AN ABSTRACT OF THE THESIS OF

PHILLIP E. PARKER for the degree of DOCTOR OF PHILOSOPHY

in MATHEMATICS presented on April 28, 1977

Title: SINGULAR GEOMETRY AND GEOMETRIC SINGULARITIES

Abstract approved: Redacted for Privacy

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In this thesis the author extends the standard techniques of differential geometry to the case of distributional (in the sense of Schwartz) pseudoriemannian structures. A new definition of flows for distributional vector fields is developed using generalized paths which are essentially nonlinear-distributional measures, and an existence theorem is proved when the coefficients are distribution extensions of real meromorphic functions. Conjugate points are defined by means of a new characterization in the smooth case, as the caustic set of the geodesic flow. The theory is applied to obtain a new interpretation of general relativity in which the Einstein equations do not break down at spacetime singularities. Fourier Integral Operator theory is used to study the propagation of spacetime singularities and the solvability of the Einstein equations in a class of algebraically special spacetimes. Explicit computations for the Schwarzschild structure show it is (modulo constants) a partial fundamental solution of the Einstein equations in the sense of partial differential equations.

Singular Geometry and Geometric Singularities

by

Phillip E. Parker

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1977

APPROVED:

Redacted for Privacy

Associate Professor of Mathematics

in charge of major

Redacted for Privacy

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Date thesis is presented April 28, 1977

Typed by Clover Redfern for Phillip E. Parker

ACKNOWLEDGMENTS

It gives me great pleasure to thank those whose influence is present, in some way, in this thesis.

I owe a great debt to my parents, who have consistently supported me in my search. Their contribution may not be readily apparent in this work, but it would not exist were it not for their unbounded love, affection and understanding.

To Cecil McDermott is due my eternal gratitude for awakening in me a love of mathematics and for his remarkable patience with me over the years.

From Richard Crownover I learned that mathematics is doing. I thank him most warmly for bearing with me while I learned how.

The knowledge that mathematics lives and breathes and the feeling of life that accompanies its beauties I owe to Aldo Andreotti. His friendship is a continuing source of inspiration.

An especial appreciation must be extended Bent Petersen, from whom I have learned much and whose stimulating conversations during the preparation of the manuscript have contributed so much to whatever clarity is present. In particular, his valuable queries and comments have contributed to the accuracy of Chapter 3. It is a pleasure to thank such a friend.

My major professor, Frank Flaherty, has shown extraordinary patience and forebearing. He selflessly allowed me the freedom I needed: to seek for myself. His wise council and sage advice are in large measure responsible for any significance that resides herein. I would like to take this opportunity to express the depth of my gratitude to him.

Without Bernard Banks, Greg Fredricks and Don Solmon the convivial atmosphere of shared mathematics would not have been the same. To these friends I give thanks. But without Gary Greenfield it would not have continued to exist. To this special friend I give most special thanks.

Finally I must acknowledge my wife, Barb, who has put up with me throughout the entire ordeal while my moods vacillated with my progress, or lack thereof. I am grateful for her faith.

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SINGULAR GEOMETRY AND GEOMETRIC SINGULARITIES

INTRODUCTION

The discovery of stars in various stages of gravitational collapse and the urge to explain the beginning (and possible end) of the universe has caused a renaissance in Einstein's general theory of relativity. It still seems to be the best candidate for a general explanation of gravitation, based on the experimental evidence to date and the weakness of competing theories. One frustrating aspect of general relativity at present is that it seems to predict that we can no longer predict. The theory apparently breaks down when it attempts to explain a completely collapsed star (black hole) or the big bang (naked singularity). These breakdowns are examples of what the physicists call "spacetime singularities." Thus the first problem that arises is to find a precise mathematical definition of a singularity.

At first it was thought that the known exact solutions of the Einstein equations had too much symmetry built into them and that generically singularities should not occur. Using geodesic incompleteness as the means of identifying smooth spacetimes from which singular points had been excised, Hawking and Penrose proceeded to prove some rather deep theorems to the effect that any physically reasonable spacetime must be geodesically incomplete [see e. g. 13]. So the Hawking-Penrose theorems together with the known exact solutions reveal that the geometric objects (structure or metric, geodesics, sectional curvature, etc.) should carry the singular information. But now the singularity is totally inaccessible, either at the end of an incomplete geodesic or at a place where curvature blows up. Because the singularity is the object we want to study, this effectively makes direct analysis impossible.

In order to reconcile such behavior, one is led to the use of distributional (in the sense of Schwartz) structures. Historically, the use of distributions has provided enormous insight in the study of singular behavior in physical phenomena. (Prime examples are the wave equation and the vibrating string or membrane.) One was immediately able to obtain distributional solutions of the relevant equations whose smoothness properties were subsequently established by regularity theorems (e.g., elliptic operators; hyperbolic Cauchy problem) or approximation methods (e.g., Sobolev techniques; Korn trick). It should be noted that distribution theory put Heaviside's operational calculus on a firm mathematical foundation. In this light, the use of distributional structures leads to clarification of ideas in the study of spacetime singularities, and places this study on a firm mathematical foundation. In summary, distributions live on smooth manifolds but need not be smooth creatures themselves, and arise naturally as solutions of partial differential equations.

The methods used are analogous to those of potential theory and geometric optics. If singularities are inevitable, as the Hawking-Penrose theorems indicate, allowance must be made for them from the beginning. Thus in Chapter 1 I collect some background material on distributions which is basic to all that follows. In particular, Section 1.2 contains the fundamental notion of WF sets [19], a refinement of the notion of singular support which eliminates the differences between local and global statements, and Section 1.3 contains a construction of flows for distributional vector fields.

In Chapter 2 a general study of distributional pseudoriemannian structures is begun, including a characterization of conjugate points which seems to be new even in the smooth case. A class of structures is presented for which all standard differential geometric constructions are valid: namely, those whose components are distribution extensions of real meromorphic functions (quotients of real analytic functions).

Since Fourier Integral Operators will be needed later, a very brief and somewhat sketchy review is included for the convenience of the reader. One whose interest is awakened by Chapter 3 should consult the seminal papers of Hormander [19] and Duistermaat and Hormander [7] for a complete exposition of the theory.

All the preceding material is then applied to general relativity in Chapter 4. A class of algebraically special spacetimes is described, the Kerr-Schild spacetimes [22], which is shown to include all the classical black hole spacetimes: Schwarzschild, Reissner-Nordström, Kerr and charged Kerr structures. The full power of Fourier Integral Operator theory is invoked to show that, in a sense made precise in Section 4.3, all the classical black holes propagate through spacetime in the same way. In addition, a new compatibility condition on candidates for stress-energy tensors is derived. Finally a comparison is made with the Hawking-Penrose singularity theorems, and it is shown that the present notion of geometric singularities includes theirs.

In the last chapter I first discuss how this theory shows that the Einstein equations do not break down in the presence of singularities. A beginning is made towards the complete analysis of propagation of singularities in Kerr-Schild structures on $\mathbb{R}^4$, and then some computations are made which show that the Schwarzschild structure is a partial fundamental solution of the Einstein equations.

Some last remarks are in order. The appearance of real meromorphic functions in a key place may lead one to suppose that the theory of hyperfunctions might also be profitably applied. An exposition of the theory may be found in [34] and a propagation of singularities theorem was proved recently [3]. Roughly speaking, hyperfunctions are a generalization of distributions and are somewhat more suitable for use with operators having analytic coefficients. In general they are defined as certain cohomology classes, but in $\mathbb{R}$ they may be most profitably regarded as boundary values of functions holomorphic in the upper half-plane. As for the inevitability of distributional structures, Isham has recently argued [20] that one must look at the superspace of distributional structures in order to obtain a consistent quantization of general relativity. Together with Souriau's work on geometric quantization [36], it seems that distributional structures have been forced on us. Fortunately, as this thesis shows, they are rather pleasant and in some ways more fruitful for geometric intuition than smooth ones.

Throughout, smooth will mean C^∞, manifolds will be smooth and paracompact, and fiber bundles will have a smooth total space, base space and projection. C^∞ and C_c^∞ (occasionally $\mathcal{E}_{\mathbb{R}}$ and $\mathcal{D}_{\mathbb{R}}$) will be real valued, $\mathcal{E}$ and $\mathcal{D}$ complex valued. The symbol $:=$ in an equation means the left hand side is defined to be the right hand side. The symbol $\square$ denotes the end of a proof or the omission of one (Halmos finality symbol).

1. DISTRIBUTIONS ON MANIFOLDS

1.1 Generalities on α-Distributional Sections

I will explicitly treat the real case, with comments on the complex case (which will be needed in any attempt to combine this theory with that of geometric quantization).

Definition. Let E be an m-dimensional real vector space and $\alpha \in \mathbb{R}$. $|\wedge^m|^\alpha E$ is the (1-dimensional) vector space of all real-valued functions f on $GL(E)$ such that $f(x) = |\det y|^\alpha f(z)$ whenever $x, y, z \in GL(E)$ and $z = yx$.

One may also use complex-valued functions to define $|\wedge^m|^\alpha_{\mathbb{C}} E$. Observe that if $f_i \in |\wedge^m|^{\alpha_i} E$, $i = 1, 2$, then $f_1 f_2 \in |\wedge^m|^{\alpha_1+\alpha_2} E$ (and $\bar{f}_i \in |\wedge^m|^{\alpha_i}_{\mathbb{C}} E$ in the complex case). If $f \geq 0$, $f \in |\wedge^m|^\alpha E$, one can define $f^\beta \in |\wedge^m|^{\alpha\beta} E$ for all $\beta \in R$ in the obvious way. $|\wedge^m|^\alpha E$ is the space of α-densities on E.

If E is now an m-plane bundle over an n-manifold X, one can form the line bundle $|\wedge^m|^\alpha E$. In particular, $X^\alpha := |\wedge^n|^\alpha T^*X$ is the bundle of α-densities over X and its space of smooth sections $\Omega_\alpha := \Gamma(|\wedge^n|^\alpha T^*X)$ will be called the space of α-densities on X. Note that 0-densities are merely smooth functions ($X^0 = \mathbb{1}$, the trivial line bundle) and that $|\wedge^n|^1 T^*X$ is the usual volume bundle. Densities provide the basis for integration without having to worry about orientation. A related notion was used by deRham [24]. Lately, $\frac{1}{2}$-densities have been of fundamental importance in Fourier Integral Operator theory [19, 7] and geometric quantization [24, 36]. By using tensor products one can define α-density sections of a vector bundle E over X.

Definition. The space of α-density sections of E is $\Gamma_\alpha(E) := \Gamma(E \otimes X^\alpha)$.

If (X, β) is a pseudoriemannian manifold, $|\det \beta|^{\alpha/2}$ defines a canonical α-density which trivializes X^α. Thus in general all the bundles X^α are trivial, but not naturally so.

Definition. The geometric α-dual of E is $E'_{1-\alpha} := E^* \otimes X^{1-\alpha}$. Similarly $E_\alpha := E \otimes X_\alpha$. The space of α-distributional sections of E is $\mathcal{B}'_\alpha(E) := (\mathcal{B}(E'_{1-\alpha}))'$ where $\mathcal{B}$ denotes smooth sections with compact support. Similarly one defines α-distributional sections with compact support $\Gamma'_\alpha(E) := (\Gamma(E'_{1-\alpha}))'$.

If U is an open subset of X and $s \in \mathcal{B}'_\alpha(E)$, I shall say that s vanishes on U whenever $s|U := s|\mathcal{B}(E'_{1-\alpha}|U) = 0$. If U is the largest open set with $s|U = 0$ then the support of s is $\operatorname{supp} s := X \setminus U$. One may consider $\Gamma'_\alpha(E) \to \mathcal{B}'_\alpha(E)$ so that s has compact support iff $s \in \Gamma'_\alpha(E)$. There is also a natural inclusion $\Gamma_\alpha(E) \to \mathcal{B}'_\alpha(E)$. To avoid confusion now and hereafter I shall use

$(\ ,\ )$ for fiberwise contractions and $\langle\ ,\ \rangle$ for the action of distributions. Thus for $s \in \Gamma_\alpha(E)$ and $\varphi \in \mathcal{D}(E'_{1-\alpha})$ one has $\langle s, \varphi\rangle := \int_X (s, \varphi)$. So s is <u>smooth on the open set</u> U iff $s|U \in \Gamma_\alpha(E|U)$, and sing supp $s := X \setminus U$ iff U is the largest open set on which s is smooth.

<u>Definition.</u> Define a multiplication $\Omega_\beta(X) \times \mathcal{D}'_\alpha(E) \to \mathcal{D}'_{\alpha+\beta}(E)$ by

$$\langle fs, \varphi\rangle := \langle s, f\varphi\rangle$$

for all $\varphi \in \mathcal{D}(E'_{1-\alpha-\beta})$, where $f \in \Omega_\beta$ and $s \in \mathcal{D}'_\alpha(E)$.

Observe that this product is continuous and that $\mathcal{D}'_\alpha(E)$ is an $\Omega_\beta(X)$-module.

<u>Definition.</u> Define a multiplication $\mathcal{D}'_\alpha(\mathbb{1}) \times \Gamma_\beta(E) \to \mathcal{D}'_{\alpha+\beta}(E)$ for $f \in \mathcal{D}'_\alpha(\mathbb{1})$ and $s \in \Gamma_\beta(E)$ by

$$\langle fs, \varphi\rangle := \langle f, (s, \varphi)\rangle$$

for all $\varphi \in \mathcal{D}(E'_{1-\alpha-\beta})$.

As usual, one may think of α-distributional sections of E as sections with α-distributional coefficients.

Let $\Gamma^0_\alpha(E)$ be the continuous sections of E with α-density coefficients (and similarly for $\Gamma^k_\alpha(E)$). Since the fiberwise contraction (s, φ) for $s \in \Gamma^0_\alpha(E)$ and $\varphi \in \mathcal{E}(E'_{1-\alpha})$ is a continuous 1-density with compact support, the equation

$$\langle s, \varphi \rangle := \int_X (s, \varphi)$$

defines a natural inclusion $\Gamma^0_\alpha(E) \to \mathcal{E}'_\alpha(E)$. More generally, if $s \in \mathcal{E}'_\alpha(E)$ I shall say that $s \in L^p_\alpha(E, \mathrm{loc})$ provided that $s|U$ has coefficients which are L^p-local α-densities in each chart U. Since the notion of null set is invariant, it is clear that $L^p_\alpha(E, \mathrm{loc})$ may be considered as equivalence classes of α-density sections of E, two such sections being equivalent iff they differ only on a null set. (Recall that the notion of sets of Legesgue measure zero or <u>null sets</u> in $\mathbb{R}^n$ is invariant under change of coordinates, hence is well-defined for manifolds.)

Now let E and F be vector bundles over X.

Recall that $L(E, F)$ is the bundle whose fiber over $x \in X$ is $L(E_x, F_x)$, the space of linear maps $E_x \to F_x$. A <u>morphism</u> $E \to F$ is an element of $\Gamma(L(E, F))$.

