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Title: Median Filters and Iterative Noise Elimination with Applications to the Sharpening of Radiographs

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A noise elimination scheme is described that uses repeated running median filters to selectively reduce noise in stages. The procedure is applied to a method for sharpening radiographs. The sharpening uses convolution with a computed tomography kernel to flatten global features and sharpen local ones. Without prior smoothing, the convolution causes a noise explosion. Examples from industry and medicine are included.

Fixed points for running median filters are characterized, and the stabilization of repeated medians is studied on a general class of linearly ordered spaces that includes the integers with counting measure and the real line with Lebesgue-Stieltjes measure, $\mu = d\alpha$, for α continuous and strictly increasing. It is shown that the results obtained for linearly ordered spaces do not generalize to R^n.

Median Filters and Iterative Noise Elimination with Applications to the Sharpening of Radiographs

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1. Introduction

Figure 1 shows two radiographs. The one at the top is a standard radiograph, and the one at the bottom has been sharpened using the procedure in [7]. The sharpening is achieved by applying a singular integro-differential operator. Used alone, this operator causes the noise in the data to explode, as is illustrated in Figure 2. In order to implement the sharpening successfully, the noise in the data must first be reduced.

In the absence of noise, a single line of data representing point to point variations in x-ray attenuation is usually a smooth function between possible jumps. Noise contributes large isolated peaks together with increasingly abundent peaks of decreasing magnitude. In many cases, important real features are represented by numerical fluctuations substantially smaller than the larger noise peaks.

In this situation, the noise can be reduced in stages. At each stage the data function f is compared to a 'smooth approximation', $T_r^{\omega}f$. Corrections are made only at points x for which $|f(x) - T_r^{\omega}f(x)|$ is large. In early stages, very few points are corrected. As the signal to noise ratio improves, more points are corrected.

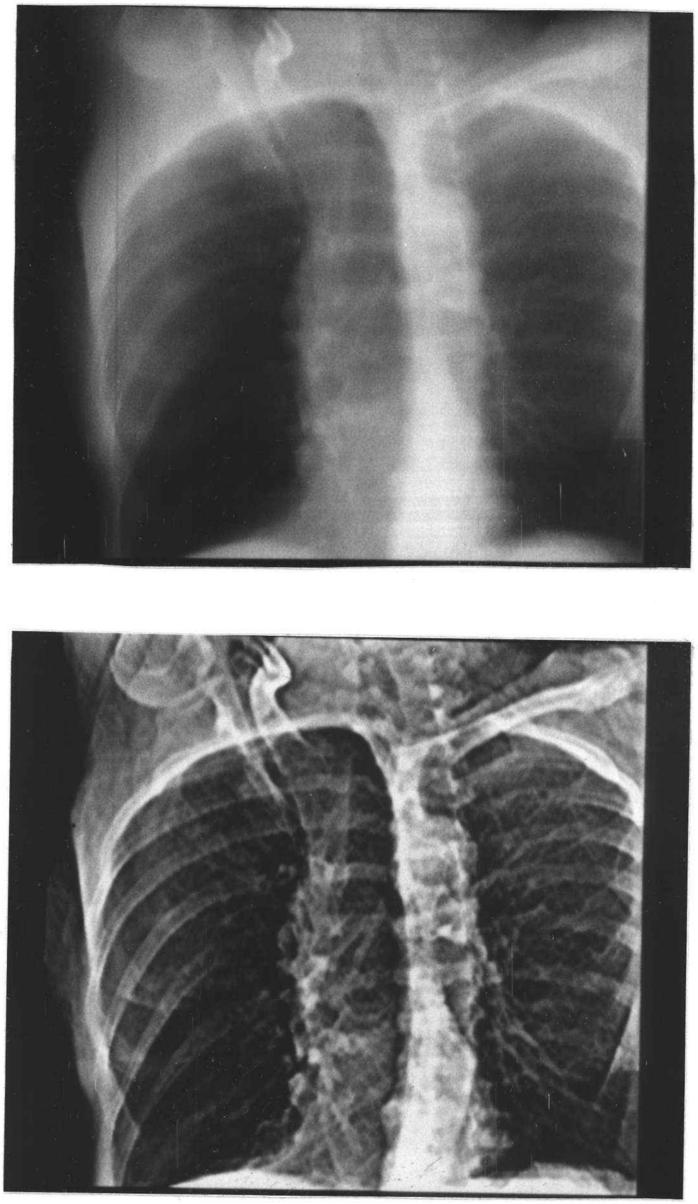

Fig. 1. Radiograph Sharpening. At the top is a standard radiograph of a chest. At the bottom is the same radiograph after sharpening.

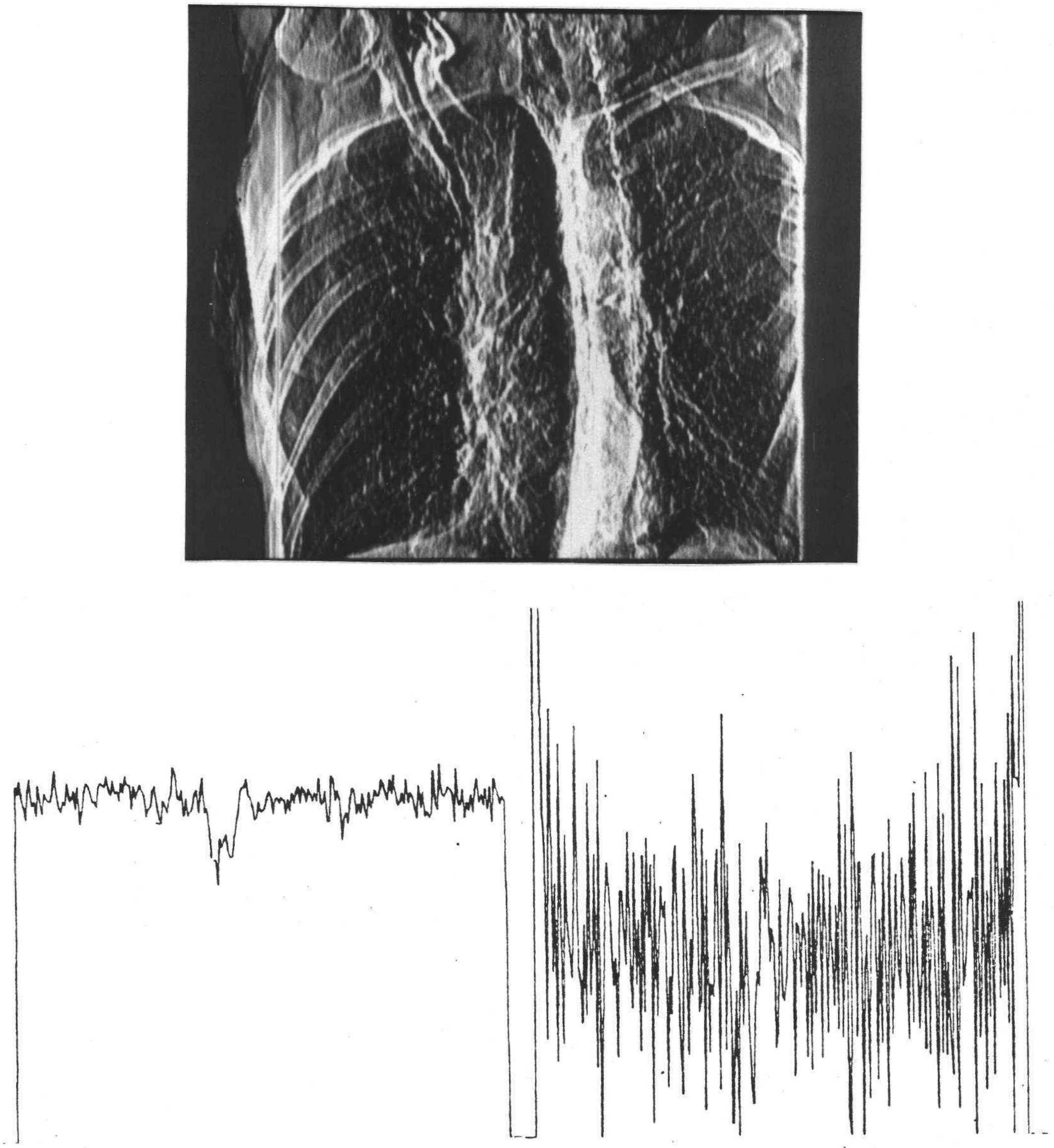

Fig. 2. Noise Explosion. At the top is the same radiograph as in Fig. 1. This time the sharpening has been run without prior smoothing. Bottom left is a line read across the radiograph of a steel shim, sandwiched in a 0.25 inch steel plate. The central dips come from two 0.01 inch radius holes in the shim with centers 0.03 inches apart. The entire radiograph of this shim appears in Fig.14. Bottom right is the same line, sharpened without prior smoothing.

An iterative procedure of this kind was introduced in [6]. A detailed example comparing results when $T_r^{\omega}f$ is produced with averages and when $T_r^{\omega}f$ is produced with medians is given in [5]. The radiograph sharpening procedure with iterative smoothing was introduced in [7].

The first part of this article contains a more detailed description of the sharpening and noise reduction procedures, and examples of their application.

Mathematical aspects of median filters have been studied by Tukey [8] and Tyan [9]. Tukey observed a class of fixed points for a specific median filter and noticed that such fixed points can be found by repeated application of the filter. Tyan classified all fixed points for more general median filters, and extended Tukey's results for obtaining fixed points. Both these studies restricted attention to functions defined on the integers.

In Part II of this article, the results of Tukey and Tyan are extended to more general spaces and additional results specific to the iterative process are developed.

Part I. Sharpening Radiographs and Noise Elimination

2. Noise Reduction

Noise is reduced in stages. At each stage, corrections are made selectively using repeated median filters.

Let f be a function on the integers. For positive integer r, the median filter on intervals of length 2r-1 is denoted by T_r and defined by

2.1 $$T_r f(x) = \text{median}\{f(x-r), f(x-r+1), \ldots, f(x+r)\} \quad .$$

Let T_r^m denote the m^{th} power of T_r (i.e., $T_r^2 f = T_r(T_r f)$). Theorem 6.2.3 shows that for large classes of functions there exist powers m such that

2.2 $$T_r^{m+1} f = T_r^m f$$

and that for such an m, $T_r^m f$ is a locally monotone function. Henceforth, ω will denote the smallest m for which 2.2 holds.

Theorem 5.2.3 does not provide a practical bound for ω, but preliminary results indicate that ω is small in practice. In producing the example in Figure 14, $T_r^{\omega} f$ was calculated 3072 times. For 1012 of these calculations ω was 0, 390 times ω was 1, 1515 times ω was 2, 149 times ω was 3, and 6 times ω was 4.

The noise reduction procedure works as follows. Let f_i denote the data after stage i. At each iteration, parameters r and L are chosen and the function f_{i+1} is obtained from f_i by

$$f_{i+1}(x) = \begin{cases} f_i(x) & \text{if } |f_i(x) - T_r^{\omega} f_i(x)| \leq L \\ T_r^{\omega} f_i(x) & \text{if } |f_i(x) - T_r^{\omega} f_i(x)| > L \end{cases}$$

The key to successful implementation of the smoothing procedure is the choice of the parameters r and L for each iteration. In early stages r is relatively large and L is large enough so that only very few points are corrected. In the later stages, smaller values of r and L are used.

Figures 3 - 7 demonstrate the stage by stage reduction of noise which has been added to a mathematical function. Six iterations were used to perform the smoothing. The interval length parameters were $r = 4,4,4,3,3,2$ respectively. The critical noise levels, L, were adjusted so that 2%, 4%, 8%, 12%, 25%, 100% of the points were corrected.

The corners in f_6 are characteristic of the use of medians in making corrections. Within the sharpening procedure, corners adversely affect the final image. For this reason, a last iteration step, which replaces each value with a three point average, is often included in the noise elimination. Thus, if N denotes the total number of iterations which make use of medians, then the final smoothing operator, S, is defined by

2.4 $$Sf_0(x) = (1/3)[f_N(x-1) + f_N(x) + f_N(x+1)]$$

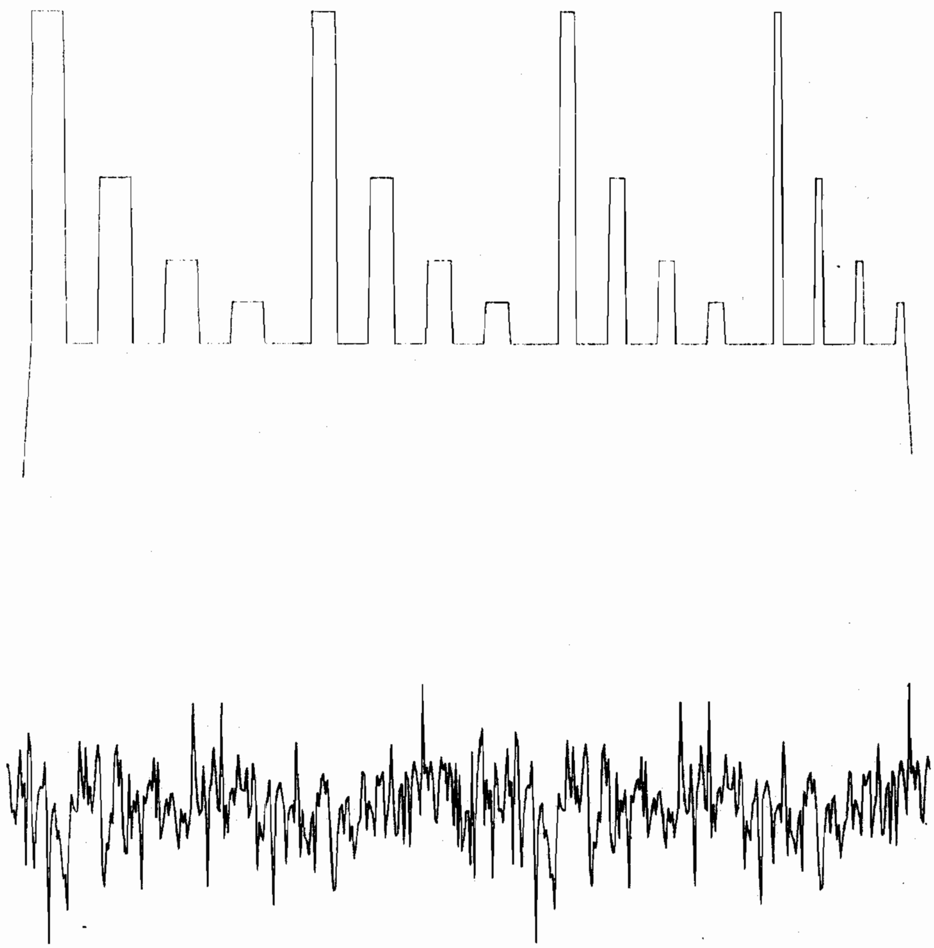

Fig. 3. Line Phantom. At the top is a mathematical function serving as a phantom. At the bottom is noise coming from an x-ray of a steel plate.

