

Mathematical
Inequalities
& **A**pplications

*With Compliments
of the Author*

Zagreb, Croatia

Volume 14, Number 1, January 2011

Larry Chen and Hung Viet Le

*Singular integrals with mixed homogeneity in
product spaces*

MATHEMATICAL INEQUALITIES & APPLICATIONS

AIMS AND SCOPE

Mathematical Inequalities & Applications (MIA, Math. Inequal. Appl.) brings together original research papers in all areas of mathematics, provided they are concerned with inequalities or their role. From time to time MIA will publish invited survey articles. Short notes with interesting results or open problems will also be accepted.

MIA is published quarterly, in January, April, July and October.

SUBMISSION

Manuscripts should be submitted electronically in PostScript or Adobe Acrobat PDF format to the MIA Editorial Office through MIA page: www.ele-math.com or by e-mail mia@ele-math.com

The author who submitted the article for publication will be denoted as a corresponding author. He/She manages all communication and correspondence with the MIA regarding the article. Authors may indicate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not assured.

In order to facilitate refereeing, copies of those papers (whether by the author or someone else) which are essential and referred to in the manuscript but are not conveniently accessible, should be enclosed.

COPYRIGHT

The acceptance of the article automatically implies the copyright transfer to MIA. Manuscripts are accepted for review with the understanding that the same work has not been published (except in the form of an abstract), that it is not under consideration for publication elsewhere, that it will not be submitted to another journal, and that its submission for publication has been approved by all the authors.

PREPARATION OF MANUSCRIPT

Manuscripts should be written in English.

The publisher strongly encourages submission of manuscripts written in TeX or one of its variants La-TeX, AMSTeX or AMSLaTeX. On acceptance of the paper, authors will be asked to send final version of file(s) to the Editorial Office.

TITLE PAGE

A title page should include the title of the article, author's name (no degrees) author's affiliation, e-mail addresses, mailing address of the corresponding author and running head less than 60 characters. Authors should be listed alphabetically.

ABSTRACT, KEY WORDS, SUBJECT CLASSIFICATION

The manuscript must be accompanied by a brief abstract, no longer than 100-150 words. It should make minimal use of mathematical symbols and displayed formulas. Mathematics Subject Classification (2000) with primary (and secondary) subject classification codes and a list of 4-5 key words must be given.

FIGURES

Figures should be prepared in a digital form suitable for direct reproduction, grey scale only, preferably at resolution of 300 dpi or higher, sized to fit a journal page, and in EPS, TIFF or JPEG format. Figures should have a short descriptive title and should be numbered consecutively in order of appearance in the text. The position of each figure in the text should be indicated.

REFERENCES

Bibliographic references should be listed alphabetically at the end of the article. The author should consult Mathematical Reviews for the standard abbreviations of journal names.

PROOFS

The author will receive proofs only once. Late return of the proofs will delay the article to a later issue.

PAGE CHARGES, OFFPRINTS

There are no page charges. Authors jointly receive 30 offprints of their articles free of charge. Additional offprints may be ordered from MIA prior to publication.

FORTHCOMING PAPERS

Papers accepted and prepared for publication will appear in the forthcoming section of Journal Web page. They are identical in form as final printed papers, except volume, issue and page numbers.

MIA is published by Publishing House **ELEMENT**, Zagreb, Croatia.

All correspondence and subscription orders should be addressed to the Editorial Office:

Mathematical Inequalities & Applications
Editorial Office
Menceticeva 2, 10000 Zagreb, Croatia
www.ele-math.com
e-mail: mia@ele-math.com
Fax: +385 1 6008799

The content of this publication is indexed in Mathematical Reviews, Zentralblatt für Mathematik, Реферативный журнал – Математика, Scopus and Science Citation Index-Expanded.

SINGULAR INTEGRALS WITH MIXED HOMOGENEITY IN PRODUCT SPACES

LARRY CHEN AND HUNG VIET LE

(Communicated by J. Marshall Ash)

Abstract. Let $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$ ($n, m \geq 2$) satisfy some cancellation conditions. We prove the L^p boundedness ($1 < p < \infty$) of the singular integral

$$Tf(x_1, x_2) = \text{p. v.} \int_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2,$$

where ρ_1, ρ_2 are some metrics which are homogeneous with respect to certain non-isotropic dilations. We also study the above singular integral along some surfaces.

1. Introduction

Consider the elliptic differential operator with constant coefficients

$$D = \sum_{i,j=1}^n a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.$$

As noted by E. Stein and S. Wainger [14], in order to study the existence and regularity results of D , one needs to consider singular integral operators with convolution kernels K satisfying the following conditions

- a) K is homogeneous of degree $-n$: $K(tx_1, \dots, tx_n) = t^{-n}K(x_1, \dots, x_n)$, $t > 0$,
- b) K is C^∞ away from the origin,
- c) $\int_{S^{n-1}} K(x) dx = 0$.

Similarly, to study the existence and regularity results of the heat equation

$$L(u) = \frac{\partial u}{\partial x_1} - \sum_{j=2}^n \frac{\partial^2 u}{\partial x_j^2},$$

one considers singular integral operators with the corresponding kernels K that satisfy

Mathematics subject classification (2010): 42B20, 42B25.

Keywords and phrases: Singular integrals, maximal functions, nonisotropic dilations, product spaces, L^p spaces.

- $\tilde{a}) K(t^2x_1, \dots, tx_n) = t^{-n-1}K(x_1, \dots, x_n), \quad t > 0,$
- $\tilde{b}) K$ is C^∞ away from the origin,
- $\tilde{c}) \int_{S^{n-1}} K(x) (2x_1^2 + x_2^2 + \dots + x_n^2) dx = 0.$

For a more general parabolic differential operator with constant coefficients, E. Fabes and N. Rivière [9] studied singular integrals with kernels K which satisfy (among some other conditions)

- $\bar{a}) K(t^{\alpha_1}x_1, \dots, t^{\alpha_n}x_n) = t^\alpha K(x_1, \dots, x_n), \quad t > 0, \quad \alpha = \sum_{i=1}^n \alpha_i,$
- $\bar{b}) \int_{S^{n-1}} K(x)J(x') dx = 0,$ where $J(x') \in C^\infty((0, 2\pi)^{n-2} \times (0, \pi))$ and without loss of generality, $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n.$

Note that the property (\bar{a}) above can be expressed as $K(A_t x) = |\det(A_t)|^{-1}K(x)$, where $A_t = \text{diag}[t^{\alpha_1}, \dots, t^{\alpha_n}]$ is a diagonal matrix. Note also that for each nonzero $x \in \mathbb{R}^n$, the function $F(x, t) = \sum_{i=1}^n t^{-2\alpha_i}x_i^2$ is a strictly decreasing function of $t > 0$.

Therefore, there exists a unique value of t which satisfies the equation $F(x, t) = 1$. If we define $\rho(x) = t$ and $\rho(0) = 0$, then it follows from [9, 14] that ρ is a metric on \mathbb{R}^n . It is well known that (\mathbb{R}^n, ρ) is a homogeneous group which admits a family of dilations $\delta_t = \exp(A \log t)$ such that $\rho(\delta_t x) = t\rho(x), t > 0$. Here A is a diagonalizable linear operator with positive eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$. By a change of variables to polar coordinates, each nonzero $x \in \mathbb{R}^n$ can be written as $x = \delta_\rho(x')$, ($x' = x/|x|$). Thus, there is a unique Radon measure (see [9], [10, p. 14]) $d\tilde{\sigma}(x') = J(x')d\sigma(x')$, where $J(x') = \alpha_1 x_1^2 + \dots + \alpha_n x_n^2$ is a C^∞ function on S^{n-1} which is bounded below and above by α_1 and α_n respectively.

Now let $Tf(x) = \text{p. v. } K * f(x)$, where the kernel $K(x) = \Omega(x')\rho^{-\alpha}(x)$, and $\alpha = \sum_{i=1}^n \alpha_i$. The following result has been obtained by E. Fabes and N. Rivière.

THEOREM 1.1. [9] *If $\Omega \in C^1(S^{n-1})$ satisfies the cancellation condition*

$$\int_{S^{n-1}} \Omega(x')J(x') d\sigma(x') = 0,$$

then the singular integral operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Subsequently, A. Nagel, N. Rivière, and S. Wainger [12] improved the above theorem by weakening the regularity condition on Ω as follows

THEOREM 1.2. [12] *If $\Omega \in L \log^+ L(S^{n-1})$ satisfies the cancellation condition above, then*

$$\|Tf\|_p \leq C\|f\|_p \text{ for } 1 < p < \infty.$$

Recently, the above result has been extended further by Y. Chen, Y. Ding, and D. Fan:

THEOREM 1.3. [4] *If $\Omega \in H^1(S^{n-1})$ (the Hardy space on the unit sphere S^{n-1}) satisfies the cancellation condition above, then*

$$\|Tf\|_p \leq C \|f\|_p \text{ for } 1 < p < \infty.$$

By the above Theorems 1.2 [12] and 1.3 [4], the L^p boundedness of singular integrals with mixed homogeneity in \mathbb{R}^n has been obtained. The extension of this result to product spaces is not simple, since the singularity now becomes two lower dimensional surfaces, instead of a singular point as in the previous case. The purpose of this paper is to extend the result in [12] to product spaces (see section 2.3). In classical harmonic analysis, it is well-known that singular integrals are dominated by Hardy-Littlewood maximal functions and square functions. We will show in section 3.2 that a singular integral along surfaces in the product domain is controlled by Hardy-Littlewood maximal functions acting on each variable along surfaces. We state our theorems in sections 2.3 and 3.2, and their proofs are given in sections 2.4 and 3.3 respectively. For recent works on the topic of singular integrals, the reader may view [1-9] among many other good references that are not listed in this paper.

