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Brian A. Grimm for the degree of Master of Science in Mechanical Engineering presented on November 30, 2012

Title: Investigation Into the Role of Strength and Toughness in Composite Materials with an Angled Incident Crack

Abstract approved

John P. Parmigiani

Understanding the mechanical behavior of composite materials requires extensive knowledge of fracture behavior as a crack approaches an interface between the bulk material and the reinforcement structure. Overall material toughness can be greatly influenced by the propensity of an impinging crack to propagate directly through the substrate or deflect along an interface boundary. As the basis for this thesis; the assertion that an impinging crack may encounter a reinforcement structure at various incident angles is explored. This requires the ability to predict crack penetration/deflection behavior not only normal to the reinforcement, but at various incident angles. Previous work in the area of interface fracture mechanics has used a stress or energy based approach, with recent advances in the field of a combined cohesive-zone method.

Work presented here investigates the interaction between strength and toughness when using the cohesive-zone method on the problem of an impinging crack not normally
incident to the interface of a composite material. Computational mechanics methods using Abaqus and user-define cohesive elements will be applied to this angled incident crack problem. A circular model based on the displacement field equations for mode-I fracture loading is introduced and verified against well-established LEFM solutions. This circular model is used to study the effects of incident crack angle on the penetration vs. deflection behavior of an impinging crack at various angles of incidence. Additionally, the effects of angle on the load applied to the model at fracture are explored. Finally, a case study investigating how the interaction between strength and toughness found using the cohesive-zone method helps to explain some of the inconsistencies seen in the interface indentation fracture test procedure.
Investigation Into the Role of Strength and Toughness in Composite Materials with an Angled Incident Crack

by
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Brian A. Grimm, Author
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Investigation Into the Role of Strength and Toughness in Composite Materials with an Angled Incident Crack
1 Introduction

1.1 Background

Composite material toughness can be greatly influenced by the propensity of an impinging micro-crack to propagate directly through the substrate or deflect along the interface boundary between substrate and bulk material. As the basis for this thesis; the assertion is that an impinging crack may encounter a reinforcement structure at various incident angles requiring the ability to predict crack penetration/deflection behavior not only normal to the interface, but at various incident angles. Predicting whether a crack will penetrate into the substrate or deflect around a reinforcement structure allows for greater accuracy when determining composite material properties.

Considering the substrate and interface material to be homogenous and isotropic, there are two distinct modes of crack propagation. As a micro crack approaches a composite interface through a bulk material under pure mode-I loading (Figure 1(a)), the crack will propagate through the reinforcement structure (penetration) (Figure 1(b)) or along the interface (deflection) (Figure 1(c)). Crack propagation behavior will depend on the material properties of the bulk material, interface and substrate, along with the incident angle of crack to interface, loading conditions, and sample geometry.
1.2 Literature Review

Predicting the penetration vs. deflection behavior of an impinging crack at varied angles of incident to an interface will be investigated in this thesis using a combined stress-energy approach. Traditionally, studies of interface fracture mechanics have taken one of two distinctly different approaches to generate failure criteria; stress-based or energy-based analysis. When utilizing a purely stress based approach the ratio of substrate strength to interface strength (critical strength ratio) is used to determine the penetration vs. deflection behavior of an impinging crack. When using an energy-based approach the ratio of critical fracture energy of the substrate to critical fracture energy of the interface (critical toughness ratio) is used to determine the penetration vs. deflection behavior of an impinging crack.
1.2.1 Stress-Based Approach

One of the early methods developed to analytically characterize the fracture mechanics of a material uses a strength-based failure criterion and is based on the Inglis approach [1]. Inglis calculated the stress field at a pre-existing crack tip and predicts crack growth when this stress exceeds a critical value, such as the yield strength of a material. Further stress-based analyses, introduced by Cook and Gordon [2], calculated the stress field immediately ahead of a pre-existing crack tip as it impinged an interface and compared the normal stresses surrounding the crack tip. A critical strength ratio at which deflection through an interface would occur (instead of penetration through the substrate) was formulated. Further refinements by Gupta et al. [3] address crack propagation behavior with anisotropic materials based on elastic property mismatch. Using a purely stress-based approach considers the strength of the constituent materials involved, but does not factor in the energy requirements (and therefore material toughness) to cause material failure.

1.2.2 Energy-Based Approach

Moving away from a stress-based failure criterion is the Griffith energy-based failure criterion [4]. Energy-based analysis compares the fracture release energy of a deflecting and penetrating crack independently using material toughness as the failure criteria. Energy-based approaches require a pre-existing kink/crack and assumption of linear elasticity for proper results. Linear elasticity assumes that material yielding
does not play a role in crack growth mechanics, or that a correction factor is applied from experimental data, such as in ductile materials.

Predicting crack growth behavior by the Griffith method requires a thorough understanding of the fracture mechanics occurring at the crack tip. This calculation can be complicated by a modulus mismatch between the interface and substrate materials or multiple crack paths [5]. While Griffith introduced his theory before the proliferation of the linear elastic fracture mechanics (LEFM) field, LEFM assumptions inherent in the Griffith method prevent accurate predictions for realistic materials outside the LEFM regime and limits the usefulness of a purely energy based approach. This approach also requires a preexisting crack assumption to account for the mathematical singularity occurring at the crack tip. In practice, Griffith’s theory provides acceptable results in the region of highly brittle materials but will overestimate the surface energy density (\(\gamma\)) of a ductile material.

Limitations of the Griffith theory to brittle materials hindered its general application until Irwin [6] modified the model to account for plastic deformation at the crack tip. By breaking Griffith’s theory into two parts, the elastic energy of the LEFM solution is combined with the energy lost due to dissipative forces such as plastic deformation [6]. The resulting equation is still used today and introduces a key parameter in LEFM solutions, the stress intensity factor (\(K\)).

Additional advancements in energy-based analysis performed by He and Hutchinson [7], compare the fracture release energy of two distinct crack propagation
geometries; one corresponding to penetration through the interface (Figure 2(a)), and one corresponding to deflection along an interface (Figure 2(b)). The ratio of interface-to-substrate energy release rates under these conditions is compared and used to formulate a critical material toughness ratio at which deflection will occur. Major limitations of a purely energy-based approach are the need for a preexisting flaw and that small scale yielding dominates the cracked body [6].

![Figure 2](image.png)

Figure 2: Two geometry approach used by He and Hutchinson for a crack impinging an interface at an oblique angle. Fracture release energy is calculated for a crack (a) penetrating the substrate or (b) deflection through the interface. The ratio of the two energy release values forms a critical toughness value.

### 1.2.3 Combined Stress-Energy-Based Approach

In order to expand the capabilities of predicting crack growth in a wide range of engineering materials, Dugdale [8] introduced the theory that a cohesive force was preventing crack growth. This cohesive force is considered to be the yield strength of a material that behaves in an elastic-ideally plastic manner [9]. The crack tip is now represented by a narrow strip of plastic deformation loaded in plane stress directly leading the crack growth region. By assuming that yield strength represents the maximum applicable local stress at the crack tip, an unrealistic singularity (as in a
strength-based analysis) is no longer required. The Dugdale strip-yield model allows us to move away from the limitations of traditional LEFM fracture mechanics, but is still limited by the use of only the material yield strength as the failure criterion.

To improve the strip-yield model, Barenblatt [10], replaces the material yield strength with a cohesive law failure criteria. This cohesive law is representative of a nonlinear force applied by the atomic bonds at the crack tip. Elastic-plastic behavior in the crack process zone is approximated by superimposing two distinct elastic solutions; crack loading under remote tension and a through crack with closure stresses acting at the tip. By superimposing those two solutions, stresses in the yield–strip model are finite at the crack tip, thereby eliminating the need for a stress singularity at said crack tip. Crack growth caused by a nonlinear force that varies by displacement between the surfaces is the basis for the traction separation law used throughout this thesis.

Crack propagation utilizing traction separation laws must satisfy two requirements; the cohesive strength as the peak stress value in the traction separation law, and the total energy to propagate the crack as the toughness or area under the traction separation law. A cohesive strength value can be compared to the peak stress for crack propagation seen in the Inglis approach. While the area under a traction separation curve may be similar to the Griffith energy based method; LEFM assumptions are not necessarily required in a combined stress-energy approach.
Work performed by Leguillon [11] shows that stress-based or energy-based failure criteria are contradictory when applied to crack onset at a notch, and that considering only one of these criterion yields results in disagreement with experimental results. Leguillon [11] states that both the stress and energy failure criteria are fulfilled simultaneously at material fracture because strength and toughness are both necessary conditions for failure. Leguillon [11] goes on to assert that for the case of interface fracture the solution space must be bounded on one side by a stress-based formulation and on the other side by an energy-based formulation, creating a complete solution for the prediction of penetration vs. deflection at a composite interface.

