are the same for the blue or green. Because there are 3 blue and 3 green.
- I: So, the 3 out of 9, what does the 3 represent?
- S: The 3 represents 3 blue marbles, out of 9 possibilities. And also 3 green marbles out of 9 possibilities.

Emily was not able to answer the question "Has your chance of winning the second day changed, or is it the same as the day before". Instead, Emily continued to compare probabilities of other colors. Emily's perpetual habit of not solving for the question of interest even appeared in Problem 6. Whenever asked about Rick's chance of winning, Emily would refer to the chance of boys winning overall, not Rick individually.

Emily showed that during the second interview, she appeared to solve the problems both rationally, and with support of probability. Previously, Emily appeared to rationalize the solutions. During the second interview, Emily had a tendency to verbally state her solutions without numerical support; however, would write her solutions using probability. While solving the problems, Emily recognized the probabilities of various events, taking into account the changes in sample space, and felt comfortable comparing probabilities. Despite Emily's ability to solve the problems rationally, supporting the solutions with probability, Emily's main problem was recognizing the question of interest and solving accordingly.

Response to Research Questions

The previous section of this chapter met the first objective: to portray the five participants prior to and after instruction. The portraits included a description of the heuristics and difficulties they encountered while trying to solve probability and conditional probability problems. The objective of the final section of this chapter is to address the three research questions of this study.

Question One: What are some of the heuristics college students use, and what are some of the difficulties they encounter solving conditional probability problems prior to receiving instruction on sample space, probability of an event, and conditional probability?

Data from the five participants administration of the Probability Knowledge Inventory and the interviews prior to instruction indicate an assortment of heuristics that the students used and a greater variety of difficulties the students encountered while solving conditional probability problems.

Heuristics

Among these five participants, rationalization, finding the odds, computing the percentages, and stating the ratio of a problem were the preferred heuristics used to solve the problems on the Probability Knowledge Inventory. Emily and Debra relied entirely on rationalization and some intuition to solve the problems, while Beth used rationalization combined with computation of odds. Of these three students, only Debra and Beth had prior instruction on probability. When asked to apply numerical reasoning

to their solutions, Emily and Debra preferred percentages, while Beth preferred finding the odds. Angela and Cathy, the two students who had the most previous exposure to probability before the course, felt more comfortable using ratios and probability to solve their problems. Angela had difficulty understanding the word problems, treating probability as another algorithm to solve word problems, while Cathy was proficient and systematic in her solutions to the problems.

Difficulties

The students encountered some difficulties while attempting to solve the conditional probability problems. Angela and Debra appeared to have difficulties understanding the questions. While Angela claimed it was her dislike of word problems overall that caused difficulty for her to start the problem. Debra seemed not to know how to start solving the last two problems on equally likely events. The conditioning event of changing the sample space of the original sample did create difficulties for Angela and Beth. Angela was not always able to recognize the change in sample space after an outcome was removed, thus had further difficulties comparing outcomes. Beth recognized the change in the sample space, but she had difficulties reassigning quantifiable statements after an outcome was removed from the original sample space. Verbal responses to their explanations of their problem solving processes illustrated the lack of probability vocabulary knowledge among the five participants. Angela preferred to use terms such as "four to nine chance", "four to ten ratio", and "one to three possibilities", using the terms chance and ratios interchangeably, while using the term possibility to indicate the odd of one situation. Beth appeared the most confused by

terminology, using terms such as "more probability in green" for choosing green overall, "four out of five" for stating the odds of choosing a Snickers bar, and "He has a one to five chance over everyone else...while his odds are better than the girls - three boys and two girls" to indicate that the chance of selecting Rick is one fifth, but the boys odds are three to two with the girls, not Rick alone. In Emily's first attempt to solve probability problems, she developed her own terminology for explaining her solutions. Emily stated there was "one more of a chance to get a Snickers bar" after a Hershey bar was removed, and her friends probability of being chosen was "one better" after the blue marble was removed. Angela and Debra had difficulties comparing probabilities after a conditioning event had occurred, Beth did not seem to understand the difference between probability and odds, and Emily was not consistent in her use of percentages, ratios, and probability.

Conclusion

Among these five participants, rationalization, finding the odds, computing the percentages, and stating the ratio of a problem were the preferred heuristics used to solve the problems on the Probability Knowledge Inventory. Overall, each participant encountered difficulties approaching the problems and stating their solutions. The difficulties the students encountered included understanding the problem; recognizing the original sample space and when it changes; lacking probability vocabulary knowledge; comparing probability after the sample space changed; understanding the difference between probability and odds; and interchanging ratio, odds, and percentages - sometimes incorrectly - to justify their solutions.

Question Two: After attending a two-week class on sample space, probability of an event, and conditional probability, in what ways did the students' heuristics of conditional probability change, and in what ways were they able to overcome difficulties they had previously encountered when solving conditional probability problems?