Let $\theta : E \to F$ be a morphism. There is an induced $\theta_\alpha : E_\alpha \to F_\alpha$ given by $\theta_\alpha := \theta \otimes 1 : E \otimes X^\alpha \to F \otimes X^\alpha$ and an induced $\theta'_\alpha : F'_\alpha \to E'_\alpha$ via $\theta'_\alpha := \theta^* \otimes 1 : F^* \otimes X^{1-\alpha} \to E^* \otimes X^{1-\alpha}$. θ'_α is the <u>geometric α-transpose</u> of θ and may also be defined by

$(\theta'_\alpha u, e) := (u, \theta_\alpha e)$ for all $u \in (F'_{1-\alpha})_x$, $e \in (E_\alpha)_x$ and $x \in X$. Now θ'_α induces a continuous linear map $\theta'_\alpha : \mathcal{B}(F'_{1-\alpha}) \to \mathcal{B}(E'_{1-\alpha})$ and its algebraic transpose $\theta_\alpha : \mathcal{B}'_\alpha(E) \to \mathcal{B}'_\alpha(F)$ clearly extends the map $\theta_\alpha : \Gamma^0_\alpha(E) \to \Gamma^0_\alpha(F)$ induced by θ. One can also extend differential operators to α-distributional sections. Let $\mathrm{Diff}_m(E, F)$ denote the differential operators of order m from sections of E to sections of F [31].

Ex. Let $\bigwedge T^*X := \oplus_k \bigwedge^k T^*X$, where $\bigwedge^k T^*X$ is the k-fold exterior product of T^*X. Then $\Omega^k := \Gamma(\bigwedge^k T^*X)$ is the space of smooth k-forms on X and $\Omega^* := \oplus_k \Omega^k$ the space of smooth forms. If d denotes the exterior derivative then $d \in \mathrm{Diff}_1(\Omega^*, \Omega^*)$, and if $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the smooth vector field $\xi \in \mathfrak{X}$ then $\mathcal{L}_\xi \in \mathrm{Diff}_1(\Omega^k, \Omega^k)$ for each k. The Laplace-Beltrami operator on a Riemannian manifold (cf. Section 3.2) and the Einstein operator considered in Section 4.3 are examples of second order differential operators.

Proposition 1. If $P \in \mathrm{Diff}_m(E_\alpha, F_\beta)$ then there is a unique $P' \in \mathrm{Diff}_m(F'_{1-\beta}, E'_{1-\alpha})$ such that

$$\int_X (P'u, s) = \int_X (u, Ps)$$

for all $u \in \Gamma(F'_{1-\beta})$ and $s \in \mathcal{B}_\alpha(E)$.

Proof: $s \mapsto \int_X (u, Ps)$ is a continuous linear functional on $\mathcal{B}_\alpha(E)$ for each $u \in \Gamma(F'_{1-\beta})$. Thus there is a unique $\mathbb{R}$-linear map $L : \Gamma(F'_{1-\beta}) \to \mathcal{B}'_{1-\alpha}(E^*) = (\mathcal{B}(E_\alpha))'$ such that $\langle Lu, s\rangle = \int_X (u, Ps)$. By computing Lu locally it is easy to see that L is a differential operator of order $\leq m$. []

Definition. P' is the geometric α-transpose of P.

It follows that $P' : \mathcal{B}(F'_{1-\beta}) \to \mathcal{B}(E'_{1-\alpha})$ is continuous so that its algebraic transpose $P : \mathcal{B}'_\alpha(E) \to \mathcal{B}'_\beta(F)$ is an $\mathbb{R}$-linear map which extends P as originally given to α-distributional sections.

Notation. When $\alpha = 0$ it will be dropped as a subscript. Similar to what is usually done, $\mathcal{E}'_\alpha(X) := \Gamma'_\alpha(\mathbb{1} \otimes \mathbb{C})$, $\mathcal{D}_\alpha(X) := \mathcal{B}_\alpha(\mathbb{1} \otimes \mathbb{C})$, etc. Thus $\mathcal{D}'(X)$ is the space of (complex) distributions or even 0-currents and $\mathcal{B}'_1(\mathbb{1})$ is the space of (real) twisted distributions or odd n-currents. $\Omega^k_\alpha(X) := \Gamma_\alpha(\Lambda^k T^*X)$, $\mathfrak{X}'_\alpha := \mathcal{B}'_\alpha(TX)$, etc.

Ex. The preceding extension process provides an exterior derivative for distributional forms and a Lie derivative for distributional tensors.

Moreover, one can extend the interior product or contraction and the usual relations among d, $\mathcal{L}$ and $\lrcorner$ continue to hold.

Proposition 2. Let $\xi \in \mathfrak{X}$ with flow c. For each $\omega \in \Omega'$

$$c_t^* \mathcal{L}_\xi \omega = \frac{d}{dt} c_t^* \omega ;$$

in particular $\mathcal{L}_\xi \omega = 0$ iff $\omega = c_t^* \omega$ whenever c_t is defined. []

Corollary 3. In this case, $c_t^* \delta_x = \delta_{c_t(x)}$ and $\xi(x) = 0$ iff $c_t^* \delta_x = \delta_x$ iff $\mathcal{L}_\xi \delta_x = 0$ where δ_x is the Dirac delta at $x \in X$. []

Observe that exterior multiplication of α-distributional forms by smooth β-density forms is also well-defined, but that exterior multiplication of α- and β-distributional forms suffers the same problems as multiplication of distributions. Whenever they are well-defined, the usual formulas relating $\wedge$ to d, $\mathcal{L}$ and $\lrcorner$ continue to hold.

One can extend the Lie derivative to $\xi \in \mathfrak{X}'$ provided that the tensor being differentiated is smooth. If t is a form, for example, one can use the equation $\mathcal{L}_\xi t = \xi \lrcorner dt + d(\xi \lrcorner t)$, and then a standard argument determines $\mathcal{L}_\xi$ on the full tensor algebra. In particular, Lie brackets are well-defined provided one of the vector fields is smooth. Subject to this restriction the usual relations continue to hold for this extended Lie derivative. If a volume element is given the divergence may be defined as usual with the Lie derivative, but now it may be a distribution.

1.2 Pullbacks, Pushouts and WF Sets

Let X and Y be manifolds and $F : X \to Y$ a smooth map. Recall that $\mathcal{E}(X)$ denotes the smooth complex-valued functions on X and $\mathcal{D}(X)$ those with compact support. Then F induces a continuous linear map $F^* : \mathcal{E}(Y) \to \mathcal{E}(X) : \varphi \mapsto \varphi \circ F$. If F is proper then $F^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ continuously. Taking transposes, one obtains continuous linear maps

$$F_* : \mathcal{E}_1'(X) \to \mathcal{E}_1'(Y) \qquad \text{for } F \text{ smooth,}$$
$$F_* : \mathcal{D}_1'(X) \to \mathcal{D}_1'(Y) \qquad \text{for } F \text{ proper,}$$

the <u>pushouts along</u> F. If one restricts F_* to 1-densities then $F_* : \mathcal{D}_1(X) \to \mathcal{E}_1'(Y)$ continuously.

<u>Proposition 1</u>. If F is a submersion then $F_* : \mathcal{D}_1(X) \to \mathcal{D}_1(Y)$ continuously.

<u>Proof</u>: Let $f \in \mathcal{D}_1(X)$. By definition,

$$\langle F_* f, \varphi \rangle = \langle f, \varphi \circ F \rangle = \int_X (\varphi \circ F) f$$

for each $\varphi \in \mathcal{E}(Y)$. Since f has compact support, by a partition of unity argument $f = \Sigma_{i=1}^N f_i$ with $\operatorname{supp} f_i$ and $\operatorname{supp} F_* f_i$ contained in suitable coordinate charts. By virtue of the local nature of submersions it suffices to consider $F : \mathbb{R}^n \to \mathbb{R}^m$ ($n = \dim X$, $m = \dim Y$) being projection on the first m coordinates.

Now if $f \in \mathcal{D}_1(\mathbb{R}^n)$ then $f = \psi dx$, where $\psi \in \mathcal{D}(\mathbb{R}^n)$ and dx is Lebesgue measure. Letting $x_1 = (x^1, \ldots, x^m)$ and $x_2 = (x^{m+1}, \ldots, x^n)$, for each $\varphi \in \mathcal{E}(\mathbb{R}^m)$

$$\begin{aligned} \langle F_* f, \varphi \rangle &= \langle f, \varphi \circ F \rangle \\ &= \int_{\mathbb{R}^n} \varphi(x_1)\psi(x)dx \\ &= \int_{\mathbb{R}^m} \varphi(y)\tilde{\psi}(y)dy , \end{aligned}$$

where $\tilde{\psi}(y) = \int_{\mathbb{R}^{n-m}} \psi(y, x_2)dx_2$, whence $F_* f = \tilde{\psi}dy \in \mathcal{D}_1(\mathbb{R}^m)$. $\square$

Hence there is a <u>pullback</u> $F^* : \mathcal{D}'(Y) \to \mathcal{D}'(X)$ which is continuous (e.g. weak $*$). Notice that for $f \in \mathcal{D}_1(X)$, $F_* f$ is essentially obtained by "integration along the fibers."

<u>Proposition 2</u>. If F is a submersion $X \to Y$ then the pullback $F^* : \mathcal{E}(Y) \to \mathcal{E}(X)$ has a unique continuous linear extension $\mathcal{D}'(Y) \to \mathcal{D}'(X)$ given by the preceding pullback of distributions.

<u>Proof</u>: Since any element of $\mathcal{D}_1(Y)$ which annihilates $\mathcal{E}(Y)$

is zero, $\mathcal{E}(Y)$ is weak * dense in $\mathcal{D}'(Y)$ by Hahn-Banach. So uniqueness is clear.

For existence, let $\varphi \in \mathcal{E}(Y)$ and $f \in \mathcal{D}_1(X)$. Observe that one may regard $F_* f \in \mathcal{E}_1'(Y)$. Thus

$$<F^*\varphi, f> = <\varphi, F_* f> = <\varphi \circ F, f>$$

where $F^*\varphi$ is defined by Proposition 1, therefore $F^*\varphi = \varphi \circ F$. []

Now let $f \in \mathcal{D}_\alpha'(X)$.

Definition. Let $x_0 \in X$, $\xi_0 \in T^*_{x_0} X \setminus 0$. Then $\xi_0 \notin WF(f)$ iff for each $\varphi_0 \in C^\infty(X)$ with $d\varphi_0(x_0) = \xi_0$ there exists a neighborhood V of φ_0 in $C^\infty(X)$ and an open neighborhood U of x_0 in X such that for each integer $N \geq 0$, each $g \in \mathcal{D}_{1-\alpha}(U)$ and each bounded set $B \subseteq V$,

$$<e^{-it\varphi} f, g> = O(t^{-N})$$

as $t \to \infty$, uniformly for $\varphi \in B$. Here O is the usual "big oh."

Note that if $\theta \in \mathcal{E}_\gamma(X)$ then $\theta f \in \mathcal{D}'_{\alpha+\gamma}(X)$ and $WF(\theta f) \subseteq WF(f)$, and that if θ never vanishes there exists $\theta^{-1} \in \mathcal{E}_{1-\gamma}(X)$ with $\theta\theta^{-1} = \theta^{-1}\theta = 1 \in \mathcal{E}_0(X)$. Hence for such θ, $WF(\theta f) = WF(f)$. Thus to study WF sets it suffices to consider $f \in \mathcal{D}'(X)$, and in this case the definition clearly agrees with the usual ones [19].

From the local theory [19] it follows that:

1) $WF(f)$ is a closed conic set in $T^*X \setminus 0$;

2) if $U \subseteq X$ is open then $WF(f|U) = WF(f)|U$;

3) $\pi WF(f) = \text{sing supp } f$, π the natural projection $T^*X \to X$.

(Recall that $C \subseteq T^*X \setminus 0$ is <u>conic</u> iff for each $\xi \in C$ and all $s > 0$, $s\xi \in C$.)

In a similar way one now defines $WF(s)$ when $s \in E'_\alpha(E)$ where E is a vector bundle over X. The three properties are then also true for $WF(s)$.

<u>Remark</u>. Observe that if ω is the symplectic form on T^*X then ω^n is nonvanishing and defines a canonical positive 1-density $\mu := \frac{1}{n!}(-1)^{n(n-1)/2}\omega^n$. In a canonical chart μ is represented by 1; it is called the <u>canonical volume element</u> (or <u>phase volume</u>) on T^*X. It may be used to identify $\mathcal{D}'(T^*X)$ and $\mathcal{D}'_\alpha(T^*X)$.

1.3 Flows

Let $C^\infty_{ploc}(\mathbb{R}, X)$ be the set of all proper smooth maps $I_p \to X$ where I_p is an interval containing 0. Define an action of this space on $\mathcal{D}(X)$ by

$$\langle p, \varphi \rangle := \int_{I_p} (p \circ \varphi)(t)\,dt. \qquad (1)$$

This allows the consideration of $C^{\infty}_{ploc}(\mathbb{R}, X)$ as a subset of $\mathcal{D}'_1(X)$.

Definition. P(X), the space of generalized proper paths in X, is the sequential closure of $C^{\infty}_{ploc}(\mathbb{R}, X)$ in $\mathcal{D}'_1(X)$.

If one thinks of a proper smooth flow as a collection of proper smooth paths, then one can define a generalized (or distributional) proper flow as a sequential limit of proper smooth flows (whenever the limit exists). It is clear from the definitions that the generalized proper flow of a distributional vector field, when it exists, is independent of the choice of a sequence of smooth vector fields converging in $\mathcal{X}'$ to the distributional vector field.

More generally, one can define $P_K(X)$ to be the sequential closure of $C^{\infty}_{ploc}(K, X) = \{p \in C^{\infty}_{ploc}(\mathbb{R}, X);\ I_p \subseteq K\}$ for each compact $K \subseteq \mathbb{R}$ with $0 \in K$. Then $\hat{P}(X) = \text{proj}\lim_K P_K(X)$ is a countable inverse (projective) limit since $\mathbb{R}$ is σ-compact and one may think of elements of $\hat{P}(X)$ as sequences in $\mathcal{D}'_1(X)$ which may not converge in $\mathcal{D}'_1(X)$. The utility of $\hat{P}(X)$ is that it allows one to consider complete smooth flows with not all paths proper, and their sequential limits. The appearance of inverse limits is neither mysterious nor surprising when one recalls that a smooth function on $\mathbb{R}^n$ is determined by its restrictions to compact sets, which merely says in words that $C^{\infty}(\mathbb{R}^n) = \text{proj}\lim_K C^{\infty}(K)$ as K runs over the compact sets in $\mathbb{R}^n$. If $p = (p_1, p_2, p_3, \ldots) \in \hat{P}(X)$ then the p_i are called the representatives of p. In order that they be well-defined, I shall henceforth use the sequence $K_i = [-i, i]$, $i \geq 1$, so that p_i is the representative of p in $P_{K_i}(X)$.

One can also inject $C^{\infty}_{ploc}(\mathbb{R}, X)$ into $\mathcal{D}'_1(TX)$, by using $\dot{\varphi}$ (the natural life of φ) instead of φ in (1) and taking $f \in \mathcal{D}(TX)$, and form its sequential closure $P^{\cdot}(X)$. There is also the accompanying $\hat{P}^{\cdot}(X)$, defined in the obvious way, which contains the generalized vector fields along generalized paths, in particular the generalized velocity field $\dot{p}$ of a generalized path $p \in \hat{P}(X)$. If $p = (p_1, p_2, \ldots) \in \hat{P}(X)$ define $\operatorname{supp} p := \cup_i \operatorname{supp} p_i$ so that $\operatorname{supp} p = \pi(\operatorname{supp} \dot{p})$. Supp p may be thought of as the geometric generalized path corresponding to p. Observe that one may consider p_i as a 1-distributional vector field on X with support $\operatorname{supp} p_i$. Thus if p is a generalized integral curve of $\xi \in \mathcal{X}'$ and X is provided with a distinguished 1-density, so that one may identify $\mathcal{X}'$ and $\mathcal{X}'_1$, then $\xi | \operatorname{supp} p_i = \dot{p}_i$ whenever the restriction is defined (roughly, if $WF(\xi) \cap N(\operatorname{supp} p_i) = \emptyset$).

