

First Quantized Electrodynamics

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First Quantized Electrodynamics

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The parametrized Dirac wave equation represents position and time as operators, and can be formulated for many particles. It thus provides, unlike field-theoretic Quantum Electrodynamics (QED), an elementary and unrestricted representation of electrons entangled in space or time. The parametrized formalism leads directly and without further conjecture to the Bethe-Salpeter equation for bound states. The formalism also yields the Uehling shift of the hydrogenic spectrum, the anomalous magnetic moment of the electron to leading order in the fine structure constant, the Lamb shift and the axial anomaly of QED.

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I. INTRODUCTION

Recent observations [1] of photons entangled in time raise the possibility that fermions could be entangled in time. Quantum Field Theory, as recently shown [2], does allow a certain sense of timelike entanglement for a massless vacuum state of unspecified spin. However, Quantum Field Theory (QFT) has only limited ability to represent entangled states. Violations of Bell's inequality in the vacuum state for a massive particle decay exponentially with increasing spacelike separation, the decay scale being the Compton wavelength. If one particle location is subjected to a timelike displacement until the separation is timelike, then violations decay exponentially with that displacement measured again in Compton wavelengths [3].

The purpose of this analysis is to show that the parametrized Dirac equation proposed by Feynman [4–6] allows the unrestricted space–time entanglement of electrons. It is also shown that the parametrized formalism leads to the Bethe–Salpeter equation [7–9] for bound states. The formalism further predicts, by use of a partial summation of the Born series, several fundamental phenomena previously predicted [10–13] by field–theoretic or second–quantized Quantum Electrodynamics (QED). Included here are the lowest–order corrections yielding the Uehling potential, the anomalous magnetic moment of the electron, the Lamb shift and the axial anomaly.

The contents of this article are as follows. The parametrized Dirac wave equation is stated in Section II. Free wavefunctions, influence functions and Møller operators for the parametrized Dirac equation are constructed in Section III. The constructions are readily extended to several particles, as outlined in Section IV. Unrestricted entanglement in space–time is shown to be an explicit contingency in the parametrized formalism. The spin–statistics connection has been proved by Jabs [14] using first quantization, and the line of proof is very briefly described here. Two–particle scattering is defined in terms of Møller operators, which lead without further conjecture to the Bethe–Salpeter equation. The construction leading to the Møller operators is given in some detail, even though it parallels those found in classic texts [15, 16], since the operators lead to precise agreement with successful predictions of field–theoretic QED. Scattering is then simplified to a single particle in an external potential. The first–order scattering of a single particle is outlined in Section V, with Mott scattering as an example. The standard one–loop corrections for scattering

of arbitrary strength are derived in Section VI by use of a substitution that is accurate for beams undergoing weak scattering. The substitution resembles the QED relation between propagators and vacuum-to-vacuum expectations. The standard QED axial anomaly is derived in Section VII for classical fields rather than fields of operators. The summary in Section VIII includes a discussion of the wide utility of semiclassical theory and the further possibilities for parametrized formalisms.

II. THE PARAMETRIZED DIRAC WAVE EQUATION

The parametrized Dirac wave equation is stated, as are its Lagrangian and its discrete symmetries.

A. covariant formulation

For a single spin-1/2 particle the wavefunction is a four-spinor $\psi(x, \tau)$. The event x is in \mathbb{R}^4 , while the parameter τ is an independent variable in \mathbb{R} . The event x is also denoted by x^μ having indices $\mu = 0, 1, 2, 3$, with $x^0 = ct$ where c is the speed of light and t is coordinate time. The Lorentz metric $g^{\mu\nu}$ on \mathbb{R}^4 has signature $(-+++)$. The position \mathbf{x} is denoted by x^j having indices $j = 1, 2, 3$. Thus $x = (ct, \mathbf{x})$.

The parametrized Dirac wave equation for ψ is

$$\frac{\hbar}{ic} \frac{\partial}{\partial \tau} \psi + \gamma^\mu \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{e}{c} A_\mu \right) \psi = 0 \quad (1)$$

where e is the charge of the particle, c is the speed of light and \hbar is the reduced Planck's constant. The γ^μ are the four Dirac matrices, while the Maxwell electromagnetic potential $A^\mu(x)$ is independent of the parameter τ . The covariance of the theory with respect to the homogeneous Lorentz transformation $(x^\mu)' = \Lambda^\mu{}_\nu x^\nu$ and $\psi'(x, \tau) = S(\Lambda)\psi(\Lambda^{-1}x, \tau)$ follows for $S(\Lambda)$ generated in the standard way [15]. No mass constant appears in (1), but masses are introduced through boundary conditions as $\tau \rightarrow \pm\infty$. Feynman's development of QED using (1) has been reviewed by Garcia Alvarez and Gaioli [4]. A simple consequence of (1) is the identity

$$\frac{\partial}{\partial \tau} \bar{\psi} \psi + \frac{\partial}{\partial x^\mu} j^\mu = 0 \quad (2)$$

where $\bar{\psi} \equiv \psi^\dagger \gamma^0$, and the τ -dependent current is $j^\mu = c\bar{\psi}\gamma^\mu\psi$. The indefiniteness of the invariant bilinear form $\bar{\psi}\psi$ has impeded [17, 18] the development of the parametrized Dirac formalism as a relativistic extension of quantum mechanics.

The energy–momentum operator is denoted $p_\mu = (\hbar/i)\partial/\partial x^\mu$, and $\pi_\mu = p_\mu - (e/c)A_\mu$. The commutator of the operators x^μ and p^ν is

$$[x^\mu, p^\nu] = x^\mu p^\nu - p^\nu x^\mu = i\hbar g^{\mu\nu}. \quad (3)$$

Henceforth the units are chosen such that $c = \hbar = 1$. The Maxwell field strength tensor is thus $F^{\mu\nu} = i(p^\mu A^\nu - p^\nu A^\mu)$. The summation convention is assumed with respect to repetitions of Greek indices such as $\mu = 0, 1, 2, 3$. The covariant and contravariant indices μ, ν, \dots will be omitted wherever convenient, as in $x = x^\mu$, $p = p^\mu$, $p \cdot x = p^\mu x_\mu$ and $F \cdot F = F_{\mu\nu} F^{\mu\nu}$.

The Newton–Wigner position operator [19] can only localize the position \mathbf{x} of, say, an electron within a distance of $\mathcal{O}(m_e^{-1})$ where m_e is the conventional rest mass. The operator is extendable in the parametrized formalism to an event operator x^μ with a footprint of the same order. However, the formalism does not restrict the state to a single or ‘sharp’ rest mass. Consider a wavefunction $\psi(x, \tau) = \exp(im\tau)\lambda(x)$. The mass m can have infinite range and so perfect localization of x^μ and of p^μ is attainable [20, 21].

B. Lagrangian and discrete symmetries

The single–particle Lagrangian is

$$\mathcal{L} = \frac{1}{4} \int d^4x F \cdot F + \int d\tau \int d^4x \bar{\psi} \left(\frac{1}{i} \frac{\partial}{\partial \tau} \psi + \not{p} \psi \right), \quad (4)$$

where d^4x is the Lorentz–invariant measure on \mathbb{R}^4 . The Feynman slash notation is for example $\not{p} = \gamma^\mu p_\mu$, and in particular $\not{p}\not{p} = -p \cdot p$. Variation of \mathcal{L} with respect to $\bar{\psi}$ yields (1), while variation of \mathcal{L} with respect to A^μ in the Lorenz gauge, where $\partial \cdot A = 0$, yields

$$\square A^\mu = p \cdot p A^\mu = -\partial \cdot \partial A = eJ^\mu \quad (5)$$

which is Maxwell’s equation for A^μ . The ‘concatenated’ current

$$J^\mu(x) = \int j^\mu(x, \tau) d\tau \quad (6)$$

in (5) is divergenceless, satisfying $\partial \cdot J = 0$.

