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An almost torus manifold M is a closed $(2n+1)$-dimensional orientable Riemannian manifold with an effective, isometric n-torus action such that the fixed point set M^T is non-empty. Almost torus manifolds are analogues of torus manifolds in odd dimension and share many of the characteristics of torus manifolds. For example, both almost torus manifolds and torus manifolds are S^1-fixed point homogenous. Just as torus manifolds are important examples of manifolds admitting so called isotropy-maximal actions, almost torus manifolds are important if one hopes to understand manifolds admitting almost isotropy-maximal actions. Recently Wiemeler classified simply connected torus manifolds with non-negative sectional curvature. To obtain this result, he proved a structure theorem for the quotient spaces of torus manifolds of non-negative curvature. In particular, he showed that non-negatively curved torus manifolds are locally standard, with orbit spaces homeomorphic to convex polytopes with acyclic lower dimensional faces. In this thesis, we obtain an analogous structure theorem for the orbit spaces of almost torus manifolds. Namely, we analyze the orbit spaces M/T of simply connected almost torus manifolds with non-negative sectional curvature. The main result we obtain is that the action of T is locally standard and although M/T is homeomorphic to a disk and shares the combinatorial structure of a polytope, lower dimensional faces are in general not acyclic.

Unlike locally standard torus manifolds, orbit spaces of locally standard almost torus manifolds need not be manifolds with corners. Nevertheless, the analysis of the orbit space structure of M/T shows that almost torus manifolds are similar enough to torus manifolds, so that the result of this thesis can then be combined with results about extending isometric almost isotropy maximal T^k actions to smooth T^{k+1} actions to obtain a classification of non-negatively curved, simply connected almost torus manifolds.

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Zheting Dong, Author

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Academic

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ON THE STRUCTURE OF THE ORBIT SPACES OF ALMOST TORUS MANIFOLDS WITH NON-NEGATIVE CURVATURE

1. INTRODUCTION

Since perhaps the Gauss-Bonnet theorem, mathematicians have been looking for a classification of compact smooth manifolds with positive or non-negative sectional curvatures. In this general context, Karsten Grove initiated a program with the following guiding principle:

Classify or determine the structure of positively or non-negatively curved manifolds with large isometry groups.

A "large" isometry group can have many different meanings. One can simply assume that the dimension of the isometry group G, which is a compact Lie group, is large relative to the dimension of the manifold M on which G acts. Or instead one can look at the dimension of a maximal torus T in G, that is, the maximal symmetry rank of G. In a perhaps more subtle way, the so called *fixed point cohomogneity* is defined as

$$\dim(M/G) - \dim(M^G),$$

where M^G is the set of fixed-points. A low fixed point cohomogeneity action reflects the property that the G action on the normal space of M^G is "large" , in the sense that the quotient M/G has small dimension. For example, if the fixed point cohomogeneity is one (that is, the action is fixed point homogenous), then G acts homogeneously on the normal space of M^G.

Historically, Grove's symmetry program has been motivated by a beautiful theorem proved by Hsiang-Kleiner in 1989. The theorem states that if M is a compact, simply connected, positively curved 4-manifold which admits an effective isometric action of a circle S^1, then M is homeomorphic to either S^4 or the complex projective space $\mathbb{C}P^2$. A classification of compact, simply connected non-negatively curved 4-manifolds was given by Kleiner in his thesis [31].

For a more general setting one assumes that the manifold M admits a smooth action by a 'large' torus T. For manifolds with positive sectional curvature, the Maximal Symmetry Rank Theorem proved by Grove-Searle [21] can be applied. The theorem states that if T acts effectively, then $\dim(T)$ is less than or equal to $\lfloor(\dim(M)+1)/2\rfloor$, with equality holding if and only if M is a sphere, or the quotient of sphere by a linear action of a compact Lie group. The key points are that if $\dim(T) > \lfloor(\dim(M)+1)/2\rfloor$ then T acts ineffectively, and if the equality holds, then there is a circle subgroup $S^1 \subset T$ such that its fixed point set $M^{S^1} \subset M$ has a totally geodesic component F of codimension 2 in M. Thus F/S^1 is embedded as the boundary of M/S^1, which is an Alexandrov space with the same lower curvature bound as M. A common practice now is to look at the set C at maximal distance from F/S^1 in M/S^1. In the situation when M has positive curvature, C is a single point in M/S^1.

We will call a G-manifold M *fixed point homogeneous* if $\dim(M/G) - \dim(M^G) = 1$. In the non-negative sectional curvature setting, the component in M^G of maximal dimension is in general only a component of the boundary of M/G. Moreover, the set C at maximal distance from F/G in M/G does not need to be a point, like in the case of positive curvature. Recently Spindeler [42] proved that, in this parallel (but more difficult) case, if M is G-fixed point homogeneous with non-negative sectional curvature, then M can in fact be obtained by gluing two disk bundles along their common boundary. In particular, we can write $M = D(F) \cup_\partial D(N)$, where F is a totally geodesic component of M^G of codimension 2 in M, and N is a closed submanifold of M.

Suppose now that M is a compact Riemannian manifold on which a torus T acts smoothly and effectively. Let T_x be the isotropy subgroup of an arbitrary point x in M. Then the inequality $\dim(T_x) + \dim(T) \leq \dim(M)$ must be satisfied, otherwise the T_x-slice representation will not be faithful. If there is a point x in M such that $\dim(T_x) = \dim(M) - \dim(T)$, that is, the isotropy subgroup is of maximal dimension, the action is called *isotropy maximal*. If we only have $\dim(T_x) = \dim(M) - \dim(T) - 1$, the action is then called *almost isotropy maximal*. In both cases one can easily check that T_x acts orthogonally on the normal sphere S of the orbit $T(x)$ of x by maximal symmetry rank. Thus there must be a circle $S^1 \subset T_x$ fixing the normal space of $T(x)$ with a codimension two fixed point component. It turns out that M is an S^1-fixed point homogeneous manifold.

If the manifold M has a metric of non-negative sectional curvature that is invariant under the T action, then Spindeler's result can be applied to conclude that M is the union of two disk bundles. Using this fact, Wiemeler classified simply connected, non-negatively curved *torus manifolds* up to equivariant diffeomorphism in [46]. Here a torus manifold, as a special example of an S^1-fixed point homogenous manifold, is defined as a $2n$-dimensional orientable manifold with an effective action of an n-dimensional torus such that $M^T \neq \emptyset$.

Recently Escher and Searle [12] generalized Wiemeler's result to simply connected non-negatively curved manifolds admitting an isotropy maximal torus action. They proved that Wiemeler's result holds as well in this case. They also showed that if a T^k action on M^n is almost isotropy-maximal with $k \geq \lfloor 2n/3 \rfloor$, then the action is in fact isotropy-maximal. This is consistent with a well-known conjecture on the maximal symmetry rank ([16], [17]) of non-negatively curved manifolds, which states that $\lfloor 2n/3 \rfloor$ should be the upper bound of the dimension of an effective, isometric torus action on a given M^n of non-negative sectional curvature.

1.1. Statement of the main result

One hopes to further generalize the previous results. We define an *almost torus manifold* to be a $(2n+1)$-dimensional orientable manifold with an effective action of an n-dimensional torus such that $M^T \neq \emptyset$. It is easy to see that the torus action on M is almost isotropy maximal. Our main result is Theorem A which gives an explicit description of the structure of the orbit spaces of almost torus manifolds. As compact manifolds with non-negative sectional curvature, orbit spaces of almost torus manifolds are similar, but more complicated, than orbit spaces of torus manifolds, which are described in Lemma 6.3 of [46]. And similar to the fundamental role that Lemma 6.3 plays in the classification of torus manifolds, Theorem A also plays a fundamental role in the classification of almost torus manifolds, as presented in [10].

Theorem A. *Let M^{2n+1} be a closed, simply connected, non-negatively curved almost torus manifold. Then the following hold.*

1. *The torus action on M is locally standard.*
2. *M/T is diffeomorphic (after smoothing the corners) to a standard disk D^{n+1}.*
3. *Any codimension one face of M/T is diffeomorphic (after smoothing the corners) to either a standard disk D^n, or $S^1 \times D^{n-1}$.*

In [46] it is shown that two nice manifolds with corners with contractible faces and isomorphic face posets are homeomorphic. This fact is then used to prove that a simply connected torus manifold M with vanishing odd degree integer cohomology is determined up to homeomorphism by the combinatorial face poset $\mathcal{P}(M/T)$ together with the characteristic map λ (Theorem 3.4 in [46]). In our case, Theorem A shows that the orbit space of a simply connected almost torus manifold will not be a nice manifold with corners. Hence we cannot apply the results of [46].

However, by using the disk bundle decomposition (Theorem 2-60) of an almost torus manifold $M^{2n+1} = D(F) \cup_\partial D(N)$, it is possible to show that M^{2n+1} can be equipped with a unique smooth S^1 action (not necessarily isometric) that commutes with the original T^n action. Thus the quotient of M/T with the extra S^1 action is a nice manifold with corners whose faces are all contractible [10]. Therefore the classification of M^{2n+1} can be obtained using similar techniques as in the classification of manifolds with isotropy-maximal action presented in [12].

1.2. Organization of the Thesis

The thesis is organized as follows. Chapter 2 is a preliminary section in which we review necessary definitions and theorems that will be used in later chapters. Contents in sections 2.1 to 2.3 are mainly gathered from [1] and [30], as we review some basic materials related to transformation groups. Sections 2.4 to 2.7 focus on torus actions, and include important definitions and results that are essential to this thesis. These sections on torus actions are throughly discussed and summarized, so that readers will have a concrete understanding of the proof of the main result. In particular, we summarize results of Spindeler on non-negatively curved fixed-point homogeneous manifolds and Wiemeler's classification of non-negatively curved torus manifolds. In sections 2.8 to 2.10 we talk about non-negative curvature and Alexandrov spaces as well as rationally-elliptic spaces, which are areas of great interest on their own. We will use some known results of those areas, however, no details will be discussed.

Chapter 3 contains propositions and lemmas that are required to prove the main theorem. We decompose the manifold M into a union of two disk bundles over two submanifolds F and N of lower dimension, using Spindeler's theorem. We prove that there are essentially three separate cases of the decomposition, depending on the properties of

F and N. The proof of the main theorem will be based on the discussion in this section.

The proof of the main theorem is presented in Chapter 4, which gathers results of the previous chapters and uses an inductive argument for each case established in Chapter 3. In Chapter 5 we summarize the conclusion of this thesis.

In Chapter 6 we will describe how our main result can be applied to obtain a classification of almost torus manifolds with non-negative sectional curvature [10].

Finally, in Chapter 7 we will outline a few future directions that are interesting and that the author would like to explore.

2. PRELIMINARIES

In this section we will gather basic results and facts about transformation groups, torus actions, torus manifolds, torus orbifolds, as well as results concerning G-invariant manifolds of positive and non-negative sectional curvature.

2.1. Compact Transformation Groups

We refer to Bredon's text [1] for most of the content in this subsection.

By a **topological transformation group** we mean a triple (G, X, Θ) where G is a topological group, X is a Hausdorff space and $\Theta : G \times X \to X$ is a continuous map such that:

1. $\Theta(g, \Theta(h, x)) = \Theta(gh, x) \quad \forall g, h \in G$ and $x \in X$;

2. $\Theta(e, x) = x \quad \forall x \in X,$ where e is the identity of G.

The map Θ is called an **action** of G on X. We will call X a **G-space** if the underlying action Θ is understood. The notation for the action will be omitted and $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$ will be written as $g(hx) = (gh)x$.

Given a G-space X, we define the **orbit** of a point $x \in X$ to be $G(x) := \{\, gx : g \in G \,\}$. The **isotropy group** at x is defined as $G_x := \{\, g \in G : gx = x \,\}$. The notation for an orbit can be also used for a set $A \subset X$, such as $G(A) := \{\, gx : g \in G\, x \in A\}$. A set A is said to be **G-invariant** if $G(A) = A$. We will sometimes denote the **fixed point set** of the G action on M by $M^G := \{\, x \in M : gx = x \,\text{for all}\, g \in G \,\}$, or sometimes we will use $\mathrm{Fix}(M, G)$ instead.

The **ineffective kernel** of the action Θ is the subgroup $ker\Theta := \cap_{x \in M} G_x$. We say that G acts **effectively** on M if $ker\Theta$ is trivial. The action is called **almost effective** if $ker\Theta$ is finite. The action is **free** if every isotropy group is trivial and **almost free** if every isotropy group is finite.

Let X and Y be two G-spaces, an **G-equivariant** map $\varphi : X \to Y$ is a map such that

$$\varphi(g(x)) = g\varphi(x) \quad \forall g \in G \text{ and } x \in X\,.$$

We say X and Y are **weakly G-equivalent** if there is an automorphism α of G and a G-equivariant homeomorphism $\varphi : X \to Y$ such that,

$$\varphi(g(x)) = \alpha(g)(\varphi(x)) \quad \forall g \in G \text{ and } x \in X\,.$$

In the case when α is the identity map, X and Y are **G-equivalent** under the G-equivalence φ.

We denote by X/G the set whose elements are the orbits $x^* = G(x)$ of G on X, that is, $x^* = y^*$ if and only if $x, y \in X$ are in the same orbit. Define the canonical projection $\pi : X \to X/G$ as $\pi(x) = G(x)$, then X/G endowed with the quotient topology is called the **orbit space** of X. The action is called **transitive** if the orbit space is a point.

So far we did not impose any restrictions on the topology of G. One of the reasons why G is usually required to be compact is because the topology of M/G may not be interesting. Here is an example of an "irrational flow" on a torus. Let $\mathbb{R}$ act on a torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$ as follows.

$$t \cdot (e^{i2\pi x}, e^{i2\pi y}) = (e^{i2\pi(x+t)}, e^{i2\pi(y+\sqrt{2}t)}).$$

Then any orbit of $\mathbb{R}$ is a dense subset of T^2, resulting in the trivial (discrete) topology on $T^2/\mathbb{R}$ which is not so interesting. However, if G is compact, we have the following properties:

Theorem 2-1. *If X is a G-space with G compact, then*

1. *X/G is Hausdorff.*

2. $\pi : X \to X/G$ *is closed and proper.*

3. X *is compact if and only if* X/G *is compact.*

Moreover, fixing $x \in X$, the natural map

$$\alpha_x : G/G_x \to G(x) \quad \text{as } g\,G_x \mapsto g\,x$$

is a continuous bijection by the definition of G/G_x. By Theorem 2-1, G/G_x is compact, and thus since M is Hausdorff, α_x is a homeomorphism.

We now assume that all groups G in this section are compact. Let X be a G-space. For two points x, y in X, we have $G(x) \simeq G/G_x$ and $G(y) \simeq G/G_y$. One can show that there exists an equivariant map $G(x) \to G(y)$ if and only if G_x is conjugate to a subgroup of G_y (see for example I.4. of [1]). We let type($G(x)$) be the **type** of the orbit of x defined as the equivalence class of $G(x)$ under equivariant homeomorphism. We also have the following partial ordering of orbit types.

$$\text{type}(G(x)) \geq \text{type}(G(y)) \quad \textit{if and only if} \quad G_x \text{ is conjugate to a subgroup of } G_y$$

We will use the following facts about lifting group actions to covering spaces, details can be found in I.9 of [1].

Let X be a G-space and suppose we are given a covering space $p : X' \to X$. It is natural to ask if we can lift the G action $\Phi : G \times X \to X$ to its covering space X'. Let G be a Lie group, and $\pi : G^* \to G$ be the universal cover of G. It turns out that the map

$$\Phi \circ (\pi \times p) : G^* \times X' \to G \times X \to X$$

can be lifted to a unique map

$$\Phi^* : G^* \times X' \to X'$$

to form an action of G^* on X'. The action Φ^* commutes with the group of deck transformations of X'. Now assume G acts *effectively*, and let G' be the effective factor group of

G^* for the action on X'. Then G' covers G and the kernel of $G' \to G$ is a subgroup of the group of deck transformations of X'.

With some more effort, we have the following theorem.

Theorem 2-2 (9.1 in [1])**.** *Let G be a connected Lie group acting effectively on a connected, locally path connected space X and let X' be any covering space of X. Then there is a covering group G' of G with an effective action of G' on X' covering the given action. Moreover, G' and its action are unique. The kernel of $G' \to G$ is a subgroup of the group of deck transformations of $X' \to X$. If H has a fixed point in X, then $G = G'$ and* $\mathrm{Fix}(X', G') = \mathrm{Fix}(X', G)$ *is the full inverse image of* $\mathrm{Fix}(X, G)$*.*

And here is a small observation that we will make use of.

Lemma 2-3. *Let $\Theta : G \times M \to M$ be the action of a compact group G on a path-connected space M. If there is a G-fixed point in M, then any G orbit in M is contractible.*

Proof. Fixing a G-fixed point $x_0 \in M$, for any $y \in M$, let $\Theta_y : G \to M$ denote the map $\Theta_y(g) = g \cdot y$. Since M is path connected, take a path $\gamma : [0, 1] \to M$ with $\gamma(0) = y$ and $\gamma(1) = x_0$. Then $\Gamma : G \times I \to M$ defined by:

$$\Gamma(g, t) = \Theta_{\gamma(t)}(g) = g \cdot \gamma(t)$$

is a homotopy between $\Gamma(g, 0) = \Theta_y(g)$ and the constant map $\Gamma(g, 1) = \Theta_{x_0}(g) = x_0$. □

2.2. Fiber bundles and G-vector bundles

This section is written based on Bredon's text [1] and Kawakubo's text [30]. Also readers may want to compare [43]. Topological group actions appear naturally in the theory of fiber bundles. In this section we give a brief review of fiber bundles, as well as a few useful tools that will be applied in later sections.

Definition 2-4. *Let G be a topological group and let F be an effective* right *G-space. Let X and B be Hausdorff spaces. A* **fiber bundle** *over B (the* **base space***) with* **total space** *X, fiber F, and* **structure group** *G is a map*

$$\pi : X \to B$$

together with a collection Φ of homeomorphisms $\varphi : F \times U \to \pi^{-1}(U)$ for U open in B, called charts over U, such that

1. *$\pi \circ \varphi$ is the coordinate projection from $F \times U$ to U, for each chart $\varphi \in \Phi$ over U.*

2. *The collection of U forms a basis for the topology of B.*

3. *If $\varphi, \psi \in \Phi$ are charts over U, then there is a continuous map $\theta : U \to G$ such that*

$$\psi(f, u) = \varphi(f \cdot \theta(u), u)$$

   for all $f \in F$ and $u \in U$. This map θ is called the transition function for the charts ψ and φ.

4. *The set Φ is maximal among collections satisfying the preceding conditions.*

It is worth noticing that for a fiber bundle with structure group G:

1. The transition functions appearing in $\psi(f, u) = \varphi(f \cdot \theta(u), u)$ are not just any homeomorphism of the fiber F, but are given by the action of some element of G.

2. The topology of G is integrated into the structure by requiring $\theta : U \to G$ to be continuous. For example, θ is automatically continuous when G has the compact-open topology (compare with I.5 of [43]).

