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Abstract - In this thesis, the stability of the discrete bilinear systems are studied. We consider not only the deterministic bilinear models but also the bilinear models operating in a stochastic environment formed by both random parameters and additive noise items. In all cases, we study the time-varying bilinear systems and with feedback control. We consider that the input or control $u(t)$ is not only a deterministic signal or a random signal but also an input with state feedback or output feedback, $u(t) = f(X(t))$ or $u(t) = f(Y(t))$. Some significant new results are given in the thesis. Almost all of the papers in the reference list deal with stability problems by finding sufficient conditions for the existence of a stabilizing feedback control. However, they do not deal with the problems of analyzing stability under a large class of inputs. Here the problems of analyzing stability under a large class of inputs are considered. Also, very little has been written on the stability of time-varying discrete bilinear systems with feedback control. And most published conditions for stability are difficult to apply in nonlinear cases. Here it is assumed that the feedback function, f, is of a larger class of functions than the class of linear functions or functions satisfying the Lipschitz condition or quadratic functions. We give the uniformly asymptotically stable and uniformly stable conditions for deterministic bilinear systems and mean-square stability or almost surely stability conditions for the stochastic models. We give these stability conditions for the stochastic models without the assumption of stationarity for the random noises. And, the derived sufficient conditions in all theorems, which assure stability for the corresponding bilinear systems only depend on the parameters of the bilinear systems. So, these results are convenient to check and easily applied in engineering and other areas. An immunology example is introduced, and analyzed by the stability theorem which is proposed in this thesis. Computer simulation adds creditability to the analysis. An immunological application and a motor control problem show the utility of the theorems.

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THE STABILITY OF DISCRETE BILINEAR SYSTEMS

I. INTRODUCTION

Bilinear systems are an important subclass of nonlinear systems. Bilinear representations have been found in various engineering areas, for example, see Mohler (1973), Espana and Landau (1978) and Koivo and Cojocariu (1977), Yang et al (1987). A simple example is as follows.

Example 1.1 Automobiles (Mohler, 1987)

The frictional force between an automobile brake shoe and drum is nearly proportional to the product of the orthogonal force u_1 between the surfaces and their relative velocity. Though actually involving Coulomb friction and velocity terms, the frictional forces generated by the mechanical brake is commonly approximated by

$$f_b = c_b u_1 \frac{dx}{dt}.$$

Then, by a summation of engine force u_2 with inertial, braking, and other frictional forces, it is seen from Newton's second law that the state of the vehicle is given by

$$\begin{aligned} F = m\frac{d^2x}{dt^2} &= -kc_f\frac{dx}{dt} - kf_b + u_2 \\ &= -kc_f\frac{dx}{dt} - kc_b u_1\frac{dx}{dt} + u_2 \\ \frac{d^2x}{dt^2} &= \frac{-kc_f}{m}\frac{dx}{dt} - \frac{-kc_b u_1}{m}\frac{dx}{dt} + \frac{u_2}{m}. \end{aligned}$$

Let $x_1 = x, x_2 = \frac{dx_1}{dt}$, then we have the state equation is as follows:

$$\frac{dX}{dt} = AX + u_1BX + Cu_2, \tag{*}$$

where X is composed of x_1, position, and x_2, velocity; $C = [0, 1/m]^T$;

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -kc_f/m \end{bmatrix};$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & -kc_b/m \end{bmatrix};$$

here k is a proportionality constant, c_f is a vehicle frictional constant, c_b is a brake constant, and m is vehicle mass. Here (*) is a bilinear systems.

Recently, more and more attention has been given to the stability of nonlinear systems, for example, see Mousa et al (1986), and including the bilinear systems, see Ionescu and Monopoli (1975), Longchamp (1980), Quinn (1980), Gutman (1981), Gounaridis and Kalouptsidis (1986), Ryan and Buckingham (1983).

Because of the random nature of the phenomena involved, for example the changes of environmental conditions, an aging of components or possible calibration errors, stochastic systems models have been suggested. Physical phenomena, which can be modelled by stochastic-parameter differential-equations include attitude control of satellites (McLane, 1971), spacecraft and missile control systems (McLane, 1969), chemical reactors, biological cells, and migrations of people (Mohler and Kolodziej, 1980). A naturally discrete macroeconomics model with random parameters was given by Aoki (1976).

Stochastic-parameter, discrete-time systems may arise by sampling a stochastic parameter, differential-equation model for the purpose of digital control. A second possibility is by random sampling deterministic or additive noise-perturbed, stochastic, continuous-time systems.

As the above, discrete bilinear system models (deterministic and stochastic) are useful in the real would. Also, the concept of stability is extremely important, since almost every workable system is designed to be stable. Stability topics are connected with topics of identification, optimal control, Kalman filter etc.

In this thesis, stability for the discrete time-varying bilinear systems (deterministic and stochastic) are studied. Also we consider these cases in which the input or control can be a deterministic , a random signal, and a feedback function. Here, we assume that the feedback function, f, is from a large class of functions (see section II) for all theorems. The class is larger but includes linear functions and these satisfying the Lipschitz conditions and quadratic functions such as usually assumed.

In this thesis, stability of deterministic discrete bilinear systems is studied in section III. Most papers consider time-invariant continuous bilinear systems with linear feedback. Almost all of the references study stability by finding a sufficient condition for the existence of a feedback control such that the resulting closed-loop system is asymptotical stable. Such studies usually do not deal with the stability problems under a large class of inputs such as the problems considered here. Stability for time-varying discrete bilinear systems with output feedback is practically absent in the literature. We consider the input, $U(t)$, as not only a signal but also a function depending on the present and the previous output. All hypotheses for stability are simplified. Those hypotheses depend on the coefficient matrices of the systems and are already given in most existing models. So, these results are very easy to check and to apply in engineering problems. Computer simulations illustrate the utility of the theorems. Comparing with almost all publications, our results in this thesis is very convenient to use.

A simple example as in paper (Gutman, 1981), the bilinear system is considered

$$\frac{dx(t)}{dt} = Ax(t) + \sum_{i=1}^{m}(B_i x + b_{i0})u_i \qquad (**)$$

If there exists a matrix $P = P^T > 0$ such that

$$\begin{bmatrix} (B_1 x + b_{10})^T Px \\ (B_2 x + b_{20})^T Px \\ \cdot \\ \cdot \\ \cdot \\ (B_m x + b_{m0})^T Px \end{bmatrix} \neq 0$$

in the set $\{x | x \neq 0,\ x^T(PA + A^T P)x \geq 0\}$, then there exists an $\alpha > 0$, such that the control

$$u_i = -\alpha(B_i x + b_{i0})^T Px \qquad i = 1, 2, \cdots, m$$

will stabilize (**).

The result is good but the problem is to find a P. The hypothesis in most other papers are much complicated than this result of this paper.

All theorems in this thesis are new, some significant results are given, although most theorems in this thesis are the local stability theorems.

For discrete deterministic bilinear systems, there are few results from the studies of stability. The main results are as follows:

Result 1: In the bilinear system, one set that the input function $u(t)$ is constant (Minaidis at al, 1986). Essentially, in this case (input is constant) bilinear systems are not big different from linear systems.

Result 2: Consider the system (Mousa at al, 1986):

$$X(k+1) = f(X(k), U(k)), X(0) = X_0$$

$$Y(k) = C(X(k), U(k)) \qquad (5)$$

where the function C is continuous from R^{m+n} to R^l and the function f, from R^{m+n} to R^n, satisfies Lipschitz condition in u. Mousa gave some results about stability. However, their conditions are strong. For example, the assumption A-2 is the one common condition for each result.

Assumption: There exists a Liapunov function $V : R^n \to R$ which satisfies the following conditions:

(i) $V(0) = 0$, there exists $K_V > 0$ such that $|x| \leq K_V V(x)$ for every $x \in R^n$

(ii) there exists $L_1 > 0$ such that for every x, $\bar{x} \in R^n$, $|V(x) - V(\bar{x})| \leq L_1 |x - \bar{x}|$

and

(iii) there exists $C \in R$ such that $0 < C < 1$ and $DV_{(E)}(x(k)) \leq (C - 1)V(x(k))$ where (E) is given by

$$x(k+1) = f(x(k), 0), \qquad x(0) = x_0 \tag{E}$$

and

$$DV_E(x(k)) = V[x(k+1)] - V[x(k)].$$

The problem is that it is difficult to find V.

In section IV and V stability of stochastic discrete bilinear systems is studied. The most traditional way to model stochastic systems is with additive noise terms such as studied by Goodwin and Sin (1984); Swamy and Tarn (1979). Another is by random-parameter components in the coefficient matrices of the state equations. The first one is easier to treat but the changes in parameter values may be quite large and inclusion of an additive noise term in the system description may not be sufficient to account for these changes. On the other hand, the nature of the process may yield parametric noise. Recently several papers consider random parameter models, for examples, Yaz (1985), Kubrusly (1986). In this thesis, we consider both cases.

Stability for continuous-time systems operating in a stochastic environment have been investigated by several authors, for examples, Kushner (1967), Kozin (1969), Curtain (1972), Has'minskii(1980), Ichikawa (1982), Cao and Ahmed (1987). A few papers have studied discrete-time stochastic systems, for examples, Swamy and Tarn (1979), Phillis (1982), Yaz (1985), Kubrusly (1986). However, very little has been written on the stability of discrete-time stochastic bilinear systems. Although there are some results about stability of stochastic systems from the recent publications, most of them are on stochastic linear systems. Also some new results are difficult to apply. For example, many sufficient conditions for stability require the existences of Liapunov functions for the systems or the existences of some negative (or positive) definite matrix P (or Q) such that the feedback input, $u(t)$, depends on P (or Q), but P and Q are not easy to find. In some papers sufficient conditions for stability are derived for very special inputs. And in most papers, the results are under the assumption of stationary random noises.

There have some good results from Kubrusly (1986), but a stochastic linear system which is called 'bilinear system' is studied:

$$x(i+1) = \left[A_0 + \sum_{k=1}^{p} A_k \omega_k(i)\right] x(i) + Bu(i),$$

where $\{A_k;\ k = 0, 1, \cdots, p\}$ and B are linear transformations, and $\{u(i); i \geq 0\}$ and $\{W(i) = \omega_1(i), \cdots, \omega_p(i); i \geq 0\}$ are second order random input sequences. The feedback input was not considered in this paper, also the results are under the assumptions of stationary random sequences.

In section IV, we study stability of bilinear systems with random parameters. We consider the input, $u(t)$, as not only a time-specified signal but also as a function depending on the present and the previous output. In section V, we study stability of bilinear systems with additive noises. We consider the input, $u(t)$, as an input in the three cases: deterministic (V-2), random (V-3), and with state feedback (V-4).

In this thesis, we give mean-square stability or almost surely conditions for the stochastic models without the stationarity assumption for the random noises. The derived sufficient conditions in all theorems, which assure stability for the corresponding bilinear systems, only depend on the parameters of the bilinear systems. So, these results can be convenient to check and easy to apply in engineering and other areas. Computer simulation adds creditability to the analysis.

In section VI, stability theory of bilinear systems with a quadratic function and more general cases are considered. All the results (from section III-V) can be developed in the case: feedback function f is a quadratic function and polynomial function (degree $p > 2$). Two examples show the utility of the theorems. An immunological application and a motor control problem are presented in section VII.

II. NOTATIONS AND PRELIMINARIES

1. Norms and Notations

Let R^n denote the usual n-dimensional vector space and the norm of a vector, $X = (x_1, \cdots, x_n)^T$, on R^n be denoted by

$$\|X\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

If A is an $n \times m$ matrix over R, then the norm of A is defined by

$$\|A\| = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}.$$

If f is a linear function from R^n to R, then the norm of f is defined by

$$\|f\| = \sup_{X \neq 0} \frac{\|f(X)\|}{\|X\|}.$$

Let $\mathbf{X^n}$ denote the n-dimensional random vector spaces.

Let $\lambda(A)$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the eigenvalues, maximal eigenvalues and minimal eigenvalues of A, respectively. Two sets A, B; $A \setminus B$ means that an arbitrary element $\alpha \in A$, but $\alpha \notin B$.

Let Z^+ denote the set of non-negative integers, that is, $Z^+ \triangleq \{0, 1, 2, \cdots\}$ and $R^+ \triangleq [0, \infty)$. Let

$$\sup_{t \in Z^+} \|A(t)\| \triangleq F_A, \qquad \sup_{t \in Z^+} \|B(t)\| \triangleq F_B,$$

$$\sup_{t \in Z^+} \|C(t)\| \triangleq F_C, \qquad \sup_{t \in Z^+} \|H(t)\| \triangleq F_H.$$

Here, $F_A < \infty$, $F_B < \infty$, $F_C < \infty$, and $F_H < \infty$, are assumed in this thesis.

2. The Assumptions of Feedback Functions

Suppose f is a measurable function (defined from R^n to R^m) and satisfies the following hypotheses:

(a1) $f(0) = 0$

(a2) There exists an integer p where $p \geq 1$, for any variable Z, $Z \in R^n$, such that

$$\|f(Z)\| \leq K_1 \|Z\|^p.$$

Specially, if $Y(t) = H(t)X(t)$ then

$$\|f(Y(t)\| \leq K_1 \|H(t)\|^p \|X(t)\|^p,$$

where K_1 is a constant which may depend on f.

The following classes satisfy the hypotheses:

1 . the linear function:

because $\|f(Z)\| \leq \|f\| \|Z\|$ where $p = 1, K_1 = \|f\|$.

2 . the function in the Lipschitz class:

take $X_1 = 0$, since $f(X_1) = 0$, then $\|f(X_2) - f(X_1)\| \leq K_u \|X_2 - X_1\|$, where $p = 1, K_1 = K_u$.

3 . the quadratic function:

Let $f(X) = X^T Q X$, where Q is a $n \times n$ matrix and $Q = \{q_{ij}\}_{n \times n}$, by Hölder's inequality, we have

$$\begin{aligned}|f(X(t)| =& |\sum_{i,j=1}^{n} q_{i,j}X_iX_j| \\ =& |\sum_{i,j=1}^{n} (\sqrt{q_{i,j}}X_i)(\sqrt{q_{i,j}}X_j)| \\ =& n\left[\left(\sum_{i=1}^{n} q_{i,j}X_i^2\right)\left(\sum_{j=1}^{n} q_{i,j}X_j^2\right)\right]^{1/2} \\ \leq& nk_q\|X(t)\|^2,\end{aligned}$$

where $K_1 = nK_q$, and $K_q = \max_{1\leq i,j\leq n} q_{i,j}$.

In section III-V, we assume that f satisfies $a1$ and $a2$, where $p = 1$, i.e. f satisfies

$$f(0) = 0, \text{ and } |f(Y(t))| \leq K_1\|Y(t)\| \leq K_1\|H(t)\|\|X(t)\|, \tag{2.1}$$

where K_1 is a constant which may depend on f, and $Y(t) = H(t)X(t)$. In section VI, we consider that f satisfies $a1$ and $a2$, where $p \geq 2$. Also we are going to discuss when $-1 < p < 0$, the stability for the bilinear systems in section VI.

3. The Definitions of Stability for Deterministic Systems

A discrete-time nonlinear system is of the form

$$X(t+1) = g(X(t),\ U(t),\ t), \tag{2.2}$$

$$Y(t) = h(X(t), t), \tag{2.3}$$

where $X(t)$ and $Y(t)$ denote the state and the output, respectively, and $U(t)$ is either an input signal or a feedback, i.e. $U(t) = f(X(t))$.

Definition 1: If for every $\epsilon > 0$ and for any $t_0 \geq 0$, there exists $\delta > 0$, (depending on ϵ, t_0) such that the inequality $\|X(t_0)\| < \delta$ implies $\|X(t)\| < \epsilon$ for all $t \geq t_0$, then the zero state for the system, (2.2), (2.3) is called stable.

Definition 2: In the above definition, if δ is independent of t_0, then the zero state for the system, (2.2), (2.3) is called uniformly stable.

Definition 3: If the zero state for the system, (2.2), is stable and there exists a $\delta > 0$ such that $\|X(t_0)\| < \delta$ implies $\lim_{t\to\infty} \|X(t)\| = 0$ then the zero state for the system, (2.2), (2.3) is called asymptotically stable.

Definition 4: The system, (2.2), (2.3), is said to be finite-gain stable if

$$\sum_{t=t_0}^{\infty} \|Y(t)\|^2 \leq k \sum_{t=t_0}^{\infty} \|U(t)\|^2 + \gamma(X(t_0))$$

where $X(t_0) \in R^n$, k is a constant, γ is a function from R^n to R.

