AN ABSTRACT OF THE THESIS OF

<u>ROBERT CARL JOHNSON</u>   for the   <u>DOCTOR OF PHILOSOPHY</u>
       (Name)                             (Degree)

in    <u>MATHEMATICS</u>   presented on <u>August 6, 1968</u>
       (Major)                            (Date)

Title: <u>EQUIVARIANT STIEFEL-WHITNEY CLASSES</u>

Abstract approved: <u>    Redacted for Privacy    </u>
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Let $\mathcal{V}$ be the category of vector bundles and $W^q$ the $q^{\underline{th}}$ Stiefel-Whitney class, regarded as a function on $\mathcal{V}$. We consider the category $\mathcal{V}_{p.\,g.}$ whose objects are pairs $(\mathcal{B}, J)$, where $\mathcal{B}$ is a vector bundle and $J$ a pseudo-group of local bundle maps on $\mathcal{B}$, and whose morphisms are equivariant bundle maps. Clearly $\mathcal{V}_{p.\,g.}$ contains $\mathcal{V}$ as a full subcategory.

In this paper we construct a maximal extension $W^q_e$ of $W^q$ to a full subcategory of $\mathcal{V}_{p.\,g.}$. The extended function $W^q_e$ is natural with respect to morphisms in $\mathcal{V}_{p.\,g.}$ and constitutes the primary obstruction to equivariant cross-sections in the associated frame bundle of appropriate dimension.

Equivariant Stiefel-Whitney Classes

by

Robert Carl Johnson

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1969

APPROVED:

Redacted for Privacy

Professor of Mathematics

in charge of major

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Date thesis is presented _____ August 6, 1968 _____

Typed by Clover Redfern for _____ Robert Carl Johnson _____

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## ACKNOWLEDGMENT

This thesis is dedicated to my wonderful, loving and very patient wife, Shirley.

# EQUIVARIANT STIEFEL-WHITNEY CLASSES

## I.  INTRODUCTION

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle,  B  its total space (bundle space),  X  its base space, and  $\pi : B \to X$  the projection.  For each  $q \geq 0$,  the $q^{\underline{th}}$ Stiefel-Whitney class $W^q(\mathcal{B})$ is an element of  $H^q(X; Z_2)$,  the singular q-dimensional cohomology group of  X  with coefficients the integers modulo two.

Let  $\mathcal{V}$  be the category of vector bundles.  The functions $W^q$ on  $\mathcal{V}$  satisfy the following properties (4, p.  281):

1) $W^0(\mathcal{B})$  is the unit class of  $H^0(X; Z_2)$;

2) $W^q(\mathcal{B}) = 0$   for   $q > n$;

3) If   $f = (\check{f}, \hat{f}) : \mathcal{B} \to \mathcal{B}'$   is a bundle map,  then
$$\check{f}^*(W^q(\mathcal{B}')) = W^q(\mathcal{B});$$

4) If  $\mathcal{B} \oplus \mathcal{B}'$  is the Whitney sum of the bundles  $\mathcal{B}$  and  $\mathcal{B}'$,   then  $W^q(\mathcal{B} \oplus \mathcal{B}') = \sum_{i+j=q} W^i(\mathcal{B}) \cup W^j(\mathcal{B}')$;

5) If  $\mathcal{B}$  is the one dimensional vector bundle over real projective 1-space, then  $W^1(\mathcal{B}) \neq 0$.

If the base space  X  of  $\mathcal{B}$  is paracompact, then these five properties characterize the functions  $W^q$  (2, p.  6-7).

In (3), J. W. Smith extends the category  $\mathcal{V}$  to a category of generalized vector bundles.  He then defines an extended Euler class on a full subcategory of this new category.  The classical Euler class

of an oriented n-dimensional vector bundle, reduced modulo two, is equal to the $n^{\text{th}}$ Stiefel-Whitney class of the bundle (2, p. 41). This leads one to consider the possibility of defining the Stiefel-Whitney classes of generalized vector bundles.

A generalized vector bundle is essentially represented by a vector bundle together with a set of local bundle maps. In the following chapters we consider vector bundles together with pseudo-groups of local bundle maps and thereby get a category $\mathcal{V}_{\text{p. g.}}$, <u>the category of vector bundles with pseudo-groups</u>. Using an obvious identification, we observe that $\mathcal{V}$ can be considered as a full subcategory of $\mathcal{V}_{\text{p. g.}}$

The functions $W^q$ on $\mathcal{V}$ can be defined in essentially two ways. One method, discussed in (2, p. 33-35) and (4, p. 281), uses the Thom isomorphism and the Steenrod squaring operations. The second method is based upon obstruction theory, and assumes that the base space $X$ is a finite complex (5, p. 148-204). For each $q$, $0 \leq q \leq n - 1$, let $\mathcal{B}^q$ be the associated bundle of $\mathcal{B}$ (5, p. 43) with fibre $V_{n, n-q}$, the Stiefel manifold of (n-q)-frames in n-space. The primary obstruction to a cross-section of $\mathcal{B}^q$ is an element $O^{q+1}(\mathcal{B}) \in H^{q+1}(X; \pi_q(V_{n, n-q}))$. The homotopy group $\pi_q(V_{n, n-q})$ is isomorphic to the group $Z$ of integers if $q$ is even or $q=n-1$, and isomorphic to the group $Z_2$ of integers modulo two if $q$ is odd and $q < n - 1$ (5, p. 132). The $(q+1)^{\text{st}}$ Stiefel-Whitney class

$W^{q+1}(\mathcal{B})$ is then equal to $O^{q+1}(\mathcal{B})$ reduced modulo two (2, p. 57-59).

Our <u>equivariant Stiefel-Whitney classes</u> $\underline{W_e^{q+1}}$, defined on a certain subcategory of $\mathcal{V}_{p.g.}$, shall be obtained by an adaptation of the obstruction theoretic approach. For $0 \le q \le n - 1$, $W_e^{q+1}$ shall constitute the primary obstruction to equivariant cross-sections in the appropriate generalized bundle.

## II.  PRELIMINARIES

Let $\Delta_q$ be the standard q-simplex with vertices $(p_0, p_1, \ldots, p_q)$ and let $e^i_{q+1} : \Delta_q \to \Delta_{q+1}$ be the linear map defined by the vertex map

$$e^i_{q+1}(p_j) = \begin{cases} p_j & \text{for } j < i \\ p_{j+1} & \text{for } j \geq i. \end{cases}$$

The image of $\Delta_q$ under $e^i_{q+1}$ is denoted by $\Delta^{(i)}_{q+1}$ and is called the $i^{\text{th}}$ q-face of $\Delta_{q+1}$. The image in $\Delta_{q+1}$ of the composition $e^i_{q+1} \circ e^j_q : \Delta_{q-1} \to \Delta_{q+1}$ is denoted by $\Delta^{(i)(j)}_{q+1}$ and is the $j^{\text{th}}$ (q-1)-face of $\Delta^{(i)}_{q+1}$.

The singular simplexes of a topological space $Y$, i. e., the continuous maps of $\Delta_q$ into $Y$, are denoted by small Greek letters, e. g., $\sigma : \Delta_q \to Y$. We shall let $S_q(Y)$ denote the set of all singular q-simplexes of $Y$. The $i^{\text{th}}$ (q-1)-face of the singular q-simplex $\sigma$ is denoted by $\sigma^{(i)}$ and is defined to be the composition $\sigma^{(i)} = \sigma \circ e^i_q : \Delta_{q-1} \to Y$. The $j^{\text{th}}$ (q-2)-face of $\sigma^{(i)}$ is denoted by $\sigma^{(i)(j)}$. We let $C_*(Y;G) = \{C_q(Y; G), \partial_q\}$ be the singular chain complex of $Y$ with coefficient group $G$. The corresponding cochain complex with coefficients in $G$ is denoted by $C^*(Y;G), = \{C^q(Y;G), \delta_q\}$. When $G = Z$, the group of integers, we shall simply write $C_*(Y)$ and $C^*(Y)$. We let $f_\#$ and $f^\#$ denote the chain and cochain maps

induced by a continuous map $f$ from one topological space to another, and $f^*$ the corresponding cohomology map.

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle.

Definition: A <u>pseudo-group</u> $J = \{u=(\check{u}, \hat{u})\}$ of local bundle maps on $\mathcal{B}$ is a set of bundle maps $u : \mathcal{B}|U \to \mathcal{B}$, where $U$ is an open subset of $X$ and $\mathcal{B}|U$ is the restriction of $\mathcal{B}$ to $U$, satisfying the following properties:

(i) the identity map $1 = (1_X, 1_B)$ is an element of $J$;

(ii) the inverse of each element of $J$ is an element of $J$;

(iii) the restriction of an element of $J$ to an open subset of

  its domain is an element of $J$;

(iv) the composition of elements of $J$, whenever defined, is

  an element of $J$.

In particular, condition ii) implies that if $(\check{u}, \hat{u}) \epsilon J$, then $\check{u}$ and $\hat{u}$ are both homeomorphisms, and the inverse of $(\check{u}, \hat{u})$ is $(\check{u}^{-1}, \hat{u}^{-1})$. We shall use the notations $\check{u} \epsilon J$ and $\hat{u} \epsilon J$ to mean that there is a $u = (\check{u}, \hat{u})$ in $J$.

Now let $\mathcal{B}$ be a vector bundle and $J$ a pseudo-group of local bundle maps on $\mathcal{B}$. We shall denote such a pair by $(\mathcal{B}, J)$; and shall call $(\mathcal{B}, J)$ a vector bundle with pseudo-group.

Let $(\mathcal{B}, J)$ and $(\mathcal{B}', J')$ be two n-dimensional vector bundles with pseudo-groups.

Definition: A bundle map $f = (\check{f}, \hat{f}) : \mathcal{B} \to \mathcal{B}'$ is called equivariant iff for each $u = (\check{u}, \hat{u}) \in J$, there is a $u' = (\check{u}', \hat{u}') \in J'$ such that $\check{f} \circ \check{u} = \check{u}' \circ \check{f}$ and $\hat{f} \circ \hat{u} = \hat{u}' \circ \hat{f}$ whenever these expressions are defined.

With this notion of map, we see that the set of vector bundles with pseudo-groups and equivariant bundle maps forms a category. This category will be called the category of vector bundles with pseudo-groups and will be denoted by $\mathcal{V}_{p. g.}$.

Let $I_{\mathcal{B}}$ be the pseudo-group of local bundle maps on $\mathcal{B}$ generated by the identity map of $\mathcal{B}$. Then we may identify the vector bundle $\mathcal{B}$ with the pair $(\mathcal{B}, I_{\mathcal{B}})$, and we note that under this identification, the category $\mathcal{V}$ of vector bundles is a full subcategory of $\mathcal{V}_{p. g.}$.

Now, let $(\mathcal{B}, J)$ be an object in $\mathcal{V}_{p. g.}$. The pseudo-group $J$ determines a relation on the set $S_q(X)$ of singular q-simplexes of $X$ as follows. For $\sigma, \sigma' \in S_q(X)$, we say $\sigma$ is J-related to $\sigma'$ iff there is a $\check{u} \in J$ such that $\sigma = \check{u} \circ \sigma'$. Because $J$ is a pseudo-group, this relation is obviously an equivalence relation. Let $J(\sigma', \sigma)$ be the (possibly empty) subset of $J$ consisting of all $(\check{u}, \hat{u}) \in J$ for which $\check{u} \circ \sigma' = \sigma$.

Definition: Two singular q-simplexes $\sigma$ and $\sigma'$ of $X$ are J-equivalent iff $J(\sigma, \sigma')$ is non-empty.

The relation of J-equivalence partitions the set $S_q(X)$ into equivalence classes. Let $A(\sigma)$ be the J-equivalence class of $\sigma \in S_q(X)$, i.e., $A(\sigma) = \{\sigma' \in S_q(X) : J(\sigma, \sigma')$ is non-empty$\}$.

Because J-equivalence preserves incidence, the singular chain complex $C_*(X)$ induces a chain complex structure on the quotient of $C_*(X)$ by J-equivalence. This new complex, denoted by $C_*^e(X)$, is called the equivariant chain complex of $X$ induced by $J$.

Definition: A q-cochain $c \in C^q(X;G)$ is called equivariant iff $c(\sigma) = c(\sigma')$ for all $\sigma, \sigma' \in S_q(X)$, with $\sigma' \in A(\sigma)$.

Thus, a q-cochain is equivariant iff it assigns the same element of the coefficient group to all J-equivalent singular q-simplexes. The equivariant cochains obviously constitute a subcomplex of the cochain complex $C^*(X;G)$ which we denote by $\underline{C_e^*(X;G)}$. The corresponding equivariant cohomology groups are denoted by $\underline{H_e^q(X;G)}$. We note that the cochain complex $C_e^*(X;G)$ is naturally isomorphic to the cochain complex corresponding to the equivariant chain complex $C_*^e(X)$ and coefficient group $G$.

Definition: A homomorphism $T_p : C_p(X;G) \to C_p(B;G)$ is called equivariant iff for each $(\breve{u}, \hat{u}) \in J$, $T_p \circ \breve{u}_\# = \hat{u}_\# \circ T_p$, whenever these expressions are defined. A set of equivariant homomorphisms $T = \{T_p : 0 \leq p \leq q, \ T_p : C_p(X;G) \to C_p(B;G)\}$ satisfying $\partial_p \circ T_p = T_{p-1} \circ \partial_p$

is called an _equivariant chain map_ on dimensions less than q+1.

Now, let $(\mathcal{B}, J)$ be an object in $\mathcal{V}_{p.g.}$, with $\mathcal{B} = (B, X, \pi)$ and the dimension of $\mathcal{B}$ equal to n. For each q, $0 \leq q \leq n - 1$, let $\mathcal{B}^q = (B^q, X, \pi^q, Y^q)$ be the associated bundle (5, p. 43) of $\mathcal{B}$ with fibre $Y^q = V_{n, n-q}$, the Stiefel manifold of (n-q)-frames in n-space. Thus, a point of $\mathcal{B}^q$ over $x \in X$ is a pair $(x, v)$ where $v = (v_1, v_2, \ldots, v_{n-q})$ is a set of (n-q) linearly independent vectors (not necessarily orthonormal) in the vector space $\pi^{-1}(x)$, and $\pi^q(x, v) = x$. The pseudo-group J induces a pseudo-group of local bundle maps on $\mathcal{B}^q$, for each q = 0, 1, ..., n-1, which we also denote by J. Thus, with each object $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ of dimension n are associated n new pairs $(\mathcal{B}^0, J), (\mathcal{B}^1, J), \ldots,$ $(\mathcal{B}^{n-1}, J)$. We note that the pair $(\mathcal{B}^q, J)$ is not an object of $\mathcal{V}_{p.g.}$. However, the notions of J-equivalence and equivariance introduced above have immediate analogues in the present context.

We shall define _the equivariant Stiefel-Whitney class $W_e^{q+1}(\mathcal{B}, J)$_ of an object $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ as the element of $H_e^{q+1}(X; Z_2)$ determined by a certain cocycle $c_\tau \in C_e^{q+1}(X; Z_2)$. The cocycle $c_\tau$ will be an obstruction to extending an equivariant chain map $\tau = \{\tau_p : C_p(X) \to C_p(B^q)\}$ on dimensions less than q+1 to an equivariant chain map on dimensions less than q+2. The equivariant chain map $\tau$ will also be required to satisfy $\pi_\# \circ \tau_p = 1_{C_p(X)}$ so that $\tau$ is a lifting of the singular chain complex of X to that of

$B^q$ on dimensions less than $q+1$.

The condition that $\tau$ be an equivariant chain lifting will require that we make some assumption concerning the holonomy induced by $J$. The holonomy group of $(\mathcal{B}, J)$ at a point $x \in X$ can be defined as follows. Let $J_x = \{(\check{u}, \hat{u}) \in J : \check{u}(x) = x\}$. Then $(\check{u}, \hat{u}) \in J_x$ implies that $\hat{u} | \pi^{-1}(x)$ is a linear isomorphism of $\pi^{-1}(x)$ onto itself. Let $\Phi_x = \{\hat{u} | \pi^{-1}(x) : (\check{u}, \hat{u}) \in J_x\}$. Then $\Phi_x$ is a subset of the group of all automorphisms of the vector space $\pi^{-1}(x)$, and because $J$ is a pseudo-group, the subset $\Phi_x$ is a subgroup. $\Phi_x$ is called the holonomy group of $(\mathcal{B}, J)$ at $x \in X$. The holonomy group $\Phi_x^q$ of $(\mathcal{B}^q, J)$ at $x \in X$ is defined similarly, and we note that $\Phi_x^q$ is induced by $\Phi_x$.

Now, suppose the holonomy group $\Phi_{x_0}$ of $(\mathcal{B}, J)$ at $x_0 \in X$ is non-trivial, i.e., contains an element other than the identity. Then there is an element $(\check{u}, \hat{u}) \in J_{x_0}$ such that $\hat{u} | \pi^{-1}(x_0) \neq 1_{\pi^{-1}(x_0)}$. If $\sigma \in S_0(X)$ is defined by $\sigma(\Delta_0) = x_0$, then the equivariance of $\tau_0 : C_0(X) \to C_0(B^q)$ requires that $\tau_0(\sigma) = \tau_0(\check{u} \circ \sigma) = \hat{u}_\# \circ \tau_0(\sigma)$. But since $\hat{u} | \pi^{-1}(x_0)$ is not the identity, this last equation is not necessarily satisfied.