Prior to attending the two-week instruction on sample space, probability of an event, and conditional probability, the five participants used a variety of heuristics to solve the problems and encountered some difficulties solving conditional probability problems.

Heuristics

Prior to instruction, rationalization, finding the odds, computing the percentages, and stating the ratio of a problem were the preferred heuristics used to solve the problems on the Probability Knowledge Inventory. After the two-week instruction, only two participants changed their use of heuristics to solve the problems on the Probability Knowledge Inventory. Beth, who originally relied on probability and odds to justify her solutions, changed her numerical justifications to ratios. However, this change in Beth's use of heuristics may be questionable since the use of ratios is similar to the concept of odds. Emily made the biggest adjustment in her use of heuristics. Previously, without prior knowledge of probability, Emily rationalized her solutions. After the instruction, Emily used probability to state her written solution, but stated her verbal solution with the rationalization of solving the problem. Among the other three participants, the two-week instruction appeared not to influence their use of heuristics. Cathy, the student with the strongest probability background, felt that the instruction "clarified the concepts in my

head better". Cathy continued using probability to justify all her solutions; however, she seemed to feel more confident in her solutions. Angela and Debra continued the use of the same heuristics after the instruction. Despite Angela's claim that the instruction clarified her understanding of probability, she continued interchanging the use of probability and ratios to justify her solutions. Believing she solved the problems the same way prior to instruction, Debra continued using rationalization to solve the problems. However, prior to instruction, if further clarification was needed, Debra would use proportions and percentages. After the two-week instruction, Debra started using probability and ratios to clarify her solutions.

Difficulties

The first interview indicated that the five participants encountered difficulties solving conditional probability problems prior to the two-week instruction. However, after the instruction, the difficulties in the students' attempts to solve the problems diminished. Angela and Debra, the two students who previously had difficulty understanding the question of interest, felt more confident in recognizing the purpose of the question and were able to give solutions to all the problems. Angela and Beth previously had difficulties recognizing the change of the sample space after a conditioning event. After the two-week instruction, when Angela responded to the Probability Knowledge Inventory, her solutions indicated that she still had difficulties recognizing the change in sample space. However, during the interview, Angela noticed the discrepancy in the sample space and corrected her solutions to the problems. Beth's difficulty with sample space did not improve. In her second interview, Beth appeared to

have more difficulty recognizing the original sample space, then she had in the first interview and she had the same difficulty recognizing the change in sample space in both interviews.

The use of probability vocabulary did improve for two of the students. Angela consistently reported her written solutions as fractions, and orally, Angela reported her probability solutions as "4 out of 9" or "1 out of 5". When Angela did chose to solve some problems using ratios, she continued using consistent terminology with her ratio responses, for example: "ratio of 3 to 2". Emily, who previously used her own terminology to explain her solutions, changed her terminology to be more consistent with probability terminology. On her Inventory, Emily's written responses would be in the form of a fraction, and she consistently reported the numerical outcomes in probability terms: " One out of 5 chances that he will be chosen, 4 out of 5 chances that he will not be chosen". Beth continued having difficulties with the terminology. She continued to confuse ratios and proportions, and her numerical responses were not consistent. Prior to instruction, Angela and Debra had difficulty comparing probabilities after the sample space changed. After the instruction, both recognized that the change in sample space also affects the probability. Finally, Beth continued having difficulties distinguishing between probability and odds. However, after the instruction, Beth first tried to rationalize her solutions, then, when she felt comfortable justifying her solutions numerically, she used ratios instead of odds. As mentioned earlier, this may not indicate a change since ratios and odds are similar concepts.

Conclusion

Overall, the two-week instruction had some influence on improving the difficulties that the five participants had solving the conditional probability problems. Some improvements included a greater ability to understand the question of interest, to recognize the change in sample space after a conditioning event, to use probability terminology consistently, and to compare probability after the sample space changed.

Question Three: How does each student's understanding of conditional probability compare to the Conditional Probability Framework developed by Tarr and Jones (1997)?

Tarr and Jones (1997) sought to develop a framework to systematically describe and predict middle school students' thinking in conditional probability based on previous work conducted by Jones, et al. (1997). Based on their research, Tarr and Jones (1997) developed an initial framework, which was then refined and validated through assessing middle school students' thinking in conditional probabilistic situations. A more in-depth description of the development of their framework can be found in Chapter II of this paper. The final framework for middle school students' thinking in conditional probabilistic situations is shown in Table 3. The framework identifies four levels