4. The Definitions of Stochastic Stability

(1) Stability for Bilinear Systems with Random Parameters

A discrete-time stochastic bilinear system with Random Parameters is of the form

$$X(t+1) = \left[A(t) + \sum_{j=1}^{p} A_j w_j(t)\right] X(t) + B(t)X(t)u(t) \tag{2.4}$$

$$Y(t) = H(t)X(t), \tag{2.5}$$

where $X(t)$ and $Y(t)$ denote the state and the output, respectively and $u(t)$ is either an input signal or a feedback, i.e. $u(t) = f(Y(t))$ and $W(t)$ are random vector defined on $\mathbf{X^n}$. The definitions of stability can be found in Agniel et al (1971) and Has'minskii(1980).

Definition 1: The zero state of system (2.4),(2.5) is said to be mean-square stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$E\left\{\|X(t)\|^2\right\} < \epsilon$$

whenever $t \geq t_0$ and $\|X_0\| < \delta$.

Definition 2: In the above definition, if δ is independent of t_0, then the zero state for the system (2.4),(2.5) is called mean-square uniformly stable.

Definition 3: If the zero state for the system (2.4),(2.5) is mean-square stable and there exists a $\delta > 0$ such that $\|X(t_0)\| < \delta$ implies $\lim_{t\to\infty} E\left\{\|X(t)\|^2\right\} = 0$, then the zero state for the system (2.4),(2.5) is called mean-square asymptotically stable.

Definition 4: The zero state of system (2.4),(2.5) is said to be almost surely (a.s.) stable if for any $\epsilon > 0$, $\epsilon_1 > 0$, there exists a $\delta > 0$ such that

$$P\left\{\|X(t)\| > \epsilon_1\right\} < \epsilon$$

whenever $t \geq t_0$ and $\|X_0\| < \delta$.

Definition 5: The zero state of system (2.4),(2.5) is said to be almost surely asymptotically stable if it is almost surely stable and $\epsilon_1 > 0$, there exists a $\delta > 0$ such that

$$lim_{t\to\infty} P\left\{\|X(t)\| > \epsilon_1\right\} = 0$$

whenever $t \geq t_0$ and $\|X_0\| < \delta$.

Definition 6: The zero state of system (2.4),(2.5) is said to be almost surely uniformly stable if δ is independent of t_0 in the above definition 4.

(2) Stability for Bilinear Systems with Additional Noises

Let $\mathbf{X^n}$ denote the n-dimensional random vector spaces. A discrete-time stochastic nonlinear system is of the form

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t) + C(t)u(t) + \sum_{i=1}^{p} \Gamma_i w_i(t), \tag{2.6}$$

$$Y(t) = H(t)X(t) + \sum_{i=1}^{q} G_i v_i(t), \tag{2.7}$$

where $X(t), Y(t)$ denote the state and the output vector respectively. $u(t)$ is an input, and we assume that the stochastic processes $W(t)$ and $V(t)$ are defined on $\mathbf{X}^n$.

The definitions of mean-square stability can be found in Agniel et al (1971) and Has'miskii'(1980).

Definition 1: The zero state of the systems (2.6), (2.7) is said to be mean-square stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$E\left\{\|X(t)\|^2\right\} < \epsilon$$

whenever $t \geq t_0$ and $\|X_0\| + \sup_{t\geq t_0} \sqrt{E\{W^T(t)W(t)\}} < \delta$.

Definition 2: In the above definition, if δ is independent of t_0, then the zero state for the system (2.6),(2.7) is called mean-square uniformly stable.

Definition 3: If the zero state for the system (2.6),(2.7) is mean-square stable and there exists a $\delta > 0$ such that $\|X(t_0)\| + \sup_{t\geq t_0} \sqrt{E\{W^T(t)W(t)\}} < \delta$ implies $\lim_{t\to\infty} E\left\{\|X(t)\|^2\right\} = 0$ then the zero state for the system (2.6),(2.7) is called mean-square asymptotically stable.

Definition 4: The zero state of the systems (2.6), (2.7) is said to be mean-square bounded if

$$\sup_{t\geq t_0} E\left\{\|x(t)\|^2\right\} < \infty$$

whenever $\|X_0\| + \sup_{t\geq t_0} \sqrt{E\{W^T(t)W(t)\}} < \delta$.

Goodwin and Sin (1984) give a definition of finite-gain stability for deterministic nonlinear systems which not involve the covariances of random noises. Here, however we extend their definition as follows to stochastic nonlinear systems.

Definition 5: The system (2.6), (2.7) is said to be mean-square finite-gain stable if

$$\sum_{t=t_0} E\left\{\|Y(t)\|^2\right\} \le k \sum_{t=t_0} \|u(t)\|^2 + E\left\{\beta(X(t_0))\right\} + \sum_{t=t_0} L[(Q_W(t)] + Z[Q_V(t)],$$

where $X(t_0) \in \mathbf{X^n}$, $k \in R$, Q_W, Q_V are covariance matrices of $W(t)$, $V(t)$ respectively. $\beta : \mathbf{X^n} \to R$ and $L, Z : R^n \to R$ are functions.

Definition 6: The zero state of system (2.6),(2.7) is said to be almost surely stable if for each $\epsilon > 0$ and $\epsilon_1 > 0$ there exists a $\delta > 0$ such that

$$P\left\{\|X(t)\| > \epsilon_1\right\} < \epsilon,$$

whenever $t \ge t_0$ and $\|X_0\| + \sup_{t \ge t_0} \sqrt{E\{W^T(t)W(t)\}} < \delta$.

Definition 7: The zero state of system (2.6),(2.7) is said to be almost surely asymptotically stable if it is almost surely stable and for any $\epsilon_1 > 0$ there exists a $\delta > 0$

$$lim_{t \to \infty} P\left\{\|X(t)\| > \epsilon_1\right\} = 0$$

for $\|X_0\| + \sup_{t \ge t_0} \sqrt{E\{W^T(t)W(t)\}} < \delta$.

Definition 8: The zero state of system (2.6),(2.7) is said to be almost surely stable if δ is independent of t_0 in the above definition 6.

III. STABILITY OF DETERMINISTIC BILINEAR SYSTEMS

1. $u(t)$ is an Input Signal

We first consider the simplest model of bilinear system

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t), \qquad t \geq t_0 \tag{3.1}$$

where $X(t)$ is an $n-$dimensional vector, $u(t)$ is a scalar input. $A(t)$, $B(t)$ are $n \times n$ matrices. Let

$$\tilde{A}(t) \triangleq A^T(t)A(t), \tag{3.2}$$

$$\tilde{B}(t) \triangleq B^T(t)B(t), \tag{3.3}$$

$$\widetilde{BA}(t) \triangleq B^T(t)A(t) + A^T(t)B(t). \tag{3.4}$$

Without loss of generality, we assume $t_0 = 0$.

Lemma 3.1: For system (3.1), the inequality,

$$b(t)\|X(t)\|^2 \leq \|X(t+1)\|^2 \leq a(t)\|X(t)\|^2, \tag{3.5}$$

holds for all $t \geq 0$, where

$$a(t) \triangleq \lambda_{\max}(\tilde{A}(t)) + \lambda_{\max}(\tilde{B}(t))u^2(t) + \max\left|\lambda(\widetilde{BA}(t))\right| |u(t)|, \tag{3.6}$$

$$b(t) \triangleq \lambda_{\min}(\tilde{A}(t)) + \lambda_{\min}(\tilde{B}(t))u^2(t) - \max\left|\lambda(\widetilde{BA}(t))\right| |u(t)|. \tag{3.7}$$

Proof: By (3.1), we have

$$X^T(t+1)X(t+1) = X^T(t)\left[A^T(t)A(t)\right]X(t) + X^T(t)\left[B^T(t)B(t)\right]X(t)u^2(t) + X^T(t)\left[B^T(t)A(t) + A^T(t)B(t)\right]X(t)|u(t)|.$$

So,

$$\begin{aligned}\|X(t+1)\|^2 &\le \lambda_{\max}(\tilde{A}(t))\|X(t)\|^2 + \lambda_{\max}(\tilde{B}(t))\|X(t)\|^2 u^2(t) \\ &\quad + \max\left|\lambda(\widetilde{BA}(t))\right| \|X(t)\|^2 |u(t)| \\ &= a(t)\|X(t)\|^2. \end{aligned} \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}\|X(t+1)\|^2 &\ge \left\{\lambda_{\min}\tilde{A}(t) + \lambda_{\min}\tilde{B}(t)u^2(t) - \max\left|\lambda(\widetilde{BA})(t)\right| |u(t)|\right\} \|X(t)\|^2 \\ &= b(t)\|X(t)\|^2. \end{aligned} \tag{3.9}$$

Here, we use the fact that the eigenvalues of the symmetric matrix of $A^T(t)A(t)$, $B^T(t)B(t)$ and $B^T(t)A(t)+A^T(t)B(t)$ exist and the eigenvalues of A^TA and B^TB are non-negative.

Theorem 3.1: (a) If $a(t) \le a_1 \le 1$ for all t, then the zero state, for the system (3.1), is stable, where $a(t)$ is defined in (3.6).

(b) If $a(t) \le a_1 < 1$ for all t, then the zero state, for the system (3.1), is asymptotically stable.

(c) If there exists $l > 0$ such that $b(t) \ge b_1 > 1$ for all $t > l$, the zero state, for the system (3.1), is not stable.

Proof: For every $\epsilon > 0$, take $\delta < \epsilon$. If $\|X(0)\| < \delta$, by hypothesis and (3.5), we have

$$\|X(1)\| \le \sqrt{a(0)}\|X(0)\| \le \|X(0)\| < \delta. \tag{3.10}$$

By mathematical induction,

$$\|X(t)\| \le \Pi_{i=0}^{t-1} a(i)^{t/2}\|X(0)\| \le a_1^{t/2}\|X(0)\| < a_1^{t/2}\delta < \epsilon. \tag{3.11}$$

So the equilibrium at the origin, for the system (3.1) is stable if $a(t) \leq a_1 \leq 1$. On the other hand, the zero state, for the system (3.1) is asymptotically stable, since

$$\|X(t)\| \leq a_1^{t/2}\|X(0)\| \to 0,$$

if $a_1 < 1$ and $t \to \infty$ for any finite initial $X(0)$.

We have proved (a), (b). By the hypotheses, there exists $l > 0$, such that if $t > l$ then $|b(t)| \geq b_1 > l$. Find an $X(t_1)$, such that $\|X(t_1)\| > 0$ and $t_1 > l$. By lemma 2.1,

$$\|X(t_1+1)\| \geq \sqrt{b(t_1)}\|X(t_1)\| \geq \sqrt{b_1}\|X(t_1)\|, \qquad t_1 > l$$

so

$$\|X(t_1+s)\| \geq b_1^{s/2}\|X(t_1)\| \to \infty \qquad \text{as} \qquad s \to \infty.$$

Now we consider the general model

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t) + C(t)u(t), \tag{3.12}$$

$$Y(t) = H(t)X(t). \tag{3.13}$$

We have the following theorem.

Theorem 3.2: In the system (3.12), (3.13), if i) $a(t) \leq a_1 < 1$ for all t, ii) $C(t)$ and $H(t)$ are uniformly bounded then there exist constants K_1 and K_2 ($0 \leq K_1 < \infty, 0 \leq K_2 < \infty$) which are independent of N such that

$$\sum_{t=0}^{N} \|Y(t)\|^2 \leq K_1\|X(0)\|^2 + K_2 \sum_{t=0}^{N} |u(t)|^2,$$

where $a(t)$ is defined in (3.6). That is, the system is finite-gain stable.

Proof: Let

$$X_1(t+1) = A(t)X(t) + B(t)X(t)u(t). \tag{3.14}$$

By lemma 2.1, and notice $X_1(0) = X(0) - C(0)u(0)$, we have

$$
\begin{aligned}
\|X_1(t+1)\|^2 &\leq a(t)\|X_1(t)\|^2 \\
&\leq a_1\|X_1(t)\|^2 \\
&\leq a_1^2\|X_1(t-1)\|^2 \\
&\leq a_1^{t+1}\|X_1(0)\|^2 \quad \text{(by successive substitution)} \\
&\leq 2a_1^{t+1}\left[\|X(0)\|^2 + \|C(0)\|^2|u(0)|^2\right],
\end{aligned} \tag{3.15}
$$

where $a(t)$ is defined in (3.6). Hence (3.12) can be written as

$$X(t+1) = X_1(t+1) + C(t)u(t).$$

So,

$$
\begin{aligned}
\|X(t+1)\|^2 &\leq 2\left[\|X_1(t+1)\|^2 + F_C^2|u(t)|^2\right] \\
&\leq 4a_1^{t+1}\left[\|X(0)\|^2 + \|C(0)\|^2|u(0)|^2\right] + 2F_C^2|u(t)|^2.
\end{aligned}
$$

By (3.13),

$$
\begin{aligned}
\sum_{t=0}^{N}\|Y(t)\|^2 &\leq F_H^2\sum_{t=0}^{N}\|X(t)\|^2 \\
&\leq 4F_H^2\sum_{t=0}^{N}{a_1}^t\left[\|X(0)\|^2 + \|C(0)\|^2|u(0)|^2\right] \\
&\quad + 2F_H^2F_C^2\sum_{t=1}^{N}|u(t-1)|^2.
\end{aligned}
$$

Here, we assume $u(t) = 0$, if $t < 0$. Thus,

$$\sum_{t=0}^{N}\|Y(t)\|^2 \leq K_1\|X(0)\|^2 + K_4|u(0)|^2 + K_3\sum_{\tau=0}^{N}|u(\tau)|^2,$$

where $K_1 = 4F_H^2/(1-a_1)$, $K_4 = \frac{4F_H^2\|C(0)\|^2}{(1-a_1)}$, $K_3 = 2F_H^2F_C^2$. So, we obtain

$$\sum_{t=0}^{N}\|Y(t)\|^2 \leq K_1\|X(0)\|^2 + K_2\sum_{\tau=0}^{N}|u(\tau)|^2.$$

2. $U(t)$ is Generated by an Output Feedback i.e. $U(t) = f(Y(t))$

Now we consider the general form of bilinear system with output feedback as follows:

$$X(t+1) = A(t)X(t) + \sum_{i=1}^{m} B_i(t)X(t)u_i(t) + C(t)U(t), \tag{3.16}$$

$$Y(t) = H(t)X(t), \tag{3.17}$$

$$U(t) \triangleq (u_1(t)\cdots,\ u_m(t))^T = f(Y(t)), \tag{3.18}$$

where $X \in R^n$, $Y \in R^p$, $p \leq n$, $U \in R^m$. $A(t)$, $B_i(t)$, $i = 1,\cdots,m$ are $n \times n$ matrices, $C(t)$ is an $n \times m$ matrix, $H(t)$ is a $p \times n$ matrix, $f : R^p \to R^m$ is defined in (2.1).

The following lemma is important for the stability theory of discrete bilinear systems.

Lemma 3.2: In the general bilinear system (3.16) – (3.18), assume that there exist $\alpha_1 > 0$, a polynomial $h(.)$ which does not include the terms of degree ≤ 3 and positive coefficients, such that the either of the following inequality

$$\|X(t+1)\| \leq \alpha_1 \|X(t)\|^2 + h(\|X(t)\|) \tag{3.19a}$$

or

$$\|X(t+1)\|^2 \leq \alpha_1 \|X(t)\|^2 + h(\|X(t)\|). \tag{3.19b}$$

is held. Then the zero state, for the system (3.16) – (3.18), is uniformly stable and asymptotically stable, if $\alpha_1 < 1$.

Proof: Here we only prove the (3.19b) case. (3.19b) can be rewritten as

$$\|X(t+1)\|^2 \leq \alpha_1 \|X(t)\|^2 + g(\|X(t)\|)\|X(t)\|^2.$$

Where polynomial g(.) has degree ≥ 1 and positive coefficients. Take $t = 0$, since $\|X(0)\| < \delta$, so

$$\|X(1)\|^2 \leq [\alpha_1 + g(\delta)]\delta^2 = \delta^2\beta,$$

where

$$\beta \triangleq \alpha_1 + g(\delta).$$

For every $\epsilon > 0$, one can take δ small enough such that $\beta < 1$ and $0 < \delta < \epsilon$. This can be done provided $\alpha_1 < 1$. Then,

$$\|X(2)\|^2 \leq [\alpha_1 + g(\delta\beta^{1/2})]\delta^2\beta \leq \delta^2\beta^2.$$

Without difficulty, by mathematical induction, one can show that

$$\|X(t)\| \leq \delta\beta^{t/2}.$$

This implies that the zero state, for the system (3.16) – (3.18), is uniformly stable and asymptotically stable, if $\beta \leq 1$ or $\beta < 1$, respectively.

Similarly, we get the same result in (3.19a).

Remark 3.1: This result does not depend on the system (3.16) – (3.18). So that the result can be generalized to the general nonlinear system (1.2) – (1.3).