Further developing the combined stress-energy approach, Parmigiani and Thouless [5] investigated the penetration vs. deflection behavior of a normally incident crack. They created a cohesive-zone method (CZM) to simulate the behavior of a single impinging crack tip normal to a composite material interface. This approach eliminated the need for two separate geometries (although if two geometries are used the results of He and Hutchinson [7] are recovered). By considering fracture energy in the interface and substrate concurrently, their results showed that the penetration vs. deflection behavior of a crack normally incident to an interface is a function of both fracture toughness and cohesive strength. The results generated by Parmigiani and Thouless [5] elucidate the importance of considering a combined stress-energy approach in the prediction of crack behavior at an interface.
1.3 Present Work

Analyzing the penetration vs. deflection behavior of a crack at a composite material interface could be solved in one of the three methods described above; stress-based, energy-based, and combined stress-energy based approaches. Considering the appropriateness of each method to analyze the same interface impinging crack situation illustrates the critical issue with using a purely stress or energy based approach. When using a stress-based approach the material strength is used as the failure criteria and material toughness is completely ignored. Conversely, for an energy-based approach the material toughness is used as the failure criteria and material strength is completely ignored. When predicting crack behavior at an interface using one of these mutually exclusive methods, one of two completely different (and unrelated) material properties is disregarded and inaccurate predictions may result. Previous work by Parmigiani and Thouless [5] has shown the importance of considering strength and toughness when predicting penetration vs. deflection behavior by applying a cohesive-zone method to a normally incident crack impinging an interface.

The work presented here will apply cohesive-zone computational methods developed by Parmigiani and Thouless [5] to explore crack penetration vs. deflection behavior with an impinging crack at varied angles of incidence to an interface. The effects of incident crack angle on the transition from penetration to deflection behavior for a composite material will be presented. Previous work has shown that comparing the resulting energy release rate of a normally incident crack for two distinct
geometries fails to capture the complex interaction of the substrate and interface at the transition from penetration to deflection. Using the cohesive-zone method allows capturing the creation of a fracture process zone in the penetration direction as well as into the interface on both sides of the impinging crack. Creating multiple process zones increases the total energy absorbed by the system prior to fracture and will therefore increase the required force to cause fracture in the composite. This increase in force required to cause fracture will be explored at the transition from penetration to deflection and how variation to the incident crack angle influences these results. Finally, in order to explore the concepts presented throughout this thesis in a real world scenario, a case study into finding the critical angle at which a given silicon nitride (Si$_3$N$_4$) material will transition from penetration to deflection will be presented.

2 Method

Computational mechanics methods combined with a stress-energy based approach, in the form of the cohesive-zone method, will be used to define the penetration vs. deflection behavior of an angled incident crack at a composite material interface.

2.1 Nomenclature

Listed below are the commonly used parameters presented throughout this thesis.

- $CZM$  Cohesive-zone method
- $\overline{E}$  Young’s modulus for plane strain
- $K$  Stress intensity factor
- $LEFM$  Linear elastic fracture mechanics
2.2 Cohesive-Zone Method

Combining strength and energy based failure criterion, the cohesive zone method (CZM) [5] used for this analysis is based on the yield-strip model introduced by Dougdale and Barenblatt [8] [10]. The CZM incorporates strength and toughness into a region of material ahead of the crack by creating a traction-separation law (TSL).

The TSL used throughout this thesis is shown in Figure 3. Strength-based failure criterion is inherent to the TSL as the cohesive-zone is loaded to a maximum cohesive strength value. Energy-based failure criterion is the area under the TSL and represents the material toughness.
Figure 3: Idealized traction-separation laws used as the basis for cohesive element modeling showing (a) mode-I and (b) mode-II failure criteria. Cohesive-zone elements used for this analysis are based on the relationship between crack tip opening displacement ($\delta_{cctod}$) and stress around the crack tip ($\sigma$ or $\tau$) (Figure 3). User defined ratios, $\delta_1/\delta_3$ and $\delta_2/\delta_3$, dictates the shape of the traction separation law. By defining the material toughness ($I$) and cohesive strength ($\sigma$), or cohesive strength ($\sigma$) and critical crack tip opening displacements ($\delta_{I,II}$), an idealized picture of a materials elastic, plastic, and failure response is created. This TSL approach is that of Thouless et. al [12], where the left portion represents a linear elastic response of the material (Figure 4($\delta_1$)), followed by perfectly plastic deformation (Figure 4($\delta_1-\delta_2$)), and finally a degradation of material strength prior to failure (Figure 4($\delta_2-\delta_3$)).
Figure 4: Traction-separation law represents ($\delta_1$) linear-elastic ($\delta_1 - \delta_2$) perfectly-plastic and ($\delta_2 - \delta_3$) degradation of strength prior to failure seen in the process zone of a crack front.

Crack tip loading in both the mode-I and mode-II direction has a large impact when simulating crack deflection into an interface. Therefore, normal (Figure 3(a)) and shear (Figure 3(b)) traction separation law properties are defined independently. Mixed-mode element failure criterion is calculated by using Equation 1 where $\Gamma_I$ and $\Gamma_{II}$ are the material toughness values in the mode-I (crack opening) and mode-II (in-plane shear) direction respectively.

$$\frac{G_I}{\Gamma_I} + \frac{G_{II}}{\Gamma_{II}} = 1$$

Equation 1

Element failure occurs when the energy release rate ($G$) equals the material toughness ($\Gamma$). Energy release rate ($G$) is the area under the appropriate traction-separation law at a given crack tip opening displacement ($\delta$) and is given by Equation 2 for mode-I loading and Equation 3 for mode-II loading.
\[ G_1 = \int_0^{\delta_3} \hat{\sigma} \, d\delta_{ctod} \]  
Equation 2

\[ G_{11} = \int_0^{\delta_3} \hat{\sigma} \, d\delta_{ctod} \]  
Equation 3

Parmigiani and Thouless [5] investigated penetration vs. deflection using a finite element (FE) cohesive-zone model to simulate the crack tip behavior at a composite material interface. Use of this approach (1) eliminates the need for two distinct crack propagation geometries and (2) inherently includes both strength and toughness criteria. Their numerical models simulate a composite interface with a single impinging crack normal to the interface. By considering fracture energy in both the interface and substrate concurrently, results have shown that the penetration vs. deflection behavior of a crack normally incident to an interface is a function of both fracture toughness and cohesive strength.

The work presented here will extend this approach to an angled incident crack. A pure mode-I loading case (Figure 5) was selected for the modeling condition to represent the most general fracture case.
A single-edge cracked semi-infinite plate with isotropic and homogeneous material properties is loaded in plane strain under a remote tensile stress. Looking at an area directly around the crack tip, the remote stress creates a displacement field near this crack tip that can be quantified through Equation 4 for material displacement parallel to the crack tip plane and Equation 5 for material displacement perpendicular to the crack tip plane [13]:

\[
\begin{align*}
    u_x &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \left( \frac{\theta}{2} \right) \left[ k - 1 + 2\sin^2 \left( \frac{\theta}{2} \right) \right] \\
    u_y &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \left( \frac{\theta}{2} \right) \left[ k + 1 + 2\cos^2 \left( \frac{\theta}{2} \right) \right]
\end{align*}
\]

where:

\[
\begin{align*}
    \mu &= \frac{E}{2(1+\nu)} & \text{for plane strain} \\
    k &= 3 - 4\nu & \text{for plane strain} \\
    K_I &= 1.12\sigma \sqrt{\pi a} & \text{for semi-infinite plate subject to remote tensile stress}
\end{align*}
\]
By creating a circular model (Figure 7) to represent the area directly around the crack tip, the mode-I displacement field equations become the outer edge displacement boundary conditions (Equation 4 & Equation 5). Circular model geometry is assumed to have a radius ($R$) less than the crack length ($a$) of the semi-infinite plate. In order to replicate a crack approaching an interface at varying incident angles ($\Theta_{inc}$), the interface cohesive zone can be tilted relative to the incident crack while the penetration cohesive zone continues along the crack tip plane. The penetration cohesive zone is assumed to always lead directly from the crack tip front along the crack tip plane because crack loading is pure mode-I.

![Figure 6: Circular model schematic; edge displacement field generated from Equation 4 & Equation 5, dashed lines indicate cohesive-zone elements for penetration vs. deflection.](image)

The bulk material is represented by Abaqus plane strain elements (CPE4R) while the cohesive zone penetration/deflection crack propagation paths are modeled with user defined cohesive elements (UEL) (Figure 7). The UEL code can be found in
Appendix A and incorporates the traction separation laws properties into the cohesive-zone method introduced previously. Mesh refinement was performed until at least 10 cohesive elements captured the fracture process zone at the smallest fracture length scale expected during analysis.

Figure 7: Circular model created using Abaqus plane strain elements (CPE4R) in the main body and user defined elements (UEL) for the cohesive zones.

Using the cohesive-zone method allows the analysis of a combined geometry approach, as opposed to the two geometry approach common in stress or energy based method. This single geometry approach captures the complex interactions occurring simultaneously in the interface and substrate during transition from penetration to deflection behavior. Numerical simulations presented here may form a process zone in up to three directions simultaneously (through the substrate and in both interface directions) as shown in Figure 8. There is an increase in the amount of energy absorbed by the composite material prior to failure due to this process zone creation,
and therefore an increase in the applied force on the material at failure. This
correlation will be further explored in section 3.3.