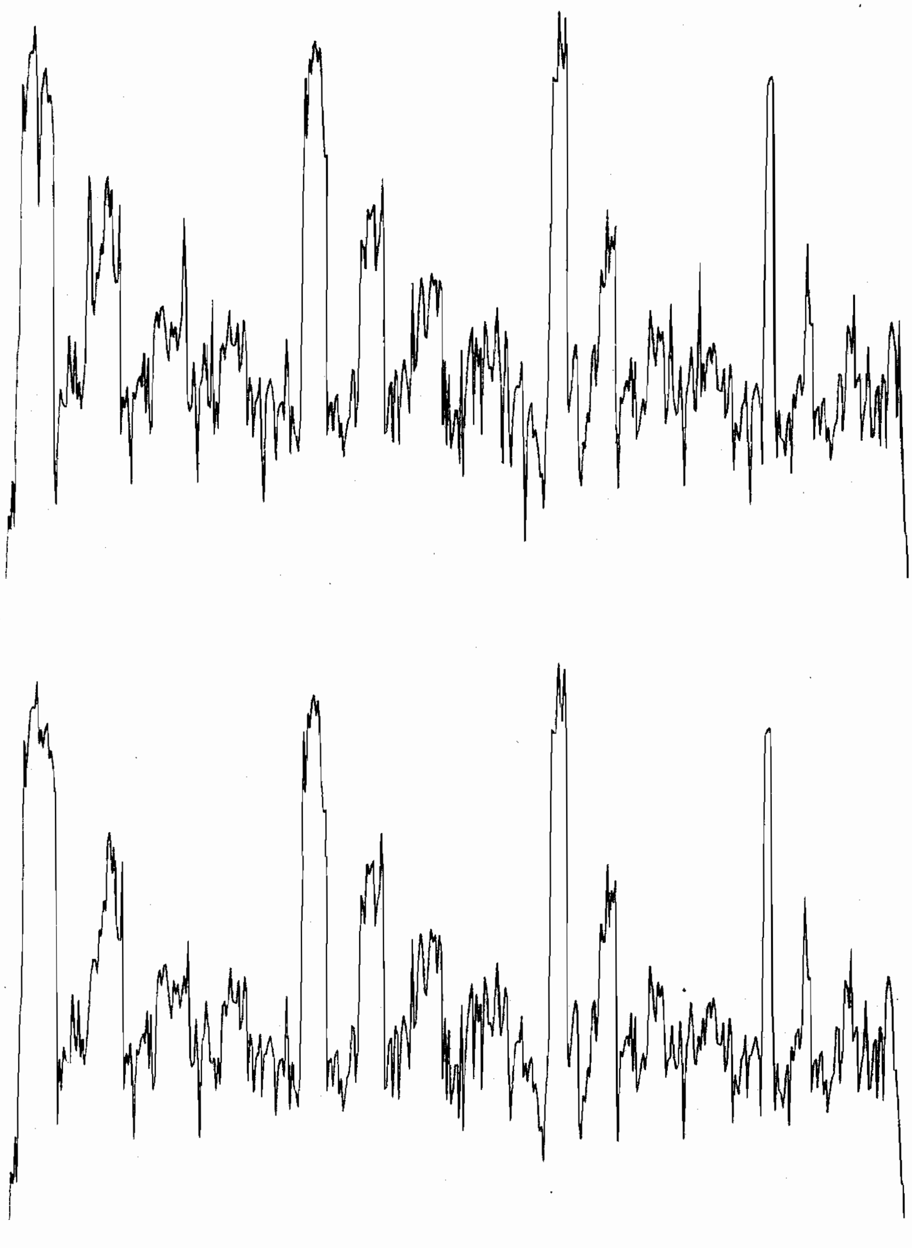

Fig. 4. First Stage Smoothing. At the top is f_0, the sum of the two lines of Fig. 3. At the bottom is f_1, the data after 1 iteration.

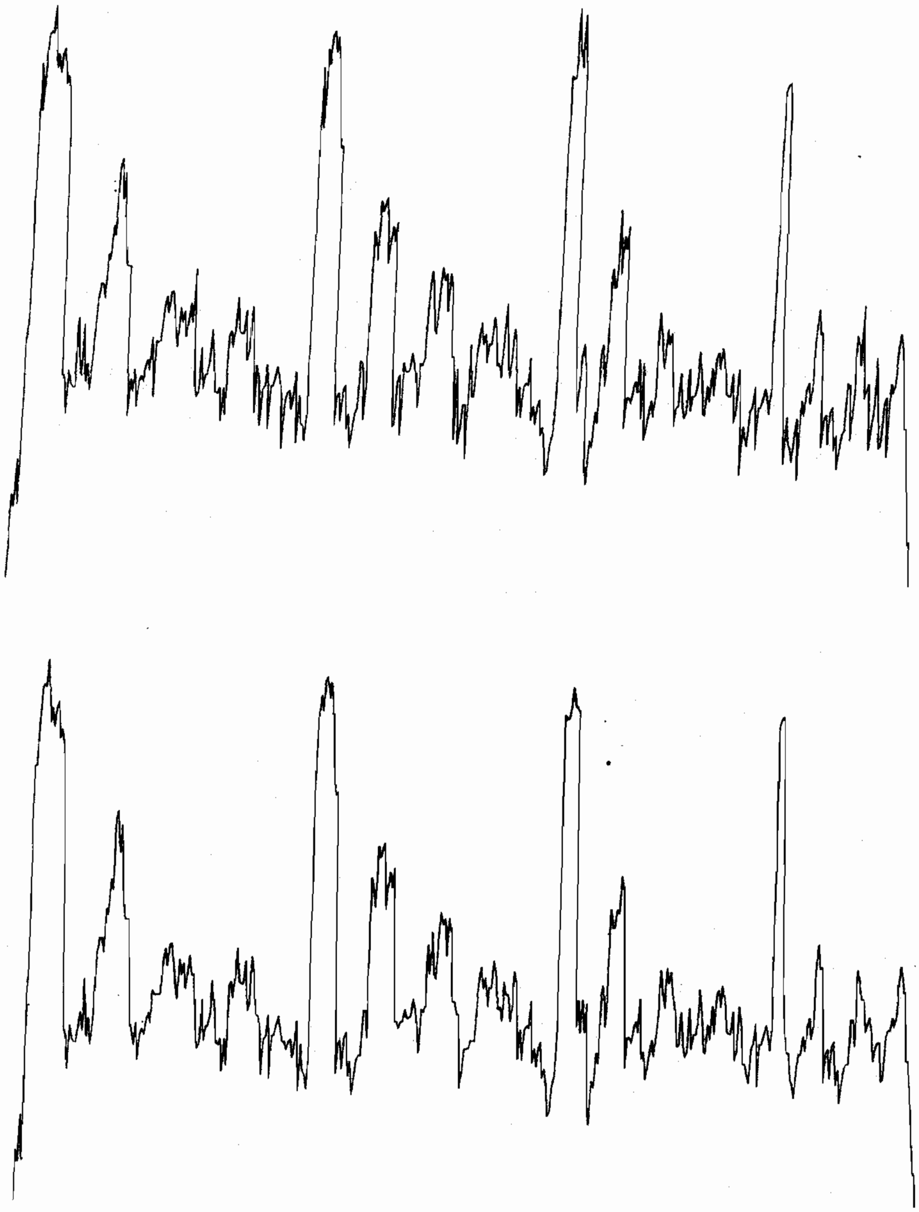

Fig. 5. Second and Third Stage Smoothing. At the top is f_2, and at the bottom is f_3.

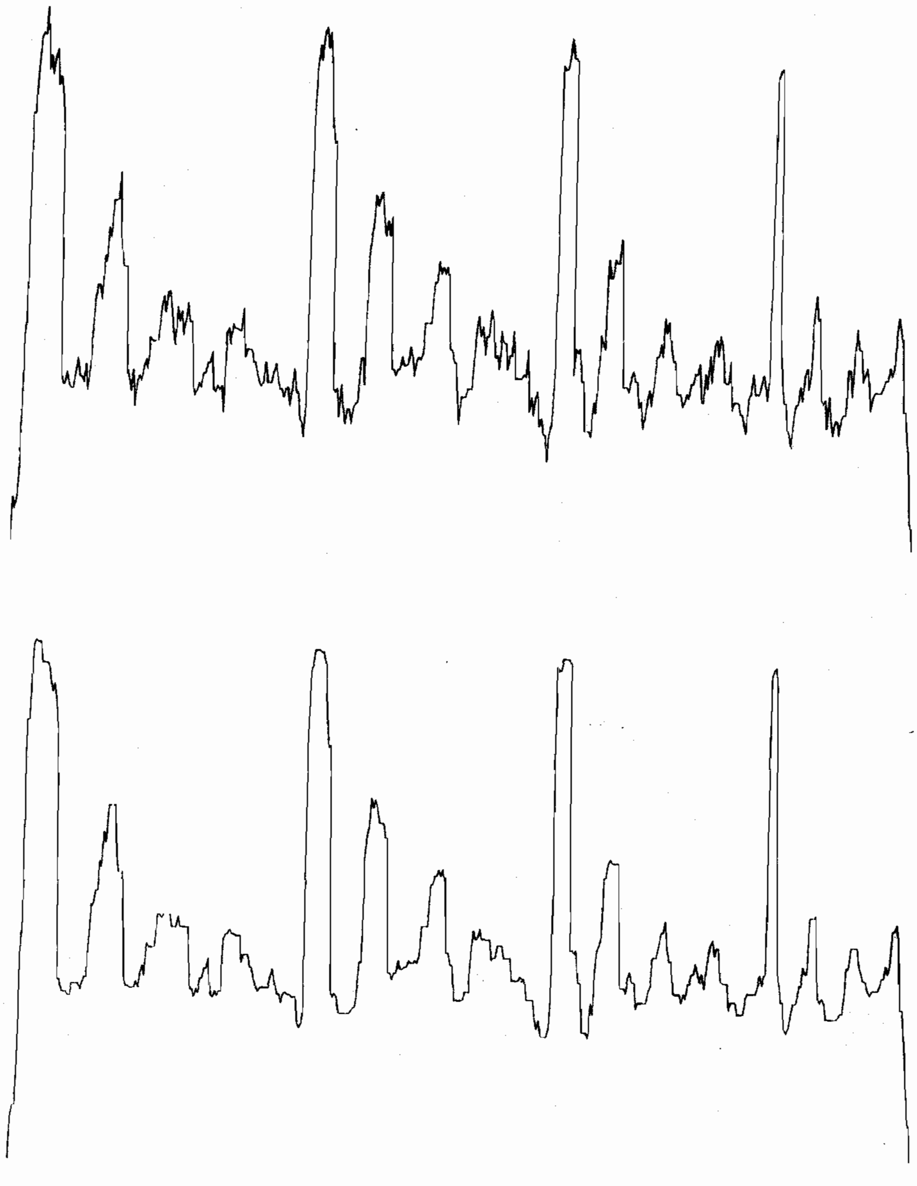

Fig. 6. Fourth and Fifth Stage Smoothing. At the top is f_4, and at the bottom is f_5.

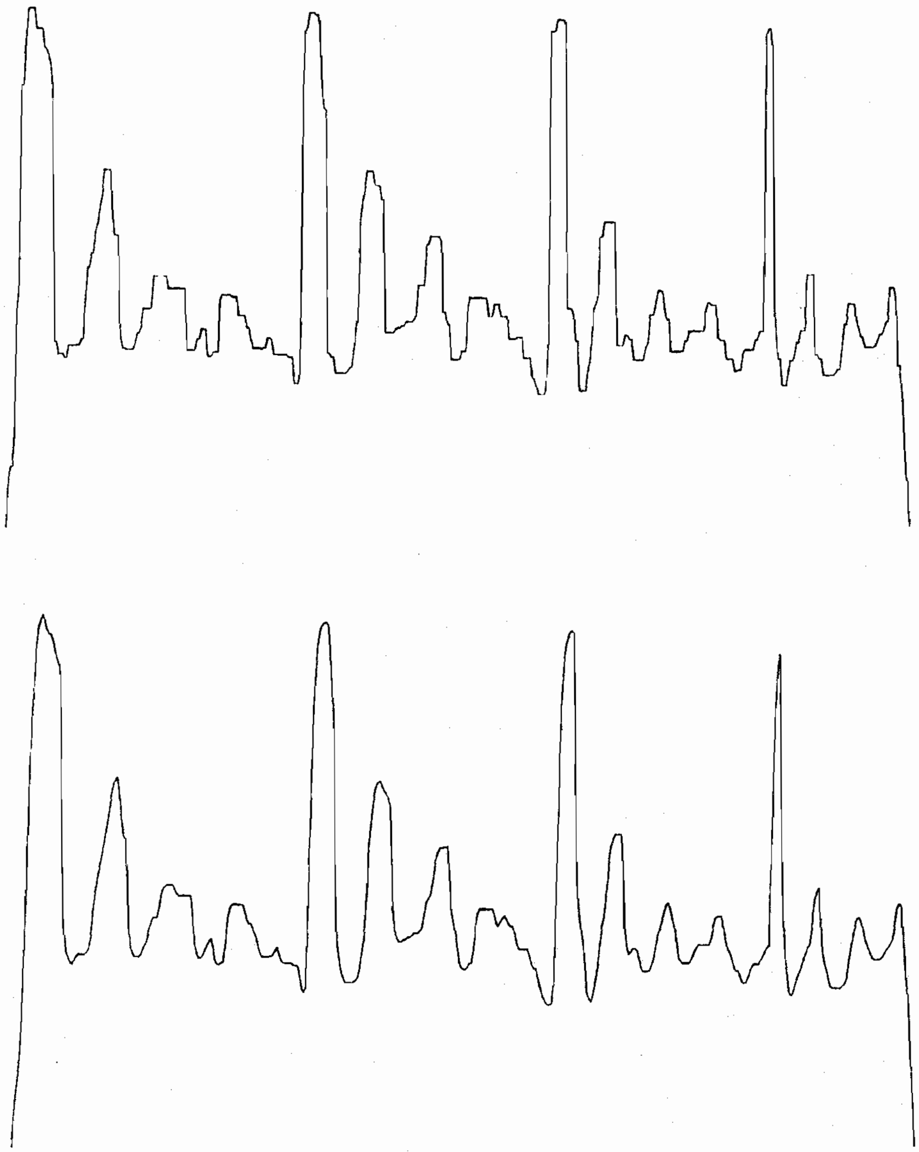

Fig. 7. Final Smoothing. At the top is f_6 and at the bottom is f_6 after the additional iteration which replaces each value by a three point average.

3. Radiograph Sharpening

Among the reasons for the difficulty in reading radiographs are masking and the following three problems. (i) Signals due to changes in x-ray attenuation are often very small compared to those due to thickness. (ii) Sharp boundaries between regions of various attenuation are blurred by the integration and averaging of the x-ray process. (iii) The appearance of features depends on the background.

The sharpening procedure depends on formulas from parallel beam computed tomography. These formulas are explained briefly below.

If g is the x-ray attenuation coefficient of an object in R^n, and θ is a direction in R^n, then the parallel beam radiograph, $P_\theta g$, is defined by

$$P_\theta g(x) = \int_{-\infty}^{\infty} g(x+t\theta)\, dt \qquad x \text{ in } \theta^\perp$$

where $\theta^\perp$ denotes the space perpendicular to θ.

The problem of computed tomography is to determine g from the radiographs $P_\theta g$, θ in S^{n-1}. In principle, this problem is solved by the inversion formula

$$g(x) = \gamma_n \Lambda \int_{S^{n-1}} P_\theta g(E_\theta x)\, d\theta$$

where E_θ denotes orthogonal projection onto $\theta^\perp$, γ_n is the constant

$$\gamma_n = \frac{\Gamma((n+1)/2)}{2(n-1)\pi^{(n+1)/2}}$$

and Λ is the operator defined by

$$\Lambda h = \sum_{j=1}^{n} (\partial/\partial x_j)\ H_j * h\ , \quad H_j = \frac{\Gamma((n+1/2))}{\pi^{(n+1)/2}} \ \frac{x_j}{|x|^{n+1}}$$

with * denoting convolution.

Alternatively, Λ is defined in Fourier space by

$$(\Lambda h)^{\wedge}(y) = |y| h^{\wedge}(y)$$

Because of the singular nature of Λ, the usual practice is to reconstruct an approximation of the form e*g, where e is an approximate delta function. In this setting, e is often referred to as the point spread function. e*g is given by

3.1 $$e*g(x) = \int_{S^{n-1}} P_\theta g * k(E_\theta x)\ d\theta\ , \qquad k = \gamma_n \Lambda\ P_\theta e$$

The function k, is called the reconstruction kernel, or sometimes the CT kernel. For dimension n = 2, formula 3.1 is due to Ramachandran and Laksminarayanan [4].

Remark.

In practice, data is commonly acquired using fan beam x-rays instead of parallel beam x-rays. Similar inversion formulas exist in this case.