Remark 3.2: This lemma is different from Liapunov' s first method for continuous case systems, except A, B_i, C are time-invariant. Liapunov' s first method is useful for continuous case ($\dot{X} = g(X)$), because it requires g is analytic and can be expanded in an infinite Taylor series, and also, it requires the remainder term beyond the first-order approximation approaches 0 faster than the linear terms in δx, δu. This lemma is useful for discrete-time nonlinear systems, and we only require f is measurable functions. Also because (3.19) is a inequality, it is very convenient to use the time-varying bilinear systems with various feedback input as Theorem 3.3-3.6. These results can not get from Liapunov' s first method, ever in the corresponding continuous case, because the corresponding perturbation equation at equilibrium origin is

$$\delta\dot{X} = A(t)\delta X + C(t)\delta U.$$

There are not such easy results of stability as in this thesis for the time-varying linear systems with various feedback input.

Now we first consider the simple form of bilinear system with output feedback.

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t), \tag{3.20}$$

$$Y(t) = H(t)X(t), \tag{3.21}$$

$$u(t) = f(Y(t)), \tag{3.22}$$

where $A(t), B(t), H(t)$ are $n \times n$ matrices, X and Y are n-vectors, $u(t)$ is scalar input.

Let

$$\lambda_1 \triangleq \sup_{t \geq 0} \lambda_{\max}[A^T(t)A(t)], \tag{3.23}$$

$$\lambda_2 \triangleq \sup_{t \geq 0} \lambda_{\max}[B^T(t)B(t)], \tag{3.24}$$

$$\lambda_3 \triangleq \sup_{t \geq 0} \max \left| \lambda[B^T(t)A(t) + A^T(t)B(t)] \right|, \qquad (3.25).$$

Theorem 3.3: In the system, (3.20) - (3.22), suppose $f : R^n \to R$ is defined as in (2.1), $H(t)$ is uniformly bounded on Z^+, and $\lambda_2 < \infty$, $\lambda_3 < \infty$. Then the zero state, for the system (3.20) - (3.22), is uniformly stable and asymptotically stable if $\lambda_1 < 1$.

Proof: Using (3.23) – (3.25), then (3.20) becomes

$$\|X(t+1)\|^2 \leq (\lambda_1 + \lambda_2 u^2(t))\|X(t)\|^2 + \lambda_3 \|X(t)\|^2 \,|u(t)|\,.$$

By (2.1), we have

$$\|X(t+1)\|^2 \le \lambda_1\|X(t)\|^2 + \lambda_3 K_1 F_H \|X(t)\|^3 + \lambda_2 K_1^2 F_H^2 \|X(t)\|^4.$$

This Theorem follows by applying Lemma 3.2.

Now we consider the more general system (3.16) – (3.18) with multiple output feedback.

Let

$$\lambda_2 \triangleq \sup_{t\ge 0} \max_{1\le i,j\le m} \{\max|\lambda[B_i^T(t)B_j(t)]|\}, \tag{3.26}$$

$$\lambda_3 \triangleq \sup_{t\ge 0} \max_{1\le i\le m} \{\max|\lambda[B_i^T(t)A(t) + A^T(t)B_i(t)]\}. \tag{3.27}$$

Theorem 3.4: In the system (3.16) – (3.18), suppose $C(t)$ and $H(t)$ are uniformly bounded on Z^+. If

$$\sqrt{\lambda_1} + K_1 F_H F_C < 1,$$

then the zero state, for the system (3.16) - (3.18), is uniformly stable and asymptotically stable, where K_1 is defined in (2.1).

Proof: Let

$$X_1(t+1) = A(t)X(t) + \sum_{i=1}^{m} B_i(t)X(t)u_i(t),$$

$$X_1^T(t+1)X_1(t+1) \le \lambda_1\|X(t)\|^2 + \sum_{i=1}^{m} \lambda_3\|X(t)\|^2|u_i(t)| + \sum_i\sum_j \lambda_2\|X(t)\|^2|u_i(t)||u_j(t)|$$

where λ_1, is defined in (3.23), By Hölder's inequality, we have

$$\sum_{i=1}^{m} |u_i(t)| \le \sqrt{m}\|U(t)\| \le \sqrt{m}K_1 F_H\|X(t)\| \tag{3.28}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{m}|u_i(t)||u_j(t)| \le m\left(\sum_{i=1}^{m}u_i^2(t)\right)^{1/2}\left(\sum_{j=1}^{m}u_j^2(t)\right)^{1/2}$$
$$= m\|U(t)\|^2 \le mK_1^2F_H^2\|X(t)\|^2.$$

So,

$$\|X_1(t+1)\|^2 \le \lambda_1\|X(t)\|^2 + \sqrt{m}\lambda_3K_1F_H\|X(t)\|^3 + mK_1^2F_H^2\lambda_2\|X(t)\|^4.$$

Hence,

$$\|X_1(t+1)\| \le \sqrt{\lambda_1}\|X(t)\| + m^{\frac{1}{4}}\sqrt{\lambda_3K_1F_H}\|X(t)\|^{\frac{3}{2}} + \sqrt{m\lambda_2}K_1F_H\|X(t)\|^2.$$

So,

$$\begin{aligned}\|X(t+1)\| &\le \|X_1(t+1)\| + F_C\|U(t)\| \le \|X_1(t+1)\| + K_1F_HF_C\|X(t)\|\\ &\le \left(\sqrt{\lambda_1} + K_1F_HF_C\right)\|X(t)\|\\ &\quad + \left(\sqrt{m}\lambda_3K_1F_H\right)^{\frac{1}{2}}\|X(t)\|^{\frac{3}{2}} + \sqrt{m\lambda_2}K_1F_H\|X(t)\|^2.\end{aligned}$$

Let $\|W(t)\|^2 \triangleq \|X(t)\|$. Substitute this into the above inequality and applying Lemma 3.2 to get the needed results.

Now let us consider the following system:

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t) + C(t)u(t), \tag{3.29}$$

$$Y(t) = H(t)X(t), \tag{3.30}$$

$$u(t) = f(Y(t),\cdots,Y(t-r+1)), \tag{3.31}$$

where the feedback system $u(t)$ depends not only on $Y(t)$, but also on $Y(t-j)$, $j = 1,\cdots,r-1$. The A, B, C, H are appropriate dimensional matrices, u is a scalar, and the function, f is defined from $R^{n\times r}$ to R.

In some time-varying bilinear systems, it is more convenient to get the norms of A, B than the eigenvalues of λ_1, λ_2. In this case, we may use the norms instead of the eigenvalues.

Theorem 3.5: Consider the system (3.29) – (3.31). Suppose $f : R^{n\times r} \rightarrow R$ is defined as in (2.1). Suppose $A(t)$, $B(t)$, $C(t)$, $H(t)$ are uniformly bounded on Z^+. If

$$F_A + F_C F_H K_1 \sqrt{r} < 1,$$

then the zero state, for the system (3.29) – (3.31), is uniformly stable and asymptotically stable, where F_A, F_C, F_H are the norms of A, C, H respectively, and K_1 is defined in (2.1).

Proof: Let

$$H \triangleq [h_{ij}]_{n\times n}, \qquad \text{and} \qquad X \triangleq [X_1(t), \cdots, X_n(t)]^T .$$

Let us write

$$\begin{aligned} Y^*(t) &= [Y(t), \cdots, Y(t-r+1)] \\ &= [H(t)X(t), \cdots, H(t-r+1)X(t-r+1)] \\ &= \begin{bmatrix} \sum_{k=1}^{n} h_{1k}(t)X_k(t) & \cdots & \sum_{k=1}^{n} h_{1k}(t-r+1)X_k(t-r+1) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} h_{nk}(t)X_k(t) & \cdots & \sum_{k=1}^{n} h_{nk}(t-r+1)X_k(t-r+1) \end{bmatrix} . \end{aligned}$$

Then

$$\|Y^*(t)\| = \left\{ \sum_{i=1}^{n} \sum_{j=0}^{r} \left[\sum_{k=1}^{n} h_{ik}(t-j) X_k(t-j) \right]^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_{j=0}^{r-1} \sum_{i=1}^{n} \left[\sum_{k=1}^{n} h_{ik}^2(t-j) \right] \left[\sum_{k=1}^{n} X_k^2(t-j) \right] \right\}^{1/2}$$

$$\leq \left\{ \sum_{j=0}^{r-1} \|H(t-j)\|^2 \|X(t-j)\|^2 \right\}^{1/2}$$

$$\leq F_H \left\{ \sum_{j=0}^{r-1} \|X(t-j)\|^2 \right\}^{1/2} .$$

Hence,

$$|u(t)| \leq K_1 F_H \left(\sum_{j=0}^{r-1} \|X(t-j)\|^2 \right)^{\frac{1}{2}},$$

where the inequality follows by the same proof as that for Lemma 2.1.

Let

$$\|X^*(t)\| \triangleq \max_{t-r+1 \leq j \leq t} \|X(j)\| \qquad \text{if} \qquad t \geq r, \tag{3.32}$$

and

$$\|X^*(t)\| = \max_{0 \leq j \leq t} \|X(j)\| \qquad \text{if} \qquad t < r.$$

So

$$|u(t)| \leq K_1 F_H \sqrt{r} \|X^*(t)\|. \tag{3.33}$$

From (3.29) – (3.31), we have

$$\begin{aligned} \|X(t+1)\| &\leq F_A \|X(t)\| + F_B \|X(t)\| |u(t)| + F_C |u(t)| \\ &\leq F_A \|X^*(t)\| + F_B K_1 F_H \sqrt{r} \|X^*(t)\|^2 + F_C K_1 F_H \sqrt{r} \|X^*(t)\| \\ &= \left(F_A + F_C K_1 F_H \sqrt{r} \right) \|X^*(t)\| + F_B K_1 F_H \sqrt{r} \|X^*(t)\|^2. \end{aligned}$$

Let

$$b_1 \triangleq F_A + F_C K_1 F_H \sqrt{r} \qquad \text{and} \qquad b_2 \triangleq F_B K_1 F_H \sqrt{r}.$$

Hence

$$\|X(t+1)\| \leq b_1\|X^*(t)\| + b_2\|X^*(t)\|^2. \tag{3.34}$$

Take δ small enough such that $b_1 + b_2(b_1\delta + b_2\delta^2) < 1$, since $b_1 < 1$. Suppose $\|X(0)\| \leq \delta$. Let

$$\nu \triangleq b_1\delta + b_2\delta^2 \qquad \text{and} \qquad \beta \triangleq b_1 + b_2(b_1\delta + b_2\delta^2) = b_1 + b_2\nu.$$

To prove this theorem, we claim that

$$\|X(t)\| \leq \nu\beta^k, \qquad \text{if} \qquad 2kr \leq t < 2(k+1)r, \quad k \geq 1$$

and

$$\|X^*(t)\| \leq \nu\beta^{k-1} \qquad \text{if} \qquad (2k-1)r \leq t < (2k+1)r, \quad k \geq 1.$$

First, let us compute

$$\|X(t)\| \qquad \text{if} \qquad 0 \leq t \leq 2r-1.$$

Since $\|X^*(0)\| = \|X(0)\| \leq \delta$, we have

$$\|X(1)\| \leq b_1\|X^*(0)\| + b_2\|X^*(0)\|^2 = \nu.$$

So

$$\|X^*(1)\| \leq \max\{\nu,\ \delta\}.$$

Continuing in this manner, we have

$$\|X(t)\| \leq \nu \qquad \text{if} \qquad t = 1, 2, \cdots, r-1$$

and

$$\|X^*(t)\| \leq \max\{\nu,\ \delta\} \qquad \text{if} \qquad t = 0, 1, 2, \cdots, r-1.$$

Next, we compute $\|X^*(t)\|$ if $r \leq t < 2r$. Since

$$\|X(r)\| \leq b_1\|X^*(r-1)\| + b_2\|X^*(r-1)\|^2 \leq \max\{\nu\beta,\ \nu\} = \nu,$$

we have

$$\|X^*(r)\| = \sup_{1\leq j\leq r} \|X(j)\| \leq \nu.$$

Again, repeating the procedure, we conclude that

$$\|X(t)\| \leq \nu \qquad \text{and} \qquad \|X^*(t)\| \leq \nu \qquad \text{if} \qquad r \leq t < 2r-1.$$

Now, we prove the claim by induction. For $k = 1$, as before, we have

$$\|X(t)\| \leq \nu\beta, \qquad \text{if} \qquad 2r \leq t < 4r.$$

Therefore,

$$\|X^*(t)\| \leq \nu \qquad \text{if} \qquad r \leq t < 3r.$$

Now, suppose the claim is true for k, i.e. we have the inequalities,

$$\|X(t)\| \leq \nu\beta^k \qquad \text{if} \qquad 2kr \leq t < 2(k+1)r \tag{3.35}$$

and

$$\|X^*(t)\| \leq \nu\beta^{k-1} \qquad \text{if} \qquad (2k-1)r \leq t < (2k+1)r. \tag{3.36}$$

We want to show that we have these inequalities at $k+1$. From (3.36) and (3.34), we have

$$\|X(t)\| \leq \nu\beta^k \qquad \text{if} \qquad (2k+1)r \leq t < (2k+2)r. \tag{3.37}$$

By (3.35), (3.37) and (3.32),

$$\|X^*(t)\| \leq \nu\beta^k \qquad \text{if} \qquad (2k+1)r \leq t < (2k+2)r. \tag{3.38}$$

From (3.34), (3.38)

$$\|X(t)\| \le \nu\beta^{k+1} \qquad \text{if} \qquad (2k+2)r \le t < (2k+3)r. \tag{3.39}$$

From (3.37), (3.39) and (3.32),

$$\|X^*(t)\| \le \nu\beta^{k} \qquad \text{if} \qquad (2k+2)r \le t < (2k+3)r. \tag{3.40}$$

Again, from (3.40) and (3.34),

$$\|X(t)\| \le \nu\beta^{k+1} \qquad \text{if} \qquad (2k+3)r \le t < (2k+4)r. \tag{3.41}$$

Therefore, the claim follows by combining (3.38), (3.39), (3.40) and (3.41).

Corollary 3.5: Suppose $f : R^n \to R$ is a linear function or function satisfying the Lipschitz condition. If

$$\|f\| \le \frac{1-F_A}{\sqrt{r}F_H F_C}, \qquad f \text{ is a linear;}$$
$$\|K_u\| \le \frac{1-F_A}{\sqrt{r}F_H F_C}, \qquad f \text{ satisfies the Lipschitz condition.}$$

Then, the zero state, for the system (3.29) – (3.31), is asymptotically stable.

Proof: This corollary follows by applying Theorem 3.5: $F_A + F_C F_H K_1 \sqrt{r} < 1$, here $K_1 = \|f\|$, if f is a linear function (see section II-2). Also $K_1 = K_u$, if f is the function satisfying the Lipschitz condition.

Now, we study the more general model (3.16), (3.17) with the feedback input vector $U(t)$, and $U(t)$ not only depends on the value of Y(t), but also depends on the past time value of $Y(t)$, i.e. $Y(t-1), \cdots, Y(t-r+1)$.

Theorem 3.6: Let

$$U(t) = f(Y(t),\ Y(t-1), \cdots, Y(t-r+1)). \tag{3.42}$$

In the system (3.16), (3.17) and (3.42), suppose $f : R^{n\times r} \to R^m$ is defined as in (2.1) and $C(t)$, $H(t)$ are uniformly bounded on Z^+. If $K_1F_HF_C\sqrt{r}+\sqrt{\lambda_1} < 1$, and $\lambda_2 < \infty$, $\lambda_3 < \infty$, where $\lambda_1, \lambda_2, \lambda_3$ are defined as (3.23),(3.26),(3.27), then the zero state, for the system (3.16), (3.17) and (3.42), is uniformly stable and asymptotically stable.

Proof: Let

$$X(t+1) \triangleq X_1(t) + C(t)U(t),$$

where $X_1(t)$ is defined as in Theorem 3.4. Let λ_i, $i = 1,2,3$, denote the same notations in (3.23), (3.26), (3.27). By (3.28) and (3.33) we have

$$\sum_{i=1}^{m} |u_i(t)| \leq \sqrt{m}\|U(t)\| \leq \sqrt{m}K_1F_H\sqrt{r}\|X^*(t)\|,$$

where $X^*(t)$ is the same as (3.32). So we have an inequality, as in Theorem 3.4,

$$\|X(t+1)\| \leq \alpha_1\|X^*(t)\| + \alpha_2\|X^*(t)\|^{\frac{3}{2}} + \alpha_3\|X^*(t)\|^2, \tag{3.43}$$

where $\alpha_1 \triangleq \sqrt{\lambda_1} + K_1F_HF_C\sqrt{r}$, $\alpha_2 \triangleq 2\left(\sqrt{mr}\lambda_3K_1F_H\right)^{1/2}$, $\alpha_3 \triangleq \sqrt{mr\lambda_2}K_1F_H$. Theorem 3.6 follows by the same proof as Theorem 3.5, only replace (3.34) by (3.43). From Theorem 3.5, we $b_1 < 1$ is the stability condition, and here the corresponding position of b_1 is α_1. Hence $\alpha_1 < 1$ is the stability condition.

