ON THE DOPPLER FREQUENCY SPECTRUM PRESENTED TO A MOVING RADAR SYSTEM BY REFLECTION FROM A ROUGH PLANE EARTH

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INTRODUCTION

If a radar system moves over a surface and transmits energy in such a manner that a portion of this energy impinges on the surface, a scattered electromagnetic field will result. In general, the field will be scattered in all directions and a portion of this field will be scattered back to the radar system. Since the scattered field arriving at the radar system will come from all parts of the surface illuminated by the transmitted beam of energy, this field will be presented to the radar system from many different directions. If the frequency of the field transmitted by the radar system is designated as $f_0$, the scattered field from a particular direction will arrive at the radar system with a particular frequency that is different from $f_0$ due to the Doppler effect. Since there will be a scattered field arriving at the radar system from many different directions, there will be a range of frequencies presented to the radar system. The Doppler shift associated with the field arriving from a particular direction will be a function of the velocity of the radar system and the direction of arrival relative to the direction of
motion of the radar system. It can be seen that for velocities much less than the speed of light, the Doppler shift is proportional to the cosine of the angle between the direction of arrival of the scattered field at the radar system and the direction of motion of the radar system.

The purpose of this work is to determine the field density spectrums received by a radar system moving in an arbitrary direction with a velocity $v$. The field density spectrum is, by definition, the frequency distribution of the field per unit frequency. Expressions will be derived from which the field density spectrums can be determined once the scattering surface is prescribed. The results are also equally valid for a random surface that satisfies certain conditions. It is only necessary to determine the electric field density spectrum to obtain a complete solution to the problem since the magnetic field density spectrum can be determined from the electric field density spectrum.

The far zone field scattered from a surface may be determined for a perfect conductor by the current distribution method of antenna theory. The far zone is usually taken to be that region where $R > 2\tau^2(D)/2$

where $\tau(D)$ is the diameter of the scattering surface,
λ is the wave length of the radiation,
R is the distance from the radar system to
the scattering surface.

The use of this method assumes that (a) there is no
shielding of any part of the surface by any other part,
(b) the current distribution over the surface is the same
as that of a plane wave incident on an infinite plane sheet
and (c), there is no current distribution over that por-
tion of the surface that is not illuminated by the radar
beam. Hence the results are restricted to the field den-
sity spectrums in the far zone scattered from perfectly
conducting surfaces of small curvature and to non grazing
angles of incidence.

This method differs from the geometrical optics
approach in that a frequency dependence of the scattered
field is introduced into the calculation of the field aris-
ing from the current distribution. Also, the field at a
given point in space is the resultant of contributions from
all points in the illuminated region rather than from a
point determined by geometrical considerations only.

The current distribution method is also the basis of
an approach by Hoffman (4, pp.291-304) and 5, pp.96-100),
in determining the scattered field from a surface. The
present approach differs from that of Hoffman in taking
account of the effect of a moving source.
NON - RANDOM SURFACE

1. FORMULATION

No generality is lost by assuming the velocity vector associated with the radar system is in the \(xz\) plane of our coordinate system and makes an angle \(\psi\) with the \(x\) axis. The origin of our coordinate system will be assumed to coincide with the radar system. If a cone with apex angle \(\theta\) is constructed with the velocity vector as its center line, the cone will intersect the plane \(z = -h\) at the curve

\[
y^2(\theta, x) = (x \cos \psi + h \sin \psi)^2 \sec^2 \theta - x^2 - h^2.
\]

Since the Doppler shift of the scattered field arriving at the radar system from the directions defined by the \(\theta\) cone is proportional to \(\cos \theta\), the portion of the field that arrives at the radar system with a frequency greater than \(f\) will be scattered from within (2.1.1). Similarly, that portion of the field at the radar system with a frequency greater than \(f - \Delta f\) will be scattered from within

\[
y^2(\theta + \Delta \theta, x) = (x \cos \psi + h \sin \psi)^2 \sec^2(\theta + \Delta \theta) - x^2 - h^2.
\]

That portion of the field with a frequency less than or equal to \(f\) but greater than \(f - \Delta f\) will be scattered from the surface in (2.1.2) but not in (2.1.1).
It is well to note that (2.1.1) is a conic and will be:

(a) an ellipse if $\theta < \psi$,
(b) a parabola if $\theta = \psi$,
(c) a hyperbola if $\theta > \psi$,
(d) a circle if $\psi = \pi/2$.

The radar beam will be considered as a uniform conical searchlight with apex angle $\alpha$ and whose center line makes an angle $\beta$ with the $xy$ plane and an angle $\gamma$ with the $xz$ plane. The $\alpha$ cone will intersect the $z = -h$ plane at the curve

$$y^2(a,x) = (xcos\beta cos\gamma + ycos\beta sin\gamma + hsin\beta)^2 sec^2\alpha-x^2-h^2.$$

This intersection will be:

(a) an ellipse if $\alpha < \beta$,
(b) a parabola if $\alpha = \beta$,
(c) a hyperbola if $\alpha > \beta$,
(d) a circle if $\beta = \pi/2$.

The scattered electrical field at the radar system in the far zone with frequencies between $f$ and $f - \Delta f$ is given by Silver (9, pp.144-149).
(2.1.6) \[ \mathbf{E}(\alpha, \beta, \gamma, r, \theta, \varphi, e_{\pm}0) = -\frac{1}{2\pi R} e^{-ikR} \]

\[ \int \left\{ (\mathbf{H} \times \mathbf{R}_1 - (\mathbf{H} \times \mathbf{R}_1 \cdot \mathbf{R}_1) \mathbf{R}_1) e^{-ik \mathbf{R}_1 \cdot \mathbf{r}} ds \right\} \]

where \( \mathbf{R}_1 \) is the vector from the origin of the coordinate system to the intersection of the \( z = -h \) plane by the center line of the cone, \( R = h / \sin \beta \), \( \mathbf{r} \) is the vector from the origin to the scattering element of the surface, \( \mathbf{N} \) is the normal to the surface, \( D_{\theta, \varphi + \pm \theta} \) is the portion of the surface from which the field that arrives at the radar system with frequencies between \( f \) and \( f - \Delta f \) is scattered, \( \mu \) is the magnetic inductive capacity, \( \mathbf{H}_1 \) is the incident magnetic vector, \( \mathbf{E} = \mathbf{w} / c \), \( \mathbf{w} = 2\pi f \), \( c \) = velocity of light, \( f - \Delta f \leq f \leq f \).
FIGURE 1
It can be seen from Fig. 1 that $\beta$ must be restricted to the range of values $\pi > \beta > 0$ to assure the intersection of the center of the $\alpha$ cone with the plane $z = -h$. It will be recalled that the theory has previously been limited to non grazing angles of incidence. In accordance with this it will also be required that $\pi > \beta > \alpha > 0$, hence the $\alpha$ cone will always intersect the plane $z = -h$ in an ellipse or circle.

The electric field density spectrum is defined to be

\begin{equation}
(2.1.7) \quad \hat{E}(\alpha, \beta, \psi, \gamma, \theta) = \lim_{\Delta \theta \to 0} \frac{1}{f-(f-\Delta f)} \hat{H}(\alpha, \beta, \psi, \gamma, \theta, \theta+\Delta \theta) .
\end{equation}

According to Silver (9, p. 149),

\begin{equation}
(2.1.8) \quad \hat{H}(\alpha, \beta, \psi, \gamma, \theta, \theta+\Delta \theta) = \left(\frac{\xi}{2}\right) \left[ -\hat{R}_1 \times \hat{E}(\alpha, \beta, \psi, \gamma, \theta) \right] .
\end{equation}

The magnetic field density spectrum is then

\begin{equation}
(2.1.9) \quad \hat{B}(\alpha, \beta, \psi, \gamma, \theta) = \left(\frac{\xi}{2}\right) \left[ -\hat{R}_1 \times \hat{E}(\alpha, \beta, \psi, \gamma, \theta) \right] ,
\end{equation}

where from Fig. 1, it is seen that

\begin{itemize}
  \item[(a)] $\hat{R}_1 = \hat{x} \cos \beta \cos \gamma + \hat{y} \cos \beta \sin \gamma - \hat{k} \sin \beta$
  \item[(b)] $\hat{r} = \hat{x} + \hat{y} + \hat{k} [z(x, y) - h]$
  \item[(c)] $\hat{n} = \hat{n}_x + \hat{n}_y + \hat{k} n_z$
\end{itemize}
and \( z(x,y)-h \) is defined to be the scattering surface.