In Chapter III, we shall treat the case of trivial holonomy. In Chapter IV, we shall consider the case where the holonomy group $\Phi_x$ at each $x \in X$ is orientation preserving and leaves at least an $(n-1)$-dimensional subspace of $\pi^{-1}(x)$ fixed. We shall also comment on why our procedure cannot be carried out for more general holonomy.

# III. THE CASE OF TRIVIAL HOLONOMY

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle and J a pseudo-group of local bundle maps on $\mathcal{B}$ such that $\Phi_x$ consists of only the identity automorphism for each $x \in X$. For each $q$, $0 \leq q \leq n - 1$, we have the associated bundle and pseudo-group $(\mathcal{B}^q, J)$ with fibre $V_{n, n-q}$. In the present case of no holonomy we note that if $(1, \hat{u}) \in J$, then $\hat{u} = 1$. Hence, if $(\check{u}, \hat{u})$ and $(\check{u}, \hat{v})$ are in J, we must have $\hat{u} = \hat{v}$.

Until stated otherwise, let $q$ be a fixed integer, $0 \leq q \leq n - 1$. In the associated bundle $\mathcal{B}^q$ we shall drop the superscript $q$ on the projection and fibre, so that $\mathcal{B}^q = (B^q, X, \pi, Y)$, with $Y = V_{n, n-q}$. The fibre over $x \in X$ will be designated as $Y_x = \pi^{-1}(x)$.

We wish to define an <u>equivariant lifting</u> of the singular chain complex of $X$ to the singular chain complex of $B^q$. More precisely, we shall define homomorphisms $T_p : C_p(X) \to C_p(B^q)$ for $0 \leq p \leq q$ satisfying the following conditions:

(o) for each $\sigma : \Delta_p \to X$, $T_p(\sigma) : \Delta_p \to B^q$;

(i) $\pi_\# \circ T_p = 1_{C_p(X)}$;

(ii) $\partial_p T_p = T_{p-1} \partial_p$;

(iii) $T_p \circ \check{u}_\# = \hat{u}_\# \circ T_p$ whenever these expression are defined.

We shall refer to condition (i) as the <u>lifting condition</u> and (iii) as the <u>equivariance condition.</u> Condition (ii) simply states that the $T_p$'s

form a chain map on dimensions less than $q+1$. A collection

$\tau = \{ \tau_p : 0 \le p \le q, \ \tau_p : C_p(X) \to C_p(B^q) \}$ satisfying conditions (o),

(i), (ii) and (iii) will be called an <u>equivariant lifting in $(\mathcal{B}^q, J)$ on</u>

<u>dimensions less than $q+1$</u>.

We shall define $\tau_p$ by finite induction on the dimension $p$.

In order to facilitate this definition, we assume that the singular sim-

plexes of $X$ on each dimension have been well ordered, and let $<$

be the order relation on $S_p(X)$. We define $\tau_p$ on $S_p(X)$ by

transfinite induction and extend $\tau_p$ to all of $C_p(X)$ by linearity.

We further assume that the pseudo-group $J$, considered as a set,

has been well ordered.

For each $\sigma \epsilon S_p(X)$, the J-equivalence class $A(\sigma)$ of $\sigma$

is a non-empty subset of the well ordered set $S_p(X)$, and conse-

quently $A(\sigma)$ has a first element. We call this first element <u>the</u>

<u>base of $A(\sigma)$</u>.

If the subset $J(\sigma, \sigma') = \{ (\check{u}, \hat{u}) \epsilon J : \check{u} \circ \sigma = \sigma' \}$ of the well or-

dered set $J$ is non-empty, then it also has a first element which we

call <u>the base of $J(\sigma, \sigma')$</u>.

Each $\sigma \epsilon S_p(X)$, being a continuous mapping of $\Delta_p$ into $X$,

induces a bundle over $\Delta_p$ (5, p. 47-48) with bundle space $B_\sigma^q$

and a bundle map $h_\sigma$ making the following diagram commutative:

(I)
$$B^q_\sigma \xrightarrow{\;h_\sigma\;} B^q$$
$$\pi_\sigma \downarrow \qquad\qquad \downarrow \pi$$
$$\Delta_p \xrightarrow[\;\sigma\;]{} X \;.$$

The bundle space $B^q_\sigma$ of the induced bundle is the subspace of the product space $\Delta_p \times B^q$ given by $(x, b) \in B^q_\sigma$ iff $\sigma(s) = \pi(b)$. The maps $\pi_\sigma$ and $h_\sigma$ are the restrictions to $B^q_\sigma$ of the natural projections to the first and second factors of $\Delta_p \times B^q$, respectively. The standard fibre of the induced bundle is again $V_{n,\,n-q}$.

The induced bundle in (I) has as its base space a finitely triangulable space. The obstruction theory for such bundles is treated in (5, p. 148-217), and we shall make use of several results proved there. In particular, the fibre $V_{n,\,n-q}$ is (q-1)-connected so that any cross-section over the boundary $\dot{\Delta}_p$ of $\Delta_p$ can be extended to a cross-section over $\Delta_p$, $0 \leq p \leq q$ (5, p. 149).

The induced bundle in (I) is a product bundle and hence it admits cross-sections. We assume that the cross-sections of the induced bundle are well-ordered for each $\sigma \in S_p(X)$.

Note that if $\sigma \in A(\sigma')$, with $\sigma = \check{u} \circ \sigma'$, then the pair $(\check{u}, \hat{u}) \in J$ induces a bundle isomorphism $(1, \hat{u}) : B^q_{\sigma'} \to B^q_\sigma$ making the following diagram commutative:

$$
\begin{array}{ccc}
B^q_{\sigma'} & \xrightarrow{\;(1,\hat{u})\;} & B^q_\sigma \\
\pi_{\sigma'} \downarrow & & \downarrow \pi_\sigma \\
\Delta_p & \xrightarrow{\;1\;} & \Delta_p
\end{array}
$$

(II)

and furthermore $(1, \hat{u})^{-1} = (1, \hat{u}^{-1})$.

We are now ready to define $\tau_0$. Let $\sigma$ be the first element of $S_0(X)$, and let $t_\sigma$ be the first cross-section of the induced bundle in (I). We define $\tau_0(\sigma) = h_\sigma \circ t_\sigma$. Then $\tau_0(\sigma) : \Delta_0 \to B^q$ and because (I) is commutative, $\pi \circ \tau_0(\sigma) = \pi \circ h_\sigma \circ t_\sigma = \sigma \circ \pi_\sigma \circ t_\sigma = \sigma$.

Now, let $\sigma$ be an arbitrary element of $S_0(X)$ and suppose $\tau_0(\sigma')$ has been defined for all $\sigma' < \sigma$ in such a manner that conditions (o) and (i) are satisfied, i.e., $\tau_0(\sigma') : \Delta_0 \to B^q$ and $\pi_\# \circ \tau_0(\sigma') = \sigma'$. We distinguish two cases in defining $\tau_0(\sigma)$:

1) $\sigma \notin A(\sigma')$ for any $\sigma' < \sigma$;

2) $\sigma \in A(\sigma'')$ for some $\sigma'' < \sigma$.

In case 1) we define $\tau_0(\sigma)$ as above, i.e., let $t_\sigma$ to be the first cross-section of the induced bundle in (I) and define $\tau_0(\sigma) = h_\sigma \circ t_\sigma$. Then $\tau_0(\sigma)$ is a lifting of $\sigma$ to $B^q$.

In case 2), let $\sigma'$ be the base of $A(\sigma'')$, i.e., the first element in $A(\sigma'')$, and let $(\check{u}, \hat{u})$ be the base of $J(\sigma', \sigma)$, so that $\sigma = \check{u} \circ \sigma'$. Since $\sigma'$ is the base of $A(\sigma'')$, $\sigma' \leqq \sigma'' < \sigma$ so that $\tau_0(\sigma')$ is defined. Let $t_{\sigma'} : \Delta_0 \to B^q_{\sigma'}$ be the cross-section induced by the map $\tau_0(\sigma') : \Delta_0 \to B^q$ and the bundle map $h_{\sigma'} : B^q_{\sigma'} \to B^q$.

The bundle map $(1, \hat{u}^{-1}) : B^q_\sigma \rightarrow B^q_{\sigma'}$ of (II) and the cross-section $t_{\sigma'}$ induce a unique cross-section $t_\sigma : \Delta_0 \rightarrow B^q_\sigma$ making the following diagram commutative (5, p. 12):

$$(\text{III}_0) \qquad \begin{array}{ccc} B^q_\sigma & \xrightarrow{\ (1, \hat{u}^{-1})\ } & B^q_{\sigma'} \\ \uparrow{\scriptstyle t_\sigma} & & \uparrow{\scriptstyle t_{\sigma'}} \\ \Delta_0 & \xrightarrow{\quad 1 \quad} & \Delta_0 \end{array} \ .$$

We define $T_0(\sigma) = h_\sigma \circ t_\sigma : \Delta_0 \rightarrow B^q$ and $\pi \circ T_0(\sigma) = \sigma$ again follows from (I).

By transfinite induction we have $T_0$ defined on all of $S_0(X)$ and satisfying condition (o) $T_0(\sigma) : \Delta_0 \rightarrow B^q$, condition (i) $\pi \circ T_0(\sigma) = \sigma$, and condition (ii) (vacuously). It remains to show that condition (iii) is satisfied.

Before showing this, we wish to make two remarks.

Firstly, the map $T_0(\sigma) : \Delta_0 \rightarrow B^q$ together with the bundle map $h_\sigma : B^q_\sigma \rightarrow B^q$ uniquely determines a cross-section $t_\sigma : \Delta_0 \rightarrow B^q_\sigma$ such that $T_0(\sigma) = h_\sigma \circ t_\sigma$. If $d : \Delta_0 \rightarrow \Delta_0 \times \Delta_0$ is the diagonal map, then $t_\sigma$ is defined by $t_\sigma = (1, T_0(\sigma)) \circ d$. This will also be true of the maps $T_p(\sigma) : \Delta_p \rightarrow B^q$ to be defined later. The cross-section determined by $T_p(\sigma)$ and $h_\sigma$ will hereafter be designated by $t_\sigma$, and we shall use the representation $T_p(\sigma) = h_\sigma \circ t_\sigma$ without additional comment.

Secondly, if $\sigma, \sigma' \in A(\sigma'')$, with $\sigma''$ the base of $A(\sigma'')$,

and if $\sigma = \check{u} \circ \sigma''$ and $\sigma' = \check{v} \circ \sigma''$, then $\sigma = \check{u} \circ \check{v}^{-1} \circ \sigma'$. By our definitions and diagram (III$_0$), $t_\sigma = (1, \hat{u}) \circ t_{\sigma''}$ and $t_{\sigma'} = (1, \hat{v}) \circ t_{\sigma''}$, so that $t_\sigma = (1, \hat{u}) \circ (1, \hat{v}^{-1}) \circ t_{\sigma'} = (1, \hat{u} \circ \hat{v}^{-1}) \circ t_{\sigma'}$. Consequently, <u>whenever we have the relation $\sigma = \check{w} \circ \sigma'$, we have the companion relation $t_\sigma = (1, \hat{w}) \circ t_{\sigma'}$</u>. This will also be true for dimensions other than zero, because our definitions will be similar, and we shall have a commutative diagram (III$_p$) corresponding to (III$_0$).

Now, let $\sigma = \check{u} \circ \sigma'$; $\sigma, \sigma' \in S_0(X)$ and let $\pi^{-1}(\sigma'(\Delta_0)) = B^q \,|\, \sigma'(\Delta_0)$. We have the commutative diagram:

$$(IV_0) \qquad
\begin{array}{ccc}
B^q \,|\, \sigma'(\Delta_0) & \xrightarrow{\ \hat{u}\ } & B^q \\[4pt]
{\scriptstyle h_{\sigma'}}\Big\uparrow & & \Big\uparrow{\scriptstyle h_\sigma} \\[4pt]
B^q_{\sigma'} & \xrightarrow[(1,\hat{u})]{} & B^q_\sigma
\end{array}
$$

Consequently, $T_0(\check{u} \circ \sigma') = T_0(\sigma) = h_\sigma \circ t_\sigma = h_\sigma \circ (1, \hat{u}) \circ (1, \hat{u}^{-1}) \circ t_\sigma$ $= \hat{u} \circ h_{\sigma'} \circ t_{\sigma'} = \hat{u} \circ T_0(\sigma')$ and condition (iii), the equivariance of $T_0$, is satisfied.

We now extend $T_0$ to all of $C_0(X)$ by linearity and note that since conditions (o), (i), (ii) and (iii) are satisfied on the generators of $C_0(X)$, namely on $S_0(X)$, and $\pi_\#$, $\partial$ and $T_0$ are homomorphisms, then conditions (o), (i), (ii) and (iii) are satisfied by the map $T_0 : C_0(X) \to C_0(B^q)$.

We now proceed to our <u>inductive step</u>. Thus, we assume that

$T_m : C_m(X) \to C_m(B^q)$ has been defined for all $m$ for which

$0 \le m < p \le q$, satisfying the conditions (o) $T_m(\sigma) : \Delta_m \to B^q$ for

each $\sigma \in S_m(X)$, (i) $\pi_\# \circ T_m = 1_{C_m(X)}$, (ii) $\partial_m T_m = T_{m-1} \partial_m$, and

(iii) $\hat{u}_\# \circ T_m = T_m \circ \check{u}_\#$ whenever these expressions are defined.

Let $\sigma$ be the first element of $S_p(X)$. For each $(p-1)$-face

$\sigma^{(i)}$ of $\sigma$, $T_{p-1}(\sigma^{(i)}) : \Delta_{p-1} \to B^q$ is defined and induces a cross-

section $t_{\sigma^{(i)}}$ in the bundle $B^q_{\sigma^{(i)}} \to \Delta_{p-1}$. Let $\Delta_p^{(i)}$ be the $i\underline{\text{th}}$

$(p-1)$-face of $\Delta_p$. We then have the commutative diagram:

$$(V)$$

$$
\begin{array}{ccccc}
B^q_{\sigma^{(i)}} & \xrightarrow{(e_p^i, 1)} & B^q_\sigma | \Delta_p^{(i)} & \subset & B^q_\sigma \\
\downarrow & & \downarrow (i) & & \downarrow \\
\Delta_{p-1} & \xrightarrow{e_p^i} & \Delta_p^{(i)} & \subset & \Delta_p
\end{array}
$$

Since $e_p^i$ is a homeomorphism, we may define a cross-section

$t_\sigma^i : \Delta_p^{(i)} \to B^q_\sigma | \Delta_p^{(i)}$ by $t_\sigma^i = (e_p^i, 1) \circ t_{\sigma^{(i)}} \circ (e_p^i)^{-1}$. Thus, $t_\sigma^i$ is a

cross-section of the part of the bundle $B^q_\sigma \to \Delta_p$ over the $i\underline{\text{th}}$ $(p-1)$-
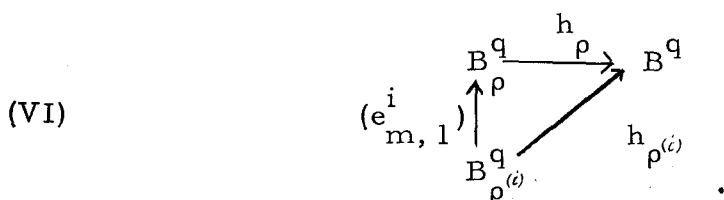
face of $\Delta_p$.

<u>If the $t_\sigma^i$'s fit together</u>, i.e., if they are equal on their com-

mon domains, then we may define a cross-section $t_\sigma$ over $\dot\Delta_p$

by $t_\sigma | \Delta_p^{(i)} = t_\sigma^i$. This condition is superfluous, of course, for

$p = 1$.

Before showing that the $t_\sigma^i$'s do in fact fit together, we verify

the following facts which are collected together as a lemma due to
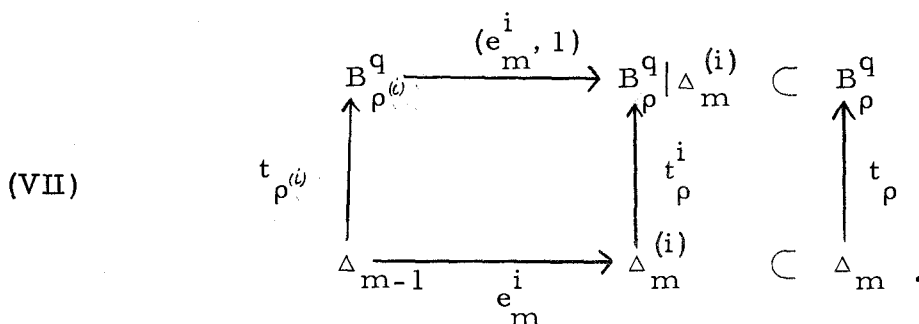
their frequent use.

    <u>Lemma III. 1:</u> <u>Let</u> $T_{m-1} : C_{m-1}(X) \to C_{m-1}(B^q)$ <u>and</u>

$T_m : C_m(X) \to C_m(B^q)$ <u>be homomorphisms satisfying</u>

$T_{m-1} : S_{m-1}(X) \to S_{m-1}(B^q)$ <u>and</u> $T_m : S_m(X) \to S_m(B^q)$. <u>Let</u>

$\rho \in S_m(X)$ <u>and let</u> $t_\rho$ <u>and</u> $h_\rho$ <u>be defined as before.</u> <u>Then</u>

    a) <u>the following diagram is commutative:</u>

(VI)

$$
\begin{array}{ccc}
& B^q_\rho \xrightarrow{\ h_\rho\ } & B^q_\rho \\
(e^i_{m,1}) \Big\uparrow & \Big\nearrow & \\
& B^q_{\rho^{(i)}} \, h_{\rho^{(i)}} &
\end{array}
$$

.

