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# Spatial determinism for a free $Z^2$-action

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*Abstract*. We extend the idea of bilateral determinism of a free $Z$-action by D. Ornstein and B. Weiss to a free $Z^2$-action. We show that we have a 'stronger' spatial determinism for $Z^2$-actions: to determine the complete $Z^2$-name of a point, it is enough to know the name of a fraction of the orbit whose density can be made arbitrarily small. Moreover, for zero-entropy $Z^2$-actions, we prove that there exists a partition $\tilde{\mathcal{P}}$ such that the $\tilde{\mathcal{P}}$-names of an arbitrarily small one-sided cone determine the points.

## 1. *Introduction*

If a dynamical system $(X, \mathcal{F}, \mu, T)$ has finite entropy, then it is well known [**7**] that there exists a finite partition $\mathcal{P}$ satisfying $\bigvee_{-\infty}^{\infty} T^i\mathcal{P} = \mathcal{F}$, and hence $h(T) = h(T, \mathcal{P})$. If a process $(X, \mathcal{P}, \mu, T)$ has entropy 0, then it has the property that the tail field of the partition $\mathcal{P}$ is the whole $\sigma$-algebra. That is,

$$\bigcap_n \bigvee_{i\le -n} T^{-i}\mathcal{P} = \bigvee_{i=-\infty}^{\infty} T^i\mathcal{P}.$$

Also, the tail field characterizes a Kolmogorov automorphism in the sense that the tail field is trivial if and only if $T$ is a Kolmogorov automorphism.

In the case of a $Z^2$-process $(X, \mathcal{P}, \mu, \phi)$, the past is defined using the lexicographic order [**3**]. And, the entropy of the system is known to be the conditional entropy of the partition given the past as in the case of a $Z$-action. The tail field of the partition $\mathcal{P}$ is defined as follows: let $\{(M_k, N_k)\}$ be a sequence going to $(\infty, \infty)$. Let

$$E_k = \{(i, j) : j \le 0 \text{ if } i \le -M_k, \text{ or } j \le -N_k \text{ if } i > -M_k\}.$$

If $\tau_0(\mathcal{P})$ denotes the $\sigma$-algebra

$$\bigcap_k \bigg\{ \bigvee_{(i,j)\in E_k} \phi^{-(i,j)}\mathcal{P} \bigg\},$$

then the tail field $\tau(\mathcal{P})$ is defined to be $\bigvee_{j=-\infty}^{\infty} \phi^{-(0,j)}\tau_0(\mathcal{P})$. We notice that $\tau_0(\mathcal{P})$ is independent of the sequence $\{(M_k, N_k)\}$ chosen and $\tau(\mathcal{P})$ is invariant under $\phi$. It was also shown in the case of a $Z^2$-action that the tail field is trivial if and only if $\phi$ is a Kolmogorov automorphism. That is, $\tau(\mathcal{P})$ is trivial if and only if $\phi$ is of completely positive entropy [**1**].

Ornstein and Weiss [**8**] have proved that every aperiodic transformation is isomorphic to a bilaterally deterministic process: given a generating partition $\mathcal{P}$, there exists a partition $\tilde{\mathcal{P}}$ such that

$$\bigvee_{|i|\geq n} T^{-i}\tilde{\mathcal{P}} = \bigvee_{-\infty}^{\infty} T^{-i}\mathcal{P}$$

for all $n$. That is, given a dynamical system $(X, \mathcal{F}, \mu, T)$, there exists a partition $\tilde{\mathcal{P}}$ such that the $\tilde{\mathcal{P}}$-name of a point $x = \{x_i\}$ for $|i| \geq n$ determines the $\mathcal{P}$-name for all $i$. Moreover, $\tilde{\mathcal{P}}$ can be chosen so that $|\mathcal{P} - \tilde{\mathcal{P}}| < \epsilon$ for any given $\epsilon > 0$. We want to generalize the Ornstein and Weiss theorem to $Z^2$-actions.

In the case of a $Z^2$-action, we are able to prove a 'stronger' assertion. For an angle $\pi/2 \leq \theta < \pi$, we let

$$G_\theta = \{(i, j) : j \leq 0, \text{ or } 0 < j < |i| \tan(\pi - \theta) \text{ for } i < 0\}.$$

We claim that given a generating partition $\mathcal{P}$ under $\phi$ and given any $\epsilon > 0$ and any $\pi/2 \leq \theta < \pi$, there exists a partition $\tilde{\mathcal{P}}$ such that:

(1) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$; and

(2) $\bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$.

In fact, we prove more. We construct a partition $\tilde{\mathcal{P}}$ independent of $\theta$: there exists a partition $\tilde{\mathcal{P}}$ such that:

(3) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$; and

(4) $\bigcap_{\pi/2\leq\theta<\pi} \bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$.

It is interesting to compare determinisms between $Z$-actions and $Z^2$-actions. We say that this $Z^2$ analogue of the Ornstein and Weiss theorem is 'stronger' in the sense that the density of $G_\theta$ in $Z^2$ is strictly less than 1, while the set $\{|i| \geq n\}$ has density 1 in $Z$. We may mention that the above results hold for other regions besides $G_\theta$. See the examples of §2 and Remark 1.