CONSTRUCT	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Qualitative	Level 4 Numerical
CONDITIONAL PROBABILITY	<ul style="list-style-type: none"> • Recognizes when "certain" and "impossible" events arise in replacement and non-replacement situations • Generally uses subjective reasoning in considering the conditional probability of any even in a "with" or "without" replacement situation • Ignores given numerical information in formulating predictions 	<ul style="list-style-type: none"> • Recognizes that the probabilities of some events change in a "without replacement" situation. Recognition is incomplete, however, and is usually confined to events that have previously occurred • Inappropriate use of numbers in determining conditional probabilities. For example, when the sample space contains two outcomes, always assumes that the two outcomes are equally likely • Representativeness acts as a confounding effect when making decisions about conditional probability. • May revert to subjective judgements 	<ul style="list-style-type: none"> • Recognizes that the probabilities of all events change in a "without replacement" situation, and that none change in a "with replacement" situation • Keeps track of the complete composition of the sample space in judging the relatedness of two events in both "with" and "without" replacement situations • Can quantify, albeit imprecisely, changing probabilities in a "without replacement" situation 	<ul style="list-style-type: none"> • Assigns numerical probabilities in "with" and "without" replacement situations • Uses numerical reasoning to compare the probabilities of events before and after each trial in "with" and "without" replacement situations • States the necessary conditions under which two events are related
INDEPENDENCE	<ul style="list-style-type: none"> • Predisposition to consider that consecutive events are always related • Pervasive belief that they can control the outcome of an event • Uses subjective reasoning which precludes any meaningful focus on the independence • Exhibits unwarranted confidence in predicting successive outcomes 	<ul style="list-style-type: none"> • Shows some recognition as to whether consecutive events are related or unrelated • Frequently uses a "representativeness" strategy, either a positive or negative recency orientation • May also revert to subjective reasoning 	<ul style="list-style-type: none"> • Recognizes when the outcome of the first event does or does not influence the outcome of the second event. In "with replacement" situations, sees the sample space as restored • Can differentiate, albeit imprecisely, independent and depended events in "with" and "without" replacement situations • May revert to the use of a representativeness strategy 	<ul style="list-style-type: none"> • Distinguishes dependent and independent events in "with" and "without" replacement situations, using numerical probabilities to justify their reasoning. • Observes outcomes of successive trials but rejects a representativeness strategy • Reluctance or refusal to predict outcomes when events are equally likely

Table 3: Refined Framework for Assessing Middle School Students' Thinking in Conditional Probability and Independence (Tarr & Jones, 1997, p. 48)

of probabilistic thinking ranging from subjective judgements to numerical reasoning.

The description for each level indicates a pattern of growth in probabilistic thinking. The framework was developed, refined, and validated using middle school students.

However, it may be interesting to observe if similar patterns are apparent in the growth of college level students thinking of conditional probability. Thus, the purpose of this section is to compare each of the five participants' growth in conditional probability thinking against the framework, noting similarities and differences in the findings.

Based on the solutions on the Probability Knowledge Inventory and the interviews prior to instruction, it is apparent that four of the participants can be classified as Level 3 students, and one participant can be classified as a Level 4 student. After the two-week instruction, despite changes in their use of heuristics and improvements in their difficulties of solving conditional probability problems, the four participants classified as Level 3 did not indicate a strong enough change in their thinking to move up to Level 4.

Angela and Beth best exemplified Level 3 students. Level 3 students are aware of the role that quantities play in forming conditional probability judgements. Although they did not assign precise numerical probabilities, they often used relative frequencies, ratios, or odds to solve conditional probability events. Level 3 students also try to keep track of the complete composition of the sample space and usually recognize that the conditional probabilities of all events change in "without replacement" situations. Angela and Beth exhibited these characteristics in their explanations of the solutions.

Unfortunately, the framework developed by Tarr and Jones (1997) does not consider a student rationalizing the solution of the problem correctly, taking into account the change in sample space, without numerical justification. Debra and Emily, who can be best described as Level 3 students, illustrated this particular characteristic not defined in the framework. However, Debra and Emily can not be considered Level 2 students according to this framework since Level 2 describes a student who uses subjective

judgements and the representativeness heuristics to solve their problems. Also, Level 2 students are prone to assuming that a probability situation containing two outcomes assumes that the two outcomes are equally likely. Debra and Emily did not exhibit these Level 2 characteristics. If quantifying solutions was not a key characteristic of Level 3 students, Debra and Emily could be clearly classified as Level 3.

Cathy was the only participant who would be considered a Level 4 student. Cathy consistently used numerical reasoning to interpret probability situations. Cathy was aware of the composition of the sample space, recognized its importance in determining conditional probability and was able to assign numerical probabilities spontaneously and with explanation. All these characteristics are defined to be Level 4 students.

It seemed that the framework developed by Tarr and Jones (1997) can apply to the five participants in the study. However, it is apparent that the characteristics associated with Level 3 may need some refining. It was not evident that the two-week instruction had influenced the participants to move to another level of thinking.

Chapter V

Discussion

Interpretation and Discussion

As the emphasis for probability in the mathematics curriculum expands, the role of understanding the teaching and learning of probability increases. The intent of this study was to look at the teaching and learning of conditional probability in a college level mathematics course. More specifically, this study considered the influence of teaching on college student's use of heuristics and difficulties they encounter solving conditional probability problems in a contemporary college level math course. By using a case study analysis, this study provided an in-depth attempt to understand five participants, by providing detailed descriptions of their solutions on the Probability Knowledge Inventory prior to and after instruction.

