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In this work, we provide a detailed analysis of a discrete time regime switching financial market model with jumps. We consider the model under two different scenarios: known and unknown initial regime. For each scenario we investigated conditions that guarantee the model's completeness. We find that the model under consideration is arbitrage-free and complete if the initial regime is known and the jump size satisfies specific condition. Formulae for a unique risk-neutral measure and arbitrage-free pricing of derivative securities are provided. Several numerical examples illustrate no-arbitrage approach to pricing of derivative securities. In the case of incomplete model the Esscher transform is considered to obtain one specific pricing measure. In particular, we show that the Esscher transformed prices are continuously differentiable as a function of the parameters at the interface of incompleteness and completeness.

Valuing Options in a Discrete Time Regime Switching Model with Jumps

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Evgenia V. Chunikhina, Author

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1: Introduction

Louis Bachelier's 1900 Ph.D. thesis "The Theory of Speculation" marked the beginning of the new scientific discipline - the mathematical theory of finance. In his thesis, Bachelier developed a theory of option pricing using Brownian motion. Despite its ultimate importance, his work was not well received and was criticized for an unconventional application of mathematics and for a luck of rigor. The rediscovery of his work began only in the middle of the twenties century with the renewed interest in the theory of option pricing. In 1952 Harry Markowitz published his Ph.D. thesis "Portfolio Selection" which became an important milestone in the development of mathematical finance. In his work, he introduced a notion of mean return and covariance for common stocks and demonstrated how to compute the mean return and the variance for a given portfolio. Later, Paul Samuelson [47] and Robert Merton [37] used the methods of stochastic calculus to investigate the pricing process in financial markets. Merton's joint work with Fischer Black and Myron Scholes won the 1997 Nobel Prize in Economics. The prize drew attention to the theory of finance and resulted in the increased popularity of this research area. Consequently, the theory of finance became more mathematical. Problems in finance now require a knowledge of statistics, probability theory, stochastic calculus, and optimization theory. On the other hand, the theory of finance did not loose its application to practical aspects, since many of the theoretical developments in finance are immediately applied in financial markets.

The 1997 Nobel Prize highlighted the importance of fair pricing of derivative securities. The derivative pricing problem can be viewed from two different perspectives: from the buyer and from the seller. The fundamental question of option pricing is to determine the initial price of the contract that is fair for both of parties. Under the assumption of the absence of the arbitrage, Black and Scholes developed an option pricing formula that allowed to find a fair price for a European call option [1]. Explicit development of no-arbitrage pricing was provided by Merton [38, 39]. Later, Harrison and Kreps [27] and Harison and Pliska [28] developed no-arbitrage pricing in continuous time models, introduced martingales and risk-neutral pricing. Recent works of Gulisashvilli and Stein [25, 26], Schweizer and Wissel [51], and Cont and Kokholm [4] indicate continued mathematical interest in pricing of derivative securities.

The no-arbitrage pricing theory approach to the derivative security pricing problem is to hedge when possible the derivative security by trading in the underlying security and the money market. More precisely, assume that the agent sells the derivative security and forms a portfolio consisting of an underlying asset and a money market account. The absence of arbitrage guarantees that there is no advantage of a price difference between two or more markets; and therefore, the return of the derivative security will be the same as the return of the synthetic trading strategy. Hence, the no-arbitrage price of the derivative security is the amount for which the derivative security must be sold at time zero in order to construct the synthetic portfolio. When all contracts can be hedged market is called complete. However, we will see this is not always the case.

In mathematical finance the no-arbitrage condition is equivalent to the existence of the risk neutral measure. Moreover, the uniqueness of the equivalent martingale measure is the mathematical condition for the completeness of the market model. In other words, a market model is complete if the return for the trading strategy involving derivative securities is equal to the return of the synthetic trading strategy. In a complete market any derivative security can be hedged. The relationship between existence and uniqueness of the risk neutral measure, the non-arbitrage condition, and market completeness is described by the following theorems given in [54].

Theorem 1. The First fundamental theorem of asset pricing: *A market* (S_n, B_n)*, on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, consisting of a collection of stocks prices* S_n *and a risk-free money market process* B_n *is arbitrage-free if and only if there exists at least one risk neutral, i.e., martingale, probability measure that is equivalent to the original probability measure* $\mathbb{P}$*.*

Theorem 2. The Second fundamental theorem of asset pricing: *An arbitrage-free market* (S_n, B_n)*, on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, consisting of a collection of stocks prices* S_n *and a risk-free money market process* B_n *is complete if and only if there exists a unique risk-neutral measure that is equivalent to* $\mathbb{P}$*.*

The next theorem provides the fundamental formula for pricing of the derivative security using the risk neutral probability measure.

Theorem 3. Risk neutral pricing: *Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let V_n be the payoff of a derivative security at time n and B_n be the money market process. If there is a risk neutral measure $\mathbb{Q}$, then the time zero price of the discounted derivative security is a martingale and is given by $V_0 = \mathbb{E}_{\mathbb{Q}}\left[\frac{V_n}{B_n}\right]$.*

Theorems 1, 2, and 3 provide a powerful tool for analyzing market models and for pricing derivative securities. If the market model is complete it guarantees that every derivative security has a unique and fair price. In order to incorporate real world features, most of the proposed models have a high level of complexity, which makes them inherently incomplete. Numerous complete pricing models have been proposed during the last forty years. However, the most profound effect on the financial industry had a Black-Scholes model. Even now, forty years later, a large amount of models that are currently used in derivative security pricing are based on the Black-Scholes model. In spite of its popularity, the Black-Scholes model makes a few unrealistic assumptions that do not explain several important empirical characteristics of option prices. For example, one of the assumptions is that the volatility of the underlying risky asset is constant and does not depend on the maturity and the strike price of the option. However, this is not the case and the volatility parameter has a specific behavior often called a volatility smile. Moreover, the model ignores human activity and information availability. If an investor, for example, has negative inside information about a company, he will try to sell the stock, knowing that its price will go down when the bad news becomes public. This, in turn, will create an arbitrage opportunity. Therefore, the assumption of the Black-Scholes model that arbitrage does not exist in the real world is somewhat impractical.

In order to depict the stock market more realistically, many alternative models have been proposed. Those models aim to capture important empirical phenomena and to reflect random market environments. Among them are constant elasticity model (CEV) [10], classical diffusion models [23], [30], jump-diffusion model [31], affine stochastic-volatility and affine jump-diffusion models [14], models based on Levy processes [20], and many more.

A different model was studied by Di Crescenzo and Pellerey [12], who used a geometric telegraph process to characterize the dynamics of the price of a risky asset. Recall, that the telegraph process describes the position of a particle that moves on the real line with velocities that randomly reverse directions according to a homogeneous Poisson process. The model allows to capture the trends in a market evolution and therefore is considered to be fairly realistic. Numerous authors studied the telegraph process based models and their various generalizations. For example, Di Masi et al. [13] used the telegraph process to model volatility of financial markets and Mazza and Rulliere [36] connected the telegraph and ruin processes.

An interesting extension of the telegraph process based model was proposed by Ratanov in [42–45]. In his model, the price of the asset is described by an inhomogeneous telegraph process with jumps at the times of velocity reversals. Ratanov [45] proved that this continuous time model is complete and obtained closed form formulas for the option prices.

In this work, we study a fusion of a discrete version of the jump telegraph model and a classical binomial tree model. Our discrete time regime switching model with jumps is based on the following intuition. We assume that the market follows a specific trend based on the economical environment. When the economical environment switches, so do the parameters of the market. These infrequent but significant switches result in drastic changes of the stock prices, i.e. jumps.

The model that we consider is somewhat similar to the continuous time regime switching models studied in [3] and [57]. However, there are important differences. First, we choose to restrict our attention to the discrete time model. We do it for the two following reasons. First, although they are much simpler to compute, N-period discrete models illustrate the same phenomenon as more complicated continuous time models. Moreover, the option valuation is discretized in practice. Modern technological progress made it possible to compute option prices using much smaller time increments. However, the derivative security pricing is still discrete in nature. The other difference is that we assume that for a given regime our model is similar to the binomial tree model. We focus on the binomial tree model because it is a simple enough instrument that provides a clear understanding of the derivative security pricing by no-arbitrage methods.

In general, the regime switching makes the market models incomplete [57]. The main objective of this work is to find under what conditions, if any, the regime switching model with jumps is complete. We demonstrate that the size of stock price jump can serve as a unique instrument to eliminate arbitrage. Moreover, we show that under certain conditions on the jump size and the initial regime the regime switching model with jumps is complete. Furthermore, we find the unique equivalent martingale measure and demonstrate how to price derivative securities in such instances. We also show that under some conditions on the initial regime the regime switching model with jumps is incomplete and there exist an infinite family of equivalent martingale measures. In this case we use the Esscher, or size-biased, transform to select a particular martingale measure from the family of measures. The Esscher transform was introduced into mathematical finance by Gerber and Shiu [22] as one of the ways for dealing with incompleteness. We provide a brief discussion of other methods proposed and give additional economical justifications for choosing the Esscher transform for our model. Furthermore, we discus the continuity of the new equivalent martingale measure (obtained using the Esscher transform) in its parameters, namely, the historic probability measure, the jump size and the Esscher parameter. We demonstrate that when parameters of the model converge to some specific values making the incomplete model a complete one, the measure obtained using the Esscher transform converges to the unique equivalent martingale measure obtained for a corresponding complete model.

Chapter 2: Definitions

We begin with the statement of some definitions from probability theory, analysis, and financial mathematics that will be used later in this work.

2.1 Probability theory

Definition 1. *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is an ordered triple comprised of a nonempty set Ω (a sample space), a σ-field $\mathcal{F}$ of subsets of Ω (where the elements of $\mathcal{F}$ are called measurable sets) and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfies the following properties:

- $\mathbb{P}(\varnothing) = 0$,
- $\mathbb{P}(\Omega) = 1$,
- $\mathbb{P}\Big(\cup_{i=1}^{\infty} A_i \Big) = \sum_{i=1}^{\infty} \mathbb{P}\,(A_i)$, $A_i \in \mathcal{F}$, $A_i \cap A_j = \varnothing$ for $i \neq j$.

Definition 2. A real-valued *random variable* is a function $X : \Omega \to \mathbb{R}$ such that $\{X \in B\} \equiv X^{-1}(B) \in \mathcal{F}$, $\forall B \in \mathcal{B}_{\mathbb{R}}$ where $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-field of subsets of $\mathbb{R}$ which contains all open sets (the Borel σ-field). We write $X \in \mathcal{F}$ for a random variable X to mean that X is measurable with respect to $\mathcal{F}$. More generally, if $(S, \mathcal{S})$ is a measurable space then $X : \Omega \to S$ such that $\{X \in B\} \in \mathcal{F}$, $\forall B \in \mathcal{S}$ is an S-valued random variable.

Definition 3. Let X be S-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the induced probability measure $\mathbb{P}_X$ on $(S, \mathcal{S})$ defined by $\mathbb{P}_X\,(B) = \mathbb{P}\,(X \in B)$, for $B \in \mathcal{S}$, is called the *distribution* of X.

Definition 4. Let Ω be a nonempty set. *A discrete time filtration* is a sequence of σ-fields $\mathcal{F}_0, \mathcal{F}_1, ...\mathcal{F}_n$ such that each σ-field in the sequence contains all the sets contained in the previous σ-field. *A continuous time filtration* is a family of σ-fields $\{\mathcal{F}_t \mid t \in [0, T]\}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s < t < T$.

Definition 5. *A stochastic process* is a family of random variables $\{X_t : t \in I\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a set S with a σ-field $\mathcal{S}$. I is called the index set, and S is a state space. A stochastic process $\{X_t\}$ is said to be *adapted* to a filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for each $t \in I$.

Definition 6. Let $\{\mathcal{F}_t\}$ be a filtration under $\mathcal{F}$ and $\{X_t\}$, $t \in I = [0, T]$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\{X_t\}$ is said to be *a Markov process* if

- $\{X_t\}$ is adapted to the filtration $\{\mathcal{F}_t\}$,
- For $s < t$, the distribution of X_t conditioned on $\mathcal{F}_s$ is the same as the distribution of X_t conditional on X_s.

A Markov process with a discrete state space is called *a Markov chain*.

Definition 7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\{\mathcal{F}_t : 0 \leqslant t \leqslant T\}$ be a filtration. A stochastic process $\{M_t\}$ is a *Martingale* with respect to the filtration $\mathcal{F}_t$ if and only if for $0 \leqslant t \leqslant T$,

- $\mathbb{E}\left[\mid M_t \mid\right] < \infty$,
- $\{M_t\}$ is adapted to $\{\mathcal{F}_t\}$,
- $\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = M_s$, for all $0 \leqslant s \leqslant t \leqslant T$.

Definition 8. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures on the same measure space $(\Omega, \mathcal{F})$. Then the probability measure $\mathbb{P}$ is said to be *absolutely continuous* with respect to the probability measure $\mathbb{Q}$, if $\mathbb{P}(A) = 0$ for every set A for which $\mathbb{Q}(A) = 0$. We write $\mathbb{P} \ll \mathbb{Q}$.

Definition 9. Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* on measure space $(\Omega, \mathcal{F})$ if the two measures are absolutely continuous with respect to each other, i.e., $\mathbb{P} \sim \mathbb{Q} \Longleftrightarrow \mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$. In other words, equivalent probability measures agree which events have probability zero: $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$.

Definition 10. Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures on measure space $(\Omega, \mathcal{F})$, and let Z be an almost surely positive random variable such that $\forall A \in \mathcal{F}$:

$$\mathbb{Q}(A) = \int_A Z d\mathbb{P}.$$

The random variable Z is called the *Radon-Nikodym derivative* of the measure $\mathbb{Q}$ with respect to the measure $\mathbb{P}$, and is written as $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. For $0 \leq t \leq T$, the process $Z_t = \mathbb{E}[Z|\mathcal{F}_t]$ is called the Radon-Nikodym derivative process.

2.2 Mathematical finance

A *derivative security* is a financial security, such as an option, whose characteristics and value depend on the characteristics and value of an *underlying security*. The most common underlying assets include stocks, bonds, commodities, currencies, interest rates and market indexes. The underlying security must be delivered once the derivative security is exercised. Stock options, warrants, and stock rights have an underlying security in the form of common stock. Derivatives are contracts and can also be used as an underlying asset. Futures contracts, forward contracts, options and swaps are the most common types of derivatives.

Stock is a type of security that signifies ownership in a corporation and represents a claim on part of the corporation's assets and earnings. Stocks sometimes are referred as shares or equity. Stocks are the foundation of nearly every portfolio.

A *risk-free* asset is an asset that has a certain future return.

A *bond* is a debt investment in which an investor loans money to an entity (corporate or governmental) that borrows the funds for a defined period of time at a fixed interest rate. Bonds are used by companies, municipalities, states and U.S. and foreign governments to finance a variety of projects and activities. Bonds are commonly referred to as fixed-income securities.

Let B_n be the bond price at time $n \in \mathbb{N}_0$. Here notation $\mathbb{N}_0$ is used for the set of natural numbers with 0. The standard model for the discrete time evolution of bond price is described by the deterministic growth at the risk-free interest rate $r \geq 0$, i.e.

$$B_{n+1} - B_n = rB_n, \quad \forall n \in \mathbb{N}_0. \tag{2.1}$$

Let $r + 1 = R \geq 1$ then we can rewrite (2.1) as follows: $B_{n+1} = RB_n, \forall n \in \mathbb{N}_0$.

A *call* option is a contract which gives the holder the right not the obligation to purchase an underlying asset at or before a specified date for a specified price. A *put* option is a contract which gives the holder the right not the obligation to sell an underlying asset at or before a specified date for a specified price. The specified date is known as the expiration date or expiry, and the specified price is often called the exercise price. There are many types of option. Two most common are the American and the European options. American option can be exercised at any time prior to and at expiry. European option can be exercised only at the expiry time.

Hedging is a strategy aimed to reduce the investment risk of adverse price movements in an asset by using derivatives, such as options contracts. In other words, hedging strategy reduces the risk by taking advantage of correlations between the asset and option price movements. To hedge a derivative security investors purchase opposite positions in the market in order to ensure a certain amount of gain or loss on a trade. A perfect hedge reduces investor's risk to nothing.

Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let $S = \{S_t : t = \overline{1,T}\}$ be a $K+1$ dimensional stochastic process of underlying *asset prices*, such that $S_t = (S_t^0, S_t^1, \cdots, S_t^K)$, with S_t^i represents the price of the security i at time t. The number $K+1$ represents the number of securities traded in the market. We specify a fixed time horizon T, at which all trading terminates.

Definition 11. A *trading strategy* is a predictable vector process $\phi = \{\phi_t : t = \overline{1,T}\}$ where $\phi_t = (\phi_t^0, \phi_t^1, \cdots, \phi_t^K)$. Here ϕ_t^i denotes the number of units of the asset i held by the investor between times $t-1$ and t. The vector ϕ_t is called the investor's *portfolio* at time t. Vector ϕ is predictable, i.e., the investor selects the time t portfolio ϕ_t after observing the asset prices S_{t-1} at time $t-1$, but before observing the asset prices S_t at time t.

Definition 12. A trading strategy ϕ is *self-financing* if $\phi_t S_t = \phi_{t+1} S_t$, $t = \overline{1, T-1}$. Here $\phi_t S_t = \sum_{i=0}^{K} \phi_t^i S_t^i$. This definition requires that funds can neither be withdrawn from or added to the value of the portfolio at any trading time $t = \overline{1, T-1}$. A more general trading strategy allows the addition or withdrawal of funds. If ϕ is self-financing, then all changes in the portfolio are results of the net gains or losses realized on investments.

Definition 13. *Value process* $V(\phi)$ is the value process for trading strategy ϕ, defined as

$$V_t(\phi) = \begin{cases} \phi_t S_t, & t = \overline{1,T}; \\ \phi_1 S_0, & t = 0. \end{cases}$$

The value process $V(\phi)$ represents the market value of the portfolio prior to time t transaction.

Definition 14. A trading strategy ϕ is called *admissible* if it is self-financing and $V(\phi)$ is a non-negative process, i.e., $V(\phi) \geq 0$.

Arbitrage is a trading strategy that guarantees riskless plan for making non-zero profit by exploiting price differences of identical or similar financial instruments on different markets or in different forms. In other words, arbitrage is a admissible trading strategy ϕ such that $V_0(\phi) = 0$ and $\mathbb{E}[V_T(\phi)] > 0$. For example, simultaneous purchase and sale of an asset in order to profit from a difference in the price is an arbitrage. Arbitrage exists as a result of market inefficiencies. It provides a mechanism to ensure that prices do not deviate substantially from the fair value for long periods of time. Given the advancement in technology it has become extremely difficult to profit from mispricing in the market. Many traders have computerized trading systems set to monitor fluctuations in similar financial instruments. Any inefficient pricing setups are usually acted upon quickly and the opportunity is often eliminated in a matter of seconds.

Definition 15. *Contingent claim* is a non-negative random variable X that represents a contract or an agreement, and pays $X(\omega)$ dollars at time T if state ω occurs. In other words, a contingent claim is a claim that can be made when certain specified outcomes occur.

Definition 16. A contingent claim X is said to be *attainable* if there exist some self-financing trading strategy ϕ such that $V_T(\phi) = X$. We say ϕ generates X.

Definition 17. Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$, equivalent to the probability measure $\mathbb{P}$ (the market measure). Let D_t be a *discount process*, such that in the discrete case, $D_t = (1+r)^t$, where $r > 0$ is a constant risk-free interest rate. Let S_t be a price of the stock at time t. If the discounted stock price $\left\{\frac{S_t}{D_t}\right\}$ is a martingale under $\mathbb{Q}$, then the measure $\mathbb{Q}$ is *a risk neutral measure*, or *an equivalent martingale measure*.

Definition 18. An arbitrage-free market model is said to be *complete* for a time horizon T if every contingent claim X with expiry T is attainable by an admissible strategy.

To *sell short* is the selling of a security that the seller does not own, or any sale that is completed by the delivery of a security borrowed by the seller. Short sellers assume that they will be able to buy the security at a lower amount than the price at which they sold short. This is an advanced trading strategy with many unique risks and pitfalls.