Definition 2-5. *A* **principal G-bundle** *is a fiber bundle with fiber G and structure group G acting by right translation. ($g \in G$ takes g' in the fiber to $g'g$.)*

Suppose X is right G-space and that Y is a left G-space. Then a left G action on $X \times Y$ is given by sending (x, y) to (xg^{-1}, gy). We define the **twisted product** of X and Y to be the orbit space

$$X \times_G Y$$

of this action. That is, $X \times_G Y$ is the quotient space of $X \times Y$ where $[xg, y] = [x, gy] \in X \times_G Y$. The following theorem is standard in the theory of fiber bundles.

Theorem 2-6. *Let $p : X \to B$ be a principal K-bundle and let F be right K-space. Then*

$$\pi : F \times_K X \to B$$

*defined by $\pi[f, x] = p(x)$ is a fiber bundle with fiber F and structure group K and is called the F-**bundle associated with this principal K-bundle**. If $\varphi : K \times U \to p^{-1}(U)$ is a chart of the principal bundle over U, then the composition*

$$\tilde{\varphi} : F \times U \xrightarrow{\simeq} (F \times_K K) \times U \xrightarrow{\simeq} F \times_K (K \times U) \xrightarrow{F \times_K \varphi} F \times_K p^{-1}(U) \xrightarrow{\simeq} \pi^{-1}(U)$$

is taken to be a chart of the associated bundle. (Note that $\tilde{\varphi}(f, u) = [f, \varphi(e, u)]$.)

Suppose the fiber F of a fiber bundle $X \to B$ is a G-space for some group G, then G acts on X if G commutes with the structure group of the bundle.

Theorem 2-7. *Suppose that $p : X \to B$ is a bundle with fiber F and structure group K. Suppose that F is also a left G-space and that the actions of G and K commute [that is, $(gf)k = g(fk)$]. Then there is a unique G action on X covering the trivial action on B and such each chart $\varphi : F \times U \to p^{-1}(U)$ is G-equivariant. [where G acts on $F \times U$ by $(g, (f, u)) \mapsto (gf, u)$).*

Corollary 2-8. *If $\pi : X \to B$ is a principal G-bundle, then there is a canonical free G action on X, which covers the identity on B (and is left translation in the fibers). The map $\pi : X \to B$ induces a homeomorphism $X/G \to B$, and thus may be regarded as the orbit map for this action.*

For our applications G is always a compact Lie group. If E is any smooth manifold on which G acts smoothly, freely and properly discontinuously, then $B := E/G$ is a smooth manifold as well, and the canonical map $\pi : E \to B$ is a principal G-bundle. Now suppose F is a smooth compact manifold and G acts on F from the left. Then there is an F-bundle associated with $E \to B$:

$$p : E \times_G F \to B$$

where the total space $E \times_G F$ is a smooth manifold. Then the tangent bundle of $E \times_G F$ is the Whitney sum:

$$\tau(E \times_G F) = p^*(\tau(B)) \oplus \tau_F$$

where τ_F is the subbundle of vectors in the tangent bundle $\tau(E \times_G F)$ that are tangent to the fibers. By [36], there exists $\psi : F \to \mathbb{R}^n$, an embedding of F into some $\mathbb{R}^n$ such that ψ is equivariant relative to a representation $\beta : G \to O(n)$ in the following sense:

$$\psi(gf) = \beta(g)\psi(f) \quad \forall f \in F, g \in G$$

Let $E \times_G \mathbb{R}^n \to B$ be the n-plane bundle associated with $E \to B$. Then ψ induces an embedding

$$\Psi : E \times_G F \to E \times_G \mathbb{R}^n \, , [e, f] \mapsto [e, \psi(f)] \, .$$

Let $\nu_\Psi(E \times_G F)$ be the normal bundle of $E \times_G F$ in $E \times_G \mathbb{R}^n$. We have the following theorem:

Theorem 2-9 ([45]).

$$\tau_F \oplus \nu_\Psi(E \times_G F) = p^*(E \times_G \mathbb{R}^n). \tag{2.1}$$

We will state an example of Theorem 2-9 as a corollary here, since it will be used in the proof of Theorem A.

Corollary 2-10. *Let ξ be a principal $O(n)$-bundle $E \to B = E/O(n)$, let $E \times_{O(n)} \mathbb{R}^n \to B$ be the associated n-plane bundle. Let S^{n-1} be embedded in $\mathbb{R}^n$ and let $p : E \times_{O(n)} S^{n-1} \to B$*

be the associated $(n-1)$-sphere bundle. Denote the induced embedding as

$$\Psi : E \times_{O(n)} S^{n-1} \to E \times_{O(n)} \mathbb{R}^n$$

Then we have

$$\tau_F(E \times_{O(n)} S^{n-1}) \oplus e^1_\Psi = p^*(E \times_{O(n)} \mathbb{R}^n)$$

where $\tau_F(E \times_{O(n)} S^{n-1})$ is the subbundle of vectors in the tangent bundle $\tau(E \times_{O(n)} S^{n-1})$ that are tangent to the fibers, and e^1_Ψ is the normal bundle of $E \times_{O(n)} S^{n-1}$ in $E \times_{O(n)} \mathbb{R}^n$, which is trivial.

The following theorem from II.6 of Bredon's text [1] is very useful.

Theorem 2-11. *Let G be a compact Lie group, and X is an path connected G-space. If there is an orbit which is connected (e.g. G is connected or $X^G \neq \emptyset$), then the fundamental group of X maps onto that of X/G. Thus if X is simply connected, then so is X/G.*

We will use the following result (6.2 in [44]):

Theorem 2-12. *Let $\pi : X \to Y$ be a principal S^1-bundle and $\alpha : T^1 \times Y \to Y$ an action of T^1 on Y. If $H^1(Y, \mathbb{Z}) = 0$, then α has a bundle lifting, that is, an action $\hat{\alpha} : T^1 \times X \to X$ such that each $g \in T^1$ acts on X as a bundle map $\hat{g}$ (that is, $\hat{g}$ commutes with the S^1 action on X, where S^1 is the structure group), and $\pi \circ \hat{g}(x) = g \circ \pi(x)$, for all $x \in X$.*

It is sufficient for us to notice that the assumptions in Theorem 2-12 can be easily generalized to a principal torus bundle whose base space admits a torus action. A generalization of the theorem can be found in [27], in which the group action on the base space is a general compact Lie group.

Next, we consider group actions on fiber bundles, in particular, the so called G-vector bundle.

Definition 2-13. *Let $p : E \to X$ be a vector bundle (that is, the fiber is a finite dimensional vector space and the structure group is the general linear group) and G acts on both E and X such that p is G-equivariant. If for each $g \in G$ and $x \in X$ the map*

$$g : p^{-1}(x) \to p^{-1}(gx)$$

is a linear isomorphism, then $p : E \to X$ is called a **G-vector bundle***.*

The tangent bundle TM of a smooth manifold M is an example of G-vector bundle where G acts via the differential of the given action on M . If a G-invariant subspace E' of E and the restriction $p' := p|_{E'} : E' \to X$ satisfy the following conditions (1) and (2), then E' is called a ***G*-vector subbundle of** E.

(1) For each $x \in X$, the fiber $E'_x = E' \cap E_x$ is a vector subspace of E_x.

(2) $p' : E' \to X$ is a G-vector bundle with respect to the vector space structure of E'_x in (1).

Then we consider the equivalence relation $\sim$ on E defined by

$$z \sim z' \quad \text{if and only if} \quad p(z) = p(z') \ \text{ and } \ z - z' \in E'_{p(z)}.$$

The quotient space $E/\sim$ is denoted as E/E'. Let $\pi : E \to E/E'$ be the canonical projection map. Then there is a well-defined map $\hat{p} : E/E' \to X$ such that $\hat{p} \circ \pi = p$, since we require the equivalence relation $\sim$ to respect the fibers of E/E', which for each $x \in X$, have the structure of the quotient vector space E_x/E'_x.

Theorem 2-14. *The map $\bar{p} : E/E' \to X$ is a G-vector bundle.*

The proof of Theorem 2-14 can be found in standard text books (see for example [30]). We will continue with this topic in next subsection, when we consider smooth actions.

2.3. Smooth actions and G-invariant tubular neighborhoods

In this subsection, let G be a compact Lie group and M a smooth manifold. If the the map $\Theta : G \times M \to M$ of a G action on M is smooth, then the action is called a **smooth G action**. Each $g \in G$ is a diffeomorphism on M and induces a natural action, $G \times TM \to TM$, of G on the tangent bundle TM given by:

$$(g, v_p) \mapsto Dg_p(v_p) \quad \forall v_p \in T_pM, g \in G$$

where Dg is the differential of g. One can check that the exponential map $exp : TM \to M$ and the canonical projection $\pi : TM \to M$ are both G-equivariant, since the G action on TM preserves the fibers of the tangent bundle.

We continue the discussion of G-vector bundles, whose definition was given in Definition 2-13. Suppose a compact Lie group acts smoothly on a manifold M. The map $g : M \to M$ is a diffeomorphism for each $g \in G$. Hence g induces a bundle isomorphism

$$Dg : T(M) \to T(M).$$

It is easy to verify that

$$g \cdot v \mapsto Dg(v) \quad \forall g \in G, v \in TM$$

defines a smooth action of G on the total space $T(M)$ of the tangent bundle of M. Thus $T(M)$ is a G-manifold and since Dg restricted to each fiber is a linear isomorphism, $T(M) \to M$ is a G-vector bundle. In particular, for each $p \in M$, since the isotropy subgroup G_p acts linearly on $T_p(M)$, $T_p(M)$ is a representation space of G_p, this representation is called the **isotropy representation at** p of the G-manifold.

Now let N be a G-invariant submanifold of M. The tangent bundle $T(N) \to N$ is a G-vector subbundle of the restriction $T(M)|_N \to N$ of the tangent G-vector bundle $T(M)$ of M. Thus by Theorem 2-14, the quotient bundle

$$\nu = (T(M)|_N)/T(N)$$

is a G-vector bundle called the **normal G-vector bundle** of N in M.

In our setting, M is a Riemannian manifold and G is a compact Lie group acting on M isometrically. If N is a G-invariant submanifold of M, the tangent bundle $T(N)$ has an orthogonal complement $T^{\perp}(N)$ in $T(M)|_N$ such that

$$T(M)|_N \simeq T(N) \oplus T^{\perp}(N) \tag{2.2}$$

where $\oplus$ is the Whitney sum. Since the metric on M and N is G-invariant, each fiber of $T^{\perp}(N)$ is a G-invariant subspace of the tangent space of M. Thus the normal bundle $T^{\perp}(N) \to N$ is a G-vector bundle. Moreover,

$$T^{\perp}(N) \simeq \nu = (T(M)|_N)/T(N). \tag{2.3}$$

This G-equivariant isomorphism can be seen from as follows: If $E \to X$ and $E' \to X$ are two G-vector bundles, then they are naturally G-vector subbundles of $E \oplus E' \to X$ and satisfy

$$E \simeq (E \oplus E')/E'. \tag{2.4}$$

The isomorphism is given by the inclusion of E into $E \oplus E'$ and then projecting to $(E \oplus E')/E'$. Now let E be $T^{\perp}(N)$ and E' be $T(N)$ and combine equations 2.2 and 2.4 to obtain equation 2.3. There are two comments we should make about the isomorphism 2.3. First, it is clear from the construction that the isomorphism is G-equivariant. Second, ν is canonically endowed with a G-invariant metric by the isomorphism.

The structure of the normal G-vector bundle can be further discussed when G is a torus.

We now return to the manifold M and a submanifold N. By a **G-invariant tubular neighborhood** of N in M we mean a G-equivariant embedding $f : \nu \to M$ such that the restriction of f to the zero section N of ν is the inclusion of N in M. The restriction of f to the unit disk bundle $f : D(N) \to M$ is called a closed G-invariant tubular neighborhood of N in M.

The following theorem is well-known:

Theorem 2-15. *Let G be a compact Lie group. For a G-invariant submanifold A of a G-manifold M, there exists a G-invariant tubular neighborhood of A in M.*

Recall that for a compact group G a G-orbit of x in M is an embedded G-invariant submanifold of M. Let ν be the normal G-vector bundle of $G(x)$ in M. Then the fiber ν_x is a representation space of the isotropy subgroup G_x. Possibly the most crucial basic result in the theory of compact transformation groups is the so-called *Slice Theorem.*

Theorem 2-16 (Slice Theorem). *Let G be a compact Lie group and M be a smooth G-manifold. For any $x \in M$, the normal G-vector bundle ν of $G(x)$ in M is isomorphic to*

$$G \times_{G_x} \nu_x \to G/G_x$$

as smooth G-vector bundles. Further, there is a G-equivariant embedding of the twisted product $G \times_{G_x} \nu_x$ onto a G-invariant tubular neighborhood U of $G(x)$ in M

$$\varphi : G \times_{G_x} \nu_x \to U \subset M.$$

A few remarks should be made with respect to the Slice Theorem.

1. The restriction of φ to the zero cross-section gives the natural G-diffeomorphism from G/G_x to $G(x)$.

2. If we take M to be a Riemannian manifold with a metric such that G acts isometrically, then the G_x action on ν_x is given by the G_x action on the orthogonal complement of $T_x(G(x))$ in T_xM.

3. The image $S_x := \varphi(\nu_x)$ is called a **slice** at x. We have $g(S_x) = S_{gx}$ for any $g \in G$.

4. The natural inclusion $S_x/G_x \to M/G$ is a homeomorphism onto the open subspace $G(S_x)/G$. Namely, at each point of the orbit space M/G there is an open neigh-

borhood that is homeomorphic to $S_x/G_x \simeq T^{\perp}(G(x))/G_x$. (See for example, [1] for details.)

The Slice Theorem is fundamental in the study of compact transformation groups. One of the many important applications of the Slice Theorem is the following well-known theorem.

Theorem 2-17 (Principal Orbit Theorem)**.** *Suppose a compact Lie group G acts smoothly on an n-dimensional smooth manifold M with connected orbit space M/G. Then there exists a maximal orbit type G/H for the action of G on M, that is, H is conjugate to a subgroup of each isotropy subgroup. The union $M(H)$ of the orbits of type G/H is open and dense in M and its image $M(H)/G$ in M/G is connected.*

Although we will not discuss this further, it is worth noticing that the Principal Orbit Theorem can be proven under the more general assumption that a compact Lie group G acts locally smoothly on a topological manifold M (see [1]). But in the setting of a smooth action (which implies local smoothness) on a Riemannian manifold, the proof can be simplified using Kleiner's isotropy lemma (see [31]).

Orbits of maximal orbit type as guaranteed by Theorem 2-17 are called **principal orbits**. The corresponding isotropy group is called the **principal isotropy group**. Depending on the relative size of their isotropy subgroups to the principal isotropy group, an orbit is called **exceptional** when its isotropy subgroup is a finite extension of the principal isotropy subgroup, and **singular** when its isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup.

The following two useful basic structure theorems (see for example [30]) will be used repeatedly:

Theorem 2-18. *Let G be a compact Lie group, suppose G acts smoothly on M. Then each connected component of the fixed point set M^G is a closed submanifold of M.*

Theorem 2-19. *Let G be a compact Lie group and M a G-manifold. Suppose that every orbit in M has type G/H. Then the orbit space M/G is a topological manifold and there is a smooth structure for M/G such that the projection $\pi : M \to M/G$ is a smooth fiber bundle with fiber G/H and structure group $N(H)/H$, where $N(H)$ is the normalizer of H in G.*

2.4. Torus Actions

Recall that a k-torus $T^k := (S^1)^k$ is nothing but a compact, connected, abelian Lie group that is diffeomorphic to a torus $T^k := (S^1)^k$. A **maximal torus** in a compact Lie group G is a torus subgroup of G which is not properly contained in any larger torus subgroup of G. The dimension of a maximal torus T in G is called **the rank** of G.

Example 2-20 (Standard maximal torus)**.** *A standard maximal torus in* $\mathrm{SO}(2n)$*, where* $(n \geq 1)$*:*

$$T^n = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & \\ & & & & \ddots & & \\ & & & & & \cos\theta_n & \sin\theta_n \\ & & & & & -\sin\theta_n & \cos\theta_n \end{pmatrix} : \theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R} \right\},$$

or equivalently, in $\mathrm{U}(n)$*, where* $(n \geq 1)$*:*

$$T^n = \left\{ \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} : \theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R} \right\},$$

A standard maximal torus in $\mathrm{SO}(2n+1)$*, where* $(n \geq 1)$*:*

$$T^n = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \cos\theta_n & \sin\theta_n & \\ & & & & & -\sin\theta_n & \cos\theta_n & \\ & & & & & & & 1 \end{pmatrix} : \theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R} \right\},$$

In general, a T^n *torus in* $\mathrm{SO}(2n)$ *is maximal, and a* T^n *torus in* $\mathrm{SO}(2n+1)$ *is maximal.*

The following well-understood proposition (see for example, [1]) describes the relation between two maximal tori.

Proposition 2-21. *Any two maximal tori of a compact Lie group* G *are conjugate in* G*.*

There are a few advantages of considering torus actions in the context of isometric group actions on Riemannian manifolds. In particular we have the following well-known results.

Theorem 2-22 ([33])**.** *Let* M *be a Riemannian manifold with an isometric torus action* T*. The fixed point set* M^T *satisfies:*

(1) Each connected component N_i *of* M^T *is a closed totally geodesic submanifold of even codimension.*

(2) The structure group of the normal bundle over N_i *can be reduced to* $\mathrm{GL}(r, \mathbb{C})$ *where* $2r =$ *codim*(N_i)*. Thus if* M *is orientable, then each* N_i *is orientable.*

(3) The Euler characteristic satisfies $\chi(M^T) = \Sigma_i \chi(N_i) = \chi(M)$*.*

We wish to share a few comments regarding Theorem 2-22. The first item of the theorem can be thought as a strengthened version of 2-56. In the case when G is a finite group, the equation of Euler characteristics in Theorem 2-22 holds if G is a cyclic subgroup of a torus:

Lemma 2-23. *Let T be a torus group and M be a compact T-manifold. Then for any finite cyclic subgroup $G \subset T$, we have:*

$$\chi(M^G) = \chi(M).$$

This formula can be found in many classical texts, for example, Corollary 5.60 of [30].

Now we will give the definition of an almost torus manifold. Let M be a connected Riemannian manifold equipped with an effective action by a torus T. For any point $x \in M$, observe that the normal subspace $T_xM/T_x(T(x))$ is a faithful T_x-representation, thus the dimension of $T_xM/T_x(T(x))$ is at least $2\dim(T_x)$. More specially, we have the following inequality:

$$\dim(T_x) \leq 1/2(\,\dim(M) - (\dim(T) - \dim(T_x))\,) \quad \forall x \in M,$$

which can be simplified as

$$\dim(T) + \dim(T_x) \leq \dim(M).$$

We have the following definition to describe those actions for which the equality holds in the above inequality.

Definition 2-24 (Isotropy-Maximal Action). *Let M^n be a connected smooth manifold with a smooth, effective torus T^k action. We call the T^k action on M^n* isotropy-maximal, *if there is a point $x \in M$ such that the dimension of its isotropy group is $n-k$. Meanwhile, an orbit with type T/H where the isotropy subgroup $H \subset T$ has the largest dimension is called a* **minimal orbit**, *that is, an orbit with dimension $k-(n-k) = 2k-n$.*

Remark: If T^k acts isotropy maximally on M^n, then since $\dim(T) \geq \dim(T_x)$ for any $x \in M$, it is necessary that $k-(n-k) \geq 0$, that is, $2k \geq n$. The dimension of the torus can not exceed half of the dimension of the manifold.