Intuitively, it is now clear how to define generalized local 1-parameter groups on X, although the lack of a suitable manifold structure on the space of local diffeomorphisms effectively stops a direct assault. Pullbacks can be handled as follows however. Let $\{\xi^n\}$ be a sequence in $\mathcal{X}$ converging in $\mathcal{X}'$ to ξ and let p^n

be the flow of ξ^n. If $p^n(x)$ is the maximal integral curve of ξ^n through x, $p(x)$ will denote the (maximal) generalized integral curve of ξ through x. I shall say that p is the <u>generalized flow</u> of ξ. If $p = (p_1, p_2, \ldots) \in \hat{P}(X)$ let $\{p_i^n\}$ be a sequence of proper restrictions of p^n such that $p_i^n \underset{n}{\to} p_i$ in $P(X)$.

<u>Definition</u>. If $\alpha \in \Omega'$ define $p_t^*\alpha := (\lim_n (p_i^n)_t^* \alpha)$, whenever each representative limit exists in Ω'.

<u>Proposition 1</u>. $\frac{d}{dt} p_t^*\alpha = p_t^* \mathcal{L}_\xi \alpha$ for $\alpha \in \Omega'$ and $\xi \in \mathcal{X}'$, where p is the generalized flow of ξ.

<u>Proof</u>: Choose $\{\xi^n\}$ in $\mathcal{X}$ with $\xi^n \to \xi$ in $\mathcal{X}'$, let p^n be the flow of ξ^n and choose p_i^n as above. Then

$$\begin{aligned} \frac{d}{dt} p_t^*\alpha &= (\lim_n \frac{d}{dt} (p_i^n)_t^* \alpha) \\ &= (\lim_n (p_i^n)_t^* \mathcal{L}_{\xi^n} \alpha) \\ &= ((p_i)_t^* \mathcal{L}_\xi \alpha) \\ &= p_t^* \mathcal{L}_\xi \alpha . \quad \square \end{aligned}$$

<u>Ex</u>. [27] Let $X = \mathbb{R}^2$ be q-space and $T^*X = \mathbb{R}^4$ be (q, p)-space as in mechanics. Denote by δ_1 the Dirac delta with respect to q^1. Consider the Hamiltonian function $E = \frac{1}{2}(p_1^2 + p_2^2) + \delta_1$.

From Hamilton's equations one obtains for $(q(t), p(t))$ an integral curve of H^E (Section 1.4), $\dot{q}^1(t) = p_1(t)$, $\dot{p}_1(t) = -\delta_1'$, $\dot{q}^2(t) = p_2(t)$, and $\dot{p}_2(t) = 0$. If, for later convenience, the initial condition (q_0, p^0) satisfies $q_0^1 > 0$, $q_0^2 < 0$, and $p_2^0/p_1^0 = -q_0^2/q_0^1 > 0$, then $q^2(t) = p_2^0 t + q_0^2$. To compute $q^1(t)$, recall that

$$\delta_1 = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + (q^1)^2}$$

[cf. e.g. 37] so that

$$-\delta_1' = \lim_{\varepsilon \downarrow 0} \frac{2}{\pi} \frac{\varepsilon q^1}{(\varepsilon^2 + (q^1)^2)^2}.$$

Thus

$$\ddot{q}^1 = \lim_{\varepsilon \downarrow 0} \frac{2}{\pi} \frac{\varepsilon q^1}{(\varepsilon^2 + (q^1)^2)^2}$$

whence, if we now fix $\varepsilon > 0$, q^1 is convex when $q^1 > 0$ and q^1 is concave when $q^1 < 0$. I shall now compute q^1 for fixed ε. Multiplying by $2\dot{q}^1$ and integrating with respect to t one obtains

$$(\dot{q}^1)^2 = (p_1^0)^2 - \frac{2}{\pi} \frac{\varepsilon}{\varepsilon^2 + (q^1)^2}.$$

Now $q^1 = 0$ implies $(\dot{q}^1)^2 = (p_1^0)^2 - \frac{2}{\pi\varepsilon}$, hence one may choose ε sufficiently small so that q^1 is never 0. Observe that for large q^1, $(\dot{q}^1)^2$ is asymptotic to $(p_1^0)^2$, hence by the convexity of q^1

the asymptotes in $\mathbb{R}^2$ are $q^2 = \pm(p_2^0/p_1^0)q^1$, $q^1 \geq 0$. If now we let $\varepsilon \downarrow 0$, then the curve $(q(t), p(t))$ consists of a ray coming in from infinity in the forth quadrant with slope $-p_2^0/p_1^0$, reflecting off the q^2-axis and then going off to infinity in the first quadrant with slope p_2^0/p_1^0. It should now be clear what happens with other initial conditions.

As another example one can take $E = \frac{1}{2}(p_1^2+p_2^2) + h(q^1)$, where h is the Heaviside function: $h(q^1) = \{0 \text{ if } q^1 < 0, 1 \text{ if } q^1 > 0\}$. By computing as in the previous example one may show that this flow is refraction by Snell's law across the q^2-axis.

The definition of generalized paths adopted here was directly inspired by L. C. Young's paper [40]. By relaxing the requirement that paths be <u>functions</u>, as opposed to some sort of distributions, the delicate measure-theoretic questions and orientability assumptions in [27] may be avoided. Instead, the assumption that the coefficients of the vector field are distribution extensions of real meromorphic functions suffices for existence. The argument uses the Łojasiewicz division theorem [26] and is straightforward. The special case of geodesic flows will be written out as 2.2.2, but it will be clear that only minor changes are necessary for the general case.

It should be noted that the generalized paths considered here are more general than currents [e.g. 8]. This was already true of

Young's paths. However the formalism here, treating paths as types of measures, is essentially that of currents.

1.4 Symplectic Geometry

Let (X, σ) be a symplectic manifold; i.e., X is a smooth manifold and σ is a nondegenerate 2-form on X.

Definition. $\xi \in \mathcal{X}^1$ is symplectic iff $\mathcal{L}_\xi \sigma = 0$.

Since $\mathcal{L}_\xi \sigma = d(\xi \lrcorner \sigma)$ it follows from the local Poincare theorem that ξ is symplectic iff $\xi \lrcorner \sigma = df$ locally for some $f \in \mathcal{B}'(\mathbb{1})$.

Definition. Let $f \in \mathcal{B}'(\mathbb{1})$. H^f given by $df = H^f \lrcorner \sigma$ is the Hamiltonian vector field of f.

Note that every Hamiltonian vector field is (globally) symplectic and that every symplectic vector field is locally Hamiltonian. Sometimes f is called the Hamiltonian distribution of H^f. It is unique modulo constants.

Definition. The Poisson bracket of $g \in C^\infty(X)$ and $f \in \mathcal{B}'(\mathbb{1})$ is $\{g, f\} = \sigma(H^g, H^f)$.

One of g and f must be smooth to avoid multiplication of distributions [cf. 19, Thm. 2.5.10].

It is easy to see that the basic relations remain valid.

Ex. $H^{\{g,f\}} = [H^f, H^g]$, $\mathcal{L}_{H^f}\sigma = 0$, $\{g,f\} = \mathcal{L}_{H^f} g$, etc.

Proposition 1. If p is the generalized flow of a symplectic vector field then $p_t^*\sigma = \sigma$.

Proof: Check that $p_t^*\sigma$ exists. □

Conservation of Energy. If $f \in \mathcal{E}'(\mathbb{I})$ and p is the generalized flow of H^f, then $p_t^* f = f$.

Proof: Check that $p_t^* f$ exists and apply Proposition 3.1. □

Recall that the symplectic manifolds (X, σ) have a natural volume element, the phase volume of mechanics, given by $\mu := \frac{1}{n!}(-1)^{n(n-1)/2}\sigma^n$ where $n = \frac{1}{2}\dim X$.

Liouville's Theorem. If p is the generalized flow of a distributional vector field then $p_t^*\mu = \mu$ whenever the left hand side is defined.

Proof: Apply Proposition 1 to the definition of μ. □

This provides the basic theorems needed to do mechanics in the distributional setting. The main application here will be to distributional pseudoriemannian structures.

Ex. Let β be a distributional pseudoriemannian structure (cf. Section 2.1) on X. Let ω be the canonical 2-form on T^*X so that (T^*X, ω) is a symplectic manifold. One may consider $\frac{1}{2}\beta \in \mathcal{D}'_{\mathbb{R}}(T^*X)$. Then the projections in X of the generalized integral curves of $H^{1/2\beta}$ are the (generalized) geodesics of β.

The projection or pushout (cf. Section 2) of $p \in \hat{P}^{\cdot}(X)$ is defined as follows. If $p = (p_1, p_2, \ldots)$ and each p_i has compact support then the sequence of pushouts of the p_i is well-defined in $\hat{P}(X)$. If p_j does not have compact support, choose $p_j^n \in \mathcal{D}_1(TX)_{\mathbb{R}}$ with $p_j^n \to p_j$ in $\mathcal{D}'_1(TX)_{\mathbb{R}}$ and define the pushout of p_j to be the limit of the pushouts of the p_j^n. The pushout thus defined is independent of the choice of $\{p_j^n\}$ [37].

2. SINGULAR GEOMETRY

2.1 Distributional Pseudoriemannian Structures

Let X be a smooth n-manifold.

Definition. A distributional pseudoriemannian structure β on X is a distributional symmetric (2,0)-tensor on X such that $\beta | X \setminus \text{sing supp } \beta$ is a (smooth) pseudoriemannian structure on $X \setminus \text{sing supp } \beta$.

Locally one can represent β by a matrix (β^{ij}) of distributions on X. Similarly to the smooth case, one may consider $\beta \in \mathcal{D}'_{\mathbb{R}}(T^*X)$ via $\langle \beta, g \rangle = \langle \pi^*\beta, \xi \otimes \xi g \rangle := \langle \pi^*\beta^{ij}, \xi_i \xi_j g \rangle$ locally, where π^* denotes the pullback and ξ_i is the ith component function in local coordinates. Here and henceforth the summation convention is in effect. Note that β is a (distributional) (2,0)-tensor rather than a (0,2)-tensor. As long as β can be inverted (cf. Theorem 2.2) there is no need. However Hamiltonian mechanics (cf. the following proposition) and Fourier Integral Operator theory (see especially the discussion at the end of Section 3.2) strongly indicate that the most natural environment of a structure is the cotangent bundle, and indeed require it if β is not invertible.

Proposition 1. If β is a smooth pseudoriemannian structure on X then the geodesics of β are the projections in X of the integral curves of $H^{1/2\beta}$.

Proof: Let (U, x) be a chart and let (x, ξ) be the induced canonical coordinates on $T^*X|U$. Consider an integral curve (q, p) of $H^{1/2\beta}$ lying over U. Since $\beta(\xi) = \beta^{ij}\xi_i\xi_j$, it follows from Hamilton's equations that $\dot{q}^i = \beta^{ij}p_j$ and

$$\dot{p}_i = -\frac{1}{2}\frac{\partial\beta^{hk}}{\partial x^i}p_h p_k = \frac{1}{2}\frac{\partial\beta_{hk}}{\partial x^i}p^h p^k$$

since $\beta^{ij}\beta_{jk} = \delta^i_k$ implies

$$\frac{\partial\beta^{ij}}{\partial x^\ell}\beta_{jk} = -\beta^{ij}\frac{\partial\beta_{jk}}{\partial x^\ell}.$$

Now compute $\ddot{q}^i$:

$$\ddot{q}^i = \frac{d}{dt}(\beta^{ij}p_j) = \frac{d}{dt}\beta^{ij}p_j + \beta^{ij}\dot{p}_j$$

$$= \frac{\partial\beta^{ij}}{\partial x^k}\dot{q}^k p_j + \frac{1}{2}\beta^{i\ell}\frac{\partial\beta_{jk}}{\partial x^\ell}\dot{q}^j\dot{q}^k$$

$$= -\frac{1}{2}\left(\beta^{i\ell}\frac{\partial\beta_{\ell j}}{\partial x^k}\dot{q}^j\dot{q}^k + \beta^{i\ell}\frac{\partial\beta_{\ell k}}{\partial x^j}\dot{q}^j\dot{q}^k\right) + \frac{1}{2}\beta^{i\ell}\frac{\partial\beta_{jk}}{\partial x^\ell}\dot{q}^j\dot{q}^k$$

$$= -\Gamma^i_{jk}\dot{q}^j\dot{q}^k,$$

whence (q,p) is a geodesic. Conversely, if q is a geodesic, then reversing the preceding calculation shows that (q,p) satisfies Hamilton's equations, where $p_i = \beta_{ij}\dot{q}^j$, hence is an integral curve of $H^{1/2\beta}$. $\square$

Thus the following definition (as indicated in Section 1.4) will agree with the classical ones in the smooth case.

Definition. The (generalized) geodesics of a distributional pseudoriemannian structure β are the pushouts (or projections) of the generalized integral curves of $H^{1/2\beta}$ (considering $\beta \in \mathcal{D}'_{\mathbb{R}}(T^*X)$).

Due to their importance in geometry and in the singularity theorems of Hawking and Penrose, a notion of conjugate points is needed. Since the version given here seems to be new even in the smooth case, I shall review that first.

Let $x_0 \in X$. Recall that one of the characterizations of the conjugate locus $C(x_0)$ of x_0 is in terms of the exponential map at x_0. Precisely, $x \in C(x_0)$ iff x is a singular value of $\exp_{x_0} : T_{x_0}X \to X$. Now if β is a smooth pseudoriemannian structure on X, $\exp_{x_0}(v) = \pi p_1(v)$ for $v \in T_{x_0}X$ sufficiently near 0, where p is the flow of the vector field on TX which corresponds to $H^{1/2\beta}$ and $\pi : TX \to X$ the natural projection. For x to be a singular value of $\exp_{x_0}$ when $\exp_{x_0}(v) = x$ means that rank $\exp_{x_0 *} = \text{rank}\, \pi_* p_{1*} < n$ at v. Since p is the flow of a spray, $\pi p_1(v) = \pi p_t(\frac{1}{t} v)$ so that $\pi_* p_{1*}$ drops rank at v iff $\pi_* p_{t*}$ drops rank at $\frac{1}{t} v$, at least when $t \neq 0$. Thus conjugate points lie below precisely those points where $\text{rank}\, \pi_* p_{t*} < n$.

In order to obtain the characterization of conjugate points referred to above, a closer analysis of the map $\pi_* p_{t*}$ is necessary. For this purpose we may return to the cotangent bundle, so now $\pi : T^*X \to X$ and p is the flow of $H^{1/2\beta}$. It will be convenient to use some terminology from symplectic geometry, which will now be recalled. If (Y, σ) and (Z, τ) are symplectic manifolds, a smooth map $F : Y \to Z$ is a symplectomorphism iff $F^*\tau = \sigma$. If Z is now an arbitrary manifold with $\dim Z = \frac{1}{2} \dim Y$, an immersion $F : Z \to Y$ is Lagrangian iff $F^*\sigma = 0$, and if Y is the cotangent space of some manifold, F is conic provided $F(Z)$ is a conic subset of Y (cf. Section 1.2). The symplectic manifold of interest here is T^*X with symplectic form ω, the canonical 2-form on T^*X. Each fiber of TT^*X is a symplectic vector space; i.e., a vector space provided with a nondegenerate alternating bilinear form. A subspace of a symplectic vector space is Lagrangian iff the natural inclusion is a Lagrangian immersion.