A statistical measure of the dispersion of returns for a given security or market is called *volatility*. In other words, volatility is a measure for variation of price of a financial instrument over time. There are several ways to measure the volatility. Volatility can either be measured by using the standard deviation or variance between returns from that same security or market. The higher the volatility, the riskier the security.

Chapter 3: Binomial tree model

In this chapter, we discuss one of the most notable models that was introduced by Cox, Ross, and Rubinstein in 1979 [6]. They proposed binomial tree model (or binomial option pricing model) that is a discrete time model for valuing options. The model is based on the evolution of the price of the option's underlying asset over a period of time. The binomial tree is constructed so that each node in the tree corresponds to a possible price of the underlying asset at a given point in time. In Fig. 3.1 we give an example of such binomial tree. Note that the tree is recombining. Authors applied the concept of replication portfolio in multiple-period binomial model, and showed that the model is complete and every derivative security has a unique and fair price [7]. Moreover, they showed that the Black-Scholes model is a special limiting case of the binomial tree model.

The framework is simple from a mathematical perspective but it provides a powerful tool to understand the arbitrage pricing theory, and it allows clearly demonstrate the relationship between no-arbitrage and risk-neutral pricing. Given the sufficient number of periods the discrete model also provides computationally tractable approximation to the continuous time model.

We start with the definition of the binomial tree model.

3.1 Binomial tree model description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space, upon which all the stochastic processes are defined. Consider the N-period model with the initial time $n = 0$ and the end time $n = N$. Let initial price of an underlying security, e.g., stock, be $S_0 > 0$. The binomial tree model assumes that at each unit of time the price of the stock may increase by an up factor $u > 0$ with probability p_u or may decrease by a down factor d, $0 < d < u$ with probability $p_d = 1 - p_u$. The stock price at times $1 \leq n \leq N$ can be described as following:

$$S_n = S_0 \prod_{i=1}^{n} Y_i, \tag{3.1}$$

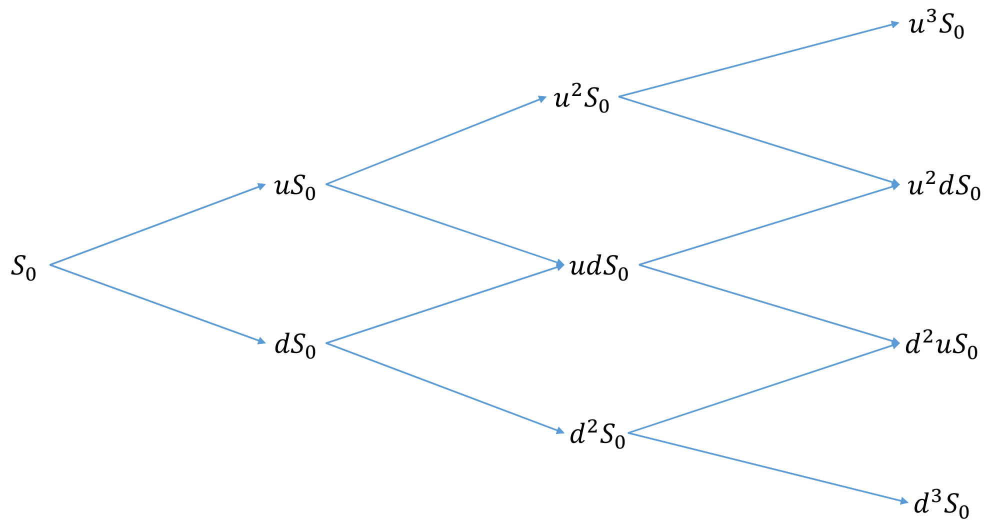

Figure 3.1: Example of the stock price dynamics for three-period binomial tree model.

where Y_i, $i = \overline{1, N}$ is an i.i.d. *factor process* such that $\mathbb{P}(Y_i = u) = p_u$ and $\mathbb{P}(Y_i = d) = p_d$. The probability measure $\mathbb{P}$ is called *the historical probability measure*.

Assume also that risk-free assets are available. Suppose there are US Treasure bonds with initial price B_0 and price at time n given by

$$B_n = (r+1)^n B_0 = R^n B_0, \tag{3.2}$$

where $r > 0$ is a risk-free interest rate and $R = r + 1 \geq 1$. In this work we use terms investments into the bonds and investments in the money market interchangeably.

Now we need to make some principal assumptions about the model. First, we assume that the interest rate r is constant. Moreover, we assume that interest rates for borrowing and investing are the same. To focus on the basic issues, we also postulate that there are no taxes, transaction costs, and margin requirements. Besides, individuals are allowed to sell short any security. We also assume that shares of stock can be subdivided for sale or purchase. Furthermore, we assume that the purchase price of the stock is the same as the selling price.

In the next section, we demonstrate that the binomial tree model is free of arbitrage.

3.2 Binomial tree model is arbitrage-free

One of the necessary features of an efficient market is the absence of arbitrage. Next proposition provides conditions for the binomial tree model to be arbitrage-free.

Proposition 1. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *be given. Let* N*-period binomial tree market model* (S_n, B_n) *be given, where* S_n *is a collection of stock prices defined by formula (3.1) and* B_n *is a risk-free money market process, defined by formula (3.2). Let the up factor* u *and the down factor* d *be given such that* $u > 1$ *and* $0 < d < 1$*. Let* $R = r + 1 > 1$*, where* $r > 0$ *is the risk-free interest rate. If*

$$0 < d < R < u \tag{3.3}$$

then the binomial tree model is arbitrage-free.

Proof. Inequality $d > 0$ is necessary for the positivity of the stock price. We will prove the other two inequalities by showing that if at least one of them is not satisfied then there appears an arbitrage opportunity. First, assume that $d \geq R$. In this case, at time zero an agent might borrow money from the money market and buy stock. Even at the worst case scenario, the stock price at time one will allow the agent not only to pay off his money market debt (if $d = R$), but also to have a nonzero profit (if $d > R$). Therefore, condition $d \geq R$ provides an arbitrage opportunity. Now, assume that $u \leq R$. In this case, the agent can sell the stock short and invest the money into the money market. Even at the best case scenario, the stock price at time one will be smaller (if $u < R$) or equal (if $u = R$) than the value of the investment into the money market. Therefore, condition $u \leq R$ also provides an arbitrage opportunity. Thus, condition $0 < d < R < u$ eliminates all arbitrage opportunities. $\square$

Note, that although simple, inequalities (3.3) provide a strict no-arbitrage condition: if inequalities (3.3) did not hold, there would be profitable riskless arbitrage opportunities involving only the stock and riskless borrowing and lending. Since under condition (3.3) binomial tree model precludes arbitrage, by Theorem 1, there exist at least one risk neutral probability measure that is equivalent to the historic probability measure $\mathbb{P}$. Next proposition shows that this measure is unique and, therefore, the binomial tree model is complete.

3.3 Binomial tree model is complete

Proposition 2. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *be given. Let* N*-period binomial tree market model* (S_n, B_n) *be given, where* S_n *is a collection of stock prices defined by formula (3.1) and* B_n *is a risk-free money market process, defined by formula (3.2). Let the up factor* u*, the down factor* d*, and the risk-free interest rate* $r > 0$ *satisfy the no-arbitrage condition* $0 < d < R < u$*, where* $R = r + 1$*. Then there exist a probability measure* $\mathbb{Q}$*, defined as follows*

$$\mathbb{Q}(Y_i = u) = q = \frac{R-d}{u-d} \qquad \mathbb{Q}(Y_i = d) = 1 - q = \frac{u-R}{u-d}, \tag{3.4}$$

such that the discounted stock price is a martingale under measure $\mathbb{Q}$*. Moreover, measure* $\mathbb{Q}$ *is a unique risk-neutral measure equivalent to the historic probability measure* $\mathbb{P}$*. Hence, the binomial tree model is complete.*

Proof. The complete proof of the Proposition 2 can be found in [54]. Here we provide a proof for one-period binomial tree model.

Note first that the discounted stock price is a martingale under the new measure $\mathbb{Q}$:

$$\begin{aligned} S_0 &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_1(Y_1) \mid Y_1\right] \\ &= \frac{1}{R}\left[qS_1(u) + (1-q)S_1(d)\right] \qquad (3.5) \\ &= \frac{1}{R}\left(uS_0\frac{R-d}{u-d} + dS_0\frac{u-R}{u-d}\right) = S_0. \qquad (3.6) \end{aligned}$$

Moreover, the equivalence of measure $\mathbb{Q}$ to the historical measure $\mathbb{P}$ follows from the no-arbitrage conditions. The uniqueness of measure $\mathbb{Q}$ follows from the equation (3.5): there exist only one measure given by q and $1-q$ that satisfies equation (3.5). ☐

The N-period binomial tree model is complete, therefore, every derivative security can be replicated by trading in the underlying stock and money market. Moreover, under this model, every derivative security has a unique price that precludes arbitrage. Consider, for example, an European call option, which gives a holder the right but not the obligation to buy one share of stock at the strike time $N \in \mathbb{N}$ for the strike price K. In other words, this option has a payoff $V_N = (S_N - K)_+$ at strike time $N \in \mathbb{N}$. One of the fundamental challenges is to determine the initial fair price of the option. Next proposition provides a core formula for pricing the derivative securities under the binomial tree model.

3.4 Pricing of the derivative security under the binomial tree model

The no-arbitrage approach to pricing derivative securities is based on the replication of the payoff of the derivative security by a synthetic portfolio. This portfolio is constructed using the money market and the underlying security such that the portfolio value matches the derivative security payoff at each time step.

Definition 19. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space. Define $V_n(Y_1, Y_2, \cdots, Y_n)$ to be the price of the derivative security at time $n = \overline{1, N}$. Denote the price of the derivative security at time zero as V_0.

Proposition 3. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let N-period binomial tree market model (S_n, B_n), $n = \overline{1, N}$ be given, where S_n is a collection of stock prices defined by formula (3.1) and B_n is a risk-free money market process, defined by formula (3.2). Let the up factor u, the down factor d, and the risk-free interest rate $r > 0$ satisfy the no-arbitrage condition $0 < d < R < u$, where $R = r + 1$. Let the risk-neutral probability measure $\mathbb{Q}$, defined by formulas (3.4), be given. Let the random variable $V_n(Y_1, Y_2, \cdots, Y_n)$ be a price of the derivative security at time n. Then the discounted prices of the derivative security is a martingale under the measure $\mathbb{Q}$:*

$$V_n\left(Y_1, Y_2, \cdots, Y_n\right) = \mathbb{E}_{\mathbb{Q}}\left[R^{-1} V_{n+1}\left(Y_1, Y_2, \cdots, Y_n, Y_{n+1}\right) \mid Y_1, Y_2, \cdots, Y_n\right], \tag{3.7}$$

and the initial price of the derivative security can be computed as follows

$$V_0 = \mathbb{E}_{\mathbb{Q}}\left[R^{-n} V_n(Y_1, Y_2, \cdots, Y_n)\right], \forall n = \overline{1, N}. \tag{3.8}$$

Moreover, if we set $W_0 = V_0$ and define recursively forward in time the portfolio values

$W_1, W_2, \cdots, W_N$ by

$$W_n = \phi_{n-1} S_n + R\left(W_{n-1} - \phi_{n-1} S_{n-1}\right), \tag{3.9}$$

where

$$\phi_{n-1}\left(Y_1, Y_2, \cdots, Y_{n-1}\right) = \frac{V_n\left(Y_1, Y_2, \cdots, Y_{n-1}, u\right) - V_n\left(Y_1, Y_2, \cdots, Y_{n-1}, d\right)}{S_n\left(Y_1, Y_2, \cdots, Y_{n-1}, u\right) - S_n\left(Y_1, Y_2, \cdots, Y_{n-1}, d\right)} \tag{3.10}$$

is the number of shares of stock held by a portfolio at time $n-1$, then

$$W_n\left(Y_1, Y_2, \cdots, Y_n\right) = V_n\left(Y_1, Y_2, \cdots, Y_n\right), \forall n = \overline{1, N}. \tag{3.11}$$

In other words, the derivative security is replicated $\forall n = \overline{1, N}$.

Proof. The complete proof of the Proposition can be found in [54]. Here we provide a for one-period binomial tree model.

Note that $V_0 = W_0$ by the definition of W_0. The objective is to show that $V_1 = W_1$. We start by demonstrating that $V_1(u) = W_1(u)$.

$$\begin{aligned} W_1(u) &= \phi_0 S_1(u) + R\left(W_0 - \phi_0 S_0\right) \\ &= \frac{V_1(u) - V_1(d)}{S_0(u-d)} S_0(u-R) + RV_0 \\ &= \frac{V_1(u) - V_1(d)}{(u-d)}(u-R) + \frac{V_1(u)(R-d)}{u-d} + \frac{V_1(d)(u-R)}{u-d} \\ &= V_1(u). \end{aligned} \tag{3.12}$$

Similarly one can show that $V_1(d) = W_1(d)$. Note now that discounter W_1 is a martingale with respect to the equivalent measure $\mathbb{Q}$:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{R} W_1\right] &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[\phi_0 S_1(u) + R\left(W_0 - \phi_0 S_0\right)\right] \\ &= S_0\phi_0 + W_0 - S_0\phi_0 \\ &= W_0. \end{aligned} \tag{3.13}$$

Since $W_0 = V_0$ and $V_1 = W_1$, it follows that discounted V_1 is also a martingale with respect to the measure $\mathbb{Q}$. $\square$

Formulas (3.7)-(3.11) permit an agent to hedge a short position in the derivative security. The derivative security that pays amount V_N at time N should be priced at time $n = 0$ by formula (3.8), that is called the *risk-neutral pricing formula* for the N-period binomial model. The initial price of the derivative security given by formula (3.8) does not introduce an arbitrage since the hedge works regardless of whether the price of the stock goes up or down. Any other price at $n = 0$ would introduce an arbitrage. Formula (3.10) is called the *delta − hedging formula* and gives the number of shares of stock that should be held by a portfolio at time n. Note that number of shares of stock ϕ_n changes during the life of an option: it must be adjusted from one time n to another. Formula (3.9) defines a *wealth equation*: it gives the value of the portfolio at time n, $\forall n = \overline{1, N}$. Synthetic portfolio constructed using formula (3.9) matches the payoff of the derivative security, as shown in equation (3.11).

Proposition 3 provides an algorithm for computing the initial price V_0 of the derivative security: the price can be computed recursively backward in time by the formula (3.7).

Note also that risk-neutral measure $\mathbb{Q}$ has the property that at any time n, the price of the stock is the discounted risk-neutral average of its two possible prices at the next time:

$$\begin{aligned} S_n(Y_1, Y_2, \cdots, Y_n) &= \frac{1}{R}\left[qS_{n+1}(Y_1, Y_2, \cdots, Y_n, u) + (1-q)S_{n+1}(Y_1, Y_2, \cdots, Y_n, d)\right] \\ &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_{n+1}(Y_1, Y_2, \cdots, Y_n, Y_{n+1}) \mid Y_1, Y_2, \cdots, Y_n\right], \qquad (3.14) \end{aligned}$$

i.e., under the risk-neutral probabilities, the mean rate of return for the stock is r, the same as the rate of return for the money market. Under the historic probability measure $\mathbb{P}$, however, the average rate of growth of the stock is typically strictly greater that the rate of growth of an investment of the money market:

$$S_n(Y_1, Y_2, \cdots, Y_n) < \frac{1}{R}\left[p_u S_{n+1}(Y_1, Y_2, \cdots, Y_n, u) + p_d S_{n+1}(Y_1, Y_2, \cdots, Y_n, d)\right]. \qquad (3.15)$$

This is the main reason why agents want to incur the risk associated with the investing in the stock. Measure $\mathbb{Q}$, however, makes the mean rate of growth of the stock to be equal to the rate of growth of the money market account, and hence makes the mean rate of growth of any portfolio of stock and money market account to be equal to the rate of growth of the money market account.

The properties (3.7) and (3.14) are refereed to as the *martingale property*, since the discounted prices $R^{-n}S_n(Y_1, Y_2, \cdots, Y_n)$ and $R^{-n}V_n(Y_1, Y_2, \cdots, Y_n)$ are martingales under the measure $\mathbb{Q}$. Such measure $\mathbb{Q}$ is also referred as an *equivalent martingale measure* (EMM).

Note also that the historical probabilities p_u, p_d do not appear in the pricing formula. The key reason is that the derivative security value is calculated in terms of the underlying stock. The probabilities of future up and down movements are already incorporated into the price of the stock. In other words, the prices of derivative securities depend on the set of possible stock price paths but not on how probable these paths are, i.e., the actual historic probabilities are irrelevant.

In the next section, we provide a numerical example of pricing a derivative security using Proposition 3.

3.5 Example

Proposition 3 can be applied to the derivative securities whose yield depends only on the final price of the stock as well as to the path-dependent options. In this section, we find a fair price of a path-dependent option. Recall the following definition.

Definition 20. *Path − dependent option* is a contract that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price during a specified time period, where the price is based on the fluctuations in the underlying asset's value during all or part of the contract term. A path dependent option's payoff is determined by the path of the underlying asset's price.

Example 1. Consider path-dependent lookback put option [54], [5]. The lookback put option payoff function is given as

$$V_T = \max_{0 \leq n \leq T} S_n - S_T,$$

where $\max_{0 \leq n \leq T} S_n$ is the asset's maximum price during the life of the option and S_T is the underlying asset's price at maturity time T. Let $T = 3$ and $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$. Let the interest rate be $r = \frac{1}{4}$. Our objective is to find the no-arbitrage price of the option at tine $n = 0$.

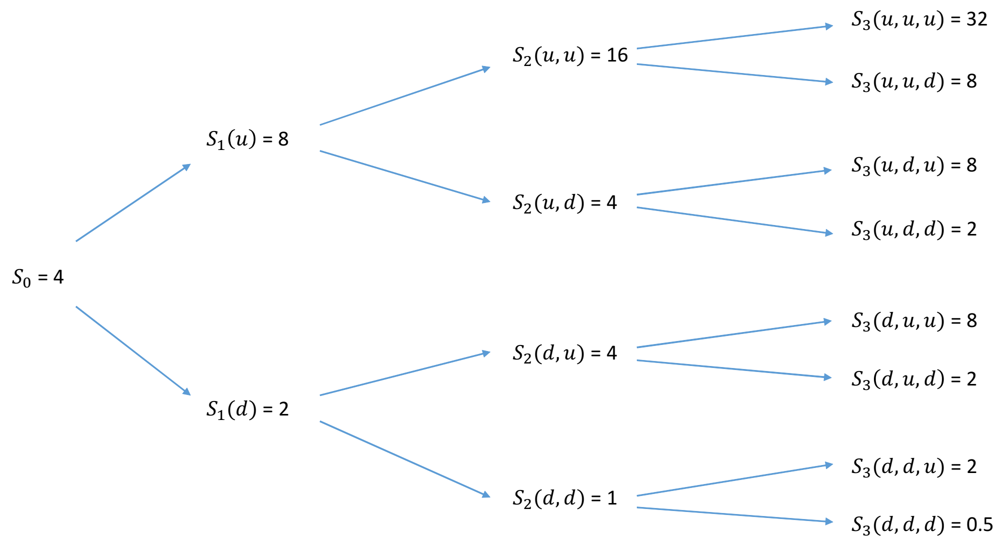

Figure 3.2: Stock price dynamics for three-period binomial tree model.