The observed math course curriculum level introduced the concept of conditional probability; hence, it is difficult to compare the results of this study with previous research on conditional probability. For example, previous research on students' use of heuristics solving conditional probability problems identified two common heuristics: time-axis fallacy (Falk, 1983, 1988) and causal bias (Tversky & Kahneman, 1980). Chapter II of this paper provides definitions and examples of these two heuristics. Due to the level of conditional probability problems on the Probability Knowledge Inventory and in the course curriculum, the five participants were not asked about conditional situations that took place "back in time". These types of conditional probability situations were related to the time-axis heuristic. However, the Falk Phenomenon problem of the two

white and two black balls was used in the classroom as an extension question for the class. The Probability Knowledge Inventory was not an instrument designed to measure the possibility of students using the time-axis fallacy heuristic. Similarly, the students in the course were not exposed to questions involving causal relationships. Hence, the use of the causal bias heuristic could not be observed on the five participants.

However, the results of this study have some similarities to the difficulties encountered in solving conditional probability problems defined in previous research. The initial difficulties that the five participants experienced while trying to solve conditional probability problems included understanding the problem; recognizing the sample space and when it changes; lacking probability vocabulary knowledge; comparing probability after the sample space changed; understanding the difference between probability and odds; and interchanging the use of ratio, odds, and percentages - sometimes incorrectly - to justify their solutions. Previous research on the difficulties students encounter solving conditional probability problems identified three common difficulties: difficulties in calculation of the inverse of the condition, difficulties in identification of the conditional event, and confusion due to the wording or framing of the conditional probability (Falk, 1989). Due to the level of conditional probability problems on the Probability Knowledge Inventory and in the course curriculum, it may be difficult to support or oppose hypotheses of previous research on solving conditional probability problems.

Overall, the results from this study may be beneficial to college mathematics instructors teaching entry-level probability courses, or courses designed for non-mathematics and science majors. Most non-mathematics and science major students

entering college mathematics receive limited instruction on the concept of probability, its laws, and its applications in real-world situations. In general, the findings of this study suggest that these students use a variety of heuristics and encounter a greater variety of difficulties solving conditional probability problems. Instruction alone may not influence their use of heuristics or help them overcome their difficulties; however, the recognition of the possible heuristics used and the difficulties they encounter may help the course instructor become more effective.

Limitations of the Study

This study has limitations that could have affected its results. Some limitations include the course curriculum, the role of the researcher in the classroom, and the research design.

First, the observed mathematics curriculum was a limitation. The probability curriculum designed for the course did not offer the students an in-depth study of the concept of conditional probability. The curriculum was designed to offer the students a general overview of conditional probability, mainly exploring conditional problems in which the removal of an outcome from a sample space affects the probability of an event, similar to the problems found on the Probability Knowledge Inventory. The students were introduced to the formal definition of conditional probability; however, the problems associated with this definition pertain to the students reading Venn Diagrams and tables to gather their data. An example of a conditional probability problem using the definition of conditional probability from the course (Parks, et al., 2000, p. 271):

A study was performed to find out how the number of defective items produced varied between the day, evening and night shifts.

	Day	Evening	Night
Defective	24	28	47
Not Defective	279	224	165

If an item is picked at random, find the probability that:

- The item is defective given that it came from the night shift
- The item is not defective given that it came from the day shift
- The evening shift produced the item given that it was not defective

The conditional probability problems the students encountered in the course curriculum did not consider the probability of cause and effect; the use of Bayes Law; changing the sample of interest; or problems in which the student must "go back in time" to compute the probability, as illustrated in the time-axis phenomena. The limitations of the course curriculum made it difficult to compare the results of this study to previous studies on conditional probability.

The role of the researcher in the classroom was a second limitation. First, the instructor was also the researcher for this study; therefore, the researcher was more than an active participant observer. Limitations associated with the instructor as the researcher include possible bias the researcher has towards the students, possible changes in the curriculum due to the researcher / instructor being aware of students difficulties, and the possibility that the students knowing they are participating in a study would act differently than if they were not participating in a study.

The use of the one group pre-test / post-test design with semi-structured interviews creates several implications to the research design of this study. First, the exposure to a pre-test may affect the student's performance in the class and on the post-

test. This administration of a pre-test may sensitize students to respond to the treatment a different way than they would if they had not been pre-tested. This is referred to as pre-test sensitization, which is a potential external-validity problem. A second limitation was in the one-group design (Ary, et al., 1990). This was a limitation because there was no control group used and the results cannot assume that the change between the pre-test and post-test was brought about by the experimental treatment. This design affects the internal validity of the study (Ary, et al., 1990). By using the same Inventory as the pre-test and post-test, the student's increased performance may not be caused by instruction, but rather by the students recalling during the course instruction how to solve a particular problem. After discussion with the curriculum developer, it was decided that the same pre-test and post-test will not have a major effect of measuring students understanding of conditional probability. Finally, although interviews provide valuable data, just as all data, this data could be susceptible to bias.