Returning to the situation considered at the beginning of the preceding paragraph, it is easy to see that since p is the flow of a symplectic vector field p_t is a symplectomorphism. Observe that $T^*_{x_0}X$ is a Lagrangian submanifold of T^*X (which is conic). Hence $p_t(T^*_{x_0}X)$, when defined, will be a (conic) Lagrangian immersion in T^*X, with tangent space at $p_t(\xi)$ equal to $p_{t*}T_\xi T^*_{x_0}X$. In any case, one may think of $p_{t*}T_\xi T^*_{x_0}X$ as the germ of a Lagrangian submanifold at $p_t(\xi)$. Thus there is associated to each ξ in $T^*_{x_0}X$ a lifting λ_ξ of the integral curve through ξ to the Grassmann bundle of Lagrangian planes ΛTT^*X given by $\lambda_\xi(t) = p_{t*}T_\xi T^*_{x_0}X$. Therefore the point $x = \pi p_t(\xi)$ is conjugate to x_0 iff $\pi_*|\lambda_\xi(t)$ drops rank.

More generally, the caustic set of a Lagrangian immersion $\iota : L \to T^*X$ is the set of all $\pi\iota(\ell) \in X$ such that $\pi_*|\iota_*T_\ell$ drops rank, where $\ell \in L$. If p is the flow of a symplectic vector field on T^*X then the caustic set of the flow p out of $\iota(L)$ is $\{\pi p_t\iota(\ell);\ \ell \in L$ and $\operatorname{rank} \pi_*|p_{t*}\iota_*T_\ell L < n\}$. Summarizing the preceding arguments there is

Theorem 2. The conjugate locus of $x_0 \in X$ is the caustic set of the geodesic flow out of $T^*_{x_0}X$. []

According to [6; 21], the set $\{p_t\iota(\ell);\ \ell \in L$ and $\operatorname{rank} \pi_*|p_{t*}\iota_*T_\ell L < n\}$ is an unfolding in the sense of Thom, the unfolding of the caustic.

Ex. Caustics in geometric optics are located at those points which are conjugate to the light source along light rays (null geodesics).

Following the procedure given in Section 1.3, if β is a distributional pseudoriemannian structure and $p(\xi)$ is the generalized integral curve through $\xi \in T^*_{x_0} X$, one can associate $\lambda_\xi \in \hat{P}(\Lambda TT^*X)$ (recall that ΛTT^*X is closed in $G_n TT^*X$, the Grassmann bundle of n-planes). ΛTT^*X is a fiber bundle over X with projection also denoted π.

Definition. The conjugate locus of x_0 is $C(x_0) = \{\pi(\ell);\ \ell \in \operatorname{supp} \lambda_\xi$ and $\operatorname{rank} \pi_* | \ell < n\}$.

This means that conjugate points of distributional pseudoriemannian structures are limits (in the appropriate sense) of sequences of conjugate points of approximating smooth pseudoriemannian structures.

2.2 Connections and Curvature

We come now to the problem of constructing the familiar geometric objects associated to a pseudoriemannian structure. In order to motivate the general definitions I shall again review the smooth case first.

Continuing with the manifold X, let β be a smooth pseudo-riemannian structure on X and let θ be the canonical 1-form and ω the canonical 2-form on T^*X. To establish the sign conventions, recall that for $z \in T_\xi T^*X$ one may define $\langle\theta(\xi), z\rangle := \langle\xi, \pi_* z\rangle$ and $\omega = -d\theta$. Observe that for all $x \in X$, if $\iota : T^*_x \to T^*X$ is the natural inclusion then $\iota^*\theta = 0$, whence $T^*_x X$ is a conic Lagrangian submanifold of T^*X. Thus the bundle of vertical vectors is a Lagrangian subbundle of TT^*X.

Proposition 1. The bundle of horizontal vectors of the Levi-Civita connection is Lagrangian.

Proof: It suffices to work at $\xi \in T^*X$. Choose normal coordinates (u^i) at $\pi(\xi)$. Then (du^i) forms a coframe near $\pi(\xi)$ whose coefficients (v_i) together with the pullbacks along π of the normal coordinates, also denoted (u^i), form a system of canonical coordinates (u^i, v_i) near ξ. Since the $\partial/\partial u^i$ are horizontal and linearly independent at ξ it follows immediately that the horizontal space at ξ is Lagrangian. ▯

Recalling the coordinatization of Lagrangian subspaces of a symplectic vector space transverse to a given Lagrangian subspace as the symmetric $n \times n$ matrices [e.g. 1], it is easy to see that each choice of horizontal bundle gives rise to a vector bundle structure of the set of all Lagrangian subspaces transverse to the vertical subspaces over T^*X. In general, however, there is no natural vector bundle structure on this bundle.

In order to define the horizontal bundle of a distributional pseudoriemannian structure, some general constructions on fiber bundles are necessary. Let $\pi : E \to B$ be a fiber bundle and denote by $\Gamma(B,E)$, or just $\Gamma(E)$ when B is clear, the smooth sections. Observe that any smooth section is a proper map $B \to E$. If B has a preferred density μ, one can define an action of $\Gamma(E)$ on $\mathcal{D}(E)$ by $<f,\varphi> := \int_B (\varphi \circ f)\mu$, where $f \in \Gamma(E)$ and $\varphi \in \mathcal{D}(E)$. As in Section 1.3, this yields an injection $\Gamma(E) \to \mathcal{D}_1'(E)$ and I shall define $Б'(E)$, the <u>distributional sections</u> of E, to be the sequential closure of $\Gamma(E)$ in $\mathcal{D}_1'(E)$. If E is a vector bundle and instead of $\mathcal{D}(E)$ one uses $\mathcal{J}(E)$, the smooth functions whose support projects to a compact set in B and which are linear on the fibers, one recovers the usual distributional sections of E. Clearly these constructions can be extended to obtain α-distributional sections of E.

Now let β be a distributional pseudoriemannian structure on X and choose a sequence of smooth structures $\{\beta_n\}$ converging to β in the distribution topology. Let $\mathcal{H}_n$ be the field of horizontal spaces Levi-Civita associated to β_n, $\mathcal{H}_n \in \Gamma(\Lambda TT^*X)$. Recall that T^*X has a preferred density μ, the phase volume. I shall say that $\mathcal{H} \in Б'(\Lambda TT^*X)$ is the <u>field of horizontal spaces Levi-Civita</u>

associated to β iff $\mathcal{H}_n \to \mathcal{H}$ in $Б'(\Lambda TT^*X)$. Somewhat loosely, we may say that $\mathcal{H}$ provides ΛTT^*X with a distributional vector bundle structure.

If E is a smooth vector bundle over X, recall that a covariant derivative ∇ on E is a first order differential operator $\Gamma(E) \to \Gamma(L(TX, E))$; i.e., $\nabla \in \mathrm{Diff}_1(E, L(TX, E))$. According to Proposition 1.1.1 and the remark following it, one may consider $\nabla : Б'(E) \to Б'(L(TX, E))$. Continuing with β_n and β as before, let ∇_n be the Levi-Civita covariant derivative of β_n. I shall say that ∇ is the Levi-Civita covariant derivative of β iff $\nabla_n \to \nabla : D \to Б'(L(TX, E))$ where $D \subseteq Б'(E)$. D may not be all of $Б'(E)$ since ∇ will be a differential operator with distributional coefficients and the usual unpleasantness of products occurs. In any case, however, $\nabla : \Gamma(E) \to Б'(L(TX, E))$.

Note that by the nature of the definitions and constructions neither $\mathcal{H}$ nor ∇ depends on the choice of $\{\beta_n\}$, essentially since convergence always takes place in a distribution topology.

It should now be clear how to define any standard geometric object associated to β: take the objects associated to the β_n and see if, as a sequence, they converge in the appropriate distributional space. This leaves the question of existence open however. Before I give a theorem, let me recall two facts from differential topology. First, any smooth map between analytic manifolds can be approximated by analytic maps [11]. Second, any smooth (even of finite class) manifold has a compatible analytic structure [11; 15]. Thus the following theorem, although perhaps not esthetically satisfying, nevertheless covers the existence question in essence. In particular, it will more than suffice for the needs of this thesis.

Theorem 2. Let X be a manifold with a distributional pseudo-riemannian structure β. If the local components β^{ij} are distributional extensions of real meromorphic functions, then all standard geometric objects associated to β exist as distributional objects.

Proof: If $\beta^{ij} | X \setminus \text{sing supp}\, \beta = f/g$, where f and g are real analytic functions, then by the Łojasiewicz division theorem [26] there exists a distinguished [2] distribution T such that $gT = f$. Saying that T extends f/g, it follows easily that for any scalar differential operator P, PT extends $P(f/g)$. Moreover there is a distinguished S which extends g/f and the products $ST = TS$ are well-defined, in fact $ST = 1$. Similarly, the products $(PS)(QT)$, $(PT)(QT)$, etc., P and Q differential operators, are all well defined. Thus one can use the classical local tensor formulation to construct the desired geometric objects. []

Remark. Atiyah's paper [2] provides a most beautiful and elegant proof of the Łojasiewicz theorem via Hironaka's theorem on the resolution of singularities. As noted by Atiyah, this latter theorem should be of particular relevance to the study of characteristic varieties (cf. Section 3.1).

Definition. A distributional pseudoriemannian structure will be called a singular geometry iff all the standard associated geometric objects exist as distributional objects.

Ex. Any (X, β) satisfying the hypotheses of Theorem 2 is a singular geometry. Let (X, β) be a singular geometry. The Christoffel symbols are distributions on X:

$$\Gamma^{i}_{jk} = \frac{1}{2}\beta^{i\ell}\left(\frac{\partial\beta_{\ell k}}{\partial x^{j}} + \frac{\partial\beta_{\ell j}}{\partial x^{k}} - \frac{\partial\beta_{jk}}{\partial x^{\ell}}\right).$$

The covariant derivative of a smooth vector field u is

$$u^{i}_{;j} = \frac{\partial u^{i}}{\partial x^{j}} + \Gamma^{i}_{jk}u^{k},$$

where the u^{i} are smooth, hence is a distributional vector field. The canonical density $|\det(\beta^{ij})|^{1/2}$ is now a 1-distribution.

3. FOURIER INTEGRAL OPERATORS

3.1 Basic Properties

The classical method of solving differential equations was to find a parametrix, an operator which inverted the differential operator in question modulo C^∞ functions. Parametrices for elliptic operators are contained in a class of singular integral operators introduced by Mihlin [28] and Calderon and Zygmund [4] which later became known as pseudodifferential operators after Friedrichs and Lax [10]. Using alternate definitions, Kohn and Nirenberg [23], Hörmander [16] and others developed the full calculus for these operators. Returning to the singular integral approach, Hörmander [19] used oscillatory integrals to construct a class of operators he had called Fourier integral operators [18] in order to extend the parametrix construction to hyperbolic, parabolic and other differential operators. Historically, singular integrals go back at least to Poincaré's research on the oblique derivative problem. A signal contribution was that of F. Noether who studied Poincaré's work and, as a result of his efforts to fill in some gaps, obtained a special case of the Index Theorem in 1921. Hilbert transforms are also pseudodifferential operators, and the history of quantum theory is replete with singular integrals. See [35] for a more complete survey, especially of the early history.

In this section I shall briefly review the definitions and calculus of Fourier Integral Operators for the convenience of the reader [cf. 19; 7].

A phase function φ on X is an element of $C^{\infty}(X \times (\mathbb{R}^N \setminus 0))$ which is positively homogeneous of degree one in the second variable and has no critical points. φ is nondegenerate iff the forms $d(\partial\varphi/\partial\theta^j)$, $1 \leq j \leq N$, are linearly independent on $C_\varphi = \{(x,\theta); d_\theta\varphi(x,\theta) = 0\}$. Denote by L_φ the image of C_φ under $\iota_\varphi : C_\varphi \to T^*X \setminus 0 : (x,\theta) \mapsto (x, d_x\varphi(x,\theta))$.

Recall that $a \in \mathcal{E}(X \times \mathbb{R}^N)$, the smooth complex-valued functions, is a symbol of order m and type ρ, denoted $a \in S^m_\rho(X \times \mathbb{R}^N)$, iff for every compact $K \subseteq X$ and all multiindices α and β there exists a constant $C_{\alpha,\beta,K}$ such that for all $(x,\theta) \in X \times \mathbb{R}^N$,

$$|D^\beta_x D^\alpha_\theta a(x,\theta)| \leq C_{\alpha,\beta,K}(1+|\theta|)^{m-\rho|\alpha|+(1-\rho)|\beta|}$$

when $X \subseteq \mathbb{R}^n$ is open. This property is invariant under diffeomorphisms so $S^m_\rho(X \times \mathbb{R}^N)$ is well-defined when X is a manifold. One can extend this to $S^m_\rho(V, E)$ where V is a cone bundle over X and E is a vector bundle over V [19, Chapt. 1].

Let E, F be (complex) vector bundles over X, Y respectively; $\dim X = n$, $\dim Y = n'$. A Fourier Integral Operator A of order m and type ρ from sections of E_α (see Section 1.1) to sections of $F_{1-\alpha}$ is determined by a conic Lagrangian immersion (see Section 2.2) $L \to T^*(Y \times X)$ together with a <u>(principal) symbol</u>

$$a \in S_\rho^{m+\frac{n+n'}{4}}(L, F \otimes E^* \otimes K_{1-\alpha})$$

where F and E are regarded (via pullbacks) as sitting over L and K is the Keller-Maslov bundle over L (cf. the Appendix or [19, Chapt. 3]). We always have $m \in \mathbb{R}$, and when $\frac{1}{2} < \rho \leq 1$ the operator is continuous $\mathcal{E}_\alpha(E) \to \mathcal{E}'_{1-\alpha}(F)$. (If one allows non-homogeneous phase functions one may delete "conic" [6], but this introduces some unpleasant features.) The set of all such operators is denoted $I_\rho^m(E_\alpha, F_{1-\alpha})$.

Locally, the distribution kernel A of the operator A can be described as follows. Choose $\{U_j\}$ an atlas of $Y \times X$ which trivializes $F \otimes E$ such that for each U_j there exists an integer $N_j > 0$ and a nondegenerate phase function φ_j defined in an open conic $V_j \subseteq U_j \times (\mathbb{R}^{N_j} \setminus 0)$ with L_{φ_j} open in L and ι_{φ_j} a diffeomorphism. Choose local representatives

$$a_j \in S_\rho^{m+\frac{n+n'-2N_j}{4}}(\mathbb{R}^{n+n'} \times \mathbb{R}^{N_j}, F \otimes E^* | U_j)$$

with

$$\operatorname{supp} a_j \subseteq \{(y, x, t\theta); t \geq 1, (y, x, \theta) \in K \text{ compact} \subseteq (\operatorname{Im} U_j) \times \mathbb{R}^{N_j}\}.$$

Then if $u \in \Gamma(F^* \otimes E)$, $dydx$ denotes Lebesgue measure in $\operatorname{Im} U_j$ and $d\theta$ Lebesgue measure in $\mathbb{R}^{N_j}$,

$$\langle A_j, u\rangle = (2\pi)^{-(n+n'+2N_j)/4} \iint e^{i(\varphi_j(y, x, \theta) - \pi N_j/4)} (a_j(y,x,\theta), u(y,x))\, dydxd\theta.$$

Finally, $A = \Sigma A_j$, a locally finite sum in the yx-variables. (Recall from Section 1.1 that (,) denotes fiberwise contraction.) In a suitable sense [19; 6], a is asymptotic to Σa_j.

A is <u>properly supported</u> iff the natural projections of $Y \times X$ onto the factors are proper when restricted to $\operatorname{supp} A$. Such an A extends uniquely to $A : \Gamma_\alpha(E) \to E'_{1-\alpha}(F)$.