2. Singular integral with rough kernels

2.1. Definitions and Notations

Most recent works dealing with singular integrals follow the ideas in [7]. In this section, we will extend two lemmas in [7]. For notational convenience, throughout the rest of this paper the dimensions n_1 and n are the same, and similarly the dimensions n_2 and m are equal to each other. Let ρ_1 be the metric on \mathbb{R}^{n_1} obtained from the unique solution of the equation $\sum_{i=1}^{n_1} t^{-2\alpha_i} x_i^2 = 1$, where $t > 0$, $x = (x_1, x_2, \dots, x_{n_1}) \in \mathbb{R}^{n_1}$, and $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{n_1}$. Similarly, let ρ_2 be the metric on \mathbb{R}^{n_2} obtained from the unique solution of the equation $\sum_{i=1}^{n_2} t^{-2\beta_i} y_i^2 = 1$, $t > 0$, $y = (y_1, y_2, \dots, y_{n_2}) \in \mathbb{R}^{n_2}$, and $0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_{n_2}$. Denote $A_{\rho_1}^{(1)} = \text{diag}[\rho_1^{\alpha_1}, \dots, \rho_1^{\alpha_{n_1}}]$ and $A_{\rho_2}^{(2)} = \text{diag}[\rho_2^{\beta_1}, \dots, \rho_2^{\beta_{n_2}}]$, $\rho_1, \rho_2 > 0$. Denote $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_{n_1}$, and $\beta = \beta_1 + \beta_2 + \dots + \beta_{n_2}$. Denote $J_1(u) = \alpha_1 u_1^2 + \dots + \alpha_{n_1} u_{n_1}^2$ and $J_2(v) = \beta_1 v_1^2 + \dots + \beta_{n_2} v_{n_2}^2$, where $u = (u_1, \dots, u_{n_1}) \in S^{n_1-1}$ and $v = (v_1, \dots, v_{n_2}) \in S^{n_2-1}$.

If f is a function defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ and $h_i \in \mathbb{R}^{n_i}$ ($i = 1, 2$), $x_i \in \mathbb{R}^{n_i}$, we define (following the notations in [5])

$$\Delta_{h_1}^1 f(x_1, x_2, x_3) = f(x_1 + h_1, x_2, x_3) - f(x_1, x_2, x_3)$$

$$\Delta_{h_2}^2 f(x_1, x_2, x_3) = f(x_1, x_2 + h_2, x_3) - f(x_1, x_2, x_3)$$

$$\Delta_{h_1, h_2}^{1,2} f(x_1, x_2, x_3) = \Delta_{h_1}^1 (\Delta_{h_2}^2 f(x_1, x_2, x_3)).$$

Given a measure $\mu \in \mathbb{R}^{n_1+n_2+n_3}$, we define the measures $\mu^{(1)} \in \mathbb{R}^{n_2+n_3}$, $\mu^{(2)} \in \mathbb{R}^{n_1+n_3}$, $\mu^{(1,2)} \in \mathbb{R}^{n_3}$ by $\mu^{(1)}(E) = \mu(\mathbb{R}^{n_1} \times E)$, $\mu^{(2)}(F) = \mu(\mathbb{R}^{n_2} \times F)$, $\mu^{(1,2)}(G) =$

$\mu(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times G)$, where E, F, G are Borel sets in $\mathbb{R}^{n_2+n_3}, \mathbb{R}^{n_1+n_3}, \mathbb{R}^{n_3}$ respectively. Finally, we write $|\sigma|$ for the total variation of the measure σ .

2.2. Preliminary Results

LEMMA 1. [10] Let $\Phi^{(i)}$ be a function in $\mathcal{S}(\mathbb{R}^{n_i})$ ($i=1, 2$) such that $\text{supp}\hat{\Phi}^{(i)} \subset \{\zeta_i \in \mathbb{R}^{n_i} : 1/2 \leq \rho_i(\zeta_i) \leq 2\}$. Define $\Psi^{(i)}$ by $\hat{\Psi}^{(i)}(\zeta_i) = \hat{\Phi}^{(i)}(\rho_i(\zeta_i))$. For $j, k \in \mathbb{Z}$, $x = (x_1, x_2, \dots, x_{n_1}) \in \mathbb{R}^{n_1}$, $y = (y_1, y_2, \dots, y_{n_2}) \in \mathbb{R}^{n_2}$, define $\Psi_j^{(1)}(x) = 2^{-j\alpha}\Psi^{(1)}(A_{2^{-j}}^{(1)}x)$ and $\Psi_k^{(2)}(y) = 2^{-k\beta}\Psi^{(2)}(A_{2^{-k}}^{(2)}y)$. Set $\Psi(x, y) = \Psi^{(1)}(x)\Psi^{(2)}(y)$ and let $\Psi_{j,k}(x, y) = \Psi_j^{(1)}(x)\Psi_k^{(2)}(y)$. Let δ be the Dirac distribution on \mathbb{R}^{n_3} . Then for $f \in L^p(\mathbb{R}^{n_1+n_2})$, $g \in L^p(\mathbb{R}^{n_1+n_2+n_3})$, we have

$$\left\| \left(\sum_{j,k} |\Psi_{j,k} * f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p,$$

and

$$\left\| \left(\sum_{j,k} |(\Psi_{j,k} \otimes \delta) * g|^2 \right)^{1/2} \right\|_p \leq C \|g\|_p \text{ for } 1 < p < \infty.$$

REMARK 1. Lemma 1 is a discrete version of Theorem 7.7 [10, p. 223] which can be extended to the product setting (see [8], [13, pp. 28–47]).

LEMMA 2. [5] Let $\sigma_{j,k}$ be Borel measures in $\mathbb{R}^{n_1+n_2+n_3}$ such that $\|\sigma_{j,k}\| \leq C$ for all $j, k \in \mathbb{Z}$. If $\sigma^*(f) = \sup_{j,k} \|\sigma_{j,k} * f\|$ is bounded in $L^q(\mathbb{R}^{n_1+n_2+n_3})$ for some $q > 1$, then the following vector value inequality holds

$$\left\| \left(\sum_{j,k} |\sigma_{j,k} * g_{j,k}|^2 \right)^{1/2} \right\|_{p_0} \leq C \left\| \left(\sum_{j,k} |g_{j,k}|^2 \right)^{1/2} \right\|_{p_0} \text{ for } \left| \frac{1}{p_0} - \frac{1}{2} \right| = \frac{1}{2q}.$$

LEMMA 3. Let $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$, ($n, m \geq 2$) satisfy

a) $\int_{S^{n-1}} \Omega(u, v) J_1(u) d\mu_1(u) = 0 \quad \forall v \in S^{m-1}$, and

b) $\int_{S^{m-1}} \Omega(u, v) J_2(v) d\mu_2(v) = 0 \quad \forall u \in S^{n-1}$, where $J_1(u) = \alpha_1 u_1^2 + \dots + \alpha_n u_n^2$

and $J_2(v) = \beta_1 v_1^2 + \dots + \beta_m v_m^2$, $u = (u_1, \dots, u_n) \in S^{n-1}$, and $v = (v_1, \dots, v_m) \in S^{m-1}$. Here μ_1, μ_2 are normalized measures on S^{n-1} and S^{m-1} respectively.

Let $E_0 = \{(u, v) \in S^{n-1} \times S^{m-1} : |\Omega(u, v)| \leq 2\}$, and for $l \in \mathbb{N}$, let $E_l = \{(u, v) \in S^{n-1} \times S^{m-1} : 2^l < |\Omega(u, v)| \leq 2^{l+1}\}$. Let $A(\Omega) = \{l \in \mathbb{N} : \mu(E_l) > 2^{-4l}\}$, where $\mu = \mu_1 \times \mu_2$ is the product measure on $S^{n-1} \times S^{m-1}$. Then Ω has a decomposition

$$\Omega = \Omega_0 + \sum_{l \in A(\Omega)} \Omega_l,$$

where Ω_o, Ω_l ($l \in A(\Omega)$) all satisfy the cancellation conditions above and

$$\|\Omega_o\|_{L^2(S^{n-1} \times S^{m-1})} \leq C, \quad \|\Omega_o\|_{L^1(S^{n-1} \times S^{m-1})} \leq C,$$

$$\|\Omega_l\|_{L^2(S^{n-1} \times S^{m-1})} \leq C 2^{2l} \|\Omega\|_{L^1(E_l)}, \quad \|\Omega_l\|_{L^1(S^{n-1} \times S^{m-1})} \leq C \|\Omega\|_{L^1(E_l)} \text{ for all } l \in A(\Omega).$$

REMARK 2. The proof of this Lemma 3 will be given in section 2.5.