The sharpening procedure is as follows. Let $F = P_\theta g$ denote the 2-dimensional radiogaph and let τ be the operator that exchanges the first and second variable. If * denotes convolution in the first variable only and S denotes the smoothing operator applied to the first variable only, then the sharpened radiograph, denoted by $F^\#$, is given by

$$F^\# = w_1 F + w_2 k*SF + w_2 \tau(k*S\tau F)$$

for appropriate weights w_1 and w_2.

The intuitive basis for the feature enhancement procedure lies in the following theorem of Shepp and Logan [2].

<u>Ellipse Theorem.</u>

Let g be the characteristic function of an ellipse in the plane. For θ in S^1, let $I = [-a,a]$ be the projection of the ellipse on the line $\theta^\perp$. Then

$$\Lambda\, P_\theta g(x) = c \begin{cases} 1 & |x| < a \\ 1 - |x|(x^2-a^2)^{-1/2} & |x| > a \end{cases}$$

where the constant c depends on the ellipse and on θ.

If k is a reconstruction kernel, then

$$k * P_\theta g = \Lambda P_\theta e * P_\theta g = P_\theta e * \Lambda P_\theta g$$

Since $P_\theta e$ is again an approximate delta function, the ellipse $P_\theta g$ appears in $k * P_\theta g$ as a rectangle on $(-a,a)$ with sharp negative dips just outside the end points. Hence global features in radiographs are suppressed and local features are sharpened, thus reducing the three problems mentioned at the beginning of this section. Figure 8 shows an example of this effect.

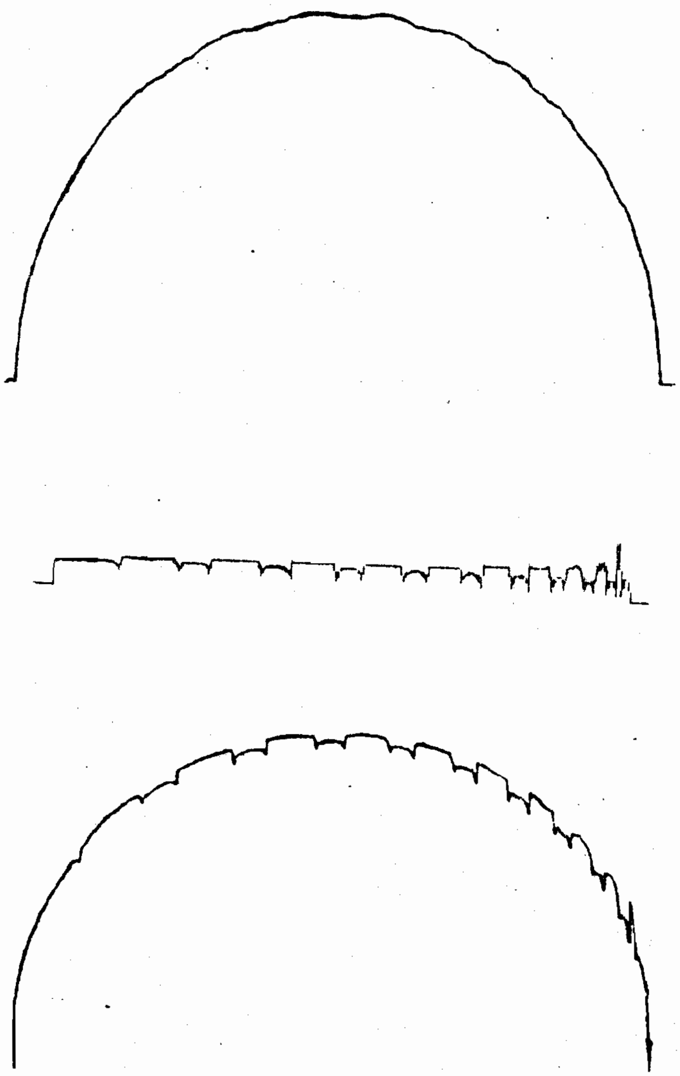

Fig. 8. Effect of Convolution on Ellipses. The function f, at the top is a large ellipse with small elliptical bumps. In the middle is k*f, and at the bottom is 0.5f + 0.5k*f.

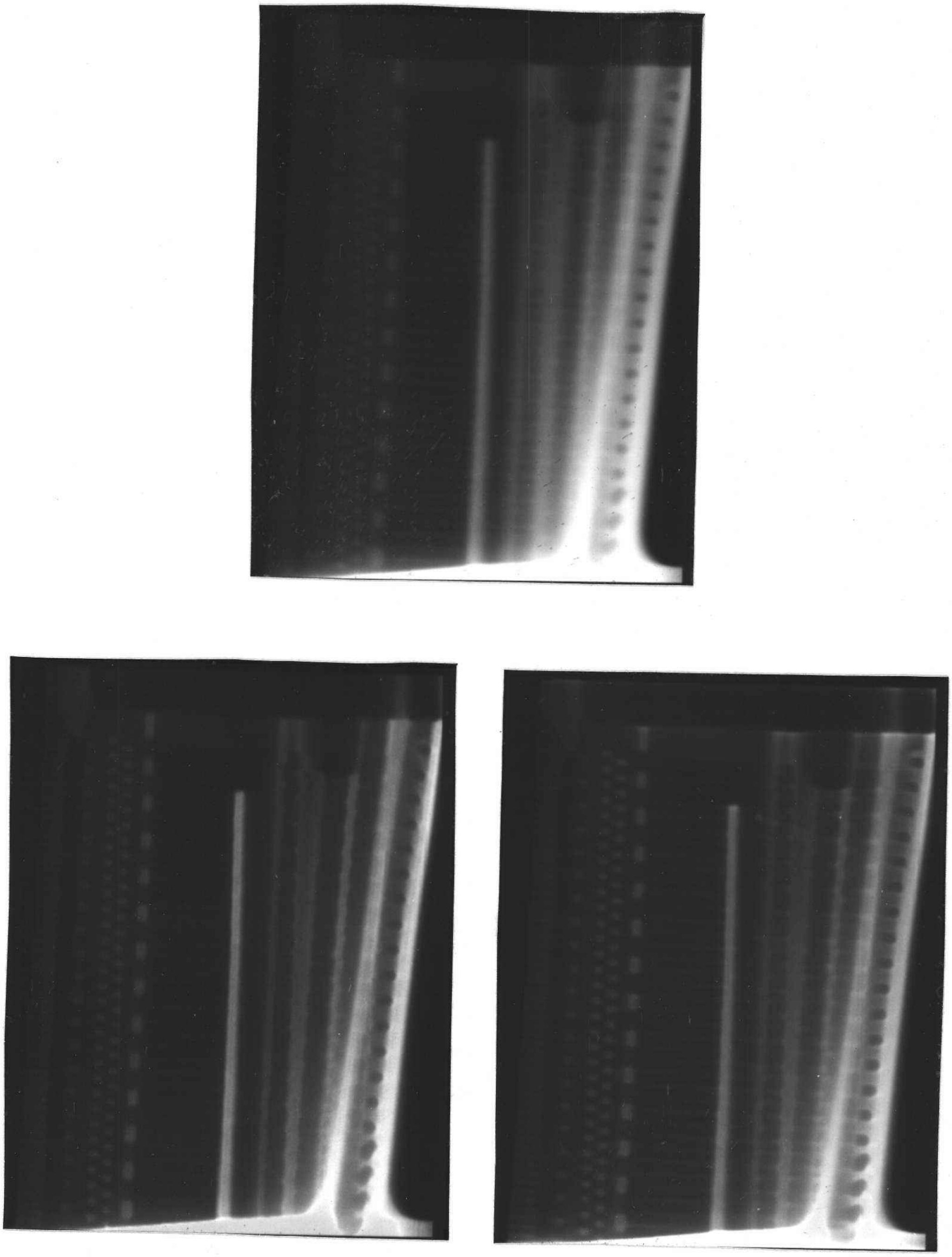

Fig. 9. One and Two Directional Smoothing. At the top is f, a standard radiograph of a jet engine blade. Bottom left is $0.5f + 0.5k*Sf$ and bottom right is $0.5f + 0.25k*Sf + 0.25\tau(k*S\tau f)$.

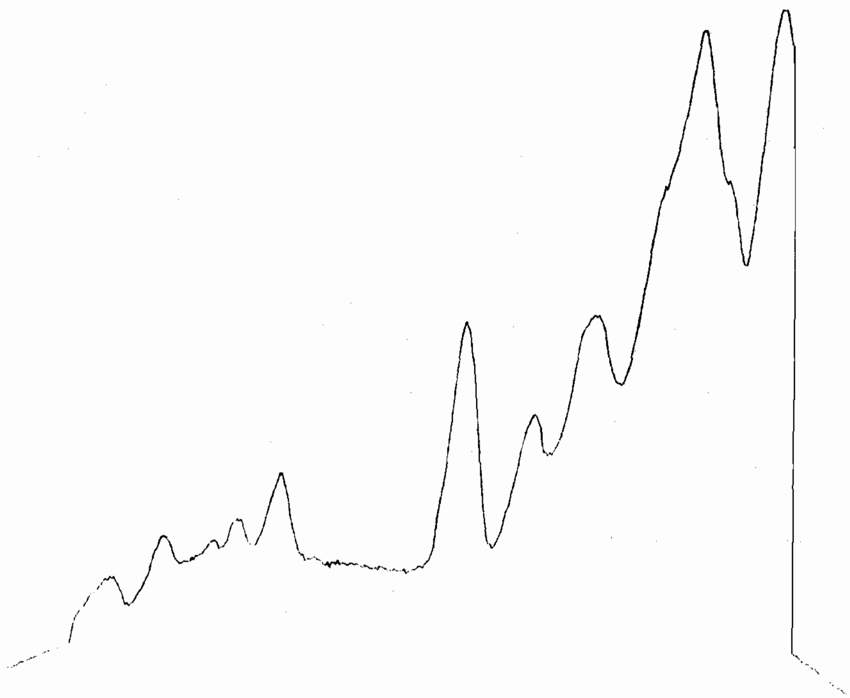

Fig. 10. Line Data. This is a single line of data f, read across the original engine blade radiograph of Fig. 9.

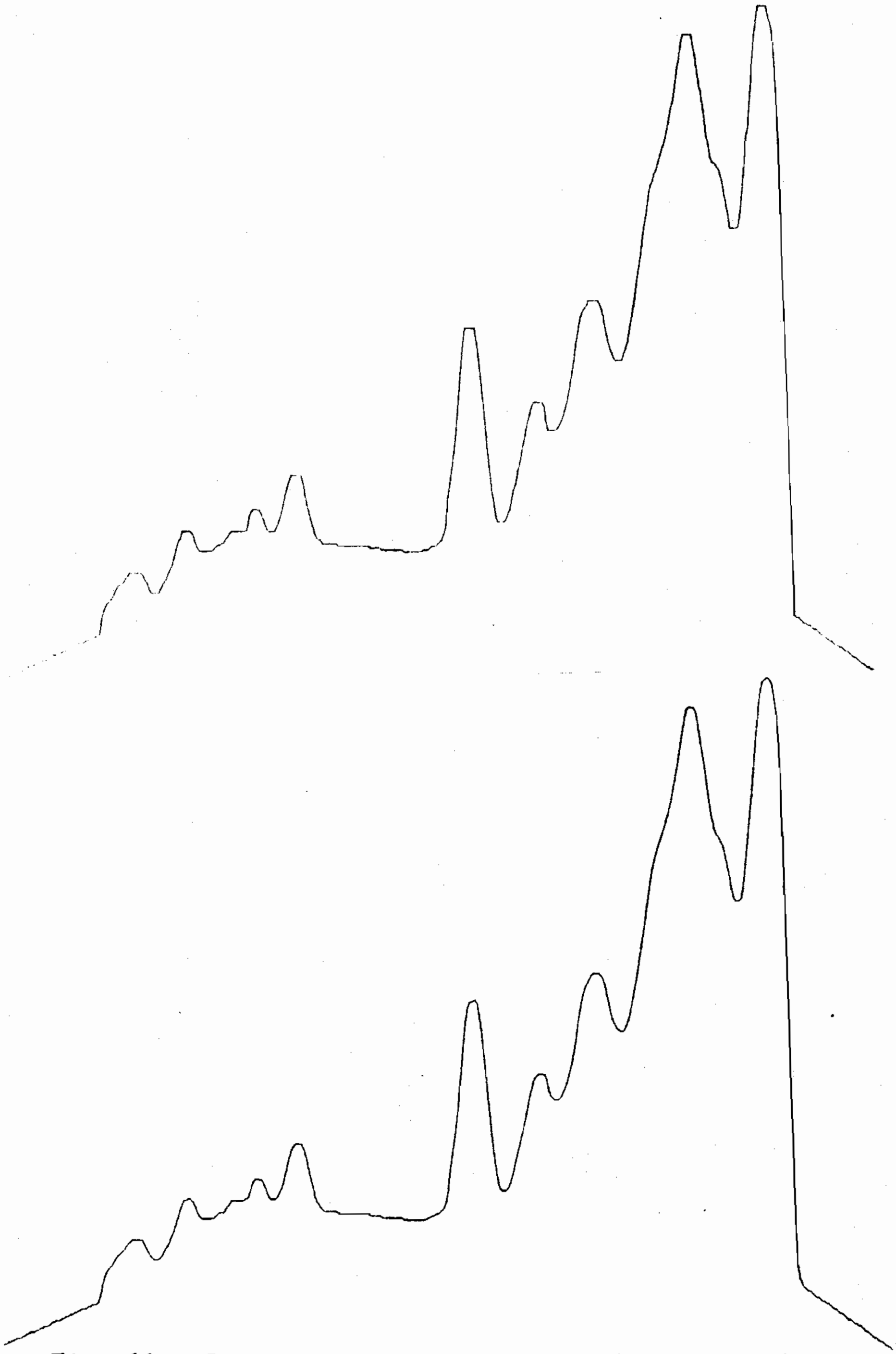

Fig. 11. Smoothing with and without Averages. At the top, is Sf with no last averaging step included, and at the bottom is Sf with an averaging step was included. Here f is as in Fig. 10.