<u>Ex</u>. If $X = Y$ and L is the normal bundle of the diagonal in $X \times X$, the operators determined by L (and any symbol) are the <u>pseudodifferential</u> operators. Among the properly supported pseudodifferential operators one finds all differential operators (cf. Section 1.1).

Properly supported operators whose domains and codomains match may be composed to give a Fourier Integral Operator, and the symbol of the composition is the product of the symbols given by a special pairing on densities [19, p. 181].

In the following sections the symbol of A will be denoted σ_A. Note that one may regard $\sigma_A : L \to L(E, F) \otimes K_{1-\alpha}$, where to be precise E and F are again regarded as over L. Thus one can consider the rank of $\sigma_A(\ell) : E_\ell \to F_\ell$, $\ell \in L$.

Definition. The characteristic variety of A, ch(A), is the set in L where σ_A fails to have maximal rank, provided σ_A is homogeneous "far out."

When $\iota : L \to T^*(Y \times X)$ is injective one may consider L and ch(A) as subsets, and when Y = X one may consider them as subsets of T^*X by projecting. This is useful, and will be done, whenever A is a pseudodifferential operator.

Ex. If A is a differential operator, for each $x \in X$, $ch(A) \cap T^*_xX$ is an algeraic variety (zero set of polynomials).

Remark. One can characterize WF(u) as $\cap\, ch(A)$, where A runs over all pseudodifferential operators which have order 0, are properly supported and with Au smooth [19].

3.2 Propagation of Singularities

Let P be a pseudodifferential operator of order m and type 1 from sections of E_α to sections of $F_{1-\alpha}$,

$P \in P\,\mathrm{Diff}_m(E_\alpha, F_{1-\alpha}) \subseteq I_1^m(E_\alpha, F_{1-\alpha})$. Remember that one must regard E, F as sitting over $L = T^*X \setminus 0$ in this case, so that precisely $P \in P\,\mathrm{Diff}_m((\pi^*E)_\alpha, (\pi^*F)_{1-\alpha})$. By using the phase volume μ on T^*X one may naturally identify all the density bundles on T^*X. It also turns out that the Keller-Maslov bundle is naturally trivial in this case. Thus one may write $P \in P\,\mathrm{Diff}_m(E, F)$ and $\sigma_P : T^*X \setminus 0 \to L(E,F)$ unambiguously. More generally, when $Y = X$ one has $I_\rho^m(E, F)$.

If $E = F = \mathbb{1} \otimes \mathbb{C}$ then one may consider $\sigma_P \in \mathcal{E}\,(T^*X \setminus 0)$. Recall that $T^*X \setminus 0$ is a symplectic manifold via the canonical 2-form $\omega | T^*X \setminus 0$. The following theorem is 6.1.1 in [7].

Theorem 1. Let $P \in P\,\mathrm{Diff}_m(X) := P\,\mathrm{Diff}_m(\mathbb{1} \otimes \mathbb{C}, \mathbb{1} \otimes \mathbb{C})$ be properly supported with real symbol σ_P homogeneous of degree m. If $Pu = f$ for $u, f \in \mathcal{D}'(X)$ then $WF(u) \setminus WF(f) \subseteq ch(P)$ and is invariant under the flow of H^{σ_P} in $ch(P) \setminus WF(f)$. $\square$

The following more general form is the one that will actually be used in the sequel. Its proof presents no additional difficulties [7].

Corollary 2. Let $P \in P\,\mathrm{Diff}_m(E, E)$ be properly supported with $\det \circ \sigma_P$ real and homogeneous of degree $m \cdot$ (fiber dim E). If $Pu = f$ then $WF(u) \setminus WF(f) \subseteq ch(P)$ and is invariant under the flow of $H^{\det \circ \sigma_P}$ in $ch(P) \setminus WF(f)$. $\square$

Recall that a subspace V of a symplectic vector space (W, σ) is <u>coisotropic</u> iff $\{w \in W; \sigma(w,v) = 0 \text{ for all } v \in V\} \subseteq V$. If (Y, τ) is a symplectic manifold then an immersion $F : Z \to Y$ is <u>coisotropic</u> iff each $F_* T_z Z$ is coisotropic in $T_{F(z)} Y$. Also recall that an algebraic variety V may be decomposed into the <u>regular part</u> $\tilde{V}$, which is a manifold, and the <u>singular part</u> $V \setminus \tilde{V}$, with $\tilde{V}$ dense in V.

In general, if $P \in P\,\mathrm{Diff}_m(E, F)$ then $\widetilde{ch}(P)$ coisotropic is a necessary condition for local solvability [17]. Thus one may as well assume that $\widetilde{ch}(P)$ is coisotropic (since here the equation $Pu = f$ is assumed to have at least one sclution), in which case it is invariantly foliated into <u>bicharacteristic strips</u>, the projections of which in X are <u>bicharacteristics</u>. If the foliation is extended to $ch(P)$ discretely over the singular set, there is a similar theorem about propagation of singularities being invariant under the foliation. Since I shall not need it here, this indication should suffice.

As an example I shall compute the propagation of singularities of a generalized wave equation.

<u>Ex</u>. Let β be a smooth pseudoriemannian structure on X and recall that one may think of $\beta \in C^\infty(T^*X)$, so that the natural lifts of geodesics correspond to the integral curves of $H^{1/2\beta}$ (Section 2.1). Let (e_i) be a local orthonormal frame field and

$u \in \mathfrak{X}$. Define $\operatorname{div} u := \varepsilon^i \beta(\nabla_{e_i} u, e_i)$ where $\varepsilon^i := \beta(e_i, e_i)$. (Recall that the summation convention is in effect.) For $f \in C^\infty(X)$ define $\Box f := \operatorname{div} \operatorname{grad} f = \varepsilon^i \beta(\nabla_{e_i} \operatorname{grad} f, e_i) = \varepsilon^i(e_i e_i(f) - \nabla_{e_i} e_i(f))$. $\Box$ is the <u>d'Alembertian</u> or <u>Laplace-Beltrami operator</u> associated to β and $\Box f = 0$ is the <u>generalized wave equation</u> on X. (The sign conventions adopted here yield $\Delta = \Sigma_i(\partial^2/\partial(x^i)^2)$ as the Laplacian on $\mathbb{R}^n$.) To compute the symbol, choose $g \in C^\infty(X)$ with $g(x) = 0$ and $dg(x) = \xi$, for some fixed but arbitrary $x \in X$ and $\xi \in T^*_x X \setminus 0$. Then

$$\sigma_\Box(\xi) = -\frac{1}{2}\Box(g^2)(x) = -\frac{1}{2}\varepsilon^i(e_i e_i(g^2) - 2g dg(\nabla_{e_i} e_i))(x)$$

$$= \varepsilon^i(\sigma_{e_i}(\xi))^2 = \beta(\xi) .$$

It follows that the bicharacteristics are the null geodesics of β, so that the singularities of f propagate along the "light rays." If β is Lorentzian, its null geodesics <u>are</u> the light rays of geometric optics.

We have already seen that the caustics of geometric optics are the conjugate points along the null geodesics (Section 2.1). Apparently, the symbol of an operator P should be regarded as some sort of (possibly degenerate) generalized pseudoriemannian structure on X, whose geometry controls the behavior of singularities of solutions u of Pu = f. In this light it seems that a more detailed study of the geometry of ch(P) should lead to more information about the behavior of singularities.

Remark. There is a refined notion of WF sets that allows one to study problems involving finite differentiability hypotheses [7, p. 201].

3.3 Parametrices

I shall discuss the scalar case first. Let $P \in P\,\mathrm{Diff}_m(X)$. P is of real principal type in X iff P is properly supported with real homogeneous symbol σ_P of order m and no complete bicharacteristic strip of P stays over a compact set in X; i.e., no complete bicharacteristic is imprisoned in the sense of Hawking and Ellis [13] (cf. Sections 4.1 and 4.3 for additional comments). If one partitions ch(P) via $ch^j(P) = \{\xi \in ch(P);\ \mathrm{rank}\,\sigma_{P*}|\mathrm{Ker}\,\pi_* = j \text{ at } \xi\}$ then the last condition is equivalent to $ch^0(P) = \emptyset$. (Those familiar with the concept will recognize $\{ch^j(P)\}$ as a stratification.) There is the following theorem [7, Thm. 6.3.3].

Theorem 1. If P is of real principal type in X, then the following are equivalent:

(1) $P : \mathcal{D}'(X) \to \mathcal{D}'(X)/\mathcal{E}(X)$ is surjective;

(2) For every compact $K \subsetneq X$ there exists a compact

$K' \subseteq X$ such that K' contains any interval on a bicharacteristic of P having both endpoints in K. []

When X satisfies condition (2) above one says that X is <u>P-pseudoconvex</u> [7, p. 217]. In this case there are criteria for the existence of genuine solutions [7, Thm. 6.3.4].

<u>Theorem 2.</u> If P is of real principal type in X which is P-pseudoconvex then these are equivalent:

(1) The equation $Pu = f$ has a solution $u \in \mathcal{D}'(X)$ for every $f \in \mathcal{D}'(X)$ such that $\langle f, v\rangle = 0$ for all $v \in \mathcal{D}(X) \cap \operatorname{Ker} {}^tP$;

(2) Same as (1) but with $u, f \in \mathcal{E}(X)$;

(3) For every compact $K \subseteq X$ there exists a compact $K' \subseteq X$ such that if $v \in \mathcal{E}'(X)$ and $\operatorname{supp} {}^tPv \subseteq K$ then ${}^tPv = {}^tPw$ for some $w \in \mathcal{E}'(X)$ with $\operatorname{supp} w \subseteq K'$;

(4) Same as (3) but with $v, w \in \mathcal{D}(X)$. []

Here ${}^tP$ is the transpose of P with respect to the distribution pairing $\langle\ ,\ \rangle$.

Finally, [7, Thm. 6.5.3] provides for the existence of (two-sided) parametrices E^+ and E^- such that $E^+ - E^- \in I_1^{(1/2)-m}(X \times X)$ whenever P is of real principal type and X is P-pseudoconvex. Somewhat loosely, one may think of

$E^{\pm}$ as being a pseudodifferential operator except at the diagonal of $ch(P) \times ch(P)$ where the symbol becomes highly singular. In general this cannot be avoided.

If now $P \in P\,Diff_m(E,E)$, where E is a vector bundle over X, is properly supported with $\det \circ \sigma_P$ real and homogeneous of degree m (fiber dim E) and $ch^0(P) = o$ then Theorems 1 and 2 remain valid and parametrices can be constructed when X is P-pseudoconvex. The arguments parallel those in [7] and are omitted. More generally, if $P \in P\,Diff_m(E,F)$ is properly supported with homogeneous symbol σ_P and $ch^0(P) = \emptyset$ then one can construct parametrices as in [17, Lemma 1.0.2] provided that $\det \circ \sigma_{P^*P}$ is real, where P^* is the adjoint of P with respect to suitably chosen Hermitian structures on E and F, and of course that X is P-pseudoconvex.

4. GENERAL RELATIVITY

4.1 Review of the Smooth Theory

In this section I shall give an outline of some of the main features of general relativity in the smooth case. Two standard references at the introductory level are Weinberg [39] and Misner, Thorne and Wheeler [30]. Both of these are aimed at physicists however, and I feel most mathematicians will prefer Sachs and Wu [33] since it is written for them in particular. At a more advanced level Hawking and Ellis [13] seems to be the best available treatment, despite its misrepresentation of the nature of distributions. With all these excellent and exhaustive references readily available, the main purpose of this section is to fix notation and conventions. In particular, recall that the summation convention is in effect (since Section 2.1).

The arena for general relativity is a smooth manifold X. At this point the physical motivation begins to make itself felt: we may as well assume X is connected since we would have no knowledge of other components anyway. The physical concepts of space and time are traditionally reflected in the geometric structure: X is assumed to have a Lorentzian structure β; i.e., a nondegenerate symmetric (2,0)-tensor (cf. Section 2.1) of constant signature $2-n$ (i.e., $+ - \ldots -$). For various reasons [13,33] arising from causality and energy density, it is physically desirable to assume that X is not compact; otherwise, e.g., one would be able to travel into one's own past. In addition there is an obstruction, the Euler characteristic, to the existence of smooth Lorentzian structures on compact manifolds. Every noncompact manifold admits a Lorentzian structure. Indeed, every noncompact manifold admits a nonvanishing one-form u. If $\tilde{\beta}$ is a Riemannian structure then β given by

$$\beta(v, w) = \tilde{\beta}(v, w) - 2\,\frac{\tilde{\beta}(u, v)\tilde{\beta}(u, w)}{\tilde{\beta}(u, u)}$$

defines a Lorentzian structure. Thus (X, β) will denote a connected noncompact Lorentzian manifold for the remainder of this section.

Remember that although it is customary that β be a $(0, 2)$-tensor, we have seen in the preceding chapter that it is more convenient for the present formulation of the theory to use a $(2, 0)$-tensor.

A co vector $\xi \in T^*X \setminus 0$ is timelike iff $\beta(\xi, \xi) > 0$, null or lightlike iff $\beta(\xi, \xi) = 0$ and spacelike iff $\beta(\xi, \xi) < 0$. One says ξ is causal iff ξ is timelike or null. The null covectors in each fiber form a double cone which separates the timelike and spacelike covectors. Recall that β determines an isomorphism $TX \cong T^*X$, which will be denoted in either direction by a tilde. Thus a vector $v \in TX$ is timelike, null or spacelike according as $\hat{\tilde{v}}$ is. A smooth path $p : \mathbb{R} \to X$ is timelike (spacelike, causal) iff its velocity field $\dot{p}$ is everywhere timelike (spacelike, causal). A Lorentzian manifold is time-orientable iff there exists a nonvanishing timelike vector field. In the appropriate sense, this means it is possible to make a smooth choice of causal cones. One defines the notion of space-orientable similarly. By using the Lorentz group instead of the orthogonal group, one can construct characteristic classes for Lorentzian manifolds [cf. 29]. Time-orientability and space-orientability may then be described by the vanishing of appropriate characteristic classes in the same way that the usual notion of orientability is described by the vanishing of the first Stiefel-Whitney class.

Most physicists agree that any reasonable candidate for spacetime should be causal; i.e., should not contain any closed causal curves. Quantum considerations further lead to the desirability of some sort of stability of this notion. Thus a Lorentzian manifold (X, β) is stably causal iff there is an open neighborhood of β in the Whitney topology (also known as the strong or fine topology) consisting of causal structures. Intuitively, a stably causal structure remains causal after small perturbations. One thing that makes this mathematically interesting is that stable causality implies time-orientability.

Proposition 1 [13, p. 198]. (X, β) is stably causal iff there is a smooth function on X with timelike gradient. □

Causality conditions are also one of the ingredients in the Hawking-Penrose signularity theorems [13, Sect. 8.2]. Locally, stable causality on a compact set implies that there are no causal curves imprisoned (see Section 3.3) in the compact set [13, Prop. 6.4.7].

On the other hand some of the singularities predicted by physicists give rise to causality violations. Thus I shall define a (smooth) spacetime to be a (smooth) connected noncompact Lorentzian manifold.

Matter and energy in a spacetime are conventionally described by a symmetric 2-tensor, the stress-energy or energy-momentum tensor T. Physically, the properties of symmetry and rank 2 are motivated by classical examples, such as an electromagnetic field or a particle flow.