First, consider the stock price dynamics for three-period binomial tree model. Since the initial price of the stock is $S_0 = 4$, and the up-factor and the down-factor are given as $u = 2$ and $d = \frac{1}{2}$, respectively, we can calculate the price of the stock for $n = \{1, 2, 3\}$. In Fig. 3.2 we depict all possible stock prices for $n = \{0, 1, 2, 3\}$. Now, we can calculate the lookback option payoffs at time $T = 3$:
$V_3(u,u,u) = S_3(u,u,u) - S_3(u,u,u) = 32 - 32 = 0,$
$V_3(u,u,d) = S_2(u,u) - S_3(u,u,d) = 16 - 8 = 8,$
$V_3(u,d,u) = S_1(u) - S_3(u,d,u) = 8 - 8 = 0,$
$V_3(u,d,d) = S_1(u) - S_3(u,d,d) = 8 - 2 = 6,$
$V_3(d,u,u) = S_3(d,u,u) - S_3(d,u,u) = 8 - 8 = 0,$
$V_3(d,u,d) = S_2(d,u) - S_3(d,u,d) = 4 - 2 = 2,$
$V_3(d,d,u) = S_0 - S_3(d,d,u) = 4 - 2 = 2,$
$V_3(d,d,d) = S_0 - S_3(d,d,d) = 4 - 0.5 = 3.5.$

Note, that $R = 1 + r = \frac{5}{4}$, and $0 < d < R < u$, i.e., the no-arbitrage condition is satisfied. Therefore, by Proposition 2, there exist a unique equivalent martingale

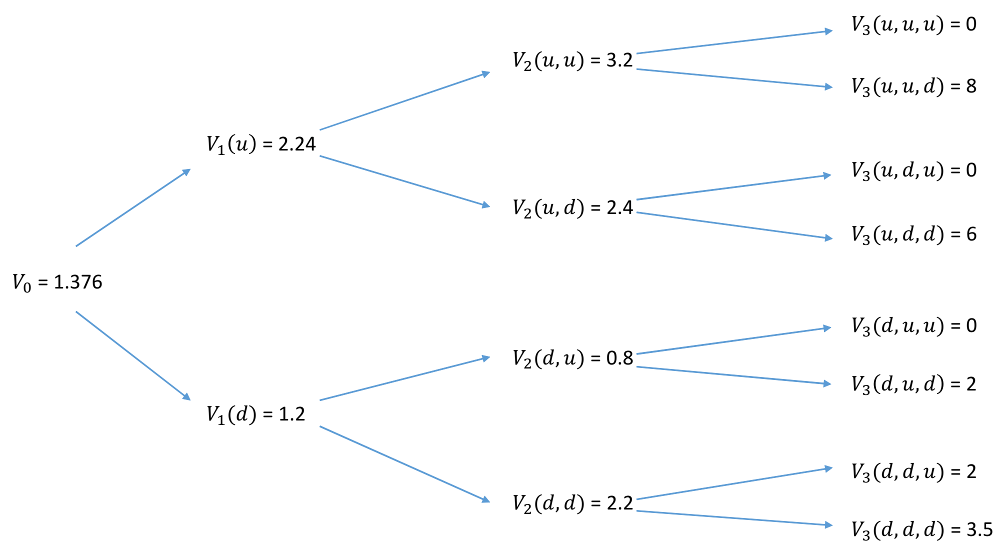

Figure 3.3: Price of the option for three-period binomial tree model.

measure $\mathbb{Q}$, defined as

$$q = \frac{u-R}{u-d} = \frac{2-1.25}{2-0.5} = \frac{1}{2},$$
$$1-q = \frac{R-d}{u-d} = \frac{1.25-0.5}{2-0.5} = \frac{1}{2}.$$

Now, using formula (3.7) from the Proposition 3, we can calculate recursively backward in time the price of the option at time $n=2$ as follows

$V_2(u,u) = \frac{4}{5}\left[\frac{1}{2}V_3(u,u,u) + \frac{1}{2}V_3(u,u,d)\right] = 3.2,$

$V_2(u,d) = \frac{4}{5}\left[\frac{1}{2}V_3(u,d,u) + \frac{1}{2}V_3(u,d,d)\right] = 2.4,$

$V_2(d,u) = \frac{4}{5}\left[\frac{1}{2}V_3(d,u,u) + \frac{1}{2}V_3(d,u,d)\right] = 0.8,$

$V_2(d,d) = \frac{4}{5}\left[\frac{1}{2}V_3(d,d,u) + \frac{1}{2}V_3(d,d,d)\right] = 2.2.$

Similarly, using backward recursion, we calculate the price of an option at time $n=1$ as follows

$V_1(u) = \frac{4}{5}\left[\frac{1}{2}V_2(u,u) + \frac{1}{2}V_2(u,d)\right] = 2.24,$

$V_1(d) = \frac{4}{5}\left[\frac{1}{2}V_2(d,u) + \frac{1}{2}V_2(d,d)\right] = 1.2.$

Finally, we compute the price of the lookback option at time $n = 0$:
$V_0 = \frac{4}{5}\left[\frac{1}{2}V_1(u) + \frac{1}{2}V_1(d)\right] = 1.376.$

In Fig. 3.3 we depict the price of the option for $n = \{0, 1, 2, 3\}$. Notice that the initial price of the option $V_0 = 1.376$ is fair and allows the seller of the options to hedge his short position in the option. Consider an agent, who sells this option at time $n = 0$ for 1.376 dollars. The agent can buy

$$\phi_0 = \frac{V_1(u) - V_1(d)}{S_1(u) - S_1(d)} = \frac{2.24 - 1.2}{8 - 2} = 0.1733$$

shares of stock for $0.1733 \times 4 = 0.6932$ dollars, and invest the remainder $1.376 - 0.6932 = 0.6828$ in the money market account with $r = \frac{1}{4}$ interest rate. This portfolio matches the payoff of the option on every time step and for every possible stock price path. Consider, for example, at time $n = 1$ the amount 0.6828 invested into the money market will yield 0.8533 dollars. If the stock price goes up, then the stock will cost 1.3876 dollars. So, the agent's total portfolio value will be 2.24 dollars, which matches $V_1(u)$. Similarly, if the stock price goes down, then at time $n = 1$ the stock will cost 0.3467 dollars, and total portfolio value will be 1.2 dollars, which matches $V_1(d)$. In other words, agents portfolio replicates the option at every time independently of the stock price paths, i.e., if a path is possible (has positive probability), then the hedge will work along this path.

This chapter has provided an introduction to the binomial tree model. Using only no-arbitrage argument, model was shown to be complete. Moreover, we provided a way to price any derivative security under the binomial tree model. It is interesting to note that no assumptions were required about the historic probabilities of up and down movements in the stock price. Example 1, provided in Section 3.5, demonstrated how to calculate the fair price of an option. More information on pricing derivative securities under the binomial tree model can be found in [6], [29], [46], [54].

In the next chapter, we examine the regime switching model with jumps that is partially based on the binomial tree model.

Chapter 4: Regime switching model with jumps

In the previous chapter, we discussed the binomial tree model, one of the main assumptions of which was that at any time the stock price could take only two possible values. This model provides rather simple framework for derivative security price analysis. In reality, pricing process is subject to many uncertain changes and abrupt economical disturbances. In this chapter, we consider a regime switching pricing model with jumps that takes into account those changes, by allowing larger moves in asset prices (jumps) caused by sudden world events or by the market responses to those events (regime switch) [31]. Suppose, for example, The Federal Reserve System lowers interest rates. This will result in an economical regime switch: it will be cheaper for people to borrow money. The more money businesses and consumers spend, the better it is for the economy. Hence, lowering interest rates will often result in the stock market going up, i.e., will result in a price jump of the stocks.

In this chapter, we introduce the regime switching model with jumps and investigate the completeness of this model. Our objective is to answer several questions: (1) under what conditions the model is arbitrage-free; (2) under what conditions, if any, there exist a unique equivalent martingale measure for a finite time horizon, i.e., completeness; (3) is the price of an underlying asset (e.g. stock price) a martingale under this measure; (4) how to price a derivative security under the regime switching model with jumps?

We start by mathematical definition of the model.

4.1 Model description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space, upon which all the stochastic processes are defined.

Definition 21. Let the *regime process* σ_n, $n = \overline{0, N}$ be defined as a finite-state stochastic process such that $\sigma_0, \sigma_1, \cdots$, are independent and identically distributed random variables with the state space $\{-1, +1\}$. The states $\{-1, +1\}$ could be interpreted as the regimes (or states) of the economy, e.g., growing economy or decreasing economy.

The initial distribution of regime process σ_n is given as follows

$$\mathbb{P}(\sigma_0 = +1) = p_0 \quad , \quad \mathbb{P}(\sigma_0 = -1) = 1 - p_0, \tag{4.1}$$

and the distribution for time n is given as

$$\mathbb{P}(\sigma_n = +1) = p_+ \quad , \quad \mathbb{P}(\sigma_n = -1) = 1 - p_+, \tag{4.2}$$

where $p_0 \in [0, 1]$ and $p_+ \in (0, 1)$ $\forall n = \overline{1, N}$. When for some $n = \overline{1, N}$ $\sigma_{n-1} = +1$ and $\sigma_n = -1$ or $\sigma_{n-1} = -1$ and $\sigma_n = +1$ we say that regime switches from one state to another. Note that, in general, we do not know a priory when, if ever, regime switching will occur.

Definition 22. Define the *jump process* $J_{\sigma_n - \sigma_{n-1}}$ as

$$J_{\sigma_n - \sigma_{n-1}} = \begin{cases} J_0 = 0, & \sigma_n = \pm 1, \sigma_{n-1} = \pm 1; \\ J_2 = h, & \sigma_n = +1, \sigma_{n-1} = -1; \\ J_{-2} = -h, & \sigma_n = -1, \sigma_{n-1} = +1, \end{cases} \tag{4.3}$$

where known quantity $h > 0$ represents the size of a jump.

Definition 23. Define the *jump-factor process* $g_n(\sigma_{n-1}, \sigma_n)$, $n = \overline{1, N}$ as follows

$$g_n(\sigma_{n-1}, \sigma_n) = \frac{u}{2}(\sigma_{n-1} + 1) - \frac{d}{2}(\sigma_{n-1} - 1) + J_{\sigma_n - \sigma_{n-1}}, \tag{4.4}$$

where up-factor u and down-factor d are given such that $u > 0$, $d > 0$, and $u > d$. More precisely, the values of the jump-factor process can be written as follows

$$g_n(\sigma_{n-1}, \sigma_n) = \begin{cases} u, & \text{if } \sigma_{n-1} = +1 \text{ and } \sigma_n = +1, & \text{w/p } \ p_+^2; \\ u - h, & \text{if } \sigma_{n-1} = +1 \text{ and } \sigma_n = -1, & \text{w/p } \ p_+(1 - p_+); \\ d + h, & \text{if } \sigma_{n-1} = -1 \text{ and } \sigma_n = +1, & \text{w/p } \ (1 - p_+)p_+; \\ d, & \text{if } \sigma_{n-1} = -1 \text{ and } \sigma_n = -1, & \text{w/p } \ (1 - p_+)^2. \end{cases} \tag{4.5}$$

Assume now, that the initial price of the stock S_0 is known and $\mathbb{P}-$a.s. is a constant, i.e., S_0 does not depend on the initial regime σ_0 and $S_0(\sigma_0) = S_0$. The price of the stock at time $n \geq 1$ is defined as follows:

$$S_n(\sigma_0, \sigma_1, \cdots, \sigma_n) = S_{n-1}(\sigma_0, \sigma_1, \cdots, \sigma_{n-1}) g_n(\sigma_{n-1}, \sigma_n) = S_0 \prod_{i=1}^{n} g_i(\sigma_{i-1}, \sigma_i). \tag{4.6}$$

Formula (4.6) shows that the price of the stock at time n is a product of the price of the stock at time $n-1$ and jump-factor process $g_n(\sigma_{n-1}, \sigma_n)$. Notice that in the case of the regime switch $\sigma_{n-1} = +1$ and $\sigma_n = -1$, the value of the jump-factor process $u - h$ is simply an up-factor u adjusted by the jump h. Similarly, in the case of the regime switch $\sigma_{n-1} = -1$ and $\sigma_n = +1$, the value of the jump-factor process $d + h$ is simply a down-factor d adjusted by the jump h. The intuition behind this adjustment is based on the fact that the moments of the regime switch are usually accompanied by a sudden moves in the asset price. Thus, the up-factor u decreased by jump h and the down-factor d increased by jump h will guarantee the moves in the asset price for the corresponding change in the regimes.

Note also that formula (4.6) is similar to the stock price formula (3.1) in the binomial tree model. However, in the binomial tree model, the factor process can take only two possible values u and d, whereas in the regime switching model with jumps, the jump-factor process can take four possible values, given by formula (4.5).

More precisely, we can rewrite formula (4.6) as follows

$$S_n(\sigma_0, \sigma_1, \cdots, \sigma_n) = S_{n-1}(\sigma_0, \sigma_1, \cdots, \sigma_{n-1}) g_n(\sigma_{n-1}, \sigma_n) = \tag{4.7}$$

$$= \begin{cases} S_{n-1}(\sigma_0, \sigma_1, \cdots, \sigma_{n-1}) U(\sigma_{n-1}), & \sigma_n = +1; \\ S_{n-1}(\sigma_0, \sigma_1, \cdots, \sigma_{n-1}) D(\sigma_{n-1}), & \sigma_n = -1, \end{cases}$$

where U is an up-factor parameter and D is a down-factor parameter defined as

$$U(\xi) = \begin{cases} u, & \xi = +1; \\ d + h, & \xi = -1, \end{cases} \tag{4.8}$$

and

$$D(\xi) = \begin{cases} u - h, & \xi = +1; \\ d, & \xi = -1. \end{cases} \tag{4.9}$$

Note that the binomial tree model can be obtained from the regime switching model with jump factors by setting the up-factor process $U(\xi) = u$ and the down-factor process $D(\xi) = d$, $\forall \xi = \{+1, -1\}$.

We also assume that risk-free assets are available, e.g., US Treasure bonds with initial price B_0 and price at time n given by

$$B_n = (r+1)^n B_0 = R^n B_0, \tag{4.10}$$

where $r > 0$ is a risk-free interest rate and $R = r + 1 \geq 1$. We assume that interest rate r has a constant value that does not depend on time step n and regime process σ_n. We also assume that a derivative security is available. We denote its payoff at time n as $V_n(\sigma_0, \sigma_1, \cdots, \sigma_n)$. We also assume that

- interest rates for borrowing and investing are the same;
- there are no taxes, transaction costs, and margin requirements;
- the purchase price of the stock is the same as the selling price;
- shares of stock can be subdivided for sale or purchase;
- individuals are allowed to sell short any security.

In the next section, we consider the regime switching model with jumps under two special scenarios. The first scenario correspond to the case when the value of the regime process is known at time $n = 0$, i.e., when σ_0 is predetermined and equal to either $+1$ or -1. The second scenario corresponds to the case when σ_0 is not known. For each of the scenarios we investigate the completeness of the model, i.e., investigate the existence and uniqueness of the equivalent martingale measure for the price process $(S_0, \cdots, S_n)$. Notice that $S_k = S_0 \prod_{i=1}^{k} g(\sigma_{n-1}, \sigma_n)$ is a function of $(\sigma_0, \cdots, \sigma_k)$. The equivalent martingale measure for $(S_0, S_1, S_2, \cdots, S_n)$ will be found in terms of the distribution of $(\sigma_0, \cdots, \sigma_k)$. This leads to a slight abuse of notation that should be clear from the context. The similar convention is typically applied even to the binomial tree model.

4.2 Special case: initial regime σ_0 is known

In this section we study the regime switching model with jumps under the assumption that the initial regime σ_0 is known. We start with a simple one-period regime switching model with jumps.

4.2.1 One-period regime switching model with jumps

Suppose $\mathbb{P}(\sigma_0 = +1) = p_0 = 1$ and $\mathbb{P}(\sigma_0 = -1) = 1 - p_0 = 0$, i.e., the initial regime is known and $\sigma_0 = +1$. Therefore, at time $n = 1$ the jump-factor process can take only two possible values:

$$g_1(\sigma_0, \sigma_1) = \begin{cases} u, & \text{if } \sigma_0 = +1 \text{ and } \sigma_1 = +1, \quad \text{w/p} \;\; p_0 p_1 = p_1; \\ u - h, & \text{if } \sigma_0 = +1 \text{ and } \sigma_1 = -1, \quad \text{w/p} \;\; p_0(1 - p_1) = 1 - p_1. \end{cases} \tag{4.11}$$

And hence, the price of the stock can also take only two possible values:

$$S_1 = \begin{cases} uS_0, & \sigma_1 = +1; \\ (u - h)S_0, & \sigma_1 = -1. \end{cases} \tag{4.12}$$

In other words, given the initial regime $\sigma_0 = +1$, at time $n = 1$ the stock price becomes $S_0 u$, if $\sigma_1 = +1$ and the stock price becomes $S_0(u - h)$, if $\sigma_1 = -1$. In Fig. 4.1(a) we schematically depict the stock price dynamics for $\sigma_0 = +1$ case. Since the stock price can take only two possible values, the regime switching model with $\sigma_0 = +1$ is equivalent to the binomial tree model with up-factor $u_{BTM} = U(\sigma_0) = u$ and down-factor $d_{BTM} = D(\sigma_0) = u - h$.

Similarly, if we assume that $\sigma_0 = -1$, the price of the stock will take only two possible values:

$$S_1 = \begin{cases} (d + h)S_0, & \sigma_1 = +1; \\ dS_0, & \sigma_1 = -1. \end{cases} \tag{4.13}$$

In Fig. 4.1(b) we schematically depict the stock price dynamics for $\sigma_0 = -1$ case. Hence, the regime switching model with $\sigma_0 = -1$ is also equivalent to the binomial tree model with up-factor $u_{BTM} = U(\sigma_0) = d + h$ and down-factor $d_{BTM} = D(\sigma_0) = d$.

Therefore, next results hold.

Proposition 4. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let the one-period regime switching market model with jumps (S_n, B_n), $n = \overline{1, N}$, $N = 1$ be given, where S_n is a collection of stock prices defined by formula (4.6) and B_n is a risk-free money market process, defined by formula (4.10). Let the initial regime σ_0 be known and be equal to*

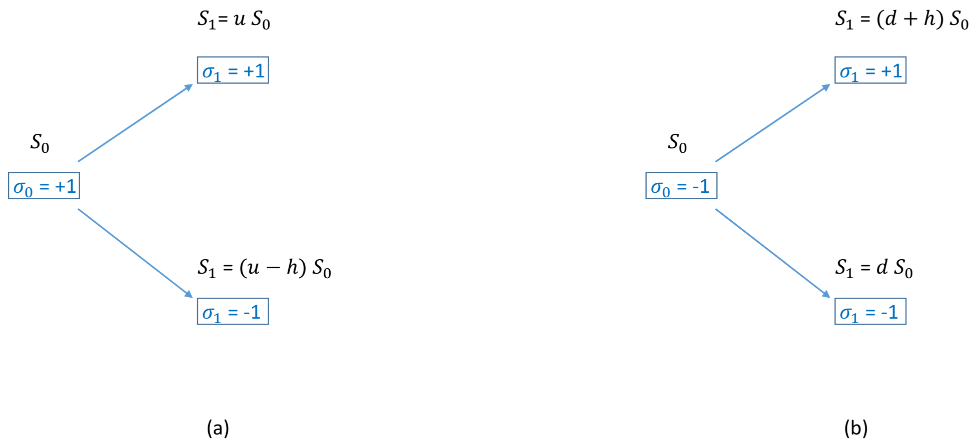

Figure 4.1: Stock price dynamics for one-period regime switching model with jumps: case (a) corresponds to $\sigma_0 = +1$, case (b) corresponds to $\sigma_0 = -1$.