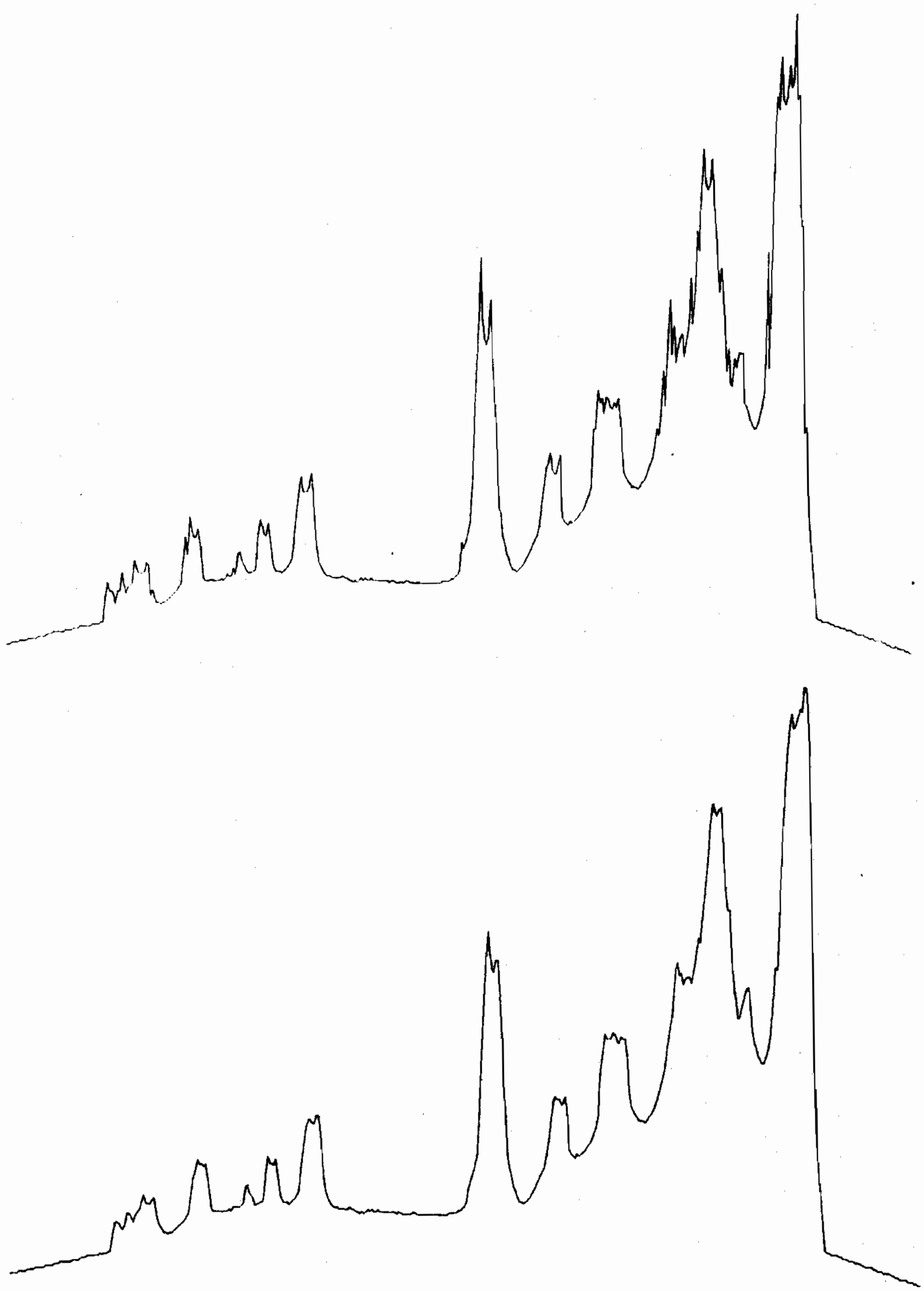

Fig. 12. Sharpened Line Data. Both lines are 0.5f + 0.5k*Sf. At the top no averaging step was included in the smoothing, and at the bottom, an averaging step was included in the smoothing.

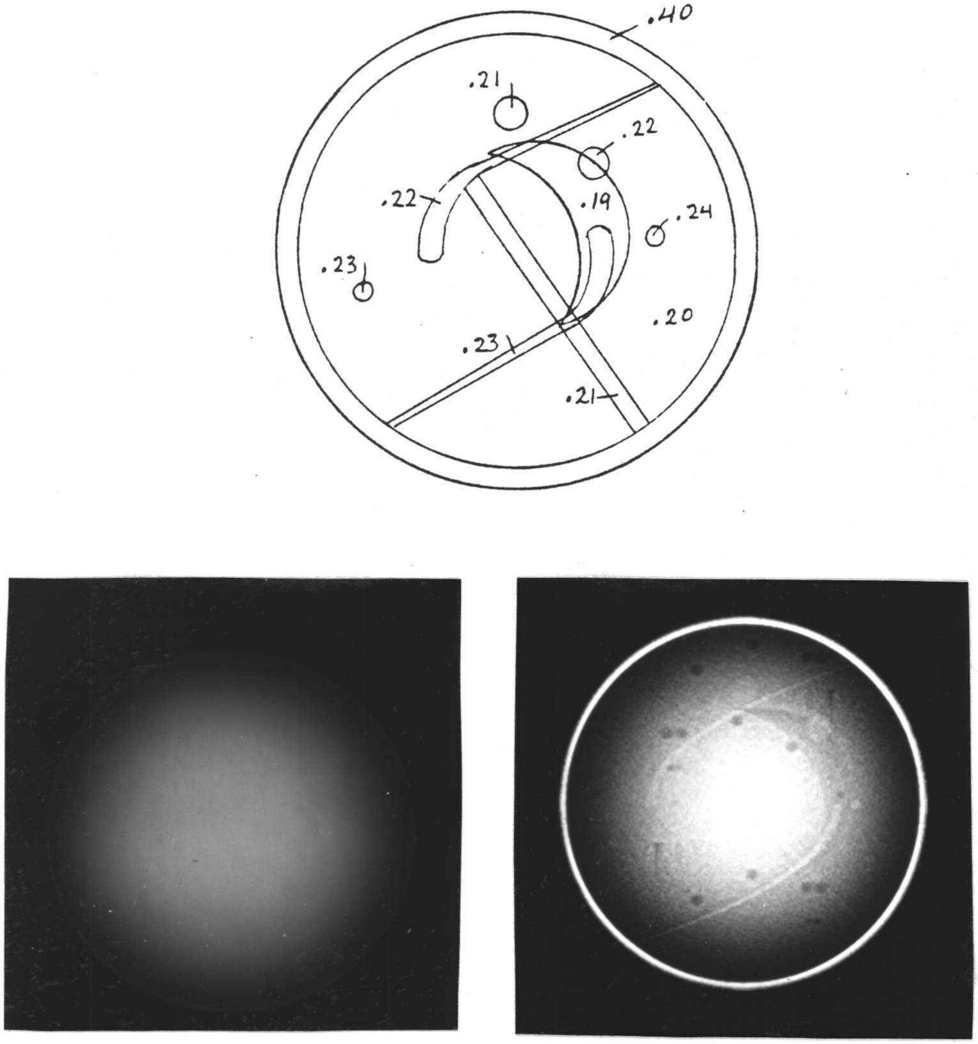

Fig. 13. Low contrast phantom. Bottom left is the projection of a 3-dimensional mathematical ellipsiod with regions having circular cross sections and various x-ray attenuations as depicted in the diagram. To provide real noise, the middle portion of the shim data in Fig. 9 was normalized to mean 0 and added to the projection. The only real features in that example are the holes, so all the positive values in the normalized version are noise. On a scale of 0-10,000, the maximum positive value is 123 and the average positive value was 18. In comparison, the attenuation change across the lightest 'artery' is about 13, from 9605 and 9607 on either side, to 9618 at the center of the artery.

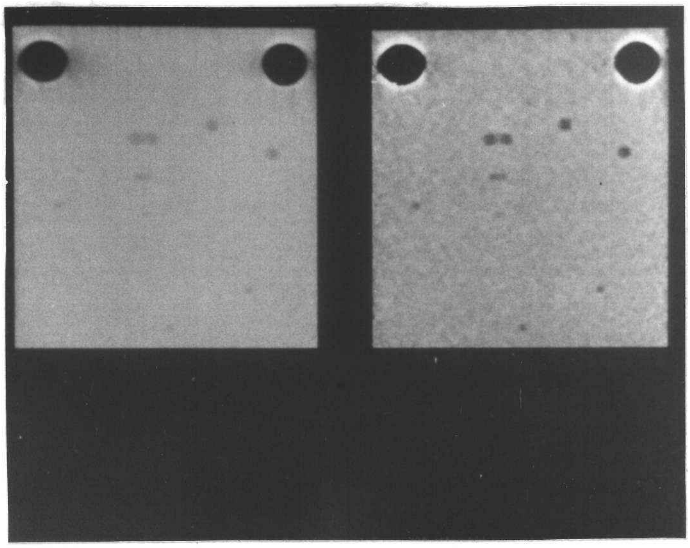

Fig. 14. Sharpening Nonelliptical Features. On the left is f, the entire radiograph of the shim described in Fig. 2. On the right is 0.5f + 0.5k*Sf. The brightness at the edges is due to the fact that the shim is rectangular and not elliptical. If f(x) is 1 for $|x| < a$, and $f(x) = 0$ for $|x| \geq a$, then

$$\Lambda f(x) = -c/(x^2 - a^2) \qquad \text{for a positive constant } c.$$

Part II. Iterated Medians

4. Medians on Measure Spaces

Let (E,μ) be a finite measure space, and let f be an integrable function on E. The distribution F of f, defined by

$$F(t) = \mu(\{x: f(x) \geq t\})$$

has the following properties.

a) F is nonincreasing

b) $F(t-0) = F(t)$

c) $\lim_{t \to -\infty} F(t) = \mu(E)$, and $\lim_{t \to \infty} F(t) = 0$.

On the interval $(0,\mu(E))$ the function

$$f^{+}(s) = \sup \{t: F(t) \geq s\}$$

has the following properties.

a) f^{+} is nonincreasing and $f^{+}(s-0) = f^{+}(s)$

b) $\lim_{s \to 0} f^{+}(s) = \text{ess sup } f$, and $\lim_{s \to \mu(E)} f^{+}(s) = \text{ess inf } f$

c) $F(t) \geq s$ if and only if $f^{+}(s) \geq t$

d) $F(f^{+}(s)+0) \leq s \leq F(f^{+}(s))$, and $f^{+}(F(t)+0) \leq t \leq f^{+}(F(t))$

e) $\mu(\{x: f(x) \leq t\}) = \mu(E) - F(t+0)$

f) If $f^{+}(s+0) \leq f^{*}(s) \leq f^{+}(s)$ on $(0,\mu(E))$, then $|\{s: f^{*}(s) \geq t\}| = F(t)$, where $|.|$ is Lebesque measure on R^{1}.

Definition 4.1

Any function f^* on $(0,\mu(E))$ satisfying

$$f^+(s+0) \leq f^*(s) \leq f^+(s)$$

is called a nonincreasing equimeasurable rearangement of f. In this article, f^* denotes the specific nonincreasing equimeasurable rearangement of f given by

$$f^*(s) = (f^+(s+0) + f^+(s))/2 = (f^+(s+0) + f^+(s-0))/2$$

Definition 4.2

Let (X,μ) be a measure space, and let f be integrable over sets of finite measure. For a measurable set E of finite measure, let f_E denote the restriction of f to E. The median of f on E is

$$\mathrm{med}_E f = f_E^*(\mu(E)/2)$$

The following proposition is clear from properties (d) and (e) above.

Proposition 4.3

a) $\mu(\{x \in E: f(x) \geq \mathrm{med}_E\, f\}) \geq \mu(E)/2$

b) $\mu(\{x \in E: f(x) \leq \mathrm{med}_E\, f\}) \geq \mu(E)/2$

5. Smoothing on Linearly Oredered Sets

5.1. Median Spaces.

Let X be a linearly ordered set with ordering $<$. The symbols $<$, $\leq$, $>$, and $\geq$, refer to the ordering on X as well as the ordering on R. The intervals $\{x: x_1 \leq x \leq x_2\}$ and $\{x: x_1 < x < x_2\}$ are denoted by $[x_1,x_2]$ and (x_1,x_2) respectively, with similar notation for half open intervals. The intervals $\{x: x \leq a\}$ and $\{x: x \geq b\}$ are denoted by $(-\infty,a]$ and $[b,\infty)$ respectively, and the set containing just the point x is denoted by $[x]$.

X is a topological space with the order topology.

<u>Definition 5.1.1.</u>

A <u>bounded set</u> in X is any set contained in an interval of the form $[x_1,x_2]$.

<u>Definition 5.1.2.</u>

X is <u>discrete</u> if every element of X that has a predecessor, has an immediate predecessor, and every element of X that has a successor, has an immediate successor.

X is <u>continuous</u> if X has both the least upper bound and greatest lower bound properties, and contains a countable set C such that for any pair of elements x_1 and x_2 in X, there exists an element in C that lies between x_1 and x_2.

Definition 5.1.3.

A median space (X,μ) is a linearly ordered set X that is either discrete or continuous, together with a measure, μ, having the following properties.

a) All intervals are measurable.

b) Bounded intervals have finite measure.

c) Nonempty open intervals have positive measure.

d) $\mu([x_1]) = \mu([x_2])$ for all x_1 and x_2 in X.

Remark.

Let X contain at least two points. If X is continuous, then (b) and (d) give that $\mu([x]) = 0$ for all x in X. If X is discrete, then (c) and (d) give that $\mu([x])$ is a fixed positive number for all x. Henceforth it is assumed that $\mu([x]) = 1$ for all x in X if X is discrete.

A dicussion of the axioms appears in section 6.

Lemma 5.1.4.

Let X be continuous, let y be in X, and let $\varepsilon > 0$ be given. If y is not the left end point of X, then there exists $c < y$ such that $\mu([c,y]) < \varepsilon$. If y is not the right end point of X, then there exists $d > y$ such that $\mu([y,d]) < \varepsilon$.

Proof. If y is not the left end point of X, then there exist elements c_n in C, the countable dense subset of X, such that

$\bigcap_n [c_n, y] = [y]$. Thus,

$$\lim_{n\to\infty} \mu([c_n, y]) = \mu(\bigcap_n [c_n, y]) = \mu([y]) = 0$$

The other statement is proved similarly.

<u>Theorem 5.1.5.</u> <u>Structure theoerm.</u>

Let a be in X and let

$$h_a(x) = \begin{cases} \mu((a,x]) & x > a \\ -\mu((x,a]) & x \leq a \end{cases}$$

If X is discrete, h_a is a 1-1, order preserving, measure preserving homeomorphism of X onto an interval in Z, and if X is continuous, h_a is a 1-1, order preserving, measure preserving homeomorphism of X onto an interval in R.

Proof. If $x_1 < x_2$, then

$$h_a(x_2) = h_a(x_1) + \mu((x_1, x_2])$$

Since half open intervals have positive measure, this gives that h_a is 1-1 and order preserving. If X is continuous, lemma 5.1.4 shows that h_a is continuous. In the discrete case, all functions are continuous.

If X is discrete, and x has an immediate predecessor, x^-, then $h_a(x^-) = h_a(x) - 1$. If x has an immediate successor, x^+, then $h_a(x^+) = h_a(x) + 1$. Thus $h_a(X)$ is an interval in Z.

If X is continuous, then for x_1 and x_2 in X, let $y_1 = h_a(x_1)$ and $y_2 = h_a(x_2)$. If $y_1 < y < y_2$, let

$$x = \sup\{z: h_a(z) \leq y\}$$

Suppose that $h_a(x) < y$. Since $y < y_2$, $x < x_2$, and by lemma 5.1.4, there exists z such that $z > x$ and $\mu((x,z]) < y - h_a(x)$. Thus

$$h_a(z) = h_a(x) + \mu((x,z]) < y$$

which contradicts the definition of x.

Now suppose that $h_a(x) > y$. By lemma 5.1.4 there exists z such that $z < x$ and $\mu([z,x)) < h_a(x) - y$. Then, since $\mu([z,x)) = \mu((z,x])$,

$$h_a(z) = h_a(x) - \mu((z,x]) > y$$

which also contradicts the definition of x. Thus $h_a(x) = y$, and so the range of h_a is an interval.

It is clear that h_a maps open intervals to open intervals, hence h_a is open.

Remark.

Discrete median spaces are identical with intervals of integers. Continuous median spaces, however, include Lebesgue-Stieltjes measure $\mu = d\alpha$, where α is continuous and strictly increasing.

Definition 5.1.6.

For a number $r > 0$, let

$$I_r = \{x \in X: \text{there exist } a \text{ and } b \text{ in } X \text{ with } \mu([a,x]) = \mu([x,b]) = r\}$$

Lemma 5.1.7.

Let I_r be nonempty, c be in X, $b = \sup\{x: \mu([c,x]) \leq r\}$, and $a = \inf\{x: \mu([x,c]) \leq r\}$. If $\mu([c,z]) \geq r$ for at least one z, then $\mu([c,b]) = r$. If $\mu([z,a]) \geq r$ for at least on z, then $\mu([a,c]) = r$.

Proof. Let $\mu([c,z]) \geq r$. If X is continuous, then $h_c(z) \geq r$. Since $h_c(X)$ is an interval, r is in the range of h_c. Thus $\{x: \mu([c,x]) \leq r\} = h_c^{-1}([0,r]) = [c,b]$, and $\mu([c,b]) = r$.

If X is discrete, then $h_c(z) \geq r - 1$. Since $h_c(X)$ is an interval in the integers, $r - 1$ is in the range of h_c. Thus $\{x: \mu([c,x]) \leq r\} = h_c^{-1}([0,r-1]) = [c,b]$, and $\mu([c,b]) = r$.

The result for $a = \inf\{x: \mu([x,y]) \leq r\}$ follows similarly.