THEOREM A. [5] Let $\mu_{j,k}$ be uniformly bounded positive measures in $\mathbb{R}^{n_1+n_2+n_3}$. Suppose that

$$|\hat{\mu}_{j,k}(\zeta)| \leq C |A_{2^l j}^{(1)} \zeta_1|^{-a/l} |A_{2^l k}^{(2)} \zeta_2|^{-b/l} \quad (1)$$

$$|\Delta_{\zeta_1}^1 \hat{\mu}_{j,k}(0, \zeta_2, \zeta_3)| \leq C |A_{2^{l(j+1)}}^{(1)} \zeta_1|^{a/l} |A_{2^l k}^{(2)} \zeta_2|^{-b/l} \quad (2)$$

$$|\Delta_{\zeta_2}^2 \hat{\mu}_{j,k}(\zeta_1, 0, \zeta_3)| \leq C |A_{2^l j}^{(1)} \zeta_1|^{-a/l} |A_{2^{l(k+1)}}^{(2)} \zeta_2|^{b/l} \quad (3)$$

$$|\Delta_{\zeta_1, \zeta_2}^{1,2} \hat{\mu}_{j,k}(0, 0, \zeta_3)| \leq C |A_{2^{l(j+1)}}^{(1)} \zeta_1|^{a/l} |A_{2^{l(k+1)}}^{(2)} \zeta_2|^{b/l} \quad (4)$$

for some $a, b > 0$ and some arbitrary positive integer $l \geq 1$, and for all $j, k \in \mathbb{Z}$. Suppose also that the maximal functions

$$\tilde{M}^{(i)} g_i = \sup_{j,k} |\mu_{j,k}^{(i)} * g_i| \quad (i = 1, 2), \quad \tilde{M}^{(1,2)} g = \sup_{j,k} |\mu_{j,k}^{(1,2)} * g|$$

are bounded in L^p for all $p > 1$. Then $Mf(x) = \sup_{j,k} |\mu_{j,k} * f(x)|$ is bounded in L^p for all $p > 1$, and the bound is independent of l .

THEOREM B. [5] Let $\sigma_{j,k}$ be Borel measures in $\mathbb{R}^{n_1+n_2+n_3}$ such that $\|\sigma_{j,k}\| \leq C$, and

$$|\hat{\sigma}_{j,k}(\zeta_1, \zeta_2, \zeta_3)| \leq C |A_{2^{l(j+1)}}^{(1)} \zeta_1|^{a/l} |A_{2^{l(k+1)}}^{(2)} \zeta_2|^{b/l} \quad (5)$$

$$\leq C |A_{2^{l(j+1)}}^{(1)} \zeta_1|^{a/l} |A_{2^l k}^{(2)} \zeta_2|^{-b/l} \quad (6)$$

$$\leq C |A_{2^l j}^{(1)} \zeta_1|^{-a/l} |A_{2^{l(k+1)}}^{(2)} \zeta_2|^{b/l} \quad (7)$$

$$\leq C |A_{2^l j}^{(1)} \zeta_1|^{-a/l} |A_{2^l k}^{(2)} \zeta_2|^{-b/l} \quad (8)$$

for some $a, b > 0$ and some arbitrary positive integer $l \geq 1$, and for all $j, k \in \mathbb{Z}$. If $\sigma^*(f) = \sup_{j,k} \|\sigma_{j,k} * f\|$ is bounded in $L^q(\mathbb{R}^{n_1+n_2+n_3})$ for some $q > 1$, then

$$Tf(x) = \sum_{j,k} \sigma_{j,k} * f(x) \quad \text{and} \quad g(f)(x) = \left(\sum_{j,k} |\sigma_{j,k} * f(x)|^2 \right)^{1/2} \quad \text{are bounded in}$$

$L^q(\mathbb{R}^{n_1+n_2+n_3})$ for $\left| \frac{1}{p} - \frac{1}{2} \right| = \frac{1}{2q}$ and the bound does not depend on l .

REMARK 3. Theorems A and B are the modified versions of Theorems 1 and 2 [5] respectively. Note that when $l = 1$, $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ and $\beta_1 = \beta_2 = \dots = \beta_n = 1$, Theorems A and B recover Theorems 1 and 2 [5]. The proofs of these theorems will be given in section 2.6. Also for the rest of this paper, we denote the letter C as a constant which may vary at different occurrences. However, it does not depend on any essential variable.

2.3. Main Theorem

Let $h \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$, where $\mathbb{R}^+ = [0, \infty)$. Let $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$, ($n, m \geq 2$) satisfy

$$a) \int_{S^{n-1}} \Omega(u, v) J_1(u) d\mu_1(u) = 0 \quad \forall v \in S^{m-1}, \text{ and}$$

$$b) \int_{S^{m-1}} \Omega(u, v) J_2(v) d\mu_2(v) = 0 \quad \forall u \in S^{n-1}, \text{ where } J_1(u) = \alpha_1 u_1^2 + \cdots + \alpha_n u_n^2$$

and $J_2(v) = \beta_1 v_1^2 + \cdots + \beta_m v_m^2$, $u = (u_1, \dots, u_n) \in S^{n-1}$, and $v = (v_1, \dots, v_m) \in S^{m-1}$.

For $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ ($n, m \geq 2$), define the singular integral Tf by

$$\begin{aligned} Tf(x_1, x_2) &= \text{p. v. } K * f(x_1, x_2) \\ &= \text{p. v. } \int \int_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2 \end{aligned}$$

Recall that $\alpha = \alpha_1 + \cdots + \alpha_n$ and $\beta = \beta_1 + \cdots + \beta_m$.

THEOREM 1. *Let $h \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ and let $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$, ($n, m \geq 2$) satisfy the cancellation conditions above. Then the singular integral Tf initially defined on the Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ has a bounded extension from $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$, provided that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_m$. The above results also hold when $0 < \alpha_1 = \alpha_2 = \cdots = \alpha_n$, or $0 < \beta_1 = \beta_2 = \cdots = \beta_m$.*

It should be remarked that $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$ ($n, m \geq 2$) is best possible, since it has been shown in [3] that $\forall \varepsilon > 0$, there is an $\Omega \in L(\log L^+)^{2-\varepsilon}(S^{n-1} \times S^{m-1})$ (satisfying certain cancellation conditions) such that the singular integral

$$T_\Omega f(x_1, x_2) = \text{p. v. } \int \int_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega(y'_1, y'_2) f(x_1 - y_1, x_2 - y_2)}{|y_1|^n |y_2|^m} dy_1 dy_2$$

is not bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for any $p \in (1, \infty)$.

2.4. Proof of Theorem 1

The decomposition of Ω in Lemma 3 induces the decomposition of the corresponding operators

$$T = T_o + \sum_{l \in A(\Omega)} T_l, \quad (9)$$

where $T_o f = \text{p. v. } K_o * f$ and $T_l f = \text{p. v. } K_l * f$. Here K_o and K_l are defined the same way as the kernel K with Ω being replaced by Ω_o and Ω_l respectively. We will show that for $1 < p < \infty$,

$$\|T_o f\|_p \leq C \|f\|_p \quad \text{and} \quad (10)$$

$$\|T_l f\|_p \leq C l^2 \|\Omega_l\|_{L^1(E_l)} \|f\|_p, \quad l \in A(\Omega). \quad (11)$$

Therefore we will obtain

$$\begin{aligned} \|Tf\|_p &\leq \|T_0f\|_p + \sum_{l \in A(\Omega)} \|T_l f\|_p \\ &\leq C \left(1 + \sum_{l \in A(\Omega)} l^2 \|\Omega\|_{L^1(E_l)} \right) \|f\|_p \\ &\leq C \left(1 + \|\Omega\|_{L(\log L^+)^2(S^{n-1} \times S^{m-1})} \right) \|f\|_p \text{ for } 1 < p < \infty. \end{aligned}$$

Since the proofs of inequalities (10) and (11) are basically the same, we will only prove (11). We will apply Theorems A and B (without the third variable) to prove (11). For $(x_1, x_2), (y_1, y_2), (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m$, we write

$$\begin{aligned} T_l f(x_1, x_2) &= \text{p. v.} \int_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega_l(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2 \\ &= \sum_{j,k} \int_{\rho_1(y_1) \cong 2^j} \int_{\rho_2(y_2) \cong 2^k} \frac{\Omega_l(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} f(x_1 - y_1, x_2 - y_2) dy_2 dy_1 \\ &\equiv \sum_{j,k} (\sigma_{j,k} * f)(x_1, x_2). \end{aligned} \quad (12)$$

(Here $\rho_1(y_1) \cong 2^j$ means $2^j \leq \rho_1(y_1) \leq 2^{l(j+1)}$ and similar definition for $\rho_2(y_2) \cong 2^k$).