Definition 2-25 (**Almost Isotropy-Maximal Action).** *Let M^n be a connected smooth manifold with a smooth, effective torus T^k action. We call the T^k action on M^n* almost isotropy-maximal*, if there is a point $x \in M$ such that the dimension of its isotropy group is $n-k-1$.*

An important subclass of smooth manifolds admitting isotropy-maximal actions are the so-called *torus manifolds*.

Definition 2-26 (**Torus Manifold).** *A* torus manifold *M is a $2n$-dimensional closed, connected, orientable, smooth manifold with an effective smooth action of an n-dimensional torus T such that $M^T \neq \emptyset$.*

For more details on torus manifolds, we refer the reader to Hattori and Masuda [26], Buchstaber and Panov [3], and Masuda and Panov [34].

Torus manifolds arose as a generalization of the concept of a *toric variety*, which is a normal algebraic variety, M, containing the algebraic torus $(C^*)^n$ as a Zariski open subset in such a way that the natural action of $(C^*)^n$ on itself extends to an action on M (see [2] for more details). In particular, in [9] a topological counterpart to non-singular projective toric varieties was introduced, now called *quasitoric manifolds*. Originally they were named “toric manifolds”, but then were renamed in [2] since the term toric manifold is reserved in algebraic geometry for a “non-singular toric variety”.

It is natural to consider the odd dimensional analogue of a torus manifold. In particular, we define:

Definition 2-27 (**Almost Torus Manifold).** *An* almost torus manifold *M is a $(2n+1)$-dimensional closed, connected, orientable, smooth manifold with an effective smooth action* *of an n-dimensional torus T such that $M^T \neq \emptyset$.*

One can easily check that the torus action on a given almost torus manifold is almost isotropy-maximal. Despite the differences in definition, the family of torus manifolds and the family of almost torus manifolds share an important property, which we will explain here. Let M be a G-fixed point homogenous manifold and recall that such a manifold satisfies

$$\dim(M/G) - \dim(M^G) = 1.$$

Notice that if the group G is a circle, then the codimension of a maximal component of M^G in M is exactly two. In particular, we define:

Definition 2-28 (Characteristic submanifold). *Let T^k act effectively and smoothly on a closed manifold M^n. Let F be a connected component of M^{S^1} for some circle subgroup $S^1 \subset T^k$. Then F is called a* characteristic submanifold *of M if it satisfies the following properties:*

1. *codim$(F) = 2$ in M;*
2. *F contains a T-fixed point.*

Proposition 2-29. *Let M be a torus manifold, or an almost torus manifold. Then every point $x \in M^T$ is contained in some characteristic submanifold of M.*

Proof. We will first prove the proposition assuming M^{2n+1} is an almost torus manifold.

The Slice Theorem 2-16 implies that if $x \in M$ is fixed by T^n, the tangent space T_xM is a representation space of T^n. Then T^n acts on $T_xM \simeq \mathbb{R}^{2n+1}$ as a maximal torus of $\mathrm{SO}(2n+1)$. Since all maximal tori in $\mathrm{SO}(2n+1)$ are conjugate, consider a standard maximal torus in $\mathrm{SO}(2n+1)$ like in Example 2-20. Observe that

1. T^n fixes only a one-dimensional subspace of T_xM. Thus the connected component of M^T containing x is one-dimensional, and being a submanifold, it is a circle. Since M is compact, M^T is a disjoint union of finitely many circles.

2. There is a circle $S^1 \subset T^n$ such that $\mathrm{codim}(\mathrm{Fix}(T_xM, S^1)) = 2$, so there is a codimension two component of M^{S^1} containing x via the exponential map.

Therefore, the fixed point x is contained in a characteristic submanifold.

In the case when we have a torus manifold M^{2n}, observe that the T^n acts on $T_xM \simeq \mathbb{R}^{2n}$ as a maximal torus of $\mathrm{SO}(2n)$. Then by a similar argument as above, the theorem holds. □

At the end of this section, we would like to make some comments concerning the definition of characteristic submanifolds. The existence of a T^n-fixed point is required for Definition 2-28 of characteristic submanifolds, but it is not necessary in the case of a torus manifold with vanishing odd degree cohomology, as was shown in Lemma 2.2 of [34]. In the case of almost torus manifolds, it is not known whether or not every maximal component fixed by a circle subgroup contains a T^n-fixed point. This will require further exploration of the structure of almost torus manifolds.

2.5. Locally Standard Torus Actions

In this section we discuss locally standard torus actions and their properties.

The following definition of a locally standard action on an even dimensional manifold was taken from [29].

Definition 2-30 (Locally Standard). *A T^n action on M^{2n} is called* locally standard *if each point in M^{2n} has an invariant neighborhood U which is equivariantly diffeomorphic to an open invariant subset of a direct sum of complex one-dimensional representation spaces of T^n.*

The definition can be made more specific, as in Davis and Januszkiewicz [9], where it was called locally isomorphic to the standard representation.

Definition 2-31 ([9] **Locally Standard**). *A T^n action on M^{2n} is called* locally standard *if each point in M^{2n} has an invariant neighborhood U which is weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ invariant under the standard T^n action on $\mathbb{C}^n$ (that is, the action of T^n as a standard maximal torus of* $\mathrm{U}(n)$*, see Example 2-20), that is, there are*

- *an automorphism $\theta : T^n \to T^n$,*
- *T^n-invariant open sets $U \subset M$ and $W \subset \mathbb{C}^n$, and*
- *a θ-equivariant diffeomorphism $f : U \to W$.*

An analogous concept for T^n actions on a $(2n+k)$-dimensional manifold can be easily defined:

Definition 2-32 (**General Locally Standard**). *A T^n action on M^{2n+k} is called* **locally standard** *if each point in M^{2n+k} has an invariant neighborhood U which is weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n \times \mathbb{R}^k$ invariant under the standard T^n action on $\mathbb{C}^n$, and the trivial action on $\mathbb{R}^k$.*

The case when $k = 1$ in the above definition corresponds to our almost torus manifolds.

Suppose (M, T) is a torus manifold and the action is locally standard. For any point $x \in M$, there is a triple (U, f, ϕ) where U is an open neighborhood containing x, together with a diffeomorphism $f : U \to W \subset \mathbb{C}^n$, as well as a T^n-equivariant embedding $\phi : T^n \to (S^1)^n \subset \mathbb{C}^n$. Denote $\phi = (\phi_1, \ldots, \phi_n)$, where the coordinate functions $\phi_i : T \to S^1 \subset \mathbb{C}$ $(i = 1, \ldots, n)$ are homomorphisms. Then U is T-invariant and f is weakly T-equivariant in the following sense:

$$f(g \cdot u) = (\phi_1(g) f_1(u), \ldots, \phi_n(g) f_n(u)) \quad \forall g \in T, u \in U.$$

A fundamental property of locally standard torus actions is stated in the following proposition.

Proposition 2-33. *If a torus action is locally standard, then all isotropy subgroups are connected, that is, all isotropy subgroups are subtori.*

Proof. Let T denote the torus group. Given a point $x \in M$, assume that the isotropy subgroup T_x is not connected, then $T_x \cong T^l \times H$ where H is a non-trivial finite group and T^l is a maximal torus as well as the identity component of T_x. Pick a non-trivial element $g \in H$. Notice that g is not contained in the identity component $T^l \times \{e\}$ of T_x. By Definition 2-30, there is an automorphism $\phi = (\phi_1, \ldots, \phi_n) : T^n \to T^n$, with each $\phi_i : T^n \to S^1 \subset \mathbb{C}$ is a homomorphism, and a ϕ-equivariant diffeomorphism $f = (f_1, \ldots, f_n) : U \to W$ where $U \subset M$ is a T^n-invariant open neighborhood of x, $W \subset \mathbb{C}^n$ and each $f_i : U \to \mathbb{C}$ is a coordinate function.

The action of g on x is given by

$$f(g \cdot x) = \phi(g) \cdot f(x) = (\phi_1(g)f_1(x), \ldots, \phi_n(g)f_n(x)).$$

Since $f(g \cdot x) = f(x)$, we have $\phi_i(g)f_i(x) = f_i(x)$ for any $i = 1, \ldots, n$. Consider the index set:

$$I := \{i \,|\, f_i(x) = 0 \in \mathbb{C}, i = 1, 2, \ldots, n\}.$$

Define a torus $(S^1)^k \subset \mathbb{C}^n$ to be

$$(S^1)^k := \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \,|\, \theta_i = 1 \text{ if } i \in I\}.$$

Clearly, the dimension $k = n - |I|$ and $(S^1)^k$ fixes $f(x)$.

On the other hand, if an index $i \notin I$ then $f_i(x) \neq 0$, and so it is necessary that $\phi_i(g) = 1$. Thus we have $\phi(g) \in (S^1)^k$. Since ϕ is an isomorphism and f is an equivariant diffeomorphism, the subtorus $T^k := \phi^{-1}((S^1)^k)$ satisfies $g \in T^k \subset T_x$. Since T^k is connected, it is contained in the identity component of T_x, but then g is contained in the identity component of T_x, a contradiction. Hence H must be trivial, and so T_x is connected. $\square$

Proposition 2-34. *Let M be a closed Riemannian manifold and let $T \subset \mathrm{Isom}(M)$ be a torus. Then the following hold.*

1. *Let N be a T-invariant closed submanifold of M. Suppose $T_1 \subset T$ such that T_1 fixes N. If N is locally standard with respect to the T/T_1 action and the T_1 action is standard (see Example 2-20) on each fiber of the normal vector space of N in M, then M is locally standard with respect to the T action on a tubular neighborhood of N.*

2. *Suppose $T_2 \subset T$ acts freely on M, and consider the principal T_2-bundle $M \to M/T_2$. If M/T_2 is locally standard with respect to the T/T_2 action, then M is locally standard with respect to T action.*

Proof. Proof of (1): Since a tubular neighborhood of N is T-equivariantly diffeomorphic to the total space of the normal T-vector bundle $p : T^{\perp}N \to N$, it suffices to show that $T^{\perp}N$ is locally standard. Firstly, since N is locally standard, for $x \in N$, by the Slice Theorem, there is a tubular neighborhood $T' \times_{T'_x} S_x$, where $S_x \subset N$ is a slice at x, that is locally standard with respect to the $T' := T/T_1$ action. Let $\phi : U \times \mathbb{R}^k \to p^{-1}(U)$ be a chart over U containing x. We can choose the slice S_x to be small enough such that $S_x \subset U$ and only consider $\phi : S_x \times \mathbb{R}^k \to p^{-1}(S_x)$. Since T_1 acts on $T_x^{\perp}N$ by differential map, define a T_1 action on $S_x \times \mathbb{R}^k$ as

$$g_1 \cdot (s, v) = (s, dg_1(v)) \quad \forall g_1 \in T_1, (s, v) \in S_x \times \mathbb{R}^k,$$

by identifying $\mathbb{R}^k$ with $T_x^{\perp}N$. Then ϕ is T_1-equivariant by construction.

Furthermore, since S_x is T'_x-invariant, there is a T'_x action on $S_x \times \mathbb{R}^k$ by letting

$$g' \cdot (s, v) = (g's, v) \quad \forall g' \in T'_x, (s, v) \in S_x \times \mathbb{R}^k.$$

We claim that ϕ is then T'_x-equivariant. This can be seen from the fact that T'_x acts isometrically and $T'_x \subset T/T_1$ does not act in the normal space of N.

All together this introduces a $T = T' \times T_1$ action on $(T' \times_{T'_x} S_x) \times \mathbb{R}^k$, and by assumption, since the T' action is locally standard on $T' \times_{T'_x} S_x$ and the T_1 action is standard on $\mathbb{R}^k$, we have a locally standard T action on $(T' \times_{T'_x} S_x) \times \mathbb{R}^k$.

Now define a map $\Phi : (T' \times_{T'_x} S_x) \times \mathbb{R}^k \to T^{\perp}N$ as follows

$$([g', s], v) \mapsto g' \cdot \phi(s, v) \quad \forall g' \in T', s \in S_x, v \in \mathbb{R}^k.$$

This map is well defined, since for all $g_0 \in T'_x$, we have

$$([g'g_0^{-1}, g_0 s], v) \mapsto g'g_0^{-1} \cdot \phi(g_0 s, v) = g'g_0^{-1}g_0 \cdot \phi(s, v).$$

Moreover, it is clear that Φ is a T-equivariant diffeomorphism. Since the T action on $(T' \times_{T'_x} S_x) \times \mathbb{R}^k$ is locally standard, the image of Φ is a locally standard open subset in $T^{\perp}N$. Therefore, $T^{\perp}N$, hence a tubular neighborhood of N in M, is locally standard via exponential map.

Proof of (2):

(2) is true by a similar argument as the proof of (1). □

There are torus manifolds that are not locally standard. The example below can be found in [29]. In the case of almost torus manifolds, one only needs to take a product $S^1 \times M^4$, where M^4 is a given torus manifold, to create an almost torus manifold that is not locally standard.

Example 2-35 (Torus 4-manifolds that are not locally standard)**.** *For a fixed positive integer $m > 1$, define a smooth action of $T^2 \subset \mathbb{C}^2$ on $T^2 \times D^2$, where $D^2 \subset \mathbb{C}$ is the unit disk, as*

$$(g_1, g_2) \star (h_1, h_2, v) = (g_1 h_1, (g_2)^m h_2, g_2^{-1} v), \tag{2.5}$$

for $(g_1, g_2) \in T^2$ and $(h_1, h_2, v) \in T^2 \times D^2$. The isotropy subgroup at (h_1, h_2, v) is trivial if $v \neq 0$ and isomorphic to $\mathbb{Z}_m$ if $v = 0$.

Consider a diffeomorphism $\phi : \partial(T^2 \times D^2) \simeq T^3 \to T^3$ *as*

$$(h_1, h_2, v) \mapsto (h_1, h_2 v^{m-1}, h_2 v^m),$$

where $v \in S^1 \subset \mathbb{C}$ *is a complex number. We equip the domain of* ϕ *with the action defined by 2.5, and notice that the action is free.*

One can easily calculate the inverse of ϕ *as the following*

$$\phi^{-1}(h_1, h_2, v) = (h_1, h_2^m v^{1-m}, h_2^{-1} v).$$

Now take a torus manifold M *of dimension* 4*, and let* x *be a point with principal orbit, that is, the isotropy group of* x *is trivial. By the Slice Theorem, there is a closed invariant neighborhood of* x*, denoted as* N*, an equivariant embedding* $\psi : N \to T^2 \times D^2$*, such that on* $T^2 \times D^2$ *the free* T^2 *action is given by the standard one:*

$$(g_1, g_2) \cdot (h_1, h_2, v) = (g_1 h_1, g_2 h_2, v).$$

One can check that the diffeomorphism $(\phi^{-1} \circ \psi)|_{\partial N} : \partial N \to \partial(T^2 \times D^2)$ *is* T^2*-equivariant. We glue* $M \setminus (int(N))$ *and* $T^2 \times D^2$ *along boundaries by* $\phi^{-1} \circ \psi$ *to obtain a manifold* M'*. That is,*

$$M' = M \setminus (int(N)) \cup_{\phi^{-1} \circ \psi} T^2 \times D^2$$

One can show that this is a smooth torus manifold with a point whose isotropy subgroup is not connected, thus by Proposition 2-33, M' *can not be locally standard.*

2.6. Orbit spaces of locally standard torus manifolds

In this section we review the structure of orbit spaces of torus manifolds (see [34]).

Let us start with an orientable manifold M with a smooth action of a torus T, and we will call M a T-manifold. There are connections between T-manifolds with vanishing odd degree integer cohomology and torus manifolds. All cohomology groups in this section have integer coefficients.

Lemma 2-36 (2.2 of [34]). *Let M be a manifold with a smooth T action. Let H be a subtorus of T and N a connected component of M^H. If $H^{odd}(M) = 0$, then $H^{odd}(N) = 0$ and $N^T \neq \emptyset$.*

Theorem 2-37 (4.1 of [34]). *A torus manifold M with $H^{odd}(M) = 0$ is locally standard.*

Given a torus manifold M^{2n} with $H^{odd}(M) = 0$, let $\pi : M \to M/T := P$ be the orbit map and P be the orbit space. Since M is locally standard, each $p \in P$ has a neighborhood which is diffeomorphic to an open subset of the positive cone:

$$\mathbb{R}^n_{\geq 0} = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0, i = 1, \ldots, n\}.$$

Thus P can be identified as a smooth n-manifold with corners. An **n-manifold with corners** is a Hausdorff space together with a maximal atlas $\{U_\alpha, \varphi_\alpha\}$, where each chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n_{\geq 0}$ is a diffeomorphism onto an open subset of $\mathbb{R}^n_{\geq 0}$. Moreover, a coordinate transformation

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

satisfies that $\varphi_\beta \circ \varphi_\alpha^{-1}$ restricted to each hypersurface of $\varphi_\alpha(U_\alpha \cap U_\beta)$ is homeomorphic to a hypersurface of $\varphi_\beta(U_\alpha \cap U_\beta)$ of the same codimension.

Connected and closed codimension one hypersurfaces of P are called the **facets** of P. Points in the relative interior of a facet F_j correspond to orbits with the same one-dimensional isotropy subgroup, which we denote by T_{F_j}. Thus $\pi^{-1}(F_j) := M_j$ is a codimension two, connected submanifold of M, that is, M_j is a maximal connected component of $M^{T_{F_j}}$. By Lemma 2-36, M_j necessarily contains a T-fixed point. Hence M_j is a characteristic submanifold. The number of characteristic submanifolds in M is finite, since the T action has only finitely many different isotropy subgroups and M is compact.

The vertices of P are codimension n faces corresponding to the T-fixed points of M through the orbit map. The intersection of k facets is called a **codimension k preface**, which does not have to be connected. **Faces** are connected components of prefaces. By the definition of manifold with corners, every codimension k face is contained in at most k facets. If the number of facets meeting at each vertex is n, then P^n is called a **nice** manifold with corners. Notice that if the torus manifold M^{2n} is locally standard, then P is nice.

The mapping $\lambda : F_j \to T_{F_j}, 1 \leq j \leq m$, is called the **characteristic function** of the torus manifold M^{2n}. We will use $\lambda(F_j)$ to represent the circle fixing F_j, following Wiemeler's notation. In general, a codimension k face corresponds to a fixed point component of a T^k subtorus. Let H be a codimension-k face of P^n. We can write H as an intersection of k facets: $H = F_{j_1} \cap \cdots \cap F_{j_k}$. Assign to each face H the subtorus $T_H = \prod_{F_i \supset H} T_{F_i}$. Then $M_H = \pi^{-1}(H)$ is a T^n-invariant submanifold of codimension $2k$ in M, and M_H is fixed under each circle subgroup $\lambda(F_{j_p})$, $1 \leq p \leq k$.

Recall that a space X is *acyclic* if the reduced homology $\widetilde{H}_i(X) = 0$ for all i. The following theorem gives an equivalent condition of vanishing odd cohomology for torus manifolds.