Proposition 5.1.8.

If I_r is nonempty, then

a) I_r is an interval.

b) If $r' \geq r$, then $I_{r'} \subset I_r$.

c) For each x in X, either there exists b in I_r with $\mu([b,x]) = r$ or there exists a in I_r with $\mu([x,a]) = r$.

To prove (a), let y_1 and y_2 be in I_r and let $y_1 < y < y_2$. For the elements a_1 and b_2 such that $\mu([a_1,y_1]) = \mu([y_2,b_2]) = r$, $\mu([y,b_2]) \geq r$ and $\mu(a_1,y]) \geq r$. Thus the result follows from lemma 5.1.7.

To prove (b), let y be in $I_{r'}$, and let a and b be such that $\mu([a,y]) = \mu([y,b]) = r'$. Since $r \leq r'$, the result follows from lemma 5.1.7.

To prove (c), let x be an element in X. Let z be in I_r, and let c and d be such that $\mu([c,z]) = \mu([z,d]) = r$. If $z = x$, the result follows trivially. If $z > x$, then $\mu([x,d]) \geq r$ and the result follows from 5.1.7. If $z < x$, then $\mu([c,x]) \geq r$ and the result again follows from 5.1.7.

<u>Proposition 5.1.9.</u>

Let a, y and b be such that $\mu([a,y]) = \mu([y,b]) = r$, and let f be continuous on [a,b]. The following two inequalities hold if and only if $L = \text{med}_{[a,b]} f$.

(a) $\mu(\{x \in [a,b]: f(x) \leq L\}) \geq r$

(b) $\mu(\{x \in [a,b]: f(x) \geq L\}) \geq r$

Proof. Suppose that $L = \text{med}_{[a,b]} f$. If X is continuous, then $\mu([a,b])/2 = r$, so the fact that (a) and (b) hold with $L = \text{med}_{[a,b]} f$ is a consequence of proposition 4.3.

If X is discrete, then $\mu([a,b])/2 = r - 1/2$ and proposition

4.3 gives that

$$\mu(\{x \in [a,b]: f(x) \leq \text{med}_{[a,b]}f\}) \geq r - 1/2 \quad \text{and}$$
$$\mu(\{x \in [a,b]: f(x) \geq \text{med}_{[a,b]}f\}) \geq r - 1/2$$

Since $\mu([a,y]) = r$, and X is discrete, r is an integer. Hence if $\mu(E) \geq r - 1/2$ for any set E, then $\mu(E) \geq r$. This proves that conditions (a) and (b) hold for $L = \text{med}_{[a,b]}f$.

To show uniqueness, suppose that (a) and (b) hold for L = s and for L = t with $s < t$. If $\mu(\{x \in [a,b]: f(x) \leq s\}) > r$, then since $\mu([a,b]) \leq 2r$, it follows that $\mu(\{x \in [a,b]: f(x) > s\}) < r$. Since $s < t$, this gives that $\mu(\{x \in [a,b]: f(x) \geq t\}) < r$. This contradiction to (b) for L = t, gives that

$$\mu(\{x \in [a,b]: f(x) \leq s\}) = r.$$

Similarly, it follows that

$$\mu(\{x \in [a,b]: f(x) \geq t\}) = r.$$

Hence the interval [a,b] contains two closed disjoint subsets of measure r, namely

$$\{x \in [a,b]: f(x) \geq t\} \quad \text{and} \quad \{x \in [a,b]: f(x) \leq s\}$$

In the case that X is discrete, this gives a contradiction since $\mu([a,b]) = 2r - 1$.

If X is continuous, this gives a contradiction since intervals are connected and a closed set cannot be contained properly in an interval of the same measure.

<u>Corollary.</u>

(a) If $\mu(\{x \in [a,b]: f(x) \leq N\}) \geq r$, then $\mathrm{med}_{[a,b]} f \leq N$.

(b) If $\mu(\{x \in [a,b]: f(x) \geq M\}) \geq r$, then $\mathrm{med}_{[a,b]} f \geq M$.

Proof. Suppose that $\mu(\{x \in [a,b]: f(x) \leq N\}) \geq r$ and $\mathrm{med}_{[a,b]} f > N$, then

$$\mu(\{x \in [a,b]: f(x) \geq N\})$$
$$\geq \mu(\{x \in [a,b]: f(x) \geq \mathrm{med}_{[a,b]} f\}) \geq r$$

Thus N satisfies conditions (a) and (b) in the proposition and so $N = \mathrm{med}_{[a,b]} f$. This contradiction proves (a), and (b) follows similarly.

5.2. Fixed Points.

Definition 5.2.1.

For x in I_r, the symbols a_x and b_x denote the elements in X with $\mu([a_x,x]) = \mu([x,b_x]) = r$. The operator T_r is defined by

$$T_r f(x) = \begin{cases} \operatorname{med}_{[a_x,b_x]} f & \text{for } x \text{ in } I_r \\ f(x) & \text{otherwise} \end{cases}$$

T_r is often called a running median filter.

Defintion 5.2.2.

Let I be an interval. f is called <u>r-monotone</u> on I if every bounded subinterval on which f is not monotone contains a closed subinterval of measure r on which f is constant.

The proofs of the next two theorems follow from a sequence of lemmas.

Theorem 5.2.3.

Let f be continuous on X with support contained in an interval (α,β) such that $[\alpha,\beta] \subset I_r$. Then $T_r f = f$ if and only if f is r-monotone.

For f with unbounded support, the result is as follows.

Theorem 5.2.4.

Let f be continuous on X. If f is r-monotone on X, then $T_r f = f$. If $T_r f = f$, then one of the following holds.

(a) f is r-monotone on I_r.

(b) For every closed interval, $J \subset I_r$ with $\mu(J) = r$, the min of f on J is the same as the min of f on I_r, and the max of f on J is the same as the max of f on I_r.

Remark.

Theorem 5.2.3 is due to Tukey [8] in the case of counting measure and $r = 2$. Theorem 5.2.3 and a version of 5.2.4 is due to Tyan [9] for counting measure and any positive integer r.

The following example illustrates a fixed point that is not r-monotone.

Example.

Let $X = Z$ with counting measure, and let $f(x) = 1$ for x even and $f(x) = 0$ for x odd. Then $T_r f = f$ if r is any odd integer.

Lemma 5.2.5.

If f is continuous on X and r-monotone on a closed interval of measure r, then f is monotone on that interval.

Lemma 5.2.6.

Let y be in I_r. If $f(x) < T_r f(y)$ for x in $[a_y, y)$, then $f(x) \geq T_r f(y)$ for x in $[y, b_y]$.

Proof. Let $D = \{x \in [a_y, b_y]: f(x) \geq T_r f(y)\}$. Since $f(x) < T_r f(y)$ on $[a_y, y)$, $D \subset [y, b_y]$. D is closed and by proposition 5.1.9 $\mu(D) \geq r$. Therefore $D = [y, b_y]$.

Corollary.

(a) If all the inequalities in the statement of lemma 5.2.6 are reversed, then the lemma still holds.

(b) If, in the statement of lemma 5.2.6, $[a_y, y)$ is replaced by $(y, b_y]$ and $[y, b_y]$ is replaced by $[a_y, y]$, then the lemma still holds.

Proof. Replace $f(x)$ by $-f(x)$ for (a) and by $f(-x)$ for (b).

Lemma 5.2.7.

Let f be continuous with $T_r f = f$. Let z be in I_r, $K(x) = \max\{f(w): w \in [z,x]\}$, and $L(x) = \min\{f(w): w \in [z,x]\}$. If f is nondecreasing on $[a_z, z]$ and $K(x) > f(z)$ for all x in $(z, b_z]$, then f is nondecreasing on $[z, b_z] \cap I_r$. If f is nonincreasing on $[a_z, z]$ and $L(x) < f(z)$ for all x in $(z, b_z]$, then f is nonincreasing on $[z, b_z] \cap I_r$.

Proof. Let x_2 be in $(z, b_z] \cap I_r$. If $K(x) > f(z)$ on $(z, b_z]$ let $x_1 = \inf\{x \in (z, b_z]: f(x) = K(x_2)\}$. Then $f(x_1) = K(x_1) = K(x_2)$, so $x_1 > z$, and for x in $[z, x_1]$, $f(x) < f(x_1) = T_r f(x_1)$. By lemma 5.2.6, $f(x) \geq f(x_1)$ on $[x_1, b_x]$, an interval that includes x_2. Hence $f(x) \leq f(x_2)$. A similar proof shows that f is nonincreasing on $[z,b_z] \cap I_r$ if $L(x) < f(z)$ on $(z,b_z]$.

Corollary.

Let f be continuous with $T_r f = f$, and let y be in I_r. If f is constant on $[a_y,y]$, then f is monotone on $[a_y,b_y] \cap I_r$.

Proof. Let $z = \sup\{x: f(x) = f(y)\}$. If z is not in I_r, the result is trivial. Suppose that z is in I_r. If for some $w > z$, $K(w) = f(z)$ and $L(w) = f(z)$, then f is constant on $[z,w]$. By the lemma, f is monotone on $[a_w,b_w] \cap I_r$, and hence on $[a_y,b_y] \cap I_r$.

Lemma 5.2.8.

Let f be continuous with $T_r f = f$. Let x_1, x_2, x_3, x_4 be such that $\mu([x_1,x_2]) = \mu([x_2,x_3]) = \mu([x_3,x_4]) = r$. If f is r-monotone on $[x_1,x_3]$, then f is r-monotone on $[x_1,x_4] \cap I_r$.

Proof. Since f is r-monotone on $[x_1,x_3]$, lemma 5.2.5 gives that f is monotone on $[x_2,x_3]$. Consider the case that f is nondecreasing on $[x_2,x_3]$. Let

5.2.9 $\quad z = \sup\{x \in [x_3,x_4]: \text{f is nondecreasing on } [x_2,x]\}$

5.2.10 $\quad y = \inf\{x \in [x_2,z]: f(w) = f(z) \text{ for all w in } [x,z]\}$

If z is not in $[x_1,x_4] \cap I_r$, there is nothing to prove. If $\mu([y,z]) \geq r$, the conclusion follows from the corollary to 5.2.7.

Suppose $\mu([y,z]) < r$ and z is in $[x_1,x_4] \cap I_r$. Then since $y > x_2$, $a_y > x_1$, so f is nondecreasing on $[a_y,z]$. By 5.2.10, $f(x) < f(y)$ for x in $[a_y,y)$, so by lemma 5.2.6,

5.2.11 $$f(x) \geq f(y) = f(z) \quad \text{for } x \text{ in } [z,b_y]$$

For w in $(z,b_y]$, let $K(w) = \max\{f(x): x \in [z,w]\}$. If $K(w) = f(z)$, then by 5.2.11, f is constant on $[z,w]$ which contradicts 5.2.9. Hence $K(w) > f(z)$ for w in $(z,b_y]$. By lemma 5.2.7, f is nondecreasing on $[z,b_y]$. This contradiction proves the lemma.

Lemma 5.2.12.

Let f be continuous with $T_r f = f$. If f is constant on any closed bounded interval J such that $\mu(J) = r$ and $J \cup I_r$ is an interval, then f is r-monotone on I_r.

Proof. Let $J = [a,b]$. If $I_r \subset J$, there is nothing to prove. Otherwise, either a or b must be in I_r. Consider the case that b is in I_r. By the corollary to 5.2.7, f is monotone on $[a,b_b] \cap I_r$. Since $[a,\infty) \cap I_r$ can be covered by countably many intervals of measure r, lemma 5.2.8 and induction give that f is r-monotone on $[a,\infty) \cap I_r$.

Similarly, f is r-monotone on $(-\infty,b] \cap I_r$.

Lemma 5.2.13.

If f is continuous and r-monotone, then $T_r f = f$.

Proof. Let y be any element in X. If y is not in I_r, then $T_r f(y) = f(y)$ by definition. Suppose y is in I_r. If f is nondecreasing on $[a_y, b_y]$, then $[a_y, y] \subset \{x \in [a_y, b_y]: f(x) \leq f(y)\}$ and $[y, b_y] \subset \{x \in [a_y, b_y]: f(x) \geq f(y)\}$. Hence,

$$\mu(\{x \in [a_y, b_y]: f(x) \leq f(y)\}) \geq \mu[a_y, y] = r$$

and

$$\mu(\{x \in [a_y, b_y]: f(x) \geq f(y)\}) \geq \mu[y, b_y] = r .$$

By proposition 5.1.9, $T_r f(y) = f(y)$. Similarly, if f is nonincreasing on $[a_y, b_y]$, then

$$\mu(\{x \in [a_y, b_y]: f(x) \leq f(y)\}) \geq \mu[y, b_y] = r$$

and

$$\mu(\{x \in [a_y, b_y]: f(x) \geq f(y)\}) \geq \mu[a_y, y] = r ,$$

and again proposition 5.1.9 gives the result.

If f is not monotone on $[a_y, b_y]$, then there exists an interval $[c,d]$ contained in $[a_y, b_y]$ with $\mu[c,d] = r$ on which f is constant. y is in $[c,d]$, so $f(x) = f(y)$ for x in $[c,d]$. Thus,

$$\mu(\{x \in [a_y, b_y]: f(x) \geq f(y)\}) \geq \mu([c,d]) = r$$

and

$$\mu(\{x \in [a_y, b_y]: f(x) \leq f(y)\}) \geq \mu([c,d]) = r$$

By proposition 5.1.9, $T_r f(y) = f(y)$.

Proof of theorem 5.2.3. By lemma 5.2.13, if f is r-monotone, then $T_r f = f$.

Suppose $T_r f = f$. The support of f is contained in an interval (α, β) with $[\alpha, \beta] \subset I_r$. f is zero on $[a_\alpha, \alpha]$, so lemma 5.2.12 gives that f is r-monotone on $[a, \beta]$ and hence on X.

Proof of theorem 5.2.4. By lemma 5.2.13, if f is r-monotone, then $T_r f = f$.

Suppose $T_r f = f$ and f is not r-monotone on X. Let I be a closed bounded interval contained in I_r and let

5.2.14 $\quad \mathrm{MAX} = \max\{f(x) : x \in I\}$

Let y be an element in I such that $f(y) = \mathrm{MAX}$. If neither a_y nor b_y is in I_r, then y is in every closed interval J with $J \subset I$ and $\mu(J) = r$. If a_y is in I, let c be such that $\mu([c, a_y]) = r$.

5.2.15 Every closed interval J contained in $[c, b_y]$ with $\mu(J) = r$ contains an element s with $f(s) = \mathrm{MAX}$.

To prove 5.2.15, suppose that $f(x) < \mathrm{MAX}$ for all x in $[a_y, y)$. $T_r f(y) = f(y)$, so lemma 5.2.6 gives that $f(x) \geq \mathrm{MAX}$ for x in $[y, b_y]$. By 5.2.14, $f(x) = \mathrm{MAX}$ on $[y, b_y]$. Thus $[y, b_y]$ is an interval of measure r on which f is constant, so lemma 5.2.12 gives that f is r-monotone on I. This contradiction shows that

$f(x) = MAX$ for some x in $[a_y, y)$. Let

5.2.16 $$w = \inf\{x \in [a_y, y): f(x) = MAX\}$$

Since f is continuous, $f(w) = MAX$. Repeating the above argument for w in place of y, gives that there exists an element x in $[a_w, w)$ such that $f(x) = MAX$. Let

$$v = \inf\{x \in [a_w, w): f(x) = MAX\}$$

Since f is continuous, $f(v) = MAX$, and by 5.2.16, v is in $[a_w, a_y)$. $\mu([c,v]) \leq \mu([c,a_y]) = r$, $\mu([v,w]) \leq \mu([a_w,w]) = r$, $\mu([w,y]) \leq \mu([a_y,y]) = r$ and $\mu([y,b_y]) = r$. Thus any closed interval J of measure r contained in $[c,b_y]$ contains at least one of v, w or y. This proves 5.2.15.