Further progress depends upon the specification of the incident magnetic field. An incident plane wave will be assumed, so that for parallel polarization

\[
H_1 = \frac{E_0}{\gamma} (isiny-jcosy) e^{i(w^*t-k^*(\vec{R}_1 \cdot \vec{j}))} ,
\]

where \( k^* = \frac{2\pi}{\lambda} f^* \), \( w^* = 2\pi f^* \),

and \( \gamma \) is the impedance characteristic of the medium above \( z(x,y)-h \). Since the velocity of the radar system is small compared to the velocity of light, (6, pp. 243-245),

\[
f^* = f_o + \frac{v}{\lambda} \cos \theta ,
\]

where \( v \) is the velocity of the radar system,
\( f_o \) is the frequency of the transmitted radiation,
\( \lambda \) is the wave length of the transmitted radiation.

For perpendicular polarization, the incident magnetic field is

\[
H_1 = \frac{E_0}{\gamma} (isin\beta \cos \gamma + jsin\beta \sin \gamma + k \cos \beta) e^{i(w^*t-k^*(\vec{R}_1 \cdot \vec{j}))} .
\]
From (2.1.6), (2.1.10), (2.1.11), and (2.1.13), omitting the factor \( e^{i\omega t} \),

\[
(2.1.14) \quad \mathbf{E}(\alpha, \beta, \psi, \gamma, \theta, \theta + \Delta \theta) = - \frac{i \omega}{2 \pi R} \mathbf{E}_0 \frac{e^{-i k R}}{\gamma}
\]

\[
\left\{ \begin{array}{l}
(\cos \gamma \sin \beta + \sin \gamma \sin \beta + \mathbf{k} \cos \beta) \text{ for parallel polarization} \\
(\sin \gamma - \mathbf{k} \cos \gamma) \text{ for perpendicular polarization}
\end{array} \right.
\]

\[
\int (n_x \cos \beta \cos \gamma + n_y \cos \beta \sin \gamma - n_z \sin \beta) e^{-i(k + k^*)} \mathbf{R}^1 dS,
\]

where

\[
(2.1.15) \quad \mathbf{R} \cdot \mathbf{R}^1 = \cos \beta \cos \gamma + [h - z(x, y)] \sin \beta.
\]

In a similar manner, it can be seen from (2.1.9) and (2.1.14) that the scattered magnetic field at the radar system with a frequency between \( f - \Delta f \) and \( f \) will be

\[
(2.1.16) \quad \mathbf{H}(\alpha, \beta, \psi, \gamma, \theta, \theta + \Delta \theta) = - \frac{i \omega}{2 \pi R} \mathbf{E}_0 \frac{e^{-i k R}}{\gamma}
\]

\[
\left\{ \begin{array}{l}
(\sin \gamma - \mathbf{k} \cos \gamma) \text{ for parallel polarization} \\
(\cos \gamma \sin \beta + \sin \gamma \sin \beta + \mathbf{k} \cos \beta) \text{ for perpendicular polarization}
\end{array} \right.
\]

\[
\int (n_x \cos \beta \cos \gamma + n_y \cos \beta \sin \gamma - n_z \sin \beta) e^{-i(k + k^*)} \mathbf{R} \cdot \mathbf{R}^1 dS.
\]

From (2.1.8), (2.1.9), (2.1.14), and (2.1.16), it is
evident that the scalar portion of the electric and magnetic field density spectrums for both parallel and perpendicular polarizations is proportional to

\[(2.1.17) \quad I(\theta) = \lim_{\Delta \theta \to 0} \frac{1}{f-(f-\Delta f)} \left\{ -\frac{iW}{2\pi R} e^{-ikR} \frac{E_0}{\eta} \right\} \int_{D_{\theta, \theta+\Delta \theta}} (n_x \cos\beta \cos\gamma + n_y \cos\beta \sin\gamma - n_z \sin\beta)e^{-i(k+k')\bar{r}\cdot\bar{R}_1} \, ds. \]

Since \( v \ll c \),

(a) \( f = f_0 + \frac{2v}{\lambda} \cos\theta \),

(b) \( f - \Delta f = f_0 + \frac{2v}{\lambda} \cos(\theta+\Delta \theta) \).

Substituting (2.1.18) into (2.1.17),

\[(2.1.19) \quad I(\theta) = \lim_{\Delta \theta \to 0} \frac{\lambda/2v}{\cos\theta - \cos(\theta+\Delta \theta)} \frac{-iW}{2\pi R} e^{-ikR} \frac{E_0}{\eta} \int_{D_{\theta, \theta+\Delta \theta}} (n_x \cos\beta \cos\gamma + n_y \cos\beta \sin\gamma - n_z \sin\beta)e^{-i(k+k')\bar{r}\cdot\bar{R}_1} \, ds. \]

From differential geometry

(a) \( n_x = -z_x(x,y)/(EG-F^2)^{1/2} \),

(b) \( n_y = -z_y(x,y)/(EG-F^2)^{1/2} \),

(c) \( n_z = 1/(EG-F^2)^{1/2} \),

(d) \( S = (EG-F^2)^{1/2} \, dydx \).
The substitution of (2.1.20) into (2.1.19) yields

\[ I(\theta) = \lim_{\Delta \theta \to 0} \left\{ \frac{\lambda}{2\nu} \cos^2 - \cos(\theta + \Delta \theta) \left\{ \frac{i\omega}{2\pi R} e^{-iK\theta} \right\} \right\} \]

\[ \iint_D (z_x(x,y)\cos\beta \cos\gamma + z_y(x,y)\cos\beta \sin\gamma + \sin\beta) \cdot e^{-i(K + K^*) R} \, dy \, dx. \]

It will be required that \( z(x,y) \) and its first derivatives exist and are finite. This requirement is not unreasonable when it is recalled that the surfaces have previously been restricted to ones that have everywhere a small curvature and where there is no shielding of any part by any other part. Also the region of integration has previously been restricted to be finite, hence equation (2.1.21) can be rewritten as

\[ I(\theta) = \frac{\text{fcBo}}{2\nu e^\gamma R} e^{-iK\theta} \lim_{\Delta \theta \to 0} \frac{1}{\cos^2 - \cos(\theta + \Delta \theta)} \]

\[ \iint_D (z_x(x,y)\cos\beta \cos\gamma + z_y(x,y)\cos\beta \sin\gamma + \sin\beta) \cdot e^{-i(K + K^*) R} \, dy \, dx. \]

It follows from (2.1.22) and (2.1.15) that

\[ I(\theta) = \lim_{\Delta \theta \to 0} \frac{1}{\cos^2 - \cos(\theta + \Delta \theta)} \]

\[ \iint_D K(x,y) \cdot e^{-i(K + K^*) (x \cos\beta \cos\gamma - z(x,y) \sin\beta)} \, dy \, dx. \]
where

\[(2.1.24) \quad S = \frac{f_{eE_0}}{2\nu \omega R} e^{-i(k+k^*)h \sin \beta},\]

\[(2.1.25) \quad K(x,y) = z_x(x,y) \cos \beta \cos \gamma + z_y(x,y) \cos \beta \sin \gamma + \sin \beta.\]

It will be recalled that the region of integration in (2.1.23) is defined by the intersection of the $\theta$ cone, the $\theta + \Delta \theta$ cone, and the $\alpha$ cone with the plane $z = -h$.
From (2.1.23) and Fig. 2, it is seen that

\[ (2.1.26) \quad I(\theta) = \lim_{\Delta \theta \to 0} \frac{1}{\cos \theta - \cos (\theta + \Delta \theta)} \]

\[ \left\{ \begin{array}{c}
\int_{x_2}^{x_1+\Delta x_1} \int_{y_1}^{y(\theta,x)} y(a,x) \\
\int_{y(\theta,x)}^{y(\theta + \Delta \theta,x)} K(x,y) e^{-i(k+k')(x\cos \gamma - y\sin \gamma)} dy dx \\
\int_{y(\theta,x)}^{y(\theta + \Delta \theta,x)} K(x,y) e^{-i(k+k')(x\cos \gamma - y\sin \gamma)} dy dx
\end{array} \right\} \]

Since the integrand in all three parts of (2.1.26) exists and is finite over the entire range of integration, it follows that

\[ (2.1.27) \quad I(\theta) = \lim_{\Delta \theta \to 0} \frac{1}{\cos \theta - \cos (\theta + \Delta \theta)} \]

\[ \int_{x_2}^{x_1} \int_{y(\theta,x)}^{y(\theta + \Delta \theta,x)} K(x,y) e^{-i(k+k')(x\cos \gamma - y\sin \gamma)} dy dx . \]

The region of integration in (2.1.27) has previously been restricted to be finite, hence it is permissable to interchange the integration with respect to \( x \) and the limit process. This results in
With the use of L'Hopital's Rule, equation (2.1.28) can be rewritten as

\[
(2.1.29) \quad I(\theta) = S \int_{x_1}^{x_2} \left\{ \lim_{\Delta \theta \to 0} \frac{1}{\cos \theta \cos \theta + \Delta \theta} \right\} \left\{ e^{-i(k^* + k^x)} e^{-ix} \left[ x \cos \theta \cos \gamma - z(x,y) \sin \beta \right] \right\} \, dy \, dx.
\]