Theorem 2-38 ([34])**.** *Let M be a torus manifold. The odd degree integer cohomology of M vanishes if and only if M is locally standard and the orbit space M/T is acyclic with acyclic faces.*

A manifold with corners is called a *homology polytope* if all its prefaces are acyclic (in particular, connected). Orbit spaces of locally standard torus manifolds need not be a homology polytope since prefaces need not be connected in general. For example, let $S^{2n} \subset \mathbb{C}^n \times \mathbb{R}$, and a locally standard T^n action on S^{2n} $(n > 0)$ can be defined as

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, y) = (e^{it_1} z_1, \ldots, e^{it_n} z_n, y).$$

Characteristics submanifolds of this action are spheres $S^{2(n-1)}$. In fact, all faces of S^{2n}/T^n correspond to spheres of even codimension, hence the orbit space S^{2n}/T^n is face-acyclic (see Theorem 2-11). The intersection of k facets is connected unless $k = n$, in which case it consists of two disjoint fixed points $\{(0,0,\ldots,0,\pm 1)\}$. On the other hand, recall that a convex n-polytope is *simple* if the number of facets meeting at each vertex is n. A simple convex polytope is an example of a nice manifold with corners and is a homology polytope. As examples of locally standard torus manifolds, *quasitoric manifolds* are defined as locally standard torus manifolds such that the orbit spaces of these manifolds are simple convex polytopes.

Proposition 2-39 (4.5 in [46])**.** *Let Q be a nice manifold with corners. If all faces of Q are acyclic and each two-dimensional face of Q has at most four vertices, then the combinatorial face poset (partially ordered set on all faces of Q), $P(Q)$, is isomorphic to the face poset of a product*

$$\prod_i \Sigma^{n_i} \times \prod_j \Delta^{n_j}. \tag{2.6}$$

Here Σ^m is the orbit space of the linear T^m action on S^{2m} and Δ^m is an m-dimensional simplex.

Notice that if M admits a T-invariant metric of non-negative sectional curvature, then the condition that each two-dimensional face of $Q = M/T$ has at most four vertices holds. (Lemma 4.2 of [46], Lemma 4.1 of [16]). In fact, in the context of non-negative sectional curvature, we have the following.

Theorem 2-40 (6.3 in [46])**.** *Let M^{2n} be a simply connected torus manifold with an invariant metric of non-negative sectional curvature. Then M^{2n} is locally standard and M^{2n}/T^n and all its faces are diffeomorphic (after smoothing the corners) to standard disks D^k. Moreover, $H^{odd}(M;\mathbb{Z}) = 0$.*

Hence for a simply connected torus manifold M with a T-invariant metric of non-negative curvature, the orbit space M/T is a face-acyclic nice manifold with corners. Applying Proposition 2-39, we know immediately that the combinatorial face poset of

M/T is a product $\prod_i \Sigma^{n_i} \times \prod_j \Delta^{n_j}$. Using this fact, Wiemeler also derived the fundamental group of a non-simply connected torus manifold.

Theorem 2-41 (7.3 of [46])**.** *Let M^{2n} be a non-negatively curved torus manifold. Then there is a $0 \leq k \leq n-1$ such that $\pi_1(M) = (\mathbb{Z}_2)^k$.*

The bound on the order of the fundamental group given in Theorem 2-41 is sharp (see Example 7.4 of [46]).

2.7. Orbit Spaces of Torus Orbifolds

In this section we review torus orbifolds based on related materials in [15]. Torus orbifolds generalize the idea of torus manifolds. In particular, we recall facts about orbit spaces of torus orbifolds.

We first recall the definition of an orbifold. For more details about orbifolds and actions of tori on orbifolds, see Haefliger and Salem [25].

Definition 2-42. *A* **local model of dimension n** *is a pair $(\widetilde{U}, \Gamma)$, where $\widetilde{U}$ is an open, connected subset of $\mathbb{R}^n$, and Γ is a finite group acting smoothly and effectively on $\widetilde{U}$. The quotient $\widetilde{U}/\Gamma$ is denoted by U.*

Let $(\widetilde{U_i}, \Gamma_i)$, $(i = 1, 2)$ be a pair of local models. If there is an injective homomorphism $\phi : \Gamma_1 \to \Gamma_2$ together with a smooth ϕ-equivariant embedding $\tilde{f} : \widetilde{U_1} \to \widetilde{U_2}$, that is, $f(\gamma \cdot \tilde{u}) = \phi(\gamma) \cdot f(\tilde{u})$, for all $\gamma \in \Gamma_1, \tilde{u} \in \widetilde{U_1}$, then the induced map $f : U_1 \to U_2$ on quotient spaces is called an **embedding***, and $(\widetilde{U_1}, \Gamma_1) \to (\widetilde{U_2}, \Gamma_2)$ is also called an embedding.*

Similar to the definition of manifolds, an **n-dimensional (smooth) orbifold**, denoted by $\mathcal{O}$, is a second-countable, Hausdorff topological space $|\mathcal{O}|$, called the underlying topological space of $\mathcal{O}$, together with an **equivalence class of n-dimensional orbifold atlases** defined in the following manner.

For each point $p \in |\mathcal{O}|$, there is a local model $(\widetilde{U}_p, \Gamma_p)$ of dimension n, such that the projection $\pi_p : \widetilde{U}_p \to U_p := \widetilde{U}_p/\Gamma_p$ gives an open neighborhood $U_p \subset |\mathcal{O}|$ of p.

An *n-dimensional orbifold atlas* is thus a collection of local charts $\mathcal{A} := \{U_\alpha\}_\alpha$ such that the neighborhoods $U_\alpha \in \mathcal{A}$ cover $\mathcal{O}$ and for any $p \in U_\alpha \cap U_\beta$, there is a local chart $U_\gamma \in \mathcal{A}$ with $p \in U_\gamma \subset U_\alpha \cap U_\beta$ and embeddings $(\widetilde{U_\gamma}, \Gamma_\gamma) \to (\widetilde{U_\alpha}, \Gamma_\alpha)$ and $(\widetilde{U_\gamma}, \Gamma_\gamma) \to (\widetilde{U_\beta}, \Gamma_\beta)$.

Definition 2-43 (**Torus Orbifold**)**.** *A* torus orbifold*, $\mathcal{O}$, is a $2n$-dimensional, closed, orientable orbifold with an effective smooth action of an n-dimensional torus T such that $\mathcal{O}^T \neq \emptyset$.*

Let $p \in \mathcal{O}$ be a point. We define the **stratum** containing p to be the connected component of the set

$$\{q \in \mathcal{O} | T_q \text{ has the same identity component as } T_p\}.$$

The projection of the closure of a stratum is an *orbifold face* in $\mathcal{O}/T$. Here are some properties ([15]) of orbifolds.

Lemma 2-44. *The fixed-point set of a torus orbifold $\mathcal{O}^{2n}$ consists of finitely many isolated points. Hence $H^{odd}(\mathcal{O}; \mathbb{Q}) = 0$ if $\mathcal{O}$ is simply connected and rationally elliptic (see Definition 2-64).*

Notice that the conclusion $H^{odd}(\mathcal{O}; \mathbb{Q}) = 0$ is weaker than $H^{odd}(\mathcal{O}, \mathbb{Z}) = 0$, which appeared in the last section concerning torus manifolds. (Compare to, say, Theorem 2-40.)

Lemma 2-45. *Let $\mathcal{O}^{2n}$ be a torus orbifold with $H^{odd}(\mathcal{O}; \mathbb{Q}) = 0$. Fix $p \in \mathcal{O}$ and let O_p be the closure of the stratum containing p in $\mathcal{O}$. Then:*

- *codim$(O_p) = 2\dim(T_p)$, and O_p is a torus orbifold with $H^{odd}(O_p; \mathbb{Q}) = 0$.*

- *p lies in the closures of exactly* $\dim(T_p)$ *strata of codimension* 2. *Equivalently, a point $p^* \in \mathcal{O}/T$ in the (relative) interior of a face of codimension k lies in exactly k faces of codimension one in $\mathcal{O}/T$.*

Following tradition, a codimension one orbifold face is called a *facet*, and a one dimensional orbifold face is called an *edge*. A zero dimensional orbifold face is called a *vertex*.

Proposition 2-46 ([15])**.** *Let $\mathcal{O}$ be a simply connected, rationally-elliptic torus orbifold. Then the face poset of $\mathcal{O}/T$ satisfies:*

1. *Each vertex-edge graph of each face is connected.*
2. *Each face of $\mathcal{O}/T$ contains at least one vertex.*
3. *Each face of $\mathcal{O}/T$ of codimension k is contained in exactly k faces of codimension* 1.
4. *Each one-dimensional face of $\mathcal{O}/T$ contains exactly two fixed points of the T action.*
5. *Every two-dimensional face of $\mathcal{O}/T$ contains at most four vertices.*

Remark:

As indicated in [15], the properties established in this proposition are precisely those required to prove that the face poset of $\mathcal{O}/T$ is combinatorially equivalent to the face poset in Equation 2.6 of Proposition 2-39. More recently, it was shown in [12] that the orbit space of a simply connected non-negatively curved manifold with isotropy maximal action has the same combinatorial structure as the one given by Equation 2.6.

2.8. Alexandrov Spaces with Curvature Bounded from Below

For the content of this section we restrict our notion to non-negatively curved Alexandrov spaces, however many definitions here make sense in a more general setting. We refer to Burago, Burago and Ivanov [5] and Burago, Gromov and Perelman [4], also Shiohama [41] for further details.

A *length space* (X, d) is a complete, locally compact metric space X with distance function d, such that for any distinct points $a, b \in X$ there exists a point $c \neq a$ and $c \neq b$ in X such that $d(a, c) + d(c, b) = d(a, b)$. Since X is complete, by iterating the previous procedure, we obtain a curve segment $\gamma : [0, d(a, b)] \to X$, joining a and b, such that the length of γ is equal to $d(a, b)$. Thus a length space can be alternatively defined as a locally compact and complete metric space such that there exists for every point $a, b \in X$ a curve joining a and b whose length realizes $d(a, b)$. Such a curve is called a **geodesic**.

In our case, let M be a complete, connected Riemannian manifold with non-negative sectional curvature and G be a compact Lie group acting isometrically on M. We can equip the orbit space $X = M/G$ with the orbital distance metric induced from M. The distance between $\overline{p}$ and $\overline{q}$ in X is the distance between the orbits $G(p)$ and $G(q)$ as subsets of M. In this way, the orbit space X becomes a length space.

The definition of the dimension of a general length space is technical. However, in our applications the dimension of M/G is clear without ambiguity. Namely, $\dim(M/G) = \dim(M) - \dim(G(x))$, where $G(x)$ is a principal orbit. Now we want to define curvature in a comparison sense.

Let $p \in X$ and let $\gamma : [0, T] \to X$ be a geodesic parametrized by arc-length. Form the *Euclidean comparison* of p and γ in $\mathbb{R}^2$ by choosing a point $\bar{p}$ and a straight segment γ_0 parametrized by arclength with end points $\bar{a}$ and $\bar{b}$ in $\mathbb{R}^2$such that

$$|\bar{a}\bar{b}| = T; \quad |\bar{p}\bar{a}| = d(p, \gamma(0)); \quad |\bar{p}\bar{b}| = d(p, \gamma(T)),$$

where $|\cdot|$ is the Euclidean norm. We will denote the triangle $\triangle\bar{a}\bar{p}\bar{b}$ in $\mathbb{R}^2$ as $\widetilde{\triangle}apb$ for convenience. Then (X,d) is said to be **non-negatively curved**, sometimes denoted as $\mathrm{Curv}(X) \geq 0$, if every point in X has a neighborhood such that whenever p and γ lie in this neighborhood, the following is satisfied:

$$|\bar{p}\gamma_0(t)| \leq d(p,\gamma(t)) \quad \forall t \in [0,T].$$

A **non-negatively curved Alexandrov space** is a finite dimensional non-negatively curved length space. One can define a general Alexandrov space with curvature bounded below by an arbitrary constant $k \in \mathbb{R}$ by replacing the comparison space $\mathbb{R}^2$ with the two-dimensional complete simply connected Riemannian manifold X_k with constant curvature k. In particular when k is positive, the comparison space is a 2-sphere with radius $1/\sqrt{k}$. If M is a Riemannian manifold with non-negative sectional curvature and G is a group of isometry, then in general M/G is an Alexandrov space with $\mathrm{Curv} \geq 0$.

Now we move on to some important definitions in Alexandrov geometry. Let X be an Alexandrov space with $\mathrm{Curv}(X) \geq 0$. Let $c_1, c_2 : [0,\epsilon) \to X$ $(\epsilon > 0)$ be two geodesics in X such that $p := c_1(0) = c_2(0)$. Then for each fixed $t \in [0,\epsilon)$ in the common domain of c_1 and c_2, we can form a comparison triangle $\widetilde{\triangle}c_1(t)pc_2(t)$ in $\mathbb{R}^2$. The Euclidean angle at p of $\widetilde{\triangle}c_1(t)pc_2(t)$ is denoted as $\widetilde{\measuredangle}c_1(t)pc_2(t)$. That is,

$$\widetilde{\measuredangle}c_1(t)pc_2(t) = \cos^{-1}\left(\frac{d(p,c_1(t))^2 + d(p,c_2(t))^2 - d(c_1(t),c_2(t))^2}{2d(p,c_1(t))d(p,c_2(t))}\right).$$

One can define **the angle between c_1 and c_2** to be

$$\measuredangle(c_1,c_2) := \lim_{t\to\infty} \widetilde{\measuredangle}c_1(t)pc_2(t),$$

if the limit exists. In a non-negatively curved Alexandrov space, one can show that the angle $\measuredangle(c_1,c_2)$ is well-defined. In fact, this is an immediate consequence of the following equivalent definition of a non-negatively curved Alexandrov space (see [5] for a proof of the equivalence of the definitions).

Definition 2-47 ("Hinge" definition of Curv ≥ 0)**.** *Let α and β be two geodesics (parametrized by arclength) starting at the same point p, let*

$$\theta(x, y) := \widetilde{\measuredangle}\alpha(x)p\beta(y)$$

be the angle at $\bar{p}$ in a comparison triangle for $\triangle\alpha(x)p\beta(y)$.

An Alexandrov space X is non-negatively curved if it can be covered by neighborhoods such that, for two any shortest segments α and β contained in this neighborhood (and starting from the same point p), upon fixing one of the variables x or y, the corresponding function $\theta(x, y)$ is non-increasing in the free variable.

Moreover, the notion of angles satisfies the triangle inequality:

Theorem 2-48. *Let X be a non-negatively curved Alexandrov space, and consider any three curves γ_1, γ_2 and γ_3 starting from the same point. Then the following holds,*

$$\measuredangle(\gamma_1, \gamma_2) \leq \measuredangle(\gamma_2, \gamma_3) + \measuredangle(\gamma_1, \gamma_3).$$

Now it is the time to introduce the *space of directions*, which is an important notion in this context.

Definition 2-49. (Space of directions) *Let X be a non-negatively curved Alexandrov space, let $\gamma : [0, \epsilon) \to X$ and $c : [0, \epsilon) \to X$ with $c(0) = p$ be geodesics starting from p. Then we say c and γ have the same* **direction** *if and only if $\measuredangle(\gamma, c) = 0$. This defines an equivalence relation on geodesics emanating from p because of the triangle inequality. The direction of γ is defined as the equivalence class of geodesics whose angle with γ is zero. Thus the set of geodesic directions at p is a metric space with distance function induced by $\measuredangle$ (see Theorem 2-48). The* **space of directions at p***, denoted as $\mathbf{\Sigma_p}$, is defined as the completion of the space of geodesic directions at $p \in X$.*

Note that the space of direction Σ_p of any point $p \in X$ is again a non-negatively curved Alexandrov space with $\dim(\Sigma_p) = \dim X - 1$. The fact that a compact Alexandrov space of dimension 1 is either a circle (without boundary) or an interval with two boundary points allows us to inductively define the boundary ∂X of an Alexandrov space X. Namely, ∂X is the set of points $p \in X$ whose space of directions Σ_p has non empty boundary. The space of directions of an n-dimensional Alexandrov space is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1 and in fact is isometric to S^{n-1} at almost every point:

Proposition 2-50 ([19]). *Let M have non-negative sectional curvature and G be a group of isometries. Let $\pi : M \to M/G$ be the orbit map. The space of directions Σ_x at $x \in M/G$ is isometric to $S_p^{\perp}/G_p$, where $p \in M$ is a point with $\pi(p) = x$ and $S_p^{\perp}$ is the unit normal sphere in $T_p^{\perp}(G(p))$.*

One can also look at subsets of Alexandrov spaces, sometimes they can be Alexandrov spaces as well.

Theorem 2-51. *A closed and connected locally convex (or convex or totally convex) subset C of an Alexandrov space A equipped with the induced intrinsic metric is an Alexandrov space with the same lower curvature bound.*

Definition 2-52 (Critical Point). *Let X be an Alexandrov space with curvature bounded from below, and let $E \subset X$ be a closed subset. Consider the distance function $dist_E : X \setminus E \to \mathbb{R}$ defined by $dist_E(p) = dist(E, p)$. A point $q \in X \setminus E$ is said to be a* critical point *for $dist_E$ if for any vector $v \in T_qX$ there is a distance minimizing geodesic c from q to X satisfying*

$$\measuredangle(v, \dot{c}(0)) \leq \pi/2.$$

A non-critical point is called a regular point.

A basic tool for our arguments is the following Soul Theorem adopted to Alexandrov geometry from [37].

Lemma 2-53. *Let A be an Alexandrov space with distance function ρ and* $\operatorname{curv}(A) \geq 0$ *and non-empty boundary $C = \partial A$. Then the distance function to the boundary, or any component of it, is concave, that is, for any $x, y \in A$, let γ be a unit speed geodesic segment joining x and y, then for any $a, b \geq 0$ and $a + b = 1$, we have:*

$$dist(C, \gamma(at_1 + bt_2)) \geq a \cdot dist(C, \gamma(t_1)) + b \cdot dist(C, \gamma(t_2)).$$

2.9. Positively and Non-negatively curved manifolds with large symmetry

Throughout this section, let M be a complete Riemannian manifold with non-negative sectional curvature. It was proved by Cheeger-Gromoll [6] that a complete manifold with non-negative sectional curvature deformation retracts to a closed, totally geodesic submanifold (the soul) such that the manifold is diffeomorphic to the normal bundle over the soul. And in the compact case, we have the Splitting Theorem of Cheeger-Gromoll.

Theorem 2-54 (**Splitting Theorem** ([6])). *Let M be a compact manifold of non-negative sectional curvature. Then $\pi_1(M)$ contains a finite normal subgroup ψ such that $\pi_1((M)/\psi$ is a finite group extended by $\mathbb{Z}^k$, and $\tilde{M}$, the universal covering of M, splits isometrically as $\bar{M} \times \mathbb{R}^k$, where $\bar{M}$ is compact.*

Recall an early, very general result on the structure of the group of isometries of a Riemannian manifold (see for example [32]).