The Fourier transform of $\sigma_{j,k}$ is

$$\hat{\sigma}_{j,k}(\xi_1, \xi_2) = \int_{\rho_1(y_1) \cong 2^j} \int_{\rho_2(y_2) \cong 2^k} \frac{e^{i(\xi_1 \cdot y_1 + \xi_2 \cdot y_2)} \Omega_l(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} dy_2 dy_1. \quad (13)$$

Let $\mu_{j,k}$ denote the total variations of $\sigma_{j,k}$ for all $j, k \in \mathbb{Z}$. In order to apply Theorems A and B, we must show that

$$\|\sigma_{j,k}\|, \|\mu_{j,k}\| \leq Cl^2 \|\Omega\|_{L^1(E_l)}, \text{ and}$$

$\hat{\sigma}_{j,k}$ (resp. $\hat{\mu}_{j,k}$) satisfy the estimates (5)–(8) (resp. (1)–(4)) for all $j, k \in \mathbb{Z}, l \in A(\Omega)$ (with the bound $Cl^2 \|\Omega\|_{L^1(E_l)}$ instead of C). Moreover, we need to show that the partial maximal functions

$$\tilde{M}^{(i)} g_i = \sup_{j,k} |\mu_{j,k}^{(i)} * g_i| \quad (i = 1, 2)$$

are bounded in L^p for all $p > 1$. By a change of variables to polar coordinates,

$$\begin{aligned} \hat{\sigma}_{j,k}(\xi_1, \xi_2) &= \int_{2^j}^{2^{l(j+1)}} \int_{2^k}^{2^{l(k+1)}} \int_{S^{n-1}} \int_{S^{m-1}} e^{i(A_{\rho_1}^{(1)} \xi_1 \cdot u + A_{\rho_2}^{(2)} \xi_2 \cdot v)} \\ &\quad \times \Omega_l(u, v) h(\rho_1, \rho_2) J_1(u) J_2(v) d\sigma(u) d\sigma(v) \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1} \end{aligned} \quad (14)$$

and note that

$$\begin{aligned} \hat{\mu}_{j,k}(\xi_1, \xi_2) &= \int_{2^l j}^{2^{l(j+1)}} \int_{2^l k}^{2^{l(k+1)}} \iint_{S^{n-1} \times S^{m-1}} e^{i(A_{\rho_1}^{(1)} \xi_1 \cdot u + A_{\rho_2}^{(2)} \xi_2 \cdot v)} \\ &\quad \times |\Omega_l(u, v) h(\rho_1, \rho_2)| J_1(u) J_2(v) d\sigma(u) d\sigma(v) \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1}. \end{aligned} \quad (15)$$

It is clear that $\|\sigma_{j,k}\| = \|\mu_{j,k}\| = \hat{\mu}_{j,k}(0, 0) \leq Cl^2 \|h\|_\infty \|\Omega\|_{L^1(E_l)}$ for all $j, k \in \mathbb{Z}$. By the cancellation conditions of Ω_l , we have

$$\begin{aligned} |\hat{\sigma}_{j,k}(\xi_1, \xi_2)| &= |\Delta_{\xi_1, \xi_2}^{1,2} \hat{\sigma}_{j,k}(0, 0)| \\ &\leq \int_{2^l j}^{2^{l(j+1)}} \int_{2^l k}^{2^{l(k+1)}} \iint_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v) h(\rho_1, \rho_2)| \\ &\quad \times |(e^{iA_{\rho_1}^{(1)} \xi_1 \cdot u} - 1)(e^{iA_{\rho_2}^{(2)} \xi_2 \cdot v} - 1)| J_1(u) J_2(v) d\sigma(u) d\sigma(v) \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1} \\ &\leq Cl^2 \|\Omega\|_{L^1(E_l)} |A_{2^l(j+1)}^{(1)} \xi_1| |A_{2^l(k+1)}^{(2)} \xi_2|. \end{aligned}$$

Also,

$$\begin{aligned} |\hat{\sigma}_{j,k}(\xi_1, \xi_2)| &= |\Delta_{\xi_1}^1 \hat{\sigma}_{j,k}(0, \xi_2)| \\ &\leq \int_{2^l j}^{2^{l(j+1)}} \int_{S^{n-1}} |I(\rho_1, u)| |e^{iA_{\rho_1}^{(1)} \xi_1 \cdot u} - 1| J_1(u) d\sigma(u) \frac{d\rho_1}{\rho_1}, \end{aligned} \quad (16)$$

where $I(\rho_1, u) = \int_{2^l k}^{2^{l(k+1)}} \int_{S^{m-1}} \Omega_l(u, v) h(\rho_1, \rho_2) e^{iA_{\rho_2}^{(2)} \xi_2 \cdot v} J_2(v) d\sigma(v) \frac{d\rho_2}{\rho_2}$. Note that

$$|I(\rho_1, u)|^2 \leq Cl \|h\|_\infty^2 \iint_{S^{m-1} \times S^{m-1}} \Omega_l(u, v) \bar{\Omega}_l(u, \tilde{v}) |K(\xi_2; v, \tilde{v})| J_2(v) J_2(\tilde{v}) d\sigma(v) d\sigma(\tilde{v}),$$

where

$$K(\xi_2; v, \tilde{v}) = \int_{2^l k}^{2^{l(k+1)}} e^{iA_{\rho_2}^{(2)} \xi_2 \cdot (v - \tilde{v})} \frac{d\rho_2}{\rho_2}. \quad (17)$$

It is clear that $|K(\xi_2; v, \tilde{v})| \leq Cl$. On the other hand,

$$\begin{aligned} |K(\xi_2; v, \tilde{v})| &\leq \sum_{j=0}^{l-1} \left| \int_{2^{l k+j}}^{2^{l k+j+1}} e^{iA_{\rho_2}^{(2)} \xi_2 \cdot (v - \tilde{v})} \frac{d\rho_2}{\rho_2} \right| \\ &= \sum_{j=0}^{l-1} \left| \int_1^2 e^{iA_{2^{l k+j}}^{(2)} \xi_2 \cdot (v - \tilde{v})} \frac{dt}{t} \right| \\ &\leq C \sum_{j=0}^{l-1} 2^{-j\beta_1/m} |A_{2^{l k}}^{(2)} \xi_2|^{-1/m} |\eta \cdot (v - \tilde{v})|^{-1/m} \\ &\leq Cl |A_{2^{l k}}^{(2)} \xi_2|^{-1/m} |\eta \cdot (v - \tilde{v})|^{-1/m} \end{aligned}$$

where $\eta = \frac{A_{2^{lk}}^{(2)} \xi_2}{|A_{2^{lk}}^{(2)} \xi_2|}$, and the next to last inequality follows from Theorem 1 [12]. Thus

$$\begin{aligned} |K(\xi_2; v, \bar{v})| &\leq Cl \min \left\{ 1, |A_{2^{lk}}^{(2)} \xi_2|^{-1/m} |\eta \cdot (v - \bar{v})|^{-1/m} \right\} \\ &\leq Cl |A_{2^{lk}}^{(2)} \xi_2|^{-1/2m} |\eta \cdot (v - \bar{v})|^{-1/2m}, \end{aligned}$$

and therefore by Hölder's inequality,

$$|I(\rho_1, u)|^2 \leq Cl^2 |A_{2^{lk}}^{(2)} \xi_2|^{-1/2m} \int_{S^{m-1}} |\Omega_l(u, v)|^2 d\sigma(v).$$

An application of Hölder's inequality to (16) produces

$$\begin{aligned} |\hat{\sigma}_{j,k}(\xi_1, \xi_2)|^2 &\leq Cl |A_{2^{l(j+1)}}^{(1)} \xi_1|^2 \int_{2^{lj}}^{2^{l(j+1)}} \int_{S^{n-1}} |I(\rho_1, u)|^2 d\sigma(u) \frac{d\rho_1}{\rho_1} \\ &\leq Cl^4 |A_{2^{l(j+1)}}^{(1)} \xi_1|^2 |A_{2^{lk}}^{(2)} \xi_2|^{-1/2m} \|\Omega_l\|_2^2. \end{aligned}$$

Thus $|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| \leq Cl^2 |A_{2^{l(j+1)}}^{(1)} \xi_1| |A_{2^{lk}}^{(2)} \xi_2|^{-1/4m} 2^{2l} \|\Omega_l\|_{L^1(E_l)}$. On the other hand, a direct integration of inequality (16) yields $|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| \leq Cl^2 |A_{2^{l(j+1)}}^{(1)} \xi_1| \|\Omega_l\|_{L^1(E_l)}$, which together with the above inequality implies that

$$|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| \leq Cl^2 \|\Omega_l\|_{L^1(E_l)} |A_{2^{l(j+1)}}^{(1)} \xi_1| |A_{2^{lk}}^{(2)} \xi_2|^{-1/4ml}.$$

By symmetry, we also obtain

$$|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| = |\Delta_{\xi_2}^2 \hat{\sigma}_{j,k}(\xi_1, 0)| \leq Cl^2 \|\Omega_l\|_{L^1(E_l)} |A_{2^{lj}}^{(1)} \xi_1|^{-1/4ml} |A_{2^{l(k+1)}}^{(2)} \xi_2|.$$