From (2.1.1) and (2.1.29)

\[
(2.1.30) \quad I(\theta) = \frac{S}{\cos^3 \theta} \int_{x_1}^{x_2} K[x, y(\theta, x)]
\]

\[
\frac{(x \cos \psi + x \sin \psi)^2}{[(x \cos \psi + x \sin \psi)^2 \sec^2 x - 2 \cdot 2 \cdot 2 \cdot 1/2} e^{-i(k^* + k^x)(x \cos \theta \cos \gamma - z(x,y) \sin \beta)} \right\} \, dx.
\]

It appears to be advantageous to make the substitution
\[(2.1.31) \quad a + dsint = x\cos\psi + h\sin\psi,\]

where

\[(a) \quad d^2 = a^2 - b^2,\]
\[(b) \quad a = \frac{h\sin\psi}{1 - \sec^2\theta \cos^2\psi},\]
\[(2.1.32) \quad (c) \quad b^2 = \frac{h^2}{1 - \sec^2\theta \cos^2\psi},\]
\[(d) \quad \theta \neq \psi,\]
\[(e) \quad \theta \neq \frac{\pi}{2}.\]

Substituting \((2.1.31)\) and \((2.1.32)\) into \((2.1.30)\),

\[(2.1.33) \quad I(\theta) = \int_{t_1}^{t_2} K_1(\theta, t) e^{-i(Msint - \rho z_1(\theta, t))} (dsint + a)^2 \, dt,\]

where

\[(2.1.34) \quad L = \frac{1}{\cos^2\theta (\cos^2\theta - \cos^2\psi)^{1/2}} \left\{ \frac{1}{2\nu^2} e^{-i(kR + \rho h)} \right\},\]

\[e^{-i(k + k^*) \cos\beta \cos\psi \sin\beta \cos\psi} \cos^2\theta - \cos^2\psi\]

\[(2.1.35) \quad M = \frac{h(k + k^*) \cos\beta \cos\psi \sin\theta \cos\theta}{\cos^2\theta - \cos^2\psi},\]
Equation (2.1.36) is an expression that is proportional to the scalar portion of the field density spectrum. Further progress depends upon the determination of the limits of integration and the specification of the scattering surface.
2. LIMITS OF INTEGRATION

It is seen that there are five cases to be considered:

Case I \[ y(\theta, x_1) \geq 0, \ y(\theta, x_2) \geq 0 \],

Case II \[ y(\theta, x_1) \geq 0, \ y(\theta, x_2) \leq 0 \],

Case III \[ y(\theta, x_1) < 0, \ y(\theta, x_2) > 0 \],

Case IV \[ y(\theta, x_1) < 0, \ y(\theta, x_2) < 0 \],

Case V No intersection.

A return to the radar system will be realized for Case V only if the intersection of the \( \theta \) cone with the \( z = -h \) plane is within the intersection of the \( \alpha \) cone with the \( z = -h \) plane. It follows from Fig. 1 that a return will be realized in the other four cases only if

\[(2.2.1) \quad \theta_1 < \theta < \theta_2,\]

where

\[(2.2.2) \quad (a) \quad \theta_1 = \delta - \alpha, \quad (b) \quad \theta_2 = \delta + \alpha.\]

In the above equations, \( \delta \) is defined to be the angle between the centers of the \( \alpha \) cone and the \( \theta \) cone, hence,

\[(2.2.3) \quad \cos \delta = \cos \alpha \cos \psi + \sin \alpha \sin \psi.\]

The situation that has been considered in all the previous work is described by Fig. 2 and corresponds to
Case I. By a process similar to that used in the development of (2.1.30), it can be seen that the following expressions are valid for the corresponding cases.

Case II

\[ (2.2.4) \quad I(\theta) = \frac{S}{\cos^3 \theta} \left[ \int_{x_1}^{m} G[x, y(\theta, x)] \, dx + \int_{x_2}^{m} F[x, y(\theta, x)] \, dx \right], \]

Case III

\[ (2.2.5) \quad I(\theta) = \frac{S}{\cos^3 \theta} \left[ \int_{x_1}^{x_2} F[x, y(\theta, x)] \, dx + \int_{x_1}^{x_2} G[x, y(\theta, x)] \, dx \right], \]

Case IV

\[ (2.2.6) \quad I(\theta) = \frac{S}{\cos^3 \theta} \left[ \int_{x_1}^{x_2} F[x, y(\theta, x)] \, dx + \int_{x_1}^{x_2} G[x, y(\theta, x)] \, dx \right], \]

Case V

\[ (2.2.7) \quad I(\theta) = \frac{S}{\cos^3 \theta} \left[ \int_{x_1}^{x_2} G[x, y(\theta, x)] + F[x, y(\theta, x)] \right] \, dx, \]

where \( \ell \) and \( m \) are the intersections of the \( \theta \) ellipse with the projection of the \( x \) axis on the \( z = -h \) plane

and from (2.1.1), it is seen that

\[ (2.2.8) \]

(a) \( \ell = h \cotn (\psi + \theta) \),

(b) \( m = h \cotn (\psi - \theta) \),

and
(a) \( G(x,y(\theta,x)) = K(x,y(\theta,x)) \)

\[
\frac{(x \cos \psi + y \sin \psi)^2}{(\cos \psi + y \sin \psi)^2 \sec^2 \theta - x^2 - h^2}^{1/2} \\
\exp[-i(k+k^*)(x \cos \psi + y \sin \psi)] \sin \beta,
\]

(b) \( F(x,y(\theta,x)) = G(x,-y(\theta,x)) \).

If the change of variable (2.1.31) is substituted in (2.2.4) through (2.2.7) inclusive, the result is:

Case II

(2.2.10) \( I(\theta) = L \left[ \int_{t_1}^{m_1} G_1(\theta,t) \, dt + \int_{t_2}^{m_1} F_1(\theta,t) \, dt \right] \),

Case III

(2.2.11) \( I(\theta) = L \left[ \int_{t_1}^{m_1} F_1(\theta,t) \, dt + \int_{t_1}^{m_1} G_1(\theta,t) \, dt \right] \),

Case IV

(2.2.12) \( I(\theta) = L \left[ \int_{t_1}^{m_1} F_1(\theta,t) \, dt + \int_{t_1}^{m_1} G_1(\theta,t) \, dt + \int_{t_2}^{m_1} F_1(\theta,t) \, dt \right] \),

Case V

(2.2.13) \( I(\theta) = L \int_{t_1}^{m_1} [G_1(\theta,t) + F_1(\theta,t)] \, dt \),

where
\[ G(\theta, t) = K_1(\theta, t)e^{-i(M\sin\theta-P_1(\theta, t))}(ds\sin\theta+a)^2, \]

\[ F(\theta, t) = G \left[ \frac{\sin\theta\cos\theta \sin\psi + \sin\psi \cos\psi}{\cos^2\theta - \cos^2\psi} \frac{\sin\theta \cos\theta}{[\cos^2\theta - \cos^2\psi]^{1/2}} \right]. \]

The limits of integration appearing in (2.2.10) through (2.2.13), inclusive, and (2.1.33) will now be determined. From (2.2.8) and (2.1.31),

\[ (2.2.14) \quad \ell_1 = \sin^{-1} \left[ \frac{(\cos^2\theta - \cos^2\psi)\cotn(\psi+\theta) - \cos\psi \sin\psi}{\cos\theta \sin\theta} \right] = -\frac{\pi}{2}, \]

\[ (2.2.15) \quad m_1 = \sin^{-1} \left[ \frac{(\cos^2\theta - \cos^2\psi)\cotn(\psi-\theta) - \cos\psi \sin\psi}{\cos\theta \sin\theta} \right] = \frac{\pi}{2}. \]

The simultaneous solution\(^1\) of (2.1.1) and (2.1.4) yields [10, pp.3-11]

\[ (2.2.16) \quad x(\cos\theta \cos\beta \cos\psi + \cos\alpha \cos\psi) + y \cos\theta \cos\beta \sin\psi \]

\[ = h(\cos \sin \psi - \cos \theta \sin \beta). \]

Taking the first sign in (2.2.16), the simultaneous solution with (2.1.1) yields a quadratic equation in \(x\). If the substitution

\[ (2.2.17) \quad u = x \cos \psi + h \sin \psi \]

\(^1\) Most of the following results on the limits of integration were taken from Boeing Mathematical Note No.35, "On the Doppler Spectrum for the 3-Dimensional Case", by W.M. Stone. It was felt that this should be included to obtain a reasonable amount of continuity.
is made, the result is a quadratic equation in $u$.