Theorem 2-55. *The group of isometries $Isom(M)$ of a compact Riemannian manifold M is a Lie group with respect to the compact-open topology. For each $x \in M$, the isotropy subgroup $Isom_x(M)$ is compact. If M is compact, $Isom(M)$ is also compact.*

The fixed point set of isometries has a very nice structure:

Theorem 2-56 (see for example [32]). *Let M be a Riemannian manifold and A be any set of isometries of M. Let F be the set of points of M which are fixed by all elements of A. Then each connected component of F is a totally geodesic submanifold of M.*

We can define the dimension of a fixed-point set to be the maximal dimension of its connected components. One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space M/G, also called the **cohomogeneity of the action**. The cohomogeneity of the action is clearly constrained by the dimension of the fixed point set M^G of G in M. In fact, $\dim(M/G) \geq \dim(M^G)+1$ for any non-trivial, non-transitive action. In light of this, the **fixed-point cohomogeneity of an action**, denoted by $\operatorname{cohomfix}(M;G)$, is defined by

$$\operatorname{cohomfix}(M;G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$

A manifold with fixed-point cohomogeneity 0 is also called a **G-fixed point homogeneous manifold**. By Proposition 2-29, if a T^k torus action on M^n is isotropy maximal or almost isotropy maximal, then M is S^1-fixed point homogenous. If a non-negatively curved S^1-fixed point homogeneous Riemannian manifold has two maximal fixed point components, we have the well-known Double Soul Theorem.

Theorem 2-57 (**Double Soul Theorem** ([40])). *Let M be a non-negatively curved S^1-fixed point homogeneous Riemannian manifold. If M^{S^1} contains at least two components F and N with maximal dimension, one of which is compact, then F and N are isometric and M is diffeomorphic to an S^2-bundle over F.*

The symmetry rank, symrank(**M**), of a Riemannian manifold M is defined as the rank of its isometry group $\operatorname{Isom}(M)$, that is, the maximal dimension of a torus $T \subset \operatorname{Isom}(M)$, where the T action is effective on M.

Theorem 2-58 (**Maximal Symmetry Rank Theorem** ([21])). *Let M^n be a closed, connected Riemannian manifold with positive sectional curvature, then* *(i)* $\operatorname{symrank}(M) \leq \lfloor \frac{n+1}{2} \rfloor$*; and*

(ii) If equality holds in (i), then M^n *is diffeomorphic to one of the following:* $S^n, \mathbb{R}P^n, L^n_{p,q}$ *or* $\mathbb{C}P^m$ *(for* $2m = n$*).*

In the proof of Theorem 2-58, there is an important structure result which shows that positively curved manifolds with maximal symmetry rank are S^1-fixed point homogenous.

Lemma 2-59 ([21])**.** *If a* T *action on* M *reaches maximal symmetry rank, then there exists* $S^1 \subset T$ *such that*

- *there is a unique component* $F \subset \operatorname{Fix}(M, S^1)$ *with codim*$(F) = 2$*;*
- *there is a unique orbit* N *at maximal distance from* F *such that* S^1 *acts freely on* $M \setminus (F \cup N)$*.*

Recall that a torus manifold is an example of an S^1-fixed point homogeneous manifold, indeed, of a nested S^1-fixed point homogeneous manifold. Fixed point homogeneous manifolds of positive curvature were classified in Grove-Searle [22].

More recently, Spindeler [42] has generalized Lemma 2-59 for G-fixed point homogeneous manifolds to the class of closed, simply connected, non-negatively curved manifolds.

Theorem 2-60 (**Spindeler's Theorem** [42])**.** *Assume that* G *acts fixed point homogeneously on a closed, non-negatively curved Riemannian manifold* M*. Let* F *be a fixed point component of maximal dimension. Then there exists a smooth submanifold* N *of* M*, without boundary, such that* M *is* G*-diffeomorphic to the normal disk bundles* $D(F)$ *and* $D(N)$ *of* F *and* N *glued together along their common boundaries;*

$$M = D(F) \cup_\partial D(N).$$

Further, N *is* G*-invariant and contains all singularities of* M *up to* F*.*

Remark 2-61. *In [42], Spindeler also showed that the submanifold N is invariant under $H = \mathrm{Isom}_F M$, the subgroup of* $\mathrm{Isom}(M)$ *which leaves F invariant. Then clearly $G \subset H$ and N will be invariant under H due to the fact that $N/G \subset M/G$ is at maximal distance from F. Then by Lemma 3.3 of [42], the gluing map $\partial : \partial D(F) \to \partial D(N)$ can be constructed to be H-equivariant.*

When M is simply connected, the codimension of N is at least two.

Proposition 2-62 ([42])**.** *Let M, N and F be as in Theorem 2-60. If in addition M is simply connected, then codim$(N) \geq 2$.*

We will also use the following theorem which discusses the fundamental groups of E, F, and N.

Theorem 2-63 ([11])**.** *Let M^n be a simply connected manifold that decomposes as the union of two disk bundles as follows:*

$$M^n = D^{k_1}(N_1) \cup_E D^{k_2}(N_2).$$

If $k_1 = k_2 = 2$, then $\pi_1(N_1)$ and $\pi_1(N_2)$ are cyclic groups.

Moreover,

1. *If $k_i = 2$, $\pi_2(N_i) = 0$, for $i = 1, 2$ and $\pi_1(N_i)$ is infinite for some $i \in \{1, 2\}$, then $\pi_1(E) \cong \mathbb{Z}^2$.*

2. *If $k_i \geq 3$, for some $i \in \{1, 2\}$, then $\pi_1(E) \cong \pi_1(N_i)$.*

2.10. Rationally Elliptic manifolds and Nonnegative Curvature

Definition 2-64. *A closed manifold M is* **rationally Ω-elliptic** *if the rational homotopy groups of the loop space ΩM satisfy:*

$$\Sigma_k \dim(\pi_k(\Omega M) \otimes \mathbb{Q}) < \infty.$$

M is called **rationally elliptic** *if it is both rationally Ω-elliptic and simply connected.*

One of the most interesting conjectures is the so-called Bott Conjecture.

Bott Conjecture. *A simply connected manifold of non-negative sectional curvature is rationally elliptic.*

If a non-negatively curved Riemannian manifold is isotropy maximal, then the Bott conjecture holds (see [42] and [15]).

Theorem 2-65 ([15])**.** *Let M be a closed, non-negatively curved Riemannian manifold admitting an effective, isometric, isotropy-maximal torus action. Then M is rationally Ω-elliptic.*

Spindeler also mentioned that from Corollary 6.1 of [23] it follows that a simply connected manifold which decomposes as a union of two disk bundles is rationally Ω-elliptic if and only if the boundary of one of the two disk bundles is rationally Ω-elliptic. Therefore, as a consequence of Theorem 2-60, we have

Theorem 2-66 ([23] and [42])**.** *Let M be a closed, simply connected, non-negatively curved, fixed point homogeneous manifold and let F be a characteristic submanifold. Then M is rationally Ω-elliptic if and only if F is rationally Ω-elliptic.*

The following corollary is parallel to Theorem 2-65 in [15], whose proof can be adopted with simple modifications.

Proposition 2-67. *Let M be a closed, non-negatively curved Riemannian manifold admitting an effective, isometric, almost isotropy-maximal torus action. Then M is rationally Ω-elliptic.*

Proof. Let T be the almost isotropy-maximal torus isometry on M. The proof will proceed by induction on the dimension $d := \dim(M/T)$ and no longer assumes that M is simply connected.

When $d = 0$, M is simply a torus, and hence rationally-Ω elliptic. Suppose now that every non-negatively curved, closed manifold admitting an isometric, almost isotropy-maximal torus action with an orbit space of dimension $d-1$ is rationally Ω-elliptic. We wish to prove that M is also rationally Ω-elliptic.

One can decompose M using Theorem 2-60, since a manifold with almost isotropy-maximal torus action is fixed point homogenous with respect to some circle subgroup of T. We have:

$$M = D^2(F) \cup_\partial D(N)$$

where F is a characteristic submanifold of M (recall definition 2-28), and we will denote the circle fixing F to be $\lambda_F \subset T$. Since F is a fixed point set, it is totally geodesic, and the induced action of T/λ_F on F is almost isotropy-maximal with an orbit space F/T of dimension $d-1$. Therefore by the induction hypothesis, F is rationally Ω-elliptic. Thus by Theorem 2-66, we know M is rationally Ω-elliptic. $\square$

3. DECOMPOSITION OF ALMOST TORUS MANIFOLD

In this chapter, we discuss the decomposition of a non-negatively curved almost torus manifold using Spindeler's Theorem 2-60. Firstly, in dimension three, using the thesis of Galaz-García [14], in which he classified fixed-point homogeneous 3-manifolds, we obtain the following result.

Proposition 3-1. *A simply connected, almost torus* 3*-manifold,* M^3*, with non-negative sectional curvature is locally standard, and its orbit space* M^3/T^1 *is diffeomorphic (after smoothing the corners) to* D^2*.*

Proof. We consider an isometric, effective action of T^1 on a closed, simply connected, non-negatively curved manifold M^3 such that $M^T \neq \emptyset$. By Theorem 2-11, the quotient space M^3/T^1 is simply connected. Since a compact, simply connected, non-negatively curved 2-dimensional Alexandrov space with non-empty boundary is homeomorphic to D^2, it follows that M^3/T^1 is diffeomorphic (after smoothing the corners) to D^2.

To show that the T^1 action is locally standard, we need to look at the T^1-fixed point homogeneous action on M^3. Using Theorem 2-60 the decomposition of M^3 is given by

$$M^3 = D^2(F) \cup_E D(N^k),$$

where clearly F is one dimensional, that is, $F = S^1$. By Proposition 2-62 we know $k \neq 2$. In fact, we will show that $k = 1$. If $k = 0$, then since N is connected and T^1-invariant, it must be a fixed point. However each component of $\mathrm{Fix}(M^3, T^1)$ has even codimension, so N can not be an isolated fixed point. By Theorem 2-60 the T^1 action is free on $M \setminus (F \cup N)$, a contradiction. Therefore k can only be 1 and thus $N = S^1$. In particular, N can be either of the following

Case (1) a T^1-fixed point component; or

Case (2) a T^1-orbit with a finite, possibly trivial, isotropy group H.

In Case (1), using the Double Soul Theorem (2-57), M is a S^2 bundle over S^1, and thus we obtain a contradiction to the hypothesis that M^3 is simply connected (see also the case 2.2.2.3 in [14]). Thus Case (1) does not occur.

In Case (2), if H is non-trivial, then we are looking at the case 2.2.2.2 in [14], where it is shown that $M^3 \simeq \mathbb{R}P^2 \times S^1$, which is not simply connected, a contradiction. Thus N is a T^1-orbit with trivial isotropy.

At last we want to show that the T^1 action is locally standard. Firstly, suppose $x \in M \setminus F$, then x has trivial isotropy.

Suppose that $x \in F$, then x is a T^1-fixed point. By the Slice Theorem, an open neighborhood of x is equivariantly diffeomorphic to T_xM, which is a standard T^1-representation space, and we are done. Otherwise if $x \in M \setminus F$, then x has principal orbit, and thus the Slice Theorem gives us a invariant neighborhood of x that is T^1-equivariantly diffeomorphic to $T^1 \times \mathbb{R}^2$, where $\mathbb{R}^2$ is a trivial T^1-representation. It is not hard to see that the $T^1 \times \mathbb{R}^2$ can be equivariantly embedded as an open solid torus in a linear T^1-representation space $\mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$.

Therefore, the T^1 action is locally standard on M^3. $\square$

However, if the dimension is larger than three, an almost torus manifold can have a different decomposition in general. Throughout the rest of this section, let M^{2n+1} ($n \geq 2$) be an almost torus manifold with an invariant metric of non-negative curvature. Let T^n denote the effective torus action on M and M^T the set of fixed-points. A connected component S of M^T is a totally geodesic submanifold of even codimension (see Theorem 2-22). At any $x \in S$, since the normal space $T_x^{\perp}S$ is a faithful T^n-representation, we can see that $\dim(T_x^{\perp}S) \geq 2n$ and thus $\dim(S) = 1$. Therefore the set of fixed-points M^T is a disjoint union of finitely many circles.

Since the T^n action on M^{2n+1} is almost isotropy maximal, there exists a *characteristic submanifold* F fixed by some circle subgroup $\lambda(F) \subset T^n$. By Theorem 2-60, we can decompose M as a $\lambda(F)$-fixed point homogenous manifold with non-negative sectional curvature, as the union of two normal disk bundles:

$$M^{2n+1} = D^2(F^{2n-1}) \cup_E D(N), \tag{3.1}$$

where $E := \partial D(F) = \partial D(N)$.

Furthermore, in Theorem 2-60, there are a few facts that only hold under the extra assumption that the group is a torus. Firstly, since a torus is abelian and F is a fixed point component, it turns out that F is invariant under T^n. By Remark 2-61, N is also T^n-invariant and the gluing of boundaries of the two disk bundles is a T^n-diffeomorphism. Thus even though the decomposition of the almost torus manifold M uses only the group $\lambda(F)$, the space $D(F) \cup_E D(N)$ is T^n-equivariantly diffeomorphic to M, such that all submanifolds F, N and the boundary E are T^n-invariant.

The following facts were used in [46] for torus manifolds, and it is not hard to see that they hold for almost torus manifolds as well. We include the proofs here for convenience and completeness.

Proposition 3-2. *Let M, F and N be defined as in Theorem 2-60. Denote by $\pi_X : E \to X$ the projection map for the sphere bundle of E over X, with $X := F, N$. If $x \in F$ is a T^n-fixed point. Then:*

1. *Both π_N and π_F are T^n-equivariant.*

2. *$\pi_F^{-1}(x)$ is a one-dimensional T^n-orbit.*

3. *If $\pi_N \circ \pi_F^{-1}(x)$ is zero dimensional, then $\pi_N \circ \pi_F^{-1}(x)$ is a T^n-fixed point in N.*

4. *If $\pi_N \circ \pi_F^{-1}(x)$ is one dimensional, then $\pi_N \circ \pi_F^{-1}(x)$ is a one-dimensional T^n-orbit on which $\lambda(F)$ acts almost freely, that is, there is a finite subgroup $H_0 \subset \lambda(F)$ such that $\pi_N \circ \pi_F^{-1}(x)$ is of type $\lambda(F)/H_0$.*

Proof. To prove Part (1), note that by Theorem 2-60, with $X = F$ or N, the bundle $D(X) \to X$ is the associated disk bundle of the normal T^n-vector bundle of X in M (see Section 2.3.), where the total space $D(X)$ is a closed T^n-invariant tubular neighborhood of X in M (compare to Theorem 2-15). Therefore by the definition of a T^n-vector bundle, both π_N and π_F are T^n-equivariant. The proofs of Parts (2)-(4) in the proposition are then a direct consequence of Part (1):

The proof of Part (2) follows from the fact that π_F is T^n-equivariant: Pick any $p \in \pi_F^{-1}(x)$, we then have $x = gx = g\pi_F(p) = \pi_F(gp)$. Thus gp is in the fiber $\pi_F^{-1}(x)$ for all $g \in T^n$, and therefore clearly $T^n(p) = \pi_F^{-1}(x)$.

To prove Part (3), suppose $\pi_N \circ \pi_F^{-1}(x) = y \in N$ is a point. Since π_N is T^n-invariant and we already showed that $T^n(p) = \pi_F^{-1}(x)$, it follows that $gy = g\pi_N(p) = \pi_N(gp) = y$ for all $g \in T^n$. Thus y must be a T^n-fixed point.

To prove Part (4), let $B := \pi_N \circ \pi_F^{-1}(x)$. By Part (2), $\pi_F^{-1}(x)$ is a T^n-orbit. Thus B is a T^n-orbit because π_N is T^n-eqivariant. In particular since B is one-dimensional and $\lambda(F)$ acts freely on $\pi_F^{-1}(x)$, the orbit type of B is $\lambda(F)/H_0$, where H_0 is a finite subgroup of $\lambda(F)$. $\square$

One hopes to understand the fundamental groups of F and N, given that M is simply connected. We have the following theorem, upon which many of our future discussions are based:

Theorem 3-3. *Let M^{2n+1} be a simply connected, non-negatively curved almost torus manifold. Let*

$$M \simeq D(F) \cup_\partial D(N).$$

as in Theorem 2-60, where F is a characteristic submanifold fixed by $T_1^1 \subset T^n$. Then the following are true:

1. *If codim$(N) > 2$;, then $\pi_1(F) = 0$ and $\pi_1(N)$ is cyclic;*

2. *If codim*$(N) = 2$ *and* N *is fixed by the same circle subgroup that fixes* F*, then* $\pi_1(F) = \pi_1(N) = 0$*;*

3. *If codim*$(N) = 2$ *and* N *is fixed by some* $T_2^1 \subset T^n$ *such that* $T_2^1 \neq T_1^1$*, then* $\pi_1(F) = 0$*;*

4. *If codim*$(N) = 2$ *and* N *is not fixed by any circle subgroup of* T^n*, then* $\pi_1(F)$ *and* $\pi_1(N)$ *are cyclic;*

5. *In the cases when* $\pi_1(F) \cong 0$ *as in Part (1)-(3), the fundamental group* $\pi_1(N)$ *is generated by any* T_1^1 *orbit in* N*, where* T_1^1 *is the subgroup fixing* F*.*

Proof. Note that if M is simply connected, then the submanifold N in the decomposition has codimension $\mathrm{codim}(N) \geq 2$, by Lemma 2-62.

Proof of Part (1)

To see Part (1), notice that since $D(F)$ and $D(N)$ are homotopy equivalent to F and N respectively, we have $\pi_1(F) \cong \pi_1(D(F)) \cong \pi_1(M \setminus D(N)) \cong \pi_1(M \setminus N)$. If $\mathrm{codim}(N) > 2$, then by transversality, we also have $\pi_1(M \setminus N) \cong \pi_1(M)$. Therefore F is simply connected if $\pi_1(M) \cong 0$. By looking at the two long exact sequences of homotopy groups induced by fibrations $S^1 \hookrightarrow E \to F$ and $S^k \hookrightarrow E \to N$ with $k \geq 2$, we know $\pi_1(N) \cong \pi_1(E)$ is cyclic. This finishes the proof of Part (1).

Proof of Part (2)

As for Part (2), if $\mathrm{codim}(N) = 2$ and N is fixed by the same T_1^1 that fixes F, then we can apply the Double Soul Theorem (2-57), which concludes that M is an S^2-bundle over F, and N is isometric to F. Thus by looking at the induced long exact sequence of homotopy groups, we have $\pi_1(N) \cong \pi_1(F) \cong \pi_1(M) \cong 0$. This finishes the proof of Part (2).

Proof of Part (3)

The proof of Part (3) is very similar to the proof of Proposition 3.6 of [18](also compare to Theorem 3.35 of [42]). Assume $\mathrm{codim}(N) = 2$ and that N is fixed by some

$T^1_2 \subset T^n$ that is not T^1_1.

Denote $T^2 := T^1_1 \oplus T^1_2$, a 2-torus in T^n. If p_0 is fixed by T^n, then $p_0 \in M^{T^2}$. Since by 2-60 the T^1_1 action is free on E, we have $E/T^1_1 \simeq F$. As for the T^1_2 action, since the T^n action is effective, T^1_2 must act freely on the circle fiber of $E \to N$. Thus we also have $E/T^1_2 \simeq N$.