![Figure 8: Region directly surrounding crack tip showing the growth of a fracture process zone in both the penetration and deflection directions through the cohesive elements for (a) penetration and (b) deflection.](image)

### 2.3 Dimensionless Groups

Dimensionless groups, created using the Buckingham pi-theorem, develop mathematical models from the complex physical phenomena occurring during interface fracture. Taking the constituent equations related to mode-I interface fracture and combining them into dimensionless groups also reduces the number of variables required to solve these constituent equations. Dimensionless groups were previously developed by Parmigiani and Thouless to fully define cohesive-zone modeling methodology [5]. The following basic material properties; elastic modulus ($\bar{E}$), Poisson ratio ($\nu$), interface toughness ($\Gamma_i$), substrate toughness ($\Gamma_s$), interface strength ($\bar{\sigma}_i$), substrate toughness ($\bar{\sigma}_s$), radius of the circular model ($R$), and angle of incident crack are used as the basis of the dimensionless groups.

$$\bar{E}, \nu, \Gamma_i, \Gamma_s, \bar{\sigma}_i, \bar{\sigma}_s, R, \theta$$
By taking these seven (7) variables and their constituent equations for mode-I loading, then subtracting the three (3) fundamental physical quantities for mass, time, and distance, we are left with four (4) critical dimensionless groups [5] [13] that must be fulfilled:

\[
\frac{E \Gamma}{\dot{\sigma}^2 R} \cdot \frac{\sigma_s}{\dot{\sigma}_i} \cdot \frac{\Gamma_s}{\Gamma_i} \cdot \frac{\Gamma}{\bar{E} \bar{R}}
\]

The fracture length scale, \( \bar{E} \Gamma / \dot{\sigma}^2 R \), scales with the length of fracture process zone ahead of the crack tip. Previous analysis has shown excellent agreement with LEFM predictions when \( \bar{E} \Gamma / \dot{\sigma}^2 R < 0.01 \) [5]. Strength ratio, \( \sigma_s / \dot{\sigma}_i \), is the ratio of substrate cohesive strength to interface cohesive strength. Toughness ratio, \( \Gamma_s / \Gamma_i \), is the ratio of substrate toughness to interface toughness. The final critical dimensionless group, \( \Gamma / \bar{E} \bar{R} \), compares fracture toughness to elastic modulus and is normalized with respect to model radius. By maintaining dimensionless group values for \( \bar{E} \Gamma / \dot{\sigma}^2 R, \sigma_s / \dot{\sigma}_i, \Gamma_s / \Gamma_i, \Gamma / \bar{E} \bar{R}, \nu \); values for \( E, \Gamma_i, \Gamma_s, \sigma_i, \sigma_s, R \) appropriate for the simulation may be selected while providing the same predictions of penetration vs. deflection.

2.4 Circular Cohesive Model Material Properties

A circular model replicating pure mode-I loading of a crack tip impinging a composite interface (Figure 7) was used to analyze the effects of varying incident angle of a crack relative to the interface. Material properties for this analysis where selected to replicate a generic ceramic material and are intended to illustrate the effects
of incident angle of an impinging crack on the penetration vs. deflection behavior with regards to material strength and toughness. Calculated and selected values of the key parameters are listed in Table 1.

Table 1 Dimensionless Group Values used for Angled Crack Circular Model

<table>
<thead>
<tr>
<th>$\frac{\bar{E} \Gamma_i}{\bar{\sigma}_l^2 R}$</th>
<th>$\frac{\Gamma_s}{\bar{E} R}$</th>
<th>$\frac{\hat{\sigma}_s}{\bar{\sigma}_l}$</th>
<th>$\frac{\Gamma_s}{\bar{\Gamma}_i}$</th>
<th>$\nu$</th>
<th>$\theta$</th>
<th>$\frac{\delta_1}{\delta_3}$</th>
<th>$\frac{\delta_1}{\delta_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1x10^{-6}</td>
<td>Varied</td>
<td>Varied</td>
<td>0.3</td>
<td>0°-90°</td>
<td>0.1</td>
<td>0.75</td>
</tr>
</tbody>
</table>

2.5 Comparing LEFM and Circular Cohesive-zone Model

Calculating the stress intensity field (K-field) of a single edge-cracked (SEC) semi-infinite plate is well established using LEFM fracture methods. A comparison is presented here in order to verify that the circular model introduced as Figure 7 would yield results comparable to those calculated using LEFM methods. For this analysis, a circular model was created with a cohesive zone in the penetration direction to ensure that an LEFM K-field could be accurately represented with user defined cohesive elements. LEFM K-field values were calculated using Equation 6 to find the stress normal to the crack plane for a linear elastic, isotropic material.

$$\sigma_{yy} = \frac{K_t}{\sqrt{2\pi r}}$$  \text{Equation 6}

To benchmark the K-field, a standard LEFM circular model was created with a collapsed node quadratic element at the crack tip. The resulting K-factor value was recorded at the crack tip to ensure that the stress comparison occurs at the same stress intensity factor value. Cohesive zone properties were selected as $\Gamma_i / \bar{E} R = 4.9E-5$,
\( \bar{E} \Gamma_s / \hat{\sigma}^2 R = 0.05 \), \( \Gamma_s = 10.0 \), \( \hat{\sigma} = 447.2 \), and \( \nu = 0.3 \) (further discussion about fracture length scale occurs in the next section). The cohesive numerical model was analyzed and compared to resulting stresses normal to the crack plane for the LEFM model (Figure 9).

![Figure 9](image)

**Figure 9:** Comparison of stress normal to the crack plane (K-field) for the circular cohesive model compared to an LEFM analytical solution.

As expected, K-field stress normal to the crack plane is approximated by cohesive elements outside of the stress singularity region generated in the analytical solution. This aspect of the cohesive-zone method is very important, because it eliminates a need for pre-existing kinks/cracks required in a purely strength or energy based analysis. One important caveat to the results presented in Figure 9 is the fracture length scale of the cohesive zone. Further explanation as to the importance of having \( \bar{E} \Gamma / \hat{\sigma}^2 R << 1.0 \) is explored in the next section.
2.6 Fracture Length Scale

Presented in this section are the effects of fracture length scale on penetration vs. deflection behavior. Fracture length scale is one of the four (4) critical dimensionless groups introduced earlier and scales the fracture process zone found in the cohesive-zone method. Evidence presented here will confirm that a small process zone is needed to accurately predict penetration vs. deflection behavior of a crack impinging an interface and reach cohesive element solutions comparable to analytical LEFM solution.

Confirming the need for small fracture length scale in order to replicate LEFM results (Equation 7), further investigation into the effects of varying $\bar{\Gamma}_i/\bar{\sigma}^2R$ and its effects on the K-field were explored.

\[ \sigma_{yy} = \frac{K_i}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] \] \hspace{1cm} \text{Equation 7}

Varying the fracture length scale of the interface while maintaining $\Gamma_i/\bar{E}R = 1.0 \times 10^{-6}$, will illustrate the importance of maintaining $\bar{\Gamma}_i/\bar{\sigma}^2R << 1.0$. Figure 10 shows the effect of increasing fracture length scale of the interface $\bar{E} \Gamma_i/\bar{\sigma}^2 R$ to 0.05, 1.0, 3.0, and finally 10.0.
Figure 10: Comparison of stress normal to the crack plane (K-field) for the circular cohesive model. Fracture length scale ($\bar{\sigma}^2 / \sigma^2 R$) is varied from small (0.05) to large (10.0) and compared to the LEFM K-field solution. Stress normal to the crack plane and distance ahead of the crack tip is normalized and presented on a log-log scale.

When compared to the LEFM solution, we can see that the cohesive elements approximate LEFM results when the fracture length scale is small. As the fracture length scale increases, we can note that the cohesive strength limits of the elements underestimate stress values calculated by LEFM methods. Because of this limitation in maximum stress applicable at the crack tip, a lower fracture length scale will represent an LEFM solution much more accurately. One final reason to maintain a small fracture length scale ($\bar{\sigma}^2 / \sigma^2 R < 0.01$) is to avoid boundary interactions, because by definition a fracture length scale greater than 1.0 is larger than the model radius.
Results of fracture length scale analysis have shown the importance of maintaining $\bar{E} \bar{\gamma} / \bar{\sigma}^2 R \ll 1.0$ in order to replicate LEFM results. The Cohesive-zone method presented here more accurately models a K-field when fracture length scales are sufficiently small to minimize the cohesive process zone to stress singularity disparity.

3 Results

Using the cohesive-zone method outlined above, a circular model will be used to define the penetration vs. deflection behavior of a crack at various incident angles to a composite interface. Also explored will be the effects of incident crack angle on the applied fracture load at failure and the role incident crack angle plays.

3.1 Failure Mechanism Map

The model used for this analysis is the circular model introduced in Figure 7. This model has no pre-existing kinks ahead of the main crack (e.g. He and Hutchinson [7]), only the impinging crack terminating at the crack tip. A strip of cohesive elements in the penetration and deflection direction are used to determine the penetration vs. deflection behavior of a particular set of material properties.