\[(2.2.18) \quad pu^2 - 2qu + sh^2 = 0,\]

where

\[(a) \quad p = (\cos\theta\cos\beta\cos\gamma - \cos\cos\psi)^2 + \cos^2\beta\sin^2\gamma(\cos^2\theta - \cos^2\psi),\]

\[(b) \quad q = \cos\theta[(\cos\theta\cos\beta\cos\gamma - \cos\cos\psi)\]

\[\cos\beta\sin\gamma\sin\psi - \sin\theta\cos\psi\]

\[(c) \quad s = \cos^2\theta[(\cos\beta\cos\gamma\sin\psi - \sin\beta\cos\psi)^2 + \cos^2\beta\sin^2\gamma]\]

\[(d) \quad q^2 - pr = \cos^2\theta\cos^2\psi\]

\[\sin^2\alpha\sin^2\delta - (\cos\cos\delta - \cos\theta)^2.\]

It follows that (10, p. 11)

\[(2.2.20) \quad \frac{u_i}{h} = \frac{\cos\theta}{(\cos\theta\cos\beta\cos\gamma - \cos\cos\psi)^2 \cos^2\beta\sin^2\gamma(\cos^2\theta - \cos^2\psi)}\]

\[\{(\cos\cos\beta\cos\gamma - \cos\cos\psi)\]

\[\cos\beta\cos\gamma\sin\psi - \sin\beta\cos\psi\]

\[+ \cos\theta\cos^2\beta\sin^2\gamma\sin\psi + R\cos\beta\sin\gamma\cos\psi\}^i\]

where $i = 1$ for the minus sign and $i = 2$ for the plus sign and
\( R^2 = \sin^2 \phi \sin^2 \phi - (\cos \phi \cos \theta - \cos \phi)^2 \).

From (2.1.31), it is seen that

\[
(2.2.22) \quad t_i = \sin^{-1} \left( \frac{u_i - a}{d} \right), \quad i = 1, 2.
\]

Equations (2.2.14), (2.2.15), and (2.2.22) are expressions for the limits of integration appearing in (2.2.10) through (2.2.13), inclusive.

In all of the following work, the situation described by Case I will be assumed. It can be seen that the only essential difference between the different cases is the limits of integration, hence all cases can be handled in the same manner as Case I. Further progress necessitates the specification of the scattering surface.
3. DOUBLY SINUSOIDAL SURFACE

The case where the scattering surface is a double sinusoid will now be considered.

Let
\[ z(x,y) = A \cos(Px) \cos(Qy) \]

Substituting (2.3.1) into (2.1.25), it is seen that
\[ K(x,y) = -(AP\sinPx\cosQy) \cos\beta \cos\gamma \]
\[ -(AQ\cosPx\sinQy) \cos\beta \sin\gamma \]
\[ + \sin\beta \]

From (2.3.2) and (2.1.31),
\[ K[\theta, t] = -[AP\sin(P(\varepsilon \sin t + \zeta) \cos(\xi \cot t)) \cos\beta \cos\gamma] \]
\[ -[AQ\cos(P(\varepsilon \sin t + \zeta) \sin(\xi \cot t)) \cos\beta \sin\gamma] \]
\[ + \sin\beta \]

where
\[ \varepsilon = \frac{h \sin \theta \cos \xi}{\cos^2 \theta - \cos^2 \psi} \]
\[ \zeta = \frac{h \cos \theta \sin \psi}{\cos^2 \theta - \cos^2 \psi} \]
\[ \xi = \frac{h \sin \theta}{(\cos^2 \theta - \cos^2 \psi)^{1/2}} \]

Substituting (2.3.3) into (2.1.33) results in
(2.3.4) \[ I(\theta) = L \int_{t_1}^{t_2} -\cos \beta \cos \gamma A \sin P(\varepsilon \sin t + j) \cos Q(\zeta \cos t) \]
\[ - \cos \beta \sin \gamma A \cos P(\varepsilon \sin t + j) \sin Q(\zeta \cos t) \]
\[ + \sin \beta \int_{t_1}^{t_2} e^{-iM \sin t - jA \cos P(\varepsilon \sin t + j) \cos Q(\zeta \cos t)} \]
\[ (\sin \theta + a)^2 \, dt \]

The scalar portion of the field density spectrums from a doubly sinusoidal scattering surface is proportional to (2.3.4). The scattering surface becomes a plane if \( A = 0 \), hence for this case

(2.3.5) \[ I(\theta) = L \sin \beta \int_{t_1}^{t_2} e^{-iM \sin t} (\sin \theta + a)^2 \, dt \]
4. SURFACE REPRESENTED BY A FOURIER SERIES

The case where the scattering surface is any surface that can be represented by a Fourier series will now be considered. Let

\[(2.4.1) \quad z(x,y) = \sum_{\rho=-\infty}^{\infty} \sum_{q=0}^{\infty} a_{\rho q} e^{i(\rho x + qy)} \]

where to make the surface real, the following condition is imposed:

\[(2.4.2) \quad a_{-\rho,-q} = \overline{a_{\rho,q}}.\]

In this case, the tilde indicates the complex conjugate.

The substitution of (2.4.1) into (2.1.25) results in

\[(2.4.3) \quad K(x,y) = i\cos\beta\cos\gamma \sum_{\rho=-\infty}^{\infty} \sum_{q=0}^{\infty} a_{\rho q} e^{i(\rho x + qy)} + i\cos\beta\sin\gamma \sum_{\rho=-\infty}^{\infty} \sum_{q=0}^{\infty} a_{\rho q} e^{i(\rho x + qy)} + \sin\beta.\]

From (2.4.3) and (2.1.31),

\[(2.4.4) \quad K(\theta,t) = i\cos\beta\cos\gamma \sum_{\rho=-\infty}^{\infty} \sum_{q=0}^{\infty} a_{\rho q} e^{i[\rho(\varepsilon\sin t + \gamma) + \gamma \cos t]} + i\cos\beta\sin\gamma \sum_{\rho=-\infty}^{\infty} \sum_{q=0}^{\infty} a_{\rho q} e^{i[\rho(\varepsilon\sin t + \gamma) + \gamma \cos t]} + \sin\beta.\]
Substituting (2.4.4) into (2.1.33) results in

\[
I(\theta) = L \int_{t_1}^{t_2} \left\{ \cos \beta \left[ \sum_{\rho=0}^{\infty} \sum_{q=0}^{\infty} (\rho \cos + q \sin y) \right. \\
- \left. a_{\rho q} e^{i[\rho(\epsilon \sin + \lambda) + \gamma \cos \theta]} \sin \beta \right] \right. \\
- \left. e^{-i(M \sin + \rho \sum_{\rho=0}^{\infty} \sum_{q=0}^{\infty} a_{\rho q} e^{i[\rho(\epsilon \sin + \lambda) + \gamma \cos \theta]}} \right) \\
\left. (\epsilon \sin \theta + a)^2 \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. dt .
\]
1. FORMULATION

Let \( z(x,y) \) be a separable random process, real and continuous in the mean over a finite region \( D \) with mean zero and covariance function \( r(x,y,x',y') \). Since \( z(x,y) \) is to represent a surface, it will be required to be twice mean square differentiable to assure the differentiability of the process (1, p.140). The surface is a separable random function if and only if there exists a bilinear representation (7, p.27):

\[
(3.1.1) \quad r(x,y,x',y') = \sum_{m,n=1}^{\infty} \frac{1}{\lambda_{mn}} \varphi_{mn}(x,y) \varphi_{mn}(x',y'),
\]

where \( \varphi_{mn} \) and \( \lambda_{mn} \) are eigenfunctions and eigenvalues of the integral equation:

\[
(3.1.2) \quad \varphi(x,y) = \iint_D r(x,y,x',y') \varphi(x',y') \, dx' \, dy'.
\]

Hence, for every \( (x,y) \in D \) (7, p.27),

\[
(3.1.3) \quad z(x,y) = \sum_{m,n=1}^{\infty} \lambda_{mn}^{-1/2} \varphi_{mn}(x,y) z_{mn},
\]

where
(a) $E' (z_{mn}) = 0$,

(3.1.4)

(b) $E' (z_{mn} z_{pq}) = \begin{cases} 
0 & \text{if } m\neq n \text{ and/or } p\neq q \\
1 & \text{if } m=n \text{ and } p=q. 
\end{cases}$

The symbol $E'$ is used to denote mathematical expectation.

The portion of the far zone electromagnetic field that arrives at the radar system with a frequency between $f - \Delta f$ and $f$ will be scattered by $D_{\theta, \theta + \Delta \theta}$ and is given by Silver (9, pp.144-149).