The discrete symmetries of charge conjugation \mathcal{C} , parity \mathcal{P} and time reversal \mathcal{T} act on the single-particle wavefunctions as follows.

$$(\mathcal{C}\psi)(x, \tau) = i\gamma^2\psi^*(x, -\tau) \quad (7)$$

$$(\mathcal{P}\psi)(t, \mathbf{x}, \tau) = \gamma^0\psi(t, -\mathbf{x}, \tau) \quad (8)$$

$$(\mathcal{T}\psi)(t, \mathbf{x}, \tau) = i\gamma^1\gamma^3\psi^*(-t, \mathbf{x}, -\tau) \quad (9)$$

$$(\mathcal{TCP}\mathcal{C}\psi)(x, \tau) = -i\gamma^5\psi(-x, \tau) \quad (10)$$

In particular, the charge-conjugate wavefunction $(\mathcal{C}\psi)(x, \tau)$ satisfies (1), subject to the charge e being replaced with $-e$.

III. WAVE FUNCTIONS AND MØLLER OPERATORS

The essential constituents of scattering theory are developed in detail for one particle. The details are routine, being close analogs of those in standard quantum mechanics. The major difference is the existence of states having ‘negative mass’ but they do not participate, even virtually, in the scattering process.

A. free wavefunctions

The free wavefunctions are the solutions of (1) in the absence of the electromagnetic potential A^μ . They have the form

$$f_p^{(+)}(x, \tau) = \frac{u(p)}{(2\pi)^2} \exp[i(p \cdot x + \varphi_p m_p \tau)] \quad (11)$$

and

$$f_p^{(-)}(x, \tau) = \frac{v(p)}{(2\pi)^2} \exp[i(p \cdot x - \varphi_p m_p \tau)] \quad (12)$$

where p^μ is a constant energy-momentum vector, while $\varphi_p = \theta(p^0) - \theta(-p^0)$ is the sign of p^0 . That is, $\varphi_p = p^0/E_p$ where $E_p = |p^0|$. The mass m_p , also denoted $m(p)$, is the positive square root of $-p \cdot p$. It is assumed that the states are not superluminal, and so $p \cdot p \leq 0$.

The phases of the complex exponentials are $p \cdot x \pm \varphi_p m_p \tau$, hence

$$\frac{dt}{d\tau} = \pm m_p/E_p \quad (13)$$

at constant phase and position, regardless of the value of φ_p . Recall that the metric signature is $(-+++)$. Thus $f_p^{(+)}$ propagates forward in coordinate time as τ increases, for both positive-energy ($\varphi_p = +1$) and negative-energy states ($\varphi_p = -1$), while $f_p^{(-)}$ propagates backward for both kinds of states.

There are two linearly independent solutions for the four-spinor $u(p)$ and two for the four-spinor $v(p)$. In the rest frame where $p = (p^0, \mathbf{0})$ they are

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (14)$$

In an arbitrary frame the solutions for u and v have the 4×2 block forms

$$\begin{aligned} \mathbf{u}(p) &= (u^{(1)}(p), u^{(2)}(p)) = K \begin{pmatrix} (m_p + E_p)I_2 \\ \varphi_p \mathbf{P} \cdot \boldsymbol{\sigma} \end{pmatrix} \\ \mathbf{v}(p) &= (v^{(1)}(p), v^{(2)}(p)) = K \begin{pmatrix} \varphi_p \mathbf{P} \cdot \boldsymbol{\sigma} \\ (m_p + E_p)I_2 \end{pmatrix} \end{aligned} \quad (15)$$

where $K = [2m_p(m_p + E_p)]^{-1/2}$, and the σ^j are the Pauli matrices ($j = 1, 2, 3$). The symbol I_n denotes the $n \times n$ unit matrix. The blocks \mathbf{u} and \mathbf{v} are eigenspinors of the ‘forward’ and ‘backward’ projections

$$\Lambda_u(p) \equiv \frac{m_p I_4 - \varphi_p \not{p}}{2m_p} = \mathbf{u}\bar{\mathbf{u}}, \quad \Lambda_v(p) \equiv \frac{m_p I_4 + \varphi_p \not{p}}{2m_p} = -\mathbf{v}\bar{\mathbf{v}} \quad (16)$$

respectively, and they obey the orthonormality conditions

$$\bar{\mathbf{u}}\mathbf{u} = I_2, \quad \bar{\mathbf{v}}\mathbf{v} = -I_2, \quad \bar{\mathbf{u}}\mathbf{v} = 0, \quad \bar{\mathbf{v}}\mathbf{u} = 0. \quad (17)$$

The free solutions having the same sense of propagation are orthonormal over \mathbb{R}^4 in the continuum normalization, with block form

$$\begin{aligned} \int d^4x \overline{\mathbf{f}_p^{(+)}(x, \tau)} \mathbf{f}_q^{(+)}(x, \tau) &= +\delta^4(p - q)I_2, \\ \int d^4x \overline{\mathbf{f}_p^{(-)}(x, \tau)} \mathbf{f}_q^{(-)}(x, \tau) &= -\delta^4(p - q)I_2, \end{aligned} \quad (18)$$

while those having the opposite sense are orthogonal, with block form

$$\begin{aligned} \int d^4x \overline{\mathbf{f}_p^{(+)}(x, \tau)} \mathbf{f}_q^{(-)}(x, \tau) &= 0, \\ \int d^4x \overline{\mathbf{f}_p^{(-)}(x, \tau)} \mathbf{f}_q^{(+)}(x, \tau) &= 0, \end{aligned} \quad (19)$$

where $\mathbf{f}_p^{(\pm)} = (f_p^{(\pm)(1)}, f_p^{(\pm)(2)})$ for $f_p^{(+)(r)}(x, \tau) = u^{(r)}(p)(1/2\pi)^2 \exp[i(p \cdot x + \varphi_p m_p \tau)]$ and $f_p^{(-)(r)}(x, \tau) = v^{(r)}(p)(1/2\pi)^2 \exp[i(p \cdot x - \varphi_p m_p \tau)]$, with $r = 1, 2$.

The spin projection operators in an arbitrary frame are

$$P(\pm s) = \frac{1}{2}(I_4 \mp \gamma^5 \not{s}). \quad (20)$$

In the rest frame, $s = (0, \mathbf{s})$ where $|\mathbf{s}| = 1$. The operators $\Lambda_u(p)$, $\Lambda_v(p)$ and $P(s)$ commute for all s and p , since $\not{s} \not{p} = -p \cdot s = 0$. The chirality projection operators are $P_{\pm} = (1 \pm \gamma^5)/2$, yielding for example the projections $\mathbf{u}_{\pm} = P_{\pm} \mathbf{u}$. The helicity operator for the unit momentum $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ is

$$\hat{\mathbf{p}} \cdot \Sigma = \begin{pmatrix} \hat{\mathbf{p}} \cdot \sigma & 0 \\ 0 & \hat{\mathbf{p}} \cdot \sigma \end{pmatrix}. \quad (21)$$

In the ultrarelativistic limit as $m_p/E_p \rightarrow 0$, the projections \mathbf{u}_{\pm} are eigenstates of the helicity operator with eigenvalues $\pm\varphi_p$ respectively. In the same limit, the projections $\mathbf{v}_{\pm} = P_{\pm} \mathbf{v}$ are helicity eigenstates also with eigenvalues $\pm\varphi_p$ respectively.