<u>Furthermore, consider the conditions (i)</u> $\pi_\# \circ T_m(\rho) = \rho$ <u>and (ii)</u>

$\partial_m T_m(\rho) = T_{m-1} \partial_m(\rho)$ <u>and the diagram:</u>

(VII)

$$
\begin{array}{ccccc}
B^q_{\rho^{(i)}} & \xrightarrow{\ (e^i_m, 1)\ } & B^q_\rho | \Delta^{(i)}_m & \subset & B^q_\rho \\
t_{\rho^{(i)}} \Big\uparrow & & t^i_\rho \Big\uparrow & & t_\rho \Big\uparrow \\
\Delta_{m-1} & \xrightarrow[\ e^i_m\ ]{} & \Delta^{(i)}_m & \subset & \Delta_m
\end{array}
$$
.

<u>Then</u> b) <u>if conditions (i) and (ii) are satisfied, then diagram (VII) is</u>

      <u>commutative;</u>

    c) <u>if diagram (VII) is commutative, then conditions (i) and (ii)</u>

are satisfied.

**Proof:** To prove a), let $(s, b) \in B^q_{\rho^{(i)}}$. Then $h_{\rho^{(i)}}((s, b)) = b$.

But $h_\rho \circ (e^i_m, 1)((s, b)) = h_\rho((e^i_m(s), b)) = b$. Thus, $h_{\rho^{(i)}} = h_\rho \circ (e^i_m, 1)$.

For b) we have from (ii) $\partial T_m(\rho) = \sum_{i=0}^{m} (-1)^i T_m(\rho) \circ e^i_m$

$= \sum_{i=0}^{m} (-1)^i T_{m-1}(\rho \circ e^i_m) = T_{m-1} \partial(\rho)$. Since $T_m$ and $T_{m-1}$ take

simplexes into simplexes, this equation implies that

$T_m(\rho) \circ e^{i_k}_m = \pm T_{m-1}(\rho \circ e^{j_k}_m)$ for some pairings $i_k$ and $j_k$. Suppose that $T_m(\rho) \circ e^{i_k}_m = \pm T_{m-1}(\rho \circ e^{j_k}_m)$ for some $i_k \neq j_k$. Then

(i) implies that $\rho \circ e^{i_k}_m = \pm \rho \circ e^{j_k}_m$ so that $T_{m-1}(\rho \circ e^{i_k}_m) = \pm T_{m-1}(\rho \circ e^{j_k}_m)$

$= T_m(\rho) \circ e^{i_k}_m$. Thus $T_m(\rho) \circ e^i_m = T_{m-1}(\rho \circ e^i_m)$ for all $i$, and

hence $h_\rho \circ t_\rho \circ e^i_m = h_{\rho^{(i)}} \circ t_{\rho^{(i)}}$. By part a) we have

$h_\rho \circ t_\rho \circ e^i_m = h_\rho \circ (e^i_m, 1) \circ t_{\rho^{(i)}}$. Let $s \in \Delta_{m-1}$. Then

$t_\rho \circ e^i_m(s) = (e^i_m(s), b)$ and $(e^i_m, 1) \circ t_{\rho^{(i)}}(s) = (e^i_m(s), \bar{b})$ for some

$b$ and $\bar{b}$ in $B^q$. Then $h_\rho \circ t_\rho \circ e^i_m(s) = b = \bar{b} = h_\rho \circ (e^i_m, 1) \circ t_{\rho^{(i)}}(s)$

so that $t_\rho \circ e^i_m = (e^i_m, 1) \circ t_{\rho^{(i)}}$, as was to be shown.

For part c) we have $\partial T_m(\rho) = \sum\limits_{i=0}^{m} (-1)^i T_m(\rho) \circ e_m^i$

$= \sum\limits_{i=0}^{m} (-1)^i h_\rho \circ t_\rho \circ e_m^i = \sum\limits_{i=0}^{m} (-1)^i h_\rho \circ (e_m^i, 1) \circ t_{\rho^{(i)}} = \sum\limits_{i=0}^{m} (-1)^i h_{\rho^{(i)}} \circ t_{\rho^{(i)}}$

$= T_{m-1}(\partial \rho).$

<div align="right">Q. E. D.</div>

Now, to show that the $t_\sigma^i$'s fit together, let $\Delta_p^{(i)}$ and $\Delta_p^{(j)}$

have a common face, say $\Delta_p^{(i)(k)} = \Delta_p^{(j)(\ell)}$. Then

$e_p^i \circ e_{p-1}^k = e_p^j \circ e_{p-1}^\ell$ so that $\sigma^{(i)(k)} = \sigma^{(j)(\ell)}$ and $t_{\sigma^{(i)(k)}} = t_{\sigma^{(j)(\ell)}}$.

By definition, $t_\sigma^i \big| \Delta_p^{(i)(k)} = (e_p^i, 1) \circ t_{\sigma^{(i)}} \circ (e_p^i)^{-1} \big| \Delta_p^{(i)(k)}$

$= (e_p^i, 1) \circ t_{\sigma^{(i)}} \big| \Delta_{p-1}^{(k)} \circ (e_p^i)^{-1} \big| \Delta_p^{(i)(k)}$. Our induction hypotheses (o),

(i) and (ii) imply by Lemma III. 1, b) that $t_{\sigma^{(i)}} \big| \Delta_{p-1}^{(k)} = t_{\sigma^{(i)}}^k$

$= (e_{p-1}^k, 1) \circ t_{\sigma^{(i)(k)}} \circ (e_{p-1}^k)^{-1}$. Also, the two preceeding equations

hold when the pair $i, k$ is replaced by the pair, $j, \ell$. Hence

$t_\sigma^i \big| \Delta_p^{(i)(k)} = (e_p^i, 1) \circ (e_{p-1}^k, 1) \circ t_{\sigma^{(i)(k)}} \circ (e_{p-1}^k)^{-1} \circ (e_p^i)^{-1} \big| \Delta_p^{(i)(k)}$

$= (e_p^i \circ e_{p-1}^k, 1) \circ t_{\sigma^{(i)(k)}} \circ (e_p^i \circ e_{p-1}^k)^{-1} \big| \Delta_p^{(i)(k)}$

$= (e_p^j \circ e_{p-1}^\ell, 1) \circ t_{\sigma^{(j)(\ell)}} \circ (e_p^j \circ e_{p-1}^\ell)^{-1} \big| \Delta_p^{(j)(\ell)} = t_\sigma^j \big| \Delta_p^{(j)(\ell)}$ and the

$t_\sigma^i$'s do in fact agree on their common domains. <u>We define</u>

$t_\sigma : \Delta_p \to B_\sigma^q$ by $t_\sigma \big| \Delta_p^{(i)} = t_\sigma^i.$

We recall that since the fibre $V_{n,\, n-q}$ of the induced bundle $B^q_\sigma \to \Delta_p$ is $(q-1)$-connected, the cross-section $t_{\dot\sigma}$ over $\mathring\Delta_p$ can be extended to a cross-section over $\Delta_p$. Let $t_\sigma$ be the first such extension, and define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$. Then condition (i) $\pi \circ T_p(\sigma) = \sigma$ follows again from the commutativity of (I).

By the definition of $t_\sigma$, the following diagrams are commutative:

$$
\begin{array}{ccccc}
B^q_{\sigma(i)} & \xrightarrow{\;(e^i_p,\, 1)\;} & B^q_\sigma | \Delta^{(i)}_p & \subset & B^q_\sigma \\[4pt]
\Big\uparrow t_{\sigma(i)} & & \Big\uparrow t^i_{\sigma(i)} & & \Big\uparrow t_\sigma \\[4pt]
\Delta_{p-1} & \xrightarrow[\;e^i_p\;]{} & \Delta^{(i)}_p & \subset & \Delta_p \ ;
\end{array}
$$

$$
\begin{array}{ccc}
B^q_\sigma & \xrightarrow{\;h_\sigma\;} & B^q \\[4pt]
(e^i_p,\, 1) \Big\uparrow & \nearrow & \\[4pt]
B^q_{\sigma(i)} & h_{\sigma(i)} &
\end{array}
\ .
$$

Hence by Lemma III. 1, (c), condition (ii) $\partial_p T_p(\sigma) = T_{p-1} \partial_p(\sigma)$ is satisfied.

Now let $\sigma$ be an arbitrary element of $S_p(X)$ and suppose $T_p(\sigma')$ has been defined for all $\sigma' < \sigma$ satisfying conditions (o), (i) and (ii). We again distinguish two cases in defining $T_p(\sigma)$:

1) $\sigma \notin A(\sigma')$ for any $\sigma' < \sigma$, i.e., $\sigma$ is not J-equivalent to any $\sigma' < \sigma$;

2) $\sigma \in A(\sigma'')$ for some $\sigma'' < \sigma$.

In case 1) we define $T_p(\sigma)$ precisely as we did above for the first element of $S_p(X)$. Hence, for each $(p-1)$-face $\sigma^{(i)}$ of $\sigma$ we get a cross-section $t_\sigma^i = (e_p^i, 1) \circ t_{\sigma(i)} \circ (e_p^i)^{-1}$ over the $(p-1)$-face $\Delta_p^{(i)}$ of $\Delta_p$ and the $t_\sigma^i$'s fit together to give a cross-section $t_{\dot\sigma}$ over the boundary $\dot\Delta_p$ of $\Delta_p$. We let $t_\sigma$ be the first cross-section over $\Delta_p$ which is an extension of $t_\sigma$ and define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$. Then conditions (i) and (ii) are satisfied exactly as before.

In case 2), let $\sigma'$ be the base of $A(\sigma'')$ and $(\check u, \hat u)$ the base of $J(\sigma', \sigma)$ so that $\sigma = \check u \circ \sigma'$. Since $\sigma'$ is the base of $A(\sigma'')$, $\sigma' \leq \sigma'' < \sigma$, and $T_p(\sigma')$ is defined. Let $t_{\sigma'} : \Delta_p \to B_{\sigma'}^q$ be the cross-section induced by $T_p(\sigma')$ so that $T_p(\sigma') = h_{\sigma'} \circ t_{\sigma'}$. We let $t_\sigma$ be the cross-section of the bundle $B_\sigma^q \to \Delta_p$ induced by the bundle map $(1, \hat u^{-1})$ and cross-section $t_{\sigma'}$, so that the following diagram is commutative:

$$(\text{III}_p) \qquad \begin{array}{ccc} B_\sigma^q & \xrightarrow{\ \ 1,\, \hat u^{-1})\ \ } & B_{\sigma'}^q \\ t_\sigma \uparrow & & \uparrow t_{\sigma'} \\ \Delta_p & \xrightarrow{\ \ 1\ \ } & \Delta_p \end{array}$$

We define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$ and as before, condition (i) follows from diagram (I).

In order to show condition (ii), we shall show that

$t_\sigma \circ e_p^i = (e_p^i, 1) \circ t_{\sigma(i)}$     and then apply Lemma III. 1, c).

We have the commutative diagram

$$(IV_p) \qquad\qquad \begin{array}{ccc} B^q \mid \sigma'(\Delta_p) & \xrightarrow{\ \hat{u}\ } & B^q \\ {\scriptstyle h_{\sigma'}}\Big\uparrow & & \Big\uparrow{\scriptstyle h_\sigma} \\ B^q_{\sigma'} & \xrightarrow[(1,\hat{u})]{} & B^q_\sigma \end{array}.$$

By our inductive hypothesis on $\sigma'$ and Lemma III. 1, b),

$t_{\sigma'} \circ e_p^i = (e_p^i, 1) \circ t_{\sigma'(i)}$.    Hence, by $(III_{p-1})$ and $(III_p)$ we have

$t_\sigma \circ e_p^i = (1, \hat{u}) \circ t_{\sigma'} \circ e_p^i = (1, \hat{u}) \circ (e_p^i, 1) \circ t_{\sigma'(i)} = (e_p^i, 1) \circ (1, \hat{u}) \circ t_{\sigma'(i)}$

$= (e_p^i, 1) \circ t_{\sigma(i)}$,    and thus    $t_\sigma \circ e_p^i = (e_p^i, 1) \circ t_{\sigma(i)}$,    as was to be

shown.

By transfinite induction, we have $T_p$ defined on all of $S_p(X)$,

satisfying conditions (o), (i) and (ii).

Now let $\sigma, \sigma' \in S_p(X)$, with $\sigma = \check{u} \circ \sigma'$. Then $t_\sigma = (1, \hat{u}) \circ t_{\sigma'}$

so that using $(IV_p)$ we have $T_p(\check{u} \circ \sigma') = T_p(\sigma) = h_\sigma \circ t_\sigma$

$= h_\sigma \circ (1, \hat{u}) \circ (1, \hat{u}^{-1}) \circ t_\sigma = \hat{u} \circ h_{\sigma'} \circ t_{\sigma'} = \hat{u} \circ T_p(\sigma')$ and condition

(iii) is satisfied.

We now extend $T_p$ to all of $C_p(X)$ by linearity and note that

since conditions (o), (i), (ii) and (iii) are satisfied on the generators

of $C_p(X)$, namely $S_p(X)$, and $\pi_\#$, $\partial_p$ and $T_p$ are homomorph-

isms, conditions (o), (i), (ii) and (iii) are satisfied by the map

$T_p : C_p(X) \to C_p(B^q)$.

The above construction of an equivariant lifting of $C_p(X)$ to $C_p(B^q)$ <u>fails on dimension $q+1$.</u> For let $\rho : \Delta_{q+1} \to X$. Then using conditions (o), (i) and (ii) for $T_q(\rho^{(i)})$ and applying Lemma III.1, b), the cross-sections $t_\rho^i$ over the $q$-faces $\Delta_{q+1}^{(i)}$ of $\Delta_{q+1}$ defined by $t_\rho^i = (e_{q+1}^i, 1) \circ t_{\rho^{(i)}} \circ (e_{q+1}^i)^{-1}$ fit together as before to give a cross-section $t_{\dot\rho}$ over the boundary $\dot\Delta_{q+1}$ of $\Delta_{q+1}$. Because $\pi_q(V_{n, n-q}) \neq 0$, the cross-section $t_{\dot\rho}$ may not be extendible to a cross-section over all of $\Delta_{q+1}$. Hence we meet with an obstruction.

<u>We now define a singular cochain $C_T \in C^{q+1}(X;Z_2)$</u> which we show is <u>equivariant,</u> is a <u>cocycle,</u> and <u>obstructs equivariant extensions of $T_q$,</u> in the sense that if there is an equivariant extension $T_{q+1}$, then $C_T = 0$.

Let $\rho : \Delta_{q+1} \to X$ be a $(q+1)$-singular simplex of $X$ and let $t_{\dot\rho}$ be the cross-section over $\dot\Delta_{q+1}$ in the bundle $B_\rho^q \to \Delta_{q+1}$, as described above. The obstruction to extending $t_{\dot\rho}$ to a cross-section over all of $\Delta_{q+1}$ is a cocycle $c(t_{\dot\rho}) \in C^{q+1}(\Delta_{q+1}, \pi_q(V_{n, n-q}))$ which is defined as follows (5, p. 148-151).

The oriented boundary $\dot\Delta_{q+1}$ of $\Delta_{q+1}$ is homeomorphic to the oriented $q$-sphere $S^q$, so that $t_{\dot\rho} : \dot\Delta_{q+1} \to B_\rho^q$ uniquely determines an element $\{t_{\dot\rho}\} \in \pi_q(B_\rho^q)$. Choosing a point $s \in \Delta_{q+1}$, say the leading vertex, and letting $Y_s$ be the fibre over $s$, the inclusion map $Y_s \subset B_\rho^q$ induces an isomorphism $\pi_q(Y_s) \approx \pi_q(B_\rho^q)$. Since

$Y_s$ is homeomorphic to $V_{n,\,n-q}$, we have an isomorphism

$\pi_q(V_{n,\,n-q}) \approx \pi_q(Y_s)$. Letting $[t_\rho]$ be the element of $\pi_q(V_{n,\,n-q})$

corresponding to $\{t_\rho\}$ under the above isomorphisms, the cocycle

$c(t_\rho)$ is defined by $c(t_\rho)(\Delta_{q+1}) = [t_\rho]$.

As noted in Chapter I, the Stiefel manifold $V_{n,\,n-q}$ had $q\underline{\text{th}}$

homotopy group isomorphic to either $Z$ or $Z_2$. Let

$\underline{r : \pi_q(V_{n,\,n-q}) \to Z_2}$ be the unique non-zero homomorphism.

For each $\rho : \Delta_{q+1} \to X$, $\underline{\text{we define }\; c_\tau(\rho) = r(c(t_\rho)(\Delta_{q+1})}$

$\underline{= r([t_\rho])}$. Extending $c_\tau$ to all of $C_{q+1}(X)$ by linearity we have a

$(q+1)$ singular cochain $c_\tau \in C^{q+1}(X; Z_2)$.

<u>Theorem III. 2:</u> <u>The cochain</u> $c_\tau \in C^{q+1}(X; Z_2)$ <u>obstructs the</u>

<u>equivariant extension of</u> $\tau_q$.