(3.1.5) \[ \mathbf{E}(a, \beta, \gamma, \rho, \delta, \theta + \Delta \theta) = \frac{-i \overline{W} \mu}{2\pi R} e^{-i \mathbf{E} \mathbf{R}} \]

\[ \int \left\{ \mathbf{W} \times \mathbf{R}_1 - (\mathbf{n} \times \mathbf{R}_1 \cdot \mathbf{R}_1) \mathbf{R}_1 \right\} e^{-i \mathbf{k} \mathbf{R}_1 \cdot \mathbf{F}} d\mathbf{S}, \]

\[ D_{\theta, \theta + \theta} \]

where $\mathbf{k}, \mathbf{n}, \mu, \mathbf{R}, \mathbf{R}_1$, and $\mathbf{F}$ are defined as in (2.1.6) and

(a) $\mathbf{n} = (-\mathbf{i} x(x,y) - \mathbf{j} z(x,y) + \mathbf{k})/(E_0 - F^2)^{1/2}$

(3.1.6) (b) $d\mathbf{S} = (E_0 - F^2)^{1/2} dy dx$.

The integral in (3.1.5) is now a stochastic integral whose existence is assured as a result of a theorem by Doob (3, pp.62-63).

As before, for the case of parallel polarization,
(3.1.7) \[ \vec{H}_1 = \frac{E_0}{\gamma} (i \sin \gamma - i \cos \gamma) e^{i(w^*t - \vec{k}^*(\vec{R}_1 \cdot \vec{p}))}, \]

and for perpendicular polarization,

(3.1.8) \[ \vec{H}_1 = \frac{E_0}{\gamma} (\vec{i} \sin \beta \cos \gamma + \vec{j} \sin \beta \sin \gamma + \vec{k} \cos \beta) e^{i(w^*t - \vec{k}^*(\vec{R}_1 \cdot \vec{p}))}. \]

Substituting the quantities defined by (2.1.6), (3.1.6), (3.1.7), and (3.1.8) into (3.1.5),

(3.1.9) \[ \vec{E}(a, \beta, \psi, \gamma, \theta, \theta + \Delta \theta) = \frac{i \vec{w}}{2\pi R} e^{-ikR} \frac{E_0}{\gamma} \]

\[ \begin{cases} \left( \vec{i} \cos \gamma \sin \beta + \vec{j} \sin \gamma \sin \beta + \vec{k} \cos \beta \right) & \text{for parallel polarization} \\ \left( \vec{i} \sin \gamma - \vec{j} \cos \gamma \right) & \text{for perpendicular polarization} \end{cases} \]

\[ \int \int_{D_{\theta, \theta + \Delta \theta}} (z_x(x,y) \cos \beta \cos \gamma + z_y(x,y) \cos \beta \sin \gamma + \sin \beta) e^{-i(\vec{k} + \vec{k}^*) \cdot \vec{R}_1} \cdot \vec{p} \ dydx. \]

From the definition of \( C(a, \beta, \nu, \gamma, \theta) \) in (2.1.7), it is seen that the scalar portion of the field density spectrums is proportional to

(3.1.10) \[ I(\theta) = \frac{i \vec{w}}{2\pi R} e^{-ikR} \frac{E_0}{\gamma} \lim_{\Delta \theta \to 0} \frac{\lambda/2\nu}{\cos \theta - \cos(\theta + \Delta \theta)} \]

\[ \int \int_{D_{\theta, \theta + \Delta \theta}} (z_x(x,y) \cos \beta \cos \gamma + z_y(x,y) \cos \beta \sin \gamma + \sin \beta) e^{-i(\vec{k} + \vec{k}^*) \cdot \vec{R}_1} \cdot \vec{p} \ ds. \]

In much the same manner that was used to derive (2.1.30),
\begin{equation}
(3.1.11) \quad I(\theta) = \frac{s}{\cos^3 \theta} \int_{x_1}^{x_2} K[x, y(\theta, x)]
\end{equation}

\begin{align*}
&\quad e^{-i(k+k^*) \{x \cos \beta \cos \gamma - z[x, y(\theta, x) \sin \beta]\}} \\
&\quad \frac{(x \cos \psi + h \sin \psi)^2 \, dx}{[(x \cos \psi + h \sin \psi)^2 \sec^2 \theta - x^2 - h^2]^{1/2}},
\end{align*}

where \( K(x, y) = z_x(x, y) \cos \beta \cos \gamma + z_y(x, y) \cos \beta \sin \gamma + \sin \beta \),

\begin{align*}
S &= \frac{ifcE_0}{2fR \nu \gamma} \quad e^{-ikR} \quad e^{-i(k+k^*) \hbar \sin \beta}.
\end{align*}

Equation (3.1.11) is an expression that is proportional to the scalar portion of the field density spectrums scattered from a random surface. The expected value of the field density spectrums and the associated covariance will now be determined. The limits of integration for all the following expressions are precisely those that were determined in Section 2 of Chapter 2.
2. EXPECTED VALUE AND COVARIANCE

From (3.1.11) and (3.1.9), it is seen that the expected value of the electric field density spectrum is

\[ E'\left\{ \mathcal{E}(\alpha, \beta, \psi, \gamma, \theta) \right\} = \frac{S \mu}{\cos^3 \theta} \]

\[ \begin{cases} (\mathbf{\hat{r}} \cos \gamma \sin \beta + \mathbf{\hat{r}} \sin \gamma \sin \beta + \mathbf{\hat{k}} \cos \beta) \text{ for parallel polarization} \\ (\mathbf{\hat{r}} \sin \gamma - \mathbf{\hat{r}} \cos \gamma) \text{ for perpendicular polarization} \end{cases} \]

\[ E'\left\{ \int_{x_1}^{x_2} K[x, y(\theta, x)] e^{-i(k+k^*)z[x, y(\theta, x)]\sin \beta} \right\} \]

\[ \frac{(x \cos \psi + h \sin \psi)^2 dx}{[(x \cos \psi + h \sin \psi)^2 \sec^2 \theta - x^2 - h^2]^{1/2}} \].

A similar expression also exists for \( E'\{\mathcal{H}(\alpha, \beta, \psi, \gamma, \theta)\} \).

Hence, using Fubini's theorem (\( \theta \), pp. 207), it is seen that the scalar portion of the expected field density spectrums is proportional to

\[ E'\left\{ I(\theta) \right\} = \]

\[ \frac{S}{\cos^3 \theta} \left\{ \int_{x_1}^{x_2} \frac{(x \cos \psi + h \sin \psi)^2 e^{-i(k+k^*)x \cos \beta \cos \gamma}}{\left[(x \cos \psi + h \sin \psi)^2 \sec^2 \theta - x^2 - h^2\right]^{1/2}} dx \right\}. \]
From the definition of \( K(x,y) \) in (3.1.11), and equation (2.1.1), it is seen that

\[
(3.2.3) \quad E \left\{ K(x,y(\theta,x)) \ e^{i(k+k')} \ z[x,y(\theta,x)] \sin \beta \right\} =
\]

\[
E \left[ \left( \cos \beta \cos \gamma + \cos \beta \sin \gamma \frac{(\cos \psi + \sin \psi)^2 \sec^2 \theta - x^2 - h^2}{2} \right) \right]
\]

\[
\frac{\partial}{\partial x} + \sin \beta \right] \ e^{i(k+k')z(x,y(\theta,x))} \sin \beta.
\]

According to Bartlett, (1, p. 140) for a mean square differentiable random process \( V(\theta,x) \),

\[
(3.2.4) \quad E \{ V_x(\theta,x) \} = \frac{\partial}{\partial x} E \{ V(\theta,x) \}.
\]

Hence,

\[
(3.2.5) \quad E \left\{ z_x[x,y(\theta,x)] \ e^{i\rho z(x,y(\theta,x))} \right\} = - \frac{1}{\rho} \frac{\partial}{\partial x} E \left\{ e^{i\rho z(x,y(\theta,x))} \right\}.
\]

From (3.2.3) and (3.2.5),

\[
(3.2.6) \quad E \left\{ K(x,y(\theta,x)) \ e^{i\rho z(x,y(\theta,x))} \right\}
\]

\[
= \left\{ - \frac{1}{\rho} \left[ \cos \beta \cos \gamma + \cos \beta \sin \gamma \frac{(\cos \psi + \sin \psi)^2 \sec^2 \theta - x^2 - h^2}{2} \right] \right] \right\}
\]

\[
\frac{\partial}{\partial x} + \sin \beta \right] \ e^{i\rho z(x,y(\theta,x))} \right\}.
\]
Substituting (2.1.31) into (3.2.6),

\[(3.2.7) \quad E' \left\{ E_1(\theta, t) e^{i\phi z_1(\theta, t)} \right\} = \left\{ - \frac{1}{R^2} [A(\theta, t) - B(\theta, t)] \frac{\partial}{\partial t} + \sin \beta \right\} E' \left\{ e^{i\phi z_1(\theta, t)} \right\} \]

where

\[A(\theta, t) = \cos \beta \cos \gamma \frac{\cos^2 \theta - \cos^2 \psi}{\sin \theta \cos \theta \cos \theta},\]

\[B(\theta, t) = \cos \beta \sin \gamma \frac{[\cos^2 \theta - \cos^2 \psi]^{1/2}}{\sin \theta \sin \theta}.\]