For the current study of penetration vs. deflection behavior, the Dundurs parameters $\alpha = \beta = 0$ (making $E_b = E_s = \bar{E}$, and $\nu_b = \nu_s = \nu$) because an elastic modulus mismatch was thoroughly explored for a normally incident crack, and is beyond the scope of this thesis in regards to an angled incident crack in a bi-material composite.
The physical explanation for framing results presented here with no elastic mismatch between substrate and interface is that self-reinforced ceramics are being considered as the primary material for the presented case study. Dimensionless group $\Gamma_i/E_R$ was fixed at a physically reasonable value of $1.0 \times 10^{-6}$ [5]. Numerical studies have shown that even fairly significant changes to $\Gamma_i/E_R$ at this level have a negligible effect on penetration vs. deflection predictions [5]. Additionally, fracture length scale of the interface was fixed at $\bar{E}\Gamma_i/\sigma_i^2 R = 0.01$ while the fracture length scale of the substrate ($\bar{E}\Gamma_s/\sigma_s^2 R$) was varied. For each selected value of the penetration fracture length scale, numerical calculations were performed for a discrete set of $\sigma_s/\sigma_i$ and $I_s/I_i$. By noting if the crack would propagate through or deflect into the interface for a set of strength and toughness values, the values at which the model switched from penetration to deflection was recorded as a transition point on a failure-mechanism map (Figure 11). The resulting transition curve reaffirms results modeled previously by Parmigiani and Thouless [5] for an edge cracked bar with the impinging crack normal to the interface.
Figure 11: Failure-mechanism map of a crack normally incident to an interface illustrating penetration vs. deflection behavior in terms of strength ratio ($\sigma_s / \sigma_i$) and toughness ratio ($\Gamma_s / \Gamma_i$) of a material in which $E \Gamma_s / \theta_s^2 R = 0.01$, $\Gamma_i / E R = 1.0 \times 10^{-6}$, and $\alpha = \beta = 0$. An impinging crack is predicted to deflect into the interface at any point right of the transition curve, while penetration into the substrate is predicted at any point to the left of the transition curve. Error bars represent the uncertainty of fracture direction near transition due to resolution of selected test data points and instability of the model near transition.

The generated transition curve represents the predicted boundary between crack penetration through the substrate (left of the curve) and crack deflection (right of the curve). It can be seen that crack deflection is more likely for higher values of $\theta_s / \theta_i$ and $\Gamma_s / \Gamma_i$. Conversely, for lower values of $\theta_s / \theta_i$ and $\Gamma_s / \Gamma_i$ a crack is more likely to penetrate through the interface and into the substrate. Two observations can be made from the shape of the transition curve: (1) at larger $E \Gamma_s / \theta_s^2 R$ values the strength ratio is the dominant parameter, while (2) at smaller $E \Gamma_s / \theta_s^2 R$ values the toughness ratio is
the dominant behavior when predicting penetration vs. deflection behavior. A vertical asymptote appears to form at lower strength ratios at which the toughness ratio would have little to no effect on the transition behavior. It’s worth noting that penetration will always occur when the strength ratio drops below this asymptotic value and would be comparable in nature to the critical strength ratio found using stress-based analysis techniques. However, at lower toughness values there does not appear to be the same asymptotic behavior. This behavior is due to the cohesive-zone method not having a pre-existing kink in the predicted crack path, but the analytical result may be replicated in a kink is created in the cohesive-zone [5] [14]. Even at high strength values, where the transition behavior is very sensitive to toughness ratio, the strength ratio still plays a role in predicting penetration vs. deflection behavior.

3.2 Variation of Incident Crack Angle

In composite materials, a crack may impinge the composite interface at any angle between 0°-90°. Previous work performed by Parmigiani and Thouless [5] illustrated the importance of incorporating strength and toughness failure criteria when predicting penetration vs. deflection behavior of a normally incident crack. By incorporating a similar cohesive-zone method, pure mode-I loading circular model, and varying incident angle of the impinging crack, the results of a combined strength-energy based computational modeling approach will be presented.
In order to generate a failure-mechanism map for the range of incident crack angles; runs where performed at each selected incident crack angle and a range of substrate fracture length scale values to generate a failure-mechanism map (Figure 12). To maintain the $\bar{E} \Gamma_s / \bar{\sigma}_s^2 R$ dimensionless value for each of the tested strength ratio values, toughness ratio is varied as well. Values for $\bar{\sigma}_s / \bar{\sigma}_i$ and $\Gamma_s / \Gamma_i$ at which transition from penetration to deflection occurs are plotted on a failure mechanism map. A transition curve was generated for each of the following values: 90°, 85°, 80°, 75°, 70°, 60°, 45°, 30°, 15°, and 0° incident crack angle.

Figure 12: Failure-mechanism map of a crack at various angles of incidence to an interface illustrating penetration vs. deflection behavior in terms of strength ratio ($\sigma_s / \sigma_i$) and toughness ratio ($\Gamma_s / \Gamma_i$) of a material in which $\bar{E} \Gamma_i / \bar{\sigma}_i^2 R = 0.01$, $\Gamma_i / \bar{E} R = 1.0 \times 10^{-6}$, and $\alpha = \beta = 0$. An impinging crack is predicted to deflect into the interface at any point right of the penetration/deflection curve, while
penetration into the substrate is predicted at any point to the left of the penetration/deflection curve. Error bars represent the uncertainty of fracture direction near transition due to resolution of selected test data points and instability of the model near transition.

As one would expect, a decrease in incident angle promotes deflection along the interface. When comparing the 90° incident crack angle to any of the shallower angles, the need for a $\hat{\sigma}_s/\hat{\sigma}_l$ and $I_s/I_l$ mismatch is reduced. At low $I_s/I_l$ values, the penetration vs. deflection behavior becomes increasing sensitive to changes in $I_s/I_l$ at shallower angles. One of the most important aspects to note from a change in incident crack angle is the decrease in strength and toughness interaction at shallower angles. As the incident crack angle is decreased, the penetration vs. deflection behavior becomes more heavily influenced by strength based failure criteria.

3.3 Applied Force Response Due to Variation of Incident Crack Angle

Using the cohesive-zone method to model interface fracture allows crack process zone growth both through the substrate and into the interface. Using a purely energy based approach such as used by He and Hutchinson [7] requires a pre-existing kink in either the penetration or deflection direction and compares the fracture energy release rate for each case (Figure 2). It stands to reason that the amount of energy absorbed to form a crack process zone, even if not in the direction of failure, will add to the total force required to cause fracture.
By looking at the reaction force of a node perpendicular to the crack tip plane during loading of the circular model, there should be an increase in the force seen at fracture when a crack process zone is forming in both penetration and deflection (transition point). Previous work by Strom [14] investigated this peak in load at fracture for a normally incident crack and developed a normalizing factor which has been adapted for use with the circular model case (Equation 8). In the following equation the nodal reaction force normal to the crack plane (\( F \)) is projected over the circular model (2R) and normalized against the change in fracture toughness of the interface (\( \Gamma_i \)).

Normalized Reaction Force = $\frac{F}{2R \sqrt{\frac{E \times \Gamma_i}}}$ \hspace{1cm} \text{Equation 8}$

Nodal reaction force values were recorded on the penetration and deflection side of the 90° incident crack case at $\bar{E} \Gamma_i / \bar{\sigma}^2 R = \bar{E} \Gamma_i / \bar{\sigma}^2 R = 0.01$ and $\Gamma_i / \bar{E} R = 1.0E^{-6}$. Incident crack angle was varied and these reaction force values were then normalized according to Equation 8 and plotted on a log/standard scale for clarity (Figure 13).
Figure 13: Normalized reaction force, $F/2R \cdot \sqrt{R/\Gamma_i}$, seen at fracture for various incident crack angles versus material toughness ratio plotted on a log/standard scale. Circular model has the following properties: $\bar{E}\Gamma_i/\bar{\sigma}_i^2R = 0.01$, $\bar{E}\Gamma_s/\bar{\sigma}_s^2R = 0.01$, $\Gamma_i/\bar{E}R = 1.0 \times 10^{-6}$.

Results from this peak reaction force per varied incident crack indicate a lower energy is absorbed in the growth of the process zone away from the main deflection direction as the incident angle becomes shallower. Similar to Figure 12, we can see that as incident angle decreases, the toughness ratio required to promote deflection is also reduced. Of special note is the penetration behavior of the model as it approaches a toughness and strength ratio of unity, where the penetration and deflection cohesive element properties are identical. At $\bar{\sigma}_s/\bar{\sigma}_i = I_s/I_i = 1$, incident crack angle of the model is irrelevant. The LEFM solution for a crack with a toughness equal to $\Gamma_i + \Gamma_s$ (to represent the interface and penetration cohesive-zone elements) in the penetration direction only is reached.

Using an analysis technique that can quantify the creation of a crack process zone in both the penetration and deflection direction simultaneously illustrates
shortcomings of using a two unique geometry approach. This complex interaction is captured quite easily using the cohesive-zone method.