Ex. The electromagnetic field is described by a one-form α called the potential. The field tensor is then $F = 2d\alpha$ yielding the energy-momentum tensor

$$T_{ij} = \frac{1}{4\pi}\left(F_{ik}F_{j}^{k} - \frac{1}{4}\beta_{ij}F_{h\ell}F^{h\ell}\right).$$

A perfect fluid may be described by a function p, the pressure, and a unit vector field v whose integral curves are the flow lines. The stress-energy tensor is

$$T^{ij} = (\mu+p)v^i v^j + p\beta^{ij} ,$$

where μ is the energy density, also a function.

That T should be a tensor may be considered as one manifestation of the famous Principle of Equivalence. (According to Weinberg [39 and references cited there] it is impossible to construct a Lorentz-invariant quantum theory for massless particles of spin 2 (e.g., gravitons) unless the corresponding classical field theory obeys the Principle of Equivalence.) Now let Ric denote the Ricci tensor and R the scalar curvature of β. Choosing units in which $c = 1 = G$ and absorbing the 8π into T, the Einstein equation becomes

$$\mathrm{Ric} - \frac{1}{2}\beta R = T.$$

Since G is not needed as a constant, I shall employ it in its other standard use as $G(\beta) := \mathrm{Ric} - \frac{1}{2}\beta R$, the Einstein operator. When T is given a priori this equation is bad enough, but as we saw in the examples T usually depends on β, so that what is really needed is a solution of $G(\beta) - T(\beta) = 0$. This extreme nonlinearity is the source of much difficulty, but there is a physically interesting class of spacetimes in which the equation is linear (see Section 3).

Many modifications of general relativity and the Einstein equation have been proposed. One of the more popular is that of Brans and Dicke, which attempts to allow for the interaction of distant matter in determining the local gravitational field by means of a scalar field incorporated into T. For more details and other examples see [30; 39]. Some use a connection with torsion; see [14] for a recent review. I shall confine myself here to the standard equation as above and always use the Levi-Civita connection.

4.2 Extension to Singular Geometries

I am now ready to formalize my concept of spacetime singularities.

Definition. A singular spacetime is a smooth connected manifold X together with a singular geometry β (see Section 2.2) such that $\beta | X \setminus \text{sing supp}\, \beta$ is a Lorentzian structure. The set $\Sigma := \text{sing supp}\, \beta$ is referred to as the singular set of the spacetime or the set of spacetime singularities and β is called a Lorentzian singular geometry.

Note that if $\Sigma = \emptyset$, (X, β) is a spacetime in the previous sense provided X is noncompact. On the other hand, if $\Sigma \neq \emptyset$ then $(X \setminus \Sigma, \beta | X \setminus \Sigma)$ is always a spacetime in the previous sense. One advantage of singular geometries is that <u>every</u> manifold has a Lorentzian singular geometry. From now on, (X, β) denotes a singular spacetime.

The extension of general relativity to singular spacetimes is largely purely formal. For example, since β is a singular geometry the Ricci tensor Ric and the scalar curvature R exist as distributional tensors and the product βR is well-defined. Thus one may require that β satisfy the Einstein equation

$$\mathrm{Ric} - \frac{1}{2}\beta R = T,$$

where T is also a distributional tensor now. In Chapter 5 a specific example of such a T will be computed.

On the macro level, the use of distributions agrees with one's physical intuition since in common experience objects have edges. On the micro level, however, the use of distributions must be regarded in certain cases as an idealization; e.g., the representation of a particle by means of a Dirac delta. Yet at the same time quantum theory is one of the places where distributions have been indispensible. One can in fact make a good case for distribution theory being a result of quantum theory. Thus it seems reasonable to expect that this formulation of general relativity will lend itself more easily to quantization, especially in light of the recent developments in geometric quantization [cf. e.g., 24; 36].

Now let us consider the causality and energy conditions. Two reasonable ways in which to formulate these conditions are: 1) require that the condition hold away from Σ; and 2) require that the condition hold in a suitable distribution sense. I shall consider energy conditions first.

Intuitively, the typical energy condition is a means of saying that gravity is always attractive. In the smooth theory this formally takes the form of an inequality such as $T \geq 0$ or $\text{Ric} \geq 0$ on causal vectors. Since the singular support of all geometric objects associated to β is contained in the singular set Σ of X, the first alternative is to require $T \geq 0$ or $\text{Ric} \geq 0$ on causal vectors over $X \setminus \Sigma$. The other is to require that $T \geq 0$ or $\text{Ric} \geq 0$ on the elements of $\mathcal{E}_1(TX \otimes TX)$ which are causal, where a vector v over Σ is _causal_ iff there is a sequence of smooth Lorentzian structures $\beta_n \to \beta$ in the distribution topology such that v is causal for each β_n. Clearly the second notion implies the first. Comparing with [13], either is a reasonable extension of their energy conditions. If Σ is nowhere dense the two notions are equivalent, but it is easy to see that if Σ has nonempty interior then they are not. If $\beta | X \setminus \Sigma$ is locally representable by real meromorphic functions and β is its distributional extension then Σ is nowhere dense. In the rest of this thesis I shall say that (X, β) _satisfies the energy condition_ if the first alternative holds for $\text{Ric}(\beta)$.

Now consider causality conditions. In this case it may be physically reasonable to require only that β satisfy a causality condition outside some neighborhood of Σ, rather than on $X \setminus \Sigma$. Since the presence of certain types of singularities precludes the time-orientability of $X \setminus \Sigma$, it would seem that imposing any sort of distributional causality condition on all of X is unnecessarily restrictive. It might even exclude the very singularity that is to be analyzed from some particular model. Thus I shall content myself for the present with considering causality conditions only in particular cases, and then only when necessary.

Finally, I shall say that two points are connected by a generalized geodesic p iff both points lie in supp p. A generalized proper geodesic is timelike, null or spacelike according as it is the limit of a sequence of timelike, null or spacelike, respectively, proper geodesics. This is equivalent to saying that the support of its generalized velocity field (cf. Section 1.3) contains only timelike, null or spacelike vectors, respectively, when it makes sense. A generalized geodesic is timelike, null or spacelike iff its representatives are (cf. Section 1.3). If $X \setminus \Sigma$ is time-orientable and if the bundles of future and past vectors can be extended continuously and disjointly across Σ I shall say that X is time-orientable. In this case the usual notion of the chronological (causal) future $I^+(S)$ $(J^+(S))$ of $S \subseteq X$ may be defined as the set of points in X that can be connected to S by a piecewise future timelike (causal) geodesic. There is the dual notion of <u>chronological (causal) past</u> $I^-(S)$ $(J^-(S))$ defined in the obvious way; such duals will not be explicitly defined again. It should now be clear how to translate any other desired concepts from Hawking and Ellis [13]. As a last example, S is <u>future trapped</u> iff $E^+(S) := J^+(S) \setminus I^+(S)$, the <u>future horismos</u> of S, is compact.

<u>4.3 Some Algebraically Special Spacetimes</u>

Consider the class of Lorentzian singular geometries on R^n of the form

$$\beta = \eta + f\lambda \otimes \lambda \tag{1}$$

where η is the Minkowski structure $\mathrm{diag}(1, -1, \ldots, -1)$, $f \in \mathcal{D}'(\mathbb{I})$ is a real distribution and λ is a distributional vector field which is null with respect to both η and β. When f and λ are smooth (1) is known as a Kerr-Schild structure after [22], and I extend the usage here. Kerr and Schild originally considered such smooth structures due to the ease of raising and lowering indices and because any vector null for one of β, η is null for both [22]. The most interesting feature of smooth Kerr-Schild structures is that the full Einstein equation is linear [12]. This means that the gravitational field is not its own source. If a singular geometry can be put into

Kerr-Schild form by a coordinate choice, then these coordinates represent an accelerated frame by the Principle of Equivalence. In some sense the problem of finding such coordinates is analogous to the classical problem of isothermal coordinates, and represents an interesting avenue for further research.

In local coordinate form, Einstein's equation becomes a system (in mixed components)

$$R^i_j - \frac{1}{2}\beta^i_j R = T^i_j \ ,$$

where R^i_j are the mixed components of Ric. If one examines the calculations in [12] one sees that they are completely formal. Hence the result is valid in the singular geometry case and the Einstein equations are

$$\partial_i \partial_j (\eta^{ij}\beta^{k\ell} - \eta^{ki}\beta^{\ell j} - \eta^{\ell i}\beta^{kj} + \eta^{k\ell}\beta^{ij}) = 2\eta^{kh} T^{\ell}_h \ , \qquad (2)$$

where ∂_i denotes covariant differentiation with respect to η and T is the stress-energy tensor of everything except the gravitational field. Observe that if η is an arbitrary smooth Lorentzian structure and $\mathbb{R}^n$ is replaced by a manifold X, then (2) will be linear when η is flat (or homaloidal). Hence it follows that for general Lorentzian singular geometries of Kerr-Schild form (1) the Einstein equations are locally given by (2) as linear equations whenever η is flat. This is the algebraically special class to which I shall henceforth restrict my attention. It is clearly strictly larger than the class obtained through the traditional linearization procedure. The following example shows that it contains those structures of greatest physical interest.

Ex. In $\mathbb{R}^4$ with coordinates (t, x, y, z) consider the following line element:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2mr^3 - r^2q^2 - r^2p^2}{r^4 + a^2z^2}\left[\frac{r(xdx+ydy) - a(xdy - ydx)}{r^2+a^2} + \frac{zdz}{r} + dt\right]^2 \tag{3}$$

where $a, m \geq 0$, q and p are constants and r is given by

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1.$$

Physically, m represents mass, q electric charge, p magnetic charge (if one believes in magnetic monopoles; otherwise $p = 0$) and a angular momentum [5]. A straightforward computation shows that this determines a Lorentzian structure of the form (1) on $\mathbb{R}^4 \setminus$ t-axis. Since all the component functions are analytic on $\mathbb{R}^4 \setminus$ t-axis, it follows from 2.2.2 that (3) extends to a Lorentzian singular geometry on $\mathbb{R}^4$. By setting various constants equal to zero, one obtains the classical black holes: $a = q = p = 0$ yields the Schwarzschild structure, $a = p = 0$ Reissner-Nordström, $q = p = 0$ Kerr, and $p = 0$ charged Kerr. Note that all are obtained as Lorentzian singular geometries on all of $\mathbb{R}^4$. In each case it is easy to see that the singular set Σ coincides with the accepted singularity locus.

Thus reassured, one may now proceed to investigate the general theory. For the rest of this section (X, β) will denote a Lorentzian singular geometry with $\beta = \eta + f\lambda \otimes \lambda$ where η is a smooth flat Lorentzian structure on X, $f \in \mathcal{B}'(\mathbb{1})$ and $\lambda \in \mathfrak{X}'$ is null for both η and β. As usual, $\tilde{\beta}$, $\tilde{\eta}$ and $\tilde{\lambda}$ will denote the covariant forms.

The left hand side of (2) locally defines a second order differential operator G on (2, 0)-tensors. In order to compute the symbol σ_G choose $(x, \xi) \in T^*X$ and $g \in C^\infty(X)$ with $dg(x) = \xi$ and $g(x) = 0$. Then locally, for s a (2, 0)-tensor,

$$
\begin{aligned}
(\sigma_G(\xi)\cdot s)^{k\ell}(x) &= \frac{1}{2}\, G(g^2 s)^{k\ell}(x) \\
&= \xi_i \xi_j (\eta^{ij} s^{k\ell} - \eta^{ki} s^{\ell j} - \eta^{\ell i} s^{kj} + \eta^{k\ell} s^{ij})(x) \qquad (4)
\end{aligned}
$$

whence

$$\sigma_G(\xi) = \eta(\xi, \xi) - 2\,\mathrm{Sym}(\xi \lrcorner\, \eta \otimes \xi \lrcorner\,) + \eta \xi \lrcorner\, \xi \lrcorner\,,$$

where Sym denotes symmetrization and $\lrcorner$ contraction. Observe that since G is a differential operator it is properly supported and that σ_G is homogeneous of degree two. Thus one can apply 3.2.2. Since the symbol depends only on η, the propagation of $WF(\beta) \setminus WF(G\beta)$ is independent of $\beta \in \mathcal{E}'(TX \otimes TX)$. This is the precise sense in which the propagation of the classical black holes is the same. Also observe that, as remarked in Section 3.1, ch(G) may be viewed as a family of real algebraic varieties parameterized by X. Each variety is the zero locus of a homogeneous polynomial and hence of codimension one and the product of $\mathbb{R} \setminus 0$ with a projective variety (i.e., one which can be algebraically embedded in a (real) projective space). At this time it seems that it will require a healthy dose of algebraic geometry in order to obtain information on the bicharacteristic strips in general.

Finally, I shall discuss the solvability of the system (2) when T is given. From 3.3.1 as extended at the end of Section 3.3 it follows that one can solve mod C^∞ for any $T \in \mathcal{E}'(TX \otimes TX)$ if G is of real principal type and X is G-pseudoconvex. Thus the notion of imprisonment (cf. Sections 3.3 and 4.1) is crucial for the complete bicharacteristics. One could define yet another causality condition, <u>bicharacteristic causality</u>, in the obvious way so that it would be a necessary condition for solvability of (2) for given $T \in \mathcal{E}'(TX \otimes TX)$. The extended version of 3.3.2 implies that the existence of genuine solutions depends on ${}^tG$. A quick inspection of (2) shows that G is its own transpose so that the relevant condition from 3.3.2(1) is that $\langle T, v \rangle = 0$ for all $v \in Б(TX \otimes TX) \cap \operatorname{Ker} G$. Clearly if T is symmetric and one seeks a symmetric solution then it suffices to consider only symmetric v. However there is no guarantee that the solution will be a Lorentzian singular geometry. Thus the theorem gives only a necessary condition for the existence of physically meaningful solutions. The condition may be interpreted, however, as a necessary condition on T in order that it be a physically reasonable stress-energy tensor. As far as I know, this is a new compatibility condition for the Einstein equation.

4.4 Singularity Theorems

This section is a discussion of singularity theorems as found for example in Hawking and Ellis [13, Sect. 8.2]. Let (X, β) be a singular spacetime (Section 2).

As stated, most of the singularity theorems can be applied only to $(X \setminus \Sigma, \beta | X \setminus \Sigma)$, provided of course it satisfies their hypotheses. The alternative version of their Theorem 2 [13, p. 267] makes good sense on X however if X is time-orientable (Section 2). For the convenience of the reader I shall state it.

Theorem 1 [13]. The following three conditions cannot all hold on X:

(1) Every maximal causal geodesic contains a pair of conjugate points (Section 2.1);

(2) X contains no closed timelike generalized paths (Section 2);

(3) There is a trapped (Section 2) achronal set S. □

Achronal means that no two points of S can be connected by a timelike generalized path. The proof given in [13] can be appropriately modified to provide a proof in the present setting.

But once one has the singularity theorems available it seems more reasonable to proceed according to the philosophy that spacetimes are singular $(\Sigma \neq \emptyset)$ until proven smooth $(\Sigma = \emptyset)$. If some sort of singularities are almost inevitable then the thing of interest would be a smoothness theorem. If they are indeed inevitable then we must learn to live with singular geometries, which is the philosophy of this thesis.

If one observes that the smooth part $p|X \setminus \Sigma$ of a generalized geodesic p of β that meets Σ is necessarily incomplete, then it is clear that the present formulation of spacetime singularity includes the geodesic incompleteness formulation [13]. A moment's reflection suffices to convince one that the inclusion is also proper (remember that every manifold has a Lorentzian singular geometry).