Suppose $h(\phi) > 0$. Let $\mathcal{P}$ be a partition such that $h(\phi, \mathcal{P}) > 0$. Since we have

$$h(\phi, \tilde{\mathcal{P}}) = H\left(\tilde{\mathcal{P}} \,\middle|\, \bigvee_{j\leq -1} \phi^{-(i,j)}\tilde{\mathcal{P}} \vee \bigvee_{i\leq -1} \phi^{-(i,0)}\tilde{\mathcal{P}}\right) > 0,$$

the $\sigma$-algebra

$$\bigvee_{j\leq -1} \phi^{-(i,j)}\tilde{\mathcal{P}} \vee \bigvee_{i\leq -1} \phi^{-(i,0)}\tilde{\mathcal{P}}$$

can never be the full $\sigma$-algebra. In particular,

$$\bigvee_{j\leq 0} \phi^{-(i,j)}\tilde{\mathcal{P}} = \phi^{(0,1)}\left(\bigvee_{j\leq -1} \phi^{-(i,j)}\tilde{\mathcal{P}}\right)$$

is never the full $\sigma$-algebra, while

$$\bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F} \quad \text{for an arbitrary } \frac{\pi}{2} \leq \theta < \pi.$$

We consider zero-entropy $Z^2$-actions. There are many $Z^2$-actions which essentially come from $Z$-actions. Many of these actions will have the property that for some $\vec{v} = (x, y)$ and some $k \geq 0$,

$$\bigvee_{(i,j)\in IP(\vec{v},k)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F},$$

where $IP(\vec{v}, k)$ denotes the double-sided infinite parallelogram in the direction of $\vec{v}$. That is,

$$IP(\vec{v}, k) = \left\{(i, j) : -k + j\frac{x}{y} \leq i \leq k + j\frac{x}{y}\right\}.$$

We note that if $\vec{v} = (p, q)$ is a rational vector and the $Z$-action $\phi^{(p,q)}$ has finite positive entropy, then the $\sigma$-algebra generated by a one-sided infinite parallelogram is not enough to capture the full $\sigma$-algebra of the $Z^2$-action. However, we show in Theorem 4 that for any given one-sided cone, there exists a partition $\tilde{\mathcal{P}}$ such that $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$ and the $\tilde{\mathcal{P}}$-names of the one-sided cone generate the full $\sigma$-algebra. This is the property for zero-entropy $Z^2$-actions analogous to the property (4). If $h(\phi) = 0$, then we may ask for what region D in $Z^2$ there exists a partition $\tilde{\mathcal{P}}$ such that the $\tilde{\mathcal{P}}$-names of D generate the full $\sigma$-algebra. See Remark 3.

We may also mention the following: if $\phi$ is a $K$-automorphism, then given $\epsilon > 0$ and a rectangle $[-n, 0] \times [-n, 0]$, there exists $k$ such that

$$\bigvee_{(i,j)\in[-n,0]\times[-n,0]} \phi^{-(i,j)}\tilde{\mathcal{P}}$$

is $\epsilon$-independent of $\bigvee_{(i,j)\in E_k} \phi^{-(i,j)}\tilde{\mathcal{P}}$ [**3**]. Meanwhile, by the property (2) of $\tilde{\mathcal{P}}$, it is easy to see that for any given $\pi/2 \leq \theta < \pi$, the $\sigma$-algebra $\bigvee_{(i,j)\in E_{k,\theta}} \phi^{-(i,j)}\tilde{\mathcal{P}}$ completely determines $\bigvee_{(i,j)\in[-n,0]\times[-n,0]} \phi^{-(i,j)}\tilde{\mathcal{P}}$, where

$$E_{k,\theta} = E_k \cup \{(i, j) : 0 < j < |i + M_k| \tan(\pi - \theta)i \leq -M_k\}.$$

It is easy to see that the partition $\tilde{\mathcal{P}}$ satisfies

$$\bigvee_{(i,j)\in B_n^c} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$$

for every $n$, where

$$B_n = \{(i, j) : 0 \leq |i| \leq n, 0 \leq |j| \leq n\}.$$

This fact can be regarded as a direct generalization of the Ornstein and Weiss theorem [**8**]. By the above property, it is clear that if $(X, \mathcal{P}, \mu, \phi)$ is Bernoulli, then $(X, \tilde{\mathcal{P}}, \mu, \phi)$ is very weak Bernoulli. However it is not weak Bernoulli. (See [**2**] for the definition.)

## 2. *Construction of* $\tilde{\mathcal{P}}$

2.1. *Positive-entropy case.* We assume that our $Z^2$-action, $(X, \mathcal{F}, \mu, \phi)$, is a free action. By the definition, for each $(i, j)$, $\phi^{(i,j)}$ is also a free $Z$-action. Through successive steps, we construct $\tilde{\mathcal{P}}$, a refinement of $\mathcal{P}$, which generates the full $\sigma$-algebra under the elements $\phi^{(i,j)}$, where $(i, j) \in G_\theta$ for every $\pi/2 \leq \theta < \pi$.

THEOREM 1. *For every generating partition $\mathcal{P}$, there exists a partition $\tilde{\mathcal{P}}$ such that:*
(1) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$; *and*
(2) $\bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$ *for any* $\pi/2 \le \theta < \pi$.