<u>Proof:</u> Let $\tau_{q+1}$ be an equivariant extension of $\tau_q$, i.e.,

$\tau_{q+1}$ is a homomorphism from $C_{q+1}(X) \to C_{q+1}(B^q)$ such that

(o) $\tau_{q+1}(\rho) : \Delta_{q+1} \to B^q$ for all $\rho \in S_{q+1}(X)$, (i) $\pi_\# \circ \tau_{q+1} = 1_{C_{q+1}(X)}$,

(ii) $\partial_{q+1}\tau_{q+1} = \tau_q \partial_{q+1}$ and (iii) $\hat{u} \circ \tau_{q+1} = \tau_{q+1} \circ \check{u}$ whenever these ex-

pressions are defined. For each $\rho : \Delta_{q+1} \to X$, $\tau_{q+1}(\rho)$ induces

a cross-section $t_\rho : \Delta_{q+1} \to B^q_\rho$. By Lemma III. 1, b), the following

diagram is commutative:

$$B^q_{\rho^{(i)}} \xrightarrow{\ (e^i_{q+1},\ 1)\ } B^q_\rho|\Delta^{(i)}_{q+1} \subset B^q_\rho$$

$$\Big\uparrow t_{\rho^{(i)}} \qquad\qquad \Big\uparrow t^i_\rho \qquad\qquad \Big\uparrow t_\rho$$

$$\Delta_q \xrightarrow[\ e^i_{q+1}\ ]{} \Delta^{(i)}_{q+1} \subset \Delta_{q+1} \ .$$

Therefore, $t_\rho$ is an extension of the cross-section $t_{\overset{\bullet}{\rho}}$ over

$\overset{\bullet}{\Delta}_{q+1}$, so that $c(t_{\overset{\bullet}{\rho}}) = 0$. Consequently, $c_T(\rho) = r(c(t_{\overset{\bullet}{\rho}})(\Delta_{q+1})) = 0$.

Q. E. D.

Because of Theorem III. 2, we call $c_T$ __the obstruction co-__

__chain__ of $T$.

__Theorem III. 3:__  __The obstruction cochain__  $c_T$  __is equivariant.__

__Proof:__ We must show that if $\rho, \rho' \in S_{q+1}(X)$ with $\rho = \check{u} \circ \rho'$,

then $c_T(\rho) = c_T(\rho')$. It therefore suffices to show that

$r(\{t_{\overset{\bullet}{\rho}}\}) = r(\{t_{\overset{\bullet}{\rho}'}\})$. Now, $\rho = \check{u} \circ \rho'$ implies that $\rho^{(i)} = \check{u} \circ \rho'^{(i)}$.

We have seen that the latter equation implies that $t_{\rho^{(i)}} = (1, \hat{u}) \circ t_{\rho'^{(i)}}$.

Therefore $t_{\overset{\bullet}{\rho}} = (1, \hat{u}) \circ t_{\overset{\bullet}{\rho}'}$. Hence, $r(\{t_{\overset{\bullet}{\rho}}\}) = r(\{(1, \hat{u}) \circ t_{\overset{\bullet}{\rho}'}\})$

$= r \circ (1, \hat{u})_*(\{t_{\overset{\bullet}{\rho}'}\}) = r(\{t_{\overset{\bullet}{\rho}'}\})$, since the homeomorphism $(1, \hat{u})$ in-

duces an isomorphism of the homotopy groups.

Q. E. D.

__Alternate Proof:__ Suppose $\rho = \check{u} \circ \rho' : \Delta_{q+1} \to X$. Then as

above $t_{\overset{\bullet}{\rho}} = (1, \hat{u}) \circ t_{\overset{\bullet}{\rho}'}$ so that $t_{\overset{\bullet}{\rho}}$ is the cross-section induced by

the bundle map $(1, \hat{u}^{-1})$ and cross-section $t_{\rho'}$. Then by the naturality of the obstruction cocycle $c(t_\rho)$ (5, p. 168),

$$c(t_\rho) = 1_{\#} c(t_{\rho'}) = c(t_{\rho'}) \quad \text{and hence} \quad c_T(\rho) = c_T(\rho').$$

Q. E. D.

Now, let $\rho : \Delta_m \to X$ be a singular m-simplex of $X$, and let $K(\rho) = \{\sigma : \sigma \text{ is a face of } \rho\}$. Let $K(\Delta_m)$ be the finite simplicial complex consisting of $\Delta_m$ and all of its proper faces. There is a natural identification of $K(\rho)$ with $K(\Delta_m)$ which we now describe. If $\Delta_p^i$ is a p-face of $\Delta_m$, there are face maps

$$e_k^{i_k} : \Delta_{k-1} \to \Delta_k \quad \text{for} \quad k=p+1, p+2, \ldots, m \quad \text{such that}$$

$$\Delta_p^i = e_m^{i_m} \circ e_{m-1}^{i_{m-1}} \circ \ldots \circ e_{p+1}^{i_{p+1}}(\Delta_p).$$ Let $\sigma^i$ be the p-face of $\rho$

defined by $\sigma^i = \rho \circ e_m^{i_m} \circ \ldots \circ e_{p+1}^{i_{p+1}}$. Then the association $\Delta_p^i \to \sigma^i$

is a bijection between $K(\Delta_m)$ and $K(\rho)$ which respects face operations in the sense that if $(\Delta_p^i)^{(k)}$ is the $k^{\text{th}}$ (p-1)-face of $\Delta_p^i$

and $(\Delta_p^i)^{(k)}$ is associated with $\sigma^k$, then $\sigma^k$ is the $k^{\text{th}}$ (p-1)-

face of $\sigma^i$.

The above allows us to identify $K(\rho)$ with the finite simplicial complex $K(\Delta_m)$. This identification is assumed for $m=q+2$ in the proof of the following theorem, and we shall use $K(\rho)$ and $K(\Delta_{q+2})$ interchangeably.

Theorem III. 4: The equivariant obstruction cochain is a cocycle.

<u>Proof</u>: Let $\rho : \Delta_{q+2} \to X$ be given. Define a $(q+1)$-simplicial cochain $\oint$ of $K(\rho)$ with values in $Z_2$ by $\oint(\rho^{(i)}) = c_T(\rho^{(i)})$. Then $\delta c_T(\rho) = c_T(\partial \rho) = \oint(\partial \rho) = \delta \oint(\rho)$, so that $\delta c_T(\rho) = 0$ if and only if $\delta \oint(\rho) = 0$.

We prove that $\delta \oint(\rho) = 0$ by showing that $\oint$ is an obstruction cochain for a cross-section $f$ over the q-skeleton of $K(\Delta_{q+2})$ in the bundle $B_\rho^q \to \Delta_{q+2}$ and is therefore a cocycle.

Let $\Delta_{q+2}^{(m)(k)} = e_{q+2}^m \circ e_{q+1}^k (\Delta_q)$ be a q-face of $\Delta_{q+1}$. We then have the following commutative diagram:

$$
\begin{array}{ccccc}
B_{\rho^{(m)(k)}}^q & \xrightarrow{(e_{q+1}^k,\, 1)} & B_{\rho^{(m)}}^q & \xrightarrow{(e_{q+2}^m,\, 1)} & B_\rho^q \\
\Big\uparrow\Big\downarrow {\scriptstyle t_{\rho^{(m)(k)}}} & & \Big\downarrow & & \Big\downarrow \\
\Delta_q & \xrightarrow[e_{q+1}^k]{} & \Delta_{q+1} & \xrightarrow[e_{q+1}^m]{} & \Delta_{q+2}
\end{array}
$$

where $t_{\rho^{(m)(k)}}$ is induced by $T_q(\rho^{(m)(k)})$. Therefore we have for each q-face $\Delta_{q+2}^{(m)(k)}$ of $\Delta_{q+2}$ a cross-section $t^{mk}$ of the part of the bundle $B_\rho^q \to \Delta_{q+2}$ over $\Delta_{q+2}^{(m)(k)}$ defined by

$$ t^{mk} = (e_{q+2}^m \circ e_{q+1}^k,\, 1) \circ t_{\rho^{(m)(k)}} \circ (e_{q+2}^m \circ e_{q+1}^k)^{-1}. $$

Now, suppose two q-faces $\Delta_q^i$ and $\Delta_q^j$ of $\Delta_{q+2}$ have a common $(q-1)$-face. Then there is a $(q+1)$-face $\Delta_{q+2}^{(m)}$ of $\Delta_{q+2}$

such that $\Delta_q^i$ and $\Delta_q^j$ are faces of $\Delta_{q+2}^{(m)}$. Thus let

$\Delta_q^i = \Delta_{q+2}^{(m)(i)}$ and $\Delta_q^j = \Delta_{q+2}^{(m)(j)}$ have a common $(q-1)$-face, $\Delta_{q-1}^{ij}$.

We define a cross-section $t_{\rho^{(m)}}$ of the part of the bundle

$B_{\rho^{(m)}}^q \to \mathring{\Delta}_{q+1}$ over $\mathring{\Delta}_{q+1}$ by $t_{\rho^{(m)}} \mid \Delta_{q+1}^{(k)} = (e_{q+1}^k, 1) \circ t_{\rho^{(m)(k)}} \circ (e_{q+1}^k)^{-1}$.

Then in the bundle $B_\rho^q \to \Delta_{q+2}$, we define a cross-section of the

part of the bundle over the boundary $\mathring{\Delta}_{q+1}^{(m)}$ of $\Delta_{q+1}^{(m)}$ by

$t^{\mathring{m}} = (e_{q+2}^m, 1) \circ t_{\rho^{(m)}} \circ (e_{q+2}^m)^{-1} \mid \Delta_{q+2}^{(\mathring{m})}$. We see that

$t^{\mathring{m}} \mid \Delta_{q+2}^{(m)(k)} = (e_{q+2}^m, 1) \circ (e_{q+1}^k, 1) \circ t_{\rho^{(m)(k)}} \circ (e_{q+1}^k)^{-1} \circ (e_{q+2}^m)^{-1} \mid \Delta_{q+2}^{(m)(k)}$

$= t^{mk}$. Consequently, $t^{mi} \mid \Delta_{q-1}^{ij} = t^{\mathring{m}} \mid \Delta_{q-1}^{ij} = t^{mj} \mid \Delta_{q-1}^{ij}$. Thus,

the cross-sections $t^{mk}$ of the parts of the bundle $B_\rho^q \to \Delta_{q+2}$

over $\Delta_{q+2}^{(m)(k)}$ fit together to give a cross-section $f$ of the part of

the bundle over the q-skeleton of $K(\Delta_{q+2})$. Furthermore, for each

$\Delta_{q+2}^{(m)}$, $f \mid \Delta_{q+2}^{(\mathring{m})} = (e_{q+2}^m, 1) \circ t_{\rho^{(\mathring{m})}} \circ (e_{q+2}^m)^{-1} \mid \Delta_{q+2}^{(\mathring{m})}$.

Now, the obstruction to extending $f$ to a cross-section over

the $(q+1)$-skeleton of $K(\Delta_{q+2})$ is a $(q+1)$-cocycle $c(f)$. Let $\Delta_{q+2}^{(m)}$

be a $(q+1)$-simplex of $K(\Delta_{q+2})$. Then $f: \Delta_{q+2}^{(\mathring{m})} \to B_\rho^q$ determines

an element of $\pi_q(B_\rho^q)$, and this element is by definition $c(f)(\Delta_{q+2}^{(m)})$.

But $f \mid \Delta_{q+2}^{(\mathring{m})} = (e_{q+2}^m, 1) \circ t_{\rho^{(\mathring{m})}} \circ (e_{q+2}^m)^{-1} \mid \Delta_{q+2}^{(\mathring{m})}$ so that

$c(f)(\Delta_{q+2}^{(m)}) = \{(e_{q+2}^m, 1) \circ t_{\rho^{(\mathring{m})}}\} \in \pi_q(B_\rho^q)$. Hence the obstruction cocycle

of $f$ with values in $Z_2$ is $r(c(f))$ and $r(c(f))(\Delta_{q+2}^{(m)})$

$= r(\{(e_{q+2}^m, 1) \circ t_{\rho^{(\mathring{m})}}\}) = r(\{t_{\rho^{(\mathring{m})}}\})$ since the homeomorphism $(e_{q+2}^m, 1)$

induces an isomorphism of the homotopy groups.

Consequently, $\not c(\rho^{(m)}) = c_T(\rho^{(m)}) = r([t_{\rho^{(\mathring{m})}}]) = r(\{t_{\rho^{(\mathring{m})}}\}) =$

$= r(c(f)(\Delta_{q+2}^{(m)}))$,　i.e.,　$\oint = r(c(f))$　and　$\oint$　is an obstruction co-cycle.　<span style="float:right">Q. E. D.</span>

We now wish to show that <u>the equivariant obstruction cocycle is natural with respect to equivariant bundle maps.</u> We recall that a vector bundle map $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ is equivariant iff for each $u = (\check{u}, \hat{u}) \in J$ there is a $u' = (\check{u}, \hat{u}') \in J'$ such that $\check{f} \circ \check{u} = \check{u}' \circ \check{f}$ and $\hat{f} \circ \hat{u} = \hat{u}' \circ \hat{f}$ whenever these expressions are defined. Also, the associated bundle maps $(\mathcal{B}^q, J) \to (\mathcal{B}'^q, J')$ are equivariant in this same sense.

Before proving naturality, we establish the following two lemmas.

<u>Lemma III. 5</u>: <u>If</u> $f = (\check{f}, \hat{f}) : \mathcal{B}^q \to \mathcal{B}'^q$ <u>is a bundle map and</u> $\mathsf{T}'_p : C_p(X') \to C_p(B'^q)$ <u>is defined for</u> $0 \leq p \leq q$ <u>satisfying</u> (o') $\mathsf{T}'_p(\sigma) : \Delta_p \to B'^q$ <u>for</u> $\sigma \in S_p(X')$, (i') $\pi'_\# \circ \mathsf{T}'_p = 1_{C_p(X')}$, <u>and</u> (ii') $\partial_p \mathsf{T}'_p = \mathsf{T}'_{p-1} \partial_p$, <u>then there is a</u> $\mathsf{T}_p : C_p(X) \to C_p(B^q)$ <u>defined for</u> $0 \leq p \leq q$ <u>and satisfying</u> (o) $\mathsf{T}_p(\sigma) : \Delta_p \to B^q$ <u>for</u> $\sigma \in S_p(X)$, (i) $\pi_\# \circ \mathsf{T}_p = 1_{C_p(X)}$, (ii) $\partial_p \mathsf{T}_p = \mathsf{T}_{p-1} \partial_p$, <u>and</u> (iii) $\hat{f}_\# \circ \mathsf{T}_p = \mathsf{T}'_p \circ \check{f}_\#$.

<u>Proof:</u> Our proof is by induction on the dimension $p$.

First, for dimension $0$, let $\sigma : \Delta_0 \to X$ be an arbitrary singular zero-simplex of $X$ and consider the commutative diagram:

$$\text{(VIII}_0) \quad \begin{array}{ccccccc} B^q & \xleftarrow{\;h_\sigma\;} & B^q_\sigma & \xrightarrow{(1,\hat{f})} & B'^q_{\check{f}\circ\sigma} & \xrightarrow{\;h^\downarrow_{\check{f}\circ\sigma}\;} & B'^q \\ \downarrow & & \downarrow & & \Uparrow\,\Big\downarrow\,t^\downarrow_{\check{f}\circ\sigma} & \nearrow^{T'_0(\check{f}\circ\sigma)} & \downarrow \\ X & \xleftarrow{\;\sigma\;} & \Delta_0 & \xrightarrow{\;\;1\;\;} & \Delta_0 & \xrightarrow{\;\check{f}\circ\sigma\;} & X' \end{array}$$

where the cross-section $t'^\downarrow_{\check{f}\circ\sigma}$ is induced by $T'_0(\check{f}\circ\sigma)$ so that

$T'_0(\check{f}\circ\sigma) = h^\downarrow_{\check{f}\circ\sigma} \circ t'^\downarrow_{\check{f}\circ\sigma}$. Let $t_\sigma : \Delta_0 \to B^q_\sigma$ be the cross-section in-

duced by $t^\downarrow_{\check{f}\circ\sigma}$ and the bundle map $(1,\hat{f})$, so that for each $s \in \Delta_0$,

with $(1,\hat{f})_s = (1,\hat{f})\big|\pi_\sigma^{-1}(s)$, we have $t_\sigma(s) = [(1,\hat{f})_s]^{-1} \circ t^\downarrow_{\check{f}\circ\sigma}$.

We define $T_0(\sigma) = h_\sigma \circ t_\sigma : \Delta_0 \to B^q$ and condition (i) follows

immediately from diagram (VIII$_0$) and the fact that $t_\sigma$ is a cross-

section. Also, condition (ii) is vacuously satisfied.

To check condition (iii), let $s \in \Delta_0$. Then $\hat{f} \circ T_0(\sigma)(s)$

$= \hat{f} \circ h_\sigma \circ t_\sigma(s) = \hat{f} \circ h_\sigma \circ [(1,\hat{f})_s]^{-1} \circ t^\downarrow_{\check{f}\circ\sigma}(s)$. But

$\hat{f} \circ h_\sigma(s,b) = \hat{f}(b) = h^\downarrow_{\check{f}\circ\sigma}(s,\hat{f}(b)) = h'^\downarrow_{\check{f}\circ\sigma} \circ (1,\hat{f})(s,b)$ for all $(s,b) \in B^q_\sigma$.