Despite taking precautionary measures to ensure the lack of bias in the final analysis, no study is without limitations.

Implications and Recommendations for Future Studies

The results of this study have implications for mathematics and statistics education at all levels, but specifically to undergraduate mathematics and statistics education. Math and statistics educators must help students understand the concept of conditional probability, teach them to develop correct heuristics that could be used to find the conditional probability, and recognize the difficulties students may encounter solving conditional probability problems. This study highlights possible variables that could be

influential in helping students gain a better understanding of conditional probability problems. The findings from this study also suggest several areas for further research.

The two-week instruction the students received in the course could not be labeled "innovative", "traditional", or "atypical". The instruction the students received was the method of instruction any student would have received if they had the opportunity to enroll in a class with the same instructor. The instructor did not change her teaching habits, teaching style, or the curriculum expectations for the course. Previous studies on the effects of classroom teaching on the learning of probability had considered "non-traditional" teaching methods on its influences the learning of probability (Austin, 1974; Castro, 1998; Fischbein & Gazit, 1984; Ojemann, et al., 1965a; Ojemann, et al., 1965b; Shaughnessy, 1977). Further research needs to be conducted on the teaching methods and their effects on student learning of probability.

The curriculum for the chapter on probability did not contain an in-depth understanding of the concept of conditional probability. As mentioned earlier in this chapter, the conditional probability problems the students encounter in the course curriculum do not consider the interpretation of the probability of cause and effect; the use of Bayes Law; changing the sample of interest; or problems in which the student must "go back in time" to compute the probability, as illustrated in the time-axis phenomena. Previous research on the heuristics used by students and the difficulties they encountered solving conditional probability problems require an understanding of these concepts (Falk, 1983, 1988, 1989; Tversky & Kahneman, 1980). Hence, another area for further investigation is the use of heuristics and the difficulties students encounter when solving conditional probability problems in a higher level probability course.

A longitudinal study could also more closely examine the long-term effects of learning conditional probability heuristics and overcoming difficulties. Jones, et al. (1997, 1999) evaluated the thinking of third grade students in relation to an instructional program in probability over the course of one year. Falk (1986) noted that students and professionals educated in the meaning of significant test results tend to misinterpret the test results over time. Probability has become a branch of mathematics with wide ramifications in scientific research, business and industry, politics, and practical daily life. Further investigation may consider how the teaching of conditional probability can influence the understanding of conditional probability years after the person learned the concept.

As noted in the transcriptions and discussing problems with the five participants, it was evident that a hindrance for learning probability was the lack of knowledge of the terminology associated with probability and the correct use of syntax. As the students explained the sample space for Problem 2, it was evident that the students did not know the difference between the words "and" or "or". Angela stated her sample space to be "1 or 3; 1 or 4; 2 or 3; 2 or 4". Further indication of the lack of proper knowledge of probability terms was mentioned in Chapter IV as a difficulty that many students encountered solving the problems. Angela preferred to use terms such as "four to nine chance", "four to ten ratio", and "one to three possibilities", using the terms chance and ratios interchangeably, while using the term possibility to indicate the odds of one situation. Beth appeared the most confused with her terminology, using terms such as "more probability in green" for choosing green overall, "four out of five" for stating the odds of choosing a Snickers bar, and "He has a one to five chance over everyone

else...while his odds are better than the girls - three boys and two girls" to indicate that the chance of selecting Rick is one fifth, but the boys odds are three to two with the girls, not Rick alone. Despite the confusion students have learning a new concept, it is just as important for them to understand the terminology and syntax associated with the concept. Further studies on the teaching and learning of probability could also explore the impact of terminology and syntax on the learning of probability.

Finally, during the course of data collection for this study, it was difficult to focus the interview on the concept of interest - conditional probability - without considering other background concepts, such as sample space and probability. Future research on the teaching and learning of probability must look at theoretical probability as a "whole" - sample space, probability, comparison probability, and conditional probability. It is very difficult to look at one concept without looking at students understanding of the underlying concepts.

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APPENDICES

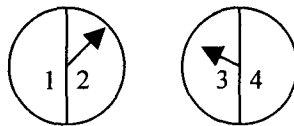
APPENDIX A: PROBABILITY KNOWLEDGE INVENTORY

Student Code Number: _____

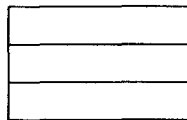
Probability Knowledge Inventory

1. In a bag, there are 4 green marbles, 3 red marbles, and 2 yellow marbles. If you close your eyes and draw a marble from the bag, what possible colors could your marble be?