Let a_n be the sequence of points in I such that

$$\mu([a_n, a_{n-1}]) = \ldots = \mu([a_1, y]) = r$$

5.2.15 gives that MAX is attained on every interval of measure r contained in $[a_2, b_y]$. Suppose that MAX is attained on every closed interval of measure r contained in $[a_n, b_y]$. If a_{n+1} is not in I, then MAX is attained on every closed interval of measure r in $I \cap (-\infty, b_y]$. If a_{n+1} is in I, let z in $[a_n, a_{n-1}]$ be such that $f(z) = MAX$. 5.2.15 gives that MAX is attained on every closed interval of measure r contained in $[c, b_z]$, where c is such that $\mu([c, a_z]) = r$. Since a_{n+1} is in $[c, a_z]$, this gives that MAX is attained on every closed interval of measure r contained in $[a_{n+1},b]$. Since $\mu(I) < \infty$, the result holds on $(-\infty,y] \cap I$. The result is similarly proved for $[y,\infty) \cap I$.

An analogous proof gives the result for MIN in place of MAX, and finishes the theorem.

5.3. Repeated Medians.

Definition 5.3.1.

Let f be a function on X. $T_r^{\omega}f$ exists if there is a number m such that $T_r^m f$ is a fixed point of T_r. In this case, ω denotes the smallest such number.

Theorem 5.3.2.

Let f be a continuous function on X with support in an open interval (α,β) such that $[\alpha,\beta] \subset I_r$. Then $T_r^{\omega}f$ exists and is r-monotone.

The proof of 5.3.2 will follow a sequence of lemmas and a proposition.

Proposition 5.3.3.

If f is continous on the interior of I_r, then $T_r f$ is continuous on the interior of I_r.

Remark.

$T_r f$ is generally not continuous at the end points of I_r, even if f is continuous everywhere.

Proof of proposition 5.3.3. If X is discrete, the result is trivial. In the case that X is continuous, let ρ be defined by

$$\rho(x,y) = \mu([x,y]) = \mu((x,y))$$

If h_a is the function introduced in 5.1.5, then

$$\rho(x,y) = |h_a(x) - h_a(y)|$$

Thus ρ is a metric on X and the topology induced by ρ is equivalent to the order topology on X.

Let y be a point in the interior of I_r and let c and d be in $I_r \cap [a_y, b_y]$ such that $c < y < d$.

f is uniformly continuous on $[a_c, b_d]$, so given $\varepsilon > 0$, let δ be such that $|f(x_2) - f(x_1)| < \varepsilon$ whenever $\rho(x_1, x_2) < \delta$. Let

$$t = \min\{\delta, \mu([c,y]), \mu([y,d])\}$$

Let $\rho(y,z) < t$ and let J denote the interval $J = [a_y, b_y] \cup [a_z, b_z]$. Then

5.3.4 $$\mu(J) = 2r + \rho(y,z)$$

Let

$$\text{MAX} = \max(T_r f(z), T_r f(y)) \quad \text{and} \quad \text{MIN} = \min(T_r f(z), T_r f(y))$$

By proposition 5.1.9, $\mu(\{x \text{ in } J\colon f(x) \geq \text{MAX}\}) \geq r$ and $\mu(\{x \in J\colon f(x) \leq \text{MIN}\}) \geq r$. Since $[c, b_y] \subset J$ and $\mu([c, b_y]) = \mu([c,y)) + \mu([y, b_y])) > \rho(y,z) + r$,

$$\mu(J - [c, b_y]) < (2r + \rho(y,z)) - (\rho(y,z) + r) = r$$

So

$$\mu(\{x \in [c,b_y]: f(x) \geq MAX\}) > 0 \quad \text{and}$$

$$\mu(\{x \in [c,b_y]: f(x) \leq MIN\}) > 0$$

Suppose that $MAX - MIN \geq \varepsilon$. Then by the uniform continuity of f, $\rho(x_1,x_2) \geq \delta$ for any x_i and x_2 in $[c,b_y]$ such that $f(x_1) \geq MAX$ and $f(x_2) \leq MIN$. $[c,b_y]$ is an interval, so $\mu(\{x \in [c,b_y]: MIN < f(x) < MAX\}) \geq \delta$. This gives that

$$\mu(J) = \mu(\{x \in J: f(x) \geq MAX\}) + \mu(\{x \in J: f(x) \leq MIN\})$$
$$+ \mu(\{x \in J: MIN < f(x) < MAX\}) \geq 2r + \delta > 2r + \rho(y,z)$$

This contradiction to 5.3.4 gives that $MAX - MIN < \varepsilon$. Thus $|T_rf(z) - T_rf(y)| = MAX - MIN < \varepsilon$ whenever $\rho(z,y) < t$, which proves the proposition.

<u>Lemma 5.3.5.</u>

Let J be a closed interval with $\mu(J) = r$. If $M_1 \leq f \leq M_2$ on J, then $M_1 \leq T_rf \leq M_2$ on J.

Proof. Let y be in J. If y is not in I_r, then $T_rf(y) = f(y)$ by definition, so the result holds. If y is in I_r, then $J \subset [a_y,b_y]$. So

$$\mu(\{x \in [a_y,b_y]: f(x) \leq M_2\}) \geq r$$

and by the corollary to proposition 5.1.9, $T_rf(y) \leq M_2$. Similarly, $T_rf(y) \geq M_1$.

Corollary.

If f is constant on an interval J of measure r, then $T_r f = f$ on J.

Lemma 5.3.6.

Let f be continuous and let y be in I_r. If J is any closed interval contained in $[a_y, b_y]$ with $\mu(J) = r$, then there exist elements c and d in J such that $f(c) \leq T_r f(y)$ and $f(d) \geq T_r f(y)$.

Proof. Suppose $f(x) < T_r f(y)$ for all x in J. Let $k = \sup\{f(x): x \in J\}$, then since J is closed, $k < T_r f(y)$. On the other hand, $\mu(\{x \in [a_y, b_y]: f(x) \leq k\}) \geq \mu(J) \geq r$, so by the corollary to 5.1.9, $T_r f(y) \leq k$. This contradiction shows that $f(d) \geq T_r f(y)$ for some d in J. The other inequality follows similarly.

Lemma 5.3.7.

Let f be continuous, let x_1 be in I_r and let x_0 and x_2 be such that $\mu([x_0, x_1]) = \mu([x_1, x_2]) = r$. If f is constant on $[x_0, x_1]$ but is not constant on any interval $[x_0, x]$ for $x > x_1$, then $T_r f$ is monotone on $[x_0, x_2]$.

Proof. Let K be the constant such that $f(x) = K$ for x in $[x_0, x_1]$. By the corollary to 5.3.5, $T_r f = K$ on $[x_0, x_1]$. Let

5.3.8 $$y = \sup\{w \geq x_0 : T_r f(x) = K \text{ on } [x_0, w]\}$$

Clearly, $y \geq x_1$, and by the continuity of $T_r f$, $T_r f(y) = K$. If $(y,\infty) \cap I_r$ is empty, the result is trivial. If $(y,\infty) \cap I_r$ is nonempty, then since I_r is an interval, y is in I_r.

5.3.9 Either $\mu(\{x \in (y,b_y]: f(x) \leq K\}) = 0$ or $\mu(\{x \in (y,b_y]: f(x) \geq K\}) = 0$.

To prove 5.3.9, suppose that

$$\mu(\{x \in (y,b_y]: f(x) \leq K\}) = s > 0 \quad \text{and}$$
$$\mu(\{x \in (y,b_y]: f(x) \geq K\}) = t > 0$$

Let $\delta = \min\{s,t\}$, and let z in I_r be such that $z > y$ and $\mu((y,z]) \leq \delta$. Then

$$\mu(\{x \in [a_z,b_y]: f(x) \geq K\}) = \mu([a_z,y]) + t$$
$$= \mu([a_y,y]) - \mu([a_y,a_z)) + t$$

Since $\mu([a_y,a_z)) = \mu((y,z]) \leq \delta$,

$$\mu(\{x \in [a_z,b_y]: f(x) \geq K\}) \geq r - \delta + t \geq r$$

Similarly,

$$\mu(\{x \in [a_z,b_y]: f(x) \leq K\}) \geq r - \delta + s \geq r$$

Since $[a_z,b_y]$ in contained in $[a_z,b_z]$, proposition 5.1.9 gives that $T_r f(z) = K$ for all z in I_r with $z > y$ and $\mu((y,z]) \leq \delta$. This contradiction to 5.3.8 proves 5.3.9.

Consider the case that $\mu(\{x \in (y,b_y]: f(x) \leq K\}) = 0$. By the continuity of f, $f(x) \geq K$ on $[y,b_y]$, and by lemma 5.3.5, $T_r f \geq K$ on $[y,b_y]$.

Let c and d in $[y,x_2] \cap I_r$ with $c < d$. Then

$$\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\} =$$
$$= [a_c,a_d) \cup [a_d,x_1] \cup \{x \in (x_1,b_c]: f(x) \leq T_r f(d)\}$$

and

$$\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\} =$$
$$= [a_d,x_1] \cup \{x \in (x_1,b_c]: f(x) \leq T_r f(d)\}$$
$$\cup \{x \in (b_c,b_d]: f(x) \leq T_r f(d)\}$$

Thus

$$\mu(\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\}) =$$

5.3.10 $$= \mu(\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\}) + $$
$$+ \mu(\{x \in (b_c,b_d]: f(x) \leq T_r f(d)\}) - \mu([a_c,a_d))$$

Since $\mu([a_c,a_d)) = \mu((b_c,b_d])$,

$$r \leq \mu(\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\})$$
$$\leq \mu(\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\})$$

where the first inequality is by proposition 5.1.9, and the second is from 5.3.10. By the corollary to proposition 5.1.9, this gives that $T_r f(c) \leq T_r f(d)$ and completes the proof.

Lemma 5.3.11.

Let f be continuous and have support contained in (α,β) such that $[\alpha,\beta]$ is contained in I_r. Let y be in I_r. If f is r-monotone on $(-\infty,y]$, then $T_r f$ is r-monotone on $(-\infty,y]$. If f is nondecreasing on $[a_y,y]$, then so is $T_r f$, and if f is nonincreasing on $[a_y,y]$, then so is $T_r f$.

Proof. Let X' denote $(-\infty,y]$, then $I'_r = (-\infty,a_y] \cap I_r$. By theorem 5.2.4, $T_r f = f$ on I'_r, so $T_r f$ is r-monotone on $(-\infty,a_y]$.

If any portion of $[a_y,y]$ is not in I_r, then since the support of f is contained in (α,β) and $[\alpha,\beta]$ is contained in I_r, f is zero on an interval of measure r which intersects $[a_y,y]$. Thus by lemma 5.3.7, the result follows.

If $[a_y,y]$ is contained in I_r, let

$$v = \inf\{s \in I_r\colon f(x) = f(a_y) \text{ for all } x \in [s,a]\}$$
$$w = \sup\{s \in I_r\colon f(x) = f(a_y) \text{ for all } x \in [a_y,s]\}$$

If $\mu([v,w]) \geq r$, then the result follows by lemma 5.3.7.

Consider the case that $\mu([v,w]) < r$. By 5.2.6, f is monotone on $[a_y,y]$, so consider the case that f is nondecreasing on $[a_y,y]$. Since $\mu[v,w] < r$ and f is r-monotone on $(-\infty,y]$, f is nondecreasing on $[a_{a_y},y]$. Thus it will suffice to show that $T_r f$ is nondecreasing on $[a_y,y]$.

$f(x) \geq f(a_y)$ on $[a_y,y]$ so by lemma 5.3.5, $T_r f(x) \geq f(a_y)$ on

$[a_y,y]$.

Let c and d be in $[a_y,y]$ with $c < d$. Then

$$\begin{aligned}&\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\}\\ &= [a_d,a_y] \cup \{x \in (a_y,b_c]: f(x) \leq T_r f(d)\}\\ &\quad \cup \{x \in (b_c,b_d]: f(x) \leq T_r f(d)\}\end{aligned}$$

and

$$\begin{aligned}&\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\}\\ &= [a_c,a_d) \cup [a_d,a_y] \cup \{x \in (a_y,b_c]: f(x) \leq T_r f(d)\}\end{aligned}$$

Combining these two equations gives

5.3.12
$$\begin{aligned}&\mu(\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\})\\ &= \mu(\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\})\\ &\quad - \mu([a_c,a_d) + \mu(\{x \in (b_c,b_d]: f(x) \leq T_r f(d)\})\end{aligned}$$

Since $\mu([a_c,a_d)) = \mu((b_c,b_d])$,

$$\begin{aligned}r &\leq \mu(\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\})\\ &\leq \mu(\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\})\end{aligned}$$

Where the first inequality is by proposition 5.1.9, and the second is the consequence of 5.3.12. By the corollary to proposition 5.1.9, $T_r f(c) \leq T_r f(d)$.

In case f is nonincreasing on $[a_y,y]$, the result follows similarly.

Lemma 5.3.13.

Let f be continuous and have support contained in (α,β), where $[\alpha,\beta]$ is contained in I_r. Suppose that a_y and b_y are in I_r, that f is r-monotone on $(-\infty,y]$ and that f is nondecreasing on $[a_y,y]$. Let

5.3.14 $$z = \sup\{x \in [a_y,b_y]: T_rf \text{ is nondecreasing on } [a_y,x]\}$$

Then $f(x) \geq T_rf(z)$ on $[a_z,y]$ and T_rf is nonincreasing on $[z,b_y]$.

If the words nonincreasing and nondecreasing are interchanged in the above, the statement holds with the reverse inequality.

Proof. By lemma 5.3.11, $z \geq y$. If $z = b_y$, there is nothing to prove, so consider the case that z is in $[y,b_y)$.

5.3.15 $$f(x) \geq T_rf(z) \text{ for } x \text{ in } [a_z,y].$$

To prove 5.3.15, lemma 5.3.6 gives that there exists some x in $[a_z,z]$ such that $f(x) \geq T_rf(z)$. Let

$$v = \inf\{x \in [a_z,z]: f(x) \geq T_rf(z)\}$$

Suppose that v is in $(a_z,z]$. Then for w in $[z,b_v] \cap I_r$

5.3.16 $$\{x \in [a_z,b_z]: f(x) \geq T_rf(z)\}$$
$$= \{x \in [v,b_z]: f(x) \geq T_rf(z)\}$$

and

5.3.17 $$\{x \in [a_w, b_w]: f(x) \geq T_r f(z)\}$$
$$= \{x \in [v, b_z]: f(x) \geq T_r f(z)\}$$
$$\cup \{x \in (b_z, b_w]: f(x) \geq T_r f(z)\}.$$

Thus

$$r \leq \mu(\{x \in [a_z, b_z]: f(x) \geq T_r f(z)\})$$
$$\leq \mu(\{x \in [a_w, b_w]: f(x) \geq T_r f(z)\})$$

where the first inequality is due to proposition 5.1.9, and the second is from 5.3.16 and 5.3.17. By the corollary to proposition 5.1.9, $T_r f(w) \geq T_r f(z)$. Thus

$$T_r f(x) \geq T_r f(z) \text{ for all } x \text{ in } [z, b_v] \cap I_r.$$

let $c < d$ be elements in $[z, b_v] \cap I_r$ and let $v_1 = \inf\{x \in [a_c, c]: f(x) \geq T_r f(c)\}$. $T_r f(c) \geq T_r f(z)$, so $v_1 \geq v$ and hence $a_c < a_d \leq v_1$. Therefore,

5.3.18 $$\{x \in [a_c, b_c]: f(x) \geq T_r f(c)\}$$
$$= \{x \in [v_1, b_c]: f(x) \geq T_r f(c)\}$$

and

5.3.19 $$\{x \in [a_d, b_d]: f(x) \geq T_r f(c)\}$$
$$= \{x \in [v_1, b_c]: f(x) \geq T_r f(c)\}$$
$$\cup \{x \in (b_c, b_d]: f(x) \geq T_r f(c)\}$$

Thus

$$r \leq \mu(\{x \in [a_c,b_c]: f(x) \geq T_r f(c)\})$$
$$\leq \mu(\{x \in [a_d,b_d]: f(x) \geq T_r f(c)\}) ,$$

where the first inequality is from proposition 5.1.9, and the second is from 5.3.18 and 5.3.19. By the corollary to proposition 5.1.9, $T_r f(d) \geq T_r f(c)$. This shows that $T_r f$ is nondecreasing on $[z,b_v] \cap I_r$, which contradicts 5.3.14 and so proves 5.3.15.