*Proof.* Let $\mathcal{P} = \{\mathcal{P}_0, \ldots, \mathcal{P}_{k-1}\}$ denote the given generating partition under our $Z^2$-action $\phi$. Without confusion, we may also consider $\mathcal{P}$ as a function from $X$ to $\{0, 1, \ldots, k-1\}$ such that $\mathcal{P}(x) = i$ if and only if $x \in \mathcal{P}_i$. Let $\{(p_1, q_1), (p_2, q_2), \ldots\}$ be a sequence of relatively prime pairs of integers. We make a list for each pair to occur infinitely often so that we construct the partition $\tilde{\mathcal{P}}$ with the generating property for any $G_\theta$, $\pi/2 \le \theta < \pi$. We are going to build a sequence of Rokhlin towers $\{R_i\}$ under the $Z$-actions $\phi^{(p_i,q_i)}$. We choose the $\epsilon_i$ such that $\sum \epsilon_i < \infty$. Let $\{n_i\}$ be a sequence going to $\infty$. We need to put some more conditions on the $\epsilon_i$ and $n_i$, which will be clear later.

For the convenience of the notation, we denote $\phi^{(p_1,q_1)}$ by $\phi_1$. Let $R_1$ denote a Rokhlin tower under $\phi_1$ of height $n_1^2$, so that

$$\mu\bigg(\bigcup_{j=0}^{n_1^2-1} \phi_1^j F_1\bigg) > 1 - \epsilon_1,$$

where $F_1$ denotes the base of the tower $R_1$. We define $\mathcal{P}^{(1)}$, which is a refinement of $\mathcal{P}$, as follows. If $x \in \bigcup_{j=n_1^2-n_1}^{n_1^2-1} \phi_1^j F_1$ and $x \in \mathcal{P}_i$, then we assign the point to $\mathcal{P}_{i,t}^{(1)}$, $0 \le i < k$. We add the information of the parity check on the bottom $n_1$ level sets $PC(n_1) = \bigcup_{j=0}^{n_1-1} \phi_1^j F_1$. If $x \in PC(n_1)$ and $x \in \mathcal{P}_i$, then we assign the point to $\mathcal{P}_{i,m}^{(1)}$, where $m = \sum_{j=0}^{n_1-2} \mathcal{P}(\phi_1^{n_1\cdot j} x)$ (mod $k$). If $x$ belongs to the rest of the tower or the error set and $x \in \mathcal{P}_i$, then we assign $x$ to $\mathcal{P}_i^{(1)}$. The new partition $\mathcal{P}^{(1)}$ is clearly a refinement of $\mathcal{P}$ and it consists of at most $k^2 + 2k$ many atoms

$$\{\mathcal{P}_0^{(1)}, \ldots, \mathcal{P}_{k-1}^{(1)}, \mathcal{P}_{0,t}^{(1)}, \ldots, \mathcal{P}_{k-1,t}^{(1)}, \mathcal{P}_{0,0}^{(1)}, \mathcal{P}_{0,1}^{(1)}, \ldots, \mathcal{P}_{0,k-1}^{(1)},$$
$$\mathcal{P}_{1,0}^{(1)}, \mathcal{P}_{1,1}^{(1)}, \ldots, \mathcal{P}_{k-1,k-2}^{(1)}, \mathcal{P}_{k-1,k-1}^{(1)}\}.$$

Given a point $x \in PC(n_1)$, we note that we add the parity check along the orbit of $x$ under $\phi_1^{n_1}$.

Suppose that we have constructed $\mathcal{P}^{(l-1)}$ using the Rokhlin tower $R_{l-1}$ under $\phi_{l-1} = \phi^{(p_{l-1},q_{l-1})}$. Let $R_l$ denote the Rokhlin tower of height $n_l^2$ with the base set $F_l$. We assume that

$$\mu\bigg(\bigcup_{i=0}^{n_l^2-1} \phi_l^i F_l\bigg) > 1 - \epsilon_l.$$

We define $\mathcal{P}^{(l)}$, which is a refinement of $\mathcal{P}$ and differs from $\mathcal{P}^{(l-1)}$ arbitrarily small by choosing $n_l$ sufficiently large. We note that the orbits of a point under $\phi_{l-1}$ and under $\phi_l$ could be completely different.

As we did on $R_1$, if $x$ belongs to the top $n_l$ level sets of $R_l$ and $x \in \mathcal{P}_i$, then we assign $x$ to $\mathcal{P}_{i,t}^{(l)}$. If $x \in PC(n_l) = \bigcup_{j=0}^{n_l-1} \phi_l^j F_l$ and $x \in \mathcal{P}_i$, then we assign $x$ to $\mathcal{P}_{i,m}^{(l)}$, where $m = \sum_{j=0}^{n_l-2} \mathcal{P}(\phi_l^{n_l \cdot j} x) \pmod{k}$. If

$$x \in \bigcup_{j=n_l}^{2n_l-1} \phi_l^j F_l \cup \bigcup_{j=n_l^2-2n_l}^{n_l^2-n_l-1} \phi_l^j F_l$$

and $x \in \mathcal{P}_i$, then we assign $x$ to $\mathcal{P}_i^{(l)}$. If $x$ belongs to the rest of the tower or the error set, then we assign $x$ to the same atom of $\mathcal{P}^{(l-1)}$. For almost every $x \in \bigcup_{j=3n_l}^{n_l(n_l-2)-1} \phi_l^j(F_l)$, if we know the $\mathcal{P}^{(l)}$-name of the point along the tower $R_l$ except for $n_l$ consecutive places containing $x$, then we can reconstruct the missing $n_l$ consecutive names from the parity check on $PC(n_l)$.