2. Spin both spinners. If you were to sum up the total of the numbers selected on the two spinners, what are all the possibilities you could you get?



3. Suppose you had three different colored pencils: red, blue, and green. Imagine you used the red, blue, and green pencils to color each section of the following figure:



- a. How many ways can you color in this figure using each color once?
- b. List them.

4. In a bag, there are 4 Snickers bars, 3 Hershey bars, and 2 Butterfinger bars. Suppose you closed your eyes and drew a candy bar.
- What kind of candy bar do you have the most chance of drawing? Why?
 - Suppose you chose a Hershey bar and ate it. If you reached into the bag again, what kind of candy bar do you have the most chance of drawing? Why?
 - Has your chance of drawing a Snickers bar in part (b) changed or is it the same chance as in part (a)? Why?
5. Suppose your teacher is going to have a drawing to see who can miss the final exam without it affecting their grade. If your color is drawn, then you do not have to take the final exam. In a bag, there are 4 blue marbles, 3 green marbles, 2 red marbles, and 1 yellow marble.
- What color do you want to be? Why?
 - Suppose you are assigned green and your friend is assigned yellow. Your teacher reached in and drew a marble. Which of the four colors do you predict will be drawn? Why?
 - Your teacher chose a blue marble, and did not replace it. Suppose your teacher has another drawing the following day. Has *your* chance of winning the second day changed or is it the same as the day before? Has your friend's chance of winning changed or is it the same as before? Why or why not?

6. The ASOSU is electing a president and a vice president. There are five people running: Beth, Jose, Maria, Rick, and Joshua. All five students are considered to have an equal chance of winning. At the end of the day, the results are announced.
- a. Is it more likely the class president will be a boy or girl? Why?
 - b. Is it more likely Rick will be chosen for one of the positions or Rick will not be chosen? Why or why not?
 - c. Suppose Beth is selected president. A vice president is selected randomly from the remaining candidates. Is it more likely the vice president will be a boy or girl? Why?
 - d. After Beth was selected, has the chance that Rick will be selected for vice president changed compared to part (b)?
7. You have a die.
- a. If you rolled the die, can you predict with certainty which number will come up? Explain your reasoning.
 - b. You rolled the die and got a 2. Roll the die again. Does the outcome of the first roll affect the possibility of rolling a 2 the second time?
 - c. Suppose you wanted to roll a 3. On average, how many rolls should it take to ensure that a 3 would come up?

APPENDIX B: CONSENT FORMS

Probabilistic Thinking in College Students Research Consent Form (Class Participation)

By signing this form below, I attest to the following:

1. I understand that I am a participating in a research study. The purpose of the research is to examine college students' probabilistic thinking knowledge. My participation will consist of taking a pre and post inventory test of my probabilistic knowledge.
2. I understand that my participation in this study is voluntary, and that I may withdraw my participation at any time with no penalty.
3. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.
4. I understand that the researcher will keep my responses confidential and will destroy all records at the completion of the research.
5. I understand that I will not receive any compensation for my participation in this study.
6. I understand that the results of the inventory tests will not have any effect to my grade.

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

Name (printed)

date

Signature

Questions concerning this research, my rights, or any research related injuries should be directed to Mary Bamberger at (541) 758-0897 or bambergm@ucs.orst.edu.

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-0670

**Probabilistic Thinking in College Students
Research Consent Form
(Interview Participation)**

By signing this form below, I attest to the following:

1. I understand that I am a participating in a research study. The purpose of the research is to examine college students' probabilistic thinking knowledge. My participation will consist of taking a pre and post inventory test of my probabilistic knowledge, and participating in two one-hour taped interviews on my responses to the inventory test.
2. I understand that my participation in this study is voluntary, and that I may withdraw my participation at any time with no penalty.
3. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.
4. I understand that the researcher will keep my responses confidential and will destroy all records at the completion of the research.
5. I understand that I will not receive any compensation for my participation in this study.
6. I understand that the results of the inventory test and the taped interviews will not have any effect to my grade.

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

Name (printed)

date

Signature

Questions concerning this research, my rights, or any research related injuries should be directed to Mary Bamberger at (541) 758-0897 or bambergm@ucs.orst.edu.

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-0670

**Probabilistic Thinking in College Students
Research Consent Form
(At Time of Interview)**

By signing this form below, I attest to the following:

1. I understand that I am a participating in a research study. The purpose of the research is to examine college students' probabilistic thinking knowledge. My participation will consist of taking part in this taped interview, where I will be asked to explain my responses to an inventory test of my probabilistic knowledge. This interview will be taped.
2. I understand that my participation in this study is voluntary, and that I may withdraw my participation at any time with no penalty.
3. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.
4. I understand that the researcher will keep my responses confidential and will destroy all records at the completion of the research.
5. I understand that I will not receive any compensation for my participation in this study.
6. I understand that the results of the interview will not have any effect to my grade.

