AN ABSTRACT OF THE THESIS OF

JOHN DAVID ELWIN for the DOCTOR OF PHILOSOPHY
(Name) (Degree)

in MATHEMATICS presented on August 1, 1969
(Major) (Date)

Title: HOMOLOGY THEORIES ON THE MAPPING CATEGORY

Abstract approved: Redacted for privacy
J. Wolfgang Smith

Let $\mathcal{C}$ denote the category of maps. An object of $\mathcal{C}$ is thus a continuous map $f : X \to Y$ between topological spaces and a morphism is a commutative square from f to f'. In this paper we give a system of axioms for a homology theory defined on $\mathcal{C}$. Consequences of these axioms are developed and it is shown that the cone theory for chain maps gives rise to an existence proof.

On the subcategory $\mathcal{C}_*$ of surjective cellular maps between finite CW-complexes, we prove a categoricity theorem.

Homology Theories on the Mapping Category

by

John David Elwin

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1970

APPROVED:

Redacted for privacy

Professor of Mathematics

in charge of major

Redacted for privacy

Acting Chairman of Department of Mathematics

Redacted for privacy

Dean of Graduate School

Date thesis is presented August 1, 1969

Typed by Clover Redfern for John David Elwin

ACKNOWLEDGMENT

I would like to express my appreciation to Dr. J. Wolfgang Smith for his encouragement and guidance in the preparation of this thesis. I would also like to thank the National Aeronautics and Space Administration for their financial support.

This thesis is dedicated to my very patient wife, Darlene.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	HOMOLOGY THEORIES ON $\mathcal{C}$	4
III.	THE CATEGORICITY THEOREM	28
	BIBLIOGRAPHY	49

HOMOLOGY THEORIES ON THE MAPPING CATEGORY

I. INTRODUCTION

Algebraic topology is principally concerned with the construction of techniques by which topological questions are transformed to algebraic ones. The resulting algebraic problem is then often easier to solve. This transition from topology to algebra has been made precise with the introduction of the notion of a category by Eilenberg and MacLane in 1945. As is common in modern works on the subject, we will use the language of categorical algebra throughout our development.

The oldest and most important functors of algebraic topology are homology theories, and numerous variations of these theories have appeared since their introduction by Poincare. Contributors such as Vietoris, Lefschetz, and Čech, introduced new constructions to solve specific problems but always at the expense of increased complexity or limited applicability. Certain pronounced similarities in the results obtained from each of these theories have led one to suspect the possibility of an axiomatic approach. This possibility was realized in 1952 by Eilenberg and Steenrod, thus making precise for the first time, the concept of a homology theory. The principal advantage of the axioms lies in the possibility of working with homology theory without recourse to the tedious machinery of any particular construction. The justification of the Eilenberg-Steenrod axioms is provided by the categoricity theorem (2) which asserts that any two homology theories must agree on the subcategory of compact polyhedra.

In the following chapters, we will construct a homology theory on the category of maps. A continuous map between topological spaces rather than a space itself will be assigned an algebraic structure. A cummutative square between two maps will induce a homomorphism of the algebraic structures. In Chapter II, we introduce an axiom system for a homology theory on the category of maps. Consequences of these axioms (e. g., the exact sequence of a triple) are developed in this chapter which concludes with an existence proof for a homology theory of maps. Chapter III contains the proof of a categoricity theorem for theories defined on a subcategory of the category of maps.

Each functor of algebraic topology measures a geometric property (or properties) at the expense of others. This "insensitivity" is essential for the transition to a simpler algebraic problem. Speaking imprecisely, the usual homology functor measures the number of n-dimensional holes in a topological space. We may make this same vague analogy for homology theories defined on the category of maps. These latter theories measure the number of n-dimensional holes either created or annihilated by a map. Hence if $f: X \to Y$ is a continuous map which neither creates nor destroys n-dimensional holes, then it is assigned a trivial algebraic structure irregardless of the complexity of X and Y. This interpretation, along with the similarity of our axiom system with that of Eilenberg and Steenrod, helps justify the term "homology theories" for these new functors.

II. HOMOLOGY THEORIES ON $\mathcal{C}$

Let $\mathcal{C}$ denote the category whose objects are continuous maps $f : X \to Y$ between topological spaces and whose morphisms $[k_1, k_2] : f \to f'$ are commutative squares

$$\begin{tikzcd} X \arrow[r, "k_1"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "k_2"] & Y' \end{tikzcd}$$

For our purposes it will be convenient to consider the category of maps between topological pairs $(f, f_A) : (X, A) \to (Y, B)$ where f_A denotes the restriction of f to A. Using ϕ for the empty set, we may regard the first category as an obvious subcategory of the second under the identification $f \mapsto (f, f_\phi)$ and we will use the same symbol for both categories.

Definition II. 1: A homology theory on $\mathcal{C}$ with coefficient group G is a pair (H, ∂) where H is a non-negative covariant functor from $\mathcal{C}$ to the category of graded groups and ∂ is a natural transformation from $H_q(f, f_A)$ to $H_{q-1}(f_A)$ satisfying the following axioms:

Axiom I. Let $f : X \to Y$ and $f' : X' \to Y'$ be objects of $\mathcal{C}$ and let $1 : I \to I$ denote the identity map on the unit interval.

Assume there exist homotopies $F : X \times I \to X'$ and $F' : Y \times I \to Y'$ between $F_0 \simeq F_1$ and $F'_0 \simeq F'_1$, respectively, making the following diagram commute:

$$\begin{tikzcd} X \times I \arrow[r, "F"] \arrow[d, "f \times 1"] & X' \arrow[d, "f'"] \\ Y \times I \arrow[r, "F'"] & Y' \end{tikzcd}$$

Then $[F_0, F'_0]_* = [F_1, F'_1]_*$ where $[F_0, F'_0]_*$ denotes the induced morphism from

$$\begin{tikzcd} X \arrow[r, "F_0"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "F'_0"] & Y' \end{tikzcd}$$

Axiom II. For any pair $(f, f_A) : (X, A) \to (Y, B)$ in $\mathcal{C}$, the inclusion morphisms $i = [i_1, i_2]$ and $j = [j_1, j_2]$ of the composition

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "f_A"] & X \arrow[r, "j_1"] \arrow[d, "f"] & (X, A) \arrow[d, "(f, f_A)"] \\ B \arrow[r, "i_2"] & Y \arrow[r, "j_2"] & (Y, B) \end{tikzcd}$$

give rise to an exact sequence

$$\cdots \to H_q(f_A) \xrightarrow{i_*} H_q(f) \xrightarrow{j_*} H_q(f, f_A) \xrightarrow{\partial} H_{q-1}(f_A) \to \cdots$$

Axiom III. Let $(f, f_A) : (X, A) \to (Y, B)$ be an object in $\mathcal{C}$. Assume U is an open subset of B such that $\overline{U}$ (closure) $\subset$ int. B and $\overline{f^{-1}(U)} \subset$ int. A. Then for each integer q,

$$H_q(f_{X-U'}, f_{A-U'}) \approx H_q(f, f_A)$$

where $U' = f^{-1}(U)$ and the isomorphism is induced by inclusion.

Axiom IV. A. Let $(f, f_A) : (X, A) \to (Y, B)$ be an object in $\mathcal{C}$ and assume that (X, A) is a contractible pair. Then for each q,

$$H_q(f, f_A) \approx H_{q+1}(Y, B; G)$$

where $H(Y, B; G)$ is the singular homology of (Y, B) with coefficients in G.

Axiom IV. B. Let $\alpha : \bigvee_{i=1}^{m} v_i \to v_0$ denote the map from a disjoint union of m vertices to a vertex v_0. Then $(G_i = G)$

$$H_q(\alpha) = \begin{cases} \bigoplus_{i=1}^{m-1} G_i & q = 0 \\ 0 & q \neq 0 \end{cases}$$

The above four axioms define a homology theory on $\mathcal{C}$ and by an obvious dual process we could define the notion of a cohomology theory on $\mathcal{C}$. A cohomology theory on $\mathcal{C}$ consists of a pair

(H, δ) where H is a non-negative contravariant functor from $\mathcal{C}$ to the category of graded groups and δ is a natural transformation from $H^q(f_A)$ to $H^{q+1}(f, f_A)$. Axioms I and III remain the same with, of course, the induced maps in the opposite direction. Axiom II gives rise to a dual exact sequence

$$\ldots \to H^q(f) \xrightarrow{i^*} H^q(f_A) \xrightarrow{\delta} H^{q+1}(f, f_A) \to \ldots$$

and Axiom IV. A states that for (X, A) a contractible pair,

$$H^q(f, f_A) \approx H^q(Y, B; G)$$

We will concern ourselves here with the notion of a homology theory, but a similar discussion could be carried out for a cohomology theory.

<u>Definition II. 2:</u> If $f : X \to Y$ and $f' : X' \to Y'$ are objects of $\mathcal{C}$ and $[k_1, k_2] : f \to f'$ is a morphism, $[k_1, k_2]$ is a <u>homotopy equivalence in</u> $\mathcal{C}$ if there exists a morphism $[\ell_1, \ell_2] : f' \to f$ and homotopies $F_1 : \ell_1 \circ k_1 \simeq 1_X$, $F_2 : \ell_2 \circ k_2 \simeq 1_Y$, $G_1 : k_1 \circ \ell_1 \simeq 1_{X'}$, $G_2 : k_2 \circ \ell_2 \simeq 1_{Y'}$ such that the following diagrams commute:

$$\begin{tikzcd} X \times I \arrow[r, "F_1"] \arrow[d, "f \times 1"'] & X \arrow[d, "f"] \\ Y \times I \arrow[r, "F_2"] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \times I \arrow[r, "G_1"] \arrow[d, "f' \times 1"'] & X' \arrow[d, "f'"] \\ Y' \times I \arrow[r, "G_2"] & Y' \end{tikzcd}$$

Clearly if $[k_1, k_2]$ is a homotopy equivalence in $\mathcal{C}$ then

$$[\ell_1 \circ k_1, \ell_2 \circ k_2]_* = [\ell_1, \ell_2]_* \circ [k_1, k_2]_* = 1_{H(f)}$$

and

$$[k_1, k_2]_* \circ [\ell_1, \ell_2]_* = 1_{H(f')}$$

by Axiom I. Hence $[k_1, k_2]_*$ is an isomorphism. We now have the following useful proposition which is a stronger form of Axiom III.