We note that $\mathcal{P}^{(l)}$ differs from $\mathcal{P}^{(l-1)}$ on a set of measure at most $4/n_l$. Hence, the set

$$E_l = \{x : \mathcal{P}^{(k)}(x) \neq \mathcal{P}^{(l)}(x) \text{ for some } k > l\}$$

has measure at most $\alpha_l = \sum_{k=l+1}^{\infty}(4/n_k)$. If we let

$$\tilde{E}_l = \bigcup_{j=0}^{n_l^2-1} \{\phi_l^j(x) : \text{there exists no } 0 \le k \le n_l^2 - 1 \text{ such that } \phi_l^k(x) \in E_l\},$$

then we have that $\mu(\tilde{E}_l^c) < n_l^2 \alpha_l + \epsilon_l$. If $x \in \tilde{E}_l$, then the $\mathcal{P}^{(k)}$-name of $x$ throughout the tower $R_l$ remains the same for all $k \ge l$. Through successive steps we choose the $n_l$ sufficiently large and the $\epsilon_l$ sufficiently small so that $\sum_l (n_l^2 \alpha_l + \epsilon_l) < \infty$. Then for almost every $x$ there exist only finitely many of the $l$ such that $x \in \tilde{E}_l^c$. Therefore, $\tilde{\mathcal{P}} = \lim \mathcal{P}^{(l)}$ is well defined and consists of

$$\{\tilde{\mathcal{P}}_0, \ldots, \tilde{\mathcal{P}}_{k-1}, \tilde{\mathcal{P}}_{0,t}, \ldots, \tilde{\mathcal{P}}_{k-1,t}, \tilde{\mathcal{P}}_{0,0}, \tilde{\mathcal{P}}_{0,1}, \ldots, \tilde{\mathcal{P}}_{0,k-1}, \ldots, \tilde{\mathcal{P}}_{1,0}, \tilde{\mathcal{P}}_{1,1}, \ldots, \tilde{\mathcal{P}}_{k-1,k-2}, \tilde{\mathcal{P}}_{k-1,k-1}\}.$$

It is now enough to show that for almost every $x \in X$, $\{\tilde{\mathcal{P}}(\phi^{(i,j)}x) : (i, j) \in G_\theta\}$ determines $\mathcal{P}(\phi^{(i,j)}x)$ for all $(i, j) \in Z^2$. For a given $\theta$, we choose $n_\theta$ such that

$$0 < \frac{p_{n_\theta}}{q_{n_\theta}} < \frac{\tan(\pi - \theta)}{2}.$$

Suppose $(i_0, j_0) \in G_\theta^c$. Notice that the line $L$ through $(i_0, j_0)$ with slope $-p_{n_\theta}/q_{n_\theta}$ hits only finitely many lattice points outside the region $G_\theta$. Let $g(i_0, j_0)$ denote the number of lattice points on the line outside $G_\theta$. Let

$$y = \phi^{(i_0,j_0)}x \quad \text{and} \quad H_l = \tilde{E}_l \cap \left( \bigcup_{i=3n_l}^{n_l(n_l-2)-1} \phi_l^i F_l \right).$$

Since $\mu(H_l^c) < n_l^2 \alpha_l + (5/n_l) + \epsilon_l$, by our choice of the $n_l$, there exists $l_0(y)$ satisfying $y \in H_l$ for all $l \ge l_0$. We choose $l \ge l_0$ such that $p_l/q_l = p_{n_\theta}/q_{n_\theta}$ and $n_l \ge g(i_0, j_0)$. By our construction, we can determine the $\mathcal{P}^{(l)}$-name and hence the $\tilde{\mathcal{P}}$-name of $x(i, j)$ for all $(i, j) \in G_\theta^c \cap L$. We repeat this for each $(i, j) \in G_\theta^c$ to recover the complete $\mathcal{P}$-name of $x$. □

Besides the $G_\theta$, there are many regions of smaller 'density' whose names generate the full $\sigma$-algebra. We have the following theorem.

THEOREM 2. *Let $B \subset Z^2$ be a region such that there exists a rational direction including $\infty$ for all $(i, j) \in B^c$ so that the line $L$ through $(i, j)$ in the direction intersects only finitely many lattice points outside $B$. Then we can construct $\tilde{\mathcal{P}}$ such that:*

(1) $|\mathcal{P} \Delta \tilde{\mathcal{P}}| < \epsilon$; *and*

(2) $\bigvee_{(i,j)\in B} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}$.

*Proof.* It is clear from our construction of $\tilde{\mathcal{P}}$ as in Theorem 1. □

Notice that our partition $\tilde{\mathcal{P}}$ satisfies the following.

*Example 1.* For any given $a > 0$, we let $B_a = \{(i, j) : j \geq ai^2 \text{ or } j \leq -ai^2\}$. The $\tilde{\mathcal{P}}$-name of a point in $B_a$ determines the point completely. Hence, we have

$$\bigcap_{a>0} \bigvee_{(i,j)\in B_a} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}.$$

*Example 2.* Given any $a > 0$ and $b > 0$, the region

$$B_{a,b} = \{(i, j) : j \geq ai^2 + b \text{ or } j \leq -ai^2 - b\}$$

also satisfies the condition of Theorem 2.