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

Name (printed)

date

Signature

Questions concerning this research, my rights, or any research related injuries should be directed to Mary Bamberger at (541) 758-0897 or bambergm@ucs.orst.edu.

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-0670

APPENDIX C: CLASSROOM EXPERIMENTS

Experiment #1

Experiment:

Roll two 4 sided dice and record the numbers on each die.

Sample Space:

Simple Experiment Table

Outcomes	Frequency	Experimental Probability	Theoretical Probability
Total:		Total:	Total:

Event:

Let E be the event of getting an even number of dots on at least one die.

Possible Outcomes of Event E:

Theoretical Probability of Event E:

Experiment #2

Experiment:

Toss a coin three times and record the results in order.

Sample Space:

Simple Experiment Table

Outcomes	Frequency	Experimental Probability	Theoretical Probability
	Total:	Total:	Total:

Event:

Let E be the event of getting a tail on the first coin.

Possible Outcomes of Event E:

Theoretical Probability of Event E:

Experiment #4

Experiment:

A bag contains 4 different colored die. Draw two die from the bag, one after another without replacing the first drawn. Record the results in order.

Sample Space:

Simple Experiment Table

Outcomes	Frequency	Experimental Probability	Theoretical Probability
	Total:	Total:	Total:

Event:

Let E be the event of one of the die is red.

Possible Outcomes of Event E:

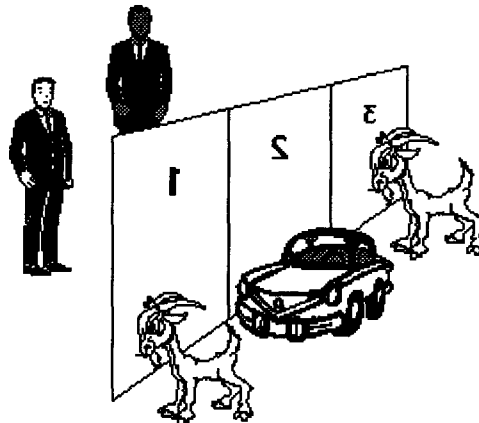
Theoretical Probability of Event E:

APPENDIX D: MONTY HALL'S DILEMMA

Name _____

Monty Hall's Dilemma
The Problem and Preliminary Analysis

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the other two doors are goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it



to your advantage to switch your choice of doors?

If you were the contestant, which of the following strategies would you choose, and **why**?

- d. Strategy 1 (stick): Stick with the original door.
- e. Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- f. Strategy 3 (switch): Switch to the other door.

APPENDIX E: MONTY HALL'S DILEMMA EXPERIMENT

Group Name: _____

Monty Hall's Dilemma Experiment

Data Collection

Group 1: Strategy 1 - **STICK**

It is now your turn to generate data for Monty Hall's Dilemma. To determine which of the three strategies yields the best chance of winning the prize, we should play each strategy many, many times and keep a record of the outcomes. Before collecting the data, make sure you have a clear understanding of the problem. For a visual representation of the problem, use the cups as the doors, the matchbox car as the prize, and find two objects you would use to represent the goats.

Understanding the Problem

Let us suppose that the prize is actually hidden behind door A (i.e. the car is under the cup labeled A).

1. Suppose that you choose door B.

a. What does Monty do?

b. What do you do?

c. Do you win or lose?

2. Suppose you choose door C.

a. What does Monty do?

b. What do you do?

c. Do you win or lose?

3. Suppose you choose door A. In this instance, Monty shows you either door B or door C.

a. What do you do?

b. Do you win or lose?

Data Collection

Use the cups, car, and objects representing the goat to run 100 trials. Assign each person in your group a job:

1. "Behind the scene" person hides the car and goats under the cups
2. Monty Hall - asks contestant which curtain he would like. After the contestant picks a curtain, you are to show the contestant the incorrect curtain.
3. Contestant - chooses which curtain the car may be under.
4. Data collector - records how many times the contestant "wins" and "loses".

Data Analysis

1. Construct a Probability Tree Diagram for your event. **Assume the car is behind Door A.**

Figure 1:
Stick-strategy Probability Tree Diagram

2. From your data and the data collected by other groups in the class, fill in the following table:

TABLE 1:
Experimental Results
(Raw Data)

Strategy	Won	Lost	Experimental Probability
Stick			
Flip			
Switch			

3. After listening to the presentations of the data collecting techniques and data analysis of the other groups, fill in the following table:

TABLE 2:
Theoretical Results
(based on Probability Tree Diagram)

Strategy	Won	Lost
Stick		
Flip		
Switch		

4. On a separate piece of paper, use the data collected and analyzed for this study to answer the following question. Can you now justify your answer? How would you explain this choice to your friends?

If you were the contestant, which of the following strategies would you choose?

- d. Strategy 1 (stick): Stick with the original door.
- e. Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- f. Strategy 3 (switch): Switch to the other door.