4 Self-Reinforced Si$_3$N$_4$ Cohesive-Zone Method Application

Ceramic materials are enticing as an engineering material because of their high heat and electrical resistance, high stiffness, wear resistance, and chemical inert nature. Possible applications range from rocket engine nozzles to turbine bearings to combustion engine pistons. However, major limitations to the proliferation of ceramic materials in engineering applications is their low fracture toughness, low strength in tension/shear, and little to no plastic deformation prior to catastrophic failure. Self-reinforced ceramics present an avenue on which to tailor engineering properties to meet the demands of new and novel applications. Understanding the mechanical behavior of ceramic composite materials requires extensive knowledge of fracture behavior as a crack approaches an interface between the bulk material and the reinforcement structure.

Ceramic composite fracture toughness is directly tied to toughening mechanisms such as crack bridging, frictional pull-out, and crack deflection around a reinforcing structure as shown in Figure 14(a-c). These reinforcement structures vary from β-phase grains to whiskers/fibers, but share the fact that they are randomly oriented within the bulk material. In this section, the prediction of critical incident crack angle to cause deflection found through the cohesive-zone method will be used to explain
some of the inaccuracies seen when using the interface indentation fracture test method.

![Image](image_url)

**Figure 14**: Material toughening mechanisms commonly seen in ceramic composites including: (a) Crack Bridging, (b) Frictional Pull-out, (c) and crack deflection around a self-reinforcing grain structure.

### 4.1 Interface Indentation Fracture Test Method

The effective use of linear elastic fracture mechanics (LEFM) requires accurate values of material fracture toughness, but obtaining toughness values for ceramic materials can be challenging. Traditional direct-measurement methods are difficult to implement and/or produce inaccurate results [15, 16, 17]. Attempts to create sharp pre-cracks without inadvertent specimen failure and erroneously high fracture toughness values lead many researchers to look into special methods [15]. Indentation fracture (IF) methods offer what has appeared to be an effective alternative.

Indentation fracture methods such as the Vickers indentation fracture (VIF) test consist of applying a sharp diamond tip to the surface of a prepared specimen and calculating fracture toughness from the resulting crack propagation. Developed during the 1970s [18], the VIF tests uses a Vickers indenter to initiate cracks in a brittle
specimen. Fracture toughness is determined from the lengths of the cracks emanating from the indentation site [19, 20]. LEFM typically assumes homogenous materials with a well-defined crack, however, in the majority of polycrystalline ceramics this assumption rarely appears [17]. More commonly, microstructure of the material plays a critical role in how a crack propagates from an initial flaw [17]. These factors can significantly affect crack growth and, therefore, have major impact on interface fracture test results.

Adapting VIF methods for determining the toughness of material interfaces is the Interface Indentation Fracture (IIF) test developed by Becher et al. [21]. This test introduces a series of indentations into the substrate at various angles of incidence to the interface. Crack growth is initiated in the material, and the tendency of each crack to penetrate through the interface or deflect (debond) along the interface is noted. Angles closer to normal incidence tend to promote penetration, while shallower incident angles tend to promote deflection. By noting the critical angle at which crack propagation changes from penetration to deflection, the IIF test makes inferences about the relative fracture toughness of an interface. However, toughness values from IIF tests have been shown to be semi-quantitative at best and their use should be relegated to rankings of material toughness or comparing process variations to the same material [15, 16, 17].

The IIF test consists of measuring (i) the length of crack propagation along the crack interface ($l_{int}$) and (ii) the crack angle of incidence to the interface ($\Theta$), as shown in Figure 15. Note that as $\Theta$ becomes greater, the tendency for a crack to propagate by
deflection along the interface is reduced and \( l_{\text{int}} \) tends towards zero. Use of this relationship between \( l_{\text{int}} \) and \( \Theta \) allows determination of the critical angle (\( \Theta_{\text{crit}} \)) at which crack propagation transitions from deflection along the interface (\( l_{\text{int}} > 0 \)) to penetration through the interface (\( l_{\text{int}} = 0 \)) into the substrate material on the other side of the interface.

![Figure 15: Vickers indenter usage in IIF test. Adapted from [21]](image)

After determining the value of the critical angle (\( \Theta_{\text{crit}} \)), a penetration vs. deflection criterion is used to determine the ratio of interface toughness to substrate material toughness. The ratio of interface-to-substrate energy release rates under these conditions is defined as the ratio, \( G_i/G_s \). Transition between penetration and deflection is predicted to occur at a critical value of \( G_i/G_s \), which is equivalent to the material toughness ratio \( \Gamma_i/\Gamma_s \), and is a function of the Dundurs relative material stiffness parameter, \( \alpha \) (Equation 9). When calculating \( \alpha \), the elastic dissimilarity between the Young’s modulus of the interface (\( E_i \)) and the substrate (\( E_s \)) is quantified. This method of calculating energy release rates for two distinct geometries also applies to the
penetration vs. deflection behavior of a wedge-loaded crack impinging an interface at an oblique angle $\Theta$ [7].

$$\alpha = \frac{E_i - E_z}{E_i + E_z}$$  \hspace{1cm} \text{Equation 9}

Based on this angled crack analysis, Becher [21] determined that $\Theta_{\text{crit}}$ to cause interface deflection in the incoming crack could be used to estimate the ratio of fracture toughness, $\Gamma_i/\Gamma_s$. Because the material in question is a self-reinforced ceramic, the Dundurs $\alpha$ stiffness ratio is considered to be zero. The resulting relationship between incident crack angle and $\Gamma_i/\Gamma_s$ is presented in Figure 16. As expected, a lower toughness mismatch is required at a comparatively shallower angle to cause deflection at the interface. Values for $\Gamma_s$ are usually found using more traditional fracture testing such as single edge pre-cracked beam or surface crack in flexure, as the substrate material is large enough for proper specimen preparation [17]. However, $\Gamma_i$ is notoriously difficult to determine due to its microscopic size [17]. This difficulty in quantifying interface fracture properties is one reason for the popularity of the IIF test method [16] [17].
Figure 16: Ratio of interface to substrate energy release rate ($\Gamma_i/\Gamma_s$) vs. critical incident crack angle ($\theta_{\text{crit}}$). Adapted from [7] [21].

### 4.2 Si$_3$N$_4$ Critical Angle Case Study

Self-Reinforced silicon nitride is the most common application for the Becher method. A case study was performed by applying the cohesive-zone method to determine the critical incident crack angle to cause deflection as a crack impinges the interface surrounding $\beta$-phase reinforcement grain. Material properties have been selected to represent a general case of self-reinforced Si$_3$N$_4$ (Table 2) with a grain size of approximately 1 $\mu$m.

<table>
<thead>
<tr>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$\Gamma_i$ (MPa\sqrt{m})</th>
<th>$\Gamma_s$ (MPa\sqrt{m})</th>
<th>$\sigma_i$ (GPa)</th>
<th>$\sigma_s$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>365</td>
<td>0.25</td>
<td>1.23</td>
<td>2.1</td>
<td>19.0</td>
<td>43.5</td>
</tr>
</tbody>
</table>
To estimate the critical angle at which an incident crack will switch from penetration through the reinforcement grain to deflection into the interface, the fracture length scale of the interface and substrate remained constant. For each tested incident crack angle, the strength ratio (and to maintain the remaining dimensionless groups, toughness ratio) was varied to values above and below the calculated modeling parameters shown in Table 3.

Table 3: Modeling Values Calculated from Material Properties

<table>
<thead>
<tr>
<th>$\tilde{E} L_i$</th>
<th>$\tilde{E} L_s$</th>
<th>$\tilde{L_s}$</th>
<th>$\tilde{L_i}$</th>
<th>$\tilde{\sigma}_s$</th>
<th>$\tilde{\sigma}_l$</th>
<th>$\Gamma_s$</th>
<th>$\Gamma_i$</th>
<th>$\nu$</th>
<th>$R$</th>
<th>$\frac{\delta_1}{\delta_3}$</th>
<th>$\frac{\delta_2}{\delta_3}$</th>
</tr>
</thead>
</table>
| 0.0042        | 0.00233        | 3.31 x 10^-5 | 2.29         | 2.915          | 0.25           | 100      | 0.05     | 0.75

By noting penetration vs. deflection behavior of the cohesive elements, the transition point for each incident angle was noted. This transition point was plotted as a function of $\tilde{\sigma}_s/\tilde{\sigma}_l$ and $\Gamma_s/\Gamma_i$ to create the failure mechanism map for the self-reinforced silicon nitride test case (Figure 17). Also plotted is a sample of the accompanying transition curve for each incident angle to show that at the selected silicon nitride material properties, strength and toughness pay a role in predicting penetration vs. deflection behavior.
Figure 17: Failure-mechanism map of self-reinforced Si$_3$N$_4$ composite illustrating penetration vs. deflection behavior in terms of strength ratio ($\sigma_s/\sigma_i$) and toughness ratio ($\Gamma_s/\Gamma_i$). $\frac{\sigma_s}{\sigma_i} = 0.00233$, $\frac{\Gamma_s}{\Gamma_i} = 3.31 \times 10^{-5}$, and $\alpha = \beta = 0$. Silicon Nitride material properties selected for this case study transition at an incident crack angle between 70° and 75°.