An application of Hölder's inequality to (14) yields

$$\begin{aligned} |\hat{\sigma}_{j,k}(\xi_1, \xi_2)|^2 &\leq Cl^2 \|h\|_\infty^2 \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \left| \int_{S^{n-1} \times S^{m-1}} \Omega_l(u, v) J_1(u) J_2(v) \right. \\ &\quad \left. \times e^{i(A_{\rho_1}^{(1)} \xi_1 \cdot u + A_{\rho_2}^{(2)} \xi_2 \cdot v)} d\sigma(u) d\sigma(v) \right|^2 \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1} \\ &\leq Cl^2 \int \int_{(S^{n-1} \times S^{m-1})^2} \Omega_l(u, v) \bar{\Omega}_l(\bar{u}, \bar{v}) \left| \left(\int_{2^{lj}}^{2^{l(j+1)}} e^{iA_{\rho_1}^{(1)} \xi_1 \cdot (u-\bar{u})} \frac{d\rho_1}{\rho_1} \right) \right. \\ &\quad \left. \times \left(\int_{2^{lk}}^{2^{l(k+1)}} e^{iA_{\rho_2}^{(2)} \xi_2 \cdot (v-\bar{v})} \frac{d\rho_2}{\rho_2} \right) \right| d\sigma(u) d\sigma(v) d\sigma(\bar{u}) d\sigma(\bar{v}) \\ &\leq Cl^4 |A_{2^{lj}}^{(1)} \xi_1|^{-1/2n} |A_{2^{lk}}^{(2)} \xi_2|^{-1/2m} \|\Omega_l\|_2^2. \end{aligned}$$

Thus $|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| \leq Cl^2 |A_{2^{lj}}^{(1)} \xi_1|^{-1/4n} |A_{2^{lk}}^{(2)} \xi_2|^{-1/4m} 2^{2l} \|\Omega\|_{L^1(E_l)}$. Also a direct integration of (14) leads to $|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| \leq Cl^2 \|\Omega\|_{L^1(E_l)}$, which together with the above inequality yields

$$|\hat{\sigma}_{j,k}(\xi_1, \xi_2)| \leq Cl^2 \|\Omega\|_{L^1(E_l)} |A_{2^{lj}}^{(1)} \xi_1|^{-1/4nl} |A_{2^{lk}}^{(2)} \xi_2|^{-1/4ml}.$$

Consequently, $\hat{\sigma}_{j,k}$ satisfy the estimates (5)–(8) in Theorem B for all $j, k \in \mathbb{Z}$. By inspecting equations (14) and (15), we infer that $\hat{\mu}_{j,k}$ also satisfy the estimates (1)–(4) in Theorem A for all $j, k \in \mathbb{Z}$.

It remains to prove the L^p boundedness of the partial maximal functions $\tilde{M}^{(i)} g_i = \sup_{j,k} |\mu_{j,k}^{(i)} * g_i|$ ($i = 1, 2$) for all $p > 1$. By symmetry it is enough to consider the case $i = 1$. We may assume $g_1 \geq 0$. For $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \tilde{M}^{(1)} g_1(x) &= \sup_{j,k} |\mu_{j,k}^{(1)} * g_1(x)| \\ &\leq \sup_{j,k} \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v) h(\rho_1, \rho_2)| g_1(x - A_{\rho_1}^{(1)} u) \\ &\quad \times J_1(u) J_2(v) d\sigma(u) d\sigma(v) \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1} \\ &\leq Cl \|h\|_\infty \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v)| \left\{ \sup_j \int_{2^{lj}}^{2^{l(j+1)}} g_1(x - A_{\rho_1}^{(1)} u) \frac{d\rho_1}{\rho_1} \right\} d\sigma(u) d\sigma(v). \end{aligned}$$

Note that

$$\sup_j \int_{2^{lj}}^{2^{l(j+1)}} g_1(x - A_{\rho_1}^{(1)} u) \frac{d\rho_1}{\rho_1} \leq 2 \sum_{i=0}^{l-1} \sup_j \left\{ \frac{1}{2^{lj+i+1}} \int_{2^{lj+i}}^{2^{l(j+i+1)}} g_1(x - A_t^{(1)} u) dt \right\},$$

where each summand in the sum above is bounded in $L^p(\mathbb{R}^n)$ for all $p > 1$ and the bound is independent of the direction vector $u \in S^{n-1}$ (see [7, Corollary 5.1]). It follows from Minskowski’s inequality that $\|\tilde{M}^{(1)} g_1\|_p \leq Cl^2 \|\Omega\|_{L^1(E_l)} \|g_1\|_p$ for $1 < p \leq \infty$. Consequently, we have proved that

$$\|T_l f\|_p \leq Cl^2 \|\Omega\|_{L^1(E_l)} \|f\|_p, \quad l \in A(\Omega).$$

Finally observe that the estimate of the function $K(\xi_2; v, \tilde{v})$ in equation (17) and Corollary 5.1 [7] both rely on Theorem 1 [14], which in turn require all coefficients α_i (and similarly all β_j) to be distinct. However, the estimate of the function $K(\xi_2; v, \tilde{v})$ and Corollary 5.1 [7] still hold in case $0 < \alpha_1 = \alpha_2 = \dots = \alpha_n$, or $0 < \beta_1 = \beta_2 = \dots = \beta_m$. The proof of Theorem 1 is complete.

2.5. Proof of Lemma 3

The decomposition of $\Omega \in L(\log L^+)(S^{n-1})$ can be found in [2]. We generalize this idea to the product setting. Let μ_1 and μ_2 be the normalized measures on S^{n-1}

and S^{m-1} respectively. Let $\mu = \mu_1 \times \mu_2$ be defined on $S^{n-1} \times S^{m-1}$. For $l \in A(\Omega)$, define

$$\Omega_l^{(1)}(u) = \int_{S^{m-1}} \Omega(u, v) \chi_{E_l}(u, v) J_2(v) d\mu_2(v)$$

and

$$\Omega_l^{(2)}(v) = \int_{S^{n-1}} \Omega(u, v) \chi_{E_l}(u, v) J_1(u) d\mu_1(u).$$

Now define a_l ($l \in A(\Omega)$) on $S^{n-1} \times S^{m-1}$ by

$$\begin{aligned} a_l(u, v) &= \frac{1}{\|J_1\|_\infty \|J_2\|_\infty \|\Omega\|_{L^1(E_l)}} \left\{ \|J_1\|_1 \|J_2\|_1 \Omega(u, v) \chi_{E_l}(u, v) - \|J_1\|_1 \Omega_l^{(1)}(u) \right. \\ &\quad \left. - \|J_2\|_1 \Omega_l^{(2)}(v) \right\} + \frac{1}{\|J_1\|_\infty \|J_2\|_\infty \|\Omega\|_{L^1(E_l)}} \int \int_{S^{n-1} \times S^{m-1}} \Omega(u, v) \chi_{E_l}(u, v) \\ &\quad \times J_1(u) J_2(v) d\mu_1(u) d\mu_2(v). \end{aligned}$$

Then each a_l satisfies the cancellation conditions as Ω does. Moreover, $\|a_l\|_{L^1(S^{n-1} \times S^{m-1})} \leq 4$ and $\|a_l\|_{L^2(S^{n-1} \times S^{m-1})} \leq 2^{2l+3}$.

Define $\tilde{\Omega}_o$ as

$$\begin{aligned} \tilde{\Omega}_o &= \|J_1\|_1 \|J_2\|_1 \Omega - \sum_{l \in A(\Omega)} \|J_1\|_\infty \|J_2\|_\infty \|\Omega\|_{L^1(E_l)} a_l \\ &\equiv \|J_1\|_1 \|J_2\|_1 \Omega - \sum_{l \in A(\Omega)} \tilde{\Omega}_l, \text{ where } \tilde{\Omega}_l = \|J_1\|_\infty \|J_2\|_\infty \|\Omega\|_{L^1(E_l)} a_l. \end{aligned}$$

Observe that $\tilde{\Omega}_o$ also satisfies the cancellation conditions. Moreover, by the cancellation conditions of Ω , we have

$$\begin{aligned} \tilde{\Omega}_o(u, v) &= \|J_1\|_1 \|J_2\|_1 \Omega(u, v) - \|J_1\|_1 \int_{S^{m-1}} \Omega(u, v) J_2(v) d\sigma(v) \\ &\quad - \|J_2\|_1 \int_{S^{n-1}} \Omega(u, v) J_1(u) d\sigma(u) \\ &\quad + \int \int_{S^{n-1} \times S^{m-1}} \Omega(u, v) J_1(u) J_2(v) d\sigma(u) d\sigma(v) - \sum_{l \in A(\Omega)} \tilde{\Omega}_l(u, v) \\ &= \|J_1\|_1 \|J_2\|_1 \Omega(u, v) - \sum_{l=0}^{\infty} \|J_1\|_1 \Omega_l^{(1)}(u) - \sum_{l=0}^{\infty} \|J_2\|_1 \Omega_l^{(2)}(v) \\ &\quad + \sum_{l=0}^{\infty} \int \int_{S^{n-1} \times S^{m-1}} \Omega(u, v) \chi_{E_l}(u, v) J_1(u) J_2(v) d\sigma(u) d\sigma(v) - \sum_{l \in A(\Omega)} \tilde{\Omega}_l(u, v) \\ &= \|J_1\|_1 \|J_2\|_1 \Omega(u, v) \sum_{l \notin A(\Omega)} \chi_{E_l}(u, v) - \sum_{l \notin A(\Omega)} \|J_1\|_1 \Omega_l^{(1)}(u) - \sum_{l \notin A(\Omega)} \|J_2\|_1 \Omega_l^{(2)}(v) \\ &\quad + \sum_{l \notin A(\Omega)} \int \int_{S^{n-1} \times S^{m-1}} \Omega(u, v) \chi_{E_l}(u, v) J_1(u) J_2(v) d\sigma(u) d\sigma(v). \end{aligned}$$

Now if $l = 0$, then $|\Omega(u, v)\chi_{E_0}(u, v)| \leq 2$, and $\int \int_{S^{n-1} \times S^{m-1}} |\Omega(u, v)|^2 \chi_{E_0}(u, v) d\sigma(u) d\sigma(v) \leq C$.