*Example 3.* Let $C(\vec{v}, \theta)$ denote the double-sided cone in the direction of $\vec{v}$ with the angle $\theta$. That is,

$$C(\vec{v}, \theta) = \left\{(x, y) : \tan(\alpha - \theta) \leq \frac{y}{x} \leq \tan(\alpha + \theta), -\infty < x, y < \infty\right\},$$

where $\alpha$ is the angle between $\vec{v}$ and the $x$ coordinate. Since it satisfies the condition of Theorem 2 for any angle $\theta$, we have

$$\bigvee_{(i,j)\in C(\vec{v},\theta)} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}.$$

In particular, if we know the $\tilde{\mathcal{P}}$-name of a point in the first and the third quadrants, then we know the full name of the point. It is clear that as $\theta$ goes to 0, the density of $C(\vec{v}, \theta)$ in $Z^2$ goes to 0, and hence it can be made arbitrarily small. Moreover, we note that, for example, $C_a(\vec{i}, \theta)$ also satisfies the condition of Theorem 2, where

$$C_a(\vec{i}, \theta) = \left\{(x, y) : -\tan\theta \leq \frac{y}{|x| - a} \leq \tan\theta, |x| \geq a\right\}$$

is a double-sided cone whose vertices are apart.

*Remark 1.* Let $B = \{(i, j) : j \leq 0 \text{ or } j \leq \log|i| \text{ for } i < 0\}$. It is not clear yet if there exists a partition $\tilde{\mathcal{P}}$ such that

$$\bigvee_{(i,j)\in B} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}.$$

2.2. *Zero-entropy case.* Recall that a zero-entropy $Z$-action has the property that if $\mathcal{P}$ is a generating partition, then the tail field of the partition $\mathcal{P}$ is the whole $\sigma$-algebra. We will prove the analogous result for zero-entropy $Z^2$-actions, which is stronger than Theorem 2. That is, we construct $\tilde{\mathcal{P}}$ so that the one-sided cone $\tilde{\mathcal{P}}$-names generate the full $\sigma$-algebra. It is known that if $h(\phi) = 0$, then, for any generating partition $\mathcal{P}$, we have

$$\bigvee_{j\le -1} \phi^{-(i,j)}\mathcal{P} \vee \bigvee_{i\le -1} \phi^{-(i,0)}\mathcal{P} = \mathcal{F}.$$

We need the following lemma, which provides us with the rough estimates of the number of elements in $\bigvee_{0\le i,j<n}\phi^{-(i,j)}\mathcal{P}$.

LEMMA 1. *Let $h(\phi) = 0$. Given a generating partition $\mathcal{P}$ and an $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n \ge N(\epsilon)$, there exists a collection $\mathcal{E}_n$, which consists of atoms of $\bigvee_{0\le i,j<n}\phi^{-(i,j)}\mathcal{P}$, where:*
(1) $|\mathcal{E}_n| < 2^{\epsilon n^2}$;
(2) $\mu(E_n) < \epsilon$, *where* $E_n = \bigcup_{C\notin\mathcal{E}_n} C$.

The above lemma follows from the Shannon–McMillan–Breiman theorem [**10**] of $Z^2$-actions.

PROPOSITION 3. *Let $h(\phi) = 0$ and let $\mathcal{P} = \{P_0, \dots, P_{k-1}\}$ be a generating partition under $\phi$. There exists a partition $\tilde{\mathcal{P}}$ satisfying the following: for a given $\epsilon > 0$ and a given $\theta$:*
(1) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$;
(2) $\bigvee_{(i,j)\in OC(\vec{v},\theta)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$, *where $OC(\vec{v},\theta)$ denotes the one-sided cone in the direction $\vec{v} = (-1,-1)$ with the angle $\theta$.*

*Proof.* We will describe the construction of $\tilde{\mathcal{P}}$ motivated by an idea of Thouvenot [**11**]. We choose $\{\epsilon_l\}$ sufficiently small and $\{n_l\}$ sufficiently large so that:
(i) $\epsilon_l < 1/2^{2l}$;
(ii) $n_l \gg 2^{2l}$ and $n_l$ is a multiple of $2^l$;
(iii) $\epsilon_l \ge \sum_{k\ge l+1} \epsilon_k$;
(iv) a Rokhlin tower $R_l$ of size $[0, n_l) \times [0, n_l)$ has fewer than $2^{\epsilon_l n_l^2}$ names with an error set of measure less than $\epsilon_l$.

We describe the induction step. Let $B_l$ denote the base of the Rokhlin tower. We partition the Rokhlin tower into columns according to their $\mathcal{P}$-names. By a column, we mean a subset of a Rokhlin tower,

$$C_{l,t} = \bigcup_{0\le i,j<n_l} \phi^{(i,j)} B_{l,t},$$

so that

$$\phi^{(i,j)} B_{l,t} \in P_{t(i,j)},$$

where $t(i, j) = 0, 1, \dots, k-1$. And, $B_{l,t}$ denotes the base of the $t$th column of the Rokhlin tower $R_l$, $t = 1, \dots, u_l$, where $u_l < 2^{\epsilon_l n_l^2}$. We encode each of the column names on the left bottom square of size $[0, n_l/2^l) \times [0, n_l/2^l)$ as follows.