Consider the projections $f_1 : E \to E/T^1_1 \simeq F$ and $f_2 : E \to E/T^1_2 \simeq N$. Notice that in fact $f_1 = \pi_F$ and $f_2 = \pi_N$, we use this alternative notation for this part of the proof. From the long exact homotopy sequences of these fibrations we obtain:

$$\cdots \pi_1(T^1_1) \xrightarrow{\pi_1(i_1)} \pi_1(E) \xrightarrow{\pi_1(f_1)} \pi_1(F) \to 1$$

and

$$\cdots \pi_1(T^1_2) \xrightarrow{\pi_1(i_2)} \pi_1(E) \xrightarrow{\pi_1(f_2)} \pi_1(N) \to 1\,,$$

where the maps i_1 and i_2 are the inclusions of the fibers over a given base point. Set $U_k = \pi_1(i_k)(\pi_1(T^1_k))$ for $k = 1, 2$. So $\pi_1(F) \cong \pi_1(E)/U_1$ and $\pi_1(N) \cong \pi_1(E)/U_2$ and we have a commutative diagram:

$$\begin{tikzcd}
\pi_1(E) \arrow[r, "\pi_1(f_2)"] \arrow[d, "\pi_1(f_1)"] & \pi_1(N) \arrow[d, "h_2"] \\
\pi_1(F) \arrow[r, "h_1"] & \pi_1(E)/U_1U_2
\end{tikzcd}$$

Here the lower map is given by $h_1 : \pi_1(F) \cong \pi_1(E)/U_1 \to \pi_1(E)/U_1U_2$, and analogously for the map on the right.

Then since $E = \partial D(F)$ is connected, by Seifert-van Kampen there exists a unique homomorphism $h : \pi_1(M) \to \pi_1(E)/U_1U_2$ making the following diagram commute:

$$\begin{tikzcd}
\pi_1(E) \arrow[r, "\pi_1(f_2)"] \arrow[d, "\pi_1(f_1)"] & \pi_1(N) \arrow[d] \arrow[ddr, bend left, "h_2"] & \\
\pi_1(F) \arrow[r] \arrow[drr, bend right, "h_1"] & \pi_1(M) \arrow[dr, "h"] & \\
& & \pi_1(E)/U_1U_2
\end{tikzcd}$$

Since h_1 and h_2 are surjective, the induced map h is also surjective. Since $\pi_1(M) \cong 0$, we have $\pi_1(E) \cong U_1U_2$. Hence $\pi_1(E)$ is generated by the orbit $T_1^1(q)$ and $T_2^1(q)$ for a given point $q \in E$. Therefore the orbit map $\tau_q : T^2 \to E$, $g \mapsto g \cdot q$ induces a surjection $\pi_1(\tau_q) : \pi_1(T^2) \to \pi_1(E)$. Pick a point $q_0 \in E$ such that $\pi_F(q_0) = x$ is fixed by T^n. Then the map $\pi_F \circ \tau_{q_0} : T^2 \to F$ induces a surjection $\pi_1(T^2) \to \pi_1(F)$ since both maps induce surjections of fundamental groups. Then since x is a fixed point, the image of $\pi_F \circ \tau_{q_0}$ is a single point. This implies that $\pi_1(F)$ is trivial. This completes the proof of Part (3).

Proof of (4)

Part (4) follows from the proof of Part (3), or follows directly from Theorem 2-63.

Proof of (5)

To show Part (5), notice that if F is simply connected, then it follows from the long exact homotopy sequence of the fibration

$$S^1 \overset{i}{\hookrightarrow} E \xrightarrow{\pi_F} F$$

that $\pi_1(i) : \pi_1(S^1) \cong \mathbb{Z} \to \pi_1(E)$ is onto. So $\pi_1(E)$ is cyclic. Moreover $\pi_1(E)$ is generated by the inclusion of a fiber of π_F.

Then it follows from the long exact homotopy sequence of the fibration $\pi_N : E \to N$ that $\pi_1(E)$ is mapped onto $\pi_1(N)$, and so $\pi_1(N)$ is generated by the inclusion of a T^n-orbit of type $\lambda(F)$ from E to N, by the loop:

$$\gamma_{x_0} : S^1 = \lambda(F) \to N,$$
$$z \mapsto zx_0,$$

where $x_0 \in N$ is any base point of N. One can also see that the fundamental group of N is in fact generated by a T^n-orbit $\pi_N(\pi_F^{-1}(x))$ (see 3-2) for any point $x \in F$. □

We will now see that the characteristic submanifold F is an almost torus manifold with non-negative sectional curvature. First note that F is totally geodesic with codim$(F) = 2$, and admits an effective $T^{n-1} \cong T^n/\lambda(F)$ action such that $F^{T^{n-1}} = M^T \cap F \neq \emptyset$. In the case of non-negatively curved torus manifolds, by [46] we know a characteristic submanifold is always simply connected, which is a very convenient property in the inductive proof of Theorem 2-40. However, in the case of almost torus manifolds, in general a characteristic submanifold F need not be simply connected. Nevertheless, according to Theorem 3-3, at least we know that the fundamental group of F is cyclic. This turns out to be sufficient for us, as we can lift the action onto the universal cover of F.

Proposition 3-4. *Let M be an almost torus manifold with nontrivial cyclic fundamental group and T the corresponding torus. Let $\widetilde{M}$ be the universal cover of M, then the T action on M lifts to a T action on $\widetilde{M}$. If M has a T-invariant metric of non-negative sectional curvature, then $\widetilde{M}$ has a T-invariant metric of non-negative sectional curvature.*

Moreover,

1. *If $\pi_1(M)$ is finite, then $\widetilde{M}$ is a closed, simply connected almost torus manifold with the lifted T^n action.*

2. *If $\pi_1(M)$ is infinite, then $\widetilde{M} = M_0 \times \mathbb{R}$ where M_0 is a compact and simply connected torus manifold with the lifted T^n action. The lifted T^n action on the $\mathbb{R}$ factor is trivial.*

Moreover, if $(\widetilde{M}, T)$ is locally standard, then (M, T) is locally standard.

Proof. Recall that the group of deck transformations Γ of $\widetilde{M}$ is isomorphic to $\pi_1(M)$. Also recall that if M is a Riemannian manifold, then the covering space $\widetilde{M}$ is also a Riemannian manifold on which Γ acts isometrically.

Let g be a T-invariant metric on M with non-negative sectional curvature and $p : \widetilde{M} \to M$ be the universal covering map. Let the pull back metric on $\widetilde{M}$ be $\widetilde{g} = p^*g$.

Using O'Neill's equation, we see that

$$K_g(\sigma) = K_{\widetilde{g}}(\widetilde{\sigma}) + 3/4\|[\widetilde{V}, \widetilde{W}]^{\perp}\|^2,$$

where σ is a plane generated by a pair of orthonormal tangent vectors V, W of M and $\widetilde{\sigma}$ is the plane generated by their horizontal lifts $\widetilde{V}, \widetilde{W}$ of tangent vectors of $\widetilde{M}$. In the case of a covering map, since p is a local diffeomorphism, $\dim(\tilde{M}) = \dim(M)$, that is, the dimension of the vertical space is zero. Therefore $[\widetilde{V}, \widetilde{W}]^{\perp} = 0$ identically and thus $\widetilde{M}$ is a manifold with non-negative sectional curvature.

Let $\widetilde{G}$ denote the covering group of T arising from Theorem 2-2, which gives us an action $\widetilde{G} \times \widetilde{M} \to \widetilde{M}$ covering the action $T \times M \to M$. It is not hard to see that $\widetilde{G}$ is invariant under the pullback metric $\widetilde{g}$, that is, $\widetilde{G}$ acts on $\widetilde{M}$ isometrically . To see this, notice that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{M} & \longrightarrow & \widetilde{M} \\ \Big\downarrow{\scriptstyle \varphi\times p} & & \Big\downarrow{\scriptstyle p} \\ T \times M & \longrightarrow & M \end{array}$$

where we use $\varphi : \widetilde{G} \to T$ to denote the covering map. Let $\widetilde{h} \in \widetilde{G}$ and $h \in T$ with $\varphi(\widetilde{h}) = h$. For any tangent vectors V, W of $\widetilde{M}$, a direct computation yields

$$\begin{aligned} \widetilde{g}(d\widetilde{h}V, d\widetilde{h}W) &= (p^*g)(d\widetilde{h}V, d\widetilde{h}W) \\ &= g(dp(d\widetilde{h}V), dp(d\widetilde{h}W)) \\ &= g(dh(dpV), dh(dpW)) \\ &= g(dpV, dpW) \\ &= \widetilde{g}(V, W). \end{aligned}$$

Thus $\widetilde{G}$ acts by isometries on $(\widetilde{M}, \widetilde{g})$.

Now we can finish the proof by considering whether the fundamental group of M is finite or infinite. We consider first the case where it is finite.

Proof of Part 1: $\pi_1(M)$ is finite.

If $\pi_1(M)$ is finite, then the projection $p : \widetilde{M} \to M$ has a finite fiber since the lifting correspondence $\pi_1(M) \to p^{-1}(x)$ is surjective. Thus since M is a compact manifold, it turns out that $\widetilde{M}$ is also a compact manifold. To see this, let $\{\widetilde{U_\beta}\}_\beta$ be an arbitrary open covering of $\widetilde{M}$. For any point $x \in M$, since $p^{-1}(x)$ is finite, we can choose an open neighborhood O_x of x in M such that $\{V_i^x : i = 1, \ldots, i_k\}$ is a finite partition of $p^{-1}(O_x)$ into slices (that is, all $V_i^x \simeq O_x$ and are pairwise disjoint) and moreover, we can require that each V_i^x is contained in some $\widetilde{U_\beta}$. Since M is compact, the open cover $\{O_x\}_{x \in M}$ of M has a finite subcover $\{O_j\}$. Thus $\{p^{-1}(O_j)\}$ is a finite covering of $\widetilde{M}$. Since by construction each $p^{-1}(O_j)$ is contained in finitely many $\widetilde{U_\beta}$, this implies that there is a finite subset of $\{\widetilde{U_\beta}\}_\beta$ that covers $\widetilde{M}$. Furthermore, since each O_j can be chosen as a chart of the manifold M, the argument also proves that $\widetilde{M}$ is a manifold as well.

We now know that the universal cover $\widetilde{M}$ is a compact manifold. Applying Theorem 2-2 to the T action on M, since the fixed point set M^T is non-empty, we know that T must be lifted to another torus $\widetilde{G} \cong T$, covering the T action on M. Moreover, Theorem 2-2 implies that $\mathrm{Fix}(\widetilde{M}, T) \neq \emptyset$. Therefore in this case $\widetilde{M}$ is a closed, simply connected, almost torus manifold. We now consider the case where the fundamental group of M is infinite.

Proof of Part 2: $\pi_1(M)$ is infinite.

In this case, since we assume that $\pi_1(M)$ is cyclic, in fact $\pi_1(M) \cong \mathbb{Z}$. By Cheeger-Gromoll's Splitting Theorem (see 2-54), the universal covering space $\widetilde{M}$ of M is a complete, non-compact, non-negatively curved manifold which has an isometric splitting into the Riemannian product $M_0 \times \mathbb{R}$, where M_0 is a compact, simply connected, non-negatively curved manifold and $\mathbb{R}$ has the flat metric of a straight line. Moreover, by Theorem 1 of Hano ([24]) (also, compare to Corollary 6.2 in [6]) the group of isometries $\mathrm{Isom}(\widetilde{M})$ of $\widetilde{M}$

also decomposes into a direct product

$$\mathrm{Isom}(\widetilde{M}) = \mathrm{Isom}(M_0) \times \mathrm{Isom}(\mathbb{R}).$$

By Theorem 2-2, the action of T on M can be lifted to an action of a Lie group $\widetilde{G}$ on $\widetilde{M}$. In particular, since the T action has a fixed point in M, the theorem implies that $\widetilde{G}$ can be chosen to be a torus isomorphic to T such that $\mathrm{Fix}(\widetilde{M}, \widetilde{G}) = \mathrm{Fix}(\widetilde{M}, T) \neq \emptyset$. Thus M_0 is a simply connected torus manifold. Furthermore, since $\mathrm{Isom}(\mathbb{R}) \cong \mathbb{R}$ is the group of rigid motion on $\mathbb{R}$, a proper connected subgroup of $\mathrm{Isom}(\mathbb{R})$ can only be the trivial group. Thus the torus $T \subset \mathrm{Isom}(M_0)$, inducing the trivial action on the $\mathbb{R}$ factor.

As a conclusion, the proof of Part 2 is complete.

Finally, suppose that the $\widetilde{G}$ action on $\widetilde{M}$ is locally standard. Let $p : \widetilde{M} \to M$ and $\varphi : \widetilde{G} \to T$ be the covering map. For any $\widetilde{x} \in \widetilde{M}$ and $\widetilde{g} \in \widetilde{G}$:

$$p(\widetilde{g} \cdot \widetilde{x}) = \varphi(\widetilde{g}) \cdot p(\widetilde{x}).$$

Let $x \in M$. If $\widetilde{U}$ is an invariant neighborhood of a point $\widetilde{x} \in p^{-1}(x)$, without loss of generality we can assume that $\widetilde{U}$ is a tubular neighborhood and $S_{\widetilde{x}} \subset \widetilde{U}$ is a slice at $\widetilde{x}$. Since p is a local diffeomorphism, we can choose $\widetilde{U}$ to be small enough such that p is one-to-one on a neighborhood of $S_{\widetilde{x}}$. Define the map $\Phi : \widetilde{U} = \widetilde{G} \times_{\widetilde{G}_{\widetilde{x}}} S_{\widetilde{x}} \to M$ by

$$\Phi([\widetilde{g}, s]) = \varphi(\widetilde{g}) \cdot p(s) \quad \forall \widetilde{g} \in \widetilde{G}, s \in S_{\widetilde{x}}.$$

Note that regardless whether the fundamental group of M is finite or infinite, the covering map $\varphi : \widetilde{G} \to T$ is an automorphism of T (we have $\widetilde{G} \cong T$). One can check that Φ is an embedding of $\widetilde{U}$ into a tubular neighborhood of x, which implies that M is locally standard around x. Since p is surjective, M is locally standard. $\square$

Now we investigate the T^n-invariant submanifold N of M^{2n+1} as in the decomposition 3.1. In the rest of this chapter, we establish necessary information on the structure of

N, so that the proof of Theorem A can be broken into cases that depend on the different structures of N.

Since F is a characteristic submanifold, there is a T^n-fixed point $x \in F$. By our discussion in 3-2, it follows that there is a T^n-orbit $\pi_N(\pi_F^{-1}(x))$ in N.

Remark 3-5. *Let $x \in F$ be a T^n-fixed point, then $\pi_N(\pi_F^{-1}(x))$ can be*

Case A*: a 0-dimensional T^n-orbit, that is, a T^n- fixed point, or*

Case B*: a one dimensional T^n-orbit, that is, a circle.*

We will see that the dimension of N and the dimension of the effective torus action on N will depend on whether N has a T^n-fixed point. In the rest of this section, we will treat the various subcases that arise from Case A and Case B above.

Case A: We now treat Case A in Remark 3-5, where $\dim \pi_N(\pi_F^{-1}(x)) = 0$. In this case N necessarily contains a T^n-fixed point. Provided the dimension of our almost torus manifold M is at least five, we can show that both F and N are simply connected in the following proposition. The reader should refer to Proposition 3-1 for the case when $\dim(M) = 3$.

Proposition 3-6. *Suppose $\dim(M) \geq 5$, and that $\dim(\pi_N(\pi_F^{-1}(x))) = 0$. Then both F and N are simply connected (and so $\dim(N) \geq 3$). Moreover, N is a closed, non-negatively curved almost torus manifold, fixed by a subtorus $T^l \subset T^n$, $1 \leq l \leq n-1$, with $2l = codim(N)$ in M.*

The proof is similar to the argument made in [46].

Proof. Let $y = \pi_N(\pi_F^{-1}(x))$ denote the T^n-fixed point. Let C be the 1-dimensional fixed point component in M^T that contains y. Recall that $\lambda(F)$ acts freely on $M \setminus (F \cup N)$, thus we have $C \subset N$.

Since T^n acts effectively on M, by the Slice Theorem, T_yM is a faithful representation space of T^n. Because $T_yM \simeq \mathbb{R}^{2n+1}$, the linear action of T^n on T_yM is equivalent to an action of a maximal torus in $\mathrm{SO}(2n+1)$ on $\mathbb{R}^{2n+1}$, and thus must be standard with respect to a certain choice of local frame. Therefore T_yM is T^n-equivariantly diffeomorphic to $\mathbb{C}^n \times \mathbb{R}$, where $\mathbb{C}^n \simeq T_y^{\perp}C$, on which the T^n action is standard, and the $\mathbb{R}$ factor is a trivial T^n-representation that is tangential to $C \simeq S^1$.

Now, because the tangent space T_yN is an invariant subspace of T_yM, it is clear that

$$T_yN \simeq \mathbb{C}^{n-l} \oplus \mathbb{R},$$

for some $l \leq n$, and there is a subtorus T^l of T^n fixing T_yN, such that $2l = \mathrm{codim}(N)$. Hence the dimension of N is $2(n-l)+1$, where the number l is at least one, since $\mathrm{codim}(N) \geq 2$, by Proposition 2-62.

As for the fundamental groups of F and N, Theorem 3-3 implies that $\pi_1(F) = 0$, and the fundamental group of N is generated by the inclusion of a $\lambda(F)$-orbit from E to N, by a loop

$$\begin{aligned} \gamma_{x_0} : S^1 = \lambda(F) &\to N, \\ z &\mapsto zx_0, \end{aligned}$$

where $x_0 \in N$ is any base point of N. By Lemma 2-3, any $\lambda(F)$-orbit is null homotopic since N has a fixed point, thus $\pi_1(N) = 0$. Notice this rules out the situation of $l = n$, that is, where T^n fixes N, as then N must be a one dimensional closed manifold, hence S^1, which is not simply connected. Hence the inequality $l \leq n-1$ in the statement of the theorem is justified.

Finally, as a fixed point component of a torus, N is totally geodesic. The induced T^n/T^l action is effective, with non-empty fixed point set. Therefore N is a simply connected almost torus manifold, with an invariant metric of non-negative sectional curvature. $\square$

Case B: Next we turn to Case B in Remark 3-5, where $\dim(\pi_N(\pi_F^{-1}(x))) = 1$.

We start with the following proposition on the dimension of N.

Proposition 3-7. *Suppose that* $\dim(\pi_N(\pi_F^{-1}(x))) = 1$. *Then for some* $k \geq 0$, *a subgroup* $T^{n-1-k} \subset T^n$ *fixes* N, *and* N *can have the following dimensions:*

B-1: $\dim(N) = 2k+2$, $k \leq n-2$ *and codim*$(N) = 2(n-k-1)+1$; *or*

B-2: $\dim(N) = 2k+1$, $k \leq n-1$ *and codim*$(N) = 2(n-k-1)+2$.

Proof. Let $x \in F$ be a T^n-fixed point and let $S := \pi_F^{-1}(x)$ be the S^1-fiber of $\pi_F : E \to F$. By Proposition 3-2, we know that S is a T^n-orbit fixed by $T' := T^n/\lambda(F)$.