If $l \in \mathbb{N} \setminus A(\Omega)$, then $|\Omega(u, v)\chi_{E_l}(u, v)| \leq 2^{l+1}$, $\mu(E_l) \leq 2^{-4l}$, and thus $\int \int_{S^{n-1} \times S^{m-1}} |\Omega(u, v)|^2 \chi_{E_l}(u, v) d\sigma(u) d\sigma(v) \leq 2^{-2l+2}$. It follows that $\|\tilde{\Omega}_o\|_{L^2(S^{n-1} \times S^{m-1})} \leq C$, which implies $\|\tilde{\Omega}_o\|_{L^1(S^{n-1} \times S^{m-1})} \leq C$.

Now let $\Omega_o = \|J_1\|_1^{-1} \|J_2\|_1^{-1} \tilde{\Omega}_o$ and $\Omega_l = \|J_1\|_1^{-1} \|J_2\|_1^{-1} \tilde{\Omega}_l$. We then have the decomposition

$$\Omega = \Omega_o + \sum_{l \in A(\Omega)} \Omega_l,$$

where Ω_o and Ω_l ($l \in A(\Omega)$) satisfy the desired conclusions stated in the lemma. Lemma 3 is proved.

2.6. Proofs of Theorems A and B

The proofs of these theorems are essentially similar to the proofs of Theorems 1 and 2 [5] respectively, except for some modifications. However for clarity, we will prove Theorem B in detail, and omit the proof of Theorem A since the proof of this Theorem A (unlike the proof of Theorem B) only requires some minor adjustments (see Remark 4 at the end of this section). We now proceed to prove Theorem B.

Let $\phi^{(i)}$ be a function in $\mathcal{S}(\mathbb{R}^{n_i})$ ($i = 1, 2$) such that $\text{supp} \hat{\phi}^{(i)} \subset \{\zeta_i \in \mathbb{R}^{n_i} : 1/2 \leq \rho_i(\zeta_i) \leq 2\}$, $0 \leq \hat{\phi}^{(i)} \leq 1$. Moreover, we require $\sum_{j=-\infty}^{\infty} \left\{ \hat{\phi}^{(i)}(2^{lj} \rho_i(\zeta_i)) \right\}^2 = 1$ for all $\zeta_i \neq 0$. For $i = 1, 2$ and $j \in \mathbb{Z}$, define $\psi^{(i)}$ and $\psi_j^{(i)}$ by $\hat{\psi}^{(i)}(\zeta_i) = \hat{\phi}^{(i)}(\rho_i(\zeta_i))$ and $\hat{\psi}_j^{(i)}(\zeta_i) = \hat{\phi}^{(i)}(2^{lj} \rho_i(\zeta_i))$ respectively. Now set $\psi(x_1, x_2) = \psi^{(1)}(x_1) \psi^{(2)}(x_2)$ ($x_i \in \mathbb{R}^{n_i}, i = 1, 2$) and let $\psi_{j,k}(x_1, x_2) = \psi_j^{(1)}(x_1) \psi_k^{(2)}(x_2)$ for $j, k \in \mathbb{Z}$. Note that $\hat{\psi}_{j,k}(\zeta_1, \zeta_2) = \hat{\psi}_j^{(1)}(\zeta_1) \hat{\psi}_k^{(2)}(\zeta_2)$ and $\sum_{j,k} \hat{\psi}_{j,k}^2(\zeta_1, \zeta_2) = 1$ for all $(\zeta_1, \zeta_2) \neq \mathbf{0} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We write

$$\begin{aligned} Tf &= \sum_{j,k} \sigma_{j,k} * f = \sum_{j,k} \sigma_{j,k} * \left(\sum_{m,n} (\psi_{j+m,k+n} \otimes \delta) * (\psi_{j+m,k+n} \otimes \delta) * f \right) \\ &= \sum_{m,n} \sum_{j,k} (\psi_{j+m,k+n} \otimes \delta) * (\sigma_{j,k} * (\psi_{j+m,k+n} \otimes \delta) * f) \equiv \sum_{m,n} T_{m,n} f. \end{aligned}$$

Then $\|Tf\|_p \leq \sum_{m,n} \|T_{m,n} f\|_p$. We estimate each of the terms of this sum by interpolating between the L^2 norm and the L^{p_o} norm (where p_o is as in Lemma 2). Observe that

$$\begin{aligned} \|T_{m,n}f\|_{p_o} &\leq C \left\| \left(\sum_{j,k} |\sigma_{j,k} * (\psi_{j+m,k+n} \otimes \delta) * f|^2 \right)^{1/2} \right\|_{p_o} \\ &\leq C \left\| \left(\sum_{j,k} |(\psi_{j+m,k+n} \otimes \delta) * f|^2 \right)^{1/2} \right\|_{p_o} \leq C \|f\|_{p_o}, \end{aligned}$$

where the first and the last inequalities follow from Lemma 1, and the second inequality follows from Lemma 2. We now calculate the L^2 estimates of $T_{m,n}f$. Notice that

$$\|T_{m,n}f\|_2^2 \leq \sum_{j,k} \int_{\mathbb{R}^{n_3}} \int_{R_{j,k}^{m,n}} |\hat{\sigma}_{j,k}(\zeta_1, \zeta_2, \zeta_3) \hat{f}(\zeta_1, \zeta_2, \zeta_3)|^2 d\zeta_1 d\zeta_2 d\zeta_3,$$

where

$$R_{j,k}^{m,n} = \left\{ (\zeta_1, \zeta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : 1/2 \leq 2^{l(j+m)} \rho_1(\zeta_1) \leq 2, 1/2 \leq 2^{l(k+n)} \rho_2(\zeta_2) \leq 2 \right\}.$$

By a change of the first two variables to polar coordinates, we obtain

$$\begin{aligned} \|T_{m,n}f\|_2^2 &\leq \sum_{j,k} \int_{\mathbb{R}^{n_3}} \int_{S^{n_1-1} \times S^{n_2-1}} \int_{D_{j,k}^{m,n}} |\hat{\sigma}_{j,k}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3) \hat{f}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3)|^2 \\ &\quad \times \rho_1^{\alpha-1} \rho_2^{\beta-1} J_1(\zeta'_1) J_2(\zeta'_2) d\rho_1 d\rho_2 d\sigma(\zeta'_1) d\sigma(\zeta'_2) d\zeta_3, \end{aligned}$$

where $D_{j,k}^{m,n} = \left\{ (\rho_1, \rho_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : 1/2 \leq 2^{l(j+m)} \rho_1 \leq 2, 1/2 \leq 2^{l(k+n)} \rho_2 \leq 2 \right\}$. For $(\rho_1, \rho_2) \in D_{j,k}^{m,n}$, we consider four cases: $m, n > 0$, $m > 0$ and $n \leq 0$, $m \leq 0$ and $n > 0$, and finally $m, n \leq 0$.

Case 1. $m, n > 0$. By inequality (5) we have

$$\begin{aligned} |\hat{\sigma}_{j,k}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3)| &\leq C |A_{2^{l(j+1)}}^{(1)} A_{\rho_1}^{(1)} \zeta'_1|^{a/l} |A_{2^{l(k+1)}}^{(2)} A_{\rho_2}^{(2)} \zeta'_2|^{b/l} \\ &\leq C 2^{-(m-1)\alpha_1} 2^{-(n-1)\beta_1} b. \end{aligned}$$

Thus

$$\begin{aligned} \|T_{m,n}f\|_2^2 &\leq C 2^{-2(m-1)\alpha_1} 2^{-2(n-1)\beta_1} b \sum_{j,k} \int_{\mathbb{R}^{n_3}} \int_{S^{n_1-1} \times S^{n_2-1}} \int_{D_{j,k}^{m,n}} |\hat{f}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3)|^2 \\ &\quad \times \rho_1^{\alpha-1} \rho_2^{\beta-1} J_1(\zeta'_1) J_2(\zeta'_2) d\rho_1 d\rho_2 d\sigma(\zeta'_1) d\sigma(\zeta'_2) d\zeta_3, \end{aligned}$$

and hence $\|T_{m,n}f\|_2 \leq C 2^{-(m-1)\alpha_1} 2^{-(n-1)\beta_1} b \|f\|_2$, $m, n > 0$.