Given a smooth spacetime that is geodesically incomplete, one must embed it as a submanifold of some larger ambient manifold having the same dimension in order to apply the present theory. Regardless of how this is done, however, there will be a distributional Lorentzian structure on the ambient space which extends the given smooth structure. This eliminates the need for the Schmidt b-boundary [13] with its unpleasant non-Hausdorff and probably non-smooth nature, as well as its basic incomputability. The problem of finding an ambient manifold is purely topological [cf. 29]. The problem of finding a Lorentzian singular geometry among the distributional Lorentzian structures extending the given structure has not been studied to my knowledge. I conjecture it will be solved in the affirmative.

5. APPLICATIONS

5.1 Geometric Singularities

By now it should be clear what is meant by letting the geometric objects carry the singular information. Although this idea was espoused by Marsden [27] in 1968, it has been apparently ignored, at least by all those authors listed in the Science Citation Index as having cited Marsden's paper. So completely in fact that it had to be rediscovered by the author, after which a thorough search of the literature in connection with the problem of flows turned up Marsden's inspiring paper.

Note that this idea has led to a view of the classical black holes in particular, and a large class of spacetimes in general, in which various unphysical features of the conventional view are absent. Moreover, the conceptual foundations of spacetime singularities have been rigorized in a natural way consistent with the results of the Hawking-Penrose theorems. Perhaps most important however is the way in which we are once again able to see the geometric nature of general relativity, and to actually discover that spacetime singularities do not lead to a breakdown of its predictive abilities. They are merely the geometric singularities that form a sufficiently abstract vantage point one would expect to be associated with a geometric theory.

5.2 Kerr-Schild Geometries in $\mathbb{R}^4$

As noted in Section 4.3, the propagation of singularities in Kerr-Schild geometries depends only on the background flat structure. In fact if the structure is given by $\beta = \eta + f\lambda \otimes \lambda$ then the bicharacteristics of the Einstein operator G depend only on η. Unfortunately for the purposes of computation this dependence is considerably more complicated than that of the geodesics. In this section I shall consider the special (but physically interesting) case where $X = \mathbb{R}^4$ and η is the Minkowski structure $\mathrm{diag}(1,-1,-1,-1)$.

In order to locate $ch(G)$ one first needs $\det(\sigma_G)$. To this end, observe that $T^*R^4 = R^4 \times R^4$ so that it suffices to work in one fiber since σ_G is independent of basepoint. Let $\mathrm{Sym}(4)$ denote the symmetric 4×4 matrices. Then one may consider σ_G as a homogeneous polynomial map $\mathbb{R}^4 \to L(\mathrm{Sym}(4), \mathrm{Sym}(4))$. Let $\beta_{11}, \ldots, \beta_{44}$ be the basis for $\mathrm{Sym}(4)$ given by

$$\beta^{ij}_{k\ell} = \begin{cases} 1 & \text{if } \{i,j\} = \{k,\ell\} \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \le k \le \ell \le 4$ and the pairs (k,ℓ) are ordered lexicographically. Substituting in the local formula for σ_G, 3.3(4), one obtains a 10×10 matrix Σ with entries

$$\Sigma_{ab}^{ij} := (\sigma_G(\xi)\cdot\beta_{ab})^{ij} = \xi_k\xi_\ell\eta^{k\ell}\beta_{ab}^{ij} - \xi_k\eta^{ki}\xi_\ell\beta_{ab}^{\ell j} - \xi_k\eta^{kj}\xi_\ell\beta_{ab}^{\ell i} + \xi_k\xi_\ell\beta_{ab}^{k\ell}\eta^{ij} ,$$

and the pairs (i,j) are also ordered lexicographically. If this ordering is used to index rows ab and columns ij and $|\xi| := \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2$, one obtains the form shown on page 67 for Σ.

To streamline the computation, write

$$\Sigma = \begin{bmatrix} \Sigma(11) & \Sigma(12) \\ \Sigma(21) & \Sigma(22) \end{bmatrix}$$

as indicated by the dashed lines so that

$$\det \Sigma = \det \Sigma(11) \det \Sigma(22) - \det \Sigma(12) \det \Sigma(21).$$

First it is obvious that $\det \Sigma(12) = 0$ since if $\xi_1 \neq 0$ then

$$(\text{row } 4) - \frac{\xi_2}{\xi_1}(\text{row } 2) - \frac{\xi_2^2}{\xi_1^2}(\text{row } 1)$$

is a row of zeros. Hence $\det \Sigma = \det \Sigma(11) \det \Sigma(22)$. In each case the computation will be via expansion by minors.

$$
\left[\begin{array}{ccccc|ccccc}
|\xi|-\xi_1^2 & \xi_2\xi_1 & \xi_3\xi_1 & \xi_4\xi_1 & -\xi_1^2 & 0 & 0 & -\xi_1^2 & 0 & -\xi_1^2 \\
0 & -\xi_3^2-\xi_4^2 & \xi_3\xi_2 & \xi_4\xi_2 & 0 & \xi_3\xi_1 & \xi_4\xi_1 & -2\xi_1\xi_2 & 0 & -2\xi_1\xi_2 \\
0 & \xi_2\xi_3 & -\xi_2^2-\xi_4^2 & \xi_4\xi_3 & -2\xi_1\xi_3 & \xi_2\xi_1 & 0 & 0 & \xi_4\xi_1 & -2\xi_1\xi_3 \\
0 & \xi_2\xi_4 & \xi_3\xi_4 & -\xi_2^2-\xi_3^2 & -2\xi_1\xi_4 & 0 & \xi_2\xi_1 & -2\xi_1\xi_4 & \xi_3\xi_1 & 0 \\
\xi_2^2 & -\xi_1\xi_2 & 0 & 0 & |\xi|+\xi_2^2 & \xi_3\xi_2 & \xi_4\xi_2 & -\xi_2^2 & 0 & -\xi_2^2 \\
\hline
2\xi_2\xi_3 & \xi_1\xi_3 & \xi_1\xi_2 & 0 & 0 & \xi_1^2-\xi_4^2 & \xi_4\xi_3 & 0 & \xi_4\xi_2 & -2\xi_2\xi_3 \\
2\xi_2\xi_4 & -\xi_1\xi_4 & 0 & -\xi_1\xi_2 & 0 & \xi_3\xi_4 & \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 & \xi_3\xi_2 & 0 \\
\xi_3^2 & 0 & -\xi_1\xi_3 & 0 & -\xi_3^2 & \xi_2\xi_3 & 0 & |\xi|+\xi_3^2 & \xi_4\xi_3 & -\xi_3^2 \\
2\xi_3\xi_4 & 0 & -\xi_1\xi_4 & -\xi_1\xi_3 & -2\xi_3\xi_4 & \xi_2\xi_4 & \xi_2\xi_3 & 0 & \xi_1^2-\xi_2^2 & 0 \\
\xi_4^2 & 0 & 0 & \xi_1\xi_4 & -\xi_4^2 & 0 & \xi_2\xi_4 & -\xi_4^2 & \xi_3\xi_4 & |\xi|+\xi_4^2
\end{array}\right]
$$

Consider $\Sigma(11)$. Expand on the first column, obtaining I and III, then expand I on row 4 and III on row 2. Thus

$$I = (|\xi| - \xi_1^2)\left(\xi_1\xi_2 \det\begin{bmatrix} \xi_3\xi_2 & \xi_4\xi_2 & 0 \\ -\xi_3^2-\xi_4^2 & \xi_4\xi_3 & -2\xi_1\xi_3 \\ \xi_3\xi_4 & -\xi_2^2-\xi_3^2 & -2\xi_1\xi_4 \end{bmatrix} \right.$$

$$\left. + (|\xi| + \xi_2^2)\det\begin{bmatrix} -\xi_3^2-\xi_4^2 & \xi_3\xi_2 & \xi_4\xi_2 \\ \xi_2\xi_3 & -\xi_2^2-\xi_4^2 & \xi_4\xi_3 \\ \xi_2\xi_4 & \xi_3\xi_4 & -\xi_2^2-\xi_3^2 \end{bmatrix} \right)$$

$$= 2\xi_1^2\xi_2^2(\xi_2^2+\xi_3^2+\xi_4^2)(\xi_3^2(\xi_2^2+\xi_4^2)+(\xi_3^2+\xi_4^2)^2)$$

and

$$II = \xi_2^2\left((\xi_3^2+\xi_4^2)\det\begin{bmatrix} \xi_3\xi_1 & \xi_4\xi_1 & -\xi_1^2 \\ -\xi_2^2-\xi_4^2 & \xi_4\xi_3 & -2\xi_1\xi_3 \\ \xi_3\xi_4 & -\xi_2^2-\xi_3^2 & -2\xi_1\xi_4 \end{bmatrix} \right.$$

$$+ \xi_3\xi_2 \det\begin{bmatrix} \xi_2\xi_1 & \xi_4\xi_1 & -\xi_1^2 \\ \xi_2\xi_3 & \xi_4\xi_3 & -2\xi_1\xi_3 \\ \xi_2\xi_4 & -\xi_2^2-\xi_3^2 & -2\xi_1\xi_4 \end{bmatrix} \quad -$$

$$- \xi_4\xi_2 \det \begin{bmatrix} \xi_2\xi_1 & \xi_3\xi_1 & -\xi_1^2 \\ \xi_2\xi_3 & -\xi_2^2-\xi_4^2 & -2\xi_1\xi_3 \\ \xi_2\xi_4 & \xi_3\xi_4 & -2\xi_1\xi_4 \end{bmatrix} \Bigg)$$

$$= -\xi_1^2\xi_2^2(\xi_2^2+\xi_3^2+\xi_4^2)[(\xi_3^2+\xi_4^2)^2+\xi_3^2(\xi_3^2+\xi_4^2)+\xi_4^2(\xi_2^2+\xi_4^2)]$$

$$- \xi_1^2\xi_2^2[(\xi_2^2+\xi_3^2)(\xi_2^2+\xi_4^2)(\xi_3^2+\xi_4^2)+\xi_2^2\xi_3^2(\xi_2^2+\xi_3^2)]$$

whence

$$\det \Sigma(11) = \mathrm{I} + \mathrm{II} + \xi_1^2\xi_2^2[(\xi_2^2+\xi_3^2+\xi_4^2)(\xi_4^2(\xi_3^2+\xi_4^2)+(2\xi_3^2-\xi_4^2)(\xi_2^2+\xi_4^2))$$

$$-(\xi_2^2+\xi_3^2)(\xi_2^2+\xi_4^2)(\xi_3^2+\xi_4^2)-\xi_2^2\xi_3^2(\xi_2^2+\xi_3^2)] . \qquad (1)$$

Now for $\det \Sigma(22)$. Expand on column 5, obtaining I, II and III, and then expand I on column 1, II on column 1 and III on row 1. This choice produces more 3×3 minors containing two zeros, somewhat simplifying the ordeal. Thus

$$\mathrm{I} = -2\xi_2\xi_3 \left(\xi_3\xi_4 \det \begin{bmatrix} 0 & |\xi|+\xi_3^2 & \xi_4\xi_3 \\ \xi_2\xi_3 & 0 & \xi_1^2-\xi_2^2 \\ \xi_2\xi_4 & -\xi_4^2 & \xi_3\xi_4 \end{bmatrix} \right. -$$

$$- \xi_2\xi_3 \det \begin{bmatrix} \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 & \xi_3\xi_2 \\ \xi_2\xi_3 & 0 & \xi_1^2-\xi_2^2 \\ \xi_2\xi_4 & -\xi_4^2 & \xi_3\xi_4 \end{bmatrix}$$

$$+ \xi_2\xi_4 \det \begin{bmatrix} \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 & \xi_3\xi_2 \\ 0 & |\xi|+\xi_3^2 & \xi_4\xi_3 \\ \xi_2\xi_4 & -\xi_4^2 & \xi_3\xi_4 \end{bmatrix} \Bigg)$$

$$= 2\xi_2^2\xi_3^2\xi_4^2(\xi_1^2\xi_4^2+\xi_2^2\xi_3^2+\xi_1^2\xi_2\xi_4-\xi_2\xi_3^2\xi_4) ,$$

$$II = -\xi_3^2 \Bigg((\xi_1^2-\xi_4^2)\det \begin{bmatrix} \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 & \xi_3\xi_2 \\ \xi_2\xi_3 & 0 & \xi_1^2-\xi_2^2 \\ \xi_2\xi_4 & -\xi_4^0 & \xi_3\xi_4 \end{bmatrix}$$

$$+ \xi_4\xi_3 \det \begin{bmatrix} \xi_3\xi_4 & -2\xi_2\xi_4 & \xi_3\xi_2 \\ \xi_2\xi_4 & 0 & \xi_1^2-\xi_2^2 \\ 0 & -\xi_4^2 & \xi_3\xi_4 \end{bmatrix}$$

$$- \xi_2\xi_4 \det \begin{bmatrix} \xi_3\xi_4 & \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 \\ \xi_2\xi_4 & \xi_2\xi_3 & 0 \\ 0 & \xi_2\xi_4 & -\xi_4^2 \end{bmatrix} \Bigg) =$$

$$= -\xi_1^2\xi_3^2\xi_4^2(\xi_1^4-\xi_1^2\xi_4^2-\xi_1^2\xi_3^2-3\xi_1^2\xi_2^2+2\xi_3^2\xi_4^2+2\xi_2^2\xi_4^2+2\xi_2^2\xi_3^2+2\xi_2^4)$$

and

$$\text{III} = (\xi_1^2-\xi_2^2-\xi_3^2)\left((\xi_1^2-\xi_4^2)\det\begin{bmatrix} \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 & \xi_3\xi_2 \\ 0 & |\xi|+\xi_3^2 & \xi_4\xi_3 \\ \xi_2\xi_3 & 0 & \xi_1^2-\xi_2^2 \end{bmatrix} \right.$$

$$- \xi_4\xi_3 \det\begin{bmatrix} \xi_3\xi_4 & -2\xi_2\xi_4 & \xi_2\xi_3 \\ \xi_2\xi_3 & |\xi|+\xi_3^2 & \xi_4\xi_3 \\ \xi_2\xi_4 & 0 & \xi_1^2-\xi_2^2 \end{bmatrix}$$

$$\left. - \xi_2\xi_4 \det\begin{bmatrix} \xi_3\xi_4 & \xi_1^2-\xi_3^2 & -2\xi_2\xi_4 \\ \xi_2\xi_3 & 0 & |\xi|+\xi_3^2 \\ \xi_2\xi_4 & \xi_2\xi_3 & 0 \end{bmatrix} \right)$$

$$= (\xi_1^2-\xi_2^2-\xi_3^2)[\xi_1^8-2\xi_1^6\xi_2^2-\xi_1^6\xi_3^2-2\xi_1^6\xi_4^2+\xi_1^4\xi_2^4+\xi_1^4\xi_4^4+\xi_1^4\xi_2^2\xi_3^2$$

$$+ 2\xi_1^4\xi_2^2\xi_4^2+\xi_1^4\xi_3^2\xi_4^2-\xi_1^2\xi_2^2\xi_3^2\xi_4^2-\xi_2^2\xi_3^2\xi_4^4]$$

whence

$$\det \Sigma(22) = \text{I} + \text{II} + \text{III} = \xi_1^{10} - 3\xi_1^8\xi_2^2 - 2\xi_1^8\xi_3^2 - 2\xi_1^8\xi_4^2 + 3\xi_1^6\xi_2^4 + \xi_1^6\xi_3^4$$

$$+ \xi_1^6\xi_4^4 + 4\xi_1^6\xi_2^2\xi_3^2 + 4\xi_1^6\xi_2^2\xi_4^2 + 2\xi_1^6\xi_3^2\xi_4^2 - \xi_1^4\xi_2^6$$

$$- 2\xi_1^4\xi_2^4\xi_3^2 - 2\xi_1^4\xi_2^4\xi_4^2 - \xi_1^4\xi_2^2\xi_3^4 - \xi_1^4\xi_2^2\xi_3^2\xi_4^2 \quad -$$

$$- \xi_1^4\xi_2^2\xi_4^4 - \xi_1^2\xi_2^4\xi_3^2\xi_4^2 + 2\xi_1^2\xi_2^3\xi_3^2\xi_4^3 - \xi_1^2\xi_2^2\xi_3^4\xi_4^2$$

$$- \xi_1^2\xi_2^2\xi_3^2\xi_4^4 - 2\xi_1^2\xi_3^4\xi_4^4 + 2\xi_2^4\xi_3^4\xi_4^2 + \xi_2^4\xi_3^2\xi_4^4$$

$$- 2\xi_2^3\xi_3^4\xi_4^3 + \xi_2^2\xi_3^4\xi_4^4 . \qquad (2)$$

Finally, the sought-for $\det \Sigma$ is the product of (1) and (2), a homogeneous polynomial of degree 20.