Defining the free wavefunctions $\mathbf{h}_p^{(\pm)}$ by

$$\mathbf{h}_p^{(\pm)}(x, \tau) = (\mathcal{TPC} \mathbf{f}_p^{(\mp)})(x, \tau) = -i\gamma^5 \mathbf{f}_p^{(\mp)}(-x, \tau), \quad (22)$$

it follows that

$$\mathbf{f}_{-p}^{(\pm)}(x, \tau) = i\mathbf{h}_p^{(\pm)}(x, \tau). \quad (23)$$

Moreover, the \mathcal{TPC} conjugate of an eigenstate of the forward projection $\Lambda_u(p)P(+s)$ is an eigenstate of $\Lambda_v(-p)P(-s)$. Thus, defining the antiparticle to be the \mathcal{TPC} conjugate of a particle, it follows that a positron is an electron having the reversed sense of propagation in time t as τ increases, the reversed energy-momentum and spin, but the same coordinate velocity dx^j/dt at constant phase and constant τ . In particular, there are positrons of positive *and* negative energy.

Again, particles and antiparticles may in general have energies of either sign, and may propagate forward or backward irrespective of the signs of their energies. The space \mathcal{S} of all subluminal states has the subspaces

\mathcal{S}_+ which is spanned by particles $f_p^{(+)(r)}(x, \tau)$ and antiparticles $h_p^{(-)(r)}(x, \tau)$, for $r = 1, 2$ and for all p such that $p^0 > 0$,

and

\mathcal{S}_- which is spanned by particles $f_p^{(+)(r)}(x, \tau)$ and antiparticles $h_p^{(-)(r)}(x, \tau)$, for $r = 1, 2$ and for all p such that $p^0 < 0$.

The two subspaces are a partition, that is, $\mathcal{S}_+ \cap \mathcal{S}_- = \{0\}$ and $\mathcal{S}_+ \cup \mathcal{S}_- = \mathcal{S}$. The case $p^0 = 0$ is precluded by admitting $m_p = 0$ only as a limit. Note that the parenthetical superscripts (\pm) on the free wavefunctions indicate the sense of propagation in time as τ increases. Also, the particles in \mathcal{S}_+ are all proportional to $\exp(+im_p\tau)$, while those in \mathcal{S}_- are all proportional to $\exp(-im_p\tau)$. Conservation of mass (see §V) prevents states in \mathcal{S}_\pm from being scattered into states in \mathcal{S}_\mp by massless photons. It is however the case that states of positive energy p^0 in \mathcal{S}_+ , for example, can be scattered into \mathcal{S}_+ states of negative energy without lower bound and so hole theory must be invoked. Equivalently, the negative-energy electron states in \mathcal{S}_+ are regarded as states of different particles, namely positive-energy positrons [22]. Pair creation and annihilation are real processes, but the formalism of second quantization is avoided here by adopting the Feynman–Stueckelberg interpretation. That is, a positron in \mathcal{S}_+ is formally represented by the \mathcal{TCP} -conjugate of an electron in \mathcal{S}_+ .

B. free influence functions

The two free influence functions $\Gamma_\pm^0(x' - x, \tau' - \tau)$ for (1) both satisfy

$$\frac{1}{i} \frac{\partial}{\partial \tau'} \Gamma_\pm^0(x' - x, \tau' - \tau) + \gamma_\mu \frac{1}{i} \frac{\partial}{\partial x'_\mu} \Gamma_\pm^0(x' - x, \tau' - \tau) = \delta^4(x' - x) \delta(\tau' - \tau), \quad (24)$$

and are given by

$$\Gamma_\pm^0(x' - x, \tau' - \tau) = (2\pi)^{-5} \int dm \int d^4 p \frac{m I_4 - \not{p}}{m^2 - (m_p^2 \pm i\epsilon)} \exp[i(p \cdot (x' - x) + m(\tau' - \tau))] \quad (25)$$

where $0 < \epsilon \ll 1$. The small positive number $+i\epsilon$ serves to remind that the inversion path is below the pole at $+m_p$ in the complex plane of m , and above the pole at $-m_p$. Contour integration in the m -plane yields

$$\Gamma_\pm^0(x' - x, \tau' - \tau) = i \int d^4 p \left\{ \theta(\tau' - \tau) \mathcal{E}_\pm - \theta(\tau - \tau') \mathcal{E}_\mp \right\}, \quad (26)$$

where θ is again the Heaviside unit step function and

$$\mathcal{E}_\pm = (2\pi)^{-4} \left(\frac{\pm m_p I_4 - \not{p}}{\pm 2m_p} \right) \exp[i(p \cdot (x' - x) \pm m_p(\tau' - \tau))]. \quad (27)$$

The two free influence functions are related by

$$\overline{\Gamma_{\pm}^0(x' - x, \tau' - \tau)} \equiv \gamma^0 (\Gamma_{\pm}^0(x' - x, \tau' - \tau))^{\dagger} \gamma^0 = \Gamma_{\mp}^0(x - x', \tau - \tau'). \quad (28)$$

Considering Γ_{+}^0 for $\tau' - \tau > 0$,

$$\Gamma_{+}^0(x' - x, \tau' - \tau) = i \int d^4p \left\{ \theta(p^0) \mathbf{f}_p^{(+)}(x', \tau') \overline{\mathbf{f}_p^{(+)}(x, \tau)} - \theta(-p^0) \mathbf{f}_p^{(-)}(x', \tau') \overline{\mathbf{f}_p^{(-)}(x, \tau)} \right\}. \quad (29)$$

As $\tau' - \tau \rightarrow +\infty$, Γ_{+}^0 evolves forward-propagating waves of positive energy and backward-propagating waves of negative energy. In terms of the forward-propagating waves and their backward-propagating \mathcal{TCP} conjugates waves, Γ_{+}^0 becomes, for $\tau' - \tau > 0$,

$$\Gamma_{+}^0(x' - x, \tau' - \tau) = i \int d^4p \theta(p^0) \left\{ \mathbf{f}_p^{(+)}(x', \tau') \overline{\mathbf{f}_p^{(+)}(x, \tau)} - \mathbf{h}_p^{(-)}(x', \tau') \overline{\mathbf{h}_p^{(-)}(x, \tau)} \right\}. \quad (30)$$

Thus as $\tau' - \tau \rightarrow +\infty$, Γ_{+}^0 evolves only states in \mathcal{S}_{+} , that is the positive-energy, forward-propagating particles (for example, electrons) and the positive-energy, backward propagating antiparticles (positrons).