+1, i.e., $\mathbb{P}(\sigma_0 = +1) = 1$*. Let the underlying asset, e.g., stock be given with the initial price* S_0*. Let* $R = r + 1$*. If* $u > R > 0$ *and the jump size* $h > u - R$ *then the no-arbitrage condition*

$$0 < u - h < R < u, \tag{4.14}$$

is satisfied and hence the one-period regime switching model with jumps is arbitrage-free. Moreover, the model is compete and there exist a unique equivalent martingale measure $\mathbb{Q}$ *given as*

$$\begin{cases} \mathbb{Q}(\sigma_0 = +1) = 1, \\ \mathbb{Q}(\sigma_0 = -1) = 0, \end{cases} \tag{4.15}$$

and

$$\begin{cases} \mathbb{Q}(\sigma_1 = +1 \mid \sigma_0 = +1) = \frac{R-(u-h)}{h}, \\ \mathbb{Q}(\sigma_1 = -1 \mid \sigma_0 = +1) = \frac{u-R}{h}, \end{cases} \tag{4.16}$$

such that the discounted stock price is a martingale with respect to the equivalent mar-

tingale measure $\mathbb{Q}$:

$$\begin{aligned} S_0 &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}S_1 \mid S_0\right] \\ &= \frac{1}{R}\left[\frac{R-(u-h)}{h}S_1(\sigma_0,\sigma_1=+1)+\frac{u-R}{h}S_1(\sigma_0,\sigma_1=-1)\right]. \end{aligned} \tag{4.17}$$

Furthermore, assume there is a derivative security that pays amount $V_1(\sigma_0, \sigma_1)$ *at time* $N = 1$. *Define* $V_0(\sigma_0)$ *to be a price of the derivative security at time* $n = 0$. *Define also*

$$\phi_0(\sigma_0) = \frac{V_1(\sigma_0,\sigma_1=+1)-V_1(\sigma_0,\sigma_1=-1)}{S_1(\sigma_0,\sigma_1=+1)-S_1(\sigma_0,\sigma_1=-1)} \tag{4.18}$$

to be the number of shares of stock required for the synthetic portfolio. If we set $W_0(\sigma_0) = V_0(\sigma_0)$ *and define the value of the synthetic portfolio at time* $N = 1$ *as follows*

$$W_1(\sigma_0,\sigma_1) = \phi_0(\sigma_0)S_1(\sigma_0,\sigma_1) + R(W_0(\sigma_0)-\phi_0(\sigma_0)S_0), \tag{4.19}$$

then $W_1(\sigma_0, \sigma_1) = V_1(\sigma_0, \sigma_1)$. *In other words, the synthetic portfolio replicates the value of the derivative security. Moreover, the values of the discounted replicating portfolio and the discounted price of the derivative security are martingales under the measure* $\mathbb{Q}$, *i.e.,*

$$\begin{aligned} W_0(\sigma_0) &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}W_1 \mid W_0\right] \\ &= \frac{1}{R}\left[\frac{R-(u-h)}{h}W_1(\sigma_0,\sigma_1=+1)+\frac{u-R}{h}W_1(\sigma_0,\sigma_1=-1)\right], \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} V_0(\sigma_0) &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}V_1 \mid V_0\right] \\ &= \frac{1}{R}\left[\frac{R-(u-h)}{h}V_1(\sigma_0,\sigma_1=+1)+\frac{u-R}{h}V_1(\sigma_0,\sigma_1=-1)\right]. \end{aligned} \tag{4.21}$$

Equation (4.21) provides a formula for the price of the derivative security at time $n = 0$.

Similar result hold for the case when initial regime is $\sigma_0 = -1$.

Proposition 5. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *be given. Let the one-period regime switching market model with jumps* (S_n, B_n), $n = \overline{1, N}$, $N = 1$ *be given, where* S_n *is a* *collection of stock prices defined by formula (4.6) and* B_n *is a risk-free money market process, defined by formula (4.10). Let the initial regime* σ_0 *be known and be equal to* -1*, i.e.,* $\mathbb{P}(\sigma_0 = -1) = 1$*. Let the underlying asset, e.g., stock be given with the initial price* S_0*. Let* $R = r + 1$*. If* $0 < d < R$ *and the jump size* $h > R - d$ *then the no arbitrage condition*

$$0 < d < R < d + h, \tag{4.22}$$

is satisfied and hence the one-period regime switching model with jumps is arbitrage-free. Moreover, the model is complete and there exist a unique equivalent martingale measure $\mathbb{Q}$ *given as*

$$\begin{cases} \mathbb{Q}(\sigma_0 = +1) = 0, \\ \mathbb{Q}(\sigma_0 = -1) = 1, \end{cases} \tag{4.23}$$

and

$$\begin{cases} \mathbb{Q}(\sigma_1 = +1 \mid \sigma_0 = -1) = \frac{R-d}{h}, \\ \mathbb{Q}(\sigma_1 = -1 \mid \sigma_0 = -1) = \frac{d+h-R}{h}. \end{cases} \tag{4.24}$$

such that the discounted stock price is a martingale with respect to the equivalent martingale measure $\mathbb{Q}$ *:*

$$\begin{aligned} S_0 &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1} S_1 \mid S_0\right] \\ &= \frac{1}{R}\left[\frac{R-d}{h} S_1(\sigma_0, \sigma_1 = +1) + \frac{d+h-R}{h} S_1(\sigma_0, \sigma_1 = -1)\right]. \end{aligned} \tag{4.25}$$

Furthermore, assume there is a derivative security that pays amount $V_1(\sigma_0, \sigma_1)$ *at time* $N = 1$*. Define* $V_0(\sigma_0)$ *to be a price of the derivative security at time* $n = 0$*. Define also*

$$\phi_0(\sigma_0) = \frac{V_1(\sigma_0, \sigma_1 = +1) - V_1(\sigma_0, \sigma_1 = -1)}{S_1(\sigma_0, \sigma_1 = +1) - S_1(\sigma_0, \sigma_1 = -1)} \tag{4.26}$$

to be the number of shares of stock required for the synthetic portfolio. If we set $W_0(\sigma_0) = V_0(\sigma_0)$ *and define the value of the synthetic portfolio at time* $N = 1$ *as follows*

$$W_1(\sigma_0, \sigma_1) = \phi_0(\sigma_0) S_1(\sigma_0, \sigma_1) + R(W_0(\sigma_0) - \phi_0(\sigma_0) S_0), \tag{4.27}$$

then $W_1(\sigma_0, \sigma_1) = V_1(\sigma_0, \sigma_1)$*. In other words, the synthetic portfolio replicates the value of the derivative security. Moreover, the values of the discounted replicating portfolio* *and the discounted price of the derivative security are martingales under the measure* $\mathbb{Q}$*, i.e.,*

$$
\begin{aligned}
W_0(\sigma_0) &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}W_1 \mid W_0, \sigma_0\right] \\
&= \frac{1}{R}\left[\frac{R-d}{h}W_1(\sigma_0, \sigma_1 = +1) + \frac{d+h-R}{h}W_1(\sigma_0, \sigma_1 = -1)\right], \quad (4.28)
\end{aligned}
$$

and

$$
\begin{aligned}
V_0(\sigma_0) &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}V_1 \mid V_0, \sigma_0\right] \\
&= \frac{1}{R}\left[\frac{R-d}{h}V_1(\sigma_0, \sigma_1 = +1) + \frac{d+h-R}{h}V_1(\sigma_0, \sigma_1 = -1)\right]. \quad (4.29)
\end{aligned}
$$

Equation (4.29) provides a formula for the price of the derivative security at time $n = 0$*.*

We omit the proofs of Propositions 4 and 5 since they follow directly from the proof of Proposition 6 that considers the more general $N-$period regime switching model with jumps under the assumption of known initial regime. Proposition 6 and its proof are given in the next section.

Propositions 4 and 5 show that under the assumption of known initial regime, the one-period regime switching model with jumps is complete and therefore every derivative security has a unique and fair price. Furthermore, the initial price of the derivative security can be computed by formulas (4.21) and (4.29), depending on the value of the initial regime σ_0. Notice that, like in the binomial tree model, the historic probability measure $\mathbb{P}$ does not appear in the pricing formulas (4.21) and (4.29). Observe also, that the no-arbitrage condition (4.14) guarantee that probabilities (4.16) define the valid probability measure. Similarly, the no-arbitrage condition (4.22) guarantee that probabilities (4.24) define the valid probability measure.

Next, we consider N-period regime switching model with jumps under the assumptions that the initial regime is known. We show that under certain conditions on the jump size the model is complete.

4.2.2 N-period regime switching model with jumps

Let the initial regime be predetermined, i.e., σ_0 is known and equals to either $+1$ or -1. In Fig. 4.2 and 4.3 we depict the dynamics of the stock price for two-period regime switching model with jumps for $\sigma_0 = +1$ and $\sigma_0 = -1$, respectively. Note, that the values of the up-factor parameter $U(\sigma_0)$ and down-factor parameter $D(\sigma_0)$ depend on the initial regime σ_0 and therefore are different for $\sigma_0 = +1$ and $\sigma_0 = -1$. Hence, under the assumption that the initial regime is known, the values $U(\sigma_0)$ and $D(\sigma_0)$ are also predetermined. However, for $n \geq 1$ the values of the up and down parameters are not predetermined and depend on the previous and current regimes. Thus, under the assumption that σ_0 is known, the regime switching model with jumps has a resemblance with the binomial tree model with stochastic volatility, a model where at each time n the up and down parameters are allowed to depend on n, but the initial up factor and down factors are not random. Unlike the Black-Sholes model that assumes that the volatility of the underlying security is constant, the stochastic volatility models take the volatility in the price of the underlying security into account and, hence, allow to model derivatives more precisely and to improve the accuracy of calculations and forecasts.

Next result provides conditions on the up-factor u, the down-factor d, the interest rate r, and the jump size h such that under the assumptions that the initial regime is known the N-period regime switching model with jumps is complete.

Proposition 6. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let the N-period regime switching market model with jumps (S_n, B_n), $n = \overline{1, N}$ be given, where S_n is a collection of stock prices defined by formula (4.6) and B_n is a risk-free money market process, defined by formula (4.10). Let the initial regime σ_0 be known, i.e., either $\mathbb{P}(\sigma_0 = +1) = 1$ or $\mathbb{P}(\sigma_0 = -1) = 1$. Let the underlying asset (e.g., stock) be given, with the initial price S_0. Let the up-factor parameter $U(\xi)$ and the down-factor parameter $D(\xi)$ be defined as follows*

$$U(\xi) = \begin{cases} u, & \xi = +1; \\ d + h, & \xi = -1, \end{cases} \tag{4.30}$$

and

$$D(\xi) = \begin{cases} u - h, & \xi = +1; \\ d, & \xi = -1. \end{cases} \tag{4.31}$$

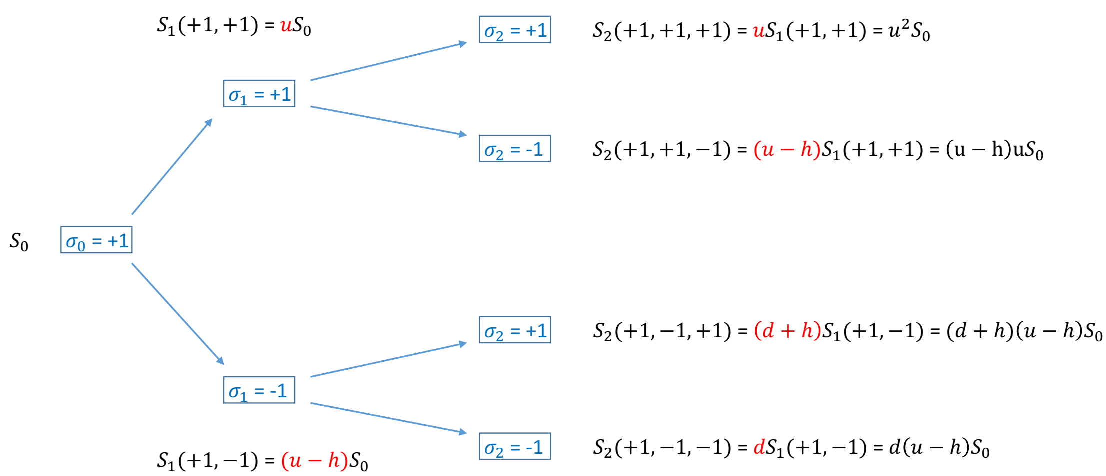

Figure 4.2: Stock price dynamics for two-period regime switching model with jumps, given $\sigma_0 = +1$. Quantities highlighted in red correspond to the values of the jump-factor process $g_n(\sigma_{n-1}, \sigma_n)$ for $n = \{1, 2\}$.

If $u > R > d > 0$ and the jump size $h > \max\{u - R, R - d\}$ then $\forall n = \overline{0, N-1}$ the up-factor parameter $U(\sigma_n)$, the down-factor parameter $D(\sigma_n)$, and the interest rate r satisfy the no-arbitrage conditions

$$0 < D(\sigma_n) < R < U(\sigma_n), \tag{4.32}$$

and hence, the N-period regime switching model with jumps is arbitrage-free. Moreover, the model is complete and there exist a unique equivalent martingale measure $\mathbb{Q}$ induced by a Markov chain as follows

$$\begin{cases} \mathbb{Q}(\sigma_0 = +1) = 1, \ \mathbb{Q}(\sigma_0 = -1) = 0 & \textit{if } \sigma_0 = +1; \\ \mathbb{Q}(\sigma_0 = +1) = 0, \ \mathbb{Q}(\sigma_0 = -1) = 1, & \textit{if } \sigma_0 = -1. \end{cases} \tag{4.33}$$

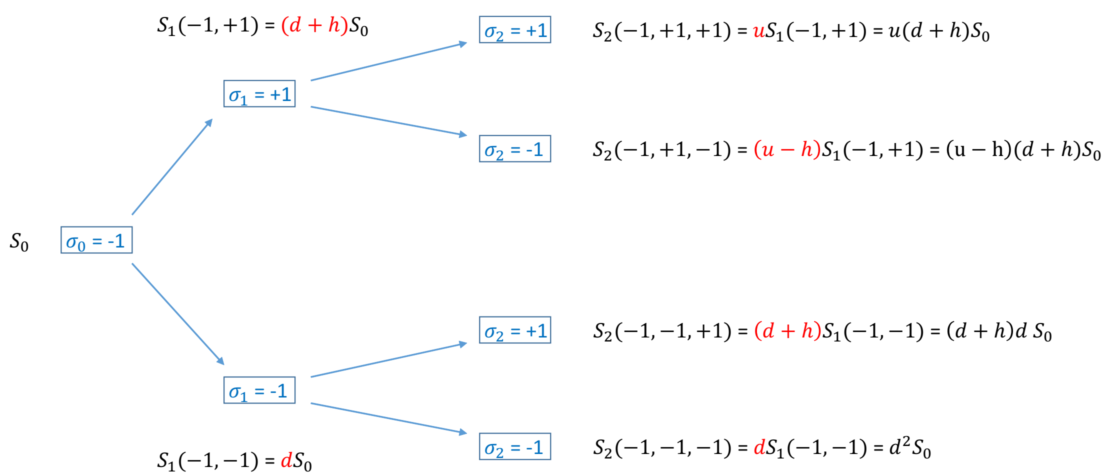

Figure 4.3: Stock price dynamics for two-period regime switching model with jumps, given $\sigma_0 = -1$. Quantities highlighted in red correspond to the values of the jump-factor process $g_n(\sigma_{n-1}, \sigma_n)$ for $n = \{1, 2\}$.

and for $n \geq 1$

$$\mathbb{Q}(\sigma_n = \xi_n | \sigma_0 = \xi_0, \cdots, \sigma_{n-1} = \xi_{n-1}, S_0 = s_0, \cdots, S_{n-1} = s_{n-1}) =$$

$$\begin{aligned} &= \quad \mathbb{Q}(\sigma_n = \xi_n | \sigma_{n-1} = \xi_{n-1}) \\ &= \begin{cases} \frac{R - D(\xi_{n-1})}{U(\xi_{n-1}) - D(\xi_{n-1})} = \frac{R-(u-h)}{h}, & \xi_{n-1} = +1, \xi_n = +1; \\ \frac{U(\xi_{n-1}) - R}{U(\xi_{n-1}) - D(\xi_{n-1})} = \frac{u-R}{h}, & \xi_{n-1} = +1, \xi_n = -1; \\ \frac{R - D(\xi_{n-1})}{U(\xi_{n-1}) - D(\xi_{n-1})} = \frac{R-d}{h}, & \xi_{n-1} = -1, \xi_n = +1; \\ \frac{U(\xi_{n-1}) - R}{U(\xi_{n-1}) - D(\xi_{n-1})} = \frac{(d+h)-R}{h}, & \xi_{n-1} = -1, \xi_n = -1, \end{cases} \end{aligned} \tag{4.34}$$

such that the discounted stock price is a martingale under the measure $\mathbb{Q}$*:*

$$S_n = \frac{1}{R} \mathbb{E}_{\mathbb{Q}} \left[S_{n+1} \mid S_0, S_1, \cdots, S_n \right]. \tag{4.35}$$

Remark 1. *Let $\mathcal{F}_n = \mathcal{F}(\sigma_0, \sigma_1, \cdots, \sigma_n, S_0, \cdots, S_n)$ be the $\sigma-$algebra, generated by the regime process and the stock prices up to time n. Let also $\mathcal{G}_n = \mathcal{F}(S_0, \cdots, S_n)$ be the $\sigma-$algebra generated by the stock prices up to time n. Clearly, $\mathcal{G}_n \subset \mathcal{F}_n$. However, if the parameters u, d, and h of the model are such that one can determine regimes $\sigma_0, \cdots, \sigma_n$ knowing the stock prices $S_0, \cdots, S_n$, then $\mathcal{G}_n = \mathcal{F}_n$. Consider, for example, a special case when $u - h \neq d + h$. Then, given stock prices $S_0, \cdots, S_n$, one can determine the regimes $\sigma_0, \cdots, \sigma_n$, and hence $\mathcal{G}_n = \mathcal{F}_n$. On the other hand, if $u - h = d + h$, one can not determine the underlying regimes $\sigma_0, \cdots, \sigma_n$. Therefore, $\mathcal{G}_n \subset \mathcal{F}_n$, $\mathcal{G}_n \neq \mathcal{F}_n$.*

Proof. (Proposition 6)
We start by showing that the no-arbitrage conditions (4.32) are satisfied $\forall n = \overline{1, N-1}$. If condition $h > \max\{u - R, R - d\}$ is satisfied then $R > u - h$ and $R < d + h$. Combining with the condition $0 < d < R < u$, we conclude that $u > R > u - h > 0$ and $d + h > R > d > 0$. Thus, the model is arbitrage-free, which guarantees the existence of at least one equivalent martingale measure.

Next we show that the discounted stock price is a martingale under the measure $\mathbb{Q}$. First note that if $h > \max\{u - R, R - d\}$ then h is greater then the average of two numbers, i.e., $h > \frac{u-d}{2}$, and hence $d + h > u - h$. Following the discussion in the Remark 1, condition $d + h > u - h$ implies $\mathcal{G}_n = \mathcal{F}_n$. Therefore, $\mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{G}_n\right] = \mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right]$. For this reason, to show that the stock price is a martingale under the measure $\mathbb{Q}$ it is enough to show that $\mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] = S_n$. Now we can rewrite conditional expectation $\frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right]$ as follows

$$
\begin{aligned}
\frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_n g_n(\sigma_{n-1}, \sigma_n) \mid \mathcal{F}_n] \\
&= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_n u \mathbb{1}_{[\sigma_n=+1]}\mathbb{1}_{[\sigma_{n+1}=+1]} + S_n(u-h)\mathbb{1}_{[\sigma_n=+1]}\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n] \\
&+ \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_n(d+h)\mathbb{1}_{[\sigma_n=-1]}\mathbb{1}_{[\sigma_{n+1}=+1]} + S_n d \mathbb{1}_{[\sigma_n=-1]}\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n] \\
&= \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=+1]}\mathbb{E}_{\mathbb{Q}}[u\mathbb{1}_{[\sigma_{n+1}=+1]} + (u-h)\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n] \\
&+ \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=-1]}\mathbb{E}_{\mathbb{Q}}[(d+h)\mathbb{1}_{[\sigma_{n+1}=+1]} + d\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n]. \qquad (4.36)
\end{aligned}
$$

Equation (4.36) was obtained using the fact that S_n, $\mathbb{1}_{[\sigma_n=+1]}$,and $\mathbb{1}_{[\sigma_n=-1]}$ are all measurable with respect to $\mathcal{F}_n$. Now, using the Markov property

$$\mathbb{1}_{[\sigma_n=\xi_n]}\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_{n+1}=\xi_{n+1}]} \mid \mathcal{F}_n] = \mathbb{1}_{[\sigma_n=\xi_n]}\mathbb{Q}(\sigma_{n+1} = \xi_{n+1} \mid \sigma_n = \xi_n)$$

we continue equation (4.36) as follows

$$= \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=+1]}(\mathbb{Q}(\sigma_{n+1}=+1|\sigma_n=+1)u + \mathbb{Q}(\sigma_{n+1}=-1|\sigma_n=+1)(u-h)) \quad (4.37)$$

$$+ \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=-1]}(\mathbb{Q}(\sigma_{n+1}=+1|\sigma_n=-1)(d+h) + \mathbb{Q}(\sigma_{n+1}=-1|\sigma_n=-1)d)\,. \quad (4.38)$$

Now, applying formulas (4.34) we obtain the values of the conditional probabilities:

$$\begin{aligned} \mathbb{Q}(\sigma_{n+1} = +1|\sigma_n = +1) &= \frac{R-(u-h)}{h}, \qquad (4.39) \\ \mathbb{Q}(\sigma_{n+1} = -1|\sigma_n = +1) &= \frac{u-R}{h}, \\ \mathbb{Q}(\sigma_{n+1} = +1|\sigma_n = -1) &= \frac{R-d}{h}, \\ \mathbb{Q}(\sigma_{n+1} = -1|\sigma_n = -1) &= \frac{d+h-R}{h}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] &= \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=+1]}\left(u\frac{R-(u-h)}{h} + (u-h)\frac{u-R}{h}\right) \\ &+ \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=-1]}\left((d+h)\frac{R-d}{h} + d\frac{d+h-R}{h}\right) \\ &= \frac{1}{R}\left(\mathbb{1}_{[\sigma_n=+1]}RS_n + \mathbb{1}_{[\sigma_n=-1]}RS_n\right) = S_n. \qquad (4.40) \end{aligned}$$

To show that the equivalent martingale measure is unique consider the equation in lines (4.37) and (4.38)

$$\begin{aligned} R &= \mathbb{1}_{[\sigma_n=+1]}\left(\mathbb{Q}(\sigma_{n+1} = +1|\sigma_n = +1)u + \mathbb{Q}(\sigma_{n+1} = -1|\sigma_n = +1)(u-h)\right) \\ &+ \mathbb{1}_{[\sigma_n=-1]}\left(\mathbb{Q}(\sigma_{n+1} = +1|\sigma_n = -1)(d+h) + \mathbb{Q}(\sigma_{n+1} = -1|\sigma_n = -1)d\right). \qquad (4.41) \end{aligned}$$

If there exist another equivalent martingale measure it must satisfy equation (4.41). However, since $h > 0$ then $u \neq u - h$ and $d \neq d + h$. Therefore, equation (4.41) has a unique solution. Thus, measure $\mathbb{Q}$ is unique.