We define $\mathcal{P}^{(l)}$ by changing the partition

$$\mathcal{P}^{(l-1)} = \{P_r^{(l-1)}, P_{r,s}^{(l-1)} : 0 \le r \le k-1, s = 0 \text{ or } 1\},$$

if necessary, to distinguish the $B_{l,t}$ by their $\mathcal{P}^{(l)}$-names on

$$BR_l = \bigcup_{0 \le i,j < n_l/2^l} \phi^{(i,j)} B_l.$$

That is, each of the $[0, n_l/2^l) \times [0, n_l/2^l)$-$\mathcal{P}^{(l)}$-names of $B_{l,t}$ uniquely determines the $[0, n_l) \times [0, n_l)$-$\mathcal{P}$-name of $B_{l,t}$. This is possible, since using $s = 0$ or 1, there are $2^{(n_l/2^l)^2}$ possible names, which is greater than $2^{\epsilon_l n_l^2}$ by the condition (i). We may need to change parts of $P_r^{(l-1)}$ into $P_{r,s}^{(l)}$ for each $r$ and also parts of $P_{r,s}^{(l-1)}$ into $P_{r,s'}^{(l)}$, where $r = 0, 1, \dots, k-1$, and $s$ or $s' = 0$ or 1 and $s \ne s'$. On the rest of the tower or on the error set, we define $\mathcal{P}^{(l)}$ to be the same as $\mathcal{P}^{(l-1)}$. Note that we have changed $\mathcal{P}^{(l-1)}$ negligibly little to get $\mathcal{P}^{(l)}$. Hence, $\lim \mathcal{P}^{(l)} = \tilde{\mathcal{P}}$ exists. Note that $\tilde{\mathcal{P}}$ also consists of $3k$ atoms $\{P_0, \dots, P_{k-1}, P_{r,s}\}_{\substack{r=0,\dots,k-1\\ s=0,1}}$ and the set $\bigcup_{\substack{r=0,\dots,k-1\\ s=0,1}} P_{r,s}$ has small measure.

We call a subcolumn a subset $C_{l,t}^o$ of a column of $R_l$,

$$C_{l,t}^o = \bigcup_{0 \le i,j < n_l} \phi^{(i,j)} B_{l,t}^o,$$

where $B_{l,t}^o$ is a subset of $B_{l,t}$. For a given $\theta$, let $A_l = \bigcup_{(i,j)\in I_l} \phi^{(i,j)} B_l$, where

$$I_l = \left[n_l\left(\tan\left(\frac{\pi}{4} - \theta\right) - \frac{1}{2^l}\right), n_l\right) \times \left[n_l\left(\tan\left(\frac{\pi}{4} - \theta\right) - \frac{1}{2^l}\right), n_l\right).$$

Note that $I_l$ is a square of size $n_l(1 - \tan(\pi/4 - \theta) + 1/2^l)$ at the upper right corner of the square $[0, n_l) \times [0, n_l)$ and, if $(i, j) \in I_l$, then the one-sided cone $OC(\vec{v}, \theta)$ starting at $(i, j)$ contains the square $[0, n_l/2^l) \times [0, n_l/2^l)$. Hence, if $x \in A_l$, then the one-sided cone name starting at $x$ determines the name of the column $C_{l,t}$ containing $x$, because the cone contains the $[0, n_l/2^l) \times [0, n_l/2^l)$-$\mathcal{P}^{(l)}$-name.

We need to show that $\mu(\bigcup_{l \ge k} A_l) = 1$ for all $k$. We note that $I_l$ takes up greater than a fixed fraction of $[0, n_l) \times [0, n_l)$ for each $l$. We choose $n_l$ sufficiently large compared with $n_{l-1}$ so that:

(v) the measure of $R_{l-1}$ in $R_l$ which intersects the boundary of $R_l$ or which intersects both $A_l$ and $R_l \backslash A_l$ is less than $\epsilon_l$;

(vi) for each column of $R_l$, the frequency of $R_{l-1}$ in $A_l$ is within $2\epsilon_l$ of the frequency of $R_{l-1}$ in $R_l \backslash A_l$ (the ergodic theorem). Therefore, the frequencies of $A_{l-1}$ in $A_l$ and in $R_l \backslash A_l$ are within $2\epsilon_l$ of each other.

We denote by $R_{l-1}^{(l)}$ the subset of $R_{l-1}$ which intersects neither the boundary of $R_l$ nor the boundary of $A_l$. By (v), we have $\mu(R_{l-1}^{(l)}) > \mu(R_{l-1}) - \epsilon_l$. Let

$$\tilde{R}_{l-1} = \bigcap_{k \ge l} {R_{l-1}}^{(k)} \quad \text{and} \quad \tilde{A}_{l-1} = \tilde{R}_{l-1} \cap A_{l-1}.$$

We note that

$$\mu(\tilde{R}_{l-1}) > \mu(R_{l-1}) - \sum_{k \ge l} \epsilon_k > \mu(R_{l-1}) - \epsilon_{l-1} \ge 1 - 2\epsilon_{l-1}.$$