Considering Γ_{+}^0 for $\tau' - \tau < 0$,

$$\Gamma_{+}^0(x' - x, \tau' - \tau) = i \int d^4p \left\{ \theta(p^0) \mathbf{f}_p^{(-)}(x', \tau') \overline{\mathbf{f}_p^{(-)}(x, \tau)} - \theta(-p^0) \mathbf{f}_p^{(+)}(x', \tau') \overline{\mathbf{f}_p^{(+)}(x, \tau)} \right\}. \quad (31)$$

Thus as $\tau' - \tau \rightarrow -\infty$, Γ_{+}^0 evolves backward-propagating wavefunctions of positive energy and forward-propagating wavefunctions of negative energy. In terms of the forward-propagating wavefunctions and their backward-propagating \mathcal{TCP} conjugate wave functions, Γ_{+}^0 becomes, for $\tau' - \tau < 0$,

$$\Gamma_{+}^0(x' - x, \tau' - \tau) = -i \int d^4p \theta(-p^0) \left\{ \mathbf{f}_p^{(+)}(x', \tau') \overline{\mathbf{f}_p^{(+)}(x, \tau)} - \mathbf{h}_p^{(-)}(x', \tau') \overline{\mathbf{h}_p^{(-)}(x, \tau)} \right\}. \quad (32)$$

Thus as $\tau' - \tau \rightarrow -\infty$, Γ_{+}^0 evolves only states in \mathcal{S}_{-} , that is, the negative-energy, forward-propagating particles and the negative-energy, backward-propagating antiparticles. Similarly it follows that for $\tau' - \tau > 0$,

$$\Gamma_{-}^0(x' - x, \tau' - \tau) = i \int d^4p \theta(-p^0) \left\{ \mathbf{f}_p^{(+)}(x', \tau') \overline{\mathbf{f}_p^{(+)}(x, \tau)} - \mathbf{h}_p^{(-)}(x', \tau') \overline{\mathbf{h}_p^{(-)}(x, \tau)} \right\}, \quad (33)$$

while for $\tau' - \tau < 0$,

$$\Gamma_{-}^0(x' - x, \tau' - \tau) = -i \int d^4p \theta(p^0) \left\{ \mathbf{f}_p^{(+)}(x', \tau') \overline{\mathbf{f}_p^{(+)}(x, \tau)} - \mathbf{h}_p^{(-)}(x', \tau') \overline{\mathbf{h}_p^{(-)}(x, \tau)} \right\}. \quad (34)$$

It is readily seen that

$$\Gamma_{\pm}^0(x' - x, \tau' - \tau) = \text{sign}(\tau' - \tau) \frac{1}{i} \int d^4x'' \Gamma_{\pm}^0(x' - x'', \tau - \tau'') \Gamma_{\pm}^0(x'' - x, \tau'' - \tau) \quad (35)$$

for $\tau' > \tau'' > \tau$ in which case $\text{sign}(\tau' - \tau) = +1$, and also for $\tau' < \tau'' < \tau$ in which case $\text{sign}(\tau' - \tau) = -1$. If either $\tau'' < \tau$ or $\tau'' > \tau'$, then the right hand side of (35) vanishes.

C. Møller operators

The influence functions $\Gamma_{\pm}(x', \tau'; x, \tau)$ are defined as solutions of

$$\frac{1}{i} \frac{\partial}{\partial \tau'} \Gamma_{\pm}(x', \tau'; x, \tau) + \gamma_{\mu} \left(\frac{1}{i} \frac{\partial}{\partial x_{\mu}'} - eA^{\mu}(x') \right) \Gamma_{\pm}(x', \tau'; x, \tau) = \delta^4(x' - x) \delta(\tau' - \tau). \quad (36)$$

They depend upon the charge e and potential A^{μ} , but in the interest of clarity the argument lists here for Γ_{\pm} include only the events x, x' and parameters τ, τ' . The boundary conditions are

$$\Gamma_{+}(x', \tau'; x, \tau) \sim \Gamma_{+}^0(x' - x, \tau' - \tau) \quad (37)$$

as $\tau', \tau \rightarrow -\infty$ with $\tau' > \tau$ and

$$\Gamma_{-}(x', \tau'; x, \tau) \sim \Gamma_{-}^0(x' - x, \tau' - \tau) \quad (38)$$

as $\tau', \tau \rightarrow +\infty$ with $\tau > \tau'$. Rearranging (36) yields [15]

$$\Gamma_{\pm}(x', \tau'; x, \tau) = \Gamma_{\pm}^0(x' - x, \tau' - \tau) + \int d\tau'' \int d^4x'' \Gamma_{\pm}^0(x' - x'', \tau' - \tau'') e\mathcal{A}(x'') \Gamma_{\pm}(x'', \tau''; x, \tau). \quad (39)$$

Expanding (39) in the Born series shows that

$$\overline{\Gamma_{\pm}(x', \tau'; x, \tau)} = \Gamma_{\mp}(x, \tau; x', \tau'). \quad (40)$$

The semigroup property (35) holds also for Γ_{\pm} .

For $\phi_i \in \mathcal{S}_+$, define $\omega_+ \phi_i$ by

$$\omega_+ \phi_i(x', \tau') = \lim_{\tau \rightarrow -\infty} \frac{+1}{i} \int d^4x \Gamma_+(x', \tau'; x, \tau) \phi_i(x, \tau). \quad (41)$$

If the support of the potential A^{μ} is bounded in \mathbb{E}^4 , and if the support of $\phi_i(x, \tau)$ is exterior to that of A^{μ} as $\tau \rightarrow \pm\infty$, then $\omega_+ \phi_i$ satisfies (1) subject to $\omega_+ \phi_i(x', \tau') \rightarrow \phi_i(x', \tau')$ as $\tau' \rightarrow -\infty$. The wavefunction $\omega_+ \phi_i$ may be expressed as

$$\omega_+ \phi_i(x', \tau') = \phi_i(x', \tau') + \int d^4x \int_{-\infty}^{+\infty} d\tau \Gamma_+^0(x' - x, \tau' - \tau) e\mathcal{A}(x) \omega_+ \phi_i(x, \tau). \quad (42)$$

It may be seen from (42) that $\omega_+ \phi_i(x, \tau) - \phi_i(x, \tau) \in \mathcal{S}_{\pm}$ as $\tau \rightarrow \pm\infty$, that is, there is ‘ τ -increasing’ scattering into positive-energy, forward-propagating particles and positive-energy, backward-propagating antiparticles, and also ‘ τ -decreasing’ scattering into negative-energy, forward-propagating particles and negative-energy, backward-propagating antiparticles. If $\phi_i \in \mathcal{S}_-$ then $\omega_+ \phi_i = 0$.

Similarly for $\phi_f \in \mathcal{S}_+$ (not \mathcal{S}_-) define $\omega_- \phi_f(x, \tau)$ by

$$\omega_- \phi_f(x', \tau') = \lim_{\tau \rightarrow +\infty} \frac{-1}{i} \int d^4x \Gamma_-(x', \tau'; x, \tau) \phi_f(x, \tau). \quad (43)$$

Then $\omega_- \phi_f$ satisfies (1), subject to $\omega_- \phi_f(x', \tau') \rightarrow \phi_f(x', \tau')$ as $\tau' \rightarrow +\infty$ and

$$\omega_- \phi_f(x', \tau') = \phi_f(x', \tau') + \int d^4x \int_{-\infty}^{+\infty} d\tau \Gamma_-^0(x' - x, \tau' - \tau) eA(x) \omega_- \phi_f(x, \tau). \quad (44)$$

It may be seen from (44) that $\omega_- \phi_f(x, \tau) - \phi_f(x, \tau) \in \mathcal{S}_\mp$ as $\tau \rightarrow \pm\infty$. If $\phi_f \in \mathcal{S}_-$ then $\omega_- \phi_f = 0$.

In summary, the Møller operators ω_\pm are defined by

$$\omega_\pm \phi(x', \tau') = \lim_{\tau \rightarrow \mp\infty} \frac{\pm 1}{i} \int d^4x \Gamma_\pm(x', \tau'; x, \tau) \phi(x, \tau) \quad (45)$$

for all $\phi \in \mathcal{S}_+$. Again, $\omega_\pm \phi = 0$ for all $\phi \in \mathcal{S}_-$.