Let

$$\lambda_H \triangleq \sup_{t\geq 0} \lambda_{\max}[H^T(t)H(t)], \tag{3.44}$$

$$\lambda_C \triangleq \sup_{t\geq 0} \lambda_{\max}[C^T(t)C(t)]. \tag{3.45}$$

Remark 3.3: Note that we still have the same results as those of Theorem 3.3 to Theorem 3.6, if we replace F_H and F_C by $\sqrt{\lambda_H}$ and $\sqrt{\lambda_C}$ respectively, and assume that f satisfies (2.1).

Since f satisfies (2.1) and $U(t) = f(y(t))$, then

$$\|U(t)\| < K_1\|Y(t)\|,$$

where

$$\begin{aligned} Y(t) &= H(t)X(t). \\ \|Y(t)\| &= \sqrt{\|Y(t)\|^2} \\ &= \sqrt{X^T(t)H^T(t)H(t)X(t)} \\ &\leq \sqrt{\lambda_H}\|X(t)\|, \end{aligned}$$

So,

$$\|U(t)\| < K_1\lambda_H\|X(t)\|. \tag{3.46}$$

Also

$$\begin{aligned} \|CX(t)\| &= \sqrt{X^T(t)C^T(t)C(t)X(t)} \\ &\leq \sqrt{\lambda_C}\|X(t)\|. \end{aligned} \tag{3.47}$$

Substitute (3.46) and (3.47) into these proofs of Theorem 3.3 – 3.6, we have the same results as above provided replace F_H and F_C by $\sqrt{\lambda_H}$ and $\sqrt{\lambda_C}$ respectively. Specially, we have

Theorem 3.10: In the general system (3.16), (3.17) and (3.42), suppose $f : R^{n\times r} \to R^m$ is defined as in (2.1). If

$$K_1\sqrt{r\lambda_H\lambda_C} + \sqrt{\lambda_1} < 1, \tag{3.48}$$

and $\lambda_2 < \infty$, $\lambda_3 < \infty$, where $\lambda_1, \lambda_2, \lambda_3$ are defined as (3.23), (3.26), (3.27), $\lambda_H and \lambda_C$ is defined as (3.44) and (3.45) respectively. Then the zero state for the system (3.16), (3.17) and (3.42) is uniformly stable and asymptotically stable.

If f is a linear function or function satisfying the Lipschitz condition. Then this sufficient condition (3.48)can be changed to:

$$\|f\|\sqrt{r\lambda_H\lambda_C} + \sqrt{\lambda_1} < 1. \tag{3.49}$$

In some case, it is easier to evaluate the norms than the eigenvalues for time-varying systems. Also sometimes a special norm may be convenient than others. For example, let

$$A = \begin{bmatrix} e^{-t} & \sin t \\ \cos t & e^{-2t} \end{bmatrix},$$

evaluation of the special norm of A is much easier than the eigenvalues.

$$\sup_{t\in Z^+} \|A\|_\infty = 1,$$

where

$$\|A\|_\infty = \sup_{i,j} |a_{i,j}|.$$

The stability theory in term of different definitions of norms is represented in section X – Appendix.

3. The Examples of Computer Simulations

Example 3.1: Consider the following time-invariant bilinear system:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} u(t) + \begin{bmatrix} -0.3 \\ 0.4 \end{bmatrix} u(t), \tag{3.50}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.8 \\ -0.9 & -0.6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3.51}$$

where

$$u(t) = 0.1[y_1(t) + y_2(t)] + 0.12[y_1(t-1) + y_2(t-1)]. \tag{3.52}$$

In Theorem 3.6, $m = 1$, $r = 2$,

$$\|f\| = \sqrt{0.1^2 + 0.12^2} \approx 0.16,$$

$$F_C = \sqrt{(-0.3)^2 + 0.4^2} = 0.5,$$

$$F_H = \sqrt{0.7^2 + 0.8^2 + (-0.9)^2 + (-0.6)^2} \approx 1.52,$$

From (3.23), and here

$$A = \begin{bmatrix} 0.2 & 0.6 \\ 0.4 & 0.4 \end{bmatrix},$$

So,

$$\lambda_1 = \lambda_{\max}[A^T(t)A(t)] \approx 0.66$$

$$\alpha_1 = K_1 F_H F_C \sqrt{r} + \sqrt{\lambda_1} \approx 0.99,$$

$\alpha_2 \approx 1.12$ and $\alpha_3 \approx 0.62$. Since $\alpha_1 < 1$, the zero state, for the system (3.50) – (3.52), is uniformly stable and asymptotically stable. The input $u(t)$ and the outputs $y_1(t)$, $y_2(t)$ are shown in Fig. 1 – Fig. 3, respectively.

Example 3.2: Consider the following time-varying bilinear system.

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0.2 & \frac{t}{2t+2} \\ 0.5\sin t & \frac{0.3t^2}{t^2+1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.4 & \frac{0.7}{t+0.0001} \\ 0.2t\cos t & \frac{t-3}{t+2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} u(t) + \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix} u(t), \tag{3.53}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0.5 & -0.6 \\ 0.7 & 0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3.54}$$

where

$$u(t) = 0.1[y_1(t) + y_2(t)] + 0.2[y_1(t-1) + y_2(t-1)]. \tag{3.55}$$

Using Theorem 3.6, here Similarly to Example 3.1, we have $m = 1$, $r = 2$, $\|f\| \approx 0.22$ (since f is linear), $F_C \approx 0.64$, $F_H \approx 1.07$, $\alpha_1 \approx 0.97$, $\alpha_2 \approx 2.36$ and $\alpha_3 \approx 6.16$. Since $\alpha_1 < 1$, the zero state for the system (3.53) – (3.55), is uniformly stable and asymptotically stable. The input $u(t)$ and the outputs $y_1(t)$, $y_2(t)$ are shown in Fig. 4 – Fig. 6, respectively. Here, it is difficult to find a Liapunov function for this time-varying bilinear system (3.53) – (3.55). Using the theorems in the thesis will be much easy.