To show that almost every $x \in X$ belongs to $\tilde{A}_l$ infinitely often, we need that $\mu(\bigcap_{l \geq m} \tilde{A}_l^c) = 0$ for all $m$. This is true by the 'almost' independence of the $A_l$ within the error of the ergodic theorem. Since there is an error set $E_l$ of measure less than $\epsilon_l$, we note that $\mu(\tilde{A}_l^c) \leq \alpha/(1-2\epsilon_l)$, where $\alpha = (1-(|I_l|/n_l^2)) < 1$. And, we may assume that the frequency of $\tilde{A}_l^c$ in $\tilde{A}_{l+1}^c$ is at most $\mu(\tilde{A}_l^c) \cdot (1+2\epsilon_{l+1})$. Hence,

$$\mu\bigg(\bigcap_{l \geq m}^{n} \tilde{A}_l^c\bigg) \leq \frac{\alpha^{m-n+1}}{1-2\epsilon_m} \prod_{k=m+1}^{n} \frac{(1+2\epsilon_k)}{(1-2\epsilon_k)} \leq \frac{\alpha^{m-n+1}}{1-2\epsilon_m} \prod_{k=m+1}^{n} (1+5\epsilon_k).$$

Since $\sum_{k \geq l+1}^{\infty} \epsilon_k < \epsilon_l$, we have

$$\mu\bigg(\bigcap_{l \geq m}^{n} \tilde{A}_l^c\bigg) < \frac{\alpha^{n-m+1}}{1-2\epsilon_m} \epsilon^{6\epsilon_m} \to 0$$

as $n \to \infty$ for all $m$.

For a given positive $m$, we let $F_l^m$ denote the set $\{(i, j) : (i, j)$ is at least distance $m$ away from the boundary of the square $[0, n_l) \times [0, n_l)\}$. Since for each $m$, the set

$$R_l^m = \bigcup_{(i,j) \in F_l^m} \phi^{(i,j)} B_l$$

has measure greater than $(1-\epsilon_l)(1-(4m/n_l))$, almost every $x$ belongs to $R_l^m$ except for only finitely many of the $l$. That is, a larger and larger square centered at $x$ is contained in $\tilde{A}_l$ as $l \to \infty$ for almost every $x$. Hence, we have the property (2). □

*Remark 2.* Note that for each $l$ we may use the $(0, 0)$ coordinate as a marker to indicate the start of $BR_{(0,0)}$. Then $\tilde{\mathcal{P}}$ may consist of $\{\mathcal{P}_0, \ldots, \mathcal{P}_{k-1}, \mathcal{P}_{r,s}\}_{\substack{r=0,\ldots,k-1\\ s=c,0,1}}$, where $\mathcal{P}_{r,c}$ indicates a marker.

THEOREM 4. *If $h(\phi) = 0$, then there exists a partition $\tilde{\mathcal{P}}$ such that for a given $\epsilon > 0$, we have the following.*

(1) $|\mathcal{P} \Delta \tilde{\mathcal{P}}| < \epsilon$.

(2) $\bigvee_{(i,j) \in OC(\vec{v},\theta)} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}$ *for every $\vec{v}$ and every angle $\theta$.*

*Proof.* Given $\epsilon > 0$, we choose $l_0$ such that $(1/2^{l_0}) \leq (\epsilon/4)$. We start from $l \geq l_0$. In the Rokhlin tower $R_l$, we consider the squares of size $(n_l/2^l) \times (n_l/2^l)$ which are located at $(0, (n_l/2^l)j)$, $((n_l/2^l)j, 0)$, $(n_l - (n_l/2^l), (n_l/2^l)j)$ or $((n_l/2^l)j, n_l - (n_l/2^l))$ for $j = 0, \ldots, 2^l - 1$. These squares meet the boundary of $R_l$. We define $\mathcal{P}^{(l)}$ different from $\mathcal{P}^{(l-1)}$ only on the boundary squares. As in Proposition 3, each of the $\mathcal{P}^{(l)}$-names of $BR_l$ uniquely determines the $\mathcal{P}$-name of the column of the Rokhlin tower $C_{l,t}$ for $t = 1, \ldots, u_l$. After we redefine the partition $\mathcal{P}^{(l-1)}$ into $\mathcal{P}^{(l)} = \{\mathcal{P}_0^{(l)}, \ldots, \mathcal{P}_{r,s}^{(l)}\}_{\substack{r=0,\ldots,k-1\\ s=0,1}}$ on $BR_l$, for each column $C_{l,t}$ we copy the $\mathcal{P}^{(l)}$-name of

$$BR_{l,t} = \bigcup_{0 \leq i,j < n_l/2^l} \phi^{(i,j)} B_{l,t}$$

on each of the boundary squares of $C_{l,t}$.

We note that we have

$$|\mathcal{P}^{(l-1)} \Delta \mathcal{P}^{(l)}| \leq \frac{(n_l/2^l)^2\,(4 \cdot 2^l)}{{n_l}^2} = \frac{4}{2^l}.$$

Let $\tilde{\mathcal{P}} = \lim \mathcal{P}^{(l)}$; we have

$$|\mathcal{P} \Delta \tilde{\mathcal{P}}| \leq \sum_{l \geq l_0+1}^{\infty} \left(\frac{4}{2^l}\right) = 4 \cdot \frac{1}{2^{l_0}} < \epsilon.$$