Proposition II. 3: Let $(f, f_A) : (X, A) \to (Y, B)$ be an object in $\mathcal{C}$. Assume $V \subset U \subset B$ are open sets such that $\overline{V} \subset \text{int. } B$ and $\overline{f^{-1}(V)} \subset \text{int. } A$. In addition, assume the inclusion morphisms $(U' = f^{-1}(U),\ V' = f^{-1}(V))$

$$\begin{tikzcd}
X-U' \arrow[r, "i_1"] \arrow[d, "f_{X-U'}"] & X-V' \arrow[d, "f_{X-V'}"] \\
Y-U \arrow[r, "i_2"] & Y-V
\end{tikzcd}$$

and

$$\begin{tikzcd}
A-U' \arrow[r, "j_1"] \arrow[d, "f_{A-U'}"] & A-V' \arrow[d, "f_{A-V'}"] \\
B-U \arrow[r, "j_2"] & B-V
\end{tikzcd}$$

are homotopy equivalences in $\mathcal{C}$. Then the inclusion morphism from $(f_{X-U'}, f_{A-U'})$ to (f, f_A) induces an isomorphism for each integer q.

Proof: The exact sequences for the pairs $(f_{X-U'}, f_{A-U'})$ and $(f_{X-V'}, f_{A-V'})$ give

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(f_{X-U'}) \arrow[r] \arrow[d, "{[i_1, i_2]_*}"] & H_q(f_{X-U'}, f_{A-U'}) \arrow[r] \arrow[d] & H_{q-1}(f_{A-U'}) \arrow[r] \arrow[d, "{[j_1, j_2]_*}"] & \cdots \\
\cdots \arrow[r] & H_q(f_{X-V'}) \arrow[r] & H_q(f_{X-V'}, f_{A-V'}) \arrow[r] & H_{q-1}(f_{A-V'}) \arrow[r] & \cdots
\end{tikzcd}$$

Since homotopy equivalences induce isomorphisms we have by the five-lemma that

$$H_q(f_{X-U'}, f_{A-U'}) \approx H_q(f_{X-V'}, f_{A-V'})$$

By Axiom III,

$$H_q(f_{X-V'}, f_{A-V'}) \approx H_q(f, f_A)$$

and the result follows by functoriality.

We have another useful proposition which implies Axiom I.

Proposition II. 4: Assume $f : X \to Y$ is an object of $\mathcal{C}$. Consider the inclusion morphisms (i_ϵ, j_ϵ), $\epsilon = 0, 1$,

$$\begin{tikzcd}
X \arrow[r, "i_\epsilon"] \arrow[d, "f"] & X \times I \arrow[d, "f \times 1"] \\
Y \arrow[r, "j_\epsilon"] & Y \times I
\end{tikzcd}$$

<u>where</u> $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$, $j_0(y) = (y, 0)$, <u>and</u> $j_1(y) = (y, 1)$. <u>Then</u> $[i_0, j_0]_* = [i_1, j_1]_*$ <u>implies Axiom I.</u>

<u>Proof</u>: This follows immediately by considering the composition

$$\begin{tikzcd} X \arrow[r, "i_\epsilon"] \arrow[d, "f"] & X \times I \arrow[r, "F"] \arrow[d, "f \times 1"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j_\epsilon"] & Y \times I \arrow[r, "G"] & Y' \end{tikzcd}$$

The following theorem, which is an analog of a basic theorem in the usual homology theory, is of great importance in the sequel. Consider the triple (f, f_A, f_B) mapping (X, A, B) to (X', A', B'). For the pair (f, f_A) we have the sequence

$$\cdots \to H_q(f) \to H_q(f, f_A) \xrightarrow{\partial} H_{q-1}(f_A) \to \cdots$$

Let $[j_1'', j_2''] : f_A \to (f_A, f_B)$ denote the inclusion morphism. Define $\bar{\partial}$ by

$$\begin{tikzcd} H_q(f, f_A) \arrow[r, "\partial"] \arrow[dr, "\bar{\partial}"'] & H_{q-1}(f_A) \arrow[d, "{[j_1'', j_2'']_*}"] \\ & H_{q-1}(f_A, f_B) \end{tikzcd}$$

and let

$$\begin{tikzcd}
(A, B) \arrow[r, "i_1"] \arrow[d, "(f_A, f_B)"] & (X, B) \arrow[r, "j_1"] \arrow[d, "(f, f_B)"] & (X, A) \arrow[d, "(f, f_A)"] \\
(A', B') \arrow[r, "i_2"] & (X', B') \arrow[r, "j_2"] & (X', A')
\end{tikzcd}$$

be inclusion morphisms. We then have the sequence

$$\cdots \to H_q(f_A, f_B) \xrightarrow{[i_1, i_2]_*} H_q(f, f_B) \xrightarrow{[j_1, j_2]_*} H_q(f, f_A) \xrightarrow{\bar{\partial}} H_{q-1}(f_A, f_B) \to \cdots$$

<u>Theorem II. 1:</u> <u>The above sequence is exact.</u>

<u>Proof:</u> We will show that $\operatorname{im} [j_1, j_2]_* = \ker \bar{\partial}$. Exactness at the other two positions follows in a similar way. Define the maps for the pairs (f, f_A), (f, f_B) and (f_A, f_B) as

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(f_B) \arrow[r, "{[i_1', i_2']_*}"] \arrow[d] & H_q(f) \arrow[r, "{[j_1', j_2']_*}"] \arrow[d] & H_q(f, f_B) \arrow[r, "\partial'"] \arrow[d] & H_{q-1}(f_B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_q(f_B) \arrow[r, "{[i_1'', i_2'']_*}"] \arrow[d] & H_q(f_A) \arrow[r, "{[j_1'', j_2'']_*}"] \arrow[d] & H_q(f_A, f_B) \arrow[r, "\partial''"] \arrow[d] & H_{q-1}(f_B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_q(f_A) \arrow[r, "{[\bar{i}_1, \bar{i}_2]_*}"] & H_q(f) \arrow[r, "{[\bar{j}_1, \bar{j}_2]_*}"] & H_q(f, f_A) \arrow[r, "\partial"] & H_{q-1}(f_A) \arrow[r] & \cdots
\end{tikzcd}$$

with $[i_1, i_2]$ inclusion morphisms and ∂'s connecting homomorphisms.

Consider the following array where the rectangle commutes by the naturality of ∂ and the triangles by functoriality.

$$\begin{tikzcd}
 & & H_q(f, f_B) \arrow[r, "\partial'"] \arrow[d, "{[j_1, j_2]_*}"] & H_{q-1}(f_B) \arrow[d, "{[i''_1, i''_2]_*}"'] \arrow[dr, "{[i'_1, i'_2]_*}"] & & \\
{} \arrow[r] & H_q(f) \arrow[ur, "{[j'_1, j'_2]_*}"] \arrow[r, "{[\bar{j}_1, \bar{j}_2]_*}"] & H_q(f, f_A) \arrow[r, "\partial"] \arrow[dr, "\bar{\partial}"'] & H_{q-1}(f_A) \arrow[r, "{[\bar{i}_1, \bar{i}_2]_*}"] \arrow[d, "{[j''_1, j''_2]_*}"'] & H_{q-1}(f) \arrow[r] & {} \\
 & & & H_{q-1}(f_A, f_B) & &
\end{tikzcd}$$

To show $\operatorname{im} [j_1, j_2]_* \subset \ker \bar{\partial}$. Assume $x \in \operatorname{im} [j_1, j_2]_*$. Then there exists a $b \in H_q(f, f_B)$ such that $[j_1, j_2]_*(b) = x$. By the above diagram,

$$\bar{\partial} \circ [j_1, j_2]_*(b) = \bar{\partial}(x) = [j''_1, j''_2]_* \circ [i''_1, i''_2]_* \circ \partial'(b)$$

which is 0 by the exactness of the pair (f_A, f_B). Hence $x \in \ker \bar{\partial}$.

To show inclusion in the opposite direction, assume $x \in \ker \bar{\partial}$. Then again by the above diagram, $\bar{\partial}(x) = [j''_1, j''_2]_* \circ \partial(x) = 0$, which implies $\partial(x) \in \ker [j''_1, j''_2]_* = \operatorname{im} [i''_1, i''_2]_*$. Hence there exists a $b \in H_{q-1}(f_B)$ such that $[i''_1, i''_2]_*(b) = \partial(x)$. Now $[i'_1, i'_2]_*(b) = [\bar{i}_1, \bar{i}_2]_* \circ [i''_1, i''_2] (b) = [\bar{i}_1, \bar{i}_2]_* \circ \partial(x) = 0$ by the exactness of the pair (f, f_A). Therefore, $b \in \ker [i'_1, i'_2]_* = \operatorname{im} \partial'$ which implies there exists a $c \in H_q(f, f_B)$ such that $\partial'(c) = b$.

$$\begin{aligned} \partial(x - [j_1, j_2]_*(c)) &= \partial(x) - \partial [j_1, j_2]_*(c) = \partial(x) - [i''_1, i''_2]_* \circ \partial'(c) \\ &= \partial(x) - \partial(x) = 0 \end{aligned}$$

Thus, $x - [j_1, j_2]_*(c) \in \ker \partial = \operatorname{im} [\bar{j}_1, \bar{j}_2]_*$ and there exists a w in $H_q(f)$ such that $[\bar{j}_1, \bar{j}_2](w) = x - [j_1, j_2]_*(c)$. Let $z = [j_1', j_2']_*(w) + c$. Then

$$[j_1, j_2]_*(z) = [j_1, j_2]_*([j_1', j_2'](w)+c) = x - [j_1, j_2]_*(c) + [j_1, j_2]_*(c) = x$$

as was to be shown.

Our next proposition deals with the direct sum decomposition of an object in $\mathcal{C}$. Assume $(f, f_A) : (X, A) \to (Y, B)$ is an object of $\mathcal{C}$ and that f and f_A are surjective. Assume further that Y admits a separation $Y = Y_1 | Y_2$ and define $X_i = f^{-1}(Y_i)$, $i = 1, 2$. Let $A_i = X_i \cap A$ and $B_i = Y_i \cap B$ and consider the objects $(f_{X_i}, f_{A_i}) : (X_i, A_i) \to (Y_i, B_i)$. We will need the following algebraic lemma, whose proof may be found in (5, p. 39).