The potential A^μ involved in the Møller operators is calculated as the particular integral of (5) for the concatenation of the Møller current

$$j_{fi}^\mu(x, \tau) = \overline{\omega_- \phi_f(x, \tau)} \gamma^\mu \omega_+ \phi_i(x, \tau) = \overline{\phi_f(x, \tau)} \gamma^\mu \phi_i(x, \tau) + \mathcal{O}(e^2), \quad (46)$$

assuming that $A^\mu = \mathcal{O}(e)$.

IV. TWO DIRAC PARTICLES

The single-particle formalism of the preceding sections is extended to two particles. Space-time entanglement is displayed in an elementary and unrestricted way. A proof of the spin-statistics theorem is very briefly described, and the Bethe-Salpeter equation for bound states is derived from the two-particle formalism without further conjecture. The mutual scattering problem for two particles is reduced to that of a single particle in the presence of an ‘external’ potential.

A. two-particle wave equation

The single-particle, spin-1/2 wave equation (1) and Lagrangian (4) are readily extended to two-particle, spin-1/2 wavefunctions in the tensor product space. The wave equation becomes

$$\frac{1}{i} \frac{\partial}{\partial \tau} \Psi + \not{x}(x) \otimes I_4 \Psi + I_4 \otimes \not{x}(y) \Psi = 0 \quad (47)$$

where $\pi^\mu(x) = (1/i)\partial/\partial x_\mu - e_1 A^\mu(x)$ and $\pi^\mu(y) = (1/i)\partial/\partial y_\mu - e_2 A^\mu(y)$.

Consider first the case of externally-sourced electromagnetic potentials $A^\mu(x)$ and $A^\mu(y)$. Two indistinguishable spin-1/2 particles entangled at the events $x = (t, \mathbf{x})$ and $y = (s, \mathbf{y})$ may be represented in the standard way with antisymmetric combinations of tensor products of single-particle 4-spinor wavefunctions, such as

$$\Psi(x, y, \tau) = \frac{1}{\sqrt{2}} \left(\psi(x, \tau) \otimes \chi(y, \tau) - \chi(x, \tau) \otimes \psi(y, \tau) \right). \quad (48)$$

This representation of entanglement, in space or time, is simpler and more general than the QFT representations.

Consider now interaction potentials and self potentials acting on both particles. The potentials are constructed by integrating over the dependences of the two-particle currents upon one of the particles. That is,

$$\square A^\mu(x) = e_1 J_1^\mu(x) + e_2 J_2^\mu(x) \quad (49)$$

where e_1 and e_2 are the charges of the two particles. The currents are, after concatenation and marginalization,

$$J_1^\mu(x) = \int d\tau \int d^4y \overline{\Psi(x, y, \tau)} \gamma^\mu \otimes I_4 \Psi(x, y, \tau) \quad (50)$$

and

$$J_2^\mu(x) = \int d\tau \int d^4y \overline{\Psi(y, x, \tau)} I_4 \otimes \gamma^\mu \Psi(y, x, \tau). \quad (51)$$

In particular, a particle may experience the potential arising from its own current. If the two particles are indistinguishable, the wavefunction Ψ is chosen to be antisymmetric as in (48). Then the corresponding quantum currents J_1^μ, J_2^μ and hence the semiclassical field A^μ are unaltered by permutation of the two single-particle states at the same parameter τ . That is, the currents and the potential are appropriately bosonic.

If the particles are distinguishable and the wavefunction is a simple tensor product $\Psi(x, y, \tau) = \psi(x, \tau) \otimes \chi(y, \tau)$, where both ψ and χ are normalized beams or packets, then the concatenated currents reduce to

$$J_1^\mu(x) = \int d\tau \overline{\psi(x, \tau)} \gamma^\mu \psi(x, \tau) \quad (52)$$

and

$$J_2^\mu(x) = \int d\tau \overline{\chi(x, \tau)} \gamma^\mu \chi(x, \tau). \quad (53)$$

B. spin and statistics

The spin–statistics connection is usually presented [10–13] as a theorem in relativistic QFT. On the other hand, the unimodular and unitary group $SU(2)$ spanned by the Pauli spin matrices is a spin–1/2 representation of the unimodular and orthogonal group $SO(3)$ of spatial rotations [23]. That is, spin is not intrinsically a relativistic phenomenon. Indeed, the single–particle, spin–1/2 Pauli wave equation [15] is covariant with respect to $SO(3)$ provided that the electric potential \mathbf{A} and the magnetic field \mathbf{B} are functions of $|\mathbf{x}|$ alone. The spin–statistics connection has furthermore been proved for both nonrelativistic and relativistic quantum mechanics of arbitrary spin [14]. The proof takes into account the phases of spin eigenstates. The phases are indeterminate since only a spin axis is specified, rather than a spin frame. Homotopically consistent permutations of the arbitrary phases between numerous otherwise indistinguishable particles, along with permutations of the conventionally observable positions and spins, lead to the standard spin–statistics connection. The proof holds in particular for relativistic spin–1/2 quantum mechanics. The phase is not a standard observable, but the universal observance of the exclusion principle impresses physical significance upon the phase indeterminacy.

C. two–particle Møller operators

There are four two–particle free influence functions. Of particular interest are Γ_{++}^0 and Γ_{--}^0 given by

$$\Gamma_{\pm\pm}^0(x' - x, y' - y, \tau' - \tau) = \frac{1}{i} \Gamma_{\pm}^0(x' - x, \tau' - \tau) \otimes \Gamma_{\pm}^0(y' - y, \tau' - \tau). \quad (54)$$

Proceeding as in Section III C leads to the two–particle Møller operators Ω_{++} and Ω_{--} acting on $\mathcal{S}_+ \otimes \mathcal{S}_+$, such that

$$\Omega_{\pm\pm} \Phi(x', y', \tau') = \lim_{\tau \rightarrow \mp\infty} \frac{\pm 1}{i} \int d^4x \int d^4y \Gamma_{\pm\pm}(x', y', \tau'; x, y, \tau) \Phi(x, y, \tau) \quad (55)$$

for all $\Phi \in \mathcal{S}_+ \otimes \mathcal{S}_+$. The dependence of the influence functions $\Gamma_{\pm\pm}$ and the Møller operators $\Omega_{\pm\pm}$ upon the two charges e_1 and e_2 , and upon the potential A^μ at x and at y , is of fundamental importance but for clarity is not made explicit here. It follows that

$\Omega_{\pm\pm}\Phi = 0$ for all $\Phi \in \mathcal{S} \otimes \mathcal{S} \setminus \mathcal{S}_+ \otimes \mathcal{S}_+$. The influence functions $\Gamma_{\pm\pm}$ satisfy

$$\begin{aligned} \Gamma_{\pm\pm}(x', y', \tau'; x, y, \tau) &= \Gamma_{\pm\pm}^0(x' - x, y' - y, \tau' - \tau) \\ &+ \int d\tau'' \int d^4x'' \int d^4y'' \Gamma_{\pm\pm}^0(x' - x'', y - y'', \tau' - \tau'') \\ &\quad \times V(x'', y'') \Gamma_{\pm\pm}(x'', y'', \tau''; x, y, \tau) \end{aligned} \quad (56)$$

The scattering potential in (56) is

$$V(x, y) = e_1 \mathcal{A}(x) \otimes I_4 + I_4 \otimes e_2 \mathcal{A}(y). \quad (57)$$