To prove the second property, it is enough to consider the vector $\overrightarrow{v} = (x, y)$, where $y > 0$. Let $\theta_0$ denote the angle such that $\tan \theta_0 = y/x$. We assume $0 \leq \theta_0 < \pi/2$. For the case of $\pi/2 \leq \theta_0 \leq \pi$, we can use similar arguments. For any given $\theta$, we let

$$\begin{aligned}\tau(\theta) = \min\{&|\tan(\pi/2 - \theta_0 + \theta) - \tan(\pi/2 - \theta_0)|, |\tan(\pi/2 - \theta_0 - \theta) - \tan(\pi/2 - \theta_0)|,\\ &|\tan(\theta_0 + \theta) - \tan\theta_0|, |\tan(\theta_0 - \theta) - \tan\theta_0|\}.\end{aligned}$$

There exists $\tilde{l}(\theta)$ such that if $l \geq \tilde{l}$, then $(1 - (1/2^l))\tau > 2/2^l$. Then the cone $C(\overrightarrow{v}, \theta)$ starting at (0, 0) contains a boundary square of size $n_l/2^l \times n_l/2^l$ for $l \geq \tilde{l}$. The $\mathcal{P}^{(l)}$-name of this boundary square determines the $\mathcal{P}$-name of the column of $R_l$. Moreover, for a given $\theta$ and $0 < c < 1$, there exists $\tilde{l}(\theta, c)$ such that if $l \geq \tilde{l}$, then $(1 - c - (1/2^l)) \cdot n_l \cdot \tau > 2 \cdot n_l/2^l$. Any cone $C(\overrightarrow{v}, \theta)$ starting at $(i, j) \in I_l = [0, cn_l) \times [0, cn_l)$ contains at least one boundary square. Hence, the $\mathcal{P}^{(l)}$-name of the cone $C(\overrightarrow{v}, \theta)$ determines the $\mathcal{P}$-name of the column of $R_l$. As in Proposition 3, it is not hard to see that almost every $x$ belongs to $A_l = \bigcup_{(i,j)\in I_l} \phi^{(i,j)} B_l$ infinitely often.

For a given $m$, we let

$$R_l^m = \bigcup_{m \leq i, j < n_l - m} \phi^{(i,j)} B_l.$$

We note that there exists $l'(m)$ such that $x \in R_l^m$ for all $l \geq l'(m)$. For each $m$, we have $x \in R_l^m \cap A_l$ infinitely often; hence, we have

$$\bigvee_{(i,j)\in OC(\vec{v},\theta)} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}$$

for every $\theta$. Since this holds for every $\vec{v}$, we have the property (2). □

*Remark 3.* If a system $(X, \mathcal{P}, \mu, \phi)$ has the property

$$\lim_{n\to\infty} \frac{1}{n^2} H\left(\bigvee_{(i,j)\in R_n} \phi^{-(i,j)} \mathcal{P}\right) = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} H\left(\bigvee_{(i,j)\in R_n} \phi^{-(i,j)} \mathcal{P}\right) = \infty,$$

where $R_n = \{(i, j) : 0 \leq i, j \leq n\}$, then it is clear that there exists no $\tilde{\mathcal{P}}$ with the property that for some $\overrightarrow{v} = (x, y)$ and some $k \geq 0$,

$$\bigvee_{(i,j)\in IP(\vec{v},k)} \phi^{-(i,j)} \tilde{\mathcal{P}} = \mathcal{F}.$$

However, the above Theorem 4 says that a one-sided cone is enough to capture the full $\sigma$-algebra. We believe that this is related to the 'complexity' of zero-entropy $Z^2$-actions. (See [**4–6**].) It is not clear yet if it is possible to refine or strengthen Proposition 3 and Theorem 4 even for the class of systems of low complexity.

*Remark 4.* It is not hard to see that the above constructions of $\tilde{\mathcal{P}}$ in Theorems 1, 2 and 4 can be generalized to $Z^n$-actions for both positive and zero entropy. However, it is not clear how to extend them to general amenable groups [**9, 10**].

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## References

**[1]** R. Burton. An asymptotic definition of K-groups of automorphisms and a non-Bernoullian counter-example. *Z. Wahr. verw. Geb.* **47** (1979), 205–212.

**[2]** R. Burton and J. Steif. Coupling surfaces and weak Bernoulli in one and higher dimensions. *Adv. Math.* **132**(1) (1997), 1–23.

**[3]** J. P. Conze. Entropie d'un groupe abelien de transformations. *Z. Wahr. verw. Geb.* **25** (1972), 11–30.

**[4]** D. Dou, W. Hwang and K. K. Park. Entropy dimension of topological dynamical systems. *Trans. Amer. Math. Soc.* **363** (2011), 659–680.

**[5]** D. Dou, W. Hwang and K. K. Park. Entropy dimension of measure preserving systems. *Preprint*.

**[6]** S. Ferenczi and K. Park. Entropy dimensions and a class of constructive examples. *J. Contin. Discrete Dyn. Syst.* **17** (2007), 133–141.

**[7]** W. Krieger. On the entropy and generators of measure preserving transformations. *Trans. Amer. Math. Soc.* **149** (1970), 453–464.

**[8]** D. Ornstein and B. Weiss. Every transformation is bilaterally deterministic. *Israel J. Math.* **21** (1975), 154–158.

**[9]** D. Ornstein and B. Weiss. Entropy and isomorphism theorem for actions of amenable groups. *J. Anal. Math.* **48** (1987), 1–141.

**[10]** D. Ornstein and B. Weiss. The Shannon–McMillan–Breiman theorem for a class of amenable groups. *Israel J. Math.* **44** (1983), 53–60.

**[11]** J.-P. Thouvenot. Personal communication.