Lemma II. 5: *Assume in the following diagram of abelian groups and homomorphisms*

$$\begin{tikzcd}
M \arrow[dr, "i"] \arrow[dd] & & M' \arrow[dl, "j"'] \arrow[dd] \\
& K \arrow[dl] \arrow[dr] & \\
N & & N'
\end{tikzcd}$$

that the triangles are commutative, that each diagonal is exact at K *and also that the two vertical maps are isomorphisms. Then for* $x \in M$, $y \in M'$, *the map* $\phi : M \oplus M' \to K$ *defined by*

$\phi(x, y) = i(x) + j(y)$ *is an isomorphism*.

Now we have

Proposition II. 6: Let $(f, f_A) : (X, A) \to (Y, B)$ *be a surjective object in* $\mathcal{C}$. *Assume* $Y = Y_1 | Y_2$ *admits a separation and* (using the notation defined above) *let*

$$\begin{tikzcd}
(X_i, A_i) \arrow[r, "t_i"] \arrow[d, "(f_{X_i}, f_{A_i})"] & (X, A) \arrow[d, "(f, f_A)"] \\
(Y_i, B_i) \arrow[r, "t_i'"] & (Y, B)
\end{tikzcd}$$

be the inclusion morphisms. Then

$$\bigoplus_{i=1}^{2} [t_i, t_i']_* : \bigoplus_{i=1}^{2} H_q(f_{X_i}, f_{A_i}) \to H_q(f, f_A)$$

is an isomorphism for each integer q.

Proof: The morphisms

$$\begin{tikzcd}
(X_1, A_1) \arrow[r] \arrow[d, "(f_{X_1}, f_{A_1})"] & (X_1 \cup A, A) \arrow[d, "(f_{X_1 \cup A}, f_A)"] \\
(Y_1, B_1) \arrow[r] & (Y_1 \cup B, B)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(X_1, A_1) \arrow[r] \arrow[d, "(f_{X_1}, f_{A_1})"] & (X, X_2 \cup A) \arrow[d, "(f, f_{X_2 \cup A})"] \\
(Y_1, B_1) \arrow[r] & (Y, Y_2 \cup B)
\end{tikzcd}$$

are excisions of $(U = B_2,\ U' = A_2)$ and $(U = Y_2,\ U' = X_2)$, respectively, and give rise to the following commutative triangle:

$$\begin{tikzcd} & (f_{X_1}, f_{A_1}) \arrow[dl, "{[j_1, j_1']}"'] \arrow[dr, "{[j_2, j_2']}"] & \\ (f_{X_1 \cup A}, f_A) \arrow[rr, "{[k_1, k_1']}"] & & (f, f_{X_2 \cup A}) \end{tikzcd}$$

By Axiom III we know that $[j_1, j_1']_*$ and $[j_2, j_2']_*$ are isomorphisms, and that $[k_1, k_1']_* = [j_2, j_2']_* \circ [j_1, j_1']_*^{-1}$ is an isomosphism. Carrying out a similar procedure for (f_{X_2}, f_{A_2}) we obtain the inclusion induced isomorphisms

$$[\ell_1, \ell_1']_* : H_q(f_{X_2 \cup A}, f_A) \to H_q(f, f_{X_1 \cup A})$$

and

$$[m, m']_* : H_q(f_{X_2}, f_{A_2}) \to H_q(f_{X_2 \cup A}, f_A).$$

We then have the array

$$\begin{tikzcd} (f_{X_1 \cup A}, f_A) \arrow[dd, "{[k_1, k_1']}"'] \arrow[dr, "{[k_2, k_2']}"] & & (f_{X_2 \cup A}, f_A) \arrow[dl, "{[\ell_2, \ell_2']}"'] \arrow[dd, "{[\ell_1, \ell_1']}"] \\ & (f, f_A) \arrow[dl] \arrow[dr] & \\ (f, f_{X_2 \cup A}) & & (f, f_{X_1 \cup A}) \end{tikzcd}$$

of inclusion morphisms which induces, for each q,

$$\begin{tikzcd}
H_q(f_{X_1 \cup A}, f_A) \arrow[dr, "{[k_2, k_2']_*}"] \arrow[dd, "{[k, k_1']_*}"'] & & H_q(f_{X_2 \cup A}, f_A) \arrow[dl, "{[\ell_1, \ell_1']_*}"'] \arrow[dd, "{[\ell_1, \ell_1']_*}"] \\
& H_q(f, f_A) \arrow[dl] \arrow[dr] & \\
H_q(f, f_{X_2 \cup A}) & & H_q(f, f_{X_1 \cup A})
\end{tikzcd}$$

where the triangles are commutative, the vertical maps are isomorphisms and the diagonals are exact (Theorem II. 1 for triples $(f, f_{X_2 \cup A}, f_A)$ and $(f, f_{X_1 \cup A}, f_A)$).

Hence, from Lemma II. 5, the map $[k_2, k_2']_* \oplus [\ell_2, \ell_2']_*$ from $H_q(f_{X_1 \cup A}, f_A) \oplus H_q(f_{X_2 \cup A}, f_A)$ to $H_q(f, f_A)$ is an isomorphism.

We then have the diagram

$$\begin{tikzcd}
H_q(f_{X_1}, f_{A_1}) \arrow[dr, "{[t_1, t_1']_*}"] \arrow[dd, "{[j_1, j_1']_*}"'] & & H_q(f_{X_2}, f_{A_2}) \arrow[dl, "{[t_2, t_2']_*}"'] \arrow[dd, "{[m, m']_*}"] \\
& H_q(f, f_A) \arrow[dl, "{[k_2, k_2']_*}"'] \arrow[dr, "{[\ell_2, \ell_2']_*}"] & \\
H_q(f_{X_1 \cup A_1}, f_A) & & H_q(f_{X_2 \cup A}, f_A)
\end{tikzcd}$$

which implies

$$[t_1, t_1']_* \oplus [t_2, t_2']_* : H_q(f_{X_1}, f_{A_1}) \oplus H_q(f_{X_2}, f_{A_2}) \to H_q(f, f_A)$$

is an isomorphism for each q.

Using the above proposition, we can prove a result which will be of use in the main theorem of the next chapter.

<u>Proposition II. 7</u>: <u>Let</u> $1_X \in \mathcal{C}$ <u>where</u> X <u>is a finite</u> CW-complex. <u>Then for each</u> q, $H_q(1_X) = 0$.

<u>Proof</u>: Assume the dimension of X is 0. Then $X = \bigvee_{i=1}^{m} v_i$ is a disjoint union of vertices, and it follows from Proposition II. 6 that

$$H_q(1_X) \approx H_q(1_{\bigvee_{i=1}^{m} v_i}) \approx \bigoplus_{i=1}^{m} H_q(1_{v_i}) = 0$$

for all q, by Axiom IV. B.

Assume now that the theorem is true for any CW-complex of dimension less than n and that X has dimension n. Denote the $n-1$ skeleton of X by X_{n-1} and consider the pair $(1_X, 1_{X_{n-1}}) : (X, X_{n-1}) \to (X, X_{n-1})$ and its sequence

$$\cdots \to H_q(1_{X_{n-1}}) \to H_q(1_X) \to H_q(1_X, 1_{X_{n-1}}) \xrightarrow{\partial} H_{q-1}(1_{X_{n-1}}) \to \cdots$$

By the inductive assumption we have, $H_q(1_{X_{n-1}}) = 0$ for all q and therefore $H_q(1_X) \approx H_q(1_X, 1_{X_{n-1}})$. Hence it suffices to consider the pair $(1_X, 1_{X_{n-1}})$.

Let X have m n-cells $\{\Delta_i^n\}$ $(i=1,\ldots,m)$ and place a smaller closed n-cell $\bar{\Delta}_i^n \subset \Delta_i^n$ for each i. We would like to excise $(\widetilde{\bigvee_{i=1}^m \bar{\Delta}_i^n})$ (complement of $\bigvee_{i=1}^m \bar{\Delta}_i^n$) from (X, X_{n-1}), but this set is too large. However, if we expand each $\bar{\Delta}_i^n$ to a slightly larger n-cell, we have from Proposition II. 3 that

$$\begin{array}{ccc} (\bigvee_{i=1}^m \bar{\Delta}_i^n, \bigvee_{i=1}^m \partial\bar{\Delta}_i^n) & \longrightarrow & (X, X'_{n-1}) \\ \downarrow & & \downarrow \\ (\bigvee_{i=1}^m \bar{\Delta}_i^n, \bigvee_{i=1}^m \partial\bar{\Delta}_i^n & \longrightarrow & (X, X'_{n-1}) \end{array}$$

induces an isomorphism, where $X'_{n-1} = X_{n-1} \cup (\widetilde{\bigvee_{i=1}^m \text{int.}\, \bar{\Delta}_i^n})$. The fact that X'_{n-1} may be substituted for X_{n-1} follows from Axiom I and the five lemma.

From the inductive assumption,

$$H_q(1_{\bigvee_{i=1}^m \partial\bar{\Delta}_i^n}) = 0$$

for each q, which implies

$$H_q(1_{\bigvee_{i=1}^m \bar{\Delta}_i^n}, 1_{\bigvee_{i=1}^m \partial\bar{\Delta}_i^n}) \approx H_q(1_{\bigvee_{i=1}^m \bar{\Delta}_i^n}) \approx \bigoplus_{i=1}^m H_q(1_{\Delta_i^n})$$

By Axioms I and IV. B, it follows that

$$H_q(1_{\Delta_i^n}) \approx H_q(1_{v_0}) = 0$$

and hence

$$H_q(1_X) \approx H_q(1_X, 1_{X_{n-1}}) = 0$$

We would like now to make some initial definitions which will lead to an existence proof of a homology theory.

Let $S(X) = \{S_q(X)\}$ denote the integral singular chain complex of X. Given a map $f : X \to Y$, let $f_\#$ denote the induced chain map $f_\# : S(X) \to S(Y)$. The mapping cone of $f_\#$ as given in (6, p. 166), is a chain complex $S(f) = \{S_q(f)\}$ defined as follows: $S_q(f) = S_q(X) \oplus S_{q+1}(Y)$, with boundary operator $\partial(x_q, y_{q+1}) = (-\partial x_q, f_\#(x_q) + \partial y_{q+1})$ for $x_q \in S_q(X)$, $y_{q+1} \in S_{q+1}(Y)$. One can readily verify that $S(f)$ is a chain complex and that there exists an exact sequence

$$\cdots \to H_{q+1}(S(Y)) \to H_q(S(f)) \to H_q(S(X)) \to \cdots$$

of singular homology groups.