Hence, $\hat{f} \circ T_0(\sigma)(s) = h'^\downarrow_{\check{f}\circ\sigma} \circ (1,\hat{f}) \circ [(1,\hat{f})_s]^{-1} \circ t^\downarrow_{\check{f}\circ\sigma}(s) = h^\downarrow_{\check{f}\circ\sigma} \circ t^\downarrow_{\check{f}\circ\sigma}(s)$

$= T'_0(\check{f}\circ\sigma)(s)$, i.e., $\hat{f} \circ T_0(\sigma) = T'_0 \circ \check{f}(\sigma)$.

We define $T_0$ on each singular zero-simplex on $X$ as above

and extend $T_0$ to all of $C_0(X)$ by linearity. Then conditions (o),

(i), (ii) and (iii) are satisfied since they are satisfied on the genera-

tors of $C_0(X)$, namely $S_0(X)$, and $\pi_\#$, $T_0$, $\partial$, $\check{f}_\#$ and $\hat{f}_\#$ are

homomorphisms.

Suppose now that $T_m : C_m(X) \to C_m(B^q)$ has been defined for

$0 \leq m < p \leq q$   satisfying (o), (i), (ii) and (iii), and let   $\sigma$   be an arbitrary singular p-simplex of   X.   We have the following commutative diagram:

$$(VIII_p)$$



where again the cross-section   $t^{\downarrow}_{f \circ \sigma}$   is induced by   $T'_p(\check{f} \circ \sigma)$   so that $T'_p(\check{f} \circ \sigma) = h^{\downarrow}_{f \circ \sigma} \circ t^{\downarrow}_{f \circ \sigma}$.   Let   $t_\sigma : \Delta_p \to B^q_\sigma$   be the cross-section induced by   $t^{\downarrow}_{f \circ \sigma}$   and   $(1, \hat{f})$,   so that for each   $s \in \Delta_p$, $t_\sigma(s) = [(1, \hat{f})_s]^{-1} \circ t^{\downarrow}_{f \circ \sigma}(s)$.

We define   $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$   and condition (i) is again immediate.   Condition (iii) follows exactly as in the case   $p = 0$, since the computations used make no reference to the dimension.

In order to show that condition (ii) is satisfied, we first recall that conditions (o'), (i') and (ii'') on   $T'_p$   imply that $t^{\downarrow}_{f \circ \sigma} \circ e^i_p = (e^i_p, 1) \circ t^{\downarrow}_{f \circ \sigma^{(i)}}$   by Lemma III. 1, b).   We also have the relation   $h^{\downarrow}_{f \circ \sigma^{(i)}} = h^{\downarrow}_{f \circ \sigma} \circ (e^i_p, 1)$   by Lemma III. 1, a) and, by the definitions,   $\hat{f} \circ h_\sigma = h^{\downarrow}_{f \circ \sigma} \circ (1, \hat{f})$.

Now let   $s \in \Delta_{p-1}$.   Then   $h_\sigma \circ t_\sigma \circ e^i_p(s) =$ $h_\sigma \circ [(1, \hat{f})_{e^i_p(s)}]^{-1} \circ t^{\downarrow}_{f \circ \sigma} \circ e^i_p(s) = h_\sigma \circ [(1, \hat{f})_{e^i_p(s)}]^{-1} \circ (e^i_p, 1) \circ t^{\downarrow}_{f \circ \sigma^{(i)}}(s)$ $= [(\hat{f}_{\sigma \circ e^i_p(s)})]^{-1} \circ h^{\downarrow}_{f \circ \sigma} \circ (e^i_p, 1) \circ t^{\downarrow}_{f \circ \sigma^{(i)}}(s) = [\hat{f}_{\sigma^{(i)}(s)}]^{-1} \circ h^{\downarrow}_{f \circ \sigma^{(i)}} \circ t^{\downarrow}_{f \circ \sigma^{(i)}}(s) =$

$= h_{\sigma(i)} \circ [(1, \hat{f})_{e_p^i(s)}]^{-1} \circ t_{\check{f} \circ \sigma(i)}(s) = h_{\sigma(i)} \circ t_{\sigma(i)}(s)$,    i. e. , we have

$h_\sigma \circ t_\sigma \circ e_p^i = h_{\sigma(i)} \circ t_{\sigma(i)}$    for each   $i = 0, 1, \ldots, p$.    Consequently,

$$\partial_p T_p(\sigma) = \sum_{i=0}^{p} (-1)^i T_p(\sigma) \circ e_p^i = \sum_{i=0}^{p} (-1)^i h_\sigma \circ t_\sigma \circ e_p^i = \sum_{i=0}^{p} (-1)^i h_{\sigma(i)} \circ t_{\sigma(i)}$$

$$= \sum_{i=0}^{p} (-1)^i T_{p-1}(\sigma^{(i)}) = T_{p-1} \partial_p(\sigma), \quad \text{and condition (ii) is satisfied.}$$

We define $T_p$ on each singular p-simplex of $X$ as above

and extend $T_p$ to all of $C_p(X)$ by linearity. Again, conditions

(o), (i), (ii) and (iii) are satisfied on the generators of $C_p(X)$ and

each of $\pi_\#$, $T_p$, $\partial$, $\check{f}_\#$ and $\hat{f}_\#$ is a homomorphism so that the condi-

tions are satisfied for $T_p : C_p(X) \to C_p(B^q)$.

<div align="right">Q. E. D.</div>

The set $T = \{T_p : C_p(X) \to C_p(B^q) : 0 \le p \le q\}$ is called <u>the in-</u>

<u>duced lifting.</u>

<u>Lemma III. 6:</u>   <u>If in Lemma III. 5,</u>   $\mathcal{B}^q$ <u>and</u> $\mathcal{B}'^q$ <u>have</u>

<u>pseudo-groups</u> J <u>and</u> J',   <u>respectively, and if</u> T' <u>and</u> $f = (\check{f}, \hat{f})$

<u>are both equivariant, then the induced lifting</u> T <u>is also equivariant.</u>

<u>Proof:</u> Let   $\sigma : \Delta_p \to X$, $0 \le p \le q$,   and suppose $\check{u} \circ \sigma$ is

defined for some   $(\check{u}, \hat{u}) \in J$.   We must show that $\hat{u} \circ T_p(\sigma) = T_p(\check{u} \circ \sigma)$,

i. e. ,   $\hat{u} \circ h_\sigma \circ t_\sigma = h_{\check{u} \circ \sigma} \circ t_{\check{u} \circ \sigma}$.   Now, by definition, for each

$s \in \Delta_p$,   $t_\sigma(s) = [(1, \hat{f})_s]^{-1} \circ t_{\check{f} \circ \sigma}$.   Let   $u' = (\check{u}', \hat{u}')$   be an element in

J'   associated with   $u = (\check{u}, \hat{u})$   in   J   by the equivariance of   f.

Then for each   $s \in \Delta_p$,   $\hat{u} \circ h_\sigma \circ t_\sigma = \hat{u} \circ h_\sigma \circ [(1, \hat{f})_s]^{-1} \circ t_{\check{f} \circ \sigma}(s) =$

$$= \hat{u} \circ [\hat{f}_{\sigma(s)}]^{-1} \circ h'_{\check{f} \circ \sigma} \circ t'_{\check{f} \circ \sigma}(s) = [\hat{f}_{\check{u} \circ \sigma(s)}]^{-1} \circ \hat{u}' \circ h'_{\check{f} \circ \sigma} \circ t'_{\check{f} \circ \sigma}(s)$$

$$= [\hat{f}_{\check{u} \circ \sigma(s)}]^{-1} \circ h'_{\check{u}' \circ \check{f} \circ \sigma} \circ t'_{\check{u}' \circ \check{f} \circ \sigma}(s) = [\hat{f}_{\check{u} \circ \sigma(s)}]^{-1} \circ h'_{\check{f} \circ \check{u} \circ \sigma} \circ t'_{\check{f} \circ \check{u} \circ \sigma}(s)$$

$$= h_{\check{u} \circ \sigma} \circ [(1, \hat{f})_s]^{-1} \circ t'_{\check{f} \circ \check{u} \circ \sigma}(s) = h_{\check{u} \circ \sigma} \circ t_{\check{u} \circ \sigma}(s). \quad \text{Thus,}$$

$$\hat{u} \circ h_{\sigma} \circ t_{\sigma} = h_{\check{u} \circ \sigma} \circ t_{\check{u} \circ \sigma}. \qquad\qquad \text{Q. E. D.}$$

$\underline{\text{Thoerem III. 7}}$: $\underline{\text{If}}$ $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ $\underline{\text{is an equivariant}}$ $\underline{\text{vector bundle map, and}}$ $T'$ $\underline{\text{is an equivariant lifting in}}$ $(\mathcal{B}'^q, J')$ $\underline{\text{on}}$ $\underline{\text{dimensions less than}}$ $q + 1$, $0 \le q \le n - 1$, $\underline{\text{and if}}$ $T$ $\underline{\text{is the induced}}$ $\underline{\text{equivariant lifting in}}$ $(\mathcal{B}^q, J)$, $\underline{\text{then}}$ $c_T = \check{f}^{\#}(c_{T'})$.

$\underline{\text{Proof}}$: Let $\rho : \Delta_{q+1} \to X$. Then $\check{f}^{\#}(c_{T'})(\rho) = c_{T'}(\check{f} \circ \rho)$

$= r(c(t'_{\check{f} \circ \rho})(\Delta_{q+1}))$ and $c_T(\rho) = r(c(t_{\rho})(\Delta_{q+1}))$. Therefore it suf-

fices to show that $c(t'_{\check{f} \circ \rho}) = c(t_{\rho})$. By the definition of the induced

lifting $T$, the following diagram is commutative:



i.e., the cross-section $t_{\rho}$ is induced by $t'_{\check{f} \circ \rho}$ and the bundle

map $(1, \hat{f})$. Consequently, by the naturality of the obstruction co-

cycle (5, p. 168-169), $c(t_{\rho}) = 1^{\#}c(t'_{\check{f} \circ \rho}) = c(t'_{\check{f} \circ \rho})$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Q. E. D.}$

We have shown above that with each equivariant lifting $T$ in

$(\mathcal{B}^q, J)$ on dimensions less than or equal to $q$, there is associated

an equivariant obstruction cocycle $c_T \in C^{q+1}(X; Z_2)$. We now wish to show that the equivariant cohomology class of $c_T$ is independent of the particular choice of equivariant lifting, i.e., if $T$ and $\overline{T}$ are equivariant liftings in $(\mathcal{B}^q, J)$ on dimensions less than $q+1$, then $c_T$ and $c_{\overline{T}}$ are equivariantly cohomologous. We do this by defining an equivariant q-cochain $d(T, \overline{T}) \in C_e^q(X; Z_2)$ which satisfies $\delta d(T, \overline{T}) = c_T - c_{\overline{T}}$.

Before we define $d(T, \overline{T})$, we note the following. Let $\sigma : \Delta_m \to X$, $m = q$ or $q+1$. Then in the induced bundle $B_\sigma^q \to \Delta_m$ any two cross-sections $t_\sigma$ and $\overline{t}_\sigma$ over the $(q-1)$-skeleton of $\Delta_m$ are homotopic (5, p. 181), and any such homotopy $k_\sigma$ gives rise to a q-cochain $d_\sigma = d(t_\sigma, k_\sigma, \overline{t}_\sigma) \in C^q(\Delta_m, \pi_q(V_{n, n-q}))$, called the deformation cochain, which satisfies the following condition. If $c(t_\sigma)$ and $c(\overline{t}_\sigma)$ are the obstruction cocycles of $t_\sigma$ and $\overline{t}_\sigma$, then $\delta d_\sigma = c(t_\sigma) - c(\overline{t}_\sigma)$ (5, p. 171-172). We suppose that for any given $\sigma : \Delta_m \to X$, $m = q$ or $q+1$, and any two cross-sections $t_\sigma$ and $\overline{t}_\sigma$ over the $(q-1)$-skeleton of $\Delta_m$ in the induced bundle $B_\sigma^q \to \Delta_m$, the set of homotopies $k_\sigma : t_\sigma \simeq \overline{t}_\sigma$ is well-ordered.

Let $\mathcal{O}$ be the set of all base elements of $S_q(X)$, $\mathcal{O} = \{\sigma \in S_q(X) : \sigma \text{ is the base of some } A(\sigma')\}$. We first define $d(T, \overline{T})$ on $\mathcal{O}$ by transfinite induction.

Thus, let $\sigma$ be the first element of $\mathcal{O}$. Then $T_q(\sigma)$ and $\overline{T}_q(\sigma)$ induce cross-sections $t_\sigma$ and $\overline{t}_\sigma$ of the bundle $B_\sigma^q \to \Delta_q$.

Let $k_\sigma$ be the first homotopy of the restrictions of $t_\sigma$ and $\bar{t}_\sigma$ to the $(q-1)$-skeleton $\dot{\Delta}_q$ of $\Delta_q$. Then $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \bar{t}_\sigma | \dot{\Delta}_q$ gives rise to $d_\sigma \in C^q(\Delta_q : \pi_q(V_{n, n-q}))$ and we define $\underline{d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q))}$, where $r$ is as before the unique epimorphism $r : \pi_q(V_{n, n-q}) \to Z_2$.

Next, let $\sigma'$ be the second element of $\mathcal{O}$ . We distinguish two cases in defining $d(T, \overline{T})(\sigma')$:

1) $\sigma'$ has no face of any dimension in common with $\sigma$;

2) $\sigma'$ has a face in common with $\sigma$.

$\underline{\text{Note that for} \quad q=0,}$ case 1) is the only possibility.

In case 1), we define $d(T, \overline{T})(\sigma')$ in the same way that we defined $d(T, \overline{T})(\sigma)$, i. e., we let $k_\sigma$ be the first homotopy of the restrictions of the induced cross-sections $t_{\sigma'}$ and $\bar{t}_{\sigma'}$ to the $(q-1)$-skeleton $\dot{\Delta}_q$ of $\Delta_q$, $d_{\sigma'} = d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'})$ the corresponding deformation cochain, and define $d(T, \overline{T})(\sigma') = r(d_{\sigma'}(\Delta_q))$.

Before discussing case 2), we prove the following lemma.

$\underline{\text{Lemma III. 8:}} \quad \underline{\text{For} \quad q \geq 1,} \quad \underline{\text{let}} \quad \dot{\Delta}_q^* \quad \underline{\text{be a proper subset of the}}$ $\underline{\text{faces of}} \quad \dot{\Delta}_q. \quad \underline{\text{Let}} \quad t_\sigma \quad \underline{\text{and}} \quad \bar{t}_\sigma \quad \underline{\text{be cross-sections over}} \quad \dot{\Delta}_q \quad \underline{\text{in the}}$ $\underline{\text{bundle}} \quad B_\sigma^q \to \Delta_q. \quad \underline{\text{Then any homotopy}} \quad k_\sigma^* : t_\sigma | \dot{\Delta}_q^* \simeq \bar{t}_\sigma | \dot{\Delta}_q^* \quad \underline{\text{can be}}$ $\underline{\text{extended to a homotopy}} \quad k_\sigma : t_\sigma \simeq \bar{t}_\sigma.$

$\underline{\text{Proof:}}$ We can extend $k_\sigma^*$ trivially to a map $k_\sigma^* : \dot{\Delta}_q^* \times I \cup \dot{\Delta}_q \times \{0\} \cup \dot{\Delta}_q \times \{1\}$ such that $k_\sigma^*(s, 0) = t_\sigma(s)$ and $k_\sigma^*(s, 1) = \bar{t}_\sigma(s)$ for all $s \in \dot{\Delta}_q$. Let $K = \dot{\Delta}_q \times I$ and

$L = \dot{\Delta}_q^* \times I \cup \dot{\Delta}_q \times \{0\} \cup \dot{\Delta}_q \times \{1\}$. Then $L$ is a closed subset of $K$, and the pair $(K, L)$ is a finitely triangulable pair. Therefore we wish to show that a cross-section over $L$ into $B_\sigma^q \times I$ can be extended to a cross-section over $K$. By (5, p. 149), since $B_\sigma^q \times I$ is $(q-1)$-connected, the cross-section $k_\sigma^*$ can be extended to a cross-section over $L \cup K^q$, where $K^q$ is the q-skeleton of $K$. But the dimension of $K$ is $q$, so that $K^q = K = L \cup K^q$ and $k_\sigma^*$ can be extended to all of $K$.

Q. E. D.

Now, in case 2), we identify $K(\sigma)$ and $K(\sigma')$ with the finite simplicial complex $K(\Delta_q)$ as in the proof of Theorem III. 4. Let $L$ be the subcomplex of $K(\Delta_q)$ consisting of all those faces of $\sigma$ which are in common with faces of $\sigma'$. Let $L'$ be the corresponding subcomplex for $\sigma'$ and let $|L|$ and $|L'|$ be the corresponding subsets of $\Delta_q$. Then $|L|$ and $|L'|$ are homeomorphic and we have the following commutative diagram:

(IX)

$$
\begin{array}{ccccccc}
B_\sigma^q & \supset & B_\sigma^q\big|\,|L| & \xleftarrow{\hat{h}} & B_{\sigma'}^q\big|\,|L'| & \subset & B_{\sigma'}^q \\
\Big\Uparrow t_\sigma \;\; \bar{t}_\sigma & & \Big\Uparrow & & \Big\Uparrow & & t_{\sigma'} \;\; \bar{t}_{\sigma'} \Big\Uparrow \\
\Delta_q & \supset & |L| & \xleftarrow[\check{h}]{} & |L'| & \subset & \Delta_q
\end{array}
$$

The map $\check{h}$ is a homeomorphism, and $\hat{h}$ is the corresponding bundle space homeomorphism. The inner vertical maps are those induced by the outer vertical maps in each outer rectangle.