From (2.1.31), (3.2.2), and (3.2.7),

\[(3.2.8) \quad E' \left\{ I(\theta) \right\} = L(\theta) \int_{t_1}^{t_2} (\sin t + a)^2 e^{-iN(\theta) \sin t} dt, \]

where

\[L(\theta) = \frac{1}{i \phi c E_0} \frac{1}{\cos^2 \theta (\cos^2 \theta - \cos^2 \psi)^{1/2}} \frac{1}{2 \sqrt{\gamma_0 R}} e^{-i(kR + \phi h)},\]

\[e^{i(k+k^*) \cos \beta \cos \sin \theta \cos \psi \cos^2 \theta - \cos^2 \psi},\]

\[N(\theta) = \frac{h(k + k^*) \cos \beta \cos \sin \theta \cos \theta}{\cos^2 \theta - \cos^2 \psi} \cdot \]

From (3.1.3) and (3.2.8),
\[(3.2.9) \ E'\{I(\theta)\} = L(\theta) \int_{t_1}^{t_2} (d\sin t + a)2e^{-iN(\theta)\sin t}
\]

\[\left\{ -\frac{i}{\rho} \left[ A(\theta, t) - B(\theta, t) \right] \frac{\partial}{\partial t} + \sin \beta \right\}
\]

\[E' \left\{ e^{i\rho} \sum_{m, n=1}^{\infty} \frac{\lambda_{mn}}{\varphi_{mn}(\theta, t)} z_{mn} \right\} dt,
\]

where \(\varphi_{mn}(\theta, t)\)

\[= \Phi_{mn} \left[ \frac{\sin \theta \cos \theta \sin t + \sin \gamma \cos \psi}{\cos^2 \theta - \cos^2 \psi} \frac{\sin \theta \cos \theta}{(\cos^2 \theta - \cos^2 \psi)^{1/2}} \right].
\]

Once the random surface has been specified, the expected field density spectrums can be determined from \((3.2.9)\) for both parallel and perpendicular polarization.

The covariance of the electric field density spectrum is by definition

\[(3.2.10) W(\alpha, \beta, \psi, \gamma, \theta, \theta') =
\]

\[E' \left\{ \left( \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta) - E'\{ \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta) \} \right) \cdot \left( \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta') - E'\{ \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta') \} \right) \right\},
\]

where the tilde indicates the complex conjugate in this case. Equation \((3.2.10)\) can be rewritten as

\[(3.2.11) W(\alpha, \beta, \psi, \gamma, \theta, \theta') =
\]

\[E' \left\{ \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta) \cdot \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta') \right\} - E' \left\{ \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta) \cdot E'\{ \tilde{\varphi}(\alpha, \beta, \psi, \gamma, \theta') \} \right\}.
\]
It is apparent in (3.2.11) that the covariance is related to

\[(3.2.12)\quad E\left\{ \hat{\mathbf{C}}(a,\beta,\psi,\gamma,\theta) \cdot \hat{\mathbf{C}}(a,\beta,\psi,\gamma,\theta) \right\} \]

\[= \frac{SS'}{\cos^3 \Theta \cos^3 \Theta'} \quad \left\{ \int_{x_1}^{x_2} K[x, y(\theta, x)] \right. \]

\[\quad \times \left. e^{-i(k+k^*)} \left\{ \cos \beta \cos \gamma - z[x, y(\theta, x)] \sin \beta \right\} \right\} \]

\[= \frac{(\cos \psi + h \sin \psi)^2 dx}{((\cos \psi + h \sin \psi)^2 \sec^2 \Theta - x^2 - h^2)^{1/2}} \]

\[\left\{ \int_{x_1}^{x_2} K[x, y(\theta, x)] \right. \]

\[\quad \times \left. e^{-i(k+k^*)} \left\{ \cos \beta \cos \gamma - z[x, y(\theta, x)] \sin \beta \right\} \right\} \]

\[\frac{(\cos \psi + h \sin \psi)^2 dx}{((\cos \psi + h \sin \psi)^2 \sec^2 \Theta - x^2 - h^2)^{1/2}} \] \}

where

\[\tilde{S}' = -i \frac{f \cdot c F_0}{2 \rho c R \nu \lambda} e^{i(kR + \phi)}\]

\[f = f_0 + \frac{2\nu}{\lambda} \cos \theta'\]

In the above equation, \(x_1'\) and \(x_2'\) are defined to be the points of intersection of the \(\theta'\) ellipse with the \(a\) ellipse.
Equation (3.2.12) can be written in the form

\[(3.2.13) \quad E' \{ \tilde{\mathcal{E}}(\alpha, \beta, \psi, \gamma, \theta) \cdot \tilde{\mathcal{E}}(\alpha, \beta, \psi, \gamma, \theta') \} \]

\[= \frac{\int_{\Omega} c^2 \mu E_o dx}{4f_o R^2 v^2 \gamma} \frac{1}{\cos^3 \theta \cos^3 \theta'} \]

\[E' \left\{ \int_{x_1}^{x_2} \int_{x_1'}^{x_2'} \mathcal{K}[x, y(\theta, x)] \mathcal{K}[x', y(\theta', x')] \right\} \]

\[= \frac{(x \cos \psi + h \sin \psi)^2}{[(x \cos \psi + h \sin \psi)^2 \sec^2 \theta - x^2 - h^2]^{1/2}} \]

\[= \frac{(x' \cos \psi + h \sin \psi)^2}{[(x' \cos \psi + h \sin \psi)^2 \sec^2 \theta' - x^2 - h^2]^{1/2}} \]

\[\exp\left\{ \int_{x_1}^{x_2} \int_{x_1'}^{x_2'} (x' - x) \cos \beta \cos \gamma \left( z[x, y(\theta, x)] - z[x', y(\theta', x')] \right) \sin \beta \right\} \]

\[dx' dx \} \]

With the use of Fubini's Theorem, the expression above becomes

\[(3.2.14) \quad E' \{ \tilde{\mathcal{E}}(\alpha, \beta, \psi, \gamma, \theta) \cdot \tilde{\mathcal{E}}(\alpha, \beta, \psi, \gamma, \theta') \} \]

\[= \frac{\int_{\Omega} c^2 \mu E_o dx}{4f_o R^2 v^2 \gamma} \frac{1}{\cos^3 \theta \cos^3 \theta'} \]

\[\int_{x_1}^{x_2} \int_{x_1'}^{x_2'} \frac{(x \cos \psi + h \sin \psi)^2}{[(x \cos \psi + h \sin \psi)^2 \sec^2 \theta - x^2 - h^2]^{1/2}} \]

\[\frac{(x' \cos \psi + h \sin \psi)^2}{[(x' \cos \psi + h \sin \psi)^2 \sec^2 \theta' - x^2 - h^2]^{1/2}} \]
\[ e^{i(k+k^*)(x'-x)\cos \beta \cos \gamma} \]

\[ E \{ K[x,y(\theta,x)]K[x',y(\theta',x')] \} \]

\[ e^{-i\rho\{z[x,y(\theta,x)]-z[x',y(\theta',x')]\}}dx'dx. \]

The following change of variable will now be made:

\[ dsint + a = x\cos \psi + h\sin \psi, \]

\[ d'sint' + a' = x'\cos \psi + h\sin \psi, \]

(a)

\[ d^2 = a^2 - b^2, \]

(b)

\[ d'^2 = a'^2 - b'^2, \]

\[ a = \frac{h\sin \psi}{1-\sec^2 \theta \cos^2 \psi}, \quad b^2 = \frac{\hbar^2}{1-\sec^2 \theta \cos^2 \psi}, \]

(c)

\[ a' = \frac{h\sin \psi}{1-\sec^2 \theta' \cos^2 \psi}, \quad b'^2 = \frac{\hbar^2}{1-\sec^2 \theta' \cos^2 \psi}. \]

Substituting (3.2.15) into (3.2.14), it follows that

(3.2.16) \[ E \{ \tilde{E}(a,\beta,\psi,\gamma,\theta) \cdot \tilde{E}'(a,\beta,\psi,\gamma,\theta') \} = \]

\[ \frac{ff' e^{\frac{2E_0}{2}}}{4f_0^2 R \nu^2 \gamma \cos^2 \theta \cos^2 \theta'} \]

\[ e^{i(k+k^')(a'-a)\cos \beta \cos \gamma} \]

\[ \frac{\cos \psi}{(\cos^2 \theta - \cos^2 \psi)^{1/2}(\cos^2 \theta' - \cos^2 \psi)^{1/2}}. \]
\[ \int_{t_1}^{t_2} \int_{t_1}^{t_2} (dsint + a)^2 (d'sint + a')^2 \]
\[ e^{i[N(\theta')sint - N(\theta)sint]} \]
\[ E' \{ K_1(\theta, t)K_1(\theta', t') \} \]
\[ e^{i\varphi[z_1(\theta, t) - z_1(\theta', t')]} \] \, dt' \, dt ,

where \( t_1 \) and \( t_2 \) are given by (2.2.22) with \( \theta \) replaced by \( \theta' \) and \( N(\theta') = (k + k') d' \cos \beta \cos \gamma \).