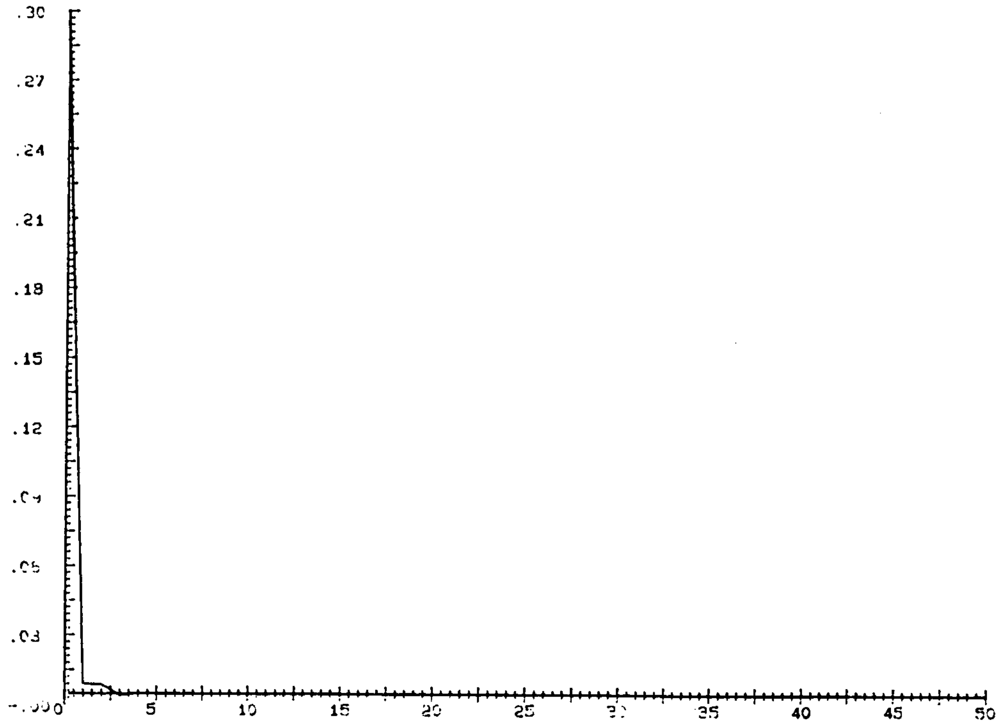

Fig.1 The Input $u(t)$ of Example 3.1

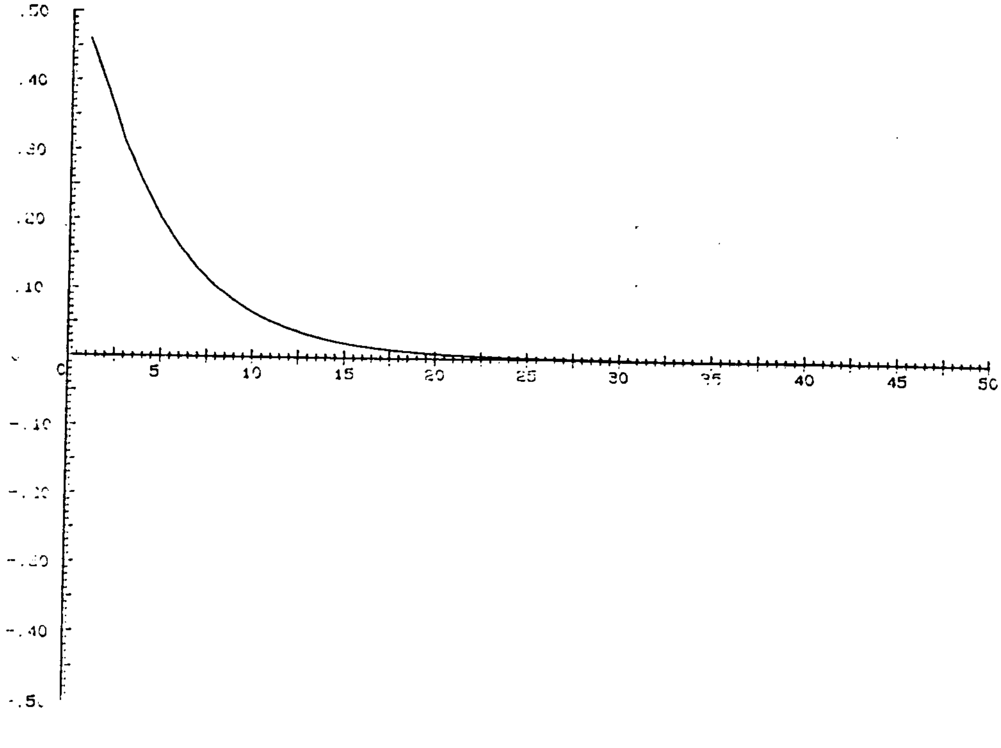

Fig.2 The Output $y_1(t)$ of Example 3.1

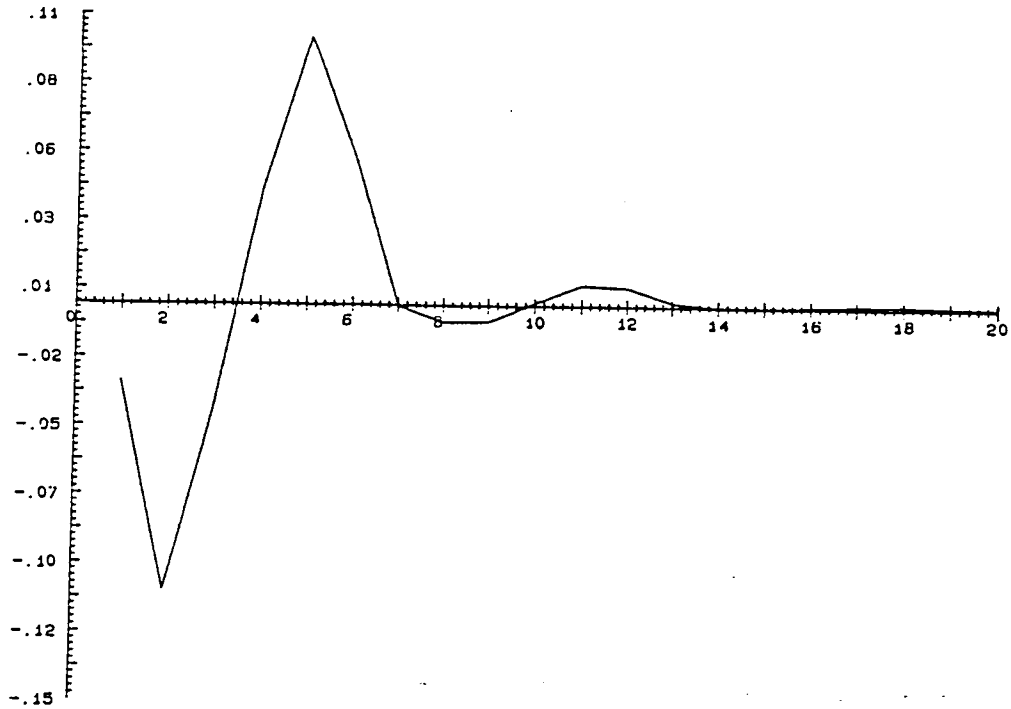

Fig.3 The Output $y_2(t)$ of Example 3.1

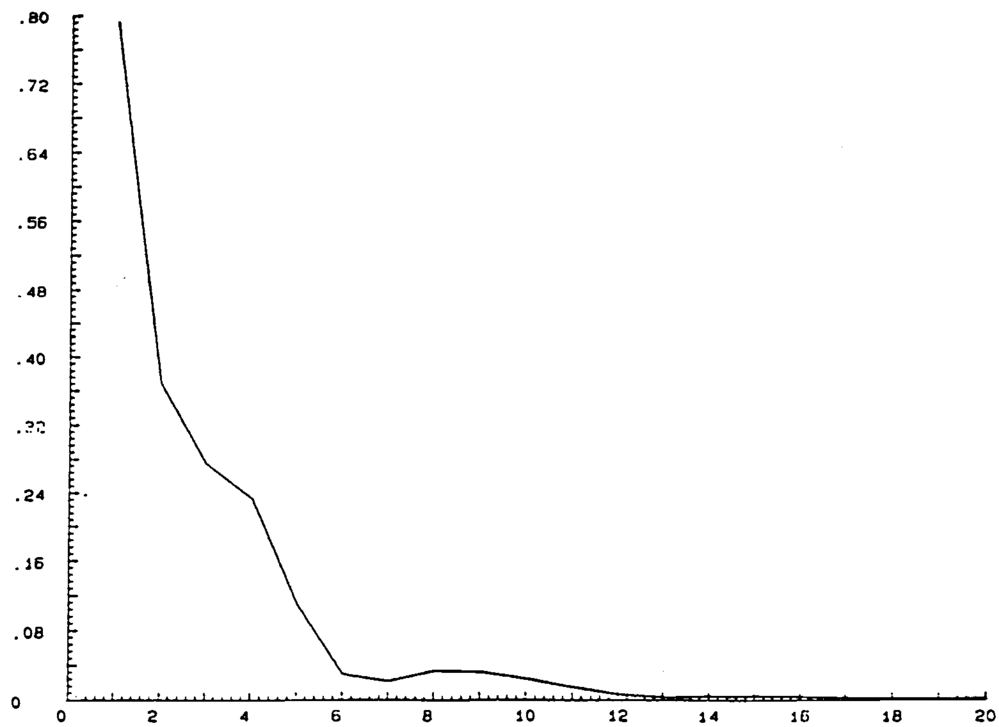

Fig.4 The Input $u(t)$ of Example 3.2

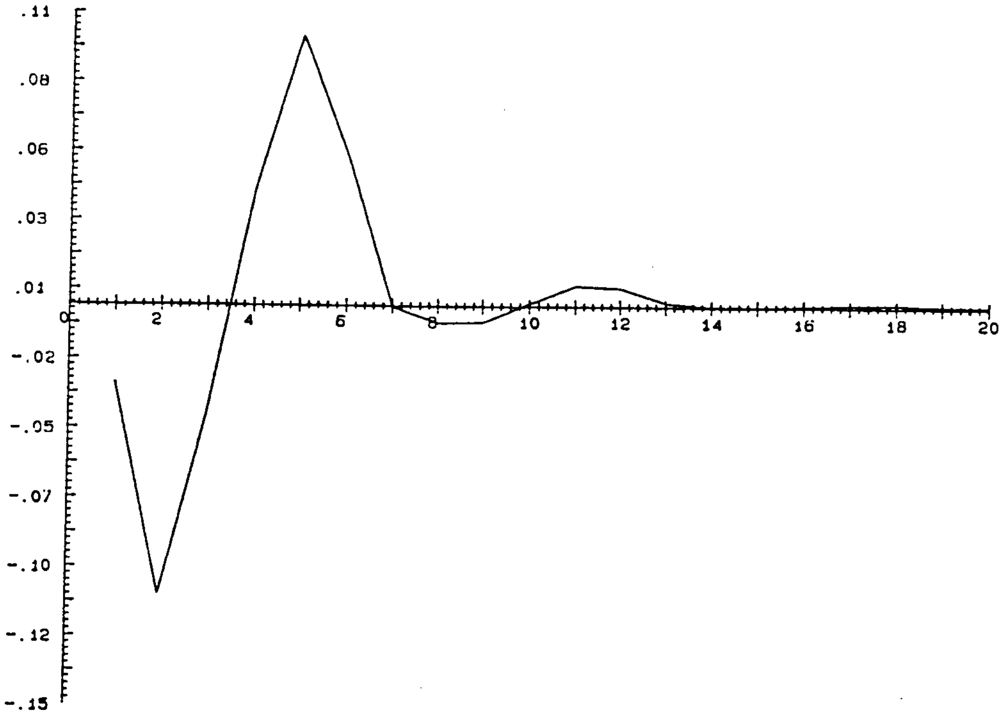

Fig.5 The Output $y_1(t)$ of Example 3.2

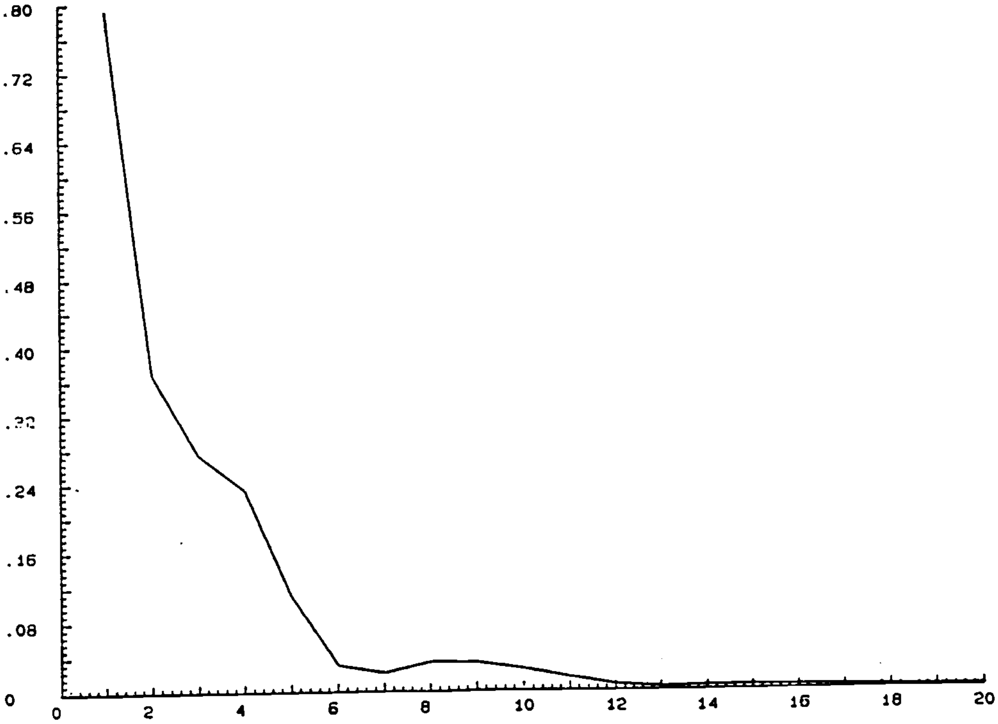

Fig.6 The Output $y_2(t)$ of Example 3.2

IV. STABILITY Of SYSTEMS WITH RANDOM PARAMETERS

1. The Assumptions of Bilinear Systems

The stochastic bilinear model which is formed by random parameters is studied. Let us consider the following stochastic discrete bilinear systems with the output feedback:

$$X(t+1) = \left[A(t) + \sum_{j=1}^{p} A_j w_j(t) \right] X(t) + B(t)X(t)u(t), \tag{4.1}$$

$$u(t) = f(Y(t)), \qquad Y(t) = H(t)X(t), \tag{4.2}$$

where $A(t)$, $B(t)$, $H(t)$ are $n \times n$ matrices, $X(t)$, $Y(t)$ are $n-$vectors, $u(t)$ is scalar input, $\{W(t),\ t \geq 0\}$ is a write noise sequence with $E\{W(t)\} = 0$ and

$$E\{w_i(t)w_j(s)\} = \begin{cases} 0, & \text{if } i \neq j \text{ or } t \neq s; \\ \sigma_i^2(t), & \text{if } i = j \text{ and } t = s, \end{cases} \tag{4.3}$$

where $W(t) = \{w_1(t), \cdots, w_p(t)\}^T$. We also assume that $X(0)$ is independent with $\{w_j(t), j = 1, 2, \cdots, n\}$.

From the model (4.1)-(4.2) and the assumption as above, it is clear that $W(t)$ has an effect on $X(t+j), j > 0$, has no effect on $X(t+j), j \leq 0$, so we have the following statements, as in the paper by Kubrusly (1986):

Independent Argument I:

$$(a) \quad E\left\{ X^T \sum_{j=1}^{p} w_j(t)X(t)u(t) \right\} = E\left\{ X^T(t)E[\sum_{j=1}^{p} w_j(t)]X(t)u(t) \right\} = 0,$$

$$(b) \quad E\left\{ X^T(t) \sum_{j=1}^{p} w_j(t)X(t) \right\} = E\left\{ X^T(t)E\left\{ \sum_{j=1}^{p} w_j(t) \right\} X(t) \right\} = 0,$$

$$(c)\quad E\left\{X^T(t)\sum_{i,j=1}^{p} w_i(t)w_j(t)X(t)\right\} = E\left\{X^T(t)E\left[\sum_{i,j=1}^{p} w_i(t)w_j(t)\right]X(t)\right\}$$
$$= \sum_{j=1}^{p}\sigma_j^2(t)E\left\{\|X(t)\|^2\right\}.$$

2. The Main Results and Proof

The following theorems are based on the next lemma.

Lemma 4.1: In the bilinear system (4.1), (4.2), if there exist positive real numbers α_1, α_2 and α_3 such that

$$E\left\{\|X(t+1)\|^2\right\} \le \alpha_1 E\left\{\|X(t)\|^2\right\} + \alpha_2 E\left\{\|X(t)\|^3\right\} + \alpha_3 E\left\{\|X(t)\|^4\right\},$$

then the zero state for the system (4.1), (4.2), is almost-surely uniformly stable and asymptotically stable if $\alpha_1 < 1$.

Proof: Let M denote the set of all $x \in \mathbf{X^n}$. Take $t = 0$, then we have

$$E\left\{\|X(1)\|^2\right\} \le \alpha_1 E\left\{\|X(0)\|^2\right\} + \alpha_2 E\left\{\|X(0)\|^3\right\} + \alpha_3 E\left\{\|X(0)\|^4\right\}.$$

Suppose $\|X(0)\| < \delta$. $\|X(0)\|^j < \delta^j, j = 2, 3, 4$. Let

$$\beta = \alpha_1 + \alpha_2\delta + \alpha_3\delta^2.$$

Assume $0 < \epsilon, \epsilon_1 < 1$, and take δ small enough such that $\beta < 1$, this can be done since $\alpha_1 < 1$. Then $E\{\|X(1)\|^2\} \le \delta^2\beta < \epsilon$.

By the Tchebycheff inequality, for any given $\epsilon_1 > 0$, we can find a $\delta(\delta \le \epsilon_1^{3/2}\sqrt{\epsilon})$ such that

$$\begin{aligned} P\{\|X(1)\| > \epsilon_1\} &\le tr\{V_{x1}\}/\epsilon_1^2 \\ &= E\left\{\|X(1)\|^2\right\}/\epsilon_1^2 \\ &= \delta^2\beta/\epsilon_1^2 < \epsilon\beta\epsilon_1 < \beta\epsilon_1 < \epsilon_1, \end{aligned} \qquad (*)$$

where V_{x1} is the covariance matrix of $X(1)$. Let

$$S_1^* = \{X \in \mathbf{X^n} : \|X(1)\| > \epsilon_1\},$$

then

$$P(S_1^*) < \epsilon_1,$$

by this above Tchebycheff inequality, for any $\epsilon_1 > 0$. Let

$$S_1 = \{X \in \mathbf{X^n} : \|X(1)\| > \delta\beta^{1/2}\},$$

then, by this above Tchebycheff inequality, we have $P(S_1) \leq \delta\sqrt{\beta}$. We have $S_1^* \subseteq S_1$, because

$$\|X(1)\| > \epsilon_1 > \epsilon_1^{3/2}\sqrt{\epsilon} \geq \delta > \delta\beta^{1/2}.$$

If $X(t) \in M \setminus S_1$, $\|X(1)\| \leq \delta\beta^{1/2}$, then $\|X(1)\|^j \leq \delta^j\beta^{j/2}$, so $E\{\|X(2)\|^2\} \leq \delta^2\beta^2$. Let

$$S = \cup_{j=1}^{\infty} S_j,$$

where

$$S_j = \{X \in \mathbf{X^n} : \|X(j)\| > \delta\beta^{j/2}\},$$

and

$$P(S_j) < \delta\beta^{j/2}.$$

Let

$$S_j^* = \{X \in \mathbf{X^n} : \|X(j)\| > \epsilon_1\},$$

then, $S_j^* \subseteq S_j$, for all j, so we obtain (see(*))

$$P(S_j^*) = P\{\|X(j)\| > \epsilon_1\} < \epsilon\beta^j\epsilon_1 < \epsilon.$$

Therefore, the zero state for the system (4.1), (4.2), is almost-surely asymptotically stable and uniformly stable if $\alpha_1 < 1$.

Let

$$\lambda_1 \triangleq \sup_{t\geq 0} \lambda_{\max}\left[A^T(t)A(t)\right], \tag{4.4}$$

$$\lambda_2 \triangleq \sup_{t\geq 0} \lambda_{\max}\left[B^T(t)B(t)\right], \tag{4.5}$$

$$\lambda_3 \triangleq \sup_{t\geq 0} \max\left|\lambda\left[B^T(t)A(t) + A^T(t)B(t)\right]\right|, \tag{4.6}$$

$$\mu_i \triangleq \lambda_{\max}\left[A_i^T A_i\right], \tag{4.7}$$

$$\mu \triangleq \sup_{t\geq 0} \sum_{i=1}^{p} \mu_i \sigma_i^2(t), \tag{4.8}$$

$$\widetilde{\lambda_1} \triangleq \lambda_1 + \mu. \tag{4.9}$$

Theorem 4.1: In the system, (4.1), (4.2), suppose $f : R^n \rightarrow R$ is defined as in (2.1) and $H(t)$ is uniformly bounded on Z^+. Then the zero state for the system, (4.1), (4.2), is almost-surely uniformly stable and almost-surely asymptotically stable if $\widetilde{\lambda_1} < 1$, $\lambda_2 < \infty$ and $\lambda_3 < \infty$.

Proof: By (4.1)

$$\begin{aligned} E\{X^T(t+1)X(t+1)\} = E\Big\{ & X^T(t)\left[A^T(t)A(t)\right]X(t) \\ & + X^T(t)\left[B^T(t)B(t)\right]X(t)u^2(t) \\ & + X^T(t)\left[B^T(t)A(t) + A^T(t)B(t)\right]X(t)u(t) \\ & + X^T(t)\sum_{j=1}^{p} A_j^T A_j w_j^2(t) X(t) + Re(t)\Big\}, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} Re(t) = {} & X^T(t)\left[B^T(t)A^*(t) + A^{*T}(t)B(t)\right]X(t)u(t) \\ & + X^T(t)\left[A^{*T}(t)A(t) + A^T(t)A^*(t)\right]X(t) \\ & + X^T(t)\sum_{\substack{i,j=1 \\ i\neq j}}^{p} A_i^T A_j w_i(t) w_j(t) X(t) \end{aligned}$$

and

$$A^*(t) \triangleq \sum_{j=1}^{p} A_j w_j(t). \tag{4.11}$$

By the independent argument, it is clear that

$$E\{Re(t)\} = 0. \tag{4.12}$$

From (4.4) - (4.8), the equality (4.10),

$$\begin{aligned} I_1 &= E\{X^T(t)\left[A^T(t)A(t)\right]X(t)\} \leq \lambda_1 E\{\|X(t)\|^2\}, \\ I_2 &= E\{X^T(t)\left[B^T(t)B(t)\right]X(t)u^2(t)\} \\ &= E\Big\{E\Big\{X^T(t)\left[B^T(t)B(t)\right]X(t)u^2(t)|X(t)\Big\}\Big\} \\ &\leq \lambda_2 E\Big\{E\Big\{\|X^T(t)\|^2 u^2(t)|X(t)\Big\}\Big\}, \end{aligned}$$

(2.1) can be used under the condition that $X(t)$ is given. Then we have

$$I_2 \leq E\Big\{\|X^T(t)\|^2\Big[\lambda_2 K_1^2 F_H^2 \|X(t)\|^2\Big]\Big\},$$

or

$$I_2 \leq \lambda_2 K_1^2 F_H^2 E\{\|X(t)\|\}^4.$$

Similarly, we have

$$\begin{aligned} I_3 &= E\{X^T(t)\left[B^T(t)A(t) + A^T(t)B(t)\right]X(t)u(t)\} \\ &= E\Big\{E\Big\{X^T(t)\left[B^T(t)A(t) + A^T(t)B(t)\right]X(t)u(t)|X(t)\Big\}\Big\} \\ &\leq 4\lambda_3 K_1 F_H\ E\{\|X(t)\|^3\}. \end{aligned}$$

We also have

$$\begin{aligned} I_4 &= E\Big\{X^T(t)\sum_{j=1}^{p} A_j^T A_j w_j^2(t) X(t)\Big\} \leq \sum_{j=1}^{p} \mu_j \sigma_j^2(t) E\Big\{\|X(t)\|^2\Big\} \\ &\leq \mu E\Big\{\|X(t)\|^2\Big\}, \end{aligned}$$

where μ is defined by (4.8). So that (4.10) becomes

$$E\{\|X(t+1)\|^2\} \leq \widetilde{\lambda_1} E\{\|X(t)\|^2\} + \lambda_3 K_1 F_H E\{\|X(t)\|^3\} \\ + \lambda_2 K_1^2 F_H^2 E\{\|X(t)\|\}^4.$$

This theorem follows by applying Lemma 4.1.

Now we consider the more general system with multiple output feedback,

$$X(t+1) = \left[A(t) + \sum_{j=0}^{p} A_j w_j(t) \right] X(t) \\ + \sum_{i=1}^{m} B_i(t) X(t) u_i(t) + C(t) U(t), \tag{4.13}$$

$$Y(t) = H(t) X(t), \tag{4.14}$$

$$U(t) \triangleq (u_1(t), \cdots, u_m(t))^T = f(Y(t)), \tag{4.15}$$

where $X \in R^n$, $Y \in R^l$, $l \leq n$, $U \in R^m$. $A(t)$, $B_i(t)$, $i = 1, \cdots, m$ are $n \times n$ matrices, $C(t)$ is an $n \times m$ matrix, $H(t)$ is a $l \times n$ matrix, $f : R^l \to R^m$ is defined in (2.1).

Let

$$\lambda_2 \triangleq \sup_{t \geq 0} \max_{1 \leq i,j \leq m} \{\max |\lambda(B_i^T(t) B_j(t))|\}, \tag{4.16a}$$

$$\lambda_3 \triangleq \sup_{t \geq 0} \max_{1 \leq i \leq m} \{\max |\lambda(B_i^T(t) A(t) + A^T(t) B_i(t))|\}. \tag{4.16b}$$

Theorem 4.2: In the system (4.13) - (4.15), suppose $C(t)$ and $H(t)$ are uniformly bounded on Z^+. If $\lambda_2 < \infty$, $\lambda_3 < \infty$, and

$$2\widetilde{\lambda_1} + 2K_1^2 F_H^2 F_C^2 < 1,$$

then the zero state for the system, (4.13) - (4.15), is almost-surely uniformly stable and almost-surely asymptotically stable where $\widetilde{\lambda_1}$, λ_2 and λ_3 are defined in (4.9), (4.16a), (4.16b) respectively; F_H and F_C are the norms of H and C respectively; K_1 is from (2.1).

Proof: Let

$$X_1(t+1) = [A(t) + A^*(t)]\, X(t) + \sum_{i=1}^{m} B_i(t) X(t) u_i(t)$$
$$= X(t+1) - C(t)U(t),$$

where $A^*(t)$ is defined in (4.11).