Let

$$\begin{array}{ccc} X & \xrightarrow{k} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{k'} & Y' \end{array}$$

by a morphism in $\mathcal{C}$. $[k, k']$ induces a chain map

$[k, k']_\# : S_q(f) \to S_q(f')$ as follows. For $(x_q, y_{q+1}) \in S_q(f)$ define $[k, k']_\#(x_q, y_{q+1}) = (k_\#(x_q), k'_\#(y_{q+1}))$. To verify that $[k, k']_\#$ is a chain map we must show the diagram

$$\begin{array}{ccc} S_q(f) & \xrightarrow{[k, k']_\#} & S_q(f') \\ \downarrow{\check{\partial}} & & \downarrow{\check{\partial}'} \\ S_{q-1}(f) & \xrightarrow{[k, k']_\#} & S_{q-1}(f') \end{array}$$

commutes. Let $(x_q, y_{q+1}) \in S_q(f)$. Then

$$[k, k']_\# \circ \check{\partial}(x_q, y_{q+1}) = [k, k']_\#(-\partial x_q, f_\#(x_q) + \partial y_{q+1})$$

$$= (-k_\#(\partial x_q), k'_\#(f_\#(x_q) + \partial y_{q+1}))$$

$$= (-\partial k_\#(x_q), k'_\# f_\#(x_q) + \partial k'_\#(y_{q+1}))$$

and by the commutativity of the morphism $[k, k']$

$$= (-\partial k_\#(x_q), f'_\# \circ k_\#(x_q) + \partial k'_\#(y_{q+1}))$$

$$= \check{\partial}'(k_\#(x_q), k'_\#(y_{q+1}))$$

$$= \check{\partial}' \circ [k, k']_\#(x_q, y_{q+1})$$

Hence, $[k, k']_\#$ is a chain map.

Let $(f, f_A) : (X, A) \to (Y, B)$ be an object in $\mathcal{C}$. Then we have the short exact sequence of chain complexes

$$0 \to S(f_A) \to S(f) \to S(f)/S(f_A) \to 0$$

Define a functor H^c from $\mathcal{C}$ to the category of graded abelian groups as follows: to an object (f, f_A) of $\mathcal{C}$, define

$$H_q^c(f, f_A) = H(S_q(f)/S_q(f_A))$$

(singular homology of the quotient chain complex) and to a morphism $[k, k'] : (f, f_A) \to (f', f'_{A'})$, define the map $[k, k']_*$ as the induced map of $[k, k']_\#$ in the singular theory. Let $\partial : H_q^c(f, f_A) \to H_{q-1}^c(f_A)$ be the connecting homomorphism of the above short exact sequence of chain complexes.

From the definitions it is obvious that H^c preserves composition and the identity morphism. To verify that ∂ is a natural transformation we must show that the diagram

$$\begin{tikzcd} H_q^c(f, f_A) \arrow[r, "\partial"] \arrow[d, "{[k, k']_*}"] & H_{q-1}^c(f_A) \arrow[d, "{[k_A, k'_B]_*}"] \\ H_q^c(f', f'_{A'}) \arrow[r, "\partial'"] & H_{q-1}^c(f'_{A'}) \end{tikzcd}$$

commutes.

Let $\{(x_q, y_{q+1}) + S_q(A) \oplus S_{q+1}(B)\}$ be the equivalence class of a cycle in $(S_q(f) / S_q(f_A))$. Then $\check{\partial}(x_q, y_{q+1}) = (-\partial x_q, f_\#(x_q) + \partial y_{q+1})$ is an element of $S_{q-1}(A) \oplus S_q(B)$. Thus we have

$$[k_A, k'_B]_* \circ \partial\{(x_q, y_{q+1}) + S_q(A) \oplus S_{q+1}(B)\}$$

$$= [k_A, k'_B]_*\{(-\partial x_q, f_\#(x_q) + \partial y_{q+1})\}$$

$$= \{(-\partial k_{A\#}(x_q), k'_{B\#}(f_\#(x_q) + \partial y_{q+1}))\}$$

$$= \{(-k_{A\#}(\partial x_q), k'_{B\#}(f_\#(x_q) + \partial y_{q+1}))\}$$

$$= \{(-k_\#(\partial x_q), k'_\#(f_\#(x_q) + \partial y_{q+1}))\}$$

$$= \{(-\partial k_\#(x_q), f'_\# \circ k_\#(x_q) + \partial k'_\#(y_{q+1}))\}$$

$$= \partial'\{(k_\#(x_q), k'_\#(y_{q+1})\}$$

$$= \partial' \circ [k, k']_*\{(x_q, y_{q+1}) + S_q(A) \oplus S_{q+1}(B)\}.$$

Out next result is important for the computations involved in the existence proof of this chapter and the categoricity theorem of Chapter III.

Proposition II. 8: Let $(f, f_A) : (X, A) \to (Y, B)$ be an object of $\mathcal{C}$. There exists an exact sequence

$$\cdots \to H_q(X, A) \to H_q(Y, B) \to H^c_{q-1}(f, f_A) \to H_{q-1}(X, A) \to \cdots$$

where $H_q(X, A)$ and $H_q(Y, B)$ are integral singular homology groups of (X, A) and (Y, B) respectively.

Proof: With $f_\# : S(X) \to S(Y)$ the induced chain map on integral singular chains, $f_{A\#} : S(A) \to S(B)$. Hence we can define a chain map,

$$\bar{f}_{\#} : \frac{S(X)}{S(A)} \to \frac{S(Y)}{X(B)}$$

on the quotient complexes. The cone sequence for the chain map $\bar{f}_{\#}$ gives

$$\ldots \to H_q(X, A) \to H_q(Y, B) \to H_{q-1}(S(\bar{f}_{\#})) \to H_{q-1}(X, A) \to \ldots$$

and we wish to show that the chain complex $\left(\frac{S(f_{\#})}{S(f_{A\#})}, \partial_1\right)$ is chain equivalent to $(S(\bar{f}_{\#}), \partial_2)$.

$$\left(\frac{S(f_{\#})}{S(f_{A\#})}\right)_q = \frac{S_q(X) \oplus S_{q+1}(Y)}{S_q(A) \oplus S_{q+1}(B)}, \quad S_q(\bar{f}_{\#}) = \frac{S_q(X)}{S_q(A)} \oplus \frac{S_{q+1}(Y)}{S_{q+1}(B)}$$

For $(x_q, y_{q+1}) \in S_q(X) \oplus S_{q+1}(Y)$, define

$$\tau_q((x_q, y_{q+1}) + S_q(A) \oplus S_{q+1}(B)) = (x_q + S_q(A), y_{q+1} + S_{q+1}(B))$$

and

$$\kappa_q(x_q + S_q(A), y_{q+1} + S_{q+1}(B)) = ((x_q, y_{q+1}) + S_q(A) \oplus S_{q+1}(B))$$

as maps

$$\tau : \frac{S(f_{\#})}{S(f_{A\#})} \to S(\bar{f}_{\#}) \quad \text{and} \quad \kappa : S(\bar{f}_{\#}) \to \frac{S(f_{\#})}{S(f_{A\#})}$$

Obviously κ is the algebraic inverse of τ and we must only show that τ is a chain map. Observe that

$$\partial_1((x_q, y_{q+1}) + S_q(A) \oplus S_{q+1}(B))$$

$$= (-\partial x_q, f_\#(x_q) + \partial y_{q+1}) + S_{q-1}(A) \oplus S_q(B)$$

and

$$\partial_2(x_q + S_q(A), y_{q+1} + S_{q+1}(B))$$

$$= (-\partial x_q + S_{q-1}(A), \overline{f}_\#(x_q + S_q(A)) + \partial y_{q+1} + S_q(B))$$

$$= (-\partial x_q + S_{q-1}(A), f_\#(x_q) + S_q(B) + \partial y_{q+1} + S_q(B))$$

$$= (-\partial x_q + S_{q-1}(A), f_\#(x_q) + \partial y_{q+1} + S_q(B))$$

Hence the diagram

$$\begin{tikzcd}
\left(\frac{S(f_\#)}{S(f_{A\#})}\right)_q \arrow[r, "\tau_q"] \arrow[d, "\partial_1"] & S_q(\overline{f}_\#) \arrow[d, "\partial_2"] \\
\left(\frac{S(f_\#)}{S(f_{A\#})}\right)_{q-1} \arrow[r, "\tau_{q-1}"] & S_{q-1}(\overline{f}_\#)
\end{tikzcd}$$

commutes, and τ is a chain map.

We are now in a position to prove

Theorem II. 2: (H^c, ∂) is a homology theory.

Proof: The fact that H^c is a functor and ∂ a natural transformation has already been shown and it remains only to verify that (H^c, ∂) satisfies our axioms.

Axiom II. For any object $(f, f_A) : (X, A) \to (Y, B)$ in $\mathcal{C}$ we have the short exact sequence of chain complexes

$$0 \to S(f_{A\#}) \to S(f_{\#}) \to S(f_{\#})/S(f_{A\#}) \to 0$$

Axiom II is verified by obtaining the long exact homology sequence and noting that ∂ is the connecting homomorphism of this sequence.

Axiom III. Let $(f, f_A) \in \mathcal{C}$ and let U and U' be the open sets as postulated in Axiom III. Then we have from Proposition II.8,

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(X-U', A-U') \arrow[r] \arrow[d] & H_q(Y-U, B-U) \arrow[r] \arrow[d] & H^c_{q-1}(f_{X-U'}, f_{A-U'}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_q(X, A) \arrow[r] & H_q(Y, B) \arrow[r] & H^c_{q-1}(f, f_A) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps are induced by inclusion. The five lemma then implies that $H^c(f_{X-U'}, f_{A-U'}) \approx H^c(f, f_A)$.

Axiom IV. IV. A follows from the sequence of Proposition II. 8 as $H_q(X, A) = 0$ for each q. Hence $H^c_q(f, f_A) \approx H_{q+1}(Y, B)$. IV. B is obtained from the absolute form of the cone sequence for $\alpha : \bigvee_{i=1}^{m} v_i \to v_0$ as given in the proof of Proposition II. 8.