It follows immediately that the bicharacteristic strips are horizontal lines. Thus $\mathbb{R}^4$ is G-pseudoconvex. Also if V is the zero set of $\det \Sigma$ in $\mathbb{R}^4\backslash 0$, then $ch(G) = \mathbb{R}^4 \times V$ and $V = V_p \times \mathbb{R}\backslash 0$ for some projective algebraic variety V_p. If one considers the sections of V in the coordinate hyperplanes, it is clear that the only non-trivial one is $\xi_3 = 0$.

I am presently programming a computer to verify this calculation and to determine the equations of the bicharacteristic strips. In particular, it should be verified that G is of real principal type in order to have solutions for arbitrary symmetric 2-tensors T satisfying the compatibility condition.

Now consider the Schwarzschild structure in $\mathbb{R}^4$ (cf. 4.3(3) with $a = p = q = 0$). In this case the singular set is the t-axis. Since each component of the structure is independent of t, it follows that each component of T may be regarded as a distribution on $\mathbb{R}^3$ with support contained in $\{0\}$. Hence each component of T is a linear combination of derivatives of the distribution δ_t defined by

$$<\delta_t, \varphi> = \int_{-\infty}^{\infty} \varphi(t, 0, 0, 0)dt$$

for $\varphi \in \mathcal{D}(\mathbb{R}^4)$. (Observe that identifying $\mathbb{R}^3$ with $\{t = 0\}$, $WF(\delta_t) \cap N(\mathbb{R}^3) = \emptyset$ so that indeed one may identify δ_t with the Dirac δ in $\mathbb{R}^3$ as indicated above.) The Schwarzschild line element is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2m}{r}\left[\frac{xdx+yxy+zdz}{r} + dt\right]^2$$

which immediately yields

$$[\beta^{jk}] = \begin{bmatrix} 1+\frac{2m}{r} & -2m\frac{x}{r^2} & -2m\frac{y}{r^2} & -2m\frac{z}{r^2} \\ -2m\frac{x}{r^2} & -1+2m\frac{x^2}{r^3} & 2m\frac{xy}{r^3} & 2m\frac{xz}{r^3} \\ -2m\frac{y}{r^2} & 2m\frac{xy}{r^3} & -1+2m\frac{y^2}{r^3} & 2m\frac{yz}{r^3} \\ -2m\frac{z}{r^2} & 2m\frac{xz}{r^3} & 2m\frac{yz}{r^3} & -1+2m\frac{z^2}{r^3} \end{bmatrix}.$$

Since each entry is locally integrable and homogeneous of degree -1, it follows from 4.3(2) that each entry in T is homogeneous of degree -3 and thus some constant multiple of δ_t. I shall compute the constants using a method suggested by Bent Petersen.

First recall that $\hat{\delta} = 1$, where $\wedge$ denotes the Fourier transform, and that in view of the preceding observations one need only compute in the space variables x, y, z to obtain $(\beta^{jk})^\wedge$. Next recall the fundamental relation $(x_j f)^\wedge = i\partial_j \hat{f}$ and the basic formulas

$$(r^{-1})^\wedge = 4\pi r^{-2}, \quad (r^{-2})^\wedge = 2\pi^2 r^{-1}, \quad (r^{-3})^\wedge = -4\pi \log r.$$

Using these one may readily compute $[\hat{\beta}^{jk}]$:

$$\begin{bmatrix} 8\pi^3\delta+8\pi m r^{-2} & i4\pi^2 mxr^{-3} & i4\pi^2 myr^{-3} & i4\pi^2 mzr^{-3} \\ i4\pi^2 mxr^{-3} & -8\pi^3\delta+8\pi m(r^2-2x^2)r^{-4} & -16\pi mxyr^{-4} & -16\pi mxzr^{-4} \\ i4\pi^2 myr^{-3} & -16\pi mxyr^{-4} & -8\pi^3\delta+8\pi m(r^2-2y^2)r^{-4} & -16\pi myzr^{-4} \\ i4\pi^2 mzr^{-3} & -16\pi mxzr^{-4} & -16\pi myzr^{-4} & -8\pi^3\delta+8\pi m(r^2-2z^2)r^{-4} \end{bmatrix},$$

where one should carefully note the symmetries in x, y, z. Together with equation 4.3(2) they imply that $T_1^2 = T_1^3 = T_1^4$, that $T_2^3 = T_2^4 = T_3^4$ and that $T_2^2 = T_3^3 = T_4^4$. Thus one need only make four particular computations:

$$\hat{T}_1^1 = \frac{1}{2}(\Box\beta^{11} - 2\partial_1\partial_j\beta^{1j} + \partial_h\partial_j\beta^{hj})^\wedge$$

$$= \frac{1}{2}[8\pi m + 8\pi mr^{-4}(x^4 - x^2y^2 - x^2z^2 - x^2y^2 + y^4 - y^2z^2 - x^2z^2 - y^2z^2 + z^4)$$

$$+ 32\pi mr^{-4}(x^2y^2 + x^2z^2 + y^2z^2)]$$

$$= 8\pi m;$$

$$\hat{T}_1^2 = \frac{1}{2}(\Box\beta^{12} - \partial_1\partial_j\beta^{2j} + \partial_2\partial_j\beta^{1j})^\wedge$$

$$= \frac{1}{2}[4\pi^2 mxr^{-1} - x(i4\pi^2 mr^{-3})(x^2+y^2+z^2)]$$

$$= 0;$$

$$\hat{T}_2^3 = -\frac{1}{2}(\Box\beta^{23} + \partial_2\partial_j\beta^{3j} + \partial_3\partial_j\beta^{2j})^\wedge$$

$$= -\frac{1}{2}[-16\pi mxyr^{-2} - x8\pi mr^{-4}(-2x^2y + x^2y - y^3 + yz^2 - 2yz^2)$$

$$- y8\pi mr^{-4}(-x^3 + xy^2 + xz^2 - 2xy^2 - 2xz^2)]$$

$$= 0;$$

$$\hat{T}_2^2 = -\frac{1}{2}(\Box\beta^{22} + 2\partial_2\partial_j\beta^{2j} - \partial_h\partial_j\beta^{hj})^\wedge$$

$$= -\frac{1}{2}[8\pi m(-x^2+y^2+z^2)r^{-2} - 2x(8\pi mr^{-4})(-x^3+xy^2+xz^2-2xy^2-2xz^2)$$

$$-8\pi m]$$

$$= 0.$$

Thus the physicists' view of the Schwarzschild structure as fundamental is somewhat justified: it is (modulo constants) a partial fundamental solution in the technical sense of partial differential equations.

In particle physics, such Dirac measures are interpreted as mathematical idealizations of a very compact entity. The true extent of its physical properties is determined by scattering experiments. This leads one to speculate that a black hole might be considered physically as an ultradense particle of nuclear or subnuclear size.

CONCLUSIONS

The main purpose of this thesis has been to provide a mathematically rigorous foundation for the study of spacetime singularities. In following the principle that geometric structures should carry the singular information we have been led to the first concrete indications of the intimate connection between geometry and singularities, and the basic geometric nature of general relativity is revealed to us anew.

Thus far we have seen differential geometry extended to distributional structures and, as a result, a new interpretation of general relativity in which the Einstein equations do not break down in the presence of spacetime singularities. The study of the propagation of spacetime singularities and the solvability of the Einstein equations in the physically important Kerr-Schild geometries has been started via the theory of Fourier Integral Operators. Explicit calculations have shown that the Schwarzschild structure is essentially a partial fundamental solution of the Einstein equations in the sense of partial differential equations theory. It has become apparent that symbols of differential (and actually more general) operators should be considered as generalized (possibly degenerate) pseudoriemannian structures whose geometry controls the behavior of singularities. Hence we must consider spacetimes as being singular until proven smooth, and thus we must learn to live comfortably with singular geometries.

Much remains to be done. Generalized flows must be studied further; in particular the relationship of generalized paths to topological curves needs to be determined. The new characterization of conjugate points should be exploited in a closer investigation of distributional geometry to decide how much of the deeper parts of the classical theory carries over. In particular, what is the notion corresponding to completeness? As I have indicated, one of the next projects will be to combine this theory with geometric quantization [24;36]. In connection with this combination, the further study of genuinely nonlinear distributions would be useful, e.g., to obtain powers of Dirac deltas. Since higher powers of the scalar curvature are felt on a small spacetime scale [39], it will be particularly interesting for singularity theory to extend the Einstein operator to a pseudodifferential operator which includes them.

The use of Fourier Integral Operators here is only the beginning. I plan to use them to construct singular solutions with specified singular behavior, and their potential applications are unlimited. It would be enlightening to determine the relation of both this and geometric quantization to Hawking's work on exploding black holes [Comm. Math. Phys. 43(1975) 199-220]. Certainly more work on the geometry of the characteristic variety is needed. The pursuit of this overly neglected area will produce deeper knowledge of the geometry-singularity connection, such as an obstruction theory for the existence of solutions of prescribed smoothness class. As a first step I shall analyze the bicharacteristics of Kerr-Schild structures, particularly the classical black holes, and the concept of bicharacteristic causality. In connection with the classical black holes, the notion of fundamental invariant solutions with respect to a Lie group action illustrates the most important method for finding explicit solutions to date.

Finally, I hope that the reader has derived as much pleasure from thinking about all these things as I have.

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APPENDIX

APPENDIX: Keller-Maslov Bundles

Denote points of $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ by $z = (x, y)$ and let $\langle \, , \, \rangle$ denote the usual inner product. $\mathbb{R}^{2n}$ has a complex structure $J : (x, y) \mapsto (-y, x)$. The usual symplectic form on $\mathbb{R}^{2n}$ is $\sigma(z_1, z_2) := \langle Jz_1, z_2 \rangle$.

The automorphisms of $\mathbb{R}^{2n}$ which preserve each of these structures are the orthogonal group $O(2n)$, the complex general linear group $GL(n, \mathbb{C})$ considered as a subgroup of the real general linear group $GL(2n)$ and the symplectic group $Sp(n)$ respectively. Observe that

$$O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(n) = Sp(n) \cap O(2n) = U(n),$$

the unitary group. (I use the embedding $A + iB \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ of $GL(n, \mathbb{C})$ in $O(2n)$.) Recall that $\det : U(n) \to S^1$, the circle group, is a homomorphism with kernel $SU(n)$ and a fibration with standard fiber $SU(n)$.

Let $\Lambda(n)$ be the set of Lagrangian subspaces (cf. Section 2.1) of $\mathbb{R}^{2n}$.

<u>Lemma 1</u>. $U(n)$ acts transitively on $\Lambda(n)$ with stationary subgroup $O(n)$, so $\Lambda(n) \cong U(n)/O(n)$.

<u>Proof</u>: By a standard theorem of Lie group theory [38] it

suffices to show the first part. Let $L \in \Lambda(n)$. By the relation of J and σ, $J(L)$ is orthogonal to L. For any $L_1 \in \Lambda(n)$ there is an $A \in O(2n)$ with $A(L) = L_1$. Now $AJ(L)$ must be orthogonal to L_1 so $AJ(L) = J(L_1)$ whence $AJ = JA$ and A is unitary. Clearly the stationary group is $O(n)$. □

Consider the plane $\{(x, 0)\}$ in $\mathbb{R}^{2n}$. Since the determinant of an orthogonal map is ± 1 there is a well-defined map $\mathrm{Det}^2 : \Lambda(n) \to S^1$, the square of the determinant of any unitary transformation carrying $\{(x, 0)\}$ into a Lagrangian subspace. If one defines $S\Lambda(n) := \{L \in \Lambda(n);\ \mathrm{Det}^2(L) = 1\}$ then arguing as in Lemma 1, $S\Lambda(n) \cong SU(n)/SO(n)$. Letting $S^0 = \{\pm 1\}$ one obtains the following commutative diagram of fibrations:

$$\begin{array}{ccccc} SO(n) & \longrightarrow & O(n) & \xrightarrow{\det} & S^0 \\ \downarrow & & \downarrow & & \downarrow \\ SU(n) & \longrightarrow & U(n) & \xrightarrow{\det} & S^1 \\ \downarrow & & \downarrow & & \downarrow w^2 \\ S\Lambda(n) & \longrightarrow & \Lambda(n) & \xrightarrow{\mathrm{Det}^2} & S^1 \end{array} .$$

Lemma 2. $\pi_1(\Lambda(n)) \cong \mathbb{Z}$ and its generator maps to the generator of $\pi_1(S^1)$.

Proof: Apply the exact homotopy sequence of a fibration to the left column and then to the bottom row. □

Corollary 3. $H_1(\Lambda(n)) \cong H^1(\Lambda(n)) \cong \mathbb{Z}$. □

Choose a generator $\alpha \in H^1(\Lambda(n))$ to be the class that gives the winding number of a map $\gamma : S^1 \to \Lambda(n)$; i.e., $\alpha[\gamma]$ is the degree of $\mathrm{Det}^2 \circ \gamma$.

Definition. α is the Arnol'd-Maslov class of $\Lambda(n)$.

More generally, if (E, σ) is a symplectic vector space and $\Lambda(E)$ its Grassmann of Lagrangian subspaces then α_E will denote the Arnol'd-Maslov class of $\Lambda(E)$. Let E_1, E_2, M_1, M_2 be Lagrangian subspaces of E with each E_j transverse to each M_k and let γ be a closed curve in $\Lambda(E)$ consisting of an arc of Lagrangian planes transverse to M_1 from E_1 to E_2 followed by an arc of Lagrangian planes transverse to M_2 from E_2 to E_1.

Definition. The Hormander index is $s(M_1, M_2; E_1, E_2) = \alpha_E[\gamma]$.

It is easy to see that s is continuous in all variables and integer valued, thus locally constant.

Now let L be a conic Lagrangian submanifold of $T^*X \setminus 0$. For $\ell \in L$ let $M_1 = T_\ell T^*_{\pi(\ell)}X$ and $M_2 = T_\ell L$ and let $\mathcal{H}_\ell$ be the set of all Lagrangian subspaces of $T_\ell T^*X$ which are transverse to both M_1 and M_2.

Definition. The Keller-Maslov bundle of L is the complex line bundle K over L with fiber

$$K_\ell = \{f : \mathcal{H}_\ell \to \mathbb{C};\ f(E_1) = i^{s(M_1, M_2; E_1, E_2)} f(E_2)\} ,$$

where the f are merely functions.

It follows that K is a smooth complex line bundle with structure group $\mathbb{Z}_4$. It can be shown that K is trivial and that if $\pi_*|TL$ has constant rank then there is a natural trivialization [19].

Clearly this construction can be extended to the case where $\iota : L \to T^*X \setminus 0$ is a conic Lagrangian immersion.