From Figure 17, the silicon nitride material properties fall on the deflection side of the 70° transition curve, but on the penetration side of the 75° transition curve. So it can be stated that for the Si$_3$N$_4$ material tested in this case study, the critical transition from penetration to deflection occurs between 70°-75° incident crack angle.

One critical observation about the penetration vs. deflection behavior around these selected material properties is the interaction between strength and toughness as shown in the transition curves. The basis for IIF test results is that the material is in a region at which strength does not play a role in penetration vs. deflection behavior, but
that a toughness ratio can be directly related to an incident crack angle. By observing that on the transition curves presented in Figure 17, a single toughness ratio value can be correlated to a range of angles simply by varying the strength ratio of the materials. Additionally, a range of toughness values for a fixed incident crack angle can be selected by varying the strength ratio of the material. It is therefore critical to incorporate both a composite materials strength and toughness values into a prediction of critical incident crack angle at which penetration vs. deflection will occur. This interaction should help to explain some of the inconsistencies seen when using the IIF testing techniques.

5 Conclusion

Interface fracture of a composite material has historically been divided into two mutually exclusive methodologies; stress-based and energy-based failure criteria. The work presented here, using a combined stress-energy approach performed through computational mechanics techniques, shows the importance of considering both strength and toughness concurrently. A circular cohesive zone model was introduced and used to illustrate the effects of varying incident crack of a composite. Of special note is the tendency of the solution space to move towards a more strength based transition behavior as the incident crack angle became shallower. Additionally, the fracture load required to cause failure in the model was analyzed close to the transition point when crack process zones where forming in all three directions (penetration and both sides of interface). This interaction between the substrate and interface cohesive
zones illustrates the shortcomings of using a two geometry approach and comparing the final energy release rates independently.

A case study was performed on a self-reinforced silicon nitride composite using the cohesive-zone method. The IIF test method was described and a comparison was made between the IIF methodology and results of the current cohesive-zone method. It was shown that the IIF method is based on a fixed relationship between a unique toughness ratio value for each incident crack angle. CZM results indicate that some of the variation seen in using the IIF method stem from the lack of considering material strength.

Additional future work is recommended in exploring an angled incident crack at a bi-material interface. Analytical solutions for such a case often require restrictive assumptions, but the cohesive-zone method would be an excellent fit for this type of complex analysis. Also, stability of the cohesive element program near transition would be beneficial to provide a more accurate transition curve. While the user defined element presented here is able to be rotated to any angle required for analysis, instability and early termination made analysis of the resulting data very time consuming. Improvements in element stability would permit runs much closer to the transition point.
6 References


7 APPENDIX A – User Defined Element Code

This code allows any element orientation, however the fracture path must be through faces 41 and 23. Also element must be rectangular. Written by J.P. Parmigiani Summer 2012

Data line inputs (in input file)
- PROPS(1),(2),(3),(4),(5),(6),(7),(8)
  1: Normal cohesive strength
  2: Normal traction law delta1/delta3
  3: Normal traction law delta2/delta3
  4: Normal toughness
  5: Shear cohesive strength
  6: Shear traction law delta1/delta3
  7: Shear traction law delta1/delta3
  8: Shear toughness
- JPROPS(1)
  1: Adhesive layer boundary condition
  2: Half-model symmetry along adhesive layer
  3: Full model, no symmetry B.C. along adhesive layer

Columns 7-72, integer i-n

SUBROUTINE UEL(RHS, AMATRX, SVARS, ENERGY, NDOFEL, NRHS, NSVARS,
  PROPS, NPROPS, COORDS, MCRD, NNODE, U, DU, V, A, JTYPE, TIME, DTIME,
  KSTEP, KINC, JELEM, PARAMS, NDLOAD, JDLTYP, ADLMAG, PREDEF, NPREDF,
  LFLAGS, MLVARX, DDLMag, MDLOAD, PNEWDT, JPROPS, NJPROP, PERIOD)

INCLUDE 'ABA_PARAM.INC'

DIMENSION RHS(MLVARX,*), AMATRX(NDOFEL,NDOFEL), PROPS(*),
  SVARS(*), ENERGY(8), COORDS(MCRD, NNODE), U(NDOFEL),
  DU(MLVARX,*), V(NDOFEL), A(NDOFEL), TIME(2), PARAMS(*),
  JDLTYP(MDLOAD,*), ADLMAG(MDLOAD), DDLMag(MDLOAD,*),
  PREDEF(2,NPREDF,NNODE), LFLAGS(*), JPROPS(*)

DOUBLE PRECISION AMATRXh(8,8), RHS(8), RHS_temp(8), L(8,8)
DOUBLE PRECISION strenN, r13N, r23N, toughN
DOUBLE PRECISION strenS, r13S, r23S, toughS
DOUBLE PRECISION d3N,d2N,d1N,d3S,d2S,d1S
DOUBLE PRECISION p1XU,p1YU,p2XU,p2YU,p3XU,p3YU,p4XU,p4YU
DOUBLE PRECISION p1XD,p1YD,p2XD,p2YD,p3XD,p3YD,p4XD,p4YD
DOUBLE PRECISION s23LU,s41LU,s23LD,s41LD,s23DN,s41DN
DOUBLE PRECISION sideA,sideB,sideC
DOUBLE PRECISION p1Angle,p2Angle,p3Angle,p4Angle
DOUBLE PRECISION s23StressN,s41StressN,s23StressS,s41StressS
DOUBLE PRECISION s23GI,s41GI,s23GII,s41GII,s23FC,s41FC
DOUBLE PRECISION eHeight,eLength
DOUBLE PRECISION y2U,y1U,x2U,x1U,y2D,y1D,x2D,x1D,thetaU,thetaD
DOUBLE PRECISION sxyDN,sxyStiffN,sxyStressN
DOUBLE PRECISION sxyDS,sxyStiffS,sxyStressS
DOUBLE PRECISION sxyGI,sxyGII
DOUBLE PRECISION s41SS,s23SS,s41DS,s23DS
DOUBLE PRECISION deltaXU,deltaYU,deltaXD,deltaYD
DOUBLE PRECISION p1DS,p2DS,p3DS,p4DS
DOUBLE PRECISION::PIOver2=1.5707963267948966D0
INTEGER i,j

VARIABLE DICTIONARY

SET ELEMENT TYPE

IF(JTYPE.EQ.3)THEN

INITIALIZE ARRAYS

DO 10 i=1,8
   RHS(h(i))=0.0D0
   RHS_temp(i)=0.0D0
   RHS(i,1)= 0.0D0
DO 20 j=1,8
   AMATRXh(i,j)=0.0D0
   AMATRX(i,j)= 0.0D0
   L(i,j)=0.0D0
20 CONTINUE
10 CONTINUE

READ INFORMATION FROM INPUT FILE

strenN=PROPS(1)
r13N = PROPS(2)
r23N = PROPS(3)
toughN = PROPS(4)
strenS = PROPS(5)
r13S = PROPS(6)
r23S = PROPS(7)
toughS = PROPS(8)

c CALCULATE TRACTION LAW PARAMETERS
c IF(JPROPS(1).EQ.1)
toughN = 0.5D0 * toughN
d3N = (2.0D0 * toughN) / ([strenN * (1.0D0 - r13N + r23N)])
d2N = r23N * d3N
d1N = r13N * d3N
d3S = (2.0D0 * toughS) / ([strenS * (1.0D0 - r13S + r23S)])
d2S = r23S * d3S
d1S = r13S * d3S
c

C CALCULATE NODAL COORDINATES
c
c Calculate the undeformed coordinates
c
p1XU = COORDS(1, 1)
p1YU = COORDS(2, 1)
p2XU = COORDS(1, 2)
p2YU = COORDS(2, 2)
p3XU = COORDS(1, 3)
p3YU = COORDS(2, 3)
p4XU = COORDS(1, 4)
p4YU = COORDS(2, 4)
c
c Calculate Deformed Coordinates
c
p1XD = p1XU + U(1)
p1YD = p1YU + U(2)
p2XD = p2XU + U(3)
p2YD = p2YU + U(4)
p3XD = p3XU + U(5)
p3YD = p3YU + U(6)
p4XD = p4XU + U(7)
p4YD = p4YU + U(8)
c