The homotopy $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \bar{t}_\sigma | \dot{\Delta}_q$ together with diagram (IX) induces a homotopy $k_{\sigma'}^* : t_{\sigma'} \big| |L'| \simeq \bar{t}_{\sigma'} \big| |L'|$. By Lemma III. 8, this homotopy can be extended to a homotopy over all of $\dot{\Delta}_q$. Let $k_{\sigma'}$ be the first such extension. Then as before, we have the deformation cochain $d_{\sigma'} \in C^q(\Delta_q; \pi_q(V_{n, n-q}))$ and we define

$$d(T, \bar{T})(\sigma') = r(d_{\sigma'}(\Delta_q)).$$

Now, let $\sigma$ be an arbitrary element of $\mathcal{O}$, and suppose that $d(T, \bar{T})(\sigma')$ has been defined as above for all $\sigma' < \sigma$ in such a manner that the following is true. If $\sigma'' < \sigma'$ and $\sigma''$ and $\sigma'$ have a common face, then the homotopy $k_{\sigma'}$ in $d_{\sigma'} = d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'})$ used to define $d(T, \bar{T})(\sigma')$ is the first extension of the partial homotopy induced by $k_{\sigma''}$ in $d_{\sigma''}$ used to define $d(T, \bar{T})(\sigma'')$. We again distinguish two cases in defining $d(T, \bar{T})(\sigma)$ :

1) $\sigma$ has no face of any dimension in common with any $\sigma' < \sigma$;

2) $\sigma$ has a face in common with some $\sigma' < \sigma$.

Note again that for $q=0$, case 1) is the only possibility.

In case 1) we define $d(T, \bar{T})(\sigma)$ exactly as in case 1) of the definition for the second element of $\mathcal{O}$. Thus, we let $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \bar{t}_\sigma | \dot{\Delta}_q$ be the first homotopy and define

$$d(T, \bar{T})(\sigma) = r(d_\sigma(\Delta_q)).$$

In case 2), suppose $\sigma$ has a certain p-face, $0 \le p \le q - 1$, in common with some preceding element of $\mathcal{O}$ and let $\sigma'$ be the first such element of $\mathcal{O}$. Let $\dot{\Delta}_q^{p*}$ be the domain of this p-face

of $\sigma$, and let $k_\sigma^{p*}$ be the homotopy of $t_\sigma$ and $\overline{t}_\sigma$ over $\overset{\bullet}{\Delta}{}_q^{p*}$ induced by the homotopy $k_{\sigma'}$ used in defining $d(\top,\overline{\top})(\sigma')$. By our inductive hypothesis, $k_\sigma^{p*}$ then coincides with the homotopy induced by $k_{\sigma''}$, $k_{\sigma''}$ used in defining $d(\top,\overline{\top})(\sigma'')$, for any $\sigma'' < \sigma$ which has the given p-face of $\sigma$ as a face. If we carry out the above procedure for each m-face of $\sigma$ which is common with an m-face of some $\sigma' < \sigma$ in $\mathcal{O}$, and let $\overset{\bullet}{\Delta}{}_q^{*}$ be the union of the domains of all such m-faces, we obtain a homotopy $k_\sigma^{*} : t_\sigma|\overset{\bullet}{\Delta}{}_q^{*} \simeq \overline{t}_\sigma|\overset{\bullet}{\Delta}{}_q^{*}$. By Lemma III. 8, this homotopy can be extended to all of $\overset{\bullet}{\Delta}{}_q$. Let $k_\sigma$ be the first such extension and let $d_\sigma = d(t_\sigma, k_\sigma, \overline{t}_\sigma)$ be the corresponding deformation cochain. We define $d(\top,\overline{\top})(\sigma) = r(d_\sigma(\Delta_q))$.

By transfinite induction, we have $d(\top,\overline{\top})$ defined on the subset $\mathcal{O}$ of $S_q(X)$ and satisfying the condition that if $\sigma < \sigma'$ in $\mathcal{O}$ and $\sigma$ and $\sigma'$ have a common p-face, $0 \le p \le q-1$, then the homotopy $k_{\sigma'}$, used to define $d(\top,\overline{\top})(\sigma')$, when restricted to the domain of this p-face is induced by the homotopy $k_\sigma$ used to define $d(\top,\overline{\top})(\sigma)$.

Now let $\sigma$ be an arbitrary element of $S_q(X)$, not in $\mathcal{O}$. Let $\sigma' \in \mathcal{O}$ be the base of $A(\sigma)$ and $(\check{u}, \hat{u})$ the base of $J(\sigma', \sigma)$. We then have the following commutative diagram:

$$(IX)$$

$$
\begin{array}{ccc}
B_\sigma^q & \xrightarrow{\quad (1, \hat{u}^{-1}) \quad} & B_{\sigma'}^q \\[4pt]
t_\sigma \Big\uparrow\Big\uparrow \overline{t}_\sigma & & t_{\sigma'} \Big\uparrow\Big\uparrow \overline{t}_{\sigma'} \\[4pt]
\Delta_q & \xrightarrow{\qquad 1 \qquad} & \Delta_q \;,
\end{array}
$$

i. e., $t_\sigma = (1, \hat{u}) \circ t_{\sigma'}$ and $\overline{t}_\sigma = (1, \hat{u}) \circ \overline{t}_{\sigma'}$. Let $k_{\sigma'}$ be the homotopy used to define $d(T, \overline{T})(\sigma')$ and let $k_\sigma : t_\sigma | \mathring{\Delta}_q \simeq \overline{t}_\sigma | \mathring{\Delta}_q$ be the induced homotopy. Then the deformation cochain

$d_\sigma = d(t_\sigma, k_\sigma, \overline{t}_\sigma) \in C^q(\Delta_q; \pi_q(V_{n, n-q}))$ is defined, and we define

$d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q))$.

We now have $d(T, \overline{T})$ defined on all of $S_q(X)$. Suppose that $\sigma$ and $\sigma'$ in $S_q(X)$ have a common p-face, $0 \le p \le q-1$, say $\sigma^p = \sigma'^p$. Let $\overline{\sigma}$ and $\overline{\sigma}'$ be the bases of $A(\sigma)$ and $A(\sigma')$, respectively, and $(\check{u}, \hat{u})$ and $(\check{v}, \hat{v})$ the bases of $J(\overline{\sigma}, \sigma)$ and $J(\overline{\sigma}', \sigma')$, respectively. Then $\overline{\sigma}$ and $\overline{\sigma}'$ in $\mathcal{O}\!\mathcal{l}$ have a common p-face, namely $\check{u}^{-1} \circ \sigma^p = \check{v}^{-1} \circ \sigma'^p$ so that if $\overline{\sigma} < \overline{\sigma}'$, then $k_{\overline{\sigma}'}$ over the common p-face is induced by $k_{\overline{\sigma}}$. By our definition, $k_\sigma$ and $k_{\sigma'}$ over the common p-face are induced by $k_{\overline{\sigma}}$ and $k_{\overline{\sigma}'}$, respectively, so that finally, since $(1, \hat{u}^{-1})$ is a homeomorphism, $\underline{k_{\sigma'}}$ $\underline{\text{over the common p-face is induced by}}$ $\underline{k_\sigma}$.

We extend $d(T, \overline{T})$ to all of $C_q(X)$ by linearity and we have $d(T, \overline{T}) \in C^q(X; Z_2)$.

Theorem III. 9: The q-cochain $d(T, \overline{T})$ is equivariant.

Proof: Let $\sigma = \check{u} \circ \sigma' \in S_q(X)$. We must show that $d(T, \overline{T})(\sigma) = d(T, \overline{T})(\sigma')$. Let $\sigma'' \in \mathcal{O}\!\mathcal{l}$ be the base of $A(\sigma) = A(\sigma')$, and $(\check{v}, \hat{v})$, $(\check{w}, \hat{w})$ be the bases of $J(\sigma'', \sigma)$ and $J(\sigma'', \sigma')$, respectively. Then $\sigma = \check{v} \circ \sigma''$ and $\sigma' = \check{w} \circ \sigma''$ so that

$\sigma = \check{v} \circ \check{w}^{-1} \circ \sigma'$.   Hence   $\check{u} = \check{v} \circ \check{w}^{-1}$   and   $(\check{u}, \hat{v} \circ \hat{w}^{-1}) \in J$.   We then have the following commutative diagram:

$$(X) \qquad \begin{array}{ccccc} B^q_\sigma & \xrightarrow{(1,\,\hat{v}^{-1})} & B^q_{\sigma''} & \xleftarrow{(1,\,\hat{w}^{-1})} & B^q_{\sigma'} \\ {\scriptstyle t_\sigma}\Big\Uparrow{\scriptstyle \bar{t}_\sigma} & & {\scriptstyle t_{\sigma''}}\Big\Uparrow{\scriptstyle \bar{t}_{\sigma''}} & & {\scriptstyle t_{\sigma'}}\Big\Uparrow{\scriptstyle \bar{t}_{\sigma'}} \\ \Delta_q & \xrightarrow{\quad 1 \quad} & \Delta_q & \xleftarrow{\quad 1 \quad} & \Delta_q \end{array} \quad .$$

By definition,   $d(T, \overline{T})(\sigma) = r(d(t_\sigma, k_\sigma, \bar{t}_\sigma)(\Delta_q))$   where $k_\sigma : t_\sigma|\dot{\Delta}_q \simeq \bar{t}_\sigma|\dot{\Delta}_q$   is induced by the left square in (X) from the homotopy   $k_{\sigma''}$   used to define   $d(T, \overline{T})(\sigma'')$,   and similarly $d(T, \overline{T})(\sigma') = r(d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'})(\Delta_q))$.   Because the bundle maps are homeomorphisms,   $k_\sigma$   is induced from   $k_{\sigma'}$   by the bundle map $(1, \hat{w} \circ \hat{v}^{-1})$.   Therefore,   $t_\sigma, k_\sigma,$   and   $\bar{t}_\sigma$   are induced from   $t_{\sigma'}$, $k_{\sigma'},$   and   $\bar{t}_{\sigma'}$.   By the naturality of the deformation cochain (5, p. 172),   $d(t_\sigma, k_\sigma, \bar{t}_\sigma) = 1^\# d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'}) = d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'})$.   Consequently,   $d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q)) = r(d_{\sigma'}(\Delta_q)) = d(T, \overline{T})(\sigma')$.

$$\text{Q. E. D.}$$

Theorem III. 10:   $\delta d(T, \overline{T}) = c_T - c_{\overline{T}}$.

Proof:   Let   $\rho : \Delta_{q+1} \to X$.   The cross-sections   $t_\rho$   and   $\bar{t}_\rho$ over the q-skeleton of   $\Delta_{q+1}$   in the bundle   $B^q_\rho \to \Delta_{q+1}$   are homotopic as maps of the (q-1)-skeleton of   $\Delta_{q+1}$,   and any such homotopy $k_\rho$   gives rise to a deformation cochain   $d_\rho = d(t_\rho, k_\rho, \bar{t}_\rho) \in$ $C^q(\Delta_{q+1}; \pi_q(V_{n, n-q}))$   satisfying   $\delta d_\rho = c(t_\rho) - c(\bar{t}_\rho)$.

Now, $\delta d(T, \overline{T})(\rho) = d(T, \overline{T})(\partial \rho) = \sum_{i=0}^{q+1} (-1)^i d(T, \overline{T})(\rho^{(i)})$

$= r(\sum_{i=0}^{q+1} (-1)^i d_{\rho^{(i)}}(\Delta_q))$. Also, $(c_T - c_{\overline{T}})(\rho) = r([c(t_\rho) - c(\overline{t}_\rho)](\Delta_{q+1}))$

$= r(\delta d_{\dot{\rho}}(\Delta_{q+1})) = r(\sum_{i=0}^{q+1} (-1)^i d_{\dot{\rho}}(\Delta_{q+1}^{(i)}))$ for any deformation cochain $d_{\dot{\rho}}$

defined as above. We shall construct a particular homotopy $k_{\dot{\rho}}$

below such that $d_{\dot{\rho}}$ satisfies the relation $e_{q+1}^{i\#} d_{\dot{\rho}} = d_{\rho^{(i)}}$ for each

$i = 0, 1, \ldots, q+1$. Then it will follow that $d_{\dot{\rho}}(\Delta_{q+1}^{(i)}) = e_{q+1}^{i\#} d_{\dot{\rho}}(\Delta_q)$

$= d_{\rho^{(i)}}(\Delta_q)$, so that $\delta d(T, \overline{T})(\rho) = r(\sum_{i=0}^{q+1} (-1)^i d_{\rho^{(i)}}(\Delta_q)) = r(\sum_{i=0}^{q+1} (-1)^i d_{\dot{\rho}}(\Delta_{q+1}^{(i)}))$

$= (c_T - c_{\overline{T}})(\rho)$, as was to be shown.

Thus, it remains only to define the homotopy $k_{\dot{\rho}}$ making $d_{\dot{\rho}}$

satisfy the relations $e_{q+1}^{i\#} d_{\dot{\rho}} = d_{\rho^{(i)}}$, $0 \le i \le q+1$. For a fixed $i$,

$0 \le i \le q+1$, consider the commutative diagram:

(XI)



Because the bundle map $(e_{q+1}^i, 1)$ is a homeomorphism, the homo-

topy $k_{\rho^{(i)}}$ used to define $d(T, \overline{T})(\rho^{(i)})$ induces a homotopy $k_{\dot{\rho}}^i$ of

the maps $t_{\dot{\rho}}^i$ and $\overline{t}_{\dot{\rho}}^i$ restricted to the $(q-1)$-skeleton of $\Delta_{q+1}^{(i)}$.

Thus, we have a homotopy of the cross-sections $t_{\dot{\rho}}$ and $\overline{t}_{\dot{\rho}}$ over

the $(q-1)$-skeleton of each $q$-face of $\Delta_{q+1}$. However, if $\Delta_{q+1}^{(i)}$ and $\Delta_{q+1}^{(j)}$ have a common $(q-1)$-face, then the homotopy $k_{\rho(i)}$ is induced by the homotopy $k_{\rho(j)}$ so that the homotopies $k_\rho^i$ and $k_\rho^j$ agree on this common $(q-1)$-face. Therefore, the homotopies $k_\rho^i$ fit together to give a homotopy $k_\rho$ of the maps $t_\rho$ and $\bar{t}_\rho$ over the $(q-1)$-skeleton of $\Delta_{q+1}$. $k_\rho$ is defined by $k_\rho | \Delta_{q+1}^{(i)(j)} \times I = k_\rho^i | \Delta_{q+1}^{(i)(j)} \times I$. Furthermore, by the naturality of the deformation cochain $d_\rho = d(t_\rho, k_\rho, \bar{t}_\rho)$ and diagram (XI), $e_{q+1}^{i*} d_\rho = d_{\rho(i)}$.

<div align="right">Q. E. D.</div>

We have shown that for each $q$, $0 \leq q \leq n-1$, we can construct an equivariant lifting in $(\mathcal{B}^q, J)$ on dimensions less than $q+1$. Using this equivariant lifting, we can define a cocycle $c_T \in C_e^{q+1}(X; Z_2)$ and the equivariant cohomology class of $c_T$ is independent of the choice of equivariant lifting.

Definition: The cohomology class $\bar{c}_e(\mathcal{B}^q, J) \in H_e^{q+1}(X; Z_2)$ of $c_T$ is called the equivariant characteristic cohomology class of $(\mathcal{B}^q, J)$. It is the primary obstruction to an equivariant lifting in $\mathcal{B}^q$.

Theorem III. 11: $\bar{c}_e(\mathcal{B}^q, J)$ obstructs equivariant liftings in $(\mathcal{B}^q, J)$ on dimension $q+1$.

Proof: Suppose there is an equivariant lifting $T_p : C_p(X) \rightarrow C_p(B^q)$ for $0 \leq p \leq q+1$. Then by Theorem III. 2, the

equivariant obstruction cocycle $c_T \in C_e^{q+1}(X; Z_2)$ is zero. There-

fore, its equivariant cohomology class $\bar{c}_e(\mathcal{B}^q, J)$ is also zero.

Q. E. D.

We now show that the equivariant characteristic cohomology

class is <u>natural</u> with respect to equivariant bundle maps.