Furthermore, measure $\mathbb{Q}$ is equivalent to the historic measure $\mathbb{P}$. Note that $\mathbb{P}(\sigma_0 = +1) = 0$ if and only if $\mathbb{Q}(\sigma_0 = +1) = 0$. Similarly, $\mathbb{P}(\sigma_0 = -1) = 0$ if and only if $\mathbb{Q}(\sigma_0 = -1) = 0$. Moreover, $\mathbb{P}(\sigma_n = +1) = p_+ > 0 \ \forall n = \overline{1, N}$ if and only if $\mathbb{Q}(\sigma_n = +1) > 0$. It follows from the fact that probability $\mathbb{Q}(\sigma_n = +1)$ can be written as follows

$$\mathbb{Q}(\sigma_n = +1) = \sum_{\xi_0, \cdots, \xi_{n-1}} \mathbb{Q}(\sigma_n = +1 \mid \sigma_{n-1} = \xi_{n-1}) \prod_{i=1}^{n-1} \mathbb{Q}(\sigma_i = \xi_i \mid \sigma_{i-1} = \xi_{i-1}) \mathbb{Q}(\sigma_0 = \xi_0), \tag{4.42}$$

where all conditional probabilities are strictly positive. This implies $\mathbb{Q}(\sigma_n = +1) > 0$. $\square$

Proposition 6 demonstrates that the equivalent measure $\mathbb{Q}$ is time-homogeneous Markov chain with transition probability matrix

$$\mathscr{Q} = \begin{pmatrix} \frac{R-(u-h)}{h} & \frac{u-R)}{h} \\ \frac{R-d}{h} & \frac{d+h-R}{h} \end{pmatrix}.$$

This is an interesting fact given that historic measure $\mathbb{P}$ is not necessarily a Markov chain. It is also important to note that under the historic probability measure $\mathbb{P}$ the value of the regime process at time n is independent of the value of the regime process at time $n-1$, i.e., $\mathbb{P}(\sigma_n \mid \sigma_{n-1}) = \mathbb{P}(\sigma_n)$. However, under the new martingale measure the value of the regime process σ_n are not independent, i.e., $\mathbb{Q}(\sigma_n \mid \sigma_{n-1}) \neq \mathbb{Q}(\sigma_n)$.

Proposition 7. *Let the assumptions of the Proposition 6 be satisfied. Then σ_n is not independent of $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ under the equivalent martingale measure $\mathbb{Q}$.*

Proof. We prove the claim of the proposition by contradiction. Suppose the opposite, i.e., σ_n is independent of $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ under measure $\mathbb{Q}$. By Proposition 6 the price of the stock is a martingale under measure $\mathbb{Q}$, i.e.,

$$S_{n-1} = \frac{1}{R} \mathbb{E}_{\mathbb{Q}} \left[S_n \mid S_0, S_1, \cdots, S_{n-1} \right]. \tag{4.43}$$

Using Remark 1 and the fact that $S_n = S_{n-1} g_n(\sigma_{n-1}, \sigma_n)$, we rewrite equation (4.43) as follows

$$
\begin{aligned}
S_{n-1} &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_n \mid \mathcal{F}_{n-1}\right] && (4.44)\\
&= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_{n-1}g_n(\sigma_{n-1},\sigma_n) \mid \mathcal{F}_{n-1}\right] && (4.45)\\
&= \begin{cases} S_{n-1}u\mathbb{Q}(\sigma_n=+1)+S_{n-1}(u-h)\mathbb{Q}(\sigma_n=-1), & \text{if } \sigma_{n-1}=+1;\\ S_{n-1}(d+h)\mathbb{Q}(\sigma_n=+1)+S_{n-1}d\mathbb{Q}(\sigma_n=-1), & \text{if } \sigma_{n-1}=-1,\end{cases} && (4.46)
\end{aligned}
$$

where system of equations (4.46) follows from the assumption that σ_n is independent of $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$. Probabilities $\mathbb{Q}(\sigma_n = +1)$ and $\mathbb{Q}(\sigma_n = -1) = 1 - \mathbb{Q}(\sigma_n = +1)$ should satisfy both equations in formula (4.46). However, the system of equations

$$
\begin{cases} 1 = \frac{1}{R}\left[u\mathbb{Q}(\sigma_n=+1)+(u-h)(1-\mathbb{Q}(\sigma_n=+1))\right],\\ 1 = \frac{1}{R}\left[(d+h)\mathbb{Q}(\sigma_n=+1)+d(1-\mathbb{Q}(\sigma_n=+1))\right],\end{cases}
$$

is overdetermined, and hence has no solution. This contradicts the results of the Proposition 6, that guarantee the existence of such equivalent martingale measure $\mathbb{Q}$. Therefore, σ_n is not independent of $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ under the equivalent martingale measure $\mathbb{Q}$. □

As we discussed in Remark 1, condition on a jump size $u - h \neq d + h$ guarantees that the $\sigma-$algebra generated by the regime process and the stock prices up to time n is the same as the $\sigma-$algebra generated by the stock prices up to time n, i.e., $\mathcal{G}_n = \mathcal{F}_n$. However, condition $u - h \neq d + h$ is also crucial for the no-arbitrage argument.

Proposition 8. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *be given. Let the* N*-period regime switching market model with jumps* (S_n, B_n)*,* $n = \overline{1, N}$ *be given, where* S_n *is a collection of stock prices defined by formula (4.6) and* B_n *is a risk-free money market process, defined by formula (4.10). Let the initial regime* σ_0 *be known, i.e., either* $\mathbb{P}(\sigma_0 = +1) = 1$ *or* $\mathbb{P}(\sigma_0 = -1) = 1$*. If* $u > R > d > 0$ *and jump size* h *satisfies equation* $u-h = d+h$*, then the regime switching model with jumps admits arbitrage and, therefore, is not complete.*

Proof. If $u - h = d + h$, then the no-arbitrage conditions $u > R > u - h > 0$ and $d + h > R > d > 0$ are not satisfied. Therefore, the model admits arbitrage and hence, by Theorems 1 and 2, there is no unique equivalent measure. Thus, the model is not complete. □

Condition on the jump size $h > \max\{u-R, R-d\}$ ensures the existence and uniqueness of equivalent martingale measure. Next proposition provides a method of pricing derivative security by constructing the replicating portfolio.

Proposition 9. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let the N-period regime switching market model with jumps (S_n, B_n), $n = \overline{1, N}$ be given, where S_n is a collection of stock prices defined by formula (4.6) and B_n is a risk-free money market process, defined by formula (4.10). Let the initial regime σ_0 be known, i.e., either $\mathbb{P}(\sigma_0 = +1) = 1$ or $\mathbb{P}(\sigma_0 = -1) = 1$. Let the underlying asset (e.g., stock) be given, with the initial price S_0. Let conditions $u > R > d > 0$ and $h > \max\{u - R, R - d\}$ be satisfied, and let the equivalent martingale measure be given by formulas (4.33) and (4.34). Let $V_N(\sigma_0, \sigma_1, \cdots, \sigma_N)$ be a payoff of the derivative security at time N. Define recursively backward in time the sequence of random variables $V_{N-1}, \cdots, V_0$ by*

$$V_n(\sigma_0, \sigma_1, \cdots, \sigma_n) =$$

$$\begin{aligned}
&= \frac{1}{R}\big[\mathbb{1}_{[\sigma_n=+1]}\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_n = +1)V_{n+1}(\sigma_0, \sigma_1, \cdots, \sigma_{n+1} = +1)\big] \\
&= \frac{1}{R}\big[\mathbb{1}_{[\sigma_n=-1]}\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_n = -1)V_{n+1}(\sigma_0, \sigma_1, \cdots, \sigma_{n+1} = +1)\big] \\
&= \frac{1}{R}\big[\mathbb{1}_{[\sigma_n=+1]}\mathbb{Q}(\sigma_{n+1} = -1 \mid \sigma_n = +1)V_{n+1}(\sigma_0, \sigma_1, \cdots, \sigma_{n+1} = -1)\big] \\
&= \frac{1}{R}\big[\mathbb{1}_{[\sigma_n=-1]}\mathbb{Q}(\sigma_{n+1} = -1 \mid \sigma_n = -1)V_{n+1}(\sigma_0, \sigma_1, \cdots, \sigma_{n+1} = -1)\big]\,.
\end{aligned} \tag{4.47}$$

Let $V_0(\sigma_0)$ to be the price of the derivative security at time $n = 0$. Define also ϕ_n, $n = \overline{0, N-1}$ to be the portfolio process

$$\phi_n(\sigma_0, \cdots, \sigma_n) = \frac{V_{n+1}(\sigma_0 \cdots, \sigma_n, \sigma_{n+1} = +1) - V_{n+1}(\sigma_0 \cdots, \sigma_n, \sigma_{n+1} = -1)}{S_{n+1}(\sigma_0 \cdots, \sigma_n, \sigma_{n+1} = +1) - S_{n+1}(\sigma_0 \cdots, \sigma_n, \sigma_{n+1} = -1)}. \tag{4.48}$$

Define the portfolio value process $W_n(\sigma_0, \cdots, \sigma_n)$ recursively forward in time as follows

$$W_n(\sigma_0, \cdots, \sigma_n) = \phi_{n-1}S_n + R(W_{n-1} - \phi_{n-1}S_{n-1}), \tag{4.49}$$

where S_n is a price of the underlying asset at time n. If we set $V_0(\sigma_0) = W_0(\sigma_0)$ then $W_N(\sigma_0 \cdots, \sigma_n, \sigma_N) = V_N(\sigma_0 \cdots, \sigma_n, \sigma_N)$, i.e., at the expiration time the payoff of the *derivative security is matched by the value of the replicating portfolio. Furthermore, the discounted portfolio value process and the discounted prices of the derivative security are martingales under the equivalent martingale measure* $\mathbb{Q}$

$$W_n\left(\sigma_0, \cdots, \sigma_n\right) = \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[W_{n+1}\left(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}\right) \mid \mathcal{F}_n\right], \tag{4.50}$$

$$V_n\left(\sigma_0, \cdots, \sigma_n\right) = \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[V_{n+1}\left(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}\right) \mid \mathcal{F}_n\right]. \tag{4.51}$$

Moreover, the initial price of the derivative security can be computed as follows

$$V_0(\sigma_0) = \frac{1}{R^n}\mathbb{E}_{\mathbb{Q}}\left[V_n(\sigma_0, \cdots, \sigma_n)\right], \forall n = \overline{1, N}. \tag{4.52}$$

Proof. We start by showing that the value of the derivative security is replicated by the synthetic portfolio constructed by using the stock and the money market. We use proof by induction to show that $W_N(\sigma_0, \cdots, \sigma_n, \sigma_N) = V_N(\sigma_0, \cdots, \sigma_n, \sigma_N)$. By construction $W_0(\sigma_0) = V_0(\sigma_0)$. Suppose that $W_n(\sigma_0, \cdots, \sigma_n) = V_n(\sigma_0, \cdots, \sigma_n,)$ is true. In order to show that

$$W_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}) = V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1})$$

it is enough to show that

$$W_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) = V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1)$$

and

$$W_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) = V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1).$$

We start by proving the first equation. By using formula (4.49) we write the value of the portfolio as follows

$$\begin{aligned} W_{n+1}(\sigma_0, \cdots, \sigma_{n+1} = +1) = {} & \phi_n(\sigma_0, \cdots, \sigma_n) S_{n+1}(\sigma_0, \cdots, \sigma_{n+1} = +1) \\ & + R(W_n(\sigma_0, \cdots, \sigma_n) - \phi_n(\sigma_0, \cdots, \sigma_n) S_n(\sigma_0, \cdots, \sigma_n)) \end{aligned} \tag{4.53}$$

Note that $S_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) - S_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) = hS_n(\sigma_0, \cdots, \sigma_n)$

no matter what the value of σ_n is and hence $\phi_n(\sigma_0, \cdots, \sigma_n)$ can be written as

$$\phi_n(\sigma_0, \cdots, \sigma_n) = \frac{V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) - V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1)}{hS_n}. \quad (4.54)$$

Using formulas (4.49) and (4.54) we obtain $W_{n+1}(\sigma_0, \cdots, \sigma_{n+1} = +1) =$

$$\begin{aligned}
&= \frac{V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1)}{hS_n(\sigma_0, \cdots, \sigma_n)} S_{n+1}(\sigma_0, \cdots, \sigma_{n+1} = +1) \\
&- \frac{V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1)}{hS_n(\sigma_0, \cdots, \sigma_n)} S_{n+1}(\sigma_0, \cdots, \sigma_{n+1} = +1) \\
&- \frac{V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1)}{hS_n(\sigma_0, \cdots, \sigma_n)} RS_n(\sigma_0, \cdots, \sigma_n) \\
&+ \frac{V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1)}{hS_n(\sigma_0, \cdots, \sigma_n)} RS_n(\sigma_0, \cdots, \sigma_n) \\
&+ RV_n(\sigma_0, \cdots, \sigma_n) \qquad (4.55)
\end{aligned}$$

Note that $S_{n+1}(\sigma_0, \cdots, \sigma_{n+1} = +1) = S_n(\sigma_0, \cdots, \sigma_n) g_{n+1}(\sigma_n, \sigma_{n+1} = +1)$. Hence, we continue equation (4.55) as follows

$$\begin{aligned}
&= V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) \frac{g_{n+1}(\sigma_n, \sigma_{n+1} = +1) - R}{h} \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) \left[\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_n = +1) \mathbb{1}_{[\sigma_n = +1]}\right] \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) \left[\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_n = -1) \mathbb{1}_{[\sigma_n = -1]}\right] \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) \frac{R - g_{n+1}(\sigma_n, \sigma_{n+1} = +1)}{h} \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) \left[\mathbb{Q}(\sigma_{n+1} = -1 \mid \sigma_n = +1) \mathbb{1}_{[\sigma_n = +1]}\right] \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) \left[\mathbb{Q}(\sigma_{n+1} = -1 \mid \sigma_n = -1) \mathbb{1}_{[\sigma_n = -1]}\right] \qquad (4.56) \\
&= V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) \mathbb{1}_{[\sigma_n = +1]} \left(\frac{u - R}{h} + \frac{R - (u - h)}{h}\right) \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1) \mathbb{1}_{[\sigma_n = -1]} \left(\frac{d + h - R}{h} + \frac{R - d}{h}\right) \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) \mathbb{1}_{[\sigma_n = +1]} \left(\frac{u - R}{h} - \frac{u - R)}{h}\right) \\
&+ V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) \mathbb{1}_{[\sigma_n = -1]} \left(\frac{d + h - R}{h} - \frac{d + h - R)}{h}\right) \qquad (4.57) \\
&= V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = +1). \qquad (4.58)
\end{aligned}$$

Similarly, one can show that $W_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1) = V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1} = -1)$. Therefore, regardless of the values of σ_{n+1}

$$W_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}) = V_{n+1}(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}).$$

Next, we show that the discounted portfolio value process W_n is a martingale under the measure $\mathbb{Q}$. Consider conditional expectation $\frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[W_{n+1}\left(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}\right) \mid \mathcal{F}_n\right]$. Using formula (4.49) we can rewrite it as follows $\frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[W_{n+1}\left(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}\right) \mid \mathcal{F}_n\right]$

$$\begin{aligned} &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[\phi_n S_{n+1} + R(W_n - \phi_n S_n) \mid \mathcal{F}_n\right] && (4.59)\\ &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[\phi_n S_{n+1} \mid \mathcal{F}_n\right] + \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[R(W_n - \phi_n S_n) \mid \mathcal{F}_n\right] && (4.60)\end{aligned}$$

By Proposition 6, S_n is a martingale with respect to the equivalent measure $\mathbb{Q}$. Moreover, quantities W_n, ϕ_n and S_n are measurable with respect to filtration $\mathcal{F}_n$, therefore we can continue equation (4.60) as follows

$$\begin{aligned} \frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[W_{n+1}\left(\sigma_0, \cdots, \sigma_n, \sigma_{n+1}\right) \mid \mathcal{F}_n\right] &= \frac{1}{R}\phi_n S_n + W_n - \frac{1}{R}\phi_n S_n \\ &= W_n. && (4.61)\end{aligned}$$

Hence, the discounted portfolio value process W_n is a martingale with respect to the measure $\mathbb{Q}$. Furthermore, the discounted prices of the derivative security is also a martingale with respect to the measure $\mathbb{Q}$. It follows from the fact that W_n is a martingale and $W_n(\sigma_0, \cdots, \sigma_n) = V_n(\sigma_0, \cdots, \sigma_n)$, $\forall n = \overline{0, N}$.

Moreover, recall that the expected value of a martingale does not change with time. Therefore, taking the expectation of V_n we obtain the derivative security pricing formula (4.52). □

Proposition 9 demonstrates that as in the case of the binomial tree model, the historical probability measure $\mathbb{P}$ do not appear in the pricing formula. This is due to the fact that prices of derivative securities depend on the set of possible stock price paths but not on the historic probabilities of those paths. Formulas (4.48) and (4.49) allow the agent to replicate the derivative security using the stock and the money market account. In the next section, we provide an example that illustrates the way to price the derivative security using Proposition 9 and formulas (4.48) and (4.49). The derivative security that we consider in the next example is the Russian option.

4.2.3 Example

Example 2. The Russian options were proposed by Shiryaev and Shepp in 1993 [53]. Also known as "reduced regret option", the Russian option gives the holder the right, but not the obligation, to buy a call or sell a put at the historical maximum value of the underlying asset. Unlike other options, Russian Options have no predetermined expiration date, so the life of the option is determined by the holder of an option. Russian option is a rare type of option since it is generally more expensive and recommended for more experienced investors. The no-arbitrage pricing of Russian options was studied in [33], [52], [15], and [32].

Let the Russian put option with the reward function

$$V_n = \max_{0 \le i \le n} S_i \tag{4.62}$$

be given, where S_i is the price of the underlying asset at time i. In this example we let the stock to be the underlying asset. Consider a simple two-period regime switching model with jump factors. Let the parameters of the model be defined as follows:

- number of periods $N = 2$,
- value of the up-factor $u = 2$,
- value of the down-factor $d = \frac{1}{5}$,
- interest rate $r = \frac{1}{5}$, and hence $R = r + 1 = \frac{6}{5}$,
- value of the jump parameter $h = \frac{7}{5}$.