Similar to Theorem 4.1, we have

$$E\{X_1^T(t+1)X_1(t+1)\} = \lambda_1 E\{\|X(t)\|^2\} + 4\sqrt{m}\lambda_3 K_1 F_H E\{\|X(t)\|^3\}$$
$$+ mK_1^2 F_H^2 \lambda_2 E\{\|X(t)\|^4\}$$
$$+ E\left\{ X^T(t) \sum_{j=1}^{p} A_j^T A_j w_j^2(t) X(t) \right\} + E\{Re(t)\},$$

where

$$Re(t) = X^T(t)\left[A^{*T}(t)A(t) + A^T(t)A^*(t)\right] + X^T(t)A^{*T}(t)\sum_{i=1}^{m} B_i(t)X(t)u_i(t)$$
$$+ X^T \sum_{j=1}^{m} B_j^T(t) u_j(t) A^*(t) X(t) + X^T(t) \sum_{\substack{i,j=1 \\ i\neq j}}^{p} A_i^T A_j w_i(t) w_j(t) X(t),$$

We have $E\{Re(t)\} = 0$ by the Independent Argument. Then,

$$E\{\|X_1(t+1)\|^2\} \leq \widetilde{\lambda_1} E\{\|X(t)\|^2\} + \sqrt{m}\lambda_3 K_1 F_H E\{\|X(t)\|^3\}$$
$$+ mK_1^2 F_H^2 \lambda_2 E\{\|X(t)\|^4\},$$

where μ_i, w_i are defined in (4.7), (4.3) respectively.

Therefore, the equation, (4.13), can be written as

$$X(t+1) = X_1(t+1) + C(t)U(t).$$

So,

$$E\{\|X(t+1)\|^2\} \leq 2E\{\|X_1(t+1)\|^2\} + 2F_c^2 E\{\|U(t)\|^2\}$$

As in the proof of Theorem 4.1, then

$$E\{\|X(t+1)\|^2\} \le \left[2\widetilde{\lambda_1} + 2K_1^2F_H^2F_C^2\right] E\left\{\|X(t)\|^2\right\}$$
$$+ 2\sqrt{m}\lambda_3 K_1 F_H E\left\{\|X(t)\|^3\right\} + 2mK_1^2F_H^2\lambda_2 E\left\{\|X(t)\|^4\right\}$$

This theorem is proved by applying Lemma 4.1.

Let

$$U(t) = f(Y(t), Y(t-1), \cdots, Y(t-r+1)), \tag{4.17}$$

where the feedback system $U(t)$ depends not only on $Y(t)$ but also on $Y(t-j)$, $j = 1, \cdots, r-1.$, $Y(t)$ is a n-dimension vector.

Notice the following fact:

Also,

$$H \triangleq [h_{ij}]_{n\times n}, \qquad \text{and} \qquad X \triangleq [X_1(t), \cdots, X_n(t)]^T.$$

Then, it is seen that

$$\begin{aligned} Y^*(t) &= [Y(t), \cdots, Y(t-r+1)] \\ &= [H(t)X(t), \cdots, H(t-r+1)X(t-r+1)] \\ &= \begin{bmatrix} \sum_{k=1}^n h_{1k}(t)X_k(t) & \dots & \sum_{k=1}^n h_{1k}(t-r+1)X_k(t-r+1) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n h_{nk}(t)X_k(t) & \dots & \sum_{k=1}^n h_{nk}(t-r+1)X_k(t-r+1) \end{bmatrix}. \end{aligned}$$

So,

$$\|Y^*(t)\|^2 = \left\{ \sum_{i=1}^n \sum_{j=0}^{r-1} \left[\sum_{k=1}^n h_{ik}(t-j)X_k(t-j) \right]^2 \right\}.$$

Then

$$E\{\|Y^*(t)\|\} \le \left[E\{\|Y^*(t)\|^2\}\right]^{1/2} = \left\{E\left[\sum_{i=1}^{n}\sum_{j=0}^{r-1}[\sum_{k=1}^{n} h_{ik}(t-j)X_k(t-j)]^2\right]\right\}^{1/2}$$

$$\le \left\{\sum_{j=0}^{r-1}\sum_{i=1}^{n} E\left[\sum_{k=1}^{n} h_{ik}^2(t-j)\right] E\left[\sum_{k=1}^{n} X_k^2(t-j)\right]\right\}^{1/2}$$

$$\le \left\{\sum_{j=0}^{r-1} \|H(t-j)\|^2 E\{\|X(t-j)\|^2\}\right\}^{1/2}$$

$$\le F_H \left\{\sum_{j=0}^{r-1} E\{\|X(t-j)\|^2\}\right\}^{1/2}.$$

Hence,

$$E\{\|U(t)\|\} \le K_1 F_H \left(\sum_{j=0}^{r-1} E\{\|X(t-j)\|^2\}\right)^{1/2}, \tag{4.19}$$

where the inequality follows by (2.1). Let

$$E\{\|X^*(t)\|\}| = \max_{t-r+1\le j\le t} E\{\|X(j)\|\} \qquad \text{if} \qquad t \ge r. \tag{4.20}$$

So,

$$E\{\|U(t)\|\} \le K_1 F_H \sqrt{r} E\{\|X^*(t)\|\}. \tag{4.21}$$

Let

$$X(t+1) \triangleq X_1(t) + C(t)U(t),$$

where $X_1(t)$ is defined as in Theorem 4.2.

By (4.19), (4.20), we have

$$E\{\sum_{i=1}^{m} |u_i(t)|\} \le \sqrt{m} E\{\|U(t)\|\} \le \sqrt{m} K_1 F_H \sqrt{r} E\{\|X^*(t)\|\},$$

where $X^*(t)$ is the same as (4.20). As in Theorem 4.2, we have a similar inequality

$$E\left\{\|X(t+1)\|^2\right\} \le \alpha_1 E\left\{\|X^*(t)\|^2\right\} + \alpha_2 E\left\{\|X^*(t)\|^3\right\} + \alpha_3 E\left\{\|X^*(t)\|^4\right\}, \tag{4.22}$$

where

$$\alpha_1 = 2\widetilde{\lambda_1} + 2K_1^2 F_H^2 F_C^2,$$

$$\alpha_2 = 2\sqrt{mr}\lambda_3 K_1 F_H,$$

$$\alpha_3 = 2mrK_1^2 F_H^2 \lambda_2.$$

Let

$$\|Z(t+1)\| = \|X(t+1)\|^2,$$

$$\|Z^*(t)\| = \|X^*(t)\|^2.$$

Then (4.22) can be written as

$$E\{\|Z(t+1)\|\} \leq \alpha_1 E\{\|Z^*(t)\|\} + \alpha_2 E\left\{\|Z^*(t)\|^{3/2}\right\} + \alpha_3 E\left\{\|Z^*(t)\|^2\right\}.$$

Let

$$\nu \triangleq \alpha_1\delta + \alpha_2\delta^{3/2} + \alpha_3\delta^2 \qquad \text{and} \qquad \beta \triangleq \alpha_1 + \alpha_2\nu^{1/2} + \alpha_3\nu.$$

Since $\alpha_1 < 1$, one takes δ small enough such that $\beta < 1$. Suppose $\|X(0)\| \leq \delta$.

We claim that

$$E\{\|Z(t)\|\} \leq \nu\beta^k, \qquad \text{if} \qquad 2kr \leq t < 2(k+1)r, \qquad k \geq 1$$

and

$$E\{\|Z^*(t)\|\} \leq \nu\beta^{k-1}, \qquad \text{if} \qquad (2k-1)r \leq t < (2k+1)r, \qquad k \geq 1.$$

So we may obtain the theorem:

Theorem 4.3: In the system (4.13), (4.14) and (4.17), suppose $f : R^{n\times r} \to R^m$ is defined as in (2.1) and $C(t)$, $H(t)$ are uniformly bounded on Z^+. If

$$\alpha_1 = 2\widetilde{\lambda_1} + 2K_1^2 F_H^2 F_C^2 < 1,$$

where $\widetilde{\lambda_1}$, λ_2 and λ_3 are defined in (4.9), (4.16a), (4.16b) respectively.

Then the zero state for the system, (4.13), (4.14) and (4.17) is almost-surely uniformly stable and almost-surely asymptotically stable. Especially, if f is a linear functional or satisfies Lipschitz condition then the results still hold provided

$$\|f\| < \frac{1}{\sqrt{2}F_H F_C}\{1 - 2\widetilde{\lambda_1}\}$$

or

$$\|K_u\| < \frac{1}{\sqrt{2}F_H F_C}\{1 - 2\widetilde{\lambda_1}\},$$

where $\|f\|, K_u$ are defined in II-2.

3. For the 2nd-Order Stationary Process

Now we study the time-invariant bilinear systems with the noise of 2nd-order stationary process, as the follows:

$$X(t+1) = \left[A + \sum_{j=1}^{p} A_j w_j\right] X(t) + BX(t)u(t), \tag{4.23}$$

$$u(t) = f(Y(t)), \qquad Y(t) = HX(t), \tag{4.24}$$

where A, B, H are $n \times n$ matrices, $X(t)$, $Y(t)$ are $n-$vectors, $u(t)$ is scalar input, $\{W,\ t \geq 0\}$ is a write 2nd-order stationary noise sequence with $E\{W\} = 0$ and

$$E\{w_i w_j\} = \begin{cases} 0, & \text{if } i \neq j \\ \sigma_i^2, & \text{if } i = j \end{cases}$$

where $W = \{w_1, \cdots, w_p\}^T$. We also assume that $X(0)$ is independent with $\{w_j, j = 1, 2, \cdots, n\}$.

We have the same results as previous, but the hypotheses will be simplified.

Theorem 4.4: In the system, (4.23), (4.24), suppose $f : R^n \to R$ is defined as in (2.1) Then the zero state for the system, (4.23), (4.24), is almost-surely uniformly stable and almost-surely asymptotically stable if $\widetilde{\lambda_1} < 1$, where

$$\lambda_1 \triangleq \lambda_{\max}\left[A^T(t)A(t)\right],$$

$$\mu_i \triangleq \lambda_{\max}\left[A_i^T A_i\right],$$

$$\mu \triangleq \sum_{i=1}^{p} \mu_i \sigma_i^2,$$

$$\widetilde{\lambda_1} \triangleq \lambda_1 + \mu. \tag{4.25}$$

This theorem corresponds theorem 4.1. Similarly, we have the following results.

For the more general system with multiple output feedback,

$$X(t+1) = \left[A + \sum_{j=0}^{p} A_j w_j\right] X(t)$$

$$+ \sum_{i=1}^{m} B_i X(t) u_i(t) + CU(t), \tag{4.26}$$

$$Y(t) = HX(t), \tag{4.27}$$

$$U(t) \triangleq (u_1(t), \cdots, u_m(t))^T = f(Y(t)),$$

where $X \in R^n$, $Y \in R^l$, $l \leq n$, $U \in R^m$. A, B_i, $i = 1, \cdots, m$ are $n \times n$ matrices, C is an $n \times m$ matrix, H is a $l \times n$ matrix, $f : R^l \to R^m$ is defined in (2.1). $w_j, j = 1, \cdots, p$ is defined as in Theorem 4.4.

Theorem 4.5: In the system (4.26) - (4.27), if

$$2\widetilde{\lambda_1} + 2K_1^2 F_H^2 F_C^2 < 1,$$

then the zero state for the system, (4.26) - (4.27), is almost-surely uniformly stable and almost-surely asymptotically stable where $\widetilde{\lambda_1}$ ise defined in (4.25); F_H and F_C are the norms of H and C respectively; K_1 is from (2.1).

Theorem 4.6: In the system (4.26), (4.27) and (4.17), suppose $f : R^{n \times r} \to R^m$ is defined as in (2.1) If

$$\alpha_1 = 2\widetilde{\lambda_1} + 2K_1^2 F_H^2 F_C^2 < 1,$$

where $\widetilde{\lambda_1}$, is defined in (4.25).

4. The Examples of Computer Simulations

Example 4.1: Consider the following time-invariant stochastic bilinear system.

$$\begin{aligned}\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} &= \left\{ \begin{bmatrix} 0.2 & 0.4 \\ 0.5 & -0.3 \end{bmatrix} + \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.4 \end{bmatrix} w(t) \right\} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} u(t) \\ &+ \begin{bmatrix} -0.3 \\ 0.4 \end{bmatrix} u(t), \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} 0.7 & 0.8 \\ -0.9 & -0.6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\end{aligned}$$

where

$$u(t) = 0.24\left[y_1(t) + y_2(t)\right] + 0.32\left[y_1(t-1) + y_2(t-1)\right],$$

and $w(t)$ is a white noise with zero mean and variance 0.2. Here, f is a linear functional, $\|f\| \approx 0.4$, $F_C = 0.5$, $F_H \approx 1.517$, $\widetilde{\lambda_1} \approx 0.41$, $\alpha_1 \approx 2[\widetilde{\lambda_1} + \|f\|^2 F_H^2 F_C^2] \approx 0.998 < 1$, $\alpha_2 \approx 4.39$, and $\alpha_3 \approx 21.58$. Also, the simulations show that the zero state of the system is almost-surely uniformly stable and almost-surely asymptotically stable. The input $u(t)$ and output $y_1(t)$, $y_2(t)$ are shown in Fig. 7 - Fig. 9 respectively.

Example 4.2: Consider the following time-varying stochastic bilinear system with nonlinear feedback.

$$\begin{aligned}\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} &= \left\{ \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & -0.3 \end{bmatrix} + \begin{bmatrix} 0.36 & -0.3 \\ 0.2 & 0.42 \end{bmatrix} w(t) \right\} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0.1 & 0.9 \\ 1.5 & 1.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} u(t) + \begin{bmatrix} -0.3t^2 \exp(-t) \\ 0.4t \exp(-t) \end{bmatrix}, \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} 0.7 \sin t & -0.9 \\ 0.8 & -0.6 \cos t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\end{aligned}$$

where

$$u(t) = 0.2\sin(y_1(t) + y_2(t)) + 0.3[y_1(t-1) + y_2(t-1)].$$

Here, f is a function which satisfies Lipschitz condition, $\|f\| \approx 0.36$, $F_C = 0.5$, $F_H \approx 1.52$, $\widetilde{\lambda_1} \approx 0.38$, $\alpha_1 \approx 0.90$, $\alpha_2 \approx 6.23$, $\alpha_3 \approx 49$. The simulations show that the zero state of the system is almost-surely uniformly stable and almost-surely mean-square asymptotically stable. The input $u(t)$ and output $y_1(t)$, $y_2(t)$ are shown in Fig. 10 - Fig. 12 respectively.