5.3.20 If for w in $[z,b_y]$, $T_r f(w) \geq T_r f(z)$, then $T_r f(x) \geq T_r f(z)$ for all x in $[z,w]$.

To prove 5.3.20, let c be in $(z,w]$. By 5.3.15, $f(x) \geq T_r f(z)$ for x in $[a_z,y]$, so

5.3.21
$$\{x \in [a_c,b_c]: f(x) \geq T_r f(z)\}$$
$$= [a_c,a_w) \cup [a_w,y] \cup \{x \in (y,b_c]: f(x) \geq T_r f(z)\}$$

and

5.3.22
$$\{x \in [a_w,b_w]: f(x) \geq T_r f(z)\}$$
$$= [a_w,y] \cup \{x \in (y,b_c]: f(x) \geq T_r f(x)\}$$
$$\cup \{x \in (b_c,b_w]: f(x) \geq T_r f(z)\}$$

Thus

$$r \leq \mu(\{x \in [a_w,b_w]: f(x) \geq T_r f(z)\})$$
$$\leq \mu(\{x \in [a_c,b_c]: f(x) \geq T_r f(z)\})$$

where the first inequality is by proposition 5.1.9, and the fact that $T_r f(z) \leq T_r f(w)$, and the second inequality is by 5.3.21, 5.3.22 and the fact that $\mu([a_c, a_w)) = \mu((b_c, b_w])$. By the corollary to proposition 5.1.9, $T_r f(c) \geq T_r f(z)$ which proves 5.3.20.

5.3.23 $\quad T_r f(x) \leq T_r f(z)$ for x in $[z, b_y]$.

To prove this statement, 5.3.15 gives that $f(x) \geq T_r f(z)$ for x in $[a_z, y]$. In the case that $\mu(\{x \in [a_z, y]: f(x) = T_r f(z)\} = 0$,

$$\mu(\{x \in (y, b_z]: f(x) \leq T_r f(z)\})$$
$$= \mu(\{x \in [a_z, b_z]: f(x) \leq T_r f(z)\}) \geq r$$

For w in $[z, b_y]$, $(y, b_z]$ is contained in $[a_w, b_w]$, so

$$\mu(\{x \in [a_w, b_w]: f(x) \leq T_r f(z)\})$$
$$\geq \mu(\{x \in (y, b_z]: f(x) \leq T_r f(z)\}) \geq r$$

By the corollary to 5.1.9, this gives that $T_r f(w) \leq T_r f(z)$.

In case $\mu(\{x \in [a_z, y]: f(x) = T_r f(z)\}) > 0$, suppose that there is an element e in $(z, b_y]$ such that $T_r f(e) \geq T_r f(z)$. Since f is nondecreasing on $[a_y, y]$, $\{x \in [a_z, y]: f(x) = T_r f(z)\} = [a_z, s]$ for some s in $(a_z, y]$. Let c and d be elements in $[z, e] \cap [z, b_s]$ with $c < d$. Since $T_r f(e) \geq T_r f(z)$, 5.3.20 gives that $T_r f(d) \geq T_r f(z)$. Thus

5.3.24 $$\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\}$$
$$= [a_c,a_d) \cup [a_d,s] \cup \{x \in (s,b_c]: f(x) \leq T_r f(d)\}$$

and

5.3.25 $$\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\}$$
$$= [a_d,s] \cup \{x \in (s,b_c]: f(x) \leq T_r f(d)\}$$
$$\cup \{x \in (b_c,b_d]: f(x) \leq T_r f(d)\}$$

Hence

$$r \leq \mu(\{x \in [a_d,b_d]: f(x) \leq T_r f(d)\})$$
$$\leq \mu(\{x \in [a_c,b_c]: f(x) \leq T_r f(d)\})$$

where the first inequality is by 5.1.9 and the second is from 5.3.24, 5.3.25 and the fact that $\mu([a_c,a_d)) = \mu((b_c,b_d])$. By the corollary to proposition 5.1.9, $T_r f(c) \leq T_r f(d)$ which shows that $T_r f$ is nondecreasing on $[a_y,e] \cap [a_y,b_s]$. This contradiction to 5.3.14 proves 5.3.23.

To finish the proof of the lemma, let c and d be in $[z,b_y]$ with $c < d$. By 5.3.23, $T_r f(c) \leq T_r f(z)$ and $T_r f(d) \leq T_r f(z)$. By 5.3.14, $f(x) \geq T_r f(z)$ for x in $[a_z,y]$, so

5.3.26 $$\{x \in [a_c,b_c]: f(x) \leq T_r f(c)\}$$
$$= \{x \in (y,b_c]: f(x) \leq T_r f(c)\}$$

and

5.3.27 $$\{x \in [a_d,b_d]: f(x) \leq T_r f(c)\}$$
$$= \{x \in (y,b_c]: f(x) \leq T_r f(c)\}$$
$$\cup \ \{x \in (b_c,b_d]: f(x) \leq T_r f(c)\}$$

Thus

$$r \leq \mu(\{x \in [a_c,b_c]: f(x) \leq T_r f(c)\})$$
$$\leq \mu(\{x \in [a_d,b_d]: f(x) \leq T_r f(c)\})$$

where the first inequality is by proposition 5.1.9, and the second follows from 5.3.26 and 5.3.27. By the corollary to proposition 5.1.9, $T_r f(c) \geq T_r f(d)$ which proves the lemma.

Lemma 5.3.28.

Let f be continuous and have support contained in (α,β) where $[\alpha,\beta]$ is contained in I_r. Let y be in I_r. If a_y and b_y are in I_r, and f is r-monotone on $(-\infty,y]$, then $T^3 f$ is r-monotone on $(-\infty,b_y]$.

Remark.

$T_r f$ need not be r-monotone on $(-\infty,b_y]$.

Proof of lemma 5.3.28. By lemma 5.3.11, $T_r^n f$ is r-monotone on $(-\infty,y]$ for $n \geq 1$. By lemma 5.2.6, f is monotone on $[a_y,y]$. Consider the case that f is nondecreasing on $[a_y,y]$. If f is constant on $[a_y,y]$, the result of the lemma follows from lemma 5.3.7. If f is not constant on $[a_y,y]$, let

5.3.29 $$z = \sup\{x \in [a_y,b_y]: T_r f \text{ is nondecreasing on } [a_y,x]\}$$

By lemma 5.3.11, $z \geq y$, and by lemma 5.3.13, $T_r f$ is nonincreasing on $[z,b_y]$. Let

5.3.30 $$z^* = \sup\{x \in [a_y,y]: T_r^2 f \text{ is nondecreasing on } [a_y,x]\}$$

By lemma 5.3.11, $z^* \geq z$, and by lemma 5.3.13, $T_r^2 f$ is nonincreasing on $[z^*,b_z]$.

5.3.31 If $T_r f(b_y) \geq T_r f(y)$, then $T_r^2 f$ is nondecreasing on $[a_y,b_y]$.

To prove this statement, observe that since Tf is nondecreasing on $[y,z]$ and nonincreasing on $[z,b_y]$, the assumption that $T_r f(b_y) \geq T_r f(y)$ gives that $T_r f(x) \geq T_r f(y)$ for x in $[y,b_y]$. Thus by lemma 5.3.6, $T_r^2 f(x) \geq T_r f(y)$ for x in $[y,b_y]$. Since $T_r f$ is nondecreasing on $[a_y,y]$, $T_r f(x) \leq T_r f(y)$ for x in $[a_y,y]$. By the corollary to proposition 5.1.9, $T_r^2 f(y) = T_r f(y)$. Thus,

$$T_r^2 f(x) \geq T_r f(y) = T_r^2 f(y) \quad \text{for } x \text{ in } [y,b_y].$$

For w in $[y,b_y]$

$$\begin{aligned} &\{x \in [a_w,b_w]: T_r f(x) \leq T_r^2 f(b_y)\} \\ &= [a_w,y) \cup \{x \in [y,b_w]: T_r f(x) \leq T_r^2 f(b_y)\} \end{aligned}$$

and

$$\{x \in [y,b_{b_y}]: T_rf(x) \le T_r^2f(b_y)\}$$

$$= \{x \in [y,b_w]: T_rf(x) \le T_r^2f(b_y)\}$$

$$\cup \{x \in (b_w,b_{b_y}]: T_rf(x) \le T_r^2f(b_y)\}$$

Together these give

5.3.32 $$\mu(\{x \in [y,b_{b_y}]: T_rf(x) \le T_r^2f(b_y)\})$$

$$= \mu(\{x \in [a_w,b_w]: T_rf(x) \le T_r^2f(b_y)\})$$

$$+ \mu(\{x \in (b_w,b_{b_y}]: T_rf(x) \le T_r^2f(b_y)\}) - \mu([a_w,y))$$

Thus

$$r \le \mu(\{x \in [y,b_{b_y}]: T_rf(x) \le T_r^2f(b_y)\})$$

$$\le \mu(\{x \in [a_w,b_w]: T_rf(x) \le T_r^2f(b_y)\})$$

where the first inequality is by proposition 5.1.9, and the second is by 5.3.32.

By the corollary to propostion 5.1.9, $T_r^2f(w) \le T_r^2f(b_y)$ for any w in $[y,b_y]$. Since Z^* is the largest element in $[a_y,y]$ so that T_r^2f is nondecreasing on $[a_y,z^*]$, and since T_r^2f is nonincreasing on $[z^*,b_y]$, this gives that $z^* = b_y$. Hence T_r^2f is nondecreasing on $[a_y,b_y]$ and 5.3.31 is proved.

Thus if $T_rf(b_y) \ge T_rf(y)$, 5.3.31 gives that T_r^2f is r-monotone on $(-\infty,b_y]$. Lemma 5.3.11, applied to T_r^2f and then to T_r^3f gives the result of the lemma in this case.

If $T_rf(b_y) < T_rf(y)$, then since T_rf is nondecreasing on

$[a_y, z]$ and nonincreasing on $[z, b_y]$, $T_r f(x) \geq T_r f(b_y)$ for all x in $[y, b_y]$.

If $T_r^2 f(b_y) \geq T_r^2 f(y)$, then 5.3.31 applied to $T_r^2 f$ in place of $T_r f$, gives that $T_r^3 f$ is nondecreasing on $[a_y, b_y]$ and hence, by lemma 5.3.11, the result of the lemma.

This leaves the case that both $T_r f(b_y) < T_r f(y)$ and $T_r^2 f(b_y) < T_r^2 f(y)$. $T_r f$ is nondecreasing on $[a_y, y]$, so by 5.1.9, $T_r^2 f(y) \leq T_r f(y)$. Hence in this case, $T_r^2 f(b_y) < T_r f(y)$. Let

$$s = \inf\{x \in [y, b_y]: T_r f(x) \leq T_r^2 f(b_y)\}$$

Since $T_r f$ in nonincreasing on $[z, b_y]$, $T_r f(x) \leq T_r^2 f(b_y)$ for x in $[s, b_y]$. Thus $\{x \in [y, b_{b_y}]: T_r f(x) \leq T_r^2 f(b_y)\} \subset [s, b_{b_y}]$. For w in $[b_y, b_s]$, $[s, b_{b_y}] \subset [a_w, b_w]$, so

$$r \leq \mu(\{x \in [y, b_{b_y}]: T_r f(x) \leq T_r^2 f(b_y)\})$$

$$\leq \mu(\{x \in [a_w, b_w]: T_r f(x) \leq T_r^2 f(b_y)\})$$

By proposition 5.1.9,

5.3.33 $\qquad T_r^2 f(w) \leq T_r^2 f(b_y)$ for all w in $[b, b_s]$.

Let

$$c = \inf\{x \in [a_y, y]: T_r f(x) \geq T_r^2 f(y)\}$$

$$d = \sup\{x \in [y, b_y]: T_r f(x) \geq T_r^2 f(y)\}$$

Since $T_r f$ is nondecreasing on $[a_y, z]$ and nonincreasing on $[z, b_y]$, $T_r f(x) \geq T_r^2 f(y)$ for x in $[c,d]$, and $T_r f(x) < T_r^2 f(y)$ for x in $[a_y, b_y] - [c,d]$. Also, by proposition 5.1.9, $\mu([c,d]) \geq r$. Thus by lemma 5.3.5,

5.3.34 $\qquad T^n f(x) \geq T_r^2 f(y) \qquad$ for x in $[c,d]$ and $n \geq 1$

$T_r^n f$ is nondecreasing on $[a_y, y]$ so

5.3.35 $\qquad T^n f(x) = T_r^2 f(y) = T^n f(y) \qquad$ for x in $[c,y]$ and $n \geq 1$

Let v be in $(y,s]$. By 5.3.35 together with 5.3.33,

$$\{x \in [a_v, b_v]: T_r^2 f(x) \leq T_r^2 f(y)\}$$
$$= [a_v, y] \cup [b_y, b_v] \cup \{x \in (y, b_y): T_r^2 f(x) \leq T_r^2 f(y)\}$$

Hence,

$$\mu(\{x \in [a_v, b_v]: T_r^2 f(x) \leq T_r^2 f(y)\})$$
$$\geq \mu([a_v, y]) + \mu([b_y, b_v]) = \mu([a_v, b_v]) - \mu((y, b_y)) \geq r$$

By the corollary to proposition 5.1.9, this gives that $T_r^3 f(x) \leq T_r^2 f(y)$ for x in $(y,s]$. Since $T_r^3 f(x) \geq T_r^2 f(y)$ on $[c,d]$, this gives that $T_r^3 f(x) = T_r^2 f(y)$ for x in $[c,d] \cap (y,s]$. By 5.3.35, $T_r^3 f(x) = T_r^2 f(y)$ for x in $[c,d] \cap [c,s]$. Since $T_r^2 f(b_y) < T_r^2 f(y)$, $s \geq d$, and so,

$$T_r^3 f(x) = T_r^2 f(y) \qquad \text{for x in } [c,d]$$

Lemma 5.3.13 applied to T_r^2f gives that T_r^3f is nonincreasing on $[d,b_y]$. Since $\mu([c,d]) \geq r$, T_r^3f is r-monotone on $(-\infty,b_y]$.

Proof of theorem 5.3.2. Since f has support contained in (α,β), f is constant on, $(-\infty,\alpha]$ and on $[\beta,\infty)$. α is in I_r, so $\mu((-\infty,\alpha]) \geq r$. By lemma 5.3.7, T_rf is monotone on $[\alpha,b_\alpha]$ and hence r-monotone on $(-\infty,b_\alpha]$. Similarly, T_rf is r-monotone on the interval $[a_\beta,\infty)$.

By lemma 5.3.28, if T_r^n is r-monotone on $(-\infty,y]$ for some y in $[b_\alpha,a_\beta]$, then $T_r^{n+3}f$ is r-monotone on $(-\infty,b_y]$. Since $\mu([\alpha,\beta]) < \infty$, T_r^mf is r-monotone on X for some power m. This completes the proof of theorem 5.3.2.

5.4 Comparison of $T_r^{\omega}f$ and f.

Theorem 5.4.1.

Let f be continuous. For an element y in X, let J(y) denote the collection of all closed intervals of measure r containing y. Let

$$M_2 = \inf_{I \text{ in } J(y)} \ \max_{x \text{ in } I} \ f(x)$$

$$M_1 = \sup_{I \text{ in } J(y)} \ \min_{x \text{ in } I} \ f(x)$$

If $T_r^{\omega}f$ esists, then $M_1 \leq T_r^{\omega}f(y) \leq M_2$.