Axiom I. We wish to prove Proposition II. 4 for H^c which implies Axiom I. Let $f : X \to Y$ be an object of $\mathcal{C}$ and consider the following inclusion morphisms as given in the hypothesis of Proposition II. 4.

$$\begin{array}{ccc} X & \xrightarrow{i_\epsilon} & X \times I \\ \downarrow f & & \downarrow f \times 1 \\ Y & \xrightarrow{j_\epsilon} & Y \times I \end{array}$$

If P represents the usual chain homotopy from $S_q(X)$ to $S_{q+1}(X \times I)$ such that $\partial P + P\partial = i_{0_\#} - i_{1_\#}$, then the following square is commutative (4, p. 46):

$$\begin{array}{ccc} S_q(X) & \xrightarrow{P} & S_{q+1}(X \times I) \\ \downarrow f_\# & & \downarrow (f \times 1)_\# \\ S_q(Y) & \xrightarrow{P} & S_{q+1}(Y \times I) \end{array}$$

We want to define a chain homotopy P^* from $S_q(f)$ to $S_{q+1}(f \times 1)$ such that $\partial P^* + P^*\partial = [i_1, j_1]_\# - [i_0, j_0]_\#$. Let $P^* = (P, -P)$. Then for $(x_q, y_{q+1}) \in S_q(f)$, $P^*(x_q, y_{q+1}) = (P(x_q), -P(y_{q+1}))$ and

$$(\partial P^* + P^*\partial)(x_q, y_{q+1})$$

$$= \partial P^*(x_q, y_{q+1}) + P^*\partial(x_q, y_{q+1})$$

$$= \partial(P(x_q), -P(y_{q+1})) + P^*(-\partial x_q, f_\#(x_q) + \partial y_{q+1})$$

$$= (-\partial P(x_q), (f \times 1)_\#[P(x_q)] - \partial P(y_{q+1})) + (-P(\partial x_q), -P[f_\#(x_q)] - P\partial y_{q+1}) =$$

$$= (-[\partial P(x_q)+P\partial x_q], -[\partial P(y_{q+1})+P\partial(y_{q+1})]+(f\times 1)_{\#}P(x_q)-Pf_{\#}(x_q))$$

$$= (-\{i_{0\#}(x_q)-i_{1\#}(x_q)\}, -\{j_0(y_{q+1})-j_1(y_{q+1})\} + 0)$$

The last term cancelling by the above commutative square. Hence

$$\begin{aligned}(\partial P^* + P^*\partial)(x_q, y_{q+1}) &= -(i_{0\#}(x_q)-i_{1\#}(x_q), j_{0\#}(y_{q+1})-j_{1\#}(y_{q+1}) \\ &= -([i_0, j_0]_{\#} - [i_1, j_1]_{\#})(x_q, y_{q+1}) \\ &= ([i_1, j_1]_{\#} - [i_0, j_0]_{\#})(x_q, y_{q+1})\end{aligned}$$

Therefore $[i_1, j_1]_* = [i_0, j_0]_*$, and Axiom I follows. This completes the proof that (H^c, ∂) is a homology theory on $\mathcal{C}$.

III. THE CATEGORICITY THEOREM

In this chapter we wish to prove a categoricity theorem for homology theories defined on a subcategory of the category $\mathcal{C}$. The proof will require several lemmas and will proceed by induction on the relative dimension of the domain of an object in this subcategory.

Definition III. 1: Let $(H, \partial, *)$ and $(H', \partial', \#)$ be two homology theories defined on $\mathcal{C}$. A homomorphism, ψ, from H to H', is a natural transformation commuting with connecting homomorphisms; that is, for any morphism $[k, k']$ between objects (f, f_A) and $(f', f'_{A'})$ in $\mathcal{C}$, the following diagrams commute for each q:

1)
$$\begin{array}{ccc} H_q(f, f_A) & \xrightarrow{[k,k']_*} & H_q(f', f'_{A'}) \\ \downarrow{\scriptstyle \psi_q} & & \downarrow{\scriptstyle \psi_q} \\ H'_q(f, f_A) & \xrightarrow{[k,k']_\#} & H'_q(f', f'_{A'}) \end{array}$$

2)
$$\begin{array}{ccc} H_q(f, f_A) & \xrightarrow{\partial} & H_{q-1}(f_A) \\ \downarrow{\scriptstyle \psi_q} & & \downarrow{\scriptstyle \psi_{q-1}} \\ H'_q(f, f_A) & \xrightarrow{\partial'} & H'_{q-1}(f_A) \end{array}$$

Definition III. 2: Let $\mathcal{C}_*$ be the subcategory of $\mathcal{C}$ whose objects are surjective cellular map pairs between finite CW-complexes and whose morphisms are commutative diagrams of cellular maps.

Let $\alpha : \bigvee_{i=1}^{m} v_i \to v$ be a map from m disjoint vertices onto a single vertex. It follows from Axiom IV. B that $H_0(\alpha) \approx \bigoplus_{i=1}^{m} G_i$ and we may state

Theorem III. 1: Let $(H, \partial, *)$ and $(H', \partial', \#)$ be two homology theories defined on $\mathcal{C}_*$ with coefficient groups G and G' respectively. Let $h : G \to G'$ be a homomorphism. There exists a unique homomorphism $\psi : H \to H'$ such that if α is the object described above, $\psi_0 : H_0(\alpha) \to H_0'(\alpha)$ is given by $\bigoplus_{i=1}^{m} h_i : \bigoplus_{i=1}^{m} G_i \to \bigoplus_{i=1}^{m} G_i'$ where, for each i, $h_i = h$.

Assuming the validity of Theorem III. 1, we have the following categoricity theorem.

Theorem III. 2: If $h : G \to G'$ is an isomorphism, then so is $\psi_q : H_q(f, f_A) \to H_q'(f, f_A)$ for each object $(f, f_A) \in \mathcal{C}_*$ and all integers q.

Proof: If we construct the homomorhpism from H' to H induced by h^{-1}, it follows that the map induced by the composition must be the identity automorphism on $H_q(f, f_A)$ for every q by the uniqueness part of Theorem 1.

Before proceeding with the proof of Theorem III. 1 we would like to make some initial constructions and prove a preliminary result. Let $f : X \to Y$ be an object of $\mathcal{C}_*$. The cone over X, denoted by $C(X)$, is the quotient space of $X \times I$ obtained by identifying the subspace $X \times \{0\}$ to a single point. The mapping cone of f, denoted by $C(f)$, is obtained in a similar fashion by identifying $(x, 1)$ with $f(x)$ in the space $C(X) \vee Y$. X and Y may obviously be regarded as subspaces of $C(X)$ and $C(f)$, respectively, under the identifications $x \mapsto (x, 1)$ and $y \mapsto y$.

Consider the map $j : C(X) \to C(f)$ given by $j(x, t) = (x, t)$, $t \neq 1$, and $j(x, 1) = f(x)$. Since X and Y are CW-complexes, it follows that $C(X)$ and $C(f)$ are CW-complexes. Hence we have the object $(j, j_X) : (C(X), X) \to (C(f), Y)$ in $\mathcal{C}_*$ and its sequence

$$\cdots \to H_q(j_X) \to H_q(j) \to H_q(j, j_X) \to H_{q-1}(j_X) \to \cdots$$

<u>Lemma III. 3:</u> $H_q(j, j_X) = 0$ <u>for all</u> q.

<u>Proof:</u> Let X and Y be expanded to J and K, respectively, where $J = X \times [1/2, 1] \subset C(X)$ and K is the corresponding expansion of Y in $C(f)$. Hence, we have the following composition of morphisms

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "j_X"] & J \arrow[r, "r_1"] \arrow[d, "j_J"] & X \arrow[d, "j_X"] \\ Y \arrow[r, "i_2"] & K \arrow[r, "r_2"] & Y \end{tikzcd}$$

where $[i_1, i_2]$ are inclusion maps and $[r_1, r_2]$ are the usual retractions. From the definitions,

$$[r_1, r_2]_* \circ [i_1, i_2]_* = 1_{H(j_X)}$$

The canonical homotopics

$$1_J \simeq i_1 \circ r_1 \quad \text{and} \quad 1_K \simeq i_2 \circ r_2$$

give the morphism

$$\begin{array}{ccc} J \times I & \xrightarrow{F} & X \\ \downarrow{\scriptstyle j_J \times 1} & & \downarrow{\scriptstyle j_X} \\ K \times I & \xrightarrow{F'} & Y \end{array}$$

Axiom I then implies that

$$[i_1, i_2]_* \circ [r_1, r_2]_* = 1_{H(j_J)}$$

and hence, $[i_1, i_2]_*$ is an isomorphism.

Consider the following array of sequences for the pairs (j, j_X) and (j, j_J):

$$\begin{array}{ccccccccc} \cdots \to & H_q(j_X) & \to & H_q(j) & \to & H_q(j, j_X) & \to & H_{q-1}(j_X) & \to \cdots \\ & \downarrow{\scriptstyle [i_1, i_2]_*} & & \downarrow{\scriptstyle 1_{H(j)}} & & \downarrow & & \downarrow & \\ \cdots \to & H_q(j_J) & \to & H_q(j) & \to & H_q(j, j_J) & \to & H_{q-1}(j_J) & \to \cdots \end{array}$$

It follows by the five lemma that the inclusion morphism from (j, j_X) to (j, j_J) induces an isomorphism and hence it suffices to consider the object $(j, j_J) : (C(X), J) \rightarrow (C(f), K)$.

Proposition II. 4 allows us to excise the open set $X \times (1/2, 1]$ from $(C(X), J)$ and the corresponding set in $(C(f), K)$. Hence, the inclusion morphism

$$\begin{array}{ccc} (C(X), X) & \longrightarrow & (C(X), J) \\ \Big\downarrow{\scriptstyle (1_{C(X)}, 1_X)} & & \Big\downarrow{\scriptstyle (j, j_J)} \\ (C(X), X) & \longrightarrow & (C(f), K) \end{array}$$

induces an isomorphism and we need only consider the object $(1_{C(X)}, 1_X)$. Proposition II. 7 and the five lemma imply that, for each integer q, $H_q(1_{C(X)}, 1_X) = 0$. Therefore, $H_q(j, j_X) = 0$ for all q, and the lemma is proved. We note that from Lemma III. 3 and the exact sequence of the pair (j, j_X), the inclusion morphism from j_X to j induces an isomorphism.