The semigroup property (35) extends to $\Gamma_{\pm\pm}$. When the potential A^μ is external, it may be shown that

$$\Gamma_{\pm\pm}(x', y', \tau'; x, y, \tau) = \frac{1}{i} \Gamma_{\pm}(x', \tau'; x, \tau) \otimes \Gamma_{\pm}(y', \tau'; y, \tau), \quad (58)$$

and that

$$\overline{\Gamma_{\pm\pm}(x', y', \tau'; x, y, \tau)} = \Gamma_{\mp\mp}(x, y, \tau; x', y', \tau'). \quad (59)$$

The separability property (58) implies that unentanglement is conserved with the passage of τ . The Møller currents are

$$J_1^\mu(x)_{fi} = \int d\tau \int d^4y \overline{\Omega_{--}\Phi_f(x, y, \tau)} \gamma^\mu \otimes I_4 \Omega_{++}\Phi_i(x, y, \tau) \quad (60)$$

and

$$J_2^\mu(x)_{fi} = \int d\tau \int d^4y \overline{\Omega_{--}\Phi_f(y, x, \tau)} I_4 \otimes \gamma^\mu \Omega_{++}\Phi_i(y, x, \tau). \quad (61)$$

D. bound states

It is now convenient to combine the influence functions in the form

$$\Gamma(x', y', \tau'; x, y, \tau) = \theta(\tau' - \tau) \Gamma_{++}(x', y', \tau'; x, y, \tau) - \theta(\tau - \tau') \Gamma_{--}(x', y', \tau'; x, y, \tau), \quad (62)$$

and similarly there is the combined free influence function $\Gamma^0(x' - x, y' - y, \tau' - \tau)$. Then for all τ' and τ , any two-particle wavefunction $\Psi(x, y, \tau)$ evolves as

$$\Psi(x', y', \tau') = \frac{1}{i} \int d^4x \int d^4y \Gamma(x', y', \tau; x, y, \tau) \Psi(x, y, \tau), \quad (63)$$

or by virtue of (56) as

$$\begin{aligned} \Psi(x', y', \tau') &= \Psi_0(x', y', \tau') \\ &+ \frac{1}{i} \int d\tau'' \int d^4x'' \int d^4y'' \Gamma^0(x' - x'', y - y'', \tau' - \tau'') V(x'', y'') \Psi(x'', y'', \tau) \end{aligned} \quad (64)$$

where the freely propagated wavefunction $\Psi_0(x', y', \tau')$ is

$$\Psi_0(x', y', \tau') = \frac{1}{i} \int d^4x \int d^4y \Gamma^0(x' - x, y' - y, \tau' - \tau) \Psi(x, y, \tau). \quad (65)$$

The Fourier transform of (64) with respect to τ is the inhomogeneous Bethe–Salpeter equation [7–9]

$$\begin{aligned} \Psi(x', y', m) &= \Psi_0(x', y', m) \\ &+ \frac{1}{i} \int d^4x'' \int d^4y'' \Gamma^0(x' - x'', y - y'', m) V(x'', y'') \Psi(x'', y'', m), \end{aligned} \quad (66)$$

in the casual notation where

$$f(m) = \int d\tau f(\tau) \exp(-im\tau). \quad (67)$$

The interaction potential $V(x, y)$ as given in (57) is determined semi-classically, that is, in terms of potentials $A^\mu(x)$ satisfying Maxwell's equation (49) for the Dirac currents (50) and (51). Bound states are defined as eigenstates of the homogeneous equation [9], that is, (66) for $\Psi_0 = 0$. Indeed, if there is binding energy in the two-particle state Ψ then the freely-propagating Ψ_0 is a kinematical impossibility [8, 9]. The Bethe–Salpeter equation is in general nonlinear but may be expanded at least formally in powers of the charges e_1 and e_2 , yielding a series of linear equations. A method of partial summation is described in §VI below.

E. two-particle scattering matrix

Let Φ_i and Φ_f be two free wavefunctions in $\mathcal{S}_+ \otimes \mathcal{S}_+$. Then the scattering matrix S_{fi} is defined by

$$S_{fi} \equiv \int d^4x \int d^4y \overline{\Omega_{--} \Phi_f(x, y, \tau)} \Omega_{++} \Phi_i(x, y, \tau). \quad (68)$$

The incident and final two-particle states for indistinguishable particles are independently fermionic. That is, the incident single-particle states may be permuted independently of the final states and vice versa, leading in each case to a reversal of the sign of S_{fi} .

It is readily shown that $\partial S_{fi}/\partial\tau = 0$, since $\not{x}^\dagger\gamma^0 = \gamma^0\not{x}$. Alternatively, it may be shown that

$$S_{fi} = \lim_{\tau \rightarrow +\infty} \int d^4x \int d^4y \overline{\Phi_f(x, y, \tau)} \Omega_{++} \Phi_i(x, y, \tau). \quad (69)$$

which is explicitly independent of τ . It follows from (69) that

$$S_{fi} = S_{fi}^{(0)} + i \int d\tau \int d^4x \int d^4y \overline{\Phi_f(x, y, \tau)} V(x, y) \Omega_{++} \Phi_i(x, y, \tau), \quad (70)$$

where

$$S_{fi}^{(0)} = \int d^4x \int d^4y \overline{\Phi_f(x, y, \tau)} \Phi_i(x, y, \tau) \quad (71)$$

which is independent of τ assuming that Φ_i and Φ_f are free particle pairs. If the particles are distinguishable then the incident and final free wavefunctions are of the form $\Phi_i = \phi_i \otimes \xi_i$ and $\Phi_f = \phi_f \otimes \xi_f$ respectively, while the Møller operators are of the form $\Omega_{\pm\pm} = \omega_{\pm} \otimes \omega_{\pm}$. The Møller currents are calculated, following (52) and (53), as

$$J_1^\mu(x) = \int d\tau \overline{\omega_- \phi_f(x, \tau)} \gamma^\mu \omega_+ \phi_i(x, \tau) \quad (72)$$

and

$$J_2^\mu(x) = \int d\tau \overline{\omega_- \xi_f(x, \tau)} \gamma^\mu \omega_+ \xi_i(x, \tau). \quad (73)$$

If the two particles are indistinguishable then both the incident and final two-particle free states Φ_i and Φ_f in (68) must be fermionic as in (48). However, both two-particle free states in the Møller currents (60) and (61) must be replaced with bosonic states of the form

$$\Phi(x, y, \tau) = \frac{1}{\sqrt{2}} \left(\phi(x, \tau) \otimes \xi(y, \tau) + \xi(x, \tau) \otimes \phi(y, \tau) \right). \quad (74)$$

The Møller currents are then bosonic, prior to marginalization. It follows eventually that the scattering matrix is appropriately fermionic with respect to the incident two-particle state, and also with respect to the final two-particle state.

The integrand for nontrivial scattering in (70) is, in the case of distinguishable particles for simplicity,

$$\begin{aligned} \overline{\Phi_f(x, y, \tau)} V(x, y) \Omega_{++} \Phi_i(x, y, \tau) = \\ \overline{\phi_f(x, \tau)} e_1 \not{A}(x) \omega_+ \phi_i(x, \tau) \overline{\xi_f(y, \tau)} \omega_+ \xi_i(y, \tau) \\ + \overline{\phi_f(x, \tau)} \omega_+ \phi_i(x, \tau) \overline{\xi_f(y, \tau)} e_2 \not{A}(y) \omega_+ \xi_i(y, \tau). \end{aligned} \quad (75)$$

Iteration on (71) by expansion in powers of the charges e_1 and e_2 proceeds the most efficiently if the Møller operators in the factors not explicitly displaying scattering, that is in the factors $\overline{\xi_f \omega_+ \xi_i}$ and $\overline{\phi_f \omega_+ \phi_i}$, are not so iterated. These factors reduce to unity, after forming the scattering cross-section and averaging over the incident and final states. It suffices therefore to consider single-particle scattering off the semiclassical potential owing to any source including the current of the particle itself. The incident and final energy-momenta p_i^μ and p_f^μ are assumed timelike, hence the ostensibly ‘external’ photon is spacelike.