C CALCULATE ELEMENT HEIGHT AND LENGTH
c Height is distance between nodes 2-3 or 4-1 since rectangular element
c Length is distance between nodes 1-2 or 3-4 since rectangular element
c
eHeight=DSQRT((p4XU-p1XU)**2.0+(p4YU-p1YU)**2.0)
eLength=DSQRT((p1XU-p2XU)**2.0+(p1YU-p2YU)**2.0)
c
c CALCULATE ELEMENT ROTATION
c
c Undefomed element
c
deltaYU= ((p2YU-p1YU)+(p3YU-p4YU))/2.0D0
deltaXU= ((p2XU-p1XU)+(p3XU-p4XU))/2.0D0
thetaU=DATAN2(deltaYU,deltaXU)
c
c Deformed element
c
deltaYD= ((p2YD-p1YD)+(p3YD-p4YD))/2.0D0
deltaXD= ((p2XD-p1XD)+(p3XD-p4XD))/2.0D0
thetaD=DATAN2(deltaYD,deltaXD)
c
c CALCULATE NORMAL AND SHEAR DISPLACEMENTS
c
c Calculate the normal displacement of each side
c
s23LU=DSQRT((p3XU-p2XU)**2.0+(p3YU-p2YU)**2.0)
s41LU=DSQRT((p1XU-p4XU)**2.0+(p1YU-p4YU)**2.0)
c
s23LD=DSQRT((p3XD-p2XD)**2.0+(p3YD-p2YD)**2.0)
s41LD=DSQRT((p1XD-p4XD)**2.0+(p1YD-p4YD)**2.0)
c
s23DN=s23LD-s23LU
s41DN=s41LD-s41LU
c
c Calculate the shear displacement of each side
c
thetaD=thetaU
p1DS=U(1)*DCOS(thetaD)+U(2)*DSIN(thetaD)
p2DS=U(3)*DCOS(thetaD)+U(4)*DSIN(thetaD)
p3DS=U(5)*DCOS(thetaD)+U(6)*DSIN(thetaD)
p4DS=U(7)*DCOS(thetaD)+U(8)*DSIN(thetaD)
c
s41DS=p4DS-p1DS
s23DS=p3DS-p2DS
c
IF(DABS(thetaU).GT.1.0)THEN
CALCULATE STIFFNESS AND STRESS FOR EACH SIDE

Calculate normal stiffness and stress for each side

CALL StN(s41DN,strenN,d1N,d2N,d3N,s41StiffN,s41StressN)
CALL StN(s23DN,strenN,d1N,d2N,d3N,s23StiffN,s23StressN)

Calculate shear stiffness and stress for each side

CALL StS(s41DS,strenS,d1S,d2S,d3S,s41StiffS,s41StressS)
CALL StS(s23DS,strenS,d1S,d2S,d3S,s23StiffS,s23StressS)

Calculate instantaneous energy release rates

CALL EnergyS(s41DS,strenS,d1S,d2S,d3S,toughS,s41GII)
CALL EnergyS(s23DS,strenS,d1S,d2S,d3S,toughS,s23GII)
CALL EnergyN(s41DN,strenN,d1N,d2N,d3N,toughN,s41GI)
CALL EnergyN(s23DN,strenN,d1N,d2N,d3N,toughN,s23GI)

Calculate failure criteria

s41FC=s41GI/toughN+s41GII/toughS
s23FC=s23GI/toughN+s23GII/toughS

If element has failed, set state variable value to -1 and set stiffness and stress to zero

IF(s41FC.GE.1.0D0)SVARS(40)=-1.0
IF(s23FC.GE.1.0D0)SVARS(41)=-1.0

IF(SVARS(40).LT.0.0)THEN
s41StiffN=0.0D0
s41StiffS=0.0D0
s41StressN=0.0D0
s41StressS=0.0D0
ENDIF
IF(SVARS(41).LT.0.0)THEN
s23StiffN=0.0D0
s23StiffS=0.0D0
s23StressN=0.0D0
s23StressS=0.0D0
ENDIF

c
DEFINE STIFFNESS MATRIX
c
For horizontal element
c
AMATRXh(1,1) = eLength*s41StiffS/3.0D0
AMATRXh(1,3) = eLength*s23StiffS/6.0D0
AMATRXh(1,5) = -eLength*s23StiffS/6.0D0
AMATRXh(1,7) = -eLength*s41StiffS/3.0D0
AMATRXh(2,2) = eLength*s41StiffN/3.0D0
AMATRXh(2,4) = eLength*s23StiffN/6.0D0
AMATRXh(2,6) = -eLength*s23StiffN/6.0D0
AMATRXh(2,8) = -eLength*s41StiffN/3.0D0
AMATRXh(3,1) = eLength*s41StiffS/6.0D0
AMATRXh(3,3) = eLength*s23StiffS/3.0D0
AMATRXh(3,5) = -eLength*s23StiffS/3.0D0
AMATRXh(3,7) = -eLength*s41StiffS/6.0D0
AMATRXh(4,2) = eLength*s41StiffN/6.0D0
AMATRXh(4,4) = eLength*s23StiffN/3.0D0
AMATRXh(4,6) = -eLength*s23StiffN/3.0D0
AMATRXh(4,8) = -eLength*s41StiffN/6.0D0
DO 30 i=1,8
AMATRXh(5,i) = -AMATRXh(3,i)
AMATRXh(6,i) = -AMATRXh(4,i)
AMATRXh(7,i) = -AMATRXh(1,i)
AMATRXh(8,i) = -AMATRXh(2,i)
30 CONTINUE

c
Define rotation matrix
c
L(1,1) = DCOS(thetaU)
L(1,2) = -DSIN(thetaU)
L(2,1) = DSIN(thetaU)
L(2,2) = DCOS(thetaU)
L(3,3) = DCOS(thetaU)
L(3,4)=DSIN(\theta_U)
L(4,3)=DSIN(\theta_U)
L(4,4)=DCOS(\theta_U)
L(5,5)=DCOS(\theta_U)
L(5,6)=DSIN(\theta_U)
L(6,5)=DSIN(\theta_U)
L(6,6)=DCOS(\theta_U)
L(7,7)=DCOS(\theta_U)
L(7,8)=DSIN(\theta_U)
L(8,7)=DSIN(\theta_U)
L(8,8)=DCOS(\theta_U)

// Rotate stiffness matrix
AMATRX = MATMUL(TRANSPOSE(L), MATMUL(AMATRXh, L))

// DEFINE RHS VECTOR
RHSh(1) = (eLength*s23StressS)/6.0D0 + (eLength*s41StressS)/3.0D0
RHSh(2) = (eLength*s23StressN)/6.0D0 + (eLength*s41StressN)/3.0D0
RHSh(3) = (eLength*s23StressS)/3.0D0 + (eLength*s41StressS)/6.0D0
RHSh(4) = (eLength*s23StressN)/3.0D0 + (eLength*s41StressN)/6.0D0
RHSh(5) = -RHSh(3)
RHSh(6) = -RHSh(4)
RHSh(7) = -RHSh(1)
RHSh(8) = -RHSh(2)

// Rotate the RHS vector
RHS_temp = MATMUL(L, RHSh)
RHS(1,1) = RHS_temp(1)
RHS(2,1) = RHS_temp(2)
RHS(3,1) = RHS_temp(3)
RHS(4,1) = RHS_temp(4)
RHS(5,1) = RHS_temp(5)
RHS(6,1) = RHS_temp(6)
RHS(7,1) = RHS_temp(7)
RHS(8,1) = RHS_temp(8)

ENDIF

// DEFINE STATE VARIABLES
cc
// Element data
SVARS(1)=eHeight
SVARS(2)=eLength
SVARS(3)=thetaU

Displacements
SVARS(11)=s41DN
SVARS(12)=s23DN
SVARS(13)=s41DS
SVARS(14)=s23DS

Stiffnesses and stresses
SVARS(21)=s41StiffN
SVARS(22)=s23StiffN
SVARS(23)=s41StiffS
SVARS(24)=s23StiffS
SVARS(25)=s41StressN
SVARS(26)=s23StressN
SVARS(27)=s41StressS
SVARS(28)=s23StressS

Energies and failure criteria
SVARS(31)=s41GI
SVARS(32)=s23GI
SVARS(33)=s41GII
SVARS(34)=s23GII
SVARS(35)=s41FC
SVARS(36)=s23FC

Element-side status flags (indicate if element side has failed)
(Used to set all stiffnesses and strengths to zero at failure)
(Also used to prevent failed elements from reforming)
SVARS(40): Failure flag for side41,0=not failed(initial value), -1=failed
SVARS(41): Failure flag for side23,0=not failed(initial value), -1=failed

WRITE DEBUGGING OUTPUT TO FILE debug.txt
IF(KINC.LT.1000)THEN
OPEN(unit=1,file="/nfs/mohr/parmigiani/grimmb/COEL_Work/Angled_crack_model/debug.txt")