By the Slice Theorem, a tubular neighborhood of $y \in S$ in M is equivariantly diffeomorphic to

$$T^n \times_{T'} T_y^{\perp} S,$$

where the normal slice $T_y^{\perp} S$ admits a T' action. Since $T^n \times_{T'} T_y^{\perp} S \to T^n/T' = \lambda(F)$ is oriented vector bundle over a circle, it is trivial. Hence the tubular neighborhood can be simplified as

$$\lambda(F) \times T_y^{\perp} S.$$

Since E has an invariant collar in $D(F)$ and $D(N)$, there is a one dimensional subspace in $T_y^{\perp} S$ that is normal to E. Thus we can decompose $T_y^{\perp} S$ into a direct sum of subspaces of the form $W \oplus \mathbb{R}$, where the $\mathbb{R}$ factor is normal to E. The orthogonal T' action is trivial on the $\mathbb{R}$ factor. Since $\dim(T') = n-1$, the T' action on $W \simeq \mathbb{R}^{2(n-1)+1}$ is equivalent to a linear action by a maximal torus in $\mathrm{SO}(2n-1)$, and thus must be standard with respect to a certain choice of basis. Thus we will identify $W = \mathbb{C}^{n-1} \oplus \mathbb{R}_0$ according to Definition 2-32, such that the T' action is standard on $\mathbb{C}^{n-1}$ and trivial on the $\mathbb{R}_0 \cong \mathbb{R}$ factor. Namely, we have a tubular neighborhood U around y of the form

$$U := \lambda(F) \times (\mathbb{C}^{n-1} \oplus \mathbb{R}_0) \oplus \mathbb{R}. \tag{3.2}$$

Recall that $\pi_N(\pi_F^{-1}(x))$ is a T^n orbit of type $T^n/(H_0 \times T')$, where $H_0 \subset \lambda(F)$ is a finite subgroup. Since π_N is an equivariant submersion (thus an open map), by choosing a smaller U if necessary, the T^n-invariant image $\pi_N(U)$ is a tubular neighborhood of $\pi_N(\pi_F^{-1}(x))$ of the form

$$\pi_N(U) \simeq \lambda(F) \times_{H_0} V,$$

where V denotes the normal space of the T^n-orbit $\pi_N(\pi_F^{-1}(x))$ in N. The differential $\mathrm{d}(\pi_N)_y : T_yM \to T_{\pi_N(y)}N$ is a linear surjection. Because π_N is T^n-equivariant, there are a few consequences. Firstly, vectors that are tangential to the orbits are preserved, and thus the normal space $T_y^{\perp}S = (\mathbb{C}^{n-1} \oplus \mathbb{R}_0) \oplus \mathbb{R}$ is mapped onto V. In fact, since the $\mathbb{R}$ factor is normal to N, thus lies in the kernel of $\mathrm{d}(\pi_N)_y$, the subspace $\mathbb{C}^{n-1} \oplus \mathbb{R}_0$ is mapped onto V. Further, each 2-dimensional $\mathbb{C}$-factor in $\mathbb{C}^{n-1} \oplus \mathbb{R}_0$ is either preserved or collapsed via $\mathrm{d}(\pi_N)_y$, in particular those collapsing $\mathbb{C}$ factors correspond to a subtorus of T' that fixes N. Therefore, depending on whether the $\mathbb{R}_0$ in $\mathbb{C}^{n-1} \oplus \mathbb{R}_0$ is in the kernel of $\mathrm{d}(\pi_N)_y$, the open neighborhood $\pi_N(U)$ in N can be either of the following

$$\lambda(F) \times_{H_0} (\mathbb{C}^k \times \mathbb{R}), \text{or} \tag{3.3}$$

$$\lambda(F) \times_{H_0} \mathbb{C}^k. \tag{3.4}$$

Thus the dimension of N is either $2 + 2k$ or $1 + 2k$, with $0 \leq k \leq n - 1$. However, in the case when the dimension is $2 + 2k$, the value of k can not be $n - 1$, since if $\pi_1(M) = 0$ the codimension of N is at least two by Proposition 2-62. $\square$

Remark 3-8. *In the above case, notice that if $k = n - 1$, then N is a codimension 2 submanifold of M that is not fixed by any circle subgroup of T^n.*

Case B-1: Now we consider the first case of Proposition 3-7, where $\dim(N)$ is even. The following proposition shows that N is an S^1-bundle over a simply connected almost torus manifold.

Proposition 3-9. *Suppose that T^{n-1-k} fixes N and* $\dim(N) = 2k+2$, $0 \leq k \leq n-2$ *(as in (i) of 3-7). Then $\lambda(F)$ acts freely on N and $N/\lambda(F)$ is a closed, simply connected almost torus manifold with an invariant metric of non-negative sectional curvature.*

Proof. First, by the proof of Theorem 2-60, the quotient space $N/\lambda(F)$ is totally convex and hence a non-negatively curved Alexandrov space. The induced $T^k := (T^n/T^{n-k-1})/\lambda(F)$ action on $N/\lambda(F)$ is effective, with a fixed point given by the image of the $\lambda(F)$-orbit $\pi_N(\pi_F^{-1}(x))$.

Moreover, by Theorem 3-3, since $\operatorname{codim}(N) > 2$, the fundamental group of $\pi_1(N)$ is generated by a $\lambda(F)$ orbit. Hence the fundamental group of the quotient space $N/\lambda(F)$ is trivial.

It remains to show that $N/\lambda(F)$ is in fact a smooth manifold. It suffices to show that the $\lambda(F)$ action on N is free. Suppose there is a point $x \in N$ such that x has non-trivial isotropy subgroup in $\lambda(F)$. Then there is a non-trivial $\mathbb{Z}_p \subset \lambda(F)$ for some prime p such that $\mathbb{Z}_p$ fixes x. We consider the induced $\mathbb{Z}_p$ action on the fiber $S^{2(n-k-1)}$ of E. By Lemma 2-23 we have $\chi(S^{2l}) = \chi(\mathrm{Fix}(S^{2l}, \mathbb{Z}_p)) \neq 0$. Hence there is a fixed point on S^{2l} $(l = n-k-1)$. Thus $\mathbb{Z}_p$ fixes a point on E, contradicting the fact that the action of $\lambda(F)$ on E is free (see 2-60). Therefore $\lambda(F)$ acts freely on N. □

Case B-2: In the second case of 3-7, when $\dim(N) = 2k+1$ is odd, the effective torus action in N^{2k+1} is $(k+1)$-dimensional. Thus N admits an isotropy maximal action. Although simply connected non-negatively curved manifolds with isotropy maximal action are classified in [12], it is not known whether N is non-negatively curved. However, we can prove that in general the quotient $N/\lambda(F)$ is a non-negatively curved, simply connected rationally elliptic torus manifold. We begin by first showing that $N/\lambda(F)$ is a non-negatively curved, simply connected, rationally elliptic torus orbifold.

Lemma 3-10. *Assume T^{n-1-k} fixes N^{2k+1} $(0 \leq k \leq n-1)$ as in the second case of Proposition 3-7, then $X^{2k} := N/\lambda(F)$ is a rationally elliptic torus orbifold.*

Proof. Firstly, the isotropy maximal T^{k+1} action on N can not have a fixed point, since the largest isotropy subgroup has dimension $(2k+1)-(k+1)=k$. Thus the $\lambda(F)$ action on N has empty fixed point set, that is, all isotropy subgroups of the $\lambda(F)$ action are finite. Let $x \in N$. Then there is a $\lambda(F)$-invariant neighborhood U_x of x such that $U_x/\lambda(F) \cong S_x/H_x$ where S_x is a slice at x and H_x is the finite isotropy group of x. Therefore $N/\lambda(F)$ is an orbifold with charts given by $\{U_x, H_x, U_x \to U_x/\lambda(F)\}$. In fact, the orbit space $X^{2k} := N/\lambda(F)$ is a torus orbifold with effective $T^k = T^{k+1}/\lambda(F)$ action.

Now we want to show that X^{2k} is rationally elliptic. First, by Theorem 3-3, the fundamental group of N is generated by a $\lambda(F)$ orbit, and thus X^{2k} is simply connected. Then it suffices to show that N is rationally Ω-elliptic (see Definition 2-64).

Recall that $\pi_N : E \to N$ is a $S^{2(n-k)-1}$-bundle with $0 \leq k \leq n-1$ and $E/\lambda(F) = F$. Thus there is a fiberation $S^{2(n-k)-1} \hookrightarrow F \to X$ which induces the following long exact sequence.

$$\cdots \to \pi_{n+1}(X) \to \pi_n(S^{2(n-k)-1}) \to \pi_n(F) \to \pi_n(X) \to \pi_{n-1}(S^{2(n-k)-1}) \to \cdots$$

We know that spheres are rationally Ω-elliptic (for example, by Theorem 2-65). Furthermore, since F is a closed, non-negatively curved Riemannian manifold admitting an effective, isometric, almost isotropy-maximal torus action, by Corollary 2-67, F is also rationally Ω-elliptic. Therefore, from the long exact sequence, we can see that X is also rationally Ω-elliptic. $\square$

Lemma 3-11. *The circle $\lambda(F)$ act freely on N. Therefore $N/\lambda(F)$ is in fact a smooth torus manifold.*

Proof. Let $\pi : N \to P := N^{2k+1}/T^{k+1}$ be the projection map. By Proposition 3-10, we know that $X^{2k} := N/\lambda(F)$ is a simply connected, rationally elliptic torus orbifold. Thus by the results in [15] (also, compare to the remark at the end of Section 2.7.), the orbit space $P = N/T = X/T$ has the face poset structure as in Equation 2.6.

In order to prove the lemma, we need the following claims:

Claim (1): Points of minimal orbits have connected isotropy subgroups, that is, a minimal orbit is a free $\lambda(F)$ orbit fixed by $T^k = T^{k+1}/\lambda(F)$.

To prove Claim (1), let $y \in N$ be a point on a minimal orbit C. Since $\dim(T^{k+1}_y) = (2k+1) - (k+1) = k$ and the $\lambda(F)$ action is almost free, it follows that $T^{k+1}_y \cong T^k \times H$ for $T^k \subset T^{k+1}$ and a finite group $H \subset \lambda(F)$. The normal space of C is isomorphic to $\mathbb{C}^k$ as a representation space of $T^k \times H$. Since a linear $T^k \times H$ action on $\mathbb{C}^k$ can not be effective if H is non-trivial, the subgroup H must be trivial. Therefore Claim (1) holds.

Claim (2): All interior points of P correspond to principal orbits.

To prove Claim (2), let $x \in N$ such that $\pi(x)$ is an interior point of P, that is, $\dim(T^{k+1}_x) = 0$ since otherwise x is fixed by at least a circle in T^{k+1} and thus lies on some facet of P. Suppose T^{k+1}_x is non-trivial, then x is an exceptional orbit. According to the proof of Theorem 2-60, X is a non-negatively curved Alexandrov space (with no boundary) on which the T^k action is isometric and effective. Since P has face poset structure as in Equation 2.6, there is a codimension two submanifold K in N^{S^1}, where S^1 is a circle in T^k, such that $\bar{K} := K/\lambda(F)/S^1$ is a boundary component of $\bar{X} := X/S^1$. Since the S^1 action is isometric, $\bar{X}$ is a non-negatively curved Alexandrov space with boundary. By Lemma 2-53, the distance function $d_{\bar{K}} : \bar{X} \to \mathbb{R}$ is concave. Note that exceptional orbits corresponding to points in the interior of P are critical points for $d_{\bar{K}}$ (see [20]). By the concavity of $d_{\bar{K}}$, any critical points for $d_{\bar{K}}$ must lie at maximal distance from $\bar{K}$ (compare to Lemma 3.21 of [42]). Thus x corresponds to a point in

$$\bar{C} := \{\bar{y} \in \bar{X} : d_{\bar{K}}(\bar{y}) \text{ is maximal}\}.$$

It suffices to show then that $\bar{C}$ corresponds to the boundary of P, that is, points on $\bar{C}$

correspond to points on N with singular orbits. However, one can adapt the proof of Lemma 6.3 in [46] to show that this is true and Claim (2) follows.

Now we want to show that all points on N corresponding to the boundary of P have connected isotropy subgroups. Now assume that $x \in N$ corresponds to an interior point of a face A of P^k. By Claim (1), if A is a vertex, then the isotropy subgroup of x is connected. Otherwise suppose $\mathrm{codim}(A) = m$, $1 \leq m < k$, then x is contained in $\hat{A}^{2(k-m)+1} := \pi^{-1}(A)$, which is a codimension $2m$ fixed point component of some subtorus $T^m \subset T^{k+1}$. By Equation 2.6, every face of P contains a vertex. Thus A contains a minimal orbit (of type $\lambda(F)$), and so the effective T^{k-m+1} action on A is isotropy maximal. In particular, x corresponds to an exceptional orbit. It follows that $\hat{A}/\lambda(F)$ is a torus orbifold. On the other hand, since $\hat{A}$ is a T^{k+1}-invariant, totally geodesic submanifold of N (see Theorem 2-22), it follows that $\hat{A}/\lambda(F)$ is a non-negatively curved Alexandrov space because $N/\lambda(F)$ is non-negatively curved. Notice that to carry out the argument in Claim (2), all we need is a non-negatively curved Alexandrov space which is also a torus orbifold. Therefore, it follows that all interior points of $\hat{A}/\lambda(F)$ correspond to principal orbits by the argument in Claim (2). Therefore the isotropy subgroup of x is T^m, a connected torus. Thus we have proved the following:

Claim (3): All points on the boundary of P have connected isotropy subgroups.

To finish the proof, note that by Claim (1), a minimal T^{k+1}-orbit has trivial isotropy, thus the principal isotropy group for the T^{k+1} action on N is trivial. Assume a non-trivial finite subgroup $H \subset \lambda(F)$ fixes a point y of N, thus $T^{k+1}_y \cong H \times T^k_y$. But then T^{k+1}_y is not connected, contradicting Claim (2) or Claim (3) . Therefore the $\lambda(F)$ action is free. $\square$

4. ORBIT SPACES OF ALMOST TORUS MANIFOLDS

4.1. The statement of the main theorem

In this section, we prove the following theorem, which extends Lemma 6.3 in Wiemeler's [46] to almost torus manifolds.

Theorem 4-1. *Let M^{2n+1} be a closed, simply connected, non-negatively curved almost torus manifold. Then the following hold.*

1. *The torus action on M is locally standard.*
2. *M/T is diffeomorphic (after smoothing the corners) to a standard disk D^{n+1}.*
3. *Any codimension one face of M/T is diffeomorphic (after smoothing the corners) to either a standard disk D^n, or $S^1 \times D^{n-1}$.*

4.2. Outline of the Proof of Theorem 4-1

The proof of Theorem 4-1 is by induction on the dimension of M^{2n+1}, a closed, simply connected, non-negatively curved almost torus manifold. We have proven the anchor of the induction in Proposition 3-1. Now assume that $\dim(M^{2n+1}) \geq 5$ ($n \geq 2$), and that the theorem holds for any closed, simply connected, non-negatively curved almost torus manifold of dimension $2k+1$ with $k < n$.

Recall that by Theorem 2-60, we have a decomposition

$$M^{2n+1} = D(N) \cup_E D^2(F^{2n-1}),$$

where F is a characteristic submanifold of M fixed by a circle $\lambda(F) \subset T^n$, and $\pi_F : E \to F$ is a principal $\lambda(F)$-bundle. Also recall that there is a T^n-fixed point $x \in F$ such that $\pi_F^{-1}(x)$ is a T^n-orbit on which the $\lambda(F)$ action is free.

Recall that we have the following cases, which are established in Chapter 3.

Case (A): $\dim \pi_N(\pi_F^{-1}(x)) = 0$. In this case N is a fixed point component of some subtorus.

Case (B-1): $\dim \pi_N(\pi_F^{-1}(x)) = 1$ and $\dim(N) = 2k+2$ is even, and there is a subtorus T^{n-1-k} $(0 \leq k \leq n-2)$ fixing N.

Case (B-2): $\dim \pi_N(\pi_F^{-1}(x)) = 1$ and $\dim(N) = 2k+1$ is odd, and there is a subtorus T^{n-1-k} $(0 \leq k \leq n-1)$ fixing N.

4.3. Proof of Theorem 4-1 in Case (A)

Proof. By Proposition 3-6, both F and N are closed, simply connected, non-negatively curved almost torus manifolds (in particular, $\dim(N) \geq 3$). Therefore, by the induction hypothesis, F and N are both locally standard, and both N/T and $F/T \simeq E/T$ are diffeomorphic, after smoothing the corners, to standard disks.

Now we want to show that M is locally standard, by showing that $D(F)$ and $D(N)$ are locally standard.

By the induction hypothesis, since $D(F) \to F$ is a D^2-bundle with $\lambda(F)$ being the structure group, the $\lambda(F)$ action on each D^2 fiber is given by a standard T^1 action on $\mathbb{C}$. Thus by Proposition 2-34, it follows that $D(F)$ is locally standard with respect to the T^n action. As for $D(N)$, since N^k is a fixed point component of a subtorus T^{n-k} with $2(n-k) = \operatorname{codim}(N)$, the T^{n-k} action on the $D^{2(n-k)}$ fibers of $D(N) \to N$ is a linear action by a maximal torus of U on $\mathbb{C}^{n-k}$, thus must be standard (see Example 2-20). Then by Proposition 2-34 it follows that $D(N)$ is locally standard. Therefore M is locally standard.

Now we want to show that M/T is a disk. Firstly, $\pi_F : D(F) \to F$ is a D^2-bundle on which $\lambda(F)$ is acting as the structure group. Since π_F is T^n-equivariant and the action of $\lambda(F)$ in each fiber is standard, the induced fiberation

$$D(F)/T \to F/T,$$

has the property that all fibers are diffeomorphic to $D^2/\lambda(F) \simeq I$. Then because F/T is diffeomorphic (after smoothing the corners) to D^n by the induction hypothesis, $D(F)/T$ is a product

$$D(F)/T \simeq F/T \times I \simeq D^n \simeq D^{n+1}.$$

Similarly, since $\pi_N : D(N) \to N$ is T^n-equivariant and N is fixed by a subtorus $T^{n-k} \subset T^n$ with $2(n-k) = \mathrm{codim}(N)$, the induced fiberation

$$D(N)/T \to N/T,$$

whose fibers are the quotient of $D^{2(n-k)}$ by a linear T^{n-k} action, which is equivalent to the standard action of a maximal torus (as in Example 2-20). Thus it is not hard to see that all fibers of $D(N)/T \to N/T$ are diffeomorphic (after smoothing the corners) to an $(n-r)$-simplex Δ^{n-r}. Thus, since $N/T \cong D^{k+1}$ by the induction hypothesis, $D(N)/T$ is in fact a product as follows,

$$D(N)/T \simeq N/T \times \Delta^{n-k} \simeq D^{k+1} \times D^{n-k} \simeq D^{n+1}.$$

Finally, since the gluing map $\partial D(F) \to \partial D(F)$ is T^n-equivariant by Remark 2-61, and $(\partial D(F))/T = E/T \simeq F/T \simeq D^n$, it follows that

$$M/T \simeq D(F)/T \cup_{E/T} D(N)/T \simeq D^{n+1} \cup_{D^n} D^{n+1} \simeq D^{n+1}.$$

Therefore M/T is diffeomorphic (after smoothing the corners) to a standard disk. $\square$

Notice we can summarize the last part of the proof of Case (A) into the following statement:

Lemma 4-2. *In the bundle decomposition $M = D(F) \cup_E D(N)$, suppose*

1. *F/T (thus E/T) and N/T are both diffeomorphic (after smoothing the corners) to a disk;*

2. *All fibers of $D(N)/T \to N/T$ are diffeomorphic (after smoothing the corners) to disks.*

Then M/T is diffeomorphic (after smoothing the corners) to a disk.