Now consider the object $(f, f_A) : (X, A) \rightarrow (Y, B)$. There exist corresponding objects $(j, j_X) : (C(X), X) \rightarrow (C(f), Y)$ and $(k, k_A) : (C(A), A) \rightarrow (C(f_A), B)$ and the inclusion morphism

$$\begin{array}{ccc} (X, A) & \longrightarrow & (C(X), C(A)) \\ \Big\downarrow {\scriptstyle (j_X, k_A)} & & \Big\downarrow {\scriptstyle (j, k)} \\ (Y, B) & \longrightarrow & (C(f), C(f_A)) \end{array}$$

We then have the sequences

$$\begin{array}{ccccccccc} \cdots \to & H_q(k_A) & \to & H_q(j_X) & \to & H_q(j_X, k_A) & \to & H_{q-1}(k_A) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_q(k) & \to & H_q(j) & \to & H_q(j, k) & \to & H_{q-1}(k) & \to \cdots \end{array}$$

with the vertical maps induced by the following array of inclusion morphisms:

$$\begin{tikzcd} & j & \\ j_X \arrow[ur] & & k \arrow[ul] \\ & k_A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The morphisms from k_A to k and j_X to j induce isomorphisms by Lemma III. 3, and hence the inclusion morphism from (j_X, k_A) to (j, k) induces an isomorphism by the five lemma. We note that $j_X = f$ and $k_A = f_A$. Since $(C(X), C(A))$ is a contractible pair, we have from Axiom IV. A that

$H_q(j, k) \approx H_{q+1}(C(f), C(f_A))$. Thus, for each q,

$$H_q(f, f_A) = H_q(j_X, k_A) \approx H_q(j, k) \approx H_{q+1}(C(f), C(f_A))$$

and we have proved

Lemma III. 4: *Assume* $(H, \partial, *)$ *and* $(H', \partial', \#)$ *are two homology theories defined on* $\mathcal{C}_*$. *Then for any object* $(f, f_A) \in \mathcal{C}_*$, *there exists an isomorphism from* $H_q(f, f_A)$ *to* $H'_q(f, f_A)$ *for each integer* q.

The lemmas in the sequel deal with an important special case necessary for the proof of Theorem 1. Let $(f, f_A) : (X, A) \to (Y, B)$ be an object of $\mathcal{C}_*$. Assume that both $X-A$ and $Y-B$ consist only of interiors of n-cells. Enumerate the n-cells of $X-A$ $(\{\Delta^n_i\}, i = 1, \ldots, m)$ and assume initially that $Y-B = \Delta^n$. Let $\bar{\Delta}^n_i \subset \Delta^n_i$, $i = 1, \ldots, m$, be smaller interior n-cells mapping onto the interior n-cell $\bar{\Delta}^n \subset \Delta^n$. Define $K = A \bigcup\limits_{i=1}^{m} \widetilde{(\text{int.}\, \bar{\Delta}^n_i)}$ and $K' = B \cup \widetilde{(\text{int.}(\bar{\Delta}^n))}$ where $\sim$ denotes complementation. We then have the triple $(f, f_K, f_A) : (X, K, A) \to (Y, K', B)$.

Using the canonical retractions for K and K' onto A and B respectively, it follows from Axiom 1 and the five lemma that the inclusion morphism from (f, f_A) to (f, f_K) induces an isomorphism. By Proposition II. 4, we may excise $\tilde{\bar{\Delta}}^n$ and its inverse from (f, f_K) to obtain

$$\begin{tikzcd}
\left(\bigvee\limits_{i=1}^{m} \bar{\Delta}^n_i, \bigvee\limits_{i=1}^{m} \partial\bar{\Delta}^n_i\right) \arrow[r, "i_1"] \arrow[d, "(g, g_\partial)"] & (X, K) \arrow[d, "(f, f_K)"] \\
(\bar{\Delta}^n, \partial\bar{\Delta}^n) \arrow[r, "i_2"] & (Y, K')
\end{tikzcd}$$

where (g, g_∂) is induced from f. Therefore, for each q, $[i_1, i_2]_*$ from $H_q(g, g_\partial)$ to $H_q(f, f_K)$ is an isomorphism. Since $H_q(f, f_A) \to H_q(f, f_K)$ is an isomorphism, so is the map from $H_q(g, g_\partial)$ to $H_q(f, f_A)$ induced by inclusion.

Assume now that Y-B consists of p n-cells $(\{\Delta_j^n\}$, $j = 1, \ldots, p)$. By the same technique applied to each n-cell in Y-B we obtain an object (g^j, g_∂^j) for each j. By the above argument, we have proved

Lemma III. 5: *Assume* $(f, f_A) : (X, A) \to (Y, B)$ *is an object of* $\mathcal{C}_*$ *with* X-A *and* Y-B *consisting only of interiors of n-cells. Further, assume the n-cells of* X-A *map onto the n-cells of* Y-B. *With* p *the number of n-cells in* Y-B *we have that the inclusion morphism from* $(g, g_\partial) = (\bigvee_{j=1}^{p} g^j, \bigvee_{j=1}^{p} g_\partial^j)$ *to* (f, f_A) *induces an isomorphism.*

Consider an object in $\mathcal{C}_*$ of the form $(f, f_A) : (X, A) \to (Y, Y)$ where X-A consists of the interior of a single n-cell. Choose K as above and form the triple

$$(f, f_K, f_A) : (X, K, A) \to (Y, Y, Y).$$

Lemma III. 6: *The inclusion morphism from* (f, f_A) *to* (f, f_K) *induces an isomorphism.*

Proof: Consider the sequence

$$\cdots \to H_q(f_K, f_A) \to H_q(f, f_A) \to H_q(f, f_K) \to H_{q-1}(f_K, f_A) \to \cdots$$

We wish to show that $H_q(f_K, f_A) = 0$ for each q. Since the singular homology groups $H_q(K, A)$ and $H_q(Y, Y)$ are zero for each q, we have from Proposition II. 8 that $H^c(f_K, f_A) = 0$. Lemma III. 4 implies, since H^c is a homology theory, that $H_q(f_K, f_A) = 0$ for each q, and the lemma follows.

Assume that for an object as considered in Lemma III. 5, $X-A$ consists of the interior of a finite number of n-cells $(\{\Delta_i^n\}, i = 1, \ldots, p)$. We may then perform the excision operation as in Lemma III. 5 and using the same notation, $(g, g_\partial) = (\bigvee_{i=1}^{p} g^i, \bigvee_{i=1}^{p} g_\partial^i)$, we have

Lemma III. 7: Let (f, f_A) be an object in $\mathcal{C}_*$ with $X-A$ consisting of the interiors of n-cells. Then the inclusion morphism from (g, g_∂) to $(f, f_A) : (X, A) \to (Y, Y)$ induces an isomorphism.

Finally, we may combine Lemmas III. 5 and III. 7 to obtain the following general result:

Lemma III. 8: Assume $(f, f_A) : (X, A) \to (Y, B)$ is an object of $\mathcal{C}_*$ with $X-A$ consisting of the interiors of $p+q$ n-cells and $Y-B$ the interiors of r n-cells. In addition, assume that p

<u>n-cells of</u> $X-A$ <u>map onto</u> $Y-B$ <u>and</u> q <u>n-cells map into</u> B. <u>Then the inclusion morphism from</u> (g, g_∂) <u>to</u> (f, f_A) <u>induces an isomorphism.</u>

<u>Proof</u>: Let D be the union of the q n-cells of $X-A$ mapped into B. Let $A^* = A \cup D$ and form the triple $(f, f_{A^*}, f_A) : (X, A^*, A) \to (Y, B, B)$. Then (f, f_{A^*}) and (f_{A^*}, f_A) are as in Lemmas III. 5 and III. 7 and the lemma follows by the sequence for the triple and the five lemma.

We now proceed with the proof of Theorem III. 1. By the relative dimension of a pair of CW-complexes (X, A), we mean the dimension of the highest dimensional cell in $X-A$. The special instance $X = A$ has relative dimension -1. We wish to construct a natural transformation $\psi : H \to H'$ commuting with connecting homomorphisms which is unique with respect to the homomorphism $h : G \to G'$ of coefficient groups.

Let $(f, f_A) : (X, A) \to (Y, B)$ be an object of $\mathcal{C}_*$ and assume that rel. dim. $(X, A) = -1$. Then from Axiom II we observe that, for each q, $H_q(f, f_A) = 0$. Hence there exists a unique homomorphism $\psi_{-1} : H_q(f, f_A) \to H'_q(f, f_A)$ which obviously satisfies all of the hypothesis of the theorem.

If rel. dim. $(X, A) = 0$, then $X-A$ consists of only vertices and $Y-B$ is either a disjoint union of vertices or is empty. Note that A and B are both open and closed in X and Y respectively and we may apply Axiom III to obtain an object of the form

$$\alpha : \bigvee_{i=1}^{m} v_i \to v, \quad H_0(\alpha) = \bigoplus_{i=1}^{m-1} G_i \quad \text{and} \quad H_0'(\alpha) = \bigoplus_{i=1}^{m-1} G_i' .$$

Therefore we have a homomorphism $\psi_0 : H_0(\alpha) \to H_0'(\alpha)$ defined by $\psi_0 = \bigoplus_{i=1}^{m-1} h_i$, where $h_i = h$, the given homomorphism of coefficient groups.

Let us now assume that for any object $(f, f_A) \in \mathcal{C}_*$ with rel. dim. $(X, A) \leq n-1$, that homomorphisms ψ have been defined. In addition, assume these homomorphisms commute with connecting homomorphisms and induced maps. Let (f, f_A) be an object with the rel. dim. of (X, A) equal to n. Denote the q-skeleton of X by X_q and define $A^q = A \cup X_q$. We wish to consider the triple $(f, f_{A^{n-1}}, f_A) : (X, A^{n-1}, A) \to (Y, B^{n-1}, B)$ where we note that $X - A^{n-1}$ consists only of interiors of n-cells. Since the maps are surjective, it follows that the rel. dim. of (Y, B) is less than or equal to n and if actually less than n, $B^{n-1} = Y$.

Hence it follows that $(f, f_{A^{n-1}})$ is one of the objects considered in either Lemma III. 7 or Lemma III. 8. It is necessary to construct homomorphisms

$$\phi_q : H_q(f, f_{A^{n-1}}) \to H_q'(f, f_{A^{n-1}})$$

for these special instances. Both cases will be treated simultaneously with the generic term $(g, g_\partial) = (\bigvee_{i=1}^{p} g^i, \bigvee_{i=1}^{p} g^i_\partial)$ used as before. $g^i_\partial : \bigvee_j \partial\Delta_j^{n_i} \to \partial\Delta^{n_i}$ and hence chose a vertex v_j and $\partial\Delta_j^{n_i}$ such that $v_j \to v_0$ for each j. We then have the triple

$$(g^i, g^i_\partial, g^i_v) : (\bigvee_j \Delta_j^{n_i}, \bigvee_j \partial\Delta_j^{n_i}, \bigvee_j v_j) \to (\Delta^{n_i}, \partial\Delta^{n_i}, v_0)$$

From Axiom I we have $H_q(g^i, g^i_v) = 0$ for each q. Hence by the sequence for the triple, we have that $\bar{\partial} : H_q(g^i, g^i_\partial) \to H_q(g^i_\partial, g^i_v)$ is an isomorphism. Thus from Proposition II. 7

$$H_q(g, g_v) \approx \bigoplus_{i=1}^{p} H_q(g^i, g^i_v) = 0$$

and it follows that $\bar{\partial} : H_q(g, g_\partial) \to H_{q-1}(g_\partial, g_v)$ is an isomorphism.