The up-factor adjusted by the jump is given as $u - h = \frac{3}{5}$ and the down-factor adjusted by the jump is given as $d + h = \frac{8}{5}$. Assume also that the initial regime is known and $\sigma_0 = +1$. Let the initial price of the stock be $S_0 = 4$. Our goal is to find the no-arbitrage price of the Russian option at time $n = 0$.

We start with the stock price dynamics for the two-period regime switching model with jumps. In Fig. 4.4 we depict all possible prices of the stock for $n = \{0, 1, 2\}$.

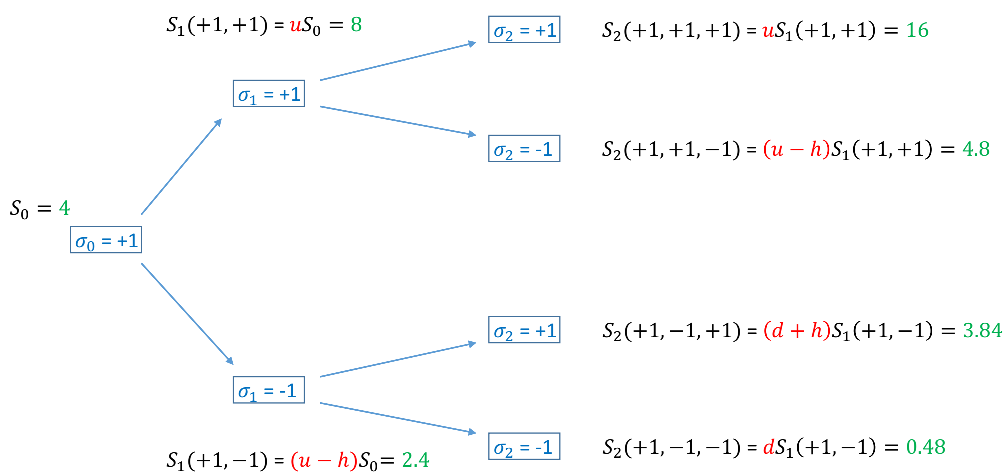

Figure 4.4: Stock price dynamics for two-period regime switching model with jumps, given $\sigma_0 = +1$. Quantities highlighted in red correspond to the values of the jump-factor process $g_n(\sigma_{n-1}, \sigma_n)$ for $n = \{1, 2\}$. Numerical values of the stock price for $n = \{0, 1, 2\}$ are highlighted in green.

Since we know all possible values for stock price, we can calculate the Russian option payoff at time $N = 2$ using formula (4.62) as follows
$V_2(\sigma_0 = +1, \sigma_1 = +1, \sigma_2 = +1) = 16,$
$V_2(\sigma_0 = +1, \sigma_1 = +1, \sigma_2 = -1) = 8,$
$V_2(\sigma_0 = +1, \sigma_1 = -1, \sigma_2 = +1) = 4,$
$V_2(\sigma_0 = +1, \sigma_1 = -1, \sigma_2 = -1) = 4.$

It is easy to check that the no-arbitrage conditions (4.32) from Proposition 6 are satisfied $\forall n = 0, 1$. Note, if $\sigma_n = +1$ then $u_n(\sigma_n) = u$ and $d_n(\sigma_n) = u - h$. Hence, condition $u > R > u - h > 0$ is satisfied since $\frac{10}{5} > \frac{6}{5} > \frac{3}{5} > 0$. Similarly, if $\sigma_n = -1$ then $d_n(\sigma_n) = d$ and $u_n(\sigma_n) = d + h$. Therefore, the condition $d + h > R > d > 0$ is also satisfied since $\frac{8}{5} > \frac{6}{5} > \frac{1}{5} > 0$. Consequently, there exist an equivalent martingale measure given as follows:

$$\mathbb{Q}(\sigma_0 = +1) = 1,$$

$$\mathbb{Q}(\sigma_0 = -1) = 0,$$

$$\mathbb{Q}(\sigma_1 = +1|\sigma_0 = +1) = q_1(\sigma_0 = +1, \sigma_1 = +1) = \tfrac{R-d_0(\sigma_0)}{h} = \tfrac{R-(u-h)}{h} = \tfrac{6/5-3/5}{7/5} = \tfrac{3}{7},$$
$$\mathbb{Q}(\sigma_1 = -1|\sigma_0 = +1) = q_1(\sigma_0 = +1, \sigma_1 = -1) = \tfrac{u_0(\sigma_0)-R}{h} = \tfrac{u-R}{h} = \tfrac{10/5-6/5}{7/5} = \tfrac{4}{7},$$
$$\mathbb{Q}(\sigma_2 = +1|\sigma_1 = +1) = q_2(\sigma_1 = +1, \sigma_2 = +1) = \tfrac{R-d_0(\sigma_0)}{h} = \tfrac{R-(u-h)}{h} = \tfrac{6/5-3/5}{7/5} = \tfrac{3}{7},$$
$$\mathbb{Q}(\sigma_2 = -1|\sigma_1 = +1) = q_2(\sigma_1 = +1, \sigma_2 = -1) = \tfrac{u_0(\sigma_0)-R}{h} = \tfrac{u-R}{h} = \tfrac{10/5-6/5}{7/5} = \tfrac{4}{7},$$
$$\mathbb{Q}(\sigma_2 = +1|\sigma_1 = -1) = q_2(\sigma_1 = -1, \sigma_2 = +1) = \tfrac{R-d_1(\sigma_1)}{h} = \tfrac{R-d}{h} = \tfrac{6/5-1/5}{7/5} = \tfrac{5}{7},$$
$$\mathbb{Q}(\sigma_2 = -1|\sigma_1 = -1) = q_2(\sigma_1 = -1, \sigma_2 = -1) = \tfrac{u_1(\sigma_1)-R}{h} = \tfrac{d+h-R}{h} = \tfrac{8/5-6/5}{7/5} = \tfrac{2}{7}.$$

Using formula (4.51) from Proposition 6 we calculate backward in time the price of the option at time $n = 1$ as follows

$$\begin{aligned} V_1(\sigma_0 = +1, \sigma_1 = +1) &= \frac{1}{R}\left[q_2(\sigma_1 = +1, \sigma_2 = +1)V_2(\sigma_0 = +1, \sigma_1 = +1, \sigma_2 = +1)\right] \\ &+ \frac{1}{R}\left[q_2(\sigma_1 = +1, \sigma_2 = -1)V_2(\sigma_0 = +1, \sigma_1 = +1, \sigma_2 = -1)\right] \\ &= \frac{5}{6}\left[\frac{3}{7}16 + \frac{4}{7}8\right] = \frac{200}{21} \approx 9.52, \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} V_1(\sigma_0 = +1, \sigma_1 = -1) &= \frac{1}{R}\left[q_2(\sigma_1 = -1, \sigma_2 = +1)V_2(\sigma_0 = +1, \sigma_1 = -1, \sigma_2 = +1)\right] \\ &+ \frac{1}{R}\left[q_2(\sigma_1 = -1, \sigma_2 = -1)V_2(\sigma_0 = +1, \sigma_1 = -1, \sigma_2 = -1)\right] \\ &= \frac{5}{6}\left[\frac{5}{7}4 + \frac{2}{7}4\right] = \frac{10}{3} \approx 3.33. \end{aligned} \tag{4.64}$$

Finally we compute the value of the Russian option at time $n = 0$ as follows

$$\begin{aligned} V_0(\sigma_0 = +1) &= \frac{1}{R}\left[q_1(\sigma_0 = +1, \sigma_1 = +1)V_1(\sigma_0 = +1, \sigma_1 = +1)\right] \\ &+ \frac{1}{R}\left[q_1(\sigma_0 = +1, \sigma_1 = -1)V_1(\sigma_0 = +1, \sigma_1 = -1)\right] \\ &= \frac{5}{6}\left[\frac{3}{7}9.52 + \frac{4}{7}3.33\right] \approx 4.9. \end{aligned} \tag{4.65}$$

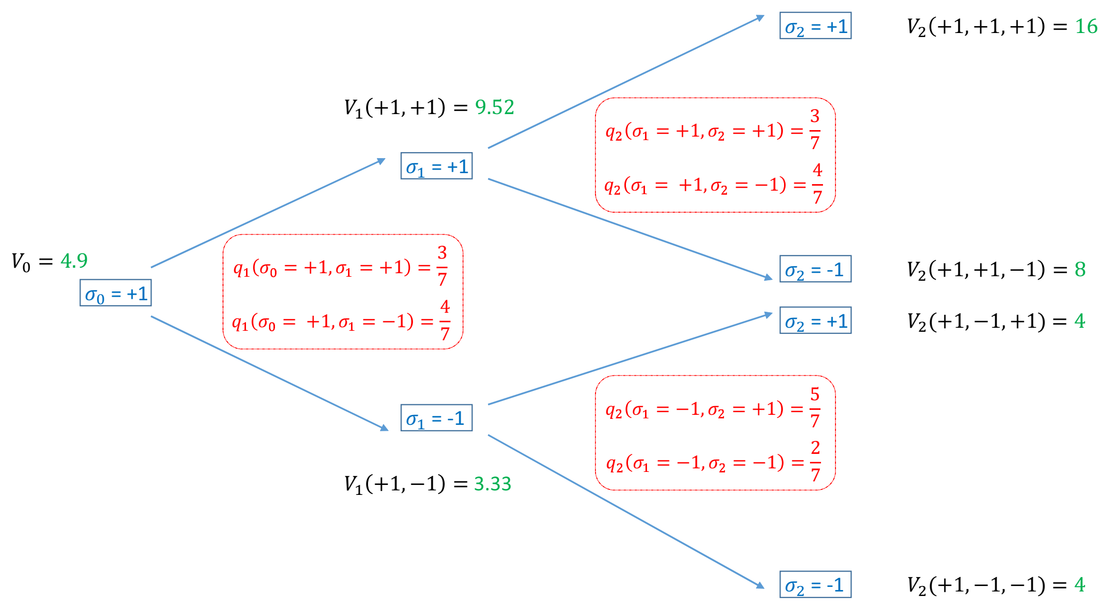

Figure 4.5: Price of the option for two-period regime switching model with jumps, given $\sigma_0 = +1$. Quantities, highlighted in green, correspond to the numerical values of the Russian option price for $n = \{0, 1, 2\}$. Corresponding numerical values of the martingale measure are highlighted in red.

In Fig. 4.5 we depict the price of the option for $n = \{0, 1, 2\}$. The initial price of the Russian option $V_0 = 4.9$ allows the seller of the option to hedge his short position. Suppose an agent sells this option at time $n = 0$ for 4.9 dollars. Formula (4.48) gives the number of shares of stock the agent should buy at time zero:

$$\phi_0(\sigma_0) = \frac{V_1(\sigma_0 = +1, \sigma_1 = +1) - V_1(\sigma_0 = +1, \sigma_1 = -1)}{S_1(\sigma_0 = +1, \sigma_1 = +1) - S_1(\sigma_0 = +1, \sigma_1 = -1)} = \frac{9.52 - 3.33}{8 - 2.4} \approx 1.1375.$$

Hence, at time $n = 0$ the agent will spend $1.1375 \times 4 = 4.55$ dollars, for buying 1.1375 shares of stock, and invest the remainder $4.9 - 4.55 = 0.35$ dollars in the money market. The money market has $r = \frac{1}{5}$ interest rate, therefore, at time $n = 1$ the money market investment will yield $0.35 \times \frac{6}{5} = 0.42$ dollars. If the stock price goes up then the agent's total portfolio value will be $0.42 + 1.1375 \times 8 = 9.52$ dollars, which matches $V_1(\sigma_0 =$

$+1, \sigma_1 = +1)$. Similarly, if the stock price goes down, then the total portfolio value will be $0.42 + 1.1375 \times 2.4 = 3.33$, which matches $V_1(\sigma_0 = +1, \sigma_1 = -1)$. In similar fashion, one can check that the hedge will fork for $n = 2$. Therefore, the agent's portfolio replicates the option independently of the stock price path.

4.3 Special case: initial regime σ_0 is not known

In this section, we study the regime switching model with jumps under the assumption that the initial regime σ_0 is not known. Although it is not intuitive, this assumption describes a real world phenomenon. As it was mentioned in [35], there is no certain way to determine current economic regime based on the observation of markets and their parameters. Moreover, there is a type of economic risk, called regime uncertainty, that is associated with uncertain future course of government policy, monetary or fiscal policy, or uncertainty over electoral outcomes. This leads to a significant decline in economic activity until this uncertainty has been resolved.

We start by considering one-period regime switching model with jumps.

4.3.1 One-period regime switching model with jumps

Suppose $0 < \mathbb{P}(\sigma_0 = +1) = p_0 < 1$, i.e., the initial regime is not known and can be either $+1$ or -1. Then at time $n = 1$ the jump-factor process can take four possible values:

$$g_1(\sigma_0, \sigma_1) = \begin{cases} d + h, & \text{if } \sigma_0 = -1 \text{ and } \sigma_1 = +1; \\ d, & \text{if } \sigma_0 = -1 \text{ and } \sigma_1 = -1; \\ u, & \text{if } \sigma_0 = +1 \text{ and } \sigma_1 = +1; \\ u - h, & \text{if } \sigma_0 = +1 \text{ and } \sigma_1 = -1. \end{cases} \tag{4.66}$$

And therefore, the price of the stock can also take four possible values:

$$S_1 = \begin{cases} (d + h)S_0, & \text{if } \sigma_0 = -1 \text{ and } \sigma_1 = +1; \\ dS_0, & \text{if } \sigma_0 = -1 \text{ and } \sigma_1 = -1; \\ uS_0, & \text{if } \sigma_0 = +1 \text{ and } \sigma_1 = +1; \\ (u - h)S_0, & \text{if } \sigma_0 = +1 \text{ and } \sigma_1 = -1. \end{cases} \tag{4.67}$$

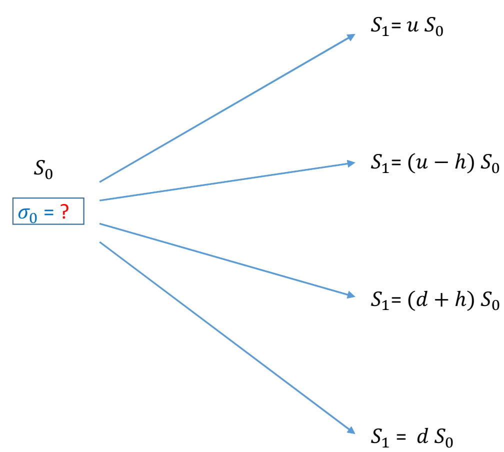

Figure 4.6: Stock price dynamics for one-period regime switching model with jumps when σ_0 is not known.

In Fig. 4.6 we schematically depict the stock price dynamics for the case when σ_0 is unknown. Since stock price can take four possible values, this one-period regime switching model with jumps has no resemblance with the binomial tree model.

Next proposition demonstrates that one-period regime switching model with jumps is not complete when the initial regime σ_0 in unknown.

Proposition 10. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *be given. Let the one-period regime switching market model with jumps* (S_n, B_n)*,* $n = \overline{1, N}$*,* $N = 1$ *be given, where* S_n *is a collection of stock prices defined by formula (4.6) and* B_n *is a risk-free money market process, defined by formula (4.10). Let the initial regime* σ_0 *be unknown, i.e.,* $0 < \mathbb{P}(\sigma_0 = +1) < 1$*. If* $u > R > d > 0$*, then there are infinitely many equivalent* *martingale measures* $\mathbb{Q}$ *described by the following equation*

$$\begin{aligned} R &= \mathbb{Q}(\sigma_0 = +1)\left(u\mathbb{Q}(\sigma_1 = +1 \mid \sigma_0 = +1) + (u-h)\mathbb{Q}(\sigma_1 = -1 \mid \sigma_0 = +1)\right) \\ &+ \mathbb{Q}(\sigma_0 = -1)\left((d+h)\mathbb{Q}(\sigma_1 = +1 \mid \sigma_0 = -1) + d\mathbb{Q}(\sigma_1 = -1 \mid \sigma_0 = -1)\right), \end{aligned} \quad (4.68)$$

where $0 < \mathbb{Q}(\sigma_1 = \xi_1 \mid \sigma_0 = \xi_0) < 1$, $\xi_0, \xi_1 \in \{+1, -1\}$ *and* $0 < \mathbb{Q}(\sigma_0 = +1) < 1$. *Thus, the model is not complete.*

Proof. Condition $u > R > d > 0$ ensures that the model does not admit an arbitrage. Therefore, by Theorem 1, there exist at least one equivalent martingale measure. Consider a stock with initial price S_0. Then under the equivalent martingale measure, the discounted stock price should be a martingale, i.e.,

$$S_0 = \mathbb{E}_{\mathbb{Q}}\left[R^{-1}S_1 | S_0\right]. \quad (4.69)$$

Following the discussion in the Remark 1, $\mathcal{G}_0 \subset \mathcal{F}_0$, but $\mathcal{G}_0 \neq \mathcal{F}_0$. Therefore, we can rewrite equation (4.69) as follows

$$\begin{aligned} S_0 &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}S_1 | S_0\right] = \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_1|S_0] \\ &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[S_1|S_0, \sigma_0]|S_0] \\ &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_0=+1]}(uS_0\mathbb{1}_{[\sigma_1=+1]} + (u-h)S_0\mathbb{1}_{[\sigma_1=-1]})|S_0, \sigma_0]|S_0] \\ &+ \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_0=-1]}((d+h)S_0\mathbb{1}_{[\sigma_1=+1]} + dS_0\mathbb{1}_{[\sigma_1=-1]})|S_0, \sigma_0]|S_0] \\ &= \frac{1}{R}\mathbb{Q}(\sigma_0 = +1)S_0\left(u\mathbb{Q}(\sigma_1 = +1|\sigma_0 = +1) + (u-h)\mathbb{Q}(\sigma_1 = -1|\sigma_0 = +1)\right) \\ &+ \frac{1}{R}\mathbb{Q}(\sigma_0 = -1)S_0\left((d+h)\mathbb{Q}(\sigma_1 = +1|\sigma_0 = -1) + d\mathbb{Q}(\sigma_1 = -1|\sigma_0 = -1)\right) \end{aligned} \quad (4.70)$$

Equation (4.70) has three unknown probabilities, namely $\mathbb{Q}(\sigma_0 = +1)$, $\mathbb{Q}(\sigma_1 = +1|\sigma_0 = +1)$, and $\mathbb{Q}(\sigma_1 = +1|\sigma_0 = -1)$. Thus, there are infinitely many way to define probabilities $\mathbb{Q}(\sigma_0 = +1) \in (0,1)$, $\mathbb{Q}(\sigma_1 = +1|\sigma_0 = +1) \in (0,1)$, and $\mathbb{Q}(\sigma_1 = +1|\sigma_0 = -1) \in (0,1)$ such that they satisfy equation (4.70). Thus, there are infinitely many ways to define measure $\mathbb{Q}$. Note, that those measures $\mathbb{Q}$ are equivalent to the historical measure $\mathbb{P}$. This follows from the fact that whenever $0 < \mathbb{P}(\sigma_i = \xi_i) < 1$, the corresponding

$0 < \mathbb{Q}(\sigma_i = \xi_i) < 1$, for $\xi_i \in \{+1, -1\}$ and $i = \{0, 1\}$. □

In the next section, we consider N-period regime switching model with jumps assuming that initial regime σ_0 is unknown. We show that the model is not complete.