Case 2. $m > 0$ and $n \leq 0$. It follows from (6) that

$$|\hat{\sigma}_{j,k}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3)| \leq C |A_{2^{l(j+1)}}^{(1)} A_{\rho_1}^{(1)} \zeta'_1|^{a/l} |A_{2^{lk}}^{(2)} A_{\rho_2}^{(2)} \zeta'_2|^{-b/l} \leq C 2^{-(m-1)\alpha_1} 2^{n\beta_1} b,$$

which leads to the inequality $\|T_{m,n}f\|_2 \leq C 2^{-(m-1)\alpha_1 a} 2^{n\beta_1 b} \|f\|_2$, $m > 0$ and $n \leq 0$.

Case 3. $m \leq 0$ and $n > 0$. By using inequality (7) we obtain

$$|\hat{\sigma}_{j,k}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3)| \leq C |A_{2^l j}^{(1)} A_{\rho_1}^{(1)} \zeta'_1|^{-a/l} |A_{2^{l(k+1)}}^{(2)} A_{\rho_2}^{(2)} \zeta'_2|^{b/l} \leq C 2^{m\alpha_1 a} 2^{-(n-1)\beta_1 b},$$

so that $\|T_{m,n}f\|_2 \leq C 2^{m\alpha_1 a} 2^{-(n-1)\beta_1 b} \|f\|_2$, $m \leq 0$ and $n > 0$.

Case 4. $m, n \leq 0$. From (8) we have

$$|\hat{\sigma}_{j,k}(A_{\rho_1}^{(1)} \zeta'_1, A_{\rho_2}^{(2)} \zeta'_2, \zeta_3)| \leq C |A_{2^l j}^{(1)} A_{\rho_1}^{(1)} \zeta'_1|^{-a/l} |A_{2^{lk}}^{(2)} A_{\rho_2}^{(2)} \zeta'_2|^{-b/l} \leq C 2^{m\alpha_1 a} 2^{n\beta_1 b},$$

and thus $\|T_{m,n}f\|_2 \leq C 2^{m\alpha_1 a} 2^{n\beta_1 b} \|f\|_2$, $m, n \leq 0$.

Consequently, $\|T_{m,n}f\|_2 \leq C 2^{-\varepsilon_1 |m|} 2^{-\varepsilon_2 |n|} \|f\|_2$ for some $\varepsilon_1, \varepsilon_2 > 0$ and for all $m, n \in \mathbb{Z}$.

Now if $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2q}$, then by interpolation we have $\frac{1}{p} = \frac{\theta}{p} + \frac{1-\theta}{p_0}$ for some $0 < \theta \leq 1$. Therefore, $\|Tf\|_p \leq \sum_{m,n} \|T_{m,n}f\|_p \leq C \sum_{m,n} 2^{-\theta\varepsilon_1 |m|} 2^{-\theta\varepsilon_2 |n|} \|f\|_p \leq C \|f\|_p$.

The proof of the L^p bounds of $g(f)$ is similar. Theorem B is proved.

REMARK 4. To prove Theorem A, we choose positive Schwartz functions $\phi^{(i)}$ in $\mathcal{S}(\mathbb{R}^{n_i})$ ($i = 1, 2$) such that $\hat{\phi}^{(1)}(0) = 1 = \hat{\phi}^{(2)}(0)$, and define $\hat{\phi}_j^{(1)}(\zeta_1) = \hat{\phi}^{(1)}(A_{2^{l(j+1)}}^{(1)} \zeta_1)$ and $\hat{\phi}_k^{(2)}(\zeta_2) = \hat{\phi}^{(2)}(A_{2^{l(k+1)}}^{(2)} \zeta_2)$ for all $j, k \in \mathbb{Z}$. Observe that for each $f_i \in L^p(\mathbb{R}^{n_i})$ ($i = 1, 2$), $\sup_{j \in \mathbb{Z}} |\phi_j^{(i)} * f_i(x_i)| \leq CM_1 \circ \dots \circ M_{n_i} f_i(x_i)$, where each M_j ($j = 1, \dots, n_i$) is the Hardy-Littlewood maximal operator acting on the j -th component of the variable x_i . The rest of the proof of this theorem is identical to the proof of Theorem 1 [5].

3. Singular integrals along surfaces

3.1. Notations, Definitions and Background

Let \mathbb{R}^+ stand for $[0, \infty)$. A function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is called a type I function if ϕ is strictly increasing on $[0, \infty)$ and ϕ' is increasing on $(0, \infty)$.

A function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ or $\phi : (0, \infty) \rightarrow \mathbb{R}$ is a type II function if ϕ is strictly decreasing on its domain and ϕ' is increasing on $(0, \infty)$.

A function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a type III function if

- i) $\phi(0) = 0$ and ϕ is strictly increasing on $[0, \infty)$,
- ii) ϕ' is decreasing on $(0, \infty)$ and
- iii) $t\phi'(t) \geq c\phi(t)$ for all $t \in (0, \infty)$ and for some fixed $c > 0$.

For $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ ($n, m \geq 2$), consider the following singular integral along surface

$$\begin{aligned} \mathfrak{S}f(x_1, x_2, x_3) = & \text{p. v.} \int_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} \\ & \times f(x_1 - y_1, x_2 - y_2, x_3 - \Gamma(\rho_1(y_1), \rho_2(y_2))) dy_1 dy_2, \end{aligned}$$

where $x_1, y_1 \in \mathbb{R}^n$, $x_2, y_2 \in \mathbb{R}^m$, $x_3 \in \mathbb{R}$, $y'_i = y_i/|y_i|$ ($i = 1, 2$), and h and Γ are measurable real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$.

Denote $\gamma_t(s) = \Gamma(s, t)$ for every fixed $t \geq 0$. Similarly, denote $\gamma_s(t) = \Gamma(s, t)$ for every fixed $s \geq 0$.

THEOREM 2. [11] *Suppose $\Gamma(s, t)$ has continuous first order partial derivatives for all $s, t > 0$. If $\gamma_t(s)$ and $\gamma_s(t)$ are either type I, type II or type III functions (with the constant c in the definition of type III function independent of both variables s and t) for each fixed $t > 0$ and for each fixed $s > 0$ respectively, then the maximal functions*

$$M_{1,2}g(x_3) = \sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} |g(x_3 - \Gamma(s, t))| ds dt \right\},$$

$$M_1 g_1(x_1, x_3) = \sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} |g_1(x_1 - s, x_3 - \Gamma(s, t))| ds dt \right\}$$

and

$$M_2 g_2(x_2, x_3) = \sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} |g_2(x_2 - t, x_3 - \Gamma(s, t))| ds dt \right\}$$

($x_1, x_2, x_3 \in \mathbb{R}$, $g \in L^p(\mathbb{R})$, and $g_1, g_2 \in L^p(\mathbb{R}^2)$) are bounded in L^p for all $p > 1$.

3.2. Theorem 2

Let Ω and h be given as in Theorem 1. Suppose $\Gamma(s, t)$ has continuous first order partial derivatives for all $s, t > 0$. If $\gamma_t(s)$ and $\gamma_s(t)$ are either type I, type II or type III functions (with the constant c in the definition of type III function independent of both variables s and t) for each fixed $t > 0$ and for each fixed $s > 0$ respectively, then the singular integral along surface $\mathfrak{S}f$ initially defined on the Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ has a bounded extension from $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for $1 < p < \infty$.

EXAMPLES.

1) Consider the surface $\Gamma(s, t) = s^a t^b$, $a, b \neq 0$ and $s, t \geq 0$ ($s > 0$ if $a < 0$ and similarly $t > 0$ if $b < 0$). For each fixed $t > 0$, the function $\gamma_t(s) = \Gamma(s, t)$ is a type I function if $a \geq 1$, a type III function if $0 < a < 1$ and a type II function if $a < 0$. Similar conclusion holds for the function $\gamma_s(t) = \Gamma(s, t)$. By Theorem 2, the singular integral along surface $\mathfrak{S}f$ is bounded in $L^p(\mathbb{R}^{n+m+1})$ for $1 < p < \infty$.

2) Let $\Gamma(s, t) = s^{a_1} e^{a_2 s} t^{b_1} e^{b_2 t}$, $a_1, b_1 \geq 1$ and $a_2, b_2 \geq 0$. Then $\gamma_t(s) = \Gamma(s, t)$ is a type I function for each fixed $t > 0$. Similarly, $\gamma_s(t) = \Gamma(s, t)$ is also a type I function for each $s > 0$. Thus, $\mathfrak{S}f$ is bounded in $L^p(\mathbb{R}^{n+m+1})$ for $1 < p < \infty$.

3) Consider the surface with a contact of infinite order at the origin, $\Gamma(s, t) = s^2 t^2 (e^{-1/s} + e^{-1/t})$ ($s, t > 0$). The functions $\gamma_t(s) = \Gamma(s, t)$ and $\gamma_s(t) = \Gamma(s, t)$ are type I functions for each fixed $s > 0$ and each fixed $t > 0$ respectively. Hence, $\mathfrak{S}f$ is bounded in $L^p(\mathbb{R}^{n+m+1})$ for $1 < p < \infty$.