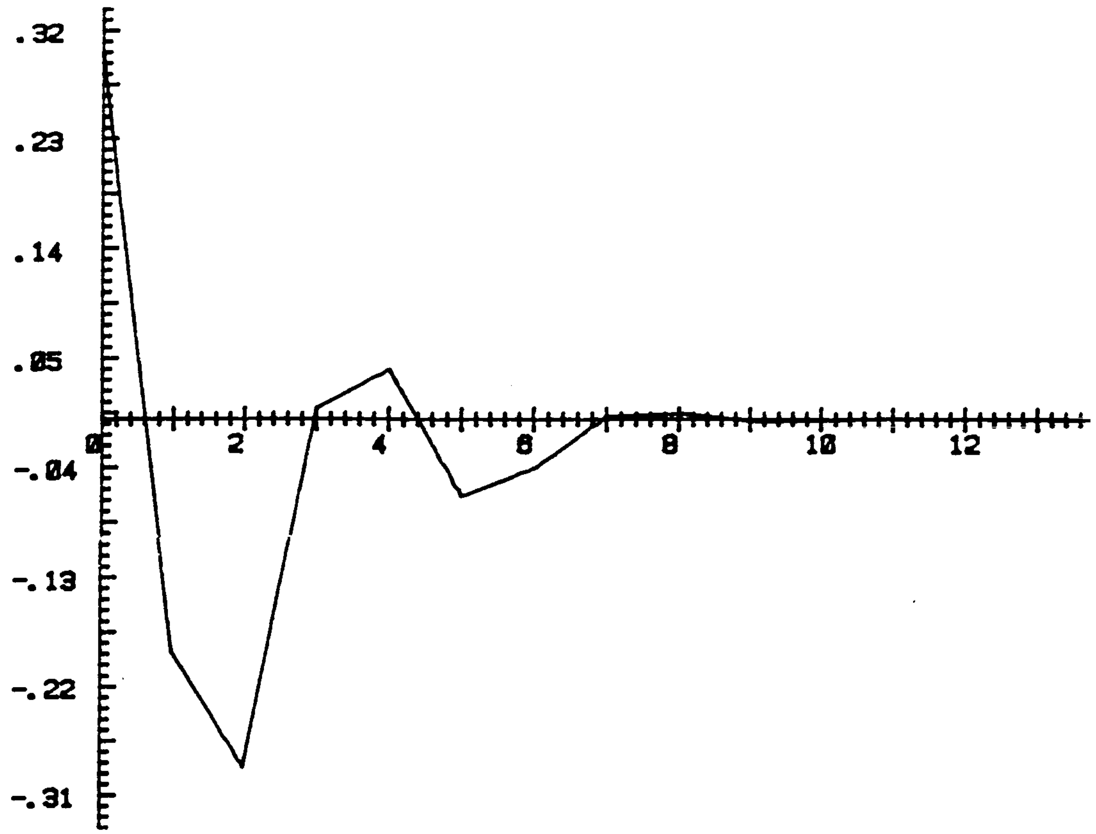

Fig.7 The Input $u(t)$ of Example 4.1

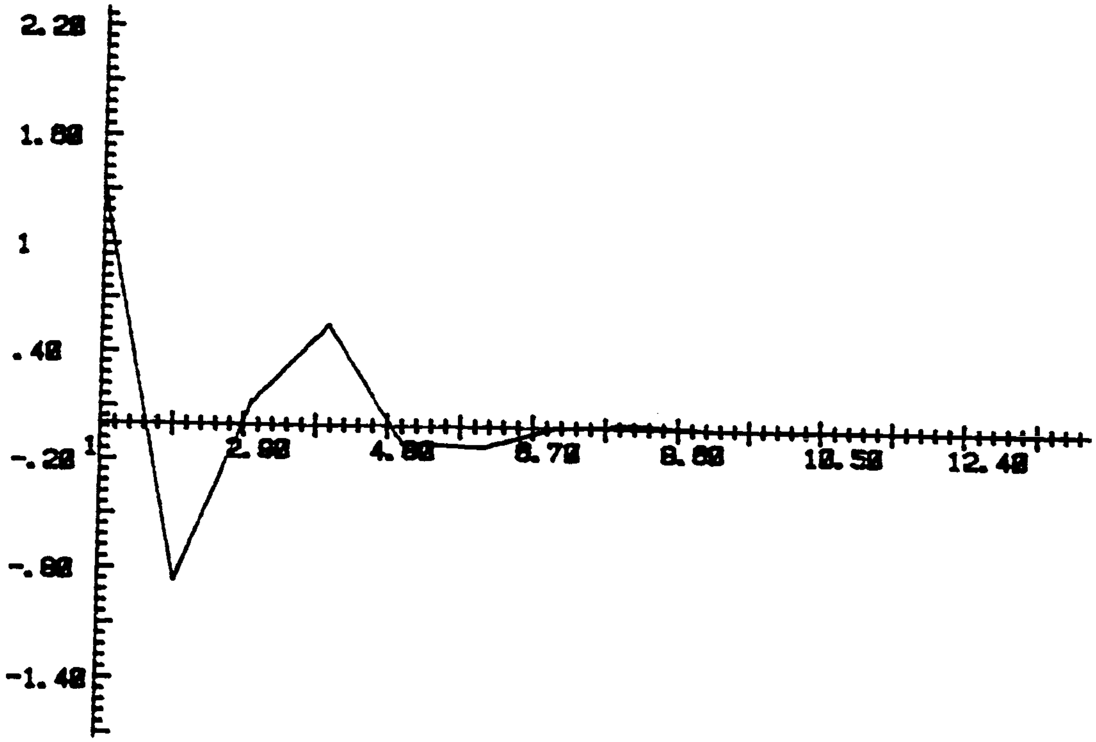

Fig.8 The Output $y_1(t)$ of Example 4.1

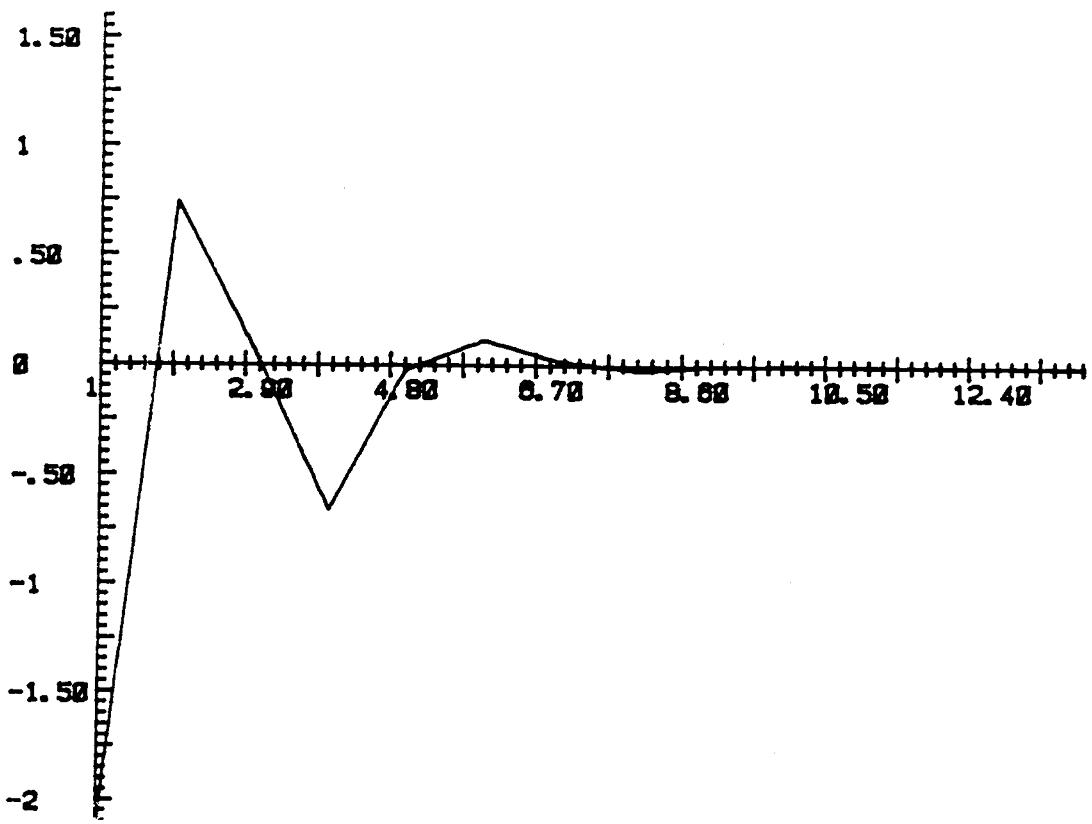

Fig.9 The Output $y_2(t)$ of Example 4.1

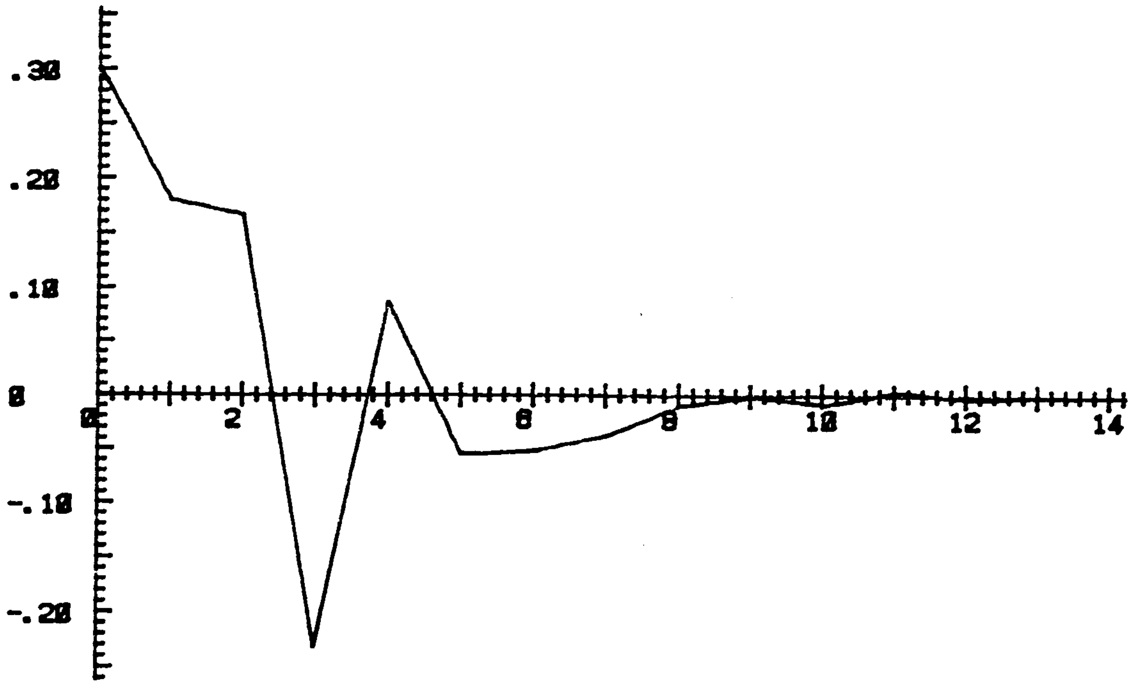

Fig. 10 The Input $u(t)$ of Example 4.2

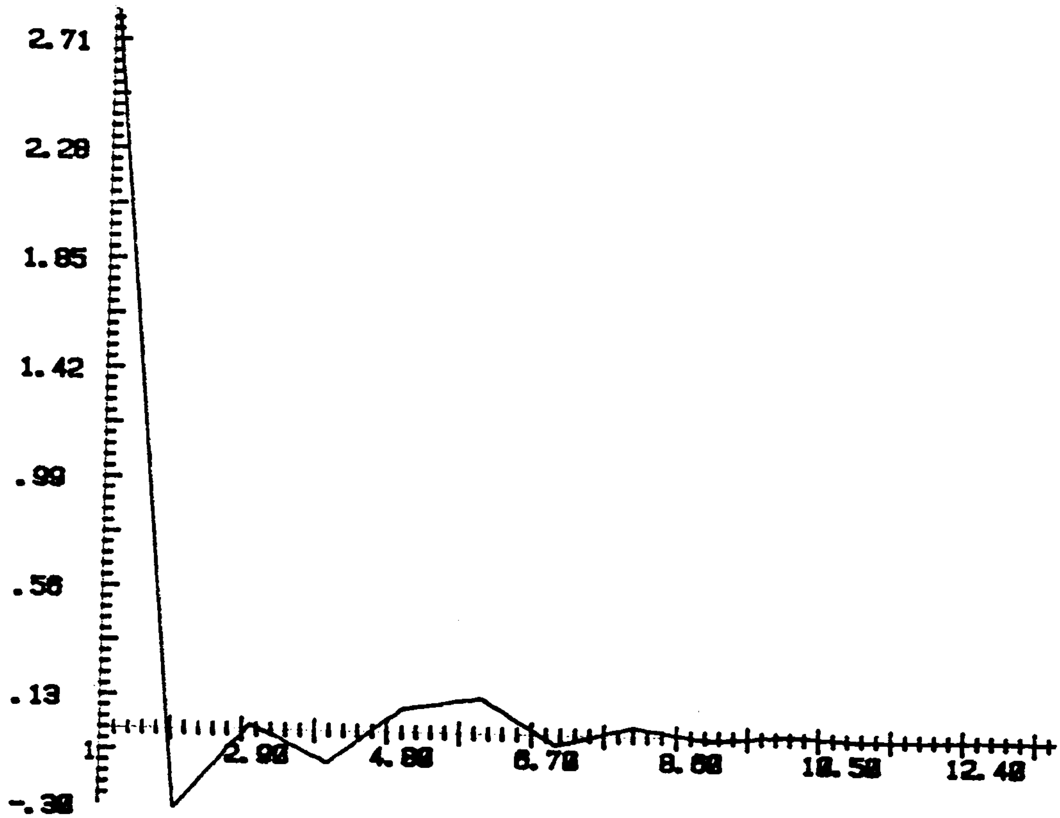

Fig. 11 The Output $y_1(t)$ of Example 4.2

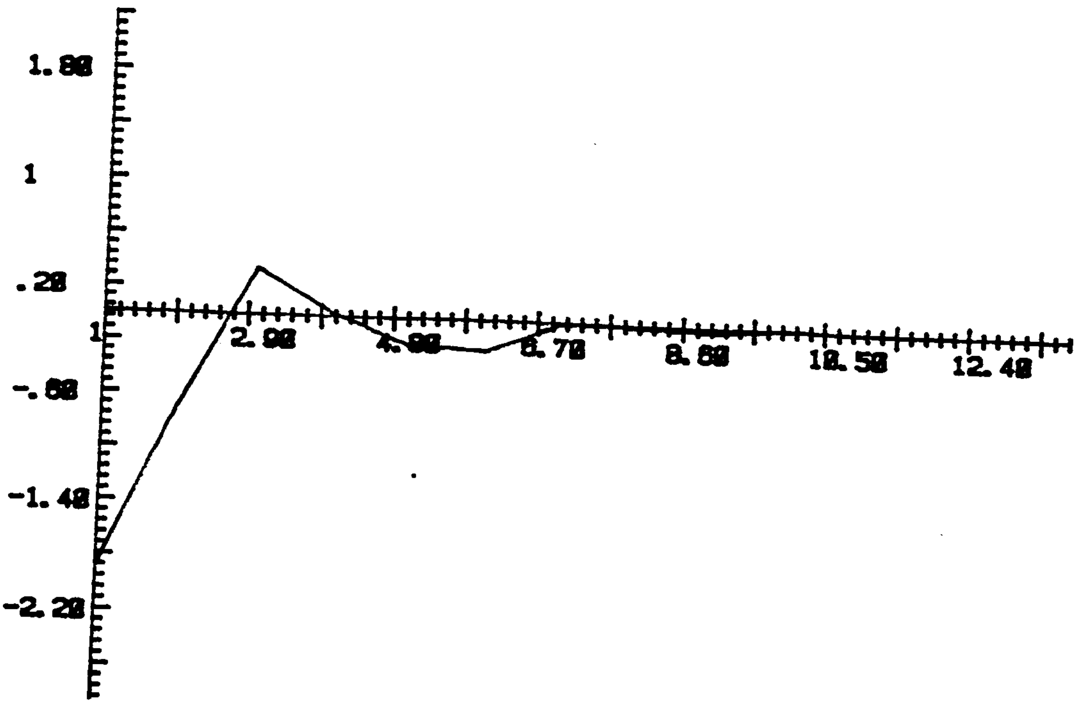

Fig. 12 The Output $y_2(t)$ of Example 4.2

V. STABILITY OF BILINEAR SYSTEMS WITH ADDITIVE NOISES

1. The Assumption of Bilinear Systems

The system is given by:

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t) + C(t)u(t) + \sum_{j=1}^{p} \Gamma_i w_i(t), \tag{5.1}$$

$$Y(t) = H(t)X(t) + \sum_{i=1}^{q} G_i v_i(t), \tag{5.2}$$

where $X(t)$ is an $n-$dimensional state vector, $Y(t)$ is an $n-$dimensional output vector, $u(t)$ is a scalar input. $A(t)$, $B(t)$, $C(t)$, $H(t)$ are time-variant $n \times n$ matrices and Γ_i,G_i are constant $n \times n$ matrices. $\{W(t), t \geq 0\}$ and $\{V(t), t \geq 0\}$ are white noise sequences defined on the probability space $\mathbf{X^n}$, and satisfy

$$E\{w_i(t)w_j(s)\} = \begin{cases} 0, & \text{if } i \neq j; \ \text{ or } t \neq s \\ \sigma_j^2(t)\delta(t-s), & \text{if } i = j \end{cases} \tag{5.3}$$

$$E\{v_i(t)v_j(s)\} = \begin{cases} 0, & \text{if } i \neq j; \ \text{ or } t \neq s \\ \gamma_j^2(t)\delta(t-s), & \text{if } i = j. \end{cases} \tag{5.4}$$

We have

$$W(t) = \{w_1(t), w_2(t), \cdots, w_p(t)\},$$

$$V(t) = \{v_1(t), v_2(t), \cdots, v_q(t)\}.$$

We assume $W(t)$ is independent with $V(t)$, i.e. we have

$$E\{w_i(t)v_j(s)\} = 0 \qquad \text{for all } i,j \text{ at all } t,s.$$

Also, we assume that both $W(t)$ and $V(t)$ are independent with $X(0)$.

Independent Argument I

From the model (5.1)-(5.2) and the above assumptions of (5.3), (5.4) etc., it is clear that $W(t)$ has no effect on $X(t+j), j \leq 0$, as in section IV; the following statement holds:

$$(a) \qquad E\left\{\sum_{j=1}^{p} w_j(t)X^T(t)X(t)\right\} = E\left\{\sum_{j=1}^{p} w_j(t)\right\} E\left\{X^T(t)X(t)\right\} = 0,$$

$$(b) \qquad E\left\{X^T(t)\sum_{j=1}^{p} w_j(t)u(t)\right\} = E\left\{X^T(t)E\left\{\sum_{j=1}^{p} w_j(t)\right\}u(t)\right\} = 0,$$

where $u(t)$ is a non-random signal.

$$(c) \quad E\left\{X^T(t)\sum_{i,j=1}^{p} w_i(t)w_j(t)X(t)\right\} = E\left\{X^T(t)E\left\{\sum_{i,j=1}^{p} w_i(t)w_j(t)\right\}X(t)\right\}$$
$$= \sum_{i=1}^{p} \sigma_i^2(t)E\left\{\|X(t)\|^2\right\}.$$

Remark 5.1: The same results hold for $V(t)$ instead of $W(t)$.