Group Name: _____

Monty Hall's Dilemma Experiment

Data Collection

Group 2: Strategy 2- FLIP

It is now your turn to generate data for Monty Hall's Dilemma. To determine which of the three strategies yields the best chance of winning the prize, we should play each strategy many, many times and keep a record of the outcomes. Before collecting the data, make sure you have a clear understanding of the problem. For a visual representation of the problem, use the cups as the doors, the matchbox car as the prize, and find two objects you would use to represent the goats.

Understanding the Problem

Let us suppose that the prize is actually hidden behind door A (i.e. the car is under the cup labeled A).

1. Suppose that you choose door B.
 - a. What does Monty do?

You flip a coin to decide whether to stick with door B or switch to door A.
What is your chance of winning the prize

2. Suppose you choose door C.
 - a. What does Monty do?

-
- b. You flip to decide between A and C. What is your chance of winning the prize?
-

3. Suppose you choose door A.
 - a. What does Monty do?

-
- b. You flip to decide between A and the other door. What is your chance of winning the prize?
-

Data Collection

Use the cups, car, and objects representing the goat to run 100 trials. Assign each person in your group a job:

1. "Behind the scene" person hides the car and goats under the cups
2. Monty Hall - asks contestant which curtain he would like. After the contestant picks a curtain, you are to show the contestant the incorrect curtain.
3. Contestant - chooses which curtain the car may be under. After Monty shows you the incorrect curtain, you flip a coin to decide which curtain you would like to choose
4. Data collector - records how many times the contestant "wins" and "loses".

Data Analysis

1. Construct a Probability Tree Diagram for your event. **Assume the car is behind Door A.**

Figure 1:
Flip-strategy Probability Tree Diagram

2. From your data and the data collected by other groups in the class, fill in the following table:

TABLE 1:
Experimental Results
(Raw Data)

Strategy	Won	Lost	Experimental Probability
Stick			
Flip			
Switch			

3. After listening to the presentations of the data collecting techniques and data analysis of the other groups, fill in the following table:

TABLE 2:
Theoretical Results
(based on Probability Tree Diagram)

Strategy	Won	Lost
Stick		
Flip		
Switch		

4. On a separate piece of paper, use the data collected and analyzed for this study to answer the following question. Can you now justify your answer? How would you explain this choice to your friends?

If you were the contestant, which of the following strategies would you choose?

- a. Strategy 1 (stick): Stick with the original door.
- b. Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- c. Strategy 3 (switch): Switch to the other door.

Group Name: _____

Monty Hall's Dilemma Experiment

Data Collection

Group 3: Strategy 3 - SWITCH

It is now your turn to generate data for Monty Hall's Dilemma. To determine which of the three strategies yields the best chance of winning the prize, we should play each strategy many, many times and keep a record of the outcomes. Before collecting the data, make sure you have a clear understanding of the problem. For a visual representation of the problem, use the cups as the doors, the matchbox car as the prize, and find two objects you would use to represent the goats.

Understanding the Problem

Let us suppose that the prize is actually hidden behind door A (i.e. the car is under the cup labeled A).

1. Suppose that you choose door B.

a. What does Monty do?

b. What do you do?

c. Do you win or lose?

2. Suppose you choose door C.

a. What does Monty do?

b. What do you do?

c. Do you win or lose?

3. Suppose you choose door A. In this instance, Monty shows you either door B or door C.

a. What do you do?

b. Do you win or lose?

Data Collection

Use the cups, car, and objects representing the goat to run 100 trials. Assign each person in your group a job:

1. "Behind the scene" person hides the car and goats under the cups
2. Monty Hall - asks contestant which curtain he would like. After the contestant picks a curtain, you are to show the contestant the incorrect curtain.
3. Contestant - chooses which curtain the car may be under. After Monty shows you which curtain the car is not under, you switch your decision.
4. Data collector - records how many times the contestant "wins" and "loses".

Data Analysis

1. Construct a Probability Tree Diagram for your event. **Assume the car is behind Door A.**

Figure 1:
Switch-strategy Probability Tree Diagram

2. From your data and the data collected by other groups in the class, fill in the following table:

TABLE 1:
Experimental Results
(Raw Data)

Strategy	Won	Lost	Experimental Probability
Stick			
Flip			
Switch			

3. After listening to the presentations of the data collecting techniques and data analysis of the other groups, fill in the following table:

TABLE 2:
Theoretical Results
(based on Probability Tree Diagram)

Strategy	Won	Lost
Stick		
Flip		
Switch		

4. On a separate piece of paper, use the data collected and analyzed for this study to answer the following question. Can you now justify your answer? How would you explain this choice to your friends?

If you were the contestant, which of the following strategies would you choose?

- g. Strategy 1 (stick): Stick with the original door.
- h. Strategy 2 (flip): Flip a coin, stick if it shows heads, and switch if it shows tails.
- i. Strategy 3 (switch): Switch to the other door.