Proof. Let the interval I be in the collection J(y). Then $f(w) \leq \max\{f(x)\colon x \text{ in } I\}$ for all w in I. By lemma 5.3.5 applied to consecutive powers $T_r^n f$, $T_r^{\omega}f(w) \leq \max\{f(x)\colon x \text{ in } I\}$ for all w in I. Since y is in every interval I in J(y), $T_r^{\omega}f(y) \leq M_1$.

Similarly, $T_r^{\omega}f(y) \geq M_1$, and this proves the theorem.

Corollary.

If I is any interval with $\mu(I) = r$, and $L \leq f(x) \leq K$ for all x in I, then $L \leq T_r^{\omega}f(x) \leq K$ for all x in I.

Proof. For every y in I, $I \subset J(y)$.

5.5 Iterative Smoothing.

Definition 5.5.1.

Let g be a function on X such that $T_r^{\omega}g$ exists and is r-monotone. Let L be a positive number. C(x) is a L-correcting function of g if C(x) is continuous and

a) $|C(x) - T_r^{\omega}g(x)| \leq L$ for all x.

b) $C(x) = g(x)$ if $|T_r^{\omega}g(x) - g(x)| = L$.

c) $\mathrm{sgn}(C(x) - T_r^{\omega}g(x)) = \mathrm{sgn}(g(x) - T_r^{\omega}g(x))$ for all x.

Iterative smoothing as discussed in section 2 incorporates running median filters on intervals of various sizes. The interval length for iteration i is denoted by r_i, and the median filter, T_{r_i}, is denoted by T_i.

Definition 5.5.2.

The i^{th} stage smoothing operator S_i, is defined by

$$S_i g(x) = \begin{cases} g(x) & \text{if } |T_i^{\omega}g(x) - g(x)| \leq L_i \\ C_i(x) & \text{if } |T_i^{\omega}g(x) - g(x)| > L_i \end{cases}$$

where C_i is a L_i-correcting function for g.

For a function f on X, $\underline{f_i}$ denotes $S_i S_{i-1} \ldots S_1 f$.

Remark.

The smoothing discussed in chapter 2 results if $C_i(x) = T_i^{\omega}g(x)$ whenever $|T_i^{\omega}g(x) - g(x)| > L_i$, and $C_i(x) = g(x)$ otherwise. This correcting function is not continuous if X is continuous.

Theorem 5.5.3.

Let f be continuous on X. Let $r_1 \geq r_2 \geq \ldots \geq r_N$. If $T_i^{\omega}f_{i-1}$ is r-monotone for each i, $i \leq N$, then $|f_N(x) - T_i^{\omega}f_{i-1}(x)| \leq L_i$ for all x, and all $i \leq N$.

Proof. The theorem may be reduced to the following statement:

If g(x) is a continuous function such that

$$g(x) = h(x) + k(x)$$

where h(x) is continuous and r_i-monotone and $|k(x)| \leq L_i$ for all x, then for each $j > i$, there exists a function $k'(x)$ with $|k'(x)| \leq L_i$ for all x such that

5.5.5 $$S_j g(x) = h(x) + k'(x)$$

The theorem is recovered by replacing h(x) by $T_i^{\omega}f_{i-1}(x)$ and applying the statement successively with $j = i+1, i+2, \ldots, N$.

If y is such that $|T_j^{\omega}g(y) - g(y)| \leq L_j$, then $S_j g(y) = g(y)$ and so 5.5.5 holds with $k'(y) = k(y)$.

Consider the case that

5.5.6 $\quad |T_j^{\omega}g(y) - g(y)| > L_j$

Suppose $|k'(y)| > L_i$. Then one of the following holds,

5.5.7 $\quad S_jg(y) - h(y) > L_i \quad$ or

5.5.8 $\quad S_jg(y) - h(y) < L_i$

The proof will follow by showing that 5.5.7 cannot hold, and separately, that 7.5.8 cannot hold.

5.5.9. Let I be any closed interval containing y such that $\mu(I) = r_j$. If 5.5.7 holds, then $g(s) > h(y) + L_i$ for some s in I. Similarly, if 5.5.8 holds, then $g(t) < h(y) - L_i$ for some t in I.

To prove this statement, suppose $g(x) \leq h(y) + L_i$ for all x in I. Then by lemma 5.3.5, $T_j^{\omega}g(x) \leq h(y) + L_i$ for all x in I. If x is such that $|T_j^{\omega}g(x) - g(x)| \leq L_j$, then $S_jg(x) = g(x) \leq h(y) + L_i$. If $|T_j^{\omega}g(x) - g(x)| > L_j$, then $S_jg(x) = C_j(x)$ and, by the definition of correcting functions, one of the following must hold depending on $\mathrm{sgn}(C_j(x) - T_j^{\omega}g(x))$. Either

$$T_j^{\omega}g(x) \leq S_jg(x) \leq T_j^{\omega}g(x) + L_j < g(x) \leq h(y) + L_i \quad \text{or}$$
$$g(x) < T_j^{\omega}g(x) - L_j \leq S_jg(x) \leq T_j^{\omega}g(x) \leq h(y) + L_i$$

Thus $S_j g(x) \leq h(y) + L_i$ for all x in I. In particular, $S_j g(y) \leq h(y) + L_i$, contradicting 5.5.7. This proves the first statement of 5.5.9, and the second is proved analogously.

Let a_y and b_y be such that $\mu([a_y,y]) = \mu([y,b_y]) = r_j$. By 5.5.6, y is in I_{r_j}, so if 5.5.7 holds, then 5.5.9 gives that there exists a in $[a_y,y]$ and b in $[y,b_y]$ such that $g(a) > h(y) + L_i$ and $g(b) > h(y) + L_i$. Since $g(y) \leq h(y) + L_i$, a is in $[a_y,y)$ and b is in $(y,b_y]$. Thus

$$h(a) + k(a) = g(a) > h(y) + L_i \quad \text{and}$$
$$h(b) + k(b) = g(b) > h(y) + L_i$$

Since $L_i \geq k(x)$ for all x, these inequalities give that

$$h(a) > h(y) \qquad \text{and} \qquad h(b) > h(y)$$

$h(x)$ is r_i-monotone, so there exists an interval $[c,d]$ contained in $[a,b]$ with $\mu([c,d]) = r_i$ such that $h(x) = \min\{h(z): z \text{ is in } [a,b]\}$ for all x in $[c,d]$. Since y lies in $[a,b]$, $h(x) \leq h(y)$ for x in $[c,d]$. Thus for x in $[c,d]$,

$$g(x) = h(x) + k(x) \leq h(y) + L_i$$

Since $\mu([c,d]) = r_i \geq r_j$, this gives a contradiction to 5.5.9. This proves that equations 5.5.6 and 5.5.7 cannot both hold. Similarly, 5.5.6 and 5.5.8 cannot both hold.

Theorem 5.5.10.

Let f be continuous on X. For y in I_r, let J(y) denote the collection of closed intervals of measure r_1 containing y. Let

$$M_2 = \inf_{I \text{ in } J(y)} \max_{x \text{ in } I} f(x)$$

$$M_1 = \sup_{I \text{ in } J(y)} \min_{x \text{ in } I} f(x)$$

If $r_1 \geq r_2 \geq \ldots \geq r_N$, then $M_1 \leq f_N(y) \leq M_2$.

The proof of Theorem 5.5.10 follows from a lemma.

Lemma 5.5.11.

Let I be a closed interval containing y with $\mu(I) = r_1$. If $g(x) \leq B$ for all x in I, then $S_j g(x) \leq B$ for all x in I.

Proof. By the corollary to theorem 5.4.1, $T_j^{\omega} g(x) \leq B$ for all x in I. If x is such that $|T_j^{\omega} g(x) - g(x)| \leq L_j$, then $S_j g(x) = g(x) \leq B$. If x is such that $|T_j^{\omega} g(x) - g(x)| > L_j$, then either

5.5.12 $\qquad g(x) > T_j^{\omega} g(x) + L_j \qquad$ or

5.5.13 $\qquad g(x) < T_j^{\omega} g(x) + L_j$

$\text{sgn}(C_j(x) - T_j^{\omega} g(x)) = \text{sgn}(g(x) - T_j^{\omega} g(x))$ and $|C_j(x) - T_j^{\omega} g(x)| \leq L_j$, so if 5.5.12 holds, then

$$S_j g(x) = C_j(x) \leq T_j^{\omega} g(x) + L_j \leq g(x) \leq B$$

If 5.5.13 holds then

$$S_j g(x) = C_j(x) \leq T_j^{\omega} g(x) \leq B$$

This proves the lemma.

To prove the theorem, let I be in J(y). $f(x) = f_0(x) \leq \max\{f(z): z \text{ in } I\}$ for all x in I. Since $r_1 \leq r_2 \leq \ldots \leq r_N$, succesive applications of the lemma give that

$$f_N(x) = f_N(x) \leq \max\{f(z): z \text{ in } I\} \quad \text{for } x \text{ in } I.$$

Since y is in I for all I in J(y), $f_N \leq M_2$. The inequality $f_N \geq M_1$ is proved analogously.

Corollary.

If $K \leq f(x) \leq L$ for all x in a closed interval I with $\mu(I) = r_1$, then $K \leq f(x) \leq L$ for all x in I.

Proof. I is contained in J(y) for all y in I.

6. Necessity of the Axioms

6.1. Linear ordering assumption.

The theorems of the last chapter do not generalize to R^n with Lebesgue measure. Let f be a function on R^n and let $B^n(x,r)$ denote the closed ball in R^n of radius r, centered at x. Let

$$T_r f(x) = \mathrm{med}_{B^n(x,r)} f$$

(For n = 1, T_r is the same as in definition 5.2.1.)

<u>Proposition 6.1.</u>

Let f be a function on R^n, $n \geq 2$, with support contained in a bounded set. Then $T_r^{\omega} f = 0$, so f is a fixed point of T_r if and only if f= 0.

The proof will follow a lemma.

<u>Lemma 6.2.</u>

Let f be a function on R^n, $n \geq 2$ and let $R_0 > 0$ be given. There exists t so that if $R \leq R_0$ and f has support in $B^n(0,R)$, then $T_r f$ has support in $B^n(0,R-t)$.

Proof. Let $V(n) = \mu(B^n(0,r))$, let y be in R^n with $|y| = R_0$, and let $A = \mu(B^n(y,r) - B^n(0,R_0))$. Then $A > V(n)/2$. Let

$$t = [A - V(n)/2]/V(n-1)$$

If f has support in $B^n(0,R)$, $R < R_0$, then for x in R^n with $R-t \leq |x| \leq R$,

$$\mu(\{z \in B^n(x,r): f(z) = 0\}) \geq A - V(n-1)t \geq V(n)/2$$

Thus $T_r f(x) = 0$. This proves the lemma.

Proof of proposition 6.1. Let R_0 be such that the support of f is contained in $B^n(0,R_0)$. Repeated application of the lemma gives that $T_r^m f = 0$ for some integer m.

6.2. Other Axioms.

Let 5.1.3' be 5.1.3 with axiom (d) replaced by

(d') I_r is an interval such that for all x in X, either there exists b in I_r such that $\mu([x,b]) = r$, or there exists a in I_r such that $\mu([a,x]) = r$.

If X is continuous, $\mu([x]) = 0$ for all x in X is a consequence of 5.1.3'. If X is discrete, μ is periodic on X in the following sense. Let x^+ and x^- be the immediate predecessor and immediate successor of x respectively. If X contains atleast two elements, then for any x, either x^+ exists, in which case there exists y such that $\mu([x^+,y]) = r$ and $\mu([x]) = \mu([y])$, or x^- exists, in which case there exists w such that $\mu([w,x^-]) = r$ and $\mu([w]) = \mu([x])$.

To prove that $\mu([x]) = 0$ if X is continuous, suppose that $\mu([x]) > 0$ for some x in X. By axiom (d'), it can be assumed that there exists b in I_r with $\mu([x,b]) = r$. By lemma 6.1.4, there exists y such that $\mu((b,y]) < \mu([x])$. For z in (x,b)

$$\mu([z,y]) = \mu([z,b]) + \mu((b,y]) < \mu([z,b]) + \mu([x]) < r$$

On the other hand, for $z \leq x$,

$$\mu([z,y]) \geq \mu([x,y]) = \mu[(x,b]) + \mu((b,y]) > r$$

Hence there is no element z in X with $\mu([z,y]) = r$. This gives that y is not in I_r, and by (d'), that there is an element b_y in I_r such that $\mu([y,b_y]) = r$. Since in b and b_y are in I_r, and $b < y < b_y$, this contradicts the fact that I_r is an interval.

To prove that μ is periodic if X is discrete, observe that (d') implies that $\mu([x]) \leq r$ for all x in X.

If x is such that $\mu([x]) = r$, then x is in I_r. By (d') it can be assumed that x^+ exists. Suppose that $\mu([x^+]) < r$. Then there is no element, v, such that $\mu([v,x^+]) = r$, so x^+ is not in I_r. By (d'), there exists an element w in I_r with $w > x^+$. This contradicts the fact that I_r is an interval. Hence $\mu([x^+]) = r$, which proves the result in the case that $\mu([x]) = r$.

If $\mu([x]) < r$, it may be assumed by (d') that there is an element b in I_r such that $\mu([x,b]) = r$. Suppose there is no element y in I_r such that $\mu([x^+,y]) = r$. Then by (d'), there must be an element a^+ in I_r such that $\mu([a^+,x^+]) = r$. Since x^+ is not in I_r, and $a^+ < x^+ < b$, this contradicts the fact that I_r is an interval. Hence there does exist y in I_r with $\mu([x^+,y]) = r$. The equations

$$\mu([x,y]) = \mu([x]) + \mu([x^+,y]) = \mu([x]) + r$$

and

$$\mu([x,y]) = \mu([x,b]) + \mu((b,y]) = r + \mu((b,y])$$

give that $\mu((b,y]) = \mu([x])$.

If $\mu([y]) \neq \mu([x])$, then (b,y) is nonempty, but if z is in (b,y), then there is no element a for which $\mu([a,z]) = r$. This contradicts the assumption that I_r is an interval.

Despite this periodicity, the main theorems of the previous sections are false without assumption (d) of 5.1.3.

Theorem 5.2.3 is false if axiom (c) in 5.1.3 is not assumed or if f is not continuous.

Example 1. (Axiom (c) is not assumed.)

Let $X = R$. Let $E = (-\infty,-2] \cup [-1,1] \cup [2,+\infty)$ and define the measure via $\mu(A) = \int_A \chi_E$. Let $f(x) = 0$ for $|x| \geq 2$, $f(x) = 1$ for $|x| \leq 1$, $f(x) = x + 2$ for x in $(-2,-1)$, and $f(x) = -x + 2$ for x in $(1,2)$. f is continuous with respect to the order topology and f is 2-monotone. X has all the properties of a defintion 6.1.3 except property (c). With $r = 2$, $Tf(x) = f(x) = 0$ for $|x| > 2$, but $Tf(x) = 1/2 \neq f(x)$ for $|x| < 2$.

Example 2. (f is not continuous)

Let $X = R$ and let μ be Lebesque measure. Let $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| > 1$. Then f is 2-monotone and $Tf(x) = f(x) = 0$ for $|x| > 1$, but $Tf(x) = 1/2 \neq f(x)$ for $|x| \leq 1$.

Theorem 5.3.2 is false if the domain of f is not bounded.

Example 3. ($T^{\omega}f$ does not exist.)

Let $X = Z$ and μ be counting measure. If $f(x) = 0$ for x even and $f(x) = 1$ for x odd, then for any even integer r, $T^n f = f$ if n is even and $T^n f = 1 - f$ for n odd.

Example 4. ($T^{\omega}f$ exists but is not r-monotone.)

Let X and f be as in example 3. If r is an odd integer, then $T^{\omega}f = f$.

Example 5. ($T^n f$ is not a fixed point for any n, but $\lim_{n \to \infty} T^n f(x)$ exists for each x.)

Let $X = Z$ and let $f(x) = 1$ for positive even integers and $f(x) = 0$ otherwise. Then for each x in X there is a number n such that $T^m f(x) = 0$ for all $m > n$, but $T^n f$ is not a fixed point for any n.

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