<u>Theorem III. 12</u>: <u>If</u> $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ <u>is an equivari-</u>

<u>ant bundle map, then for each</u> $q$, $0 \leq q \leq n - 1$,

$$\bar{c}_e(\mathcal{B}^q, J) = \check{f}^*(\bar{c}_e(\mathcal{B}'^q, J')).$$

<u>Proof</u>: Let $q$ be a fixed integer $0 \leq q \leq n - 1$. Let

$T_p' : C_p(X') \to C_p(B'^q)$ be an equivariant lifting in $(\mathcal{B}'^q, J')$ for

$0 \leq p \leq q$ and let $T_p$ be the induced lifting in $(\mathcal{B}^q, J)$. Then

$c_{T'}$ and $c_T$ are representative cocycles of $\bar{c}_e(\mathcal{B}'^q, J')$ and

$\bar{c}_e(\mathcal{B}^q, J)$. By Theorem III.7, $c_T = \check{f}^{\#}(c_{T'})$, so that

$$\bar{c}_e(\mathcal{B}^q, J) = f^*(\bar{c}_e(\mathcal{B}'^q, J')).$$

Q. E. D.

Let $\mathcal{V}_{p.g.}^0$ be the full subcategory of the category $\mathcal{V}_{p.g.}$

consisting of those objects $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ which have trivial

holonomy. For each $q \geq 0$, we define the function $\underline{W_e^{q+1}}$ on

$\mathcal{V}_{p.g.}^0$ by

$$W_e^{q+1}(\mathcal{B}, J) = \begin{cases} \bar{c}_e(\mathcal{B}^q, J) & \text{if the dimension of } \mathcal{B} \text{ is greater than } q; \\ 0 \in H_e^{q+1}(X; Z_2) & \text{otherwise.} \end{cases}$$

Definition: The element $W_e^{q+1}(\mathcal{B}, J) \in H_e^{q+1}(X; Z_2)$ is called the $(q+1)^{\text{st}}$ equivariant Stiefel-Whitney class of the object $(\mathcal{B}, J)$.

For $0 \le q \le n - 1$, $W_e^{q+1}(\mathcal{B}, J)$ is by definition the primary obstruction to a lifting in $(\mathcal{B}^q, J)$.

Theorem III. 13: The equivariant Stiefel-Whitney classes are natural with respect to the morphisms in $\mathcal{V}_{\text{p. g.}}^0$.

Proof: Let $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ be a morphism between n-dimensional objects of $\mathcal{V}_{\text{p. g.}}^0$. Then $W_e^{q+1}(\mathcal{B}', J') = 0$ $= W_e^{q+1}(\mathcal{B}, J)$ for $q > n - 1$. For $0 \le q \le n - 1$, $\check{f}^*(W_e^{q+1}(\mathcal{B}', J'))$ $= \check{f}^*(\overline{c}_e(\mathcal{B}'^q, J')) = \overline{c}_e(\mathcal{B}^q, J) = W_e^{q+1}(\mathcal{B}, J)$ by Theorem III. 12.

Q. E. D.

If $(\mathcal{B}, J)$ is an object in the subcategory $\mathcal{V}$ of $\mathcal{V}_{\text{p. g.}}^0$, then $J = I_{\mathcal{B}}$ and singular cohomology and equivariant cohomology of the base space coincide. Furthermore, every lifting $\tau$ in $(\mathcal{B}^q, I_{\mathcal{B}})$ is equivariant and gives rise to an equivariant obstruction cocycle $c_\tau$.

For any two such liftings $\tau$ and $\overline{\tau}$ in $(\mathcal{B}^q, I_{\mathcal{B}})$ the co-chain $d(\tau, \overline{\tau})$ is equivariant. Therefore, our definition of $W_e^{q+1}(\mathcal{B}, I_{\mathcal{B}})$ coincides with the definition of $W^{q+1}(\mathcal{B})$ and we have the following.

Theorem III. 14: The function $W_e^{q+1}$ on $\mathcal{V}_{\text{p. g.}}^0$ is an extension of the function $W^{q+1}$ on $\mathcal{V}$.

## IV.   THE CASE OF ALLOWABLE HOLONOMY

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle and J a pseudo-group of local bundle maps on $\mathcal{B}$ such that for each $x \in X$, the holonomy group $\Phi_x$ of $(\mathcal{B}, J)$ at $x$ is orientation preserving and leaves a subspace $B_x^*$ of $\pi^{-1}(x)$ of dimension at least $(n-1)$ fixed. With this restriction on the holonomy, we note that if $(\breve{u}, \hat{u})$, $(\breve{u}, \hat{v}) \in J$, then for each $x \in$ domain $\breve{u}$, $\hat{u} | \pi^{-1}(x)$ and $\hat{v} | \pi^{-1}(x)$ differ only on a one-dimensional subspace of $\pi^{-1}(x)$, and $\hat{u} | \pi^{-1}(x)$ preserves orientation if and only if $\hat{v} | \pi^{-1}(x)$ preserves orientation.

For each $q$, $0 \leq q \leq n - 1$, the corresponding holonomy group of $(\mathcal{B}^q, J)$ at $x$ will be denoted by $\Phi_x^q$. An element of $\Phi_x^q$ acts only on one and the same component of each vector in the $(n-q)$-frames of the fibre $\pi^{-1}(x) = Y_x$. Now, $Y_x$ is homeomorphic to $V_{n, n-q}$. For $q > 0$, the subset $Y_x^*$ of $Y_x$ which is left pointwise fixed by the action of $\Phi_x^q$ can therefore be identified with $V_{n-1, (n-1)-(q-1)}$. Therefore, the first non-zero homotopy group of $Y_x^*$ is $\pi_{q-1}(Y_x^*)$ and $\pi_{q-1}(Y_x^*) \approx \pi_{q-1}(V_{n-1, (n-1)-(q-1)})$ is isomorphic to either $Z$ or $Z_2$.

Now let $q$ be a fixed integer, $0 \leq q \leq n - 1$, until stated otherwise, and let $B^{q*} = \bigcup_{x \in X} Y_x^*$. Then the pseudo-group $J$ acts on the bundle $B^{q*} \to X$, denoted by $\mathcal{B}^{q*}$, with trivial holonomy.

Consequently, <u>the pair $(\mathcal{B}^{q*}, J)$ is of the type considered in Chapter III</u>, and we may apply the results of that chapter. In particular, we may define an equivariant lifting $T_p : C_p(X) \to C_p(B^{q*})$ for $0 \le p \le q - 1$ and we suppose we have done so. We identify $C_p(B^{q*})$ with its image in $C_p(B^q)$ under the monomorphism induced by the inclusion $B^{q*} \subset B^q$.

Thus, for each $p$, $0 \le p \le q - 1$, we have a homomorphism $T_p : C_p(X) \to C_p(B^q)$ satisfying:

(o) $T_p(\sigma) : \Delta_p \to B^q$ for each $\sigma \in S_p(X)$;

(i) $\pi_\# \circ T_p = 1_{C_p(X)}$;

(ii) $\partial_p T_p = T_{p-1} \partial_p$;

(iii) $T_p \circ \check{u}_\# = \hat{u}_\# \circ T_p$ whenever these expressions are defined.

We shall define an extension (not necessarily equivariant) $T_q : C_q(X) \to C_q(B_q)$ of $T_{q-1}$ in such a manner that the obstruction cochain $c_T$, defined as before, is an equivariant cocycle. Thus, we shall define a homomorphism $T_q : C_q(X) \to C_q(B^q)$ satisfying:

(o) $T_q(\sigma) : \Delta_q \to B^q$ for $\sigma \in S_q(X)$;

(i) $\pi_\# \circ T_q = 1_{C_q(X)}$;

(ii) $\partial_q T_q = T_{q-1} \partial_q$;

(iii) $\delta c_T = 0$;

(iv) $c_T(\check{w} \circ \rho) = c_T(\rho)$ for all $\rho : \Delta_{q+1} \to X$ and $\check{w} \in J$ for which $\check{w} \circ \rho$ is defined.

In order to define $T_q$, we shall again use transfinite induction.

Hence, as in Chapters II and III, suppose $S_q(X) = \{\sigma : \Delta_q \to X\}$ is well ordered, $A(\sigma) = \{\sigma' \in S_q(X) : \sigma'$ is J-equivalent to $\sigma\}$, and each set $A(\sigma)$ has as base the first element of $A(\sigma)$. Also, let J be well-ordered, $J(\sigma, \sigma') = \{(\check{u}, \hat{u}) \in J : \check{u} \circ \sigma = \sigma'\}$, and if $J(\sigma, \sigma')$ is non-empty, then its first element is the base of $J(\sigma, \sigma')$.

Let $\mathcal{O}$ be the set of all base elements in $S_q(X)$, $\mathcal{O} = \{\sigma \in S_q(X) : \sigma$ is the base of some $A(\sigma')\}$. We first define $\tau_q$ on the subset $\mathcal{O}$ of $S_q(X)$.

Let $\sigma \in \mathcal{O}$ and consider the induced bundle $B_\sigma^q \to \Delta_q$ and bundle map $h_\sigma : B_\sigma^q \to B^q$ as before. Then exactly as in the induction step of the definition of $\tau_p$ in Chapter III, for each (q-1)-face $\sigma^{(i)}$ of $\sigma$ we get a cross-section $t_\sigma^i = (e_q^i, 1) \circ t_{\sigma^{(i)}} \circ (e_q^i)^{-1}$ over the (q-1)-face $\Delta_q^{(i)}$ in the induced bundle, and these fit together to give a cross-section $t_{\dot{\sigma}}$ over $\dot{\Delta}_q$, the boundary of $\Delta_q$. Because the fibre $V_{n, n-q}$ is (q-1)-connected, $t_{\dot{\sigma}}$ can be extended to a cross-section over all of $\Delta_q$. Let $t_\sigma$ be the first such extension and define $\tau_q(\sigma) = h_\sigma \circ t_\sigma : \Delta_q \to B^q$. Then also as before, $\pi \circ \tau_q(\sigma) = \sigma$ and $\partial_q \tau_q(\sigma) = \tau_{q-1}(\partial_q \sigma)$.

This defines $\tau_q$ on the subset $\mathcal{O}$ of $S_q(X)$. Now let $\sigma : \Delta_q \to X$ be an arbitrary element of $S_q(X)$. If $\sigma \in \mathcal{O}$, i.e., if $\sigma$ is the base of some $A(\sigma')$, then $\tau_q(\sigma)$ has been defined above. If $\sigma \notin \mathcal{O}$, then there is a unique element $\sigma' \in \mathcal{O}$ for which $\sigma \in A(\sigma')$. Let $(\check{u}, \hat{u})$ be the base of $J(\sigma', \sigma)$ and define

$T_q(\sigma) = \hat{u} \circ T_q(\sigma') : \Delta_q \to B^q$. Then $\pi \circ T_q(\sigma) = \pi \circ \hat{u} \circ T_q(\sigma')$

$= \check{u} \circ \pi \circ T_q(\sigma') = \check{u} \circ \sigma' = \sigma$ and condition (i) is satisfied.

Also, since $T_{q-1}$ is equivariant, we have $\hat{u}_{\#} \circ T_{q-1}(\partial\sigma')$

$= T_{q-1} \circ \check{u}_{\#}(\partial\sigma')$ so that $\partial T_q(\sigma) = \partial(\hat{u} \circ T_q(\sigma') = \hat{u}_{\#} \circ \partial T_q(\sigma')$

$= \hat{u}_{\#} \circ T_{q-1}(\partial\sigma') = T_{q-1} \circ \check{u}_{\#}(\partial\sigma') = T_{q-1}\partial(\check{u} \circ \sigma') = T_{q-1}\partial(\sigma)$ and condition (ii) is satisfied.

The above defines $T_q$ on $S_q(X)$ satisfying conditions (o), (i) and (ii). We extend $T_q$ to all of $C_q(X)$ by linearity and note that conditions (o), (i) and (ii) are satisfied since $\pi_{\#}$, $T_q$ and $\partial$ are homomorphisms.

Before discussing conditions (iii) and (iv) concerning the cochain $c_T$, we note the following. In the definition of $T_q$, we relied on the equivariance of $T_{q-1}$ to show that condition (ii) was satisfied. The fact that $T_{q-1}$ could be defined equivariantly depended on the fact that the holonomy group $\Phi_x$ of $(\mathcal{B}, J)$ at $x \in X$ leaves an (n-1)-dimensional subspace of $\pi^{-1}(x)$ pointwise fixed. Therefore, we see that the construction of this section does not extend to vector bundles with pseudo-group $(\mathcal{B}, J)$ where the holonomy groups do not leave at least an (n-1)-dimensional subspace of the fibre pointwise fixed.

Now, the cochain $c_T \in C^{q+1}(X; Z_2)$ is defined precisely as in Chapter III. Thus, for $\rho : \Delta_{q+1} \to X$, the cross-sections over the q-faces $\Delta_{q+1}^{(i)}$ of $\Delta_{q+1}$ in the bundle $B_\rho^q \to \Delta_{q+1}$, induced by

the cross-sections $t_{\rho^{(i)}} : \Delta_q \to B^q_{\rho(i)}$, fit together to give a cross-

section $t_\rho : \dot{\Delta}_{q+1} \to B^q_\rho$. The obstruction cocycle

$c(t_\rho) \in C^{q+1}(\Delta_{q+1}, \pi_q(V_{n, n-q}))$ is then defined by $c(t_\rho)(\Delta_{q+1})$

$= [t_\rho] \in \pi_q(V_{n, n-q})$, and we define $c_T(\rho) = r(c(t_\rho)(\Delta_{q+1}) = r([t_\rho])$.

Theorem IV.1: The cochain $c_T$ obstructs the equivariant ex-

tension of $T_q$.

Proof: The proof is identical to that of Theorem III.2, since the

equivariance of $T_q$ played no part in the proof.

We state condition (iii) in the form of a theorem.

Theorem IV.2: The cochain $c_T$ is a cocycle.

Proof: The proof is identical to that of Theorem III.4, since

the equivariance of $T_q$ played no part in the proof.        Q. E. D.

We precede the proof of condition (iv) by the following lemma.

Lemma IV.3: Let $\rho : \Delta_{q+1} \to X$ and $\rho' = \check{u} \circ \rho$, $(\check{u}, \hat{u}) \in J$.

Then under the present holonomy assumptions, the following diagram

is homotopy commutative:

$$
\begin{array}{ccc}
B^q_\rho & \xrightarrow{\;(1, \hat{u})\;} & B^q_{\rho'} \\[1ex]
{\scriptstyle t_\rho} \Big\uparrow & & \Big\uparrow {\scriptstyle t_{\rho'}} \\[1ex]
\dot{\Delta}_{q+1} & \xrightarrow[\;1\;]{} & \dot{\Delta}_{q+1} \; .
\end{array}
$$

<u>Proof</u>: In order to show $(1, \hat{u}) \circ t_{\dot{\rho}} \simeq t_{\dot{\rho}'}$, it suffices to show that for each $i$, $0 \leq i \leq q$, $(1, \hat{u}) \circ t_{\dot{\rho}} | \Delta_{q+1}^{(i)} \simeq t_{\check{u} \circ \dot{\rho}} | \Delta_{q+1}^{(i)}$ rel $(\Delta_{q+1}^{(i)})^{\bullet}$, where $(\Delta_{q+1}^{(i)})^{\bullet}$ is the boundary of $\Delta_{q+1}^{(i)}$, for since $T_{q-1}$ is equivariant, we actually have $(1, \hat{u}) \circ t_{\dot{\rho}} | (\Delta_{q+1}^{(i)})^{\bullet} = t_{\check{u} \circ \dot{\rho}} | (\Delta_{q+1}^{(i)})^{\bullet}$. Recall that $t_{\dot{\rho}} | \Delta_{q+1}^{(i)} = t_{\rho}^{i} = (e_{q+1}^{i}, 1) \circ t_{\rho^{(i)}} \circ (e_{q+1}^{i})^{-1}$ so that it suffices to show $(1, \hat{u}) \circ (e_{q+1}^{i}, 1) \circ t_{\rho^{(i)}} \simeq (e_{q+1}^{i}, 1) \circ t_{\check{u} \circ \rho^{(i)}}$ rel $\dot{\Delta}_q$. Thus, let $i$ be a fixed integer, $0 \leq i \leq q$.

For each $\sigma: \Delta_q \to X$, $t_{\sigma} = (1, T_q(\sigma)) \circ d$, where $d: \Delta_q \to \Delta_q \times \Delta_q$ is the diagonal map. Hence it is sufficient to show that

$(1, \hat{u}) \circ (e_{q+1}^{i}, 1) \circ (1, T_q(\rho^{(i)})) = (e_{q+1}^{i}, \hat{u} \circ T_q(\rho^{(i)}))$

$\simeq (e_{q+1}^{i}, T_q(\check{u} \circ \rho^{(i)}))$ rel $d(\dot{\Delta}_q)$ and hence to show $\hat{u} \circ T_q(\rho^{(i)})$

$\simeq T_q(\check{u} \circ \rho^{(i)})$ rel $\dot{\Delta}_q$.

Now, let $\sigma$ be the base of $A(\rho^{(i)})$, $(\check{w}, \hat{w})$ the base of $J(\sigma, \rho^{(i)})$. Then by definition, $T_q(\rho^{(i)}) = \hat{w} \circ T_q(\sigma)$. Also, $\check{u} \circ \rho^{(i)} \in A(\rho^{(i)})$ so that $\check{u} \circ \rho^{(i)} = \check{v} \circ \sigma$, and if $(\check{v}, \hat{v})$ is the base of $J(\sigma, \check{u} \circ \rho^{(i)})$, then $T_q(\check{u} \circ \rho^{(i)}) = \hat{v} \circ T_q(\sigma)$. Therefore $T_q(\rho^{(i)}) = \hat{w} \circ \hat{v}^{-1} \circ T_q(\check{u} \circ \rho^{(i)})$. But, $\rho^{(i)} = \check{w} \circ \sigma = \check{w} \circ \check{v}^{-1} \circ \check{u} \circ \rho^{(i)}$ so that $\check{w} \circ \check{v}^{-1} = \check{u}^{-1}$ and we have $(\check{u}, \hat{u})$ and $(\check{u}, \hat{v} \circ \hat{w}^{-1})$ in $J$ and $\hat{v} \circ \hat{w}^{-1} \circ T_q(\rho^{(i)}) = T_q(\check{u} \circ \rho^{(i)})$. Let $\hat{v} \circ \hat{w}^{-1} = \hat{\tilde{u}}$ so that $(\check{u}, \hat{u}), (\check{u}, \hat{\tilde{u}}) \in J$ and $\hat{\tilde{u}} \circ T_q(\rho^{(i)}) = T_q(\check{u} \circ \rho^{(i)})$

By the above, it is therefore sufficient to show that if $(\check{u}, \hat{u})$, $(\check{u}, \hat{\tilde{u}}) \in J$, then $\hat{u}^{-1} \circ \hat{\tilde{u}} \circ T_q(\rho^{(i)}) \simeq T_q(\rho^{(i)})$ rel $\dot{\Delta}_q$. We do this by exhibiting a homotopy.