4.3.2 N-period regime switching model with jumps

Proposition 11. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let the $N-$period regime switching market model with jumps (S_n, B_n), $n = \overline{1, N}$ be given, where S_n is a collection of stock prices defined by formula (4.6) and B_n is a risk-free money market process, defined by formula (4.10). Let $u > R > d > 0$. Let the initial regime σ_0 be unknown, i.e., $0 < \mathbb{P}(\sigma_0 = +1) < 1$. Then, the model is not complete.*

Proof. First note that condition $u > R > d > 0$ guarantees that the model is arbitrage-free. Therefore, the exist at least one equivalent martingale measure. We will demonstrate that this measure is not unique, by considering two cases: when $h \geq \max\{u - R, R - d\}$ and when $u - h = d + h$. Let the jump size h satisfy condition $h \geq \max\{u - R, R - d\}$. Then, following the discussion in the Remark 1, we conclude that $\mathcal{G}_n = \mathcal{F}_n$ for $n \geq 1$. Let a stock with the initial price S_0 be given. Under the assumption that the discounted stock price should be a martingale under the equivalent martingale measure, we obtain the following

$$\begin{aligned}
\frac{1}{R}\mathbb{E}_{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_n g_n(\sigma_{n-1}, \sigma_n) \mid \mathcal{F}_n] \\
&= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_n u \mathbb{1}_{[\sigma_n=+1]}\mathbb{1}_{[\sigma_{n+1}=+1]} + S_n(u-h)\mathbb{1}_{[\sigma_n=+1]}\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n] \\
&+ \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_n(d+h)\mathbb{1}_{[\sigma_n=-1]}\mathbb{1}_{[\sigma_{n+1}=+1]} + S_n d\mathbb{1}_{[\sigma_n=-1]}\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n] \\
&= \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=+1]}\mathbb{E}_{\mathbb{Q}}[u\mathbb{1}_{[\sigma_{n+1}=+1]} + (u-h)\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n] \\
&+ \frac{1}{R}S_n\mathbb{1}_{[\sigma_n=-1]}\mathbb{E}_{\mathbb{Q}}[(d+h)\mathbb{1}_{[\sigma_{n+1}=+1]} + d\mathbb{1}_{[\sigma_{n+1}=-1]} \mid \mathcal{F}_n]. \qquad (4.71)
\end{aligned}$$

Equation (4.71) leads to unique transition probabilities given as

$$\mathbb{Q}(\sigma_n = \xi_n | \sigma_0 = \xi_0, \cdots, \sigma_{n-1} = \xi_{n-1}, S_0 = s_0, \cdots, S_n = s_n) =$$

$$
\begin{aligned}
&= \mathbb{Q}(\sigma_n = \xi_n | \sigma_{n-1} = \xi_{n-1}) \\
&= \begin{cases}
\frac{R-D(\xi_{n-1})}{U(\xi_{n-1})-D(\xi_{n-1})} = \frac{R-(u-h)}{h}, & \xi_{n-1} = +1, \xi_n = +1; \\
\frac{U(\xi_{n-1})-R}{U(\xi_{n-1})-D(\xi_{n-1})} = \frac{u-R}{h}, & \xi_{n-1} = +1, \xi_n = -1; \\
\frac{R-D(\xi_{n-1})}{U(\xi_{n-1})-D(\xi_{n-1})} = \frac{R-d}{h}, & \xi_{n-1} = -1, \xi_n = +1; \\
\frac{U(\xi_{n-1})-R}{U(\xi_{n-1})-D(\xi_{n-1})} = \frac{(d+h)-R}{h}, & \xi_{n-1} = -1, \xi_n = -1,
\end{cases}
\end{aligned} \tag{4.72}
$$

$\forall n \geq 2$. However, by Proposition 10, probabilities $\mathbb{Q}(\sigma_0 = +1)$, $\mathbb{Q}(\sigma_1 = +1|\sigma_0 = +1)$, and $\mathbb{Q}(\sigma_1 = +1|\sigma_0 = -1)$ are not defined uniquely, and are described by the following equation

$$
\begin{aligned}
R &= \mathbb{Q}(\sigma_0 = +1)\left(u\mathbb{Q}(\sigma_1 = +1 \mid \sigma_0 = +1) + (u-h)\mathbb{Q}(\sigma_1 = -1 \mid \sigma_0 = +1)\right) \\
&+ \mathbb{Q}(\sigma_0 = -1)\left((d+h)\mathbb{Q}(\sigma_1 = +1 \mid \sigma_0 = -1) + d\mathbb{Q}(\sigma_1 = -1 \mid \sigma_0 = -1)\right).
\end{aligned} \tag{4.73}
$$

Therefore, under the assumption that $h \geq \max\{u-R, R-d\}$, there are infinitely many measures that satisfy equation (4.73), and therefore, the model is not complete.

Now, let jump size h satisfy equation $u - h = d + h$. In this case, $\mathcal{G}_n \subset \mathcal{F}_n$ for $n \geq 1$, $\mathcal{G}_n \neq \mathcal{F}_n$. Thus, equation $S_n = \mathbb{E}_{\mathbb{Q}}\left[R^{-1}S_{n+1}|S_0, S_1, \cdots, S_n\right]$ can be rewritten as follows

$$
\begin{aligned}
S_n &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}S_{n+1} \mid S_0, \cdots, S_n\right] = \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[S_{n+1} \mid S_0, \cdots, S_n] \\
&= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[S_{n+1} \mid S_0, \cdots, S_n, \sigma_0, \cdots, \sigma_n] \mid S_0, \cdots, S_n] \\
&= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_n=+1]}(uS_n\mathbb{1}_{[\sigma_{n+1}=+1]}) \mid S_0, \cdots, S_n, \sigma_0, \cdots, \sigma_n] \mid S_0, \cdots, S_n] \\
&+ \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_n=+1]}((u-h)S_n\mathbb{1}_{[\sigma_{n+1}=-1]}) \mid S_0, \cdots, S_n, \sigma_0, \cdots, \sigma_n] \mid S_0, \cdots, S_n] \\
&+ \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_n=-1]}((d+h)S_n\mathbb{1}_{[\sigma_{n+1}=+1]}) \mid S_0, \cdots, S_n, \sigma_0, \cdots, \sigma_n] \mid S_0, \cdots, S_n] \\
&+ \frac{1}{R}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{[\sigma_n=-1]}(dS_n\mathbb{1}_{[\sigma_{n+1}=-1]}) \mid S_0, \cdots, S_n, \sigma_0, \cdots, \sigma_n] \mid S_0, \cdots, S_n] \\
&= \frac{1}{R}\mathbb{Q}(\sigma_n = +1)S_n u\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_0, \cdots, \sigma_n = +1) \\
&+ \frac{1}{R}\mathbb{Q}(\sigma_n = +1)S_n(u-h)\mathbb{Q}(\sigma_{n+1} = -1 \mid \sigma_0, \cdots, \sigma_n = +1) \\
&+ \frac{1}{R}\mathbb{Q}(\sigma_n = -1)S_n(d+h)\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_0, \cdots, \sigma_n = -1) \\
&+ \frac{1}{R}\mathbb{Q}(\sigma_n = -1)S_n d\mathbb{Q}(\sigma_{n+1} = -1 \mid \sigma_0, \cdots, \sigma_n = -1).
\end{aligned} \tag{4.74}
$$

Equation (4.74) has three unknown probabilities $\mathbb{Q}(\sigma_n = -1) \in (0,1)$, $\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_0, \cdots, \sigma_n = +1) \in (0,1)$, and $\mathbb{Q}(\sigma_{n+1} = +1 \mid \sigma_0, \cdots, \sigma_n = -1) \in (0,1)$. Hence, there are infinitely many measures that satisfy equation (4.74), and therefore, the model is not complete. Notice that, in both cases, measures $\mathbb{Q}$ are equivalent to the historical measure $\mathbb{P}$. This follows from the fact that $0 < \mathbb{P}(\sigma_i = \xi_i) < 1$ and $0 < \mathbb{Q}(\sigma_i = \xi_i) < 1$, for $\xi_i \in \{+1, -1\}$ and $i = \{0, 1\}$. $\square$

Propositions 10 and 11 imply that the regime switching model with jumps is not complete if the initial regime is not known. In other words, there are some derivative securities that can not be priced under the model and some that can be priced. Next examples will illustrate both scenarios.

4.3.3 Examples

Example 3. Consider a one-period regime switching model with jumps. Assume that the initial regime is not known. Consider a contract with a payoff function at time $N = 1$

$$V_1 = S_1 - K, \tag{4.75}$$

where S_1 is the price of the underlying asset at time $N = 1$ and K is a strike price. Let the parameters of the model be given as follows:

- the number of periods $N = 1$,
- the strike price $K = 2/5$,
- the value of the up-factor $u = 2$,
- the value of the down-factor $d = \frac{1}{5}$,
- the interest rate $r = \frac{1}{5}$, and hence $R = r + 1 = \frac{6}{5}$,
- the value of the jump parameter $h = \frac{7}{5}$,
- the value of the up-factor adjusted by the jump $u - h = \frac{3}{5}$,
- the value of the down-factor adjusted by the jump $d + h = \frac{8}{5}$,

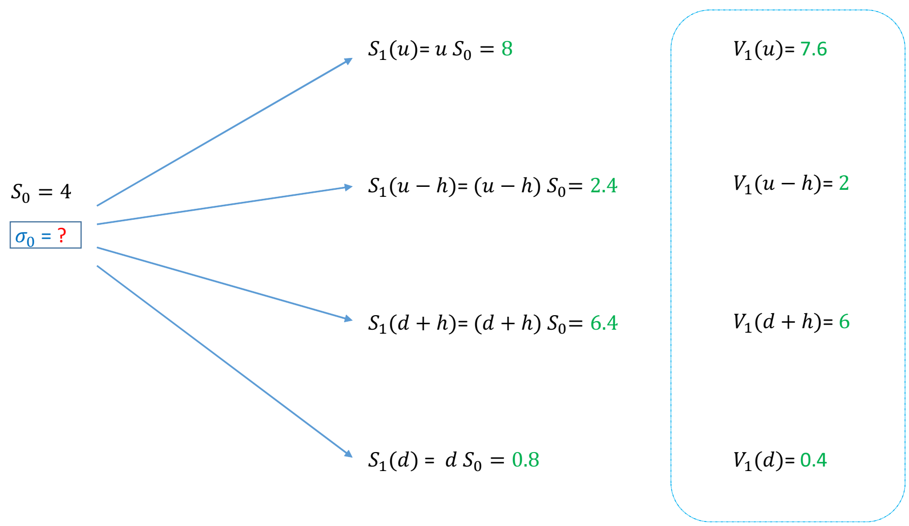

Figure 4.7: Stock price dynamics for the case when initial regime σ_0 is not known. Values of the contract at time $N = 1$ are depicted on the right. Numerical values of the stock and contract are highlighted in green.

Let the initial price of the stock be $S_0 = 4$. Our goal is to find the price of this contract at time $n = 0$. We accomplish that by constructing the replication portfolio that matches the payoff of the contract.

We start with the stock price dynamics for the one-period model. In Fig. 4.7 we depict all possible prices of the stock for $n = \{0, 1\}$. Given the payoff function defined in (4.75), we calculate the value of the contract at time $N = 1$ as follows

$V_1(u) = 7.6,$

$V_1(u - h) = 2,$

$V_1(d + h) = 6,$

$V_1(d) = 0.4.$

Let $W_1 = \phi_0 S_1 + R(W_0 - \phi_0 S_0)$ be the value of the portfolio at time $N = 1$. In order for this portfolio to replicate the contract's payoff for every possible value of the regime process sequence σ_0, σ_1, we need to equate the value of the portfolio and the payoff of the contract for every possible value of the regime process sequence σ_0, σ_1. Therefore, we obtain the following system of equations

$$\begin{cases} W_1(u) = \phi_0 u S_0 + R(W_0 - \phi_0 S_0) = V_1(u) = (S_0 u - K), \\ W_1(u-h) = \phi_0(u-h)S_0 + R(W_0 - \phi_0 S_0) = V_1(u-h) = (S_0(u-h) - K), \\ W_1(d+h) = \phi_0(d+h)S_0 + R(W_0 - \phi_0 S_0) = V_1(d+h) = (S_0(d+h) - K), \\ W_1(d) = \phi_0 d S_0 + R(W_0 - \phi_0 S_0) = V_1(d) = (S_0 d - K). \end{cases} \tag{4.76}$$

Note, that if the system of equations (4.76) with two unknowns ϕ_0 and W_0 has a unique solution, then the initial price of the contract V_0 is unique and is equal to the initial value of the replicating portfolio W_0. Solving the system of equations (4.76) we obtain a unique solution $\phi_0 = 1$ and $W_0 = 11/3$. Hence, the contract of interest has a unique initial price $V_0 = 11/3$.

Next example illustrates that there are derivative securities that can not be priced under the assumption that the initial regime is unknown.

Example 4. Consider a one-period regime switching model with jumps with initial regime being unknown. Consider an European call option with a payoff function at time $N = 1$

$$V_1 = (S_1 - K)_+, \tag{4.77}$$

where S_1 is the price of the underlying asset at time $N = 1$ and K is a strike price. Let the parameters of the model be the same as in the example 3 with only one exception. Let the strike price of the option be $K = 2.4$. The objective is to find the price of the option at time $n = 0$.

Given the payoff function given by formula (4.77), we calculate the value of the option at time $N = 1$ as follows
$V_1(u) = 5.6$,
$V_1(u-h) = 0$,
$V_1(d+h) = 4$,
$V_1(d) = 0$.
In Fig. 4.8 we depict all possible prices of the stock for $n = \{0, 1\}$ and the values of the payoff function of the European call option at time $N = 1$.

We construct a replicating portfolio that matches the payoff of the option, and obtain

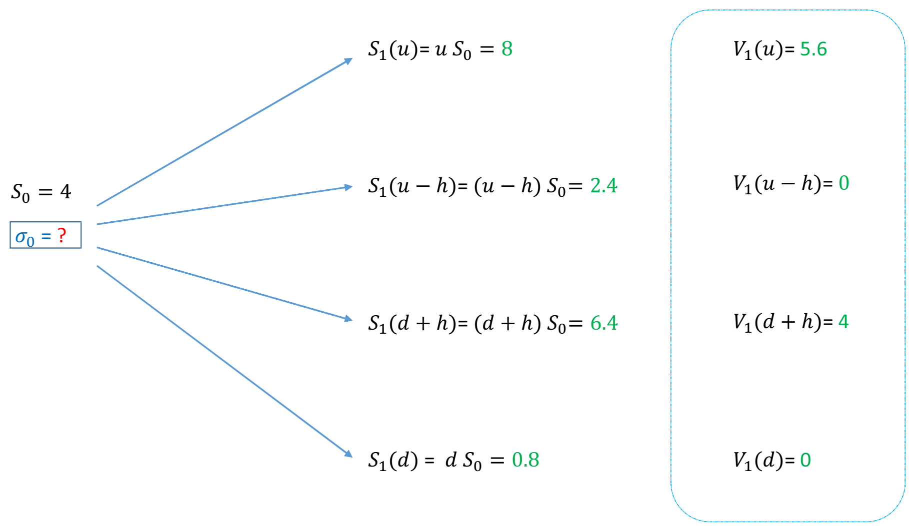

Figure 4.8: Stock price dynamics for the case when initial regime σ_0 is not known. Values of the European call option at time $N = 1$ are depicted on the right. Numerical values of the stock and European call option are highlighted in green

the following system of equations with two unknowns W_0 and ϕ_0

$$\begin{cases} W_1(u) = \phi_0 u S_0 + R(W_0 - \phi_0 S_0) = V_1(u) = (S_0 u - K)_+, \\ W_1(u-h) = \phi_0 (u-h) S_0 + R(W_0 - \phi_0 S_0) = V_1(u-h) = (S_0(u-h) - K)_+, \\ W_1(d+h) = \phi_0 (d+h) S_0 + R(W_0 - \phi_0 S_0) = V_1(d+h) = (S_0(d+h) - K)_+, \\ W_1(d) = \phi_0 d S_0 + R(W_0 - \phi_0 S_0) = V_1(d) = (S_0 d - K)_+. \end{cases} \tag{4.78}$$

However, the system of equations (4.78) is overdetermined and does not have a solution. This implies that under the described setting the initial price of the European call option V_0 can not be computed.

4.3.4 Selection of one specific equivalent martingale measure

In the previous sections we showed that under the assumption of unknown initial regime the regime switching model with jumps is incomplete and there exist infinitely many martingale measures equivalent to the historical measure $\mathbb{P}$. In this chapter we restrict our attention to one-period regime switching model with jumps and denote $\mathscr{M}$ to be the family of equivalent martingale measures described by the following equation

$$R = q_1 u + q_2(u-h) + q_3(d+h) + q_4 d, \tag{4.79}$$

where $0 < q_i < 1$, $\forall i = [1, 2, 3, 4]$.

Recall that in Example 3 we showed that even thought the market is incomplete, there exist an attainable contingent claim. Example 4, however, described a contingent claim that is not attainable. Therefore, there appears a reasonable question on how to value a non-attainable contingent claim. Note that any measure from the family $\mathscr{M}$ can be used to evaluate contingent claim. But how to choose the "right" equivalent martingale measure? How to select one specific martingale measure that is the "closest" (in some sense) to the historical measure $\mathbb{P}$? The no-arbitrage pricing principle alone is not sufficient to answer those questions. In this section, we are interested in choosing one specific martingale measure from the family $\mathscr{M}$ that can be used to value a derivative security in our incomplete model. The price of the derivative security will be calculated as the expectation with respect to the equivalent martingale measure of the discounted payoff.

There have been several approaches proposed to select one particular measure form the family $\mathscr{M}$, for example, a variance minimizing hedging [49, 50], minimal martingale measure approach [17] and Merton's method [39]. However, in some cases minimal martingale measure may not be equivalent to the historical measure $\mathbb{P}$ [19].

Utility based models provide another way of choosing the right equivalent martingale measure. These models are based on the assumption that the investor's preferences can be expressed by choosing a suitable utility function. By maximizing expected utility one can obtain the optimal (in the particular utility function sense) equivalent martingale measure. There are several types of utility function used, for example, exponential utility, logarithmic utility and power utility functions.

Another approach is based on the minimal entropy criterion. Application of this criterion can be found in various disciplines such as statistical physics, information theory and theory of large deviation. In financial mathematics, it was applied, for example, by Csiszar in [8], who studied the problem of existence of probability measure that minimizes the relative entropy on a set of probability measures under some linear constraints. Frittelli [19] demonstrated that measure in the class $\mathscr{M}$ that minimizes the relative entropy with respect to the historical measure $\mathbb{P}$ can be a good selection for the pricing measure. It was noticed in several papers [11,19,24,48] that the problem of finding the minimal entropy martingale measure is dual to the problem of the exponential utility maximization. Moreover, the minimal entropy measure coincides with the martingale measure obtained by utility maximization principle for exponential utility function [16,19]. It has also been shown [18] that under the no-arbitrage assumption minimal entropy martingale measure always exists and is equivalent to the historic measure $\mathbb{P}$. Moreover, Miyahara [40] connected the minimal entropy martingale measure with the large deviation theory though the Sanov's theorem, in particular, concluded that the minimal entropy martingale measure "is the most possible empirical probability measure of paths of price process in the class of the equivalent martingale measures." Furthermore, Stutzer [55] showed that minimal entropy measure is Esscher transform of original measure $\mathbb{P}$. This leads us to one of the most popular methods for selecting equivalent martingale measure.

Esscher transform was introduces by F. Esscher in 1932 and had been extensively used in actuarial pricing. Gerber and Shiu [22] used it to define a possible pricing measure in incomplete markets. In complete markets Esscher measure is equal to unique equivalent measure in each market [22]. It was shown [21] that the Esscher parameter is unique such that the discounted stock price is a martingale under the new probability measure. The measure obtained by applying Esscher transform to the historical measure is often justified through its connection to the minimal entropy measure, i.e., the "closest" equivalent martingale measure to the historic measure $\mathbb{P}$ in the sense of Kullback-Leibler distance. Buhlmann et al. [2] provided an additional economical rationale behind the Esscher transform. Authors considered a problem of Pareto optimal allocation of resources (i.e., the discounted increase in total aggregate market values of all assets) among investors. Under the assumption that each investor had an exponential utility function, Esscher transform provided optimal allocation of resources. Moreover, Esscher measure was connected to the exponential utility maximization measure in [9, 41].

In the next section we demonstrate how to chose an equivalent martingale measure from a family $\mathscr{M}$ using the Esscher transform.

4.3.4.1 Esscher transform approach

Consider one-period regime switching model with jumps under the assumption of unknown initial regime. Recall that the price of the stock at time $n = 1$ is defined as

$$S_1 = S_0 g_1,$$

where the g_1 is the jump-factor process with probability mass function $p_{g_1}(x)$ given as follows

$$p_{g_1}(x) = \begin{cases} p_1, & \text{when } \; x = u; \\ p_2, & \text{when } \; x = u - h; \\ p_3, & \text{when } \; x = d + h; \\ p_4, & \text{when } \; x = d, \end{cases} \tag{4.80}$$

such that $p_i > 0 \; \forall i = [1, 2, 3, 4]$. Denote corresponding moment generating function $M(z) = \mathbb{E}\left[g_1^z\right].$

Definition 24. Let $f(x)$ be a probability density function. Let θ be a real number such that

$$T(\theta) = \int_{-\infty}^{+\infty} exp(\theta x) f(x) dx$$

exists. A probability density function

$$f(x; \theta) = \frac{exp(\theta x) f(x)}{\int_{-\infty}^{+\infty} exp(\theta x) f(x) dx}$$

is called the Esscher transform (with parameter θ) of the original distribution.