The potential owing to an ‘external’ source $\mathcal{Z}^\mu(x)$ is now denoted $\mathcal{A}^\mu(x)$, with

$$\square \mathcal{A}^\mu = \mathcal{Z}^\mu. \quad (76)$$

If the external source \mathcal{Z}^μ is in fact the current $e_2 J_2^\mu$ of a second and possibly distinguishable spin-1/2 particle then, owing to the parity symmetry of the D’Alembertian \square , it is readily shown that the single-particle scattering matrix is symmetric with respect to the two particles. That is, the scattering of the second particle by the first is the same as the scattering of the first particle by the second.

V. FIRST-ORDER SCATTERING OF A SINGLE PARTICLE

The scattering matrix for a single particle resembles (68) or equivalently (70). To leading order in powers of the charge e , the nontrivial contribution to the analog of (70) is

$$S_{fi}^{(1)} = i \int d\tau \int d^4x \overline{\phi_f(x, \tau)} e \mathcal{A}(x) \phi_i(x, \tau). \quad (77)$$

Assume that both ϕ_i and ϕ_f have the form (11) with both p_i^0 and p_f^0 positive, hence both wavefunctions are in \mathcal{S}_+ . It follows that

$$S_{fi}^{(1)} = \frac{ie}{(2\pi)^3} \delta(\Delta m_p) \overline{u_f} \mathcal{A}(\Delta p) u_i \quad (78)$$

where $\Delta p = p_f - p_i$ and $\Delta m_p = m(p_f) - m(p_i)$, while $u_i = \mathbf{u}(p_i) \mathbf{a}_i$ and $u_f = \mathbf{u}(p_f) \mathbf{a}_f$. The coefficients \mathbf{a}_i and \mathbf{a}_f are complex 2×1 matrices. Finally, $\mathcal{A}^\mu(\Delta p)$ is the Fourier transform of $\mathcal{A}^\mu(x)$ defined by

$$\mathcal{A}^\mu(\Delta p) = \int \frac{d^4x}{(2\pi)^4} \mathcal{A}^\mu(x) \exp[-i\Delta p \cdot x]. \quad (79)$$

Note that the mass of the particle is conserved, that is, $m_f = m(p_f) = m(p_i) = m_i$ which may as well be assigned the notation m_e for, say, an electron.

The Coulomb potential for a point charge $-Ze > 0$ is defined by $\mathcal{A}^0(x) = -Ze/(4\pi|\mathbf{x}|)$ where $x = (x^0, \mathbf{x})$, and by $\mathcal{A}^j(x) = 0$ for $j = 1, 2, 3$. Hence $A^0(\Delta p) = -2\pi Ze \delta(\Delta p^0)/|\Delta \mathbf{p}|^2$. Conservation of energy p^0 as well as mass $m(p)$ implies that the magnitude of momentum \mathbf{p} is also conserved, and so a scattering parameter κ may be defined by $\mathbf{p}_f \cdot \mathbf{p}_i = |\mathbf{p}|^2 \cos \kappa$. It follows from (78) in the standard way [15, 24] that the effect of spin is to modify the κ -dependence of the Rutherford cross section by the Mott factor $\mathcal{M} = 1 - |\mathbf{p}/m_e|^2 \sin^2(\kappa/2)$.

It may be remarked in passing that, for the purposes of calculating cross-sections, the continuum normalizations (11) and (12) are replaced with 4-cube normalizations rather than the standard 3-cube. Specifically, $(2\pi)^4$ is replaced with L^4 where L is the edge of the 4-cube. The cross-section as calculated is then a volume rather than the standard area. In the case of Coulomb scattering, where the potential is independent of coordinate time x^0 , the cross-sectional volume is proportional to $T_0 = 2\pi\delta(p_i^0 - p_i^0)$ which is interpreted in the standard way [15] as a large interval in x^0 during which scattering is observed. The incident wavefunction is in practice a beam having finite energy spread. The beam accordingly takes a finite time to sweep through an external scattering potential, although it must be conceded that such a time is undefined for an inverse-distance potential. Thus it is in general appropriate when the potential is time-independent to divide the cross-sectional volume by T_0 , yielding a cross-sectional area [25].

VI. ONE-LOOP CORRECTIONS

Assuming as in Coulomb scattering that \mathcal{A} is $\mathcal{O}(e)$, the first-order external scattering $S_{fi}^{(1)}$ given by (77) is $\mathcal{O}(e^2)$. One-loop corrections $S_{fi}^{(1)}$ arise from (i) self-scattering corrections to (77) and (ii) external-scattering corrections to first-order self scattering. There are two contributions to (i), obtained by iteration on (39) yielding Γ_+ correct to $\mathcal{O}(e^2)$. There are also two contributions to (ii), obtained by iteration on (45) yielding ω_{\pm} correct to $\mathcal{O}(e^2)$. The contributions to (ii) are, by virtue of (28), identical to those to (i). Hence there are in all four relevant corrections of $\mathcal{O}(e^4)$ with total $M_4 = 2M_{12} + 2M_{21}$ where

$$M_{12} = i \int d\tau \int d^4x \int d\tau' \int d^4x' \overline{\phi_f(x, \tau)} e \mathcal{A}(x) \Gamma_+^0(x - x', \tau - \tau') e \mathcal{A}(x') \phi_i(x', \tau'), \quad (80)$$

while M_{21} differs from M_{12} only by exchanging the potentials \mathcal{A} and \mathcal{A} in (80). The self potential A^μ in (80) satisfies

$$\square A^\mu(x) = eJ^\mu(x) = e \int d\tau \overline{\phi_f(x, \tau)} \gamma^\mu \phi_i(x, \tau), \quad (81)$$

and is obtained as

$$A^\mu(x) = e \int d^4x' D_F(x - x') J^\mu(x') \quad (82)$$

where the Feynman photon propagator D_F is [15]

$$D_F(x - x') = \int \frac{d^4k}{2\pi^4} \frac{1}{k^2 - i\epsilon} \exp[ik \cdot (x - x')]. \quad (83)$$

Again, the metric here has the signature $(-+++)$.

A change to the $\mathcal{O}(e^4)$ scattering matrix contribution $M_4 = M_{12} + M_{21}$ is now made, by the substitution

$$\phi_i(x, \tau) \overline{\phi_f(x', \tau')} \Rightarrow \frac{1}{iT_\tau} \exp[i(m_i\tau - m_f\tau')] \int \frac{d^4r}{(2\pi)^4} \frac{\overline{m}I_4 - \not{r}}{\overline{m}^2 + r^2 - i\epsilon} \exp[ir \cdot (x - x')] \quad (84)$$

where $\overline{m} = (m_f + m_i)/2$, but the substitution (84) is made *without* regard to the τ -ordering. The standard additional minus sign is introduced for a closed electron loop [15]. The substitution is accurate for incident and final free-wavefunction beams ϕ_i and ϕ_f , if the necessarily spacelike impact parameter $\Delta p = p_f - p_i$ for the beam axes is small. The scale $T_\tau = 2\pi\delta(m_i - m_f)$, for the parameter τ , is the inverse of the small mass spread of the beams. In the far field the beams consist of free particles all on the same mass shell $m_i = m_f = m_e$, but the scattering takes place off mass shell and so an $\mathcal{O}(m_e e^2)$ mass spread is defined for all internal fermion lines. The quantum-mechanical substitution (84) has an analog in QFT [11, 12], where the free fermion propagator is a two-event correlation. Making the substitution (84) in the expansion of the Bethe-Salpeter equation (66) yields all of the standard diagrams [9].