Let $F$ be defined on $\Delta_q \times I$ by $F(s,t) = t(\hat{u}^{-1} \circ \overset{\wedge}{u} \circ T_q(\rho^{(i)})(s))$
$+ (1-t)(T_q(\rho^{(i)})(s))$. Thus $F(s,0) = T_q(\rho^{(i)})(s)$, $F(s,1) = \hat{u}^{-1} \circ \overset{\wedge}{u} \circ T_q(\rho^{(i)})(s)$
so that $F$ is a homotopy between $T_q(\rho^{(i)})$ and $\hat{u}^{-1} \circ \overset{\wedge}{u} \circ T_q(\rho^{(i)})$.
Furthermore $F$ is a homotopy relative to $\overset{\bullet}{\Delta}_q$ because for
$s \in \overset{\bullet}{\Delta}_q$, $T_q(\rho^{(i)})(s)$ is in $Y^*_{\rho^{(i)}(s)}$, the subspace of $Y_{\rho^{(i)}(s)}$ fixed
under the action of the holonomy group, and $\hat{u}^{-1} \circ \overset{\wedge}{u}\, |\, Y_{\rho^{(i)}(s)} \in J_{\rho^{(i)}(s)}$.
Hence for $s \in \overset{\bullet}{\Delta}_q$, $\hat{u}^{-1} \circ \overset{\wedge}{u} \circ T_q(\rho^{(i)})(s) = T_q(\rho^{(i)})(s)$ and
$F(s,t) = T_q(\rho^{(i)})(s)$ for all $t \in I$.

We wish to show that $F$ is a homotopy in $B^q$, i.e., that
$F : \Delta_q \times I \rightarrow B^q$ or $F(s,t)$ is an $(n-q)$-frame for each $s \in \Delta_q$,
$t \in I$. Let $B_{\rho^{(i)}(s)}$ be the fibre in $B$ over $\rho^{(i)}(s)$. $B_{\rho^{(i)}(s)}$ is
homeomorphic to $R^n$. The holonomy group $\Phi_{\rho^{(i)}(s)}$ leaves an
$(n-1)$-dimensional subspace $B^*_{\rho^{(i)}(s)}$ fixed. Let $B_{\rho^{(i)}(s)} = R^{n-1} \times R$
and $B^*_{\rho^{(i)}(s)} = R^{n-1} \times \{0\}$.

Now, the $(n-q)$-frame $T_q(\rho^{(i)})(s)$ can be written
$$T_q(\rho^{(i)})(s) = (v_1 + \overline{v}_1, v_2 + \overline{v}_2, \ldots, v_{n-q} + \overline{v}_{n-q}) \quad \text{where}$$
$v_k = (v_{1k}, \ldots, v_{n-1,k}, 0)$ and $\overline{v}_k = (0, 0, \ldots, 0, \overline{v}_{nk})$. Let
$\phi = \hat{u}^{-1} \circ \overset{\wedge}{u}\, |\, Y_{\rho^{(i)}(s)} \in \Phi_{\rho^{(i)}(s)}$. By our assumption concerning holonomy,
$\phi(T_q(\rho^{(i)}(s))) = (v_1 + a\overline{v}_1, \ldots, v_{n-q} + a\overline{v}_{n-q})$ for some positive real number $a$.

Thus, $F(s,t) = t(v_1 + a\overline{v}_1, \ldots, v_{n-q} + a\overline{v}_{n-q}) + (1-t)(v_1 + \overline{v}_1, \ldots, v_{n-q} + \overline{v}_{n-q})$
$= (v_1 + (at + 1-t)\overline{v}_1, \ldots, v_{n-q} + (at+1-t)\overline{v}_{n-q})$. To show $F(s,t) \in B^q$,
it suffices to show that the following matrix is of maximal rank $(n-q)$:

(a)

$$\begin{bmatrix} v_{11} & \cdots & v_{1,n-1} & (ta+1-t)\overline{v}_{1n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,1} & \cdots & v_{n-q,n-1} & (ta+1-t)\overline{v}_{n-q,n} \end{bmatrix} \, .$$

We know that the matrix

($\beta$)

$$\begin{bmatrix} v_{11} & \cdots & v_{1,n-1} & \overline{v}_{1n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,1} & \cdots & v_{n-q,n-1} & \overline{v}_{n-q,n} \end{bmatrix}$$

is of rank $(n-q)$, so that ($\beta$) has an $(n-q)$ by $(n-q)$ submatrix

$V$ with non-zero determinant. If this submatrix is of the form

$$V = \begin{bmatrix} v_{1,i_1} & \cdots & v_{1,i_{n-q}} \\ \vdots & & \vdots \\ v_{n-q,i_1} & \cdots & v_{n-q,i_{n-q}} \end{bmatrix}$$

then $V$ is also a submatrix of (a) and (a) has rank $(n-q)$.

If the submatrix $V$ is of the form

$$V = \begin{bmatrix} v_{1,i_1} & \cdots & v_{1,i_{n-q-1}} & \overline{v}_{1n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,i_1} & \cdots & v_{n-q,i_{n-q-1}} & \overline{v}_{n-q,n} \end{bmatrix}$$

then consider the following submatrix of (a):

$$V_t = \begin{bmatrix} v_{1,i_1} & \cdots & v_{1,i_{n-q-1}} & (ta+1-t)\bar{v}_{1,n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,i_1} & \cdots & v_{n-q,i_{n-q-1}} & (ta+1-t)\bar{v}_{n-q,n} \end{bmatrix} .$$

Then, determinant of $V_t = \det V_t = (ta+1-t) \det V$ so that $\det V_t = 0$ if and only if $ta + 1 - t = t(a-1) + 1 = 0$. But, $a > 0$ and $t \in I$ implies that $t(a-1) > -1$. Therefore, $\det V_t \neq 0$ and $(a)$ is of rank $(n-q)$.  Q. E. D.

We now prove condition (iv) as a theorem.

**Theorem IV. 4:** <u>The obstruction cocycle $c_T$ is equivariant.</u>

<u>Proof:</u> Let $\rho' = \check{u} \circ \rho : \Delta_{q+1} \to X$. Then the elements $\{(1,\hat{u}) \circ t_\rho\}$ and $\{t_{\rho'}\}$ in $\pi_q(B^q_{\rho'})$ are equal by Lemma IV. 3. Hence, the elements $[(1,\hat{u}) \circ t_\rho]$ and $[t_{\rho'}]$ are equal in $\pi_q(V_{n,n-q})$. Because $(1,\hat{u})$ is a homeomorphism, it induces an isomorphism $(1,\hat{u})_*$ on the homotopy groups. Therefore, $c_T(\rho) = r(c(t_\rho)(\Delta_{q+1}))$ $= r([(1,\hat{u}) \circ t_\rho]) = r \circ (1,\hat{u})_*([t_\rho]) = r([t_\rho])$. But also, $r([(1,\hat{u}) \circ t_\rho])$ $= r([t_{\rho'}]) = c_T(\rho')$. Thus, $c_T(\rho) = c_T(\rho')$.  Q. E. D.

We next show that <u>the equivariant obstruction cocycle is natural</u> with respect to equivariant bundle maps. This fact is contained in the following theorem.

**Theorem IV. 5:** <u>Let $f = (\check{f},\hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ be an</u>

equivariant bundle map. Let $T'_p : C_p(X') \to C_p(B'^q)$ be an equivariant lifting for $0 \le p \le q - 1$ and let $T'_q : C_q(X') \to C_q(B'^q)$ be an extension of $T'_{q-1}$ which yields an equivariant obstruction cocycle $c_{T'} \in C_e^{q+1}(X; Z_2)$. Then the induced lifting $T_p : C_p(X) \to C_p(B^q)$, $0 \le p \le q$, given by Lemma III. 5 is such that $T_p$ is equivariant for $0 \le p \le q - 1$, $\check{f}^{\#}(c_{T'}) = c_T$, and $c_T$ is equivariant.

Proof: The fact that $T_p$ is equivariant for $0 \le p \le q - 1$ follows exactly as in Lemma III. 6.

Let $\rho : \Delta_{q+1} \to X$. By the definition of the induced lifting, the following diagram is commutative:

$$
\begin{array}{ccc}
B^q_\rho & \xrightarrow{\;(1, \hat{f})\;} & B'^q_{\check{f} \circ \rho} \\
{\scriptstyle t_{\dot\rho}}\Big\uparrow & & \Big\uparrow{\scriptstyle t^{\nu}_{\check{f} \circ \dot\rho}} \\
\Delta_{q+1} & \xrightarrow{\quad 1 \quad} & \Delta_{q+1}
\end{array}
$$

Therefore, as in the proof of Theorem III. 7, $c(t_{\dot\rho}) = c(t^{\nu}_{\check{f} \circ \dot\rho})$ so that $c_T(\rho) = r(c(t_{\dot\rho})(\Delta_{q+1})) = r(c(t^{\nu}_{\check{f} \circ \dot\rho})(\Delta_{q+1})) = c_{T'}(\check{f} \circ \rho)$, i. e., $c_T = \check{f}^{\#}(c_{T'})$.

Finally, let $\rho' = \check{u} \circ \rho : \Delta_{q+1} \to X$. Since $f$ is equivariant, there is a $\check{u}' \in J'$ such that $\check{f} \circ \check{u} = \check{u}' \circ \check{f}$. Therefore $c_T(\check{u} \circ \rho) = c_{T'}(\check{f} \circ \check{u} \circ \rho) = c_{T'}(\check{u}' \circ \check{f} \circ \rho) = c_{T'}(\check{f} \circ \rho)$ since $c_{T'}$ is equivariant. Consequently, $c_T(\check{u} \circ \rho) = c_T(\rho)$. Q. E. D.

We have now shown that with every equivariant lifting $T$ in

$(\mathcal{B}^q, J)$ on dimensions less than or equal to $q - 1$, we may define

an extension $T_q$ which yields an equivariant obstruction cocycle

$c_T \in C^{q+1}(X; Z_2)$. We now wish to show that the equivariant cohomol-

ogy class of $c_T$ is independent of the particular equivariant lifting

on dimension less than $q$ and the extension $T_q$, as long as $T_q$

yields an equivariant obstruction cocycle.

Thus, suppose $T_p, \overline{T}_p : C_p(X) \to C_p(B^q)$ are equivariant lift-

ings for $0 \le p \le q - 1$, and $T_q, \overline{T}_q : C_q(X) \to C_q(B^q)$ are exten-

sions which yield equivariant obstruction cocycles $c_T, c_{\overline{T}}$. We

shall define an equivariant $q$-cochain $d(T, \overline{T}) \in C^q(X; Z_2)$ such that

$\delta d(T, \overline{T}) = c_T - c_{\overline{T}}$.

The cochain $\underline{d(T, \overline{T})}$ is defined precisely as in Chapter III.

Thus, for $\sigma \in \mathcal{A}$, $\sigma : \Delta_q \to X$, we associate the cross-sections

$t_\sigma$ and $\overline{t}_\sigma$ induced by $T_q(\sigma)$ and $\overline{T}_q(\sigma)$ and let

$k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \overline{t}_\sigma | \dot{\Delta}_q$ be the first homotopy satisfying the condition

that if $\sigma' < \sigma$ and $\sigma'$ and $\sigma$ have a common face, then the

homotopy $k_\sigma$ restricted to the domain of this common face is in-

duced by the homotopy $k_{\sigma'}$ used to define $d(T, \overline{T})(\sigma')$. Associated

with $t_\sigma$, $k_\sigma$ and $\overline{t}_\sigma$ is the deformation cochain

$d_\sigma \in C^q(\Delta_q, \pi_q(V_{n, n-q})$ and we define $d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q))$.

If $\sigma \in S_q(X) - \mathcal{A}$, i.e., $\sigma$ is not a base element in $S_q(X)$,

then there is a unique $\sigma' \in \mathcal{A}$ such that $\sigma' \in A(\sigma)$. Let $(\check{u}, \hat{u})$ be

the base element of $J(\sigma', \sigma)$. Then $k_{\sigma'}$, the homotopy used to

define $d(T, \overline{T})(\sigma')$, and the bundle map $(1, \hat{u}^{-1}) : B^q_\sigma \rightarrow B^q_{\sigma'}$, induce a homotopy $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \overline{t}_\sigma | \dot{\Delta}_q$. This yields the deformation co-chain $d_\sigma$, and we define $d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q))$.

This defines $d(T, \overline{T})$ on $S_q(X)$, satisfying the condition that if $\sigma$ and $\sigma'$ have a common face and $k_\sigma$ and $k_{\sigma'}$ are the homotopies used to define $d(T, \overline{T})(\sigma)$ and $d(T, \overline{T})(\sigma')$, then the homotopy $k_\sigma$ restricted to the domain of the common face, is in-duced by $k_{\sigma'}$.

We extend $d(T, \overline{T})$ to all of $C_q(X)$ by linearity, and we have $d(T, \overline{T}) \in C^q(X; Z_2)$.

Because the definition of $d(T, \overline{T})$ depends upon what occurs on dimension $(q-1)$, and $T_{q-1}$ and $\overline{T}_{q-1}$ are equivariant, the proof of the following theorem is identical to the proof of Theorem III. 9.


Theorem IV. 6: The q-cochain $d(T, \overline{T})$ is equivariant.


Since $c_T$, $c_{\overline{T}}$ and $d(T, \overline{T})$ are defined precisely as in Chap-ter III, the following theorem is proved exactly as is Theorem III. 10.


Theorem IV. 7: $\delta d(T, \overline{T}) = c_T - c_{\overline{T}}$.


We have now shown that for each $q$, $0 \leq q \leq n - 1$, we can con-struct an equivariant lifting in $(\mathcal{B}^q, J)$ on dimensions less than $q$, and an extension to dimension $q$ which yields a cocycle

$c_T \in C_e^{q+1}(X; Z_2)$.   Furthermore, the equivariant cohomology class of

$c_T$   is independent of the particular equivariant lifting and extension

to dimension   q.

Definition:   The cohomology class   $\bar{c}_e(\mathcal{B}^q, J) \in H_e^{q+1}(X; Z_2)$   of

$c_T$   is called the equivariant characteristic cohomology class of

$(\mathcal{B}^q, J)$.   It is the primary obstruction to an equivariant lifting in

$\mathcal{B}^q$.

Theorem IV. 8:   If   $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$   is an equivariant bundle map, then for each   q,   $0 \leq q \leq n-1$,

$$\bar{c}_e(\mathcal{B}^q, J) = \check{f}^*(\bar{c}_e(\mathcal{B}'^q, J')).$$

Proof:   The proof is identical to that of Theorem III. 12, except

that we refer to Theorem IV. 5 rather than Theorem III. 7.

Q. E. D.

Let   $\mathcal{V}_{p.g.}^1$   be the full subcategory of the category   $\mathcal{V}_{p.g.}$

consisting of those objects   $(\mathcal{B}, J)$   in   $\mathcal{V}_{p.g.}$   for which the holo-

nomy group   $\Phi_x$   of   $(\mathcal{B}, J)$   at each   $x \in X$   is orientation preserv-

ing and leaves a subspace of   $\pi^{-1}(x)$   of dimension at least   (n-1)

pointwise fixed, when the dimension of   $(\mathcal{B}, J)$   is   n.   For each

$q \geq 0$,   we define the function   $W_e^{q+1}$   on   $\mathcal{V}_{p.g.}^1$   by

$$W_e^{q+1}(\mathcal{B}, J) = \begin{cases} \bar{c}_e(\mathcal{B}^q, J) & \text{if the dimension of } (\mathcal{B}, J) \text{ is greater} \\ & \text{than } q; \\ 0 \in H_e^{q+1}(X; Z_2) & \text{otherwise.} \end{cases}$$

Definition: The element $W_e^{q+1}(\mathcal{B}, J) \in H_e^{q+1}(X; Z_2)$ is called the $(q+1)^{st}$ equivariant Stiefel-Whitney class of the object $(\mathcal{B}, J)$.

Theorem IV. 9: The equivariant Stiefel-Whitney classes are natural with respect to the morphisms in $\mathcal{V}_{p.g.}^1$.

Proof: The proof is identical to the proof of Theorem III. 13, except that we refer to Theorem IV. 8 rather than Theorem III. 12.

Q. E. D.

As in Chapter III, the definition of $W_e^{q+1}(\mathcal{B}, I_{\mathcal{B}})$ for an object $(\mathcal{B}, I_{\mathcal{B}}) \in \mathcal{V}$ coincides with the definition of $W^{q+1}(\mathcal{B})$. Therefore we have the following.

Theorem IV. 10: The function $W_e^{q+1}$ on $\mathcal{V}_{p.g}^1$ is an extension of the function $W^{q+1}$ on $\mathcal{V}$.

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