3.3. Proof of Theorem 2

The proof of this theorem is essentially similar to the proof of Theorem 1. It is enough to prove the L^p boundedness of the operator \mathfrak{S}_l , where $\mathfrak{S}_l f$ is obtained from $\mathfrak{S}f$ by replacing the kernel Ω by Ω_l . As in the proof of Theorem 1, we write $\mathfrak{S}_l f = \sum_{j,k} \sigma_{j,k} * f$, where

$$\begin{aligned} & \hat{\sigma}_{j,k}(\xi_1, \xi_2, \xi_3) \\ &= \int_{\rho_1(y_1) \cong 2^{lj}} \int_{\rho_2(y_2) \cong 2^{lk}} \frac{e^{i(\xi_1 \cdot y_1 + \xi_2 \cdot y_2)} \Omega_l(y'_1, y'_2) h(\rho_1(y_1), \rho_2(y_2)) e^{i\xi_3 \Gamma(\rho_1(y_1), \rho_2(y_2))}}{\rho_1^\alpha(y_1) \rho_2^\beta(y_2)} dy_2 dy_1. \end{aligned}$$

Let $\mu_{j,k}$ denote the total variations of $\sigma_{j,k}$ for all $j, k \in \mathbb{Z}$. In view of Theorems A and B, we need to show that

$$\|\sigma_{j,k}\|, \|\mu_{j,k}\| \leq C l^2 \|\Omega\|_{L^1(E_l)}, \text{ and}$$

$\hat{\sigma}_{j,k}$ (resp. $\hat{\mu}_{j,k}$) satisfy the estimates (5)–(8) (resp. (1)–(4)) for all $j, k \in \mathbb{Z}$, $l \in A(\Omega)$ (with the bound $C l^2 \|\Omega\|_{L^1(E_l)}$). Moreover, we need to prove that the partial maximal functions $\tilde{M}^{(1)} g_1$, $\tilde{M}^{(2)} g_2$, and $\tilde{M}^{(1,2)} g$ are bounded in L^p for all $p > 1$.

It is clear that the measures $\sigma_{j,k}$ and $\mu_{j,k}$ are uniformly bounded by $C l^2 \|\Omega\|_{L^1(E_l)}$. The proof for the estimates of $\hat{\sigma}_{j,k}$ and $\hat{\mu}_{j,k}$ is the same as those in the proof of Theorem 1 (simply replace $h(\rho_1(y_1), \rho_2(y_2))$ in the proof of Theorem 1 by $e^{i\xi_3 \Gamma(\rho_1(y_1), \rho_2(y_2))} h(\rho_1(y_1), \rho_2(y_2))$).

Now observe that for $g \in L^p(\mathbb{R})$, $g \geq 0$, we have

$$\begin{aligned} \tilde{M}^{(1,2)} g(x_3) &= \sup_{j,k} |\mu_{j,k}^{(1,2)} * g(x_3)| \\ &\leq \sup_{j,k} \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v) h(\rho_1, \rho_2) g(x_3 - \Gamma(\rho_1, \rho_2))| \\ &\quad \times J_1(u) J_2(v) d\sigma(u) d\sigma(v) \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1} \\ &\leq C l^2 \|h\|_\infty \int \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v)| \\ &\quad \times \sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} g(x_3 - \Gamma(s, t)) ds dt \right\} d\sigma(u) d\sigma(v). \\ &\leq C l^2 \|h\|_\infty \int \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v)| M^H g(x_3) d\sigma(u) d\sigma(v), \end{aligned}$$

where the last inequality follows from Theorem 2 [11] and $M^H g$ denotes the Hardy-Littlewood maximal function. Therefore, $\|\tilde{M}^{(1,2)} g\|_p \leq C l^2 \|\Omega\|_{L^1(E_l)} \|M^H g\|_p \leq C l^2 \|\Omega\|_{L^1(E_l)} \|g\|_p$ for $1 < p \leq \infty$.

For $g_1 \in L^p(\mathbb{R}^n \times \mathbb{R})$, $g_1 \geq 0$, we have

$$\begin{aligned} \tilde{M}^{(1)} g_1(x_1, x_3) &= \sup_{j,k} |\mu_{j,k}^{(1)} * g_1(x_1, x_3)| \\ &\leq \sup_{j,k} \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v) h(\rho_1, \rho_2)| \\ &\quad \times g_1(x_1 - A_{\rho_1}^{(1)} u, x_3 - \Gamma(\rho_1, \rho_2)) |J_1(u) J_2(v) d\sigma(u) d\sigma(v)| \frac{d\rho_2}{\rho_2} \frac{d\rho_1}{\rho_1} \\ &\leq C l^2 \|h\|_\infty \int_{S^{n-1} \times S^{m-1}} |\Omega_l(u, v)| \\ &\quad \times \sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_1} \int_0^{r_2} g_1(x_1 - A_s^{(1)} u, x_3 - \gamma_s(t)) dt ds \right\} d\sigma(u) d\sigma(v). \end{aligned}$$

Note that $\frac{1}{r_2} \int_0^{r_2} g_1(x_1 - A_s^{(1)} u, x_3 - \gamma_s(t)) dt \leq M_3^H g_1(x_1 - A_s^{(1)} u, x_3)$ for all $r_2 > 0$ and for each fixed $s > 0$ (please see the proof of Theorem 2 [11]). Here $M_3^H g_1$ is the Hardy-Littlewood maximal function acting on the variable x_3 . Thus

$$\begin{aligned} &\sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_1} \int_0^{r_2} g_1(x_1 - A_s^{(1)} u, x_3 - \gamma_s(t)) dt ds \right\} \\ &\leq \sup_{r_1 > 0} \left\{ \frac{1}{r_1} \int_0^{r_1} M_3^H g_1(x_1 - A_s^{(1)} u, x_3) ds \right\}, \end{aligned}$$

which is bounded in L^p for all $p > 1$ (see [7, Corollary 5.1]) and the bound is independent of $u \in S^{n-1}$. Consequently,

$$\|\tilde{M}^{(1)} g_1\|_{L^p(\mathbb{R}^{n+1})} \leq C l^2 \|\Omega\|_{L^1(E_l)} \|g_1\|_{L^p(\mathbb{R}^{n+1})} \text{ for } 1 < p \leq \infty.$$

By symmetry we also obtain

$$\|\tilde{M}^{(2)} g_2\|_{L^p(\mathbb{R}^{m+1})} \leq C l^2 \|\Omega\|_{L^1(E_l)} \|g_2\|_{L^p(\mathbb{R}^{m+1})} \text{ for } 1 < p \leq \infty.$$

The proof of Theorem 2 is complete.

REFERENCES

- [1] AHMAD AL-SALMAN, *Maximal functions along surfaces on product domains*, Anal. Math., **34**, 3 (2008), 163–175.
- [2] AHMAD AL-SALMAN AND YIBIAO PAN, *Singular integrals with rough kernels in $LlogL(S^{n-1})$* , J. London Math. Soc. (2), **66** (2002), 153–174.
- [3] AHMAD AL-SALMAN, H. AL-QASSEM AND YIBIAO PAN, *Singular integrals on product domains*, Indiana Univ. Math. J., **55**, 1 (2006), 369–387.
- [4] YANPING CHEN, YONG DING AND DASHAN FAN, *A parabolic singular integral operator with rough kernel*, submitted.
- [5] J. DUOANDIKOETXEA, *Multiple singular integrals and maximal functions along hypersurfaces*, Annales de l'institut Fourier, **36**, 4 (1986), 185–206.

- [6] DASHAN FAN AND YIBIAO PAN, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math., **119**, 4 (1997), 799–839.
- [7] JAVIER DUOANDIKOETXEA, JOSÉ L. RUBIO DE FRANCIA, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., **84**, 3 (1986), 541–561.
- [8] ROBERT FEFFERMAN, *Singular integrals on product spaces*, Advances in Mathematics, **45** (1982), 117–143.
- [9] E. B. FABES AND N. M. RIVIÈRE, *Singular integrals with mixed homogeneity*, Studia Mathematica, **27** (1966), 19–38.
- [10] G. B. FOLLAND AND E. M. STEIN, *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton University Press and University of Tokyo Press, 1982.
- [11] H. V. LE, *Maximal function along surfaces in product spaces*, J. Math. Anal. Appl., **316** (2006), 422–432.
- [12] A. NAGEL, N. M. RIVIÈRE AND S. WAINGER, *On Hilbert transforms along curves*, II, Amer. J. Math., **98** (1976), 395–403.
- [13] ELIAS M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.
- [14] E. M. STEIN, S. WAINGER, *Problems in Harmonic Analysis related to curvature*, Bull. Amer. Math. Soc., **84** (1978), 1239–1295.

(Received May 15, 2009)

Larry Chen
 Department of Mathematics
 Oregon State University
 Corvallis, OR 97330
 USA

e-mail: chen@science.oregonstate.edu

Hung Viet Le
 Faculty of Science and Technology
 Hoa Sen University
 Quang Trung Software Park, Section 10
 Ward Tan Chanh Hiep, District 12
 Ho Chi Minh City
 Viet Nam

e-mail: lvhung@hoasen.edu.vn

or

e-mail: hv1e2008@yahoo.com