Applying the Esscher transform to the historic probability measure $\mathbb{P}$ we obtain a new probability mass function specified by the parameter θ

$$p_{g_1}(x;\theta) = \begin{cases} \frac{p_1 u^\theta}{M(\theta)}, & \text{when } \ x = u; \\ \frac{p_2(u-h)^\theta}{M(\theta)}, & \text{when } \ x = u-h; \\ \frac{p_3(d+h)^\theta}{M(\theta)}, & \text{when } \ x = d+h; \\ \frac{p_4 d^\theta}{M(\theta)}, & \text{when } \ x = d. \end{cases} \tag{4.81}$$

Note that no-arbitrage conditions $0 < d < R < u$ and $h > \max\{u-R, R-d\}$ guarantee that the new probability measure is equivalent to measure $\mathbb{P}$. Denote the corresponding moment generating function as $M(z;\theta)$. We need to make sure that the discounted stock price process is a martingale with respect to the equivalent probability measure parameterized by θ. In particular, we require that

$$S_0 = \mathbb{E}_{\mathbb{Q}(\theta)}\left[R^{-1}S_1\right] = R^{-1}S_0\mathbb{E}_{\mathbb{Q}(\theta)}\left[g_1\right]. \tag{4.82}$$

Let $\theta^\star$ be the value of the parameter θ for which equation (4.82) holds. Note that $M(z;\theta) = \frac{M(z+\theta)}{M(z)}$ [21], and hence equation (4.82) can be equivalently rewritten as

$$\begin{aligned} R &= \frac{M(1+\theta)}{M(\theta)} \\ &= \frac{p_1 u^{\theta+1} + p_2(u-h)^{\theta+1} + p_3(d+h)^{\theta+1} + p_4 d^{\theta+1}}{p_1 u^\theta + p_2(u-h)^\theta + p_3(d+h)^\theta + p_4 d^\theta}. \end{aligned} \tag{4.83}$$

Multiplying both sides of equation (4.83) by the denominator of the right-hand side and regrouping similar terms we obtain equation

$$p_1(u-R)u^\theta + p_2(u-h-R)(u-h)^\theta + p_3(d+h-R)(d+h)^\theta + p_4(d-R)d^\theta = 0. \tag{4.84}$$

Next proposition demonstrates that there exist unique $\theta^\star$ that satisfies equation (4.84).

Proposition 12. *Let u, d, R, and h be given such that the no-arbitrage conditions $u > R > d > 0$ and $u > h > \max\{u-R, R-d\}$ are satisfied. Let $1 \geq p_i \geq 0$ $\forall i = \{1,2,3,4\}$ such that $\sum_{i=1}^4 p_i = 1$. Then there exist unique $\theta^\star$ that satisfies equation*

$$p_1(u-R)u^\theta + p_2(u-h-R)(u-h)^\theta + p_3(d+h-R)(d+h)^\theta + p_4(d-R)d^\theta = 0. \tag{4.85}$$

Proof. We start by considering function

$$f(\theta) = p_1(u-R)u^\theta + p_2(u-h-R)(u-h)^\theta + p_3(d+h-R)(d+h)^\theta + p_4(d-R)d^\theta. \quad (4.86)$$

We need to show that there exist a unique $\theta^\star$ such that $f(\theta^\star) = 0$. Let $\rho > \max\{u-h, d\}$. Define auxiliary function

$$\tilde{f}(\theta) = \frac{f(\theta)}{\rho^\theta}$$

$$= p_1(u-R)\left(\frac{u}{\rho}\right)^\theta + p_2(u-h-R)\left(\frac{u-h}{\rho}\right)^\theta + p_3(d+h-R)\left(\frac{d+h}{\rho}\right)^\theta + p_4(d-R)\left(\frac{d}{\rho}\right)^\theta. \quad (4.87)$$

Note that

$$f(\theta^\star) = 0 \Leftrightarrow \frac{f(\theta^\star)}{\rho^{\theta^\star}} = 0.$$

Now consider the derivative of $\tilde{f}(\theta)$

$$\begin{aligned} \tilde{f}'(\theta) &= p_1(u-R)\left(\frac{u}{\rho}\right)^\theta \ln\left(\frac{u}{\rho}\right) + p_2(u-h-R)\left(\frac{u-h}{\rho}\right)^\theta \ln\left(\frac{u-h}{\rho}\right) \\ &+ p_3(d+h-R)\left(\frac{d+h}{\rho}\right)^\theta \ln\left(\frac{d+h}{\rho}\right) + p_4(d-R)\left(\frac{d}{\rho}\right)^\theta \ln\left(\frac{d}{\rho}\right). \quad (4.88) \end{aligned}$$

From the no-arbitrage conditions and from the fact that $\rho > \max\{u-h, d\}$ it follows that $\tilde{f}'(\theta) > 0$. Therefore, $\tilde{f}(\theta)$ is a strictly increasing function. Now consider the limit of $\tilde{f}(\theta)$ when $\theta \to -\infty$:

$$\begin{aligned} \lim_{\theta\to-\infty} \tilde{f}(\theta) &= \lim_{\theta\to-\infty} p_1(u-R)\left(\frac{u}{\rho}\right)^\theta + \lim_{\theta\to-\infty} p_2(u-h-R)\left(\frac{u-h}{\rho}\right)^\theta \\ &+ \lim_{\theta\to-\infty} p_3(d+h-R)\left(\frac{d+h}{\rho}\right)^\theta + \lim_{\theta\to-\infty} p_4(d-R)\left(\frac{d}{\rho}\right)^\theta \\ &= -\infty. \quad (4.89) \end{aligned}$$

Formula (4.89) follows from the fact that $\frac{u}{\rho} > 1$, $\frac{u-h}{\rho} < 1$, $\frac{d+h}{\rho} > 1$, $\frac{d}{\rho} < 1$, and $d-R < 0$,

$u - h - R < 0$. Similarly, we consider the limit of $\tilde{f}(\theta)$ when $\theta \to \infty$:

$$\begin{aligned}
\lim_{\theta\to\infty} \tilde{f}(\theta) &= \lim_{\theta\to\infty} p_1(u-R)\left(\frac{u}{\rho}\right)^{\theta} + \lim_{\theta\to\infty} p_2(u-h-R)\left(\frac{u-h}{\rho}\right)^{\theta} \\
&+ \lim_{\theta\to\infty} p_3(d+h-R)\left(\frac{d+h}{\rho}\right)^{\theta} + \lim_{\theta\to\infty} p_4(d-R)\left(\frac{d}{\rho}\right)^{\theta} \\
&= \infty. \qquad (4.90)
\end{aligned}$$

Formula (4.90) follows from the fact that $\frac{u}{\rho} > 1$, $\frac{u-h}{\rho} < 1$, $\frac{d+h}{\rho} > 1$, $\frac{d}{\rho} < 1$, and $u-R > 0$, $d + h - R > 0$. Therefore, by Bolzano's Intermediate Value Theorem [34, p. 120] there exist a unique $\theta^\star$ such that $\tilde{f}(\theta^\star) = 0$, and hence $f(\theta^\star) = 0$. □

Although $\theta^\star$ exist and is unique, there is no closed-form solution for $\theta^\star$ for one-period regime switching model with jump. However, it can always be found numerically (iteratively) as a solution of equation (4.84).

Parameter $\theta^\star$ uniquely determines the equivalent martingale measure. Denote this measure as $\mathbb{Q}$. Consider now the properties of measure $\mathbb{Q}$ given by probabilities

$$\begin{cases}
q_1 = \frac{p_1 u^{\theta^\star}}{M(\theta^\star)}, & \text{when } x = u; \\
q_2 = \frac{p_2 (u-h)^{\theta^\star}}{M(\theta^\star)}, & \text{when } x = u-h; \\
q_3 = \frac{p_3 (d+h)^{\theta^\star}}{M(\theta^\star)}, & \text{when } x = d+h; \\
q_4 = \frac{p_4 d^{\theta^\star}}{M(\theta^\star)}, & \text{when } x = d,
\end{cases} \qquad (4.91)$$

where $\theta^\star$ is a solution to equation

$$p_1(u-R)u^{\theta} + p_2(u-h-R)(u-h)^{\theta} + p_3(d+h-R)(d+h)^{\theta} + p_4(d-R)d^{\theta} = 0. \quad (4.92)$$

As we stated earlier, because of the no-arbitrage conditions this measure $\mathbb{Q}$ is equivalent to the historic measure $\mathbb{P}$. Moreover, measure $\mathbb{Q}$ is in the class $\mathscr{M}$ described by the equation (4.79). In order to check it, one can divide equation (4.92) by $M(\theta^\star)$ and regroup similar terms to obtain equation (4.79). Note also that measure $\mathbb{Q}$ is a function of historical probability distribution $\mathbb{P}$, a jump size h and the Esscher parameter $\theta^\star$. Next proposition demonstrates that the four probabilities q_1, q_2, q_3, and q_4 defining new equivalent martingale measure $\mathbb{Q}$ are continuously differentiable functions of parameters

p_1, p_2, p_3, p_4, and h.

Proposition 13. *Let u, d, R, and h be given such that the no-arbitrage conditions $u > R > d > 0$ and $u > h > \max\{u - R, R - d\}$ are satisfied. Let $1 \geq p_i \geq 0$ $\forall i = \{1, 2, 3, 4\}$ such that $\sum_{i=1}^{4} p_i = 1$. Let $\theta^\star$ be the solution to the equation*

$$p_1(u-R)u^\theta + p_2(u-h-R)(u-h)^\theta + p_3(d+h-R)(d+h)^\theta + p_4(d-R)d^\theta = 0. \quad (4.93)$$

Define measure $\mathbb{Q}$ by probabilities

$$\begin{cases} q_1 = \frac{p_1 u^{\theta^\star}}{M(\theta^\star)}, & \text{when} \quad x = u; \\ q_2 = \frac{p_2 (u-h)^{\theta^\star}}{M(\theta^\star)}, & \text{when} \quad x = u - h; \\ q_3 = \frac{p_3 (d+h)^{\theta^\star}}{M(\theta^\star)}, & \text{when} \quad x = d + h; \\ q_4 = \frac{p_4 d^{\theta^\star}}{M(\theta^\star)}, & \text{when} \quad x = d. \end{cases} \quad (4.94)$$

Then probabilities q_1, q_2, q_3 and q_4 are continuously differentiable functions of parameters p_1, p_2, p_3, p_4, and h.

To prove Proposition 13 we use The Implicit Function Theorem 9.7.2 from [56, p. 269].

Proof. (of Proposition 13) In Proposition 12 we have demonstrated that for fixed values of parameters u, d, R and h that satisfy the no-arbitrage conditions and probabilities p_1, p_2, p_3, p_4 there exist a unique $\theta^\star$ such that $\tilde{f}(\theta^\star) = 0$. Observe also that function $\tilde{f}$ depends on p_1, p_2, p_3, p_4 and h and hence we can write function $\tilde{f}$ as $\tilde{f}(\theta; p_1, p_2, p_3, p_4, h)$. We have also showed that $\frac{\partial \tilde{f}}{\partial \theta}(\theta^\star; p_1, p_2, p_3, p_4, h) > 0$. Moreover, function $\tilde{f}$ is continuously differentiable with respect to θ. Therefore, by the Implicit Function Theorem there exist a continuously differentiable function $\hat{f}$ such that $\theta = \hat{f}(p_1, p_2, p_3, p_4, h)$ and $\tilde{f}(\hat{f}(p_1, p_2, p_3, p_4, h); p_1, p_2, p_3, p_4, h) = 0$. Hence, probabilities q_1, q_2, q_3 and q_4 are continuously differentiable functions of parameters p_1, p_2, p_3, p_4, and h. $\square$

Next we consider two special cases: when $h = 0$ and when $p_3 = p_4 = 0$.

Consider the case when $h = 0$. Equation (4.84) can be rewritten as

$$(p_1 + p_2)(u - R)u^\theta + (p_3 + p_4)(d - R)d^\theta = 0. \quad (4.95)$$

Solving this equation for θ we obtain

$$\theta^\star = \frac{\log\left(\frac{(p_3+p_4)(R-d)}{(p_1+p_2)(u-R)}\right)}{\log\left(\frac{u}{d}\right)}. \tag{4.96}$$

For $h = 0$ factor process g_1 can take two possible values u and d. Historic probability mass function is given as follows

$$p_{g_1}(x) = \begin{cases} p_1 + p_2, & \text{when } \; x = u; \\ p_3 + p_4, & \text{when } \; x = d, \end{cases} \tag{4.97}$$

and new probability mass function is given by

$$p_{g_1}(x;\theta^\star) = \begin{cases} \frac{(p_1+p_2)u^{\theta^\star}}{M(\theta^\star)}, & \text{when } \; x = u; \\ \frac{(p_3+p_4)d^{\theta^\star}}{M(\theta^\star)}, & \text{when } \; x = d, \end{cases} \tag{4.98}$$

where $M(\theta^\star) = (p_1 + p_2)u^{\theta^\star} + (p_3 + p_4)d^{\theta^\star}$. We plug in $\theta^\star$ into the formula (4.98) and obtain

$$p_{g_1}(x;\theta^\star) = \begin{cases} \frac{R-d}{u-d}, & \text{when } \; x = u; \\ \frac{u-R}{u-d}, & \text{when } \; x = d, \end{cases} \tag{4.99}$$

Note that when $h = 0$ model completely coincides with the classical binomial tree model. And the newly obtained equivalent martingale measure is the unique martingale measure provided in Chapter 3.

Consider now the case when $p_3 = p_4 = 0$. This corresponds to the scenario when initial regime is $\sigma_0 = +1$. Equation (4.84) can be rewritten as

$$p_1(u - R)u^\theta = p_2(R - (u - h))(u - h)^\theta. \tag{4.100}$$

Solving this equation for θ we obtain

$$\theta^\star = \frac{\log\left(\frac{p_2(R-(u-h))}{p_1(u-R)}\right)}{\log\left(\frac{u}{u-h}\right)}. \tag{4.101}$$

Therefore, new probability measure is found as

$$p_{g_1}(x;\theta^\star) = \begin{cases} \frac{R-(u-h)}{h}, & \text{when } \ x = u; \\ \frac{u-R}{h}, & \text{when } \ x = u-h; \\ 0, & \text{when } \ x = d+h; \\ 0, & \text{when } \ x = d. \end{cases} \tag{4.102}$$

Note that in the case when $p_3 = p_4 = 0$, new probability measure obtained using the Esscher transform coincides with a unique equivalent martingale measure provided for the regime switching model with $\sigma_0 = +1$ in Section 4.2.1. This example is a good illustration of continuity of probabilities q_1, q_2, q_3, and q_4 in parameters: when $p_3 \to 0$ and $p_4 \to 0$ probabilities q_1, q_2, q_3, and q_4 converges to corresponding probabilities given by (4.102). Similarly, one can show that for $p_1 = p_2 = 0$, new probability measure obtained using the Esscher transform will result in a unique equivalent martingale measure obtained for the regime switching model with $\sigma_0 = -1$ in section 4.2.1.

In the next example we will show how to price a non-attainable contingent claim from example 4.

Example 5. Consider now the non-attainable contingent claim from example 4. We have demonstrated that the European call option with a payoff $V_1 = (S_1 - K)_+$ can not be replicated by the stock and money market accounts. In order to price this contract we need to choose one equivalent martingale measure form a family of measures $\mathscr{M}$. We obtain this measure by applying the Esscher transform to the historic measure $\mathbb{P}$ given, for example, by $p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$. As we stated earlier, equation

$$p_1(u-R)u^\theta + p_2(u-h-R)(u-h)^\theta + p_3(d+h-R)(d+h)^\theta + p_4(d-R)d^\theta = 0 \tag{4.103}$$

has no closed-form solution. However, using numerical methods (e.g., Newton method) we obtain $\theta^\star = 0.16$. Therefore, measure $\mathbb{Q}$ is given by probabilities

$$\begin{cases} q_1 = \frac{p_1 u^{\theta^\star}}{M(\theta^\star)} = 0.2872, & \text{when } \ x = u; \\ q_2 = \frac{p_2 (u-h)^{\theta^\star}}{M(\theta^\star)} = 0.2369, & \text{when } \ x = u-h; \\ q_3 = \frac{p_3 (d+h)^{\theta^\star}}{M(\theta^\star)} = 0.2771, & \text{when } \ x = d+h; \\ q_4 = \frac{p_4 d^{\theta^\star}}{M(\theta^\star)} = 0.1988, & \text{when } \ x = d. \end{cases} \tag{4.104}$$

The initial price of the contract is the discounted expectation with respect to measure $\mathbb{Q}$ of contract's payoff, i.e.,

$$\begin{aligned} V_0 &= \mathbb{E}_{\mathbb{Q}}\left[R^{-1}V_1\right] \\ &= \frac{5}{6}\left(0.2872 \times V_1(u) + 0.2369 \times V_1(u-h) + 0.2771 \times V_1(d+h) + 0.1988 \times V_1(d)\right) \\ &= 2.2639. \end{aligned} \tag{4.105}$$

Thus, the Esscher transform allowed us to price a non-attainable contingent claim.

In this section, we have considered the regime switching model with jumps. We investigated the properties of this model under different scenarios. First, we considered the case of known initial regime σ_0. We have concluded that the model is arbitrage-free and complete under the condition that the jump size h satisfies inequality $h > \max\{u - R, R - d\}$. We provided formulas for pricing any derivative security and demonstrated the pricing process with the example. Next, we considered the case of unknown initial regime. In this case, both one-period model and N-period model are incomplete since there are infinitely many equivalent martingale measures. We have demonstrated that some derivative securities can be replicated under such assumptions and some derivative securities can not. In order to price derivative securities that can not be replicated by the stock and money market accounts, we used the Esscher transform to obtain an equivalent martingale pricing measure. Example 5 demonstrated how to price a non-attainable contingent claim using the Esscher transform.

Chapter 5: Conclusion

In this work, we considered discrete time regime switching model with jumps. This model is a combination of a discrete telegraph model with jumps and a classical binomial tree model. This model is more realistic than the standard binomial tree model. First, our model of interest incorporates regime switches. This leads to more truthful modeling of changing economic environment. Second, usage of jumps enables large movements of the model parameters. Thus, regime switching model with jumps is rich enough to capture the effect of economical, political, and other significant events.

The purpose of this work was to provide a comprehensive completeness analysis of the discrete time regime switching model with jumps. First, we briefly described useful definitions from probability theory and financial mathematics that are used throughout the text. We started with the description of the classical binomial tree model. We showed that under some conditions the model is arbitrage-free and complete. The numerical example 1 demonstrates how to construct replicating portfolio and price derivative security in the binomial tree model. Next, we described the discrete time regime switching model with jumps. We considered two scenarios: when initial regime is known and when initial regime is unknown. In the first case, we proved that if jump size h satisfies condition $h > \max\{R - d, u - R\}$ then the model is complete and hedging is perfect. This result emphasizes the importance of the jump size; jumps serve as an instrument to avoid arbitrage and complete the model. Moreover, we found a closed form formula for the unique equivalent martingale measure. In numerical example 2 we provided a way to price a derivative security under regime switching model with jumps and known initial regime. Next, we considered the case of unknown initial regime. We demonstrated that both one-period and N-period models are not complete even when jump size h satisfies condition $h > \max\{R - d, u - R\}$. Incompleteness followed from the fact that when initial regime is unknown there are infinitely many equivalent martingale measures. This implies that initial regime is also crucial for the model completeness. Numerical examples 3 and 4 demonstrated that there are derivative securities that can be priced under the regime switching model with jumps and unknown initial regime and there are derivative securities that can not be priced. Furthermore, we have discussed the Esscher transform as a tool to select a particular martingale measure for pricing derivative securities that can not be replicated with the stock and money market accounts. The numerical example 5 illustrated the way of applying the Esscher transform to price a European call option in the incomplete one-period regime switching model with jumps.

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