Proof (summary). If $F/T \simeq D^n$, then $\pi_F : D(F) \to F$ is a disk bundle over a disk with group $\lambda(F)$, hence π_F is a trivial bundle with $D(F)/T \simeq F/T \times I \simeq D^{n+1}$.

If $N/T \simeq D^m$ and all fibers of $D(N)/T \to N/T$ are diffeomorphic (after smoothing the corners) to disks, then $D(N)/T \to N/T$ is a disk bundle over a disk, hence a trivial bundle. Thus $D(N)/T$ is a product of disks, that is, a disk.

Finally, M/T is the gluing of two disks along a disk E/T, therefore a disk. $\square$

4.4. Proof of Theorem 4-1 in Case (B-1)

Proof. By Proposition 3-9 the $\lambda(F)$ action on N is free and $(N^*)^{2k+1} := N/\lambda(F)$ is a closed, simply connected almost torus manifold with an invariant metric of non-negative sectional curvature. Moreover, since $\dim(N)$ is even, we have $\mathrm{codim}(N) > 2$, and thus by Theorem 3-3, F is simply connected. It then follows by the induction hypothesis that F is locally standard, hence $D(F)$ is locally standard by Proposition 2-34. Now we turn back to N, whose properties will be demonstrated via a series of claims.

Claim: *N is orientable.*

Recall that a smooth manifold N is orientable if and only if the tangent bundle τ_N is an oriented bundle.

Firstly, $N^* := N/\lambda(F)$ is orientable since it is simply connected. To see this, assume N^* is not orientable, then the orientable double covering $\phi : \widetilde{N^*} \to N^*$ has connected total space $\widetilde{N^*}$. Moreover since $\widetilde{N^*}$ is a manifold, it is path-connected. Therefore the lifting correspondence $\pi_1(N^*) \to \phi^{-1}(b)$ is surjective for any base point $b \in N^*$. Since $\pi_1(N^*) = 0$, it turns out that ϕ is a homeomorphism, which is a contradiction. Thus the tangent bundle τ_{N^*} is an orientable bundle.

Let $\tau_F N$ be the subbundle of τ_N consisting vectors that are tangential to the fibers of the principal S^1-bundle $p : N \to N^*$. Thus we can write τ_N as a Whitney sum:

$$\tau_N \simeq p^*(\tau_{N^*}) \oplus \tau_F N,$$

where $p^*(\tau_{N^*})$ is the pull-back bundle. Let ζ be the 2-plane vector bundle associated with $N \to N/\lambda(F)$ defined as

$$\pi_\zeta : N \times_{\lambda(F)} \mathbb{R}^2 \to N/\lambda(F).$$

Since the structure group $\lambda(F)$ preserves the orientation on N, ζ is orientable. Corollary 2-10 implies that

$$p^*(\zeta) \simeq \tau_F N \oplus e^1(N),$$

where $e^1(N)$ is the normal bundle of N in $N \times_{\lambda(F)} \mathbb{R}^2$, which is trivial since the normal bundle of $\lambda(F)$ in $\mathbb{R}^2$ has a non-vanishing cross section. Therefore, we have bundle isomorphisms:

$$\tau_N \oplus e^1(N) \simeq (p^*\tau_{N^*} \oplus \tau_F N) \oplus e^1(N) \simeq p^*\tau_{N^*} \oplus p^*(\zeta) \simeq p^*(\tau_{N^*} \oplus \zeta).$$

Thus by looking at the first Stiefel-Whitney classes, we obtain:

$$\begin{aligned} w_1(\tau_N) &= w_1(\tau_N \oplus e^1(N)) \\ &= p^* w_1(\tau_{N^*} \oplus \zeta) \\ &= p^*(w_1(\tau_{N^*}) + w_1(\zeta)). \end{aligned}$$

Recall that the first Stiefel-Whitney class vanishes if and only if the vector bundle is orientable (see for example, Proposition 4.36 of [7]). Thus both $w_1(\zeta)$ and $w_1(\tau_{N^*})$ are zero, and therefore $w_1(\tau_N) = 0$, and thus τ_N is an oriented bundle, that is, N is orientable.

Claim: *The structure group of the bundle $E \to N$ can be reduced to a torus.*

Denote $l = n - k - 1$, and recall that $\mathrm{codim}(N) = 2l + 1$ and N^{2k+2} is fixed by an l-torus. By the proof of Proposition 3-7 and Theorem 2-22, there is a connected component N' in $M^{T'}$ such that $N \subset N'$ with codimension 1. Since $\mathrm{codim}(N') = 2\dim(T')$, the induced T' action on the normal space of N' is a linear action of a maximal torus in $\mathrm{GL}(l, \mathbb{C})$. By looking at the T'-representation space $T_x^{\perp} M$, for a point $x \in N$ (see Example 2-20), one can see that there is a $T^1 \subset T'$ fixing a submanifold N_1 with $\mathrm{codim}(N_1) = 2$ and $N \subset N' \subset N_1$. We have

$$\nu(N_1) \oplus \tau N_1 \cong \tau_{N_1} M,$$

where $\tau_{N_1} M$ is the tangent bundle of M restricts to N_0 and the normal bundle $\nu(N_1)$ is a complex line bundle with structure group $\mathrm{GL}(1, \mathbb{C})$ (By Theorem 2-22), which can be reduced to $\mathrm{U}(1) \cong T^1$ since T^1 is isometric and fixes N_1.

In fact, it is not hard to see that there is a nested sequence

$$N \subset N' = N_l \subset \cdots \subset N_1 \subset M$$

of submanifolds such that each N_j is the fixed point component of a $T^j \subset T' \cong T^l$ with $\dim(N_j) - \dim(N_{j+1}) = 2$. By the previous argument, the normal bundle of each N_{j+1} in N_j is a complex line bundle with the structure group isomorphic to a circle T^{j+1}/T^j. Thus the normal bundle $T^{\perp} N' = T^{\perp} N_l$ splits into a sum of complex line bundles

$$T^{\perp}(N') = E_1 \oplus \cdots \oplus E_l,$$

and thus the structure group of $T^{\perp}(N')$ can be reduced to the product of the structure group of these line bundles, which is exactly T'.

Since N is an orientable hypersurface of N', the normal bundle of N in N' is trivial.

Thus the structure group of the normal bundle $T^{\perp}(N)$ in M is also given by T', and thus the group of the associated bundle $E \to N$ is also given by T'.

Claim: $D(N)$ *is locally standard.*

Since $\pi_N : E \to N$ is $\lambda(F)$-equivariant and $\lambda(F)$ acts freely on both E and N, there is an induced S^{2l}-bundle

$$\pi_{N^*} : E/\lambda(F) = F \to N^*,$$

whose structure group is still T'. Let $p : P \to N^*$ be the principal T'-bundle associated with π_{N^*} and denote $T'' = T^n/T'$ as the complementary subtorus of T', and notice that $\lambda(F) \subset T''$. By Theorem 2-12, since N^* is simply connected and thus $H^1(N^*, \mathbb{Z}) = 0$, there is a bundle lift of the T'' action to P, commuting with the canonical T' action on P.

Let the associated S^{2l}-bundle:

$$\pi_P : P \times_{T'} S^{2l} \to N^* \tag{4.1}$$

be defined, where the T^n action on $P \times_{T'} S^{2l}$ is given by

$$g \cdot [p, v] = [p \cdot g, v] \quad \forall g \in T^n, p \in P, v \in S^{2l}.$$

Then since the two bundles π_P and π_{N^*} share the same fiber space and the same associated principal bundle, they are isomorphic. In fact, we have a T^n-equivariant diffeomorphism

$$F \simeq P \times_{T'} S^{2l}, \tag{4.2}$$

by letting the unspecified T' action on each S^{2l} fiber to be the action induced by the T' action on F.

Now we show that $D(N)$ is locally standard. Firstly, notice that the $\lambda(F)$ action is free on $M \setminus (F \cup N)$ by Theorem 2-60, and we have shown that the action is free on N. Thus the $\lambda(F)$ action is free on $D(N)$. The bundle $D(N) \to N$ descends to a bundle $D(N)/\lambda(F) \to N^*$ with same structure group T' and fiber D^{2l+1}. Thus since $\dim(T') = l$, the torus T' acts on D^{2l+1} by a maximal torus (as Example 2-20). Thus since N^* is locally standard by the induction hypothesis, $D(N)/\lambda(F)$, hence $D(N)$, is locally standard by Proposition 2-34.

To show that M/T is a disk, first notice by the inductive hypothesis, $F/T = E/T$ and $N/T \simeq N^*/T^{n-1}$ are diffeomorphic, after smoothing the corners, to a standard disk. We want to show that the fibers of $D(N)/T \to N/T$ are diffeomorphic (after smoothing the corners) to disks, then by using Lemma 4-2, we can conclude that M/T is a disk.

Observe that since the bundle

$$F \simeq P \times_{T'} S^{2l} \to N^*$$

has the same fiber D^{2l+1} and structure group T' as the bundle $D(N) \to N$. Then because we have shown that M is locally standard, the $T' \cong T^l$ action on D^{2l+1} is standard, and hence the quotient D^{2l+1}/T' is a disk. This completes the proof of Theorem 4-1 for Case (B-1). $\square$

4.5. Proof of Theorem 4-1 in Case (B-2)

Proof. In this case, recall that N^{2k+1} is fixed by a subtorus T^{n-k-1} with $0 \leq k \leq n-1$. By Lemma 3-10 and Lemma 3-11, the $\lambda(F)$ action on N is free, and $N/\lambda(F)$ is a simply connected, non-negatively curved torus manifold.

Therefore, since the effective T^{k+1} action on N^{2k+1} induces an effective $T^k \cong T^{k+1}/\lambda(F)$ action on $X^{2k} := N^{2k+1}/\lambda(F)$ with non-empty fixed point set, by the classification of torus manifolds with non-negative curvature (see [46]), we know:

1. X^{2k} is locally standard. Note that this implies that $D(N)$ is locally standard which we can see as follows. Similar to Case (B-1), $D(N) \to N$ is a D^{2l} bundle, the T^l fixing N acts on each fiber by a maximal torus, thus the action is standard (see Example

2-20) and each fiber of $D(N) \to N$ is diffeomorphic (after smoothing the corners) to a disk. Furthermore, $N \to X$ is a principal $\lambda(F)$-bundle, using Proposition 2-34, we can see that $D(N)$ is locally standard.

2. $X/T = N/T$ is a disk and all its faces are contractible.

Now, we turn to F and $D(F)$, this time we have two cases to consider.

Case B-2-i, where $\dim(N) < 2n - 1$.

The fundamental group of F is trivial if $\dim(N) < 2n-1$ (by Theorem 3-3), thus F is a simply connected, non-negatively curved almost torus manifold. By the induction hypothesis, F is locally standard and $F/T \simeq E/T \simeq D^n$. The previous discussion on $D(N) \to N$ allows us to apply Lemma 4-2 to conclude that M/T is a disk. The Proposition 2-34 implies that $D(F)$ is locally standard, and hence M is locally standard.

Case B-2-ii, where $\dim(N) = 2n - 1$.

Since $\mathrm{codim}(N) = 2$ and N is not fixed by any circle in T^n, the S^1-bundle $\pi_N : E \to N$ induces a fiberation $E/\lambda(F) \simeq F/\lambda(F) \to N/\lambda(F)$ where we have shown that $N/\lambda(F) \simeq D^{n-1}$. Thus, by considering the long exact sequence of homotopy groups

$$\pi_1(S^1) \to \pi_1(F/T) \to \pi_1(D^{n-1}),$$

we know that $\pi_1(S^1) \cong \mathbb{Z}$ is mapped onto $\pi_1(F/T)$, and thus $\pi_1(F/T) \cong$. Recall that $\pi_1(F)$ is cyclic and thus because $\pi_1(F)$ generates $\pi_1(F/T)$ (see 2-11), the fundamental group of F can only be $\mathbb{Z}$. Thus F satisfies Proposition 3-4. The universal cover $\widetilde{F}$ of F splits into a product $F_0 \times \mathbb{R}$ where F_0 is a simply connected, non-negatively curved $2(n-1)$-torus manifold, admitting an $(n-1)$-torus action lifted from $T^n/\lambda(F)$. The group of deck transformation $\Gamma \cong \pi_1(F) \cong \mathbb{Z}$ acts on $\mathbb{R}$ exclusively. Then it follows that

$$F/T \simeq (\widetilde{F}/\Gamma)/T \simeq F_0/T \times \mathbb{R}/\mathbb{Z} \simeq D^{n-1} \times S^1,$$

where $F_0/T \simeq D^{n-1}$ follows from Lemma 6.3 of [46]. Since the structure group of $D(F) \to F$ is $\lambda(F) \subset T^n$, $D(F)/T \to F/T$ is a trivial fiber bundle because it has trivial structure group. Thus $D(F)/T$ is a product

$$D(F)/T \simeq F/T \times I \simeq D^{n-1} \times S^1 \times I,$$

where I is the quotient of a D^2-fiber by $\lambda(F)$.

On the other hand. $D(N)/T \to N/T$ is a D^2-bundle over N/T, which is a trivial bundle since we have shown that N/T is a disk. Thus $D(N)$ is a product

$$D(N)/T = N/T \times D^2 = D^{n-1} \times D^2.$$

We obtain

$$M/T = (D^{n-1} \times S^1 \times I) \times_{D^{n-1} \times S^1} (D^{n-1} \times D^2) = D^{n+1},$$

where one of the boundary component of $S^1 \times I$ is glued to the boundary of D^2. We conclude that the orbit space M/T is, as desired, a standard disk.

In either of the above cases, F is locally standard, thus Proposition 2-34 implies that $D(F)$ is locally standard. Therefore, as we already shown that $D(N)$ is locally standard, M is locally standard.

At this point, the proof of Theorem 4-1 is complete. $\square$

5. CONCLUSIONS

Given a simply connected, non-negatively curved, almost torus manifold M^{2n+1}, in this dissertation we first obtain a thorough description of the disk bundle decomposition of M according to Spindeler's Theorem 2-60. There are several factors affecting the fundamental groups of F and N. The codimension of N is affected by two factors: (1) The existence of T^n-fixed point in N. (2) The rank of the subtorus that fixes N. The three basic cases that occur can be summarized as follows.

Case (A): If N contains a T^n-fixed point, then $\dim(N) = 2k+1$ (N has even codimension) with $1 \leq k \leq n-1$ ($N \simeq S^1$ is impossible in this case), and N is a fixed component of a nontrivial T^{n-k}-subtorus. In this case both N and F are simply connected non-negatively curved almost torus manifolds. In particular, F/T and N/T are contractible faces of M/T.

Case (B-1): If N does not contain a T^n-fixed point and $\dim(N) = 2k+2$ is even, then there is a subtorus T^{n-1-k} that fixes N ($0 \leq k \leq n-2$, so the codimension of N is strictly greater than two). In this case we do not know whether N admits a T^n-invariant metric of non-negative curvature. Nevertheless, F is a simply connected non-negatively curved almost torus manifold. We also obtain that the $\lambda(F)$ action is free on N, and it turns out that N is a principal $\lambda(F)$-bundle over a simply connected non-negatively curved almost torus manifold $N/\lambda(F)$.

Case (B-2): If N does not contain a T^n-fixed point and $\dim(N) = 2k+1$ is odd, then there is a subtorus T^{n-1-k} ($0 \leq k \leq n-1$) fixing N. We prove that $N/\lambda(F)$ is a simply connected non-negatively curved torus manifold.

If $\mathrm{codim}(N) < 2$, then F is a simply connected non-negatively curved almost torus manifold. In the situation that $\mathrm{codim}(N) = 2$, we prove that F is a non-negatively curved almost torus manifold with infinite cyclic fundamental group. This is the only case where F is a characteristic submanifold that does not project to a contractible facet in M/T.

After a case by case argument, we conclude that

1. The torus action on M is locally standard, in particular, all isotropy subgroups are connected.
2. M/T is diffeomorphic (after smoothing the corners) to a standard disk D^{n+1}.
3. Any codimension one face of M/T is diffeomorphic (after smoothing the corners) to either a standard disk D^n, or $S^1 \times D^{n-1}$.

6. FURTHER RESULTS AND IMPLICATIONS

We will outline how the contents of Chapters 3 and 4 contribute to the classification of almost torus manifolds in [10]. In particular, we will state the results we obtain by using the work of this thesis.

Firstly, it has been shown in [18] that an isometric T^1 fixed point homogeneous action on a non-negatively curved 3-manifold can be extended to a smooth T^2 action. We can prove the following general theorem using induction on the dimension of M, in particular, the contents of Chapter 3 were used for the inductive argument.

Theorem 6-1 ([10]). *Let M^{2n+1} be a simply connected, non-negatively curved almost torus manifold. Then there exists a unique, smooth T^1 action on M^{2n+1} that commutes with the isometric almost isotropy-maximal T^n action on M^{2n+1}.*

Moreover, using the extended action we can prove:

Proposition 6-2 ([10]). *The extended, smooth and effective T^{n+1} action on M^{2n+1}, a simply- connected, non-negatively curved almost torus manifold, has only connected isotropy subgroups, and the quotient space and all of its faces are diffeomorphic (after smoothing the corners) to disks.*

Finally, it is possible to obtained a classification of almost torus manifolds.

Theorem 6-3 (**Classification of Almost Torus Manifolds** [10]). *Let T^n act isometrically and effectively on a simply-connected, non-negatively curved, almost torus manifold, M^{2n+1}. Then M^{2n+1} is equivariantly diffeomorphic to the quotient by a free, linear torus action on a product of spheres.*

We want to briefly mention the idea of the proof of Theorem 6-3. Observe that the T^{n+1} smooth isotropy-maximal action on M^{2n+1} has free rank 1 by Proposition 5.4 in

[12], and the action is isotropy-maximal. Therefore there is a $T^1 \subset T^{n+1}$ that acts almost freely on M^{2n+1}. The situation breaks into two cases: Case (1), where the T^1 action is free, then M^{2n+1}/T^1 is a torus manifold, and Case (2), where the T^1 action is almost free. In both cases, M^{2n+1}/T^1 is rationally elliptic, since M^{2n+1} is rationally elliptic. Therefore Theorem 6-3 follows by results of [46] and [12].

7. FUTURE DIRECTIONS

There are a few questions that are closely related to almost torus manifolds. In particular, answering these questions may lead to generalizations of the techniques that are used in the proving of the main theorem of this thesis.

1. Let S^1 be a circle of T. One can ask: Does any codimension two component of M^{S^1} contain a T-fixed point? We know this is true for locally standard torus manifolds with acyclic faces, which is equivalent to the condition that the odd degree cohomology groups of the manifold vanish. One can then ask: What would be a parallel topological condition for locally standard almost torus manifolds? In particular, how can we generalize the connection between locally standard torus actions to rationally elliptic manifolds? We refer the reader to the results in [42], [46], [34] and [15].

2. In [15], they considered *torus orbifolds* and proved some results that are similar to those in [46]. We can define an almost torus orbifold in a similar way, and it would be interesting to see if we can get parallel results as in [15]. This research is also interesting as the concept of orbifolds arises in many industrial applications that involve the study of the quotient of manifolds by groups of symmetry.

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