\[ K_1(\theta, t) = [A(\theta, t) - B(\theta, t)] \frac{\partial z_1(\theta, t)}{\partial t} + \sin \beta . \]

From the expression for \( K_1(\theta, t) \) in (3.2.16), it is seen that

\[ (3.2.17) \quad E' \{ K_1(\theta, t)K_1(\theta', t') e^{i\varphi[z_1(\theta, t) - z_1(\theta', t')]} \} = \]
\[ E' \{ ([A(\theta, t) - B(\theta, t)][A(\theta', t') - B(\theta', t')]) \]
\[ \frac{\partial z_1(\theta, t)}{\partial t} \frac{\partial z_1(\theta', t')}{\partial t'} \]
\[ + \sin \beta \{ [A(\theta', t') - B(\theta', t') \] \frac{\partial z_1(\theta', t')}{\partial t'} \]
\[ + [A(\theta, t) - B(\theta, t)] \frac{\partial z_1(\theta, t)}{\partial t} \}
\[ + \sin^2 \beta \} e^{i\varphi[z_1(\theta, t) - z_1(\theta', t')]} \} . \]
As a result of (3.2.4), the following equations are valid:

(3.2.18) (a) \( E' \left\{ z_1(t, \theta, \phi) e^{i \phi \left[ z_1(\theta, t) - z_1(\theta', t') \right]} \right\} = -\frac{1}{\rho} \frac{\partial}{\partial t} E' \left\{ e^{i \phi \left[ z_1(\theta, t) - z_1(\theta', t') \right]} \right\}; \\

(b) \( E' \left\{ z_1'(t', \theta', \phi') e^{i \phi \left[ z_1(\theta, t) - z_1(\theta', t') \right]} \right\} = \frac{1}{\rho} \frac{\partial}{\partial t'} E' \left\{ e^{i \phi \left[ z_1(\theta, t) - z_1(\theta', t') \right]} \right\}; \\

(c) \( E' \left\{ z_1(t, \theta, \phi) z_1'(t', \theta', \phi') e^{i \phi \left[ z_1(\theta, t) - z_1(\theta', t') \right]} \right\} = \frac{1}{\rho^2} \frac{\partial^2}{\partial t \partial t'} E' \left\{ e^{i \phi \left[ z_1(\theta, t) - z_1(\theta', t') \right]} \right\}. \\

The substitution of (3.2.18) and (3.2.17) into (3.2.16) yields

(3.2.19) \( E' \left\{ \hat{F}^{(a, \beta, \gamma, \phi, \theta)} \cdot \hat{F}^{(a, \beta, \gamma, \phi, \theta')} \right\} = \frac{ff' c^2 \mu^2 E_0^2}{4f_0^2 R^2 \gamma \cos^2 \theta \cos^2 \theta'} \frac{1}{(\cos^2 \theta - \cos^2 \psi)^{1/2}(\cos^2 \theta' - \cos^2 \psi)^{1/2}} \frac{1}{(k+k^+) (a'-a) \cos \beta \cos \gamma} \cos \psi \).
\[
\int_{t_1}^{t_2} \int_{t'_1}^{t'_2} (\mathrm{dsint} + a)^2 (\mathrm{d'sint'} + a')^2 e^{i[N(\theta')\mathrm{sint'} - N(\theta)\mathrm{sint}]} \\
\left\{ [A(\theta,t) - B(\theta,t)] [A(\theta',t') - B(\theta',t')] \right\} \frac{1}{\rho^2} \frac{\partial^2}{\partial t \partial t'} \\
+ \frac{\mathrm{i} \sin \beta}{\rho} \left( [A(\theta,t) - B(\theta,t)] \right) \frac{\partial}{\partial t'} \\
\left\{ [A(\theta',t') - B(\theta',t')] \right\} \frac{\partial}{\partial t} \\
- \left( [A(\theta,t) - B(\theta,t)] \right) \frac{\partial}{\partial t} \\
\mathrm{dt}' \mathrm{dt} .
\]

From (3.2.19), (3.2.9), (3.2.1), and (3.1.3),

(3.2.20) \ W(a, \beta, \psi, \gamma, \theta, \theta', \gamma) = \frac{ff' e^2 \mu^2 E_0^2}{4f_0^2 \gamma^2 R^2} \\
\frac{1(k+k')(a'-a)}{\cos \beta \cos \gamma} \frac{\cos \psi}{\cos \psi} \\
\frac{1}{\cos^2 \theta \cos^2 \theta'} (\cos^2 \theta - \cos^2 \psi)^{1/2} \frac{1}{(\cos^2 \theta' - \cos^2 \psi)^{1/2}} \\
\left\{ \int_{t_1}^{t_2} \int_{t'_1}^{t'_2} (\mathrm{dsint} + a)^2 (\mathrm{d'sint'} + a')^2 e^{i[N(\theta')\mathrm{sint'} - N(\theta)\mathrm{sint}]} \\
\left\{ [A(\theta,t) - B(\theta,t)] [A(\theta',t') - B(\theta',t')] \right\} \frac{1}{\rho^2} \frac{\partial^2}{\partial t \partial t'} \\
+ \frac{\mathrm{i} \sin \beta}{\rho} \left( [A(\theta,t) - B(\theta,t)] \right) \frac{\partial}{\partial t'} \\
\left\{ [A(\theta',t') - B(\theta',t')] \right\} \frac{\partial}{\partial t} \\
- \left( [A(\theta,t) - B(\theta,t)] \right) \frac{\partial}{\partial t} \right\} \mathrm{dt}' \mathrm{dt} .
\]
\begin{align*}
\{ e^{i\rho} \sum_{m,n=1}^{\infty} \lambda_{mn} \Phi_{1mn}(\theta,t)z_{mn} \} \\
&\quad \times \left\{ e^{-i\rho} \sum_{m,n=1}^{\infty} \lambda_{mn} \Phi_{1mn}(\theta',t')z_{mn} \right\} \\
&\quad - \int_{t_1}^{t_2} \int_{t_1}^{t_2} (dsint + a)^2 (dsint' + a')^2 \\
&\quad \times \left\{ i[N(\theta')sint' - N(\theta)sint] \right\} \\
&\quad \times \frac{1}{\rho^2} \frac{\partial}{\partial t} \frac{\partial}{\partial t'} \\
&\quad + \frac{isin\beta}{\rho} \left\{ A(\theta',t') - B(\theta',t') \right\} \frac{\partial}{\partial t} \\
&\quad + \sin^2 \beta \} E' \\
&\quad \times \left\{ e^{i\rho} \sum_{m,n=1}^{\infty} \lambda_{mn} \Phi_{1mn}(\theta,t)z_{mn} \right\} \\
&\quad - \frac{1}{\rho^2} \frac{\partial}{\partial t} \frac{\partial}{\partial t'} \\
&\quad - \left\{ A(\theta,t) - B(\theta,t) \right\} \frac{\partial}{\partial t} \\
&\quad + \sin^2 \beta \} E' \\
&\quad \times \left\{ e^{-i\rho} \sum_{m,n=1}^{\infty} \lambda_{mn} \Phi_{1mn}(\theta',t')z_{mn} \right\} \\
&\quad \times dt'dt \\
\end{align*}

Equation (3.2.20) is an expression for the covariance of the electric field density spectrum that is valid for both perpendicular and parallel polarization. It can be seen that the covariance goes to zero as it should for the case of a random surface where the motion of any particular part is independent of all other parts. The
covariance for the magnetic field spectrum is identical to (3.2.20) except for the quantity $\mu^2$ being replaced by $\varepsilon \mu$. The particular case where the scattering surface is described by a Gaussian statistic will now be considered.
3. GAUSSIAN SURFACE

Let \( z(x, y) \) be a Gaussian surface with mean zero and standard deviation \( \sigma^2(x, y) \). From the definition of mathematical expectation,

\[
E \left\{ e^{i\varphi z(x, y)} \right\} = \frac{1}{\sqrt{2\pi} \sigma(x, y)}
\]

(3.3.1)

\[
\int_{-\infty}^{\infty} e^{i\varphi z(x, y)} e^{-\frac{z^2(x, y)}{2\sigma^2(x, y)}} dz ,
\]

(b) \( E \left\{ e^{-i\varphi z(x, y)} \right\} = \frac{1}{\sqrt{2\pi} \sigma(x, y)}
\]

\[
\int_{-\infty}^{\infty} e^{-i\varphi z(x, y)} e^{-\frac{z^2(x, y)}{2\sigma^2(x, y)}} dz .
\]

Hence,

(a) \( E \left\{ e^{i\varphi z(x, y)} \right\} = e^{-\frac{\varphi^2 \sigma^2(x, y)}{2}} \)