Consider the following diagram:

(1)
$$\begin{tikzcd}
 & H_q(f, f_{A^{n-1}}) \arrow[dd, dashed, "\phi"] & \\
H_q(g, g_\partial) \arrow[ur, "{[i,i']_*}"] \arrow[rr, "\bar{\partial}"] \arrow[dd] & & H_q(g_\partial, g_v) \arrow[dd, "\psi_{n-1}"] \\
 & H'_q(f, f_{A^{n-1}}) & \\
H'_q(g, g_\partial) \arrow[ur, "{[i,i']_\#}"] \arrow[rr, "\bar{\partial}'"] & & H'_q(g_\partial, g_v)
\end{tikzcd}$$

Since the object (g_∂, g_v) has rel. dim. $(n-1)$, ψ_{n-1} has already been defined. Lemmas III. 5 and III. 6 guarantee that

$[i, i']_*$ and $[i, i']_\#$ are isomorphisms. Hence we may define $\phi : H_q(f, f_{A^{n-1}}) \to H'_q(f, f_{A^{n-1}})$ by $\phi = [i, i']_\# \circ \bar{\partial}'^{-1} \circ \psi_{n-1} \circ \bar{\partial} \circ [i, i']_*^{-1}$. Note that by Lemma III. 5 and Proposition II. 8, ϕ is non-trivial only on dimension n.

The triple $(f, f_{A^{n-1}}, f_A)$ gives the array:

$$
(2) \qquad
\begin{tikzcd}
\cdots \arrow[r] & H_q(f_{A^{n-1}}, f_A) \arrow[r] & H_q(f, f_A) \arrow[r] \arrow[d, dashed, "\psi_n"] & H_q(f, f_{A^{n-1}}) \arrow[r, "\bar{\partial}"] \arrow[d, "\phi_n"] & H_{q-1}(f_{A^{n-1}}, f_A) \arrow[r] \arrow[d, "\psi_{n-1}"] & H_{q-1}(f, f_A) \arrow[r] & \cdots \\
\cdots \arrow[r] & H'_q(f_{A^{n-1}}, f_A) \arrow[r] & H'_q(f, f_A) \arrow[r] & H'_q(f, f_{A^{n-1}}) \arrow[r, "\bar{\partial}'"] & H'_{q-1}(f_{A^{n-1}}, f_A) \arrow[r] & H'_{q-1}(f, f_A) \arrow[r] & \cdots
\end{tikzcd}
$$

Homomorphisms ϕ have already been defined and since rel. dim. $(A^{n-1}, A) < n$, so have the homomorphisms ψ_{n-1}. It remains to construct homomorphisms ψ_n.

For this purpose it is necessary to show $\bar{\partial}' \circ \phi_n = \psi_{n-1} \circ \bar{\partial}$ in the above diagram. For $q \neq n$, we have observed that $H_q(f, f_{A^{n-1}}) \approx H'_q(f, f_{A^{n-1}}) = 0$ and commutativity is trivial. Hence consider the following sequence for the triple

$$(f_{A^{n-1}}, f_{A^{n-2}}, f_A) : (A^{n-1}, A^{n-2}, A) \to (B^{n-1}, B^{n-2}, B)$$

and the diagram

(3)

$$
\begin{tikzcd}
 & & & & H_{n-2}(f_{A^{n-2}}, f_A) \arrow[dd] \\
 & & & H_{n-1}(f_{A^{n-1}}, f_{A^{n-2}}) \arrow[ur] \arrow[dd] & \\
H_n(f, f_{A^{n-1}}) \arrow[rr] \arrow[urrr, dashed] \arrow[dd] & & H_{n-1}(f_{A^{n-1}}, f_A) \arrow[ur, "j_*"] \arrow[dd] & & H'_{n-2}(f_{A^{n-2}}, f_A) \\
 & H_{n-1}(f_{A^{n-2}}, f_A) \arrow[ur] \arrow[dd] & & H'_{n-1}(f_{A^{n-1}}, f_{A^{n-2}}) \arrow[ur] & \\
H'_n(f, f_{A^{n-1}}) \arrow[rr] \arrow[urrr, dashed] & & H'_{n-1}(f_{A^{n-1}}, f_A) \arrow[ur, "j_\#"] & & \\
 & H'_{n-1}(f_{A^{n-2}}, f_A) \arrow[ur] & & &
\end{tikzcd}
$$

Since the rel. dim. of (A^{n-2}, A) is less than $n-1$, Proposition II. 8 and Lemma III. 5 imply that j_* and $j_\#$ in the above diagram are monomorphisms. Then we have (where commutativity of the left rectangle is what we wish to show) the diagram

(4)

$$
\begin{tikzcd}
H_n(f, f_{A^{n-1}}) \arrow[rr, dashed, "\check{\partial}"] \arrow[dr, "\bar{\partial}"] \arrow[dd, "\phi_n"] & & H_{n-1}(f_{A^{n-1}}, f_{A^{n-2}}) \arrow[dd, "\psi_{n-1}"] \\
 & H_{n-1}(f_{A^{n-1}}, f_A) \arrow[ur, "j_*"] \arrow[dd, "\psi_{n-1}"] & \\
H'_n(f, f_{A^{n-1}}) \arrow[rr, dashed, "\check{\partial}'"] \arrow[dr, "\bar{\partial}'"] & & H_{n-1}(f_{A^{n-1}}, f_{A^{n-2}}) \\
 & H'_{n-1}(f_{A^{n-1}}, f_A) \arrow[ur, "j_\#"] &
\end{tikzcd}
$$

where $\check{\partial}$ and $\check{\partial}'$ are connecting homomorphisms for the triple

$(f, f_{A^{n-1}}, f_{A^{n-2}})$. The triangles commute by virtue of the diagram

$$\begin{tikzcd}
& H_{n-1}(f_{A^{n-1}}, f_A) \arrow[dr, "j_*"] & \\
H_{n-1}(f_{A^{n-1}}) \arrow[ur, "i_*"] \arrow[rr, "\ell_*"] & & H_{n-1}(f_{A^{n-1}}, f_{A^{n-2}}) \\
H_n(f, f_{A^{n-1}}) \arrow[u, "\partial"] \arrow[urr, "\partial"] & &
\end{tikzcd}$$

where the top triangle is induced by inclusion and commutes by functoriality. By definition,

$$\ell_* \circ \partial = \check{\partial}, \quad i_* \circ \partial = \bar{\partial} \quad \text{and hence} \quad j_* \circ \bar{\partial} = \check{\partial}\,.$$

$\psi_{n-1} \circ j_* = j_\# \circ \psi_{n-1}$ follows by the inductive assumption and we must now verify that $\psi_{n-1} \circ \check{\partial} = \check{\partial}' \circ \phi_n$.

The definition of the maps ϕ_n give us the following cubical array:

(5)
$$\begin{tikzcd}
& & H_n(f, f_{A^{n-1}}) \arrow[rrr] \arrow[dd, "\phi_n"] & & & H_n(f_{A^{n-1}}, f_{A^{n-2}}) \arrow[dd, "\psi_{n-1}"] \\
0 \arrow[r] & H_n(g, g_\partial) \arrow[ur, "i_*"] \arrow[rr, "\bar{\partial}"] \arrow[dd, "\gamma"] & & H_{n-1}(g_\partial, g_v) \arrow[r] \arrow[urr, "i_*"] \arrow[dd] & 0 & \\
& & H'_n(f, f_{A^{n-1}}) \arrow[rrr, "\partial'"] & & & H'_n(f_{A^{n-1}}, f_{A^{n-2}}) \\
0 \arrow[r] & H'_n(g, g_\partial) \arrow[ur, "i_\#"] \arrow[rr, "\bar{\partial}'"] & & H'_{n-1}(g_\partial, g_v) \arrow[r] \arrow[urr, "i_\#"] & 0 &
\end{tikzcd}$$

The top and bottom of the cube commute by functoriality of H and H'. The left side and near side commute by construction and the right side commutes by the inductive assumption on ψ. Hence

$$\check{\partial}' \circ \phi_n \circ i_* = \check{\partial}' \circ i_\# \circ \gamma = i_\# \circ \bar{\partial}' \circ \gamma = i_\# \circ \psi_{n-1} \circ \bar{\partial}$$
$$= \psi_{n-1} \circ i_* \circ \bar{\partial} = \psi_{n-1} \circ \check{\partial} \circ i_*$$

Since i_* is an isomorphism, $\check{\partial}' \circ \phi_n = \psi_{n-1} \circ \check{\partial}$.

Returning to diagram 4, we have

$$j_\# \circ \psi_{n-1} \circ \bar{\partial} = \psi_{n-1} \circ j_* \circ \bar{\partial} = \psi_{n-1} \circ \check{\partial} = \check{\partial}' \circ \phi_n = j_\# \circ \bar{\partial}' \circ \phi_n$$

Therefore $j_\# \circ \bar{\partial}' \circ \phi_n = j_\# \circ \psi_{n-1} \circ \bar{\partial}$, and since $j_\#$ is a monomorphism, $\bar{\partial}' \circ \phi_n = \psi_{n-1} \circ \bar{\partial}$. This is the commutativity we wanted to verify for diagram 2. We observe from this configuration

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(f_{A^{n-1}}, f_A) \arrow[r] & H_n(f, f_A) \arrow[r, "i_*"] \arrow[d, "\psi_n"] & H_n(f, f_{A^{n-1}}) \arrow[r, "\bar{\partial}"] \arrow[d, "\phi_n"] & H_{n-1}(f_{A^{n-1}}, f_A) \arrow[r] \arrow[d, "\psi_{n-1}"] & \cdots \\
\cdots \arrow[r] & H'_n(f_{A^{n-1}}, f_A) \arrow[r] & H'_n(f, f_A) \arrow[r, "i_\#"] & H'_n(f, f_{A^{n-1}}) \arrow[r, "\bar{\partial}'"] & H'_{n-1}(f_{A^{n-1}}, f_A) \arrow[r] & \cdots
\end{tikzcd}$$

that Proposition II. 8 and Lemma III. 4, imply

$$H_n(f_{A^{n-1}}, f_A) \approx H'_n(f_{A^{n-1}}, f_A) = 0.$$

Thus i_* and $i_\#$ are monomorphisms. Now

$$\psi_{n-1} \circ \bar{\partial} \circ i_* = \bar{\partial}' \circ \phi_n \circ i_* = 0$$

and hence, for x in $H_n(f, f_A)$, $\phi_n \circ i_*(x) \in \ker \bar{\partial}' = \operatorname{im} i_\#$. We may then define $\psi_n(x) = i_\#^{-1}(\phi_n \circ i_*(x))$ and easily observe that ψ_n is uniquely defined and a homomorphism.