IF(JELEM.GE.10216.AND.JELEM.LE.10220)THEN
  IF(DABS(s41DS).NE.0.0D0)THEN
    WRITE(1,*), "**************************************************"
    WRITE(1,*), "Element number=", JELEM
    WRITE(1,*), "Increment number=", KINC
    WRITE(1,*), "strenN=", strenN
    WRITE(1,*), "r13N=", r13N
    WRITE(1,*), "r23N=", r23N
    WRITE(1,*), "toughN=", toughN
    WRITE(1,*), "stenS=", strenS
    WRITE(1,*), "r13S=", r13S
    WRITE(1,*), "r23S=", r23S
    WRITE(1,*), "toughS=", toughS
    WRITE(1,*), "d1N=", d1N
    WRITE(1,*), "d2N=", d2N
    WRITE(1,*), "d3N=", d3N
    WRITE(1,*), "d1S=", d1S
    WRITE(1,*), "d2S=", d2S
    WRITE(1,*), "d3S=", d3S
    WRITE(1,*), "U(1)=", U(1)
    WRITE(1,*), "U(2)=", U(2)
    WRITE(1,*), "U(3)=", U(3)
    WRITE(1,*), "U(4)=", U(4)
    WRITE(1,*), "U(5)=", U(5)
    WRITE(1,*), "U(6)=", U(6)
    WRITE(1,*), "U(7)=", U(7)
    WRITE(1,*), "U(8)=", U(8)
    WRITE(1,*), "p1XU=", p1XU
    WRITE(1,*), "p1YU=", p1YU
  END IF
END IF
WRITE(1,*)"p2XU=",p2XU
WRITE(1,*)"p2YU=",p2YU
WRITE(1,*)"p3XU=",p3XU
WRITE(1,*)"p3YU=",p3YU
WRITE(1,*)"p4XU=",p4XU
WRITE(1,*)"p4YU=",p4YU
WRITE(1,*,"")
WRITE(1,*)"p1XD=",p1XD
WRITE(1,*)"p1YD=",p1YD
WRITE(1,*)"p2XD=",p2XD
WRITE(1,*)"p2YD=",p2YD
WRITE(1,*)"p3XD=",p3XD
WRITE(1,*)"p3YD=",p3YD
WRITE(1,*)"p4XD=",p4XD
WRITE(1,*)"p4YD=",p4YD
WRITE(1,*,"")
WRITE(1,*)"eHeight=",eHeight
WRITE(1,*)"eLength=",eLength
WRITE(1,*)"thetaU=",thetaU
WRITE(1,*)"thetaD=",thetaD
WRITE(1,*,"")
WRITE(1,*)"U(7)-U(1)=",U(7)-U(1)
WRITE(1,*)"s41LU=",s41LU
WRITE(1,*)"s41DN=",s41DN
WRITE(1,*)"s41LD=",s41LD
WRITE(1,*)"s41DN-(U(7)-U(1))=",s41DN-(U(7)-U(1))
WRITE(1,*,"")
WRITE(1,*)"U(5)-U(3)=",U(5)-U(3)
WRITE(1,*)"s23LU=",s23LU
WRITE(1,*)"s23DN=",s23DN
WRITE(1,*)"s23LD=",s23LD
WRITE(1,*)"s23DN-(U(5)-U(3))=",s23DN-(U(5)-U(3))
WRITE(1,*,"")
WRITE(1,*)"s41DS=",s41DS
WRITE(1,*)"s23DS=",s23DS
WRITE(1,*,"")
WRITE(1,*)"s41StiffS=",s41StiffS
WRITE(1,""")"s23StiffS=s23StiffS"
WRITE(1,""")"s41StressS=s41StressS"
WRITE(1,""")"s23StressS=s23StressS"
WRITE(1,""")""
WRITE(1,""")"s41StiffN=s41StiffN"
WRITE(1,""")"s23StiffN=s23StiffN"
WRITE(1,""")"s41StressN=s41StressN"
WRITE(1,""")"s23StressN=s23StressN"
WRITE(1,""")""
 WRITE(1,""")"AMATRXh"
 DO 100 i=1,8
  WRITE(1,"")AMATRXh(1,i)
  WRITE(1,"")AMATRXh(2,i)
  WRITE(1,"")AMATRXh(3,i)
  WRITE(1,"")AMATRXh(4,i)
  WRITE(1,"")AMATRXh(5,i)
  WRITE(1,"")AMATRXh(6,i)
  WRITE(1,"")AMATRXh(7,i)
  WRITE(1,"")AMATRXh(8,i)
WRITE(1,""")"
100      CONTINUE
WRITE(1,""")"AMATRX"
 DO 110 i=1,8
  WRITE(1,"")AMATRX(1,i)
  WRITE(1,"")AMATRX(2,i)
  WRITE(1,"")AMATRX(3,i)
  WRITE(1,"")AMATRX(4,i)
  WRITE(1,"")AMATRX(5,i)
  WRITE(1,"")AMATRX(6,i)
  WRITE(1,"")AMATRX(7,i)
  WRITE(1,"")AMATRX(8,i)
WRITE(1,""")"
110      CONTINUE
WRITE(1,""")"RHSh"
 DO 120 i=1,8
  WRITE(1,"")RHSh(i)
WRITE(1,""")"
120      CONTINUE
WRITE(1,""")"RHS"
c DO 130 i=1,8

WRITE(1,*)RHS(i,1)
WRITE(1,*)"

130 CONTINUE

ENDIF
ENDIF
CLOSE(1)
ENDIF

RETURN
END

SUBROUTINES

Calculate normal stiffness and stress

SUBROUTINE StN(sxyDN, strenN, d1N, d2N, d3N, sxyStiffN, sxyStressN)
IMPLICIT DOUBLE PRECISION(A-H, O-Z)
IF(sxyDN.LE.d1N)THEN
sxyStiffN=strenN/d1N
sxyStressN=strenN*sxyDN/d1N
ELSEIF(d1N.LT.sxyDN.AND.sxyDN.LE.d2N)THEN
sxyStiffN=0.0D0
sxyStressN=strenN
ELSEIF(d2N.LT.sxyDN.AND.sxyDN.LE.d3N)THEN
sxyStiffN=strenN/(d3N-d2N)
ELSEIF(sxyDN.GT.d3N)THEN
sxyStiffN=0.0D0
sxyStressN=0.0D0
ENDIF
RETURN
END

Calculate shear stiffness and stress

SUBROUTINE StS(sxyDS, strenS, d1S, d2S, d3S, sxyStiffS, sxyStressS)
IMPLICIT DOUBLE PRECISION(A-H, O-Z)
IF(DABS(sxyDS).LE.d1S)THEN
sxyStiffS=strenS/d1S
sxyStressS=SIGN(strenS*sxyDS/d1S,sxyDS)
ELSEIF(d1S.LT.DABS(sxyDS).AND.DABS(sxyDS).LE.d2S)THEN
sxyStiffS=0.0D0
sxyStressS=SIGN(strenS,sxyDS)
ELSEIF(d2S.LT.DABS(sxyDS).AND.DABS(sxyDS).LE.d3S)THEN
sxyStiffS=-strenS/(d3S-d2S)
sxyStressS=SIGN(strenS*(d3S-DABS(sxyDS))/(d3S-d2S),sxyDS)
ELSEIF(DABS(sxyDS).GT.d3S)THEN
sxyStiffS=0.0D0
sxyStressS=0.0D0
ENDIF
RETURN
END

c Calculate instantaneous normal energy release rate
c
SUBROUTINE EnergyN(sxyDN,strenN,d1N,d2N,d3N,toughN,sxyGI)
IMPLICIT DOUBLE PRECISION(A-H, O-Z)
IF(sxyDN.LE.0.0D0)THEN
sxyGI=0.0D0
ELSEIF(0.0D0.LE.sxyDN.AND.sxyDN.LE.d1N)THEN
sxyGI=0.5D0*sxyDN*(strenN/d1N)*sxyDN
ELSEIF(d1N.LT.sxyDN.AND.sxyDN.LE.d2N)THEN
sxyGI=0.5D0*d1N*strenN + (sxyDN - d1N)*strenN
ELSEIF(d2N.LT.DABS(sxyDN).AND.sxyDN.LE.d3N)THEN
sxyGI=0.5D0*d1N*strenN+(d2N-d1N)*strenN+0.5D0*(d3N-d2N)*strenN-0.5D0*(d3N-sxyDN)*(strenN/(d3N-d2N))*(d3N-sxyDN)
ELSEIF(sxyDN.GT.d3N)THEN
sxyGI=toughN
ENDIF
RETURN
END

c Calculate instantaneous shear energy release rate
c
SUBROUTINE EnergyS(sxyDS,strenS,d1S,d2S,d3S,toughS,sxyGII)
IMPLICIT DOUBLE PRECISION(A-H, O-Z)
IF(DABS(sxyDS).LE.d1S)THEN
sxyGII=0.5D0*DABS(sxyDS)*(strenS/d1S)*DABS(sxyDS)
ELSEIF(d1S.LT.DABS(sxyDS).AND.DABS(sxyDS).LE.d2S)THEN
sxyGII=0.5D0*d1S*strenS+(DABS(sxyDS)-d1S)*strenS
ELSEIF(d2S.LT.DABS(sxyDS).AND.DABS(sxyDS).LE.d3S)THEN
sxyGII=0.5D0*d1S*strenS+(d2S-d1S)*strenS+0.5D0*(d3S-d2S)*strenS-0.5D0*(d3S-DABS(sxyDS))*(strenS/(d3S-d2S))*(d3S-DABS(sxyDS))
ELSEIF(DABS(sxyDS).GT.d3S)THEN
sxyGII=toughS
ENDIF
RETURN
END