(3.3.2)

(b) \( E \left\{ e^{-i\varphi z(x, y)} \right\} = e^{-\frac{\varphi^2 \sigma^2(x, y)}{2}} \)

According to Cramer (2, pp.287-288),

(3.3.3) \( E \left\{ e^{i\varphi[z(x, y) - z(x', y')]} \right\} \)

\[
= e^{-1/2 \left( \sigma^2(x, y) - 2r(x, y, x', y') + \sigma^2(x', y') \right)}. \]
It is now possible to write

\[(a) \quad E' \left\{ e^{i\varphi z_1(\theta, t)} \right\} = e^{-\frac{\rho^2 \sigma_1^2(\theta, t)}{2}},\]

\[(3.3.4) \quad (b) \quad E' \left\{ e^{-i\varphi z_1(\theta, t)} \right\} = e^{-\frac{\rho^2 \sigma_1^2(\theta, t)}{2}},\]

\[(c) \quad E' \left\{ e^{i\varphi [z_1(\theta, t) - z_1(\theta', t')]} \right\} = e^{-1/2 \rho^2 [\sigma_1^2(\theta, t) - 2r_1(\theta, t, \theta', t') + \rho^2(\theta', t')]} \]

where

\[\sigma_1^2(\theta, t) = \sigma^2 \left[ \frac{\sin \theta \cos \theta \sin t + \sin \theta' \cos \theta'}{\cos^2 \theta - \cos^2 \psi} \right],\]

\[\frac{\sin \theta \cos \theta}{(\cos^2 \theta - \cos^2 \psi)^{1/2}} \]

\[r_1(\theta, t; \theta', t') = r \left[ \frac{\sin \theta \cos \theta \sin t + \sin \theta' \cos \theta'}{\cos^2 \theta - \cos^2 \psi} \right],\]

\[\frac{\sin \theta' \cos \theta' \sin t' + \sin \theta' \cos \theta'}{\cos^2 \theta' - \cos^2 \psi} \]

\[\frac{\sin \theta' \cos \theta'}{(\cos^2 \theta' - \cos^2 \psi)^{1/2}} \]

The substitution of (3.3.4) into (3.2.8) and (3.2.20) results in
(3.3.5) \[ E \{ I(\theta) \} = L(\theta) \int_{t_1}^{t_2} (\text{dsint} + a)^2 e^{-iN(\theta)\text{sint}} \]

\[ \left\{ -i \frac{\partial}{\partial t} \sigma_1^2(\theta, t) \right\} [A(\theta, t) - B(\theta, t)] + \sin \beta \]

\[ e^{-i \frac{\sigma_1^2(\theta, t)}{2}} \int_{t_1}^{t_2} dt ; \]

(3.3.6) \[ W(a, \beta, \gamma, \theta, \theta) = \frac{\pi e^{\frac{1}{2} \mu_0 E_0^2}}{4 \pi v^2 \gamma R} \]

\[ i(k+k') (a'-a) \frac{\cos \theta \cos \psi}{\cos \psi} \]

\[ \frac{\cos^2 \theta \cos^2 \psi (\cos^2 \theta - \cos^2 \psi)^{1/2}}{(\cos^2 \theta' - \cos^2 \psi)^{1/2}} \]

\[ \int_{t_1}^{t_2} \int_{t_1}^{t_2'} (\text{dsint} + a)^2 (\text{dsint} + a')^2 \]

\[ e^{-iN(\theta)\text{sint} - N(\theta)sint} \]

\[ \left\{ [A(\theta, t) - B(\theta, t)][A(\theta', t') - B(\theta', t')] \right\} \]

\[ (-i \frac{\sigma_1^2(\theta, t)}{2} \left[ \frac{\partial}{\partial t} \sigma_1^2(\theta, t) - 2 \frac{\partial r_1(\theta, t, \theta', t')}{\partial t} \right] \]

\[ \left[ \frac{\partial}{\partial t'} \sigma_1^2(\theta', t') - 2 \frac{\partial r_1(\theta, t, \theta', t')}{\partial t'} \right] \]

\[ + \frac{\partial r_1(\theta, t, \theta', t')}{\partial t} \frac{\partial r_1(\theta, t, \theta', t')}{\partial t'} - \frac{i \rho \sin \beta}{2} \left[ (A(\theta', t') - B(\theta', t')) \right] \]

\[ \left( \frac{\partial}{\partial t'} - 2 \frac{\partial r_1(\theta, t, \theta', t')}{\partial t'} \right) - (A(\theta, t) - B(\theta, t)) \]
\[
\left( \frac{\partial \sigma^2_1(\theta, t)}{\partial t} - 2 \frac{\partial r_1(\theta, t, \theta', t')}{\partial t} \right) + \sin^2 \beta \right)
\]
\[
e^{-1/2 \beta^2 \left[ \sigma^2_1(\theta, t) - 2r_1(\theta, t, \theta', t') + \sigma^2_1(\theta', t') \right] dt'dt}
\]
\[
\int_{t_1}^{t_2} \int_{t_1}^{t'} (\text{d}s\text{int} + a)^2 (d's\text{int}' + a')^2
\]
\[
e^{i[N(\theta)\text{sint} - N(\theta')\text{sint}']} \]
\[
\left\{ \frac{\sigma^2}{4} [A(\theta, t) - B(\theta, t)][A(\theta', t') - B(\theta', t')] \right. \\
\frac{\partial \sigma^2_1(\theta, t)}{\partial t} \frac{\partial \sigma^2_1(\theta', t')}{\partial t'} \\
\left. - \frac{i\rho \sin \beta}{2} \left( [A(\theta', t') - B(\theta', t')] \frac{\partial \sigma^2_1(\theta', t')}{\partial t} \\
- [A(\theta, t) - B(\theta, t)] \frac{\partial \sigma^2_1(\theta, t)}{\partial t} \right) \right. \\
+ \sin^2 \beta \right\} e^{-1/2 \left[ \sigma^2_1(\theta, t) + \sigma^2_1(\theta', t') \right] dt'dt}.
\]

If \( \sigma^2(x, y) = \sigma_x^2 \) a constant, over the entire surface, equations (3.3.5) and (3.3.6) reduce to

(3.3.7) \quad E'[I(\theta)] = L(\theta) \sin \beta e^{-\frac{\rho^2 \sigma_x^2}{2}}
\]
\[
\int_{t_1}^{t_2} (\text{d}s\text{int} + a)^2 e^{iN(\theta)\text{sint}} dt ;
\]
\[(3.3.8) \quad W(\alpha, \beta, \psi, \gamma, \theta, \theta') = \frac{ff_c^2 \mu B_0^2}{4f_o \nu^2 \gamma^2 R^2} \]

\[1(k+k^*)(a' - a) \cos \beta \cos \gamma \cos^2 \theta \cos^2 \theta' \left( \cos^2 \theta - \cos^2 \theta' \right)^{1/2} \]

\[\cos^2 \theta \cos^2 \theta' \left( \cos^2 \theta - \cos^2 \theta' \right)^{1/2} \left( \cos^2 \theta - \cos^2 \theta' \right)^{1/2} \]

\[\{ e^{-\frac{r^2_0}{2}} \int_{t_1}^{t_2} \int_{t_1'}^{t_2'} (\sin t + a)^2 (\sin t' + a')^2 \]

\[e^{i(N(\theta') \sin t' - N(\theta) \sin t)} \]

\[\{ [A(\theta, t) - B(\theta, t)] [A(\theta', t') - B(\theta', t')] \]

\[\left( - \frac{\partial^2 r_1(\theta, t, \theta', t')}{\partial t} \frac{\partial r_1(\theta, t, \theta', t')}{\partial t'} \right) \]

\[+ \frac{\partial^2 r_1(\theta, t, \theta', t')}{\partial t \partial t'} \]

\[+ i \gamma \sin \beta [A(\theta', t') - B(\theta', t')] \frac{\partial r_1(\theta, t, \theta', t')}{\partial t'} \]

\[+ i \gamma \sin \beta [A(\theta, t) - B(\theta, t)] \frac{\partial r_1(\theta, t, \theta', t')}{\partial t} \]

\[+ \sin^2 \beta \} \ dt' \ dt \]

\[- \sin^2 \beta \ e^{-\frac{r^2_0}{2}} \int_{t_1}^{t_2} \int_{t_1'}^{t_2'} (\sin t + a)^2 (\sin t' + a')^2 \]

\[e^{i(N(\theta') \sin t' - N(\theta) \sin t)} \ dt' \ dt \}

The scattering surface degenerates to a plane if \(\sigma = 0\).

It is noted that there is agreement between (3.3.7) and (2.3.4) for this particular case.


10. Stone, W. M. On the Doppler spectrum for the 3-dimensional case. Seattle, Boeing airplane co., 1954. 36p. (Boeing mathematical note no. 35.)