To define $\psi_n : H_{n-1}(f, f_A) \to H'_{n-1}(f, f_A)$ we have the following array:

$$\begin{array}{ccccccccc}
\cdots \to & H_n(f,f_{A^{n-1}}) & \xrightarrow{\bar{\partial}} & H_{n-1}(f_{A^{n-1}}, f_A) & \xrightarrow{j_*} & H_{n-1}(f, f_A) & \to & H_{n-1}(f, f_{A^{n-1}}) & \to \cdots \\
& \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \psi_{n-1}} & & \downarrow{\scriptstyle \psi_n} & & & \\
\cdots \to & H'_n(f,f_{A^{n-1}}) & \xrightarrow{\bar{\partial}'} & H'_{n-1}(f_{A^{n-1}}, f_A) & \xrightarrow{j_\#} & H'_{n-1}(f, f_A) & \to & H'_{n-1}(f, f_{A^{n-1}}) & \to \cdots
\end{array}$$

Since $H_{n-1}(f, f_{A^{n-1}}) \approx H'_{n-1}(f, f_{A^{n-1}}) = 0$, j_* and $j_\#$ are onto. Therefore, for any x in $H_{n-1}(f, f_A)$, choose a y in $H_{n-1}(f_{A^{n-1}}, f_A)$ such that, $j_*(y) = x$. Define $\psi_n(x) = j_\# \circ \psi_{n-1}(y)$. If y' is in $H_{n-1}(f_{A^{n-1}}, f_A)$ such that, $j_*(y') = x$, we have $j_*(y-y') = 0$. Thus there exists a z in $H_n(f, f_{A^{n-1}})$ such that $\bar{\partial}(z) = y - y'$. It then follows easily that y' gives the same value as y and consequently ψ_n is well defined. As before we observe that ψ_n is also a homomorphism.

For $q \neq n, n-1$, we have

$$\begin{tikzcd} 0 \arrow[r] & H_q(f_{A^{n-1}}, f_A) \arrow[r, "j_*"] \arrow[d, "\psi_{n-1}"] & H_q(f, f_A) \arrow[r] \arrow[d, "\psi_n"] & 0 \\ 0 \arrow[r] & H'_q(f_{A^{n-1}}, f_A) \arrow[r, "j_\#"] & H'_q(f, f_A) \arrow[r] & 0 \end{tikzcd}$$

and we may define ψ_n as $j_\# \circ \psi_{n-1} \circ j_*^{-1}$. Maps ψ_n have now been defined on each dimension for an arbitrary pair (f, f_A) of relative dimension n under the assumption that maps ψ_q, for $q < n$, have been defined satisfying the hypothesis of the theorem. It remains now to show that the maps ψ_n satisfy the commutativity relations with induced maps and connecting homomorphisms.

Let $(f, f_A) : (X, A) \to (Y, B)$ be an object of $\mathcal{C}_*$ with the relative dimension of (X, A) equal to n. Let $k = [k_1, k_2] : (f, f_A) \to (f', f'_{A'})$ be a morphism. Denote by

$$\bar{k} = [\bar{k}_1, \bar{k}_2] : (f, f_{A^{n-1}}) \to (f', f'_{A'^{n-1}})$$

and

$$\bar{\bar{k}} = [\bar{\bar{k}}_1, \bar{\bar{k}}_2] : (f_{A^{n-1}}, f_A) \to (f'_{A'^{n-1}}, f'_{A'})$$

the morphisms induced by k. Using the notation (g, g_∂) and $(g', g'_{\partial'})$ as defined previously, we have the following diagram:

$$
\begin{tikzcd}
& H_q(f, f_{A^{n-1}}) \arrow[rr, "\bar{k}_*"] \arrow[dd, "\phi_n"] & & H_q(f', f'_{A'^{n-1}}) \arrow[dd, "\phi_n"] \\
H_q(g, g_\partial) \arrow[ur, "i_*"] \arrow[rr] \arrow[dd] & & H_q(g', g'_{\partial'}) \arrow[ur, "i_*"'] \arrow[dd] & \\
& H'_q(f, f_{A^{n-1}}) \arrow[rr, "\bar{k}_\#"] & & H'_q(f', f'_{A'^{n-1}}) \\
H'_q(g, g_\partial) \arrow[ur, "i_\#"] \arrow[rr] & & H'_q(g', g'_{\partial'}) \arrow[ur, "i_\#"'] &
\end{tikzcd}
$$

By the commutativity of the above cube and the fact that i_* is an isomorphism, it follows that ϕ_n commutes with induced maps.

Thus we have diagram (6) where we wish to show that $\psi'_n \circ k_* = k_\# \circ \psi_n$. The top and bottom of the diagram commute by functoriality and the sides by construction. From the above remark,

$$\phi'_n \circ \bar{k}_* = \bar{k}_\# \circ \phi_n$$

and by our inductive assumption we have

$$\psi'_{n-1} \circ \bar{\bar{k}}_* = \bar{\bar{k}}_\# \circ \psi_{n-1}$$

Hence, for $q \neq n$

$$\psi'_n \circ k_* \circ j_* = \psi'_n \circ j'_* \circ \bar{\bar{k}}_* = j'_\# \circ \psi'_{n-1} \circ \bar{\bar{k}}_* = j'_\# \circ \bar{\bar{k}}_\# \circ \psi_{n-1}$$

$$= k_\# \circ j_\# \circ \psi_{n-1} = k_\# \circ \psi_n \circ j_*$$

However, for $q \neq n$, $H_q(f, f_{A^{n-1}}) \approx H_q(f', f'_{A'^{n-1}}) = 0$ and therefore

(6)

$$
\begin{tikzcd}[column sep=small]
{} \arrow[r] & H_q(f_{A^{n-1}}, f_A) \arrow[rr, "j_*"] \arrow[dr, "\bar{k}_*"] \arrow[dd, "\psi_{n-1}"] & & H_q(f, f_A) \arrow[rr, "i_*"] \arrow[dr, "k_*"] \arrow[dd, "\psi_n"] & & H_q(f, f_{A^{n-1}}) \arrow[r] \arrow[dr, "\bar{k}_*"] \arrow[dd, "\phi_n"] & {} & \\
& {} \arrow[r] & H_q(f'_{A'^{n-1}}, f_{A'}) \arrow[rr, "j'_*"] \arrow[dd, "\psi'_{n-1}"] & & H_q(f', f'_{A'}) \arrow[rr, "i'_*"] \arrow[dd, "\psi'_n"] & & H_q(f', f'_{A'^{n-1}}) \arrow[r] \arrow[dd, "\phi'_n"] & {} \\
{} \arrow[r] & H'_q(f_{A^{n-1}}, f'_{A'}) \arrow[rr, "j_\#"] \arrow[dr, "\bar{\bar{k}}_\#"] & & H'_q(f, f_A) \arrow[rr, "i_\#"] \arrow[dr, "k_\#"] & & H'_q(f, f_{A^{n-1}}) \arrow[r] \arrow[dr, "\bar{k}_\#"] & {} & \\
& {} \arrow[r] & H'_q(f'_{A'^{n-1}}, f'_{A'}) \arrow[rr, "j'_\#"] & & H'_q(f', f'_{A'}) \arrow[rr, "i'_\#"] & & H'_q(f', f'_{A'^{n-1}}) \arrow[r] & {}
\end{tikzcd}
$$

j_* is an epimorphism. This implies that $\psi'_n \circ k_* = k_\# \circ \psi_n$.

For $q = n$, we have $H'_n(f'_{A'n-1}, f'_{A'}) = 0$ and hence $i'_\#$ is a monomorphism. Thus

$$i'_\# \circ \psi'_n \circ k_* = \phi'_n \circ i'_* \circ k_* = \phi'_n \circ \overline{k}_* \circ i_* = \overline{k}_\# \circ \phi_n \circ i_*$$

$$= \overline{k}_\# \circ i_\# \circ \psi_n = i'_\# \circ k_\# \circ \psi_n$$

and, since $i'_\#$ is a monomorphism,

$$\psi'_n \circ k_* = k_\# \circ \psi_n.$$

At this point we have constructed homomorphisms ψ_n which commute with induced maps. To establish the fact that ψ_n commutes with connecting homomorphisms, we would first construct a diagram similar to (5). Then with the corresponding diagram for (6), we would repeat the same argument to obtain the result.

Finally, to establish uniqueness, assume we have another homomorphism $\overline{\psi} : H \to H'$ compatible with the homomorphism $h : G \to G'$ of coefficient groups. Then our inductive assumption is that $\overline{\psi}_q = \psi_q$ for q less than n and by an entirely similar procedure, we can show that $\overline{\psi}_n = \psi_n$. Therefore $\overline{\psi} = \psi$ and the proof of Theorem III. 1 is complete.

BIBLIOGRAPHY

1. Burghelea, Dan. Despre sirurile exacte asociate aplicatiilor. Academia Republicii Popular Romîne, Studii şi Cercelări de matematica 14:661-667. 1963.

2. Dold, Von Albrecht and René Thom. Quasifaserungen und unendliche symmetrische Produkte. Annals of Mathematics 67: 239-281. 1958.

3. Eilenberg, Samuel and Norman Steenrod. Foundations of algebraic topology. Princeton, Princeton University Press, 1952. 328 p.

4. Greenberg, Marvin. Lectures on algebraic topology. New York, W. A. Benjamin, 1967. 235 p.

5. Hu, Sze-Tsen. Homology theory. San Francisco, Holden Day, 1966. 247 p.

6. Spanier, Edwin H. Algebraic topology. New York, McGraw-Hill, 1966. 528 p.