The substitution (84) having been made in the two copies of the scattering matrices M_{12} and M_{21} in all four topologically different ways, it is the case that the vertex is modified by a factor \mathcal{F} that is a sum of the four standard Feynman diagrams in Fig. 8–10 (b), (c), (d) and (e) of Bjorken and Drell [15]. In order of renormalization the diagrams are the vacuum polarization of the external field (e), the mass-renormalization counter term (d) and self-mass (c) on both the incident and final fermion lines, and the vertex correction (b). The four diagrams are different, even though they arise here from two duplicated integrals, hence

the symmetry factor for each diagram is $S = 1$. Each of the four additive contributions to the modification factor \mathcal{F} is independent of the parameter scale T_τ . The contributions are, relative to $S_{fi}^{(1)}$, precisely the standard wavenumber integrals in Eqs (8.8), (8.34) and (8.49) of Bjorken and Drell [15]. All the integrals derived here have the signs as deduced from QFT [10, 12, 13]. That is, there is no need to introduce a sign correction by fiat, as is done by Bjorken and Drell [15] who assume the connection between spin and statistics. It is concluded that the Uehling potential, the anomalous magnetic moment of the electron and the Lamb shift have the respective leading-order values as eventually deduced by Bjorken and Drell [15] and as accurately observed [15].

The above-mentioned diagrams, integrals and resulting one-loop corrections are obtained by Bjorken and Drell [15] from the standard Dirac wave equation for a single particle. Instead of making the substitution (84), those authors replace the potential product $A^\mu(x)A^\nu(y)$ in the second-order scattering matrix with a superposition of products of leading-order Møller currents. The resulting matrix is then symmetric in its dependence upon the two particles. Bjorken and Drell refer to their text on QFT [26] for strict justification.

VII. AXIAL ANOMALY

The parametrized Dirac equation (1) leads to the vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$ obeying the identity (2), and also to the axial current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ obeying

$$\bar{\psi}\gamma_5\frac{\partial\psi}{\partial\tau} - \frac{\partial\bar{\psi}}{\partial\tau}\gamma_5\psi + \frac{\partial}{\partial x^\mu}\bar{\psi}\gamma^\mu\gamma_5\psi = 0. \quad (85)$$

The concatenated vector current J^μ is again divergenceless, that is,

$$\partial_\mu J^\mu = \int j^\mu d\tau = 0, \quad (86)$$

while for the sharp mass value m_e the divergence of the concatenated axial current J_5^μ is

$$\partial_\mu J_5^\mu = \int j_5^\mu d\tau = -2im_e \int \bar{\psi}\gamma_5\psi d\tau. \quad (87)$$

The preceding identities also hold for Møller charges and currents, that is, if the replacements $\bar{\psi} \rightarrow \overline{\omega_- \phi_f}$ and $\psi \rightarrow \omega_+ \phi_i$ are made. In particular the Møller identities must hold at all orders in e , although verification order by order is tedious. It follows immediately from the Møller analog of (2) that the concatenated vector Møller current $\int \overline{\omega_- \phi_f} \gamma^\mu \omega_+ \phi_i d\tau$ is divergenceless.

The divergence of the axial Møller current $\int \overline{\omega_- \phi_f} \gamma^\mu \gamma_5 \omega_+ \phi_i d\tau$ is next recalculated correctly to $\mathcal{O}(e^2)$ for a particle in an external field. The particle is assumed for simplicity to be massless, that is, $m_e \rightarrow 0$. Fermion loops and triangles are again closed using the substitution (84). It is straightforward to show that the concatenated axial Møller divergence vanishes at $\mathcal{O}(1)$, and also at $\mathcal{O}(e)$. At $\mathcal{O}(e^2)$ the divergence is the sum of three terms, each of which is an integral over the energy–momenta of photons at all the vertices of a triangle. The three terms are found to be identical, and so a symmetry factor of $S = 3$ is assigned. The assignment is not $S = 3!$ since the factor of 2 is already incorporated into each of the three original integrals, following symmetrization with respect to the photons at the two vector vertices. Hence the required divergence is the value of just one such term, which is precisely equal to the divergence of the QED coordinate–time–ordered, vacuum–to–vacuum amplitude of the product of an axial current and two vector currents [12]. The ‘anomalous’ divergence is accordingly found to have the standard value [10, 12, 13]

$$\partial_\lambda J_5^\lambda(x) = -\frac{e^2}{(4\pi)^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x). \quad (88)$$

Here, however, the fields are classically valued rather than operator valued.

VIII. SUMMARY AND DISCUSSION

A. summary

The parametrized Dirac equation with a semiclassical Maxwell potential reproduces the standard one–loop corrections and axial anomaly of QED. The correct Feynman diagrams are the consequences here of the substitution (84), which is justified for Δp small. The correctness of the Feynman diagrams for arbitrary Δp implies that the substitution is an effective partial summation of the perturbative series. Parametrized relativistic quantum mechanics provides, unlike field–theoretic QED, an elementary and unrestricted representation of entanglement in space or time. The Bethe–Salpeter equation for bound states is a direct consequence of the parametrized Dirac equation for multiple particles.

B. discussion

The electromagnetic radiation considered here is semiclassical. That is, the electromagnetic four-potential is a classical solution of Maxwell's equation possibly in the presence of a Dirac current. Semiclassical explanations of the photoelectric effect, variously attributed to Jaynes and to Lamb and Scully, are now accepted [27, 28] although informal controversy continues over the quantum mechanics of the photodetector. The semiclassical solutions of Maxwell's equation are typically subject to an homogeneous boundary condition, and so the classical vacuum is therefore a null potential. The Rayleigh–Jeans law, with its ultraviolet catastrophe, is of course another prediction of the classical theory. The Quantum Theory of Radiation (QTR) predicts [10–13] that the vacuum is a zero-point quantum field of energy density $\hbar\omega/2$ per mode of circular frequency ω , leading in particular to the Planck law or blackbody spectrum. Consider, however, Maxwell's equation subject to a boundary condition which is a Lorentz-invariant classical free potential, which is statistically stationary and isotropic, and which has the same energy density $\hbar\omega/2$ per mode. The classical solution can [29–31] then account not only for the blackbody spectrum, but also for the Van der Waals forces, the Casimir effect and the Einstein 'A' coefficient for spontaneous emission. A random classical potential cannot [27], however, explain antibunching for light beams passing through the two arms of an interferometer. That is, owing to the Cauchy–Schwarz inequality, the correlation of intensity for classical beams must be greater than unity. Correlations less than unity are routinely observed [32] at very low levels of illumination, and such antibunching can be explained [27] with QTR. It appears that only QTR can explain the sub-Poisson statistics which are detected again in very weak light beams [27]. Yet there is no immediate prospect of observing antibunching or sub-Poisson statistics for the W or Z gauge bosons, while a weak beam of free gluons is thus far a theoretical impossibility.

It remains to consider the wider utility of parametrized relativistic quantum mechanics. The Higgs mechanism [10], for example, is essentially unaffected. The semiclassical massless gauge potential is independent of the parameter τ , while the scalar Higgs wavefunction obeys the parametrized wave equation of Stueckelberg [33–35] with the standard 'sombbrero' self-interaction.

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