2. $u(t)$ is a Deterministic Signal

Let us consider the following bilinear system:

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t) + \sum_{j=1}^{p} \Gamma_i w_i(t). \qquad (5.5)$$

Lemma 5.1: For system (5.5), the following inequality,

$$b(t)E\{\|X(t)\|^2\} + \sum_{j=1}^{p} \beta_j\sigma_j^2(t) \leq E\{\|X(t+1)\|^2\} \leq a(t)E\{\|X(t)\|^2\} + \sum_{j=1}^{p} \alpha_j\sigma_j^2(t)$$

holds for all $t \geq 0$, where

$$\widetilde{A}(t) \triangleq A^T(t)A(t),$$

$$\tilde{B}(t) \triangleq B^T(t)B(t),$$

$$\widetilde{BA}(t) \triangleq B^T(t)A(t) + A^T(t)B(t),$$

$$a(t) \triangleq \lambda_{\max}(\tilde{A}(t)) + \lambda_{\max}(\tilde{B}(t))u^2(t) + \max\left|\lambda(\widetilde{BA}(t))\right|, \tag{5.6}$$

$$b(t) \triangleq \lambda_{\min}(\tilde{A}(t)) + \lambda_{\min}(\tilde{B}(t))u^2(t) - \max\left|\lambda(\widetilde{BA}(t))\right| |u(t)|, \tag{5.7}$$

$$\alpha_i \triangleq \lambda_{\max}(\Gamma_i^T\Gamma_i), \tag{5.8}$$

$$\beta_i \triangleq \lambda_{\min}(\Gamma_i^T\Gamma_i). \tag{5.9}$$

Proof: From (5.5), we have

$$\begin{aligned} E\{X^T(t+1)X(t+1)\} = E\{X^T(t)&\left[A^T(t)A(t)\right]X(t) + X^T(t)\left[B^T(t)B(t)\right]X(t)u^2(t) \\ &+ X^T(t)\left[B^T(t)A(t) + A^T(t)B(t)\right]X(t)u(t) \\ &+ \sum_{j=1}^{p} \Gamma_j^T\Gamma_j w_j^2(t) + Re(t)\}, \end{aligned}$$

where

$$\begin{aligned} Re(t) = &\sum_{j=1}^{p} \Gamma_j^T w_j(t)[A(t)X(t) + B(t)X(t)u(t)] \\ &+ [X^T(t)A^T(t) + X^T(t)B^T(t)u(t)]\sum_{j=1}^{p} \Gamma_j w_j(t) \\ &+ \sum_{\substack{i,j=1 \\ i\neq j}}^{p} \Gamma_i^T\Gamma_j w_i(t)w_j(t). \end{aligned}$$

In the following proof, we use the fact that the eigenvalues of the symmetric matrices of $A^T(t)A(t)$, $B^T(t)B(t)$ and $B^T(t)A(t) + A^T(t)B(t)$ exist, and the eigenvalues of A^TA and B^TB are non-negative. Therefore,

$$E\{\|X(t+1)\|^2\} \leq \lambda_{\max}(\tilde{A}(t))E\{\|X(t)\|^2\} + \lambda_{\max}(\tilde{B}(t))E\{\|X(t)\|^2\}u^2(t)$$

$$+\max\left|\lambda(\widetilde{BA}(t))\right| E\{\|X(t)\|^2\}|u(t)|$$
$$+\sum_{j=1}^{P}\lambda_{\max}(\Gamma_j^T\Gamma_j)\sigma_j^2(t).$$

Using the Independent Argument I, it is easy to show that $E\{Re(t)\} = 0$, and taking the expectation on both sides, we have

$$E\{\|X(t+1)\|^2\} \leq a(t)E\{\|X(t)\|^2\} + \sum_{j=1}^{p}\alpha_j\sigma_j^2(t).$$

On the other hand, we have

$$E\{\|X(t+1)\|^2\} \geq \left\{\lambda_{\min}\tilde{A}(t) + \lambda_{\min}\tilde{B}(t)u^2(t) - \max\left|\lambda(\widetilde{BA})(t)\right| |u(t)|\right\} E\{\|X(t)\|^2\}$$
$$+\sum_{i=1}^{p}\lambda_{\min}(\Gamma_i^T\Gamma_i)E\{w_i^2(t)\}$$
$$= b(t)E\{\|X(t)\|^2\} + \sum_{j=1}^{p}\beta_j^2\sigma_j^2(t).$$

Theorem 5.1: In the system (5.1), (5.2), if i) $a(t) \leq a_1 < 1$ for all t, ii) $C(t)$ and $H(t)$ are uniformly bounded, then there exist non-negative constants K_1, K_2, K_3, K_4 which are independent of N such that

$$\sum_{t=0}^{N} E\{\|Y(t)\|^2\} \leq K_1E\{\|X(0)\|^2\} + K_2\sum_{t=0}^{N}|u(t)|^2$$
$$+K_3\sum_{t=0}^{N}\|Q_w(t)\|^2 + K_4\sum_{t=0}^{N}\|Q_v(t)\|^2,$$

where $Q_w(t) = \{\sigma_1(t),\cdots,\sigma_p(t)\}^T$, $Q_v = \{\gamma_1(t),\cdots,\gamma_l(t)\}^T$, and $a(t)$ is defined in (5.6), That is, the system is mean-square finite-gain stable.

Proof: Let

$$X(t) = X_1(t) + C(t)u(t),$$

where

$$X_1(t) = A(t)X(t) + B(t)X(t)u(t) + \sum_{j=1}^{p} \Gamma_j w_j(t).$$

By Lemma 5.1, we have

$$\begin{aligned} E\{\|X_1(t+1)\|^2\} &\le a(t)E\|X_1(t)\|^2 + \sum_{j=1}^{p} \alpha_j \sigma_j^2(t) \\ &\le a_1 E\|X_1(t)\|^2 + \sum_{j=1}^{p} \alpha_j \sigma_j^2(t) \\ &\le a_1^2 E\|X_1(t-1)\|^2 + a_1 \sum_{j=1}^{p} \alpha_j \sigma_j^2(t-1) + \sum_{j=1}^{p} \alpha_j \sigma_j^2(t) \\ &\le a_1^{t+1} E\|X_1(0)\|^2 + \sum_{j=1}^{t} (a_1)^{j-1} \sum_{i=1}^{p} \alpha_i \sigma_i^2(t+1-j). \end{aligned}$$

The successive substitution is utilized by the last inequality. Notice $X_1(0) = X(0) - C(0)u(0)$ and

$$E\{\|X_1(0)\|^2\} \le 2E\left\{\|X(0)\|^2 + \|C(0)\|^2|u(0)|^2\right\}.$$

So,

$$\begin{aligned} E\{\|X_1(t)\|^2\} \le\ & 2a_1^t E\left\{\|X(0)\|^2 + \|C(0)\|^2|u(0)|^2\right\} \\ &+ \sum_{j=1}^{t} (a_1)^{j-1} \sum_{i=1}^{p} \alpha_i \sigma_i^2(t-j). \end{aligned}$$

Then,

$$\begin{aligned} E\{\|X(t)\|^2\} &\le 2E\{\|X_1(t)\|^2\} + 2F_C^2|u(t-1)|^2 \\ &\le 4a_1^t E\left\{\|X(0)\|^2 + \|C(0)\|^2|u(0)|^2\right\} \\ &\quad + \sum_{j=1}^{t} (a_1)^{j-1} \sum_{i=1}^{p} \alpha_i \sigma_i^2(t-j) + 2F_C^2|u(t-1)|^2. \end{aligned}$$

Let $h = \sup_{t \geq 0} \lambda_{\max}(H^T(t)H(t))$. By the Independent Argument, we have

$$
\begin{aligned}
E\{\|Y(t)\|^2\} &\leq E\{\|X(t)\|^T H^T(t)H(t)X(t)\} + \sum_{i,j=1}^{q} G_i^T E\{v_i(t)v_j(t)\}G_j \\
&\leq hE\{\|X(t)\|^2\} + \sum_{i,j=1}^{q} G_i^T E\{v_i(t)v_j(t)\}G_j \\
&\leq hE\{\|X(t)\|^2\} + d\sum_{j=1}^{q} \gamma_j^2(t),
\end{aligned}
$$

where $d = \max_{1 \leq i \leq q} \lambda_{\max}(D_i^T G_i)$, and γ_j is defined by (5.4).

$$
\begin{aligned}
E\{\|Y(t)\|\}^2 &\leq K_1 E\{\|X(0)\|^2\} + K_4|u(0)|^2 + K_2 \sum_{\tau=1}^{N} u(\tau - 1)|^2 \\
&\quad + d\sum_{t=1}^{N}\sum_{j=1}^{q} \gamma_j^2(t) + K_p \sum_{\tau=0}^{N-1}\sum_{t=\tau+1}^{N} (a_1)^{t-\tau-1}\|Q_w(\tau)\|^2 \\
&\leq K_1 E\{\|X(0)\|^2\} + K_2 \sum_{\tau=0}^{N-1} |u(\tau)|^2 \\
&\quad + K_3 \sum_{\tau=0}^{N-1} \|Q_w(\tau)\|^2 + K_4 \sum_{\tau=0}^{N-1} \|Q_v(\tau)\|^2,
\end{aligned}
$$

here $K_1 = 4a_1h/(1-2a_1)$, $K_2 = 2hF_C^2$; $K_3 = K_p/(1-a_1)$, where $K_p = \max_{1 \leq i \leq p} \alpha_i$; and $K_4 = d$. So we obtain

$$
\begin{aligned}
\sum_{t=0}^{N} E\{\|Y(t)\|\}^2 &\leq K_1 E\{|X(0)\|\}^2 + K_2 \sum_{\tau=0}^{N} |u(\tau)|^2 \\
&\quad + K_3 \sum_{t=0}^{N} \|Q_w(t)\|^2 + K_4 \sum_{t=1}^{N} \|Q_v(t)\|^2.
\end{aligned}
$$

3. $u(t)$ is a random signal

Consider the system as follows:

$$
X(t+1) = A(t)X(t) + \sum_{j=1}^{m} B_i(t)X(t)u_i(t) + W(t). \tag{5.11}
$$

Here, $W(t)$ is independent with $U(t)$, $X \in R^n$, $U \in R^m$, and

$$U(t) = (u_1(t), u_2(t), \cdots, u_m(t))^T = f(X(t)).$$

$A(t)$, $B_i(t)$, $i = 1, \cdots, m$ are $n \times n$ matrices. $f : R^l \to R^m$ is defined in (2.1). $\{W(t), t \geq 0\}$ is a white noise with zero mean and covariance matrix $Q_w(t)$. We assume that $W(t), U(t)$ are independent with $X(0)$; $Q_u(t)$ is the covariance matrix of $U(t)$.

The Independent Argument II

From the model (5.11) and the assumption as above, it is clear that the following statement holds

$$(a) \qquad E\{X^T(t)A^T(t)W(t)\} = E\left\{X^T(t)A^T(t)E\{W(t)\}\right\} = 0,$$

$$(b) \qquad E\{X^T(t)B_i^T(t)u_i(t)W(t)\} = E\Big\{X^T(t)B_i^T(t)u_i(t)E\{w(t)\}\Big\} = 0.$$

By (5.11), we have

$$\begin{aligned} E\{X^T(t+1)X(t+1)\} =E\Big\{ & X^T(t)\left[A^T(t)A(t)\right]X(t) \\ & + X^T(t)\left[\sum_{i,j=1}^{m} B_i^T(t)B_j(t)\right] X(t)u_i(t)u_j(t) \\ & + X^T(t)\left[\sum_{i=1}^{m} B_i^T(t)A(t) + A^T(t)B_i(t)\right] X(t)u_i(t) \\ & + W^T(t)W(t)\Big\}, \end{aligned} \tag{5.12}$$

where we use the hypothesis of $U(t)$ and $W(t)$, and the Independent Argument II.

Let

$$\widetilde{A}(t) \triangleq A^T(t)A(t),$$

$$\tilde{B}_{ij}(t) \triangleq B_i^T(t)B_j(t),$$

$$\widetilde{AB}_i(t) \triangleq B_i^T(t)A(t) + A^T(t)B_i(t).$$

Let

$$\lambda_1(t) \triangleq \lambda_{\max}(\widetilde{A}(t)), \tag{5.13}$$

$$\lambda_2(t) \triangleq \max_{1\leq i,j\leq m}\{\max|\lambda(\tilde{B}_{ij}(t))|\}, \tag{5.14}$$

$$\lambda_3(t) \triangleq \max_{1\leq i\leq m}\{\max|\lambda(\widetilde{AB}_i(t))|\}. \tag{5.15}$$

Also,

$$I_1 = E\{X^T(t)\left[A^T(t)A(t)\right]X(t)\} \leq \lambda_1(t)E\{\|X(t)\|^2\}.$$

$$\begin{aligned}
I_2 =& E\{X^T(t)\left[\sum_{i,j=1}^{m} B_i^T(t)B_j(t)\right]X(t)u_i(t)u_j(t)\} \\
\leq& \lambda_2(t)E\left\{X^T(t)\left[\sum_{i,j=1}^{m} u_i(t)u_j(t)\right]X(t)\right\} \\
\leq& \lambda_2(t)\sum_{i,j=1}^{m} R_{ij}(t)E\{\|X(t)\|^2\},
\end{aligned} \tag{5.16}$$

where

$$R_{ij}(t) = E\{u_i(t)u_j(t)\}, \tag{5.16a}$$

and the fact, that $U(t)$ has no effect on $X(t+j), j \leq 0$, was employed for the last inequality.

Similarly, we have

$$\begin{aligned}
I_3 =& E\{X^T(t)\left[\sum_{i=1}^{m} B_i^T(t)A(t) + A^T(t)B_i(t)\right]X(t)u_i(t)\} \\
\leq& \lambda_3(t)\sum_{i=1}^{m} E\{u_i(t)\}E\{\|X(t)\|^2\}.
\end{aligned} \tag{5.17}$$

Then (5.12) along with (5.13)-(5.17) becomes

$$
\begin{aligned}
E\{\|X(t+1)\|^2\} &\leq \left[\lambda_1(t) + \lambda_2(t)\sum_{i,j=1}^{m} R_{ij}(t) + \lambda_3(t)\sum_{i=1}^{m} E\{u_i(t)\}\|\}\right] \\
&\qquad \bullet\, E\{\|X(t)\|^2\} + tr(Q_w(t)) \\
&= \alpha(t)E\{\|X(t)\|^2\} + tr(Q_w(t)), \qquad (5.18)
\end{aligned}
$$

where

$$\alpha(t) \triangleq \lambda_1(t) + \lambda_2(t)\sum_{i,j=1}^{m} R_{ij}(t) + \lambda_3(t)\sum_{i=1}^{m} E\{u_i(t)\}.$$

For zero mean $U(t)$, then

$$\alpha(t) = \lambda_1(t) + \lambda_2(t)\sum_{i,j=1}^{m} R_{ij}(t),$$

where $R_{ij}(t)$ is the correlated function of $u_i(t)$ with $u_j(t)$, and defined by (5.16a). Specially, if $\{U(t), t \geq 0\}$ is independent random sequence and with zero mean, then $\alpha(t) = \lambda_1(t)$.

Obviously,

$$tr\{Q_w(t)\} = E\{W^T(t)W(t)\},$$

where $Q_w(t) = E\{W(t)W^T(t)\}$ is the covariance of $W(t)$.

Consequently, we have the following Theorem:

Theorem 5.2: For the system (5.11) suppose $U(t)$ is independent with $W(t)$, which is defined as above with (5.1). If $\alpha(t) \leq \alpha_1$, for all t. Then,

1. The zero state for the system (5.11) is mean-square uniformly stable if $\alpha_1 < 1$.
2. If $\alpha_1 < 1$, $\sup_{t\geq 0} tr(Q_w(t)) < \infty$, then the zero state of the system (5.11) is mean-square bounded.
3. If $\alpha_1 < 1$, then the system (5.11) is mean-square finite-gain stable.

Proof: By (5.18), with $t = 0$, then

$$E\{\|X(1)\|^2\} \leq \alpha_1 E\{\|X(0)\|^2\} + tr(Q_w(0)),$$

$$E\{\|X(2)\|^2\} \leq \alpha_1^2 E\{\|X(0)\|^2\} + \alpha_1 tr(Q_w(0)) + tr(Q_w(1)).$$

By successive substitution, we have

$$E\{\|X(t)\|^2\} \leq \alpha_1^t E\{\|X(0)\|^2\} + \sum_{j=1}^{t} \alpha_1^{j-1} tr(Q_w(t-j)). \tag{5.19}$$

Suppose $\|X(0)\| + \sup_{t\geq 0} \sqrt{tr(Q_w(t))} < \delta$; then it is obviously that $\|X(0)\| < r_1\delta$, $\sup_{t\geq 0} tr(Q_w(t)) < r_2^2\delta^2$, and $0 < r_1, r_2 < 1$. By (5.19) and hypothesis $\alpha_1 < 1$,

$$E\{\|X(t)\|^2\} \leq \alpha_1^t r_1^2 \delta^2 + K_2 r_2^2 \delta^2 \leq \delta^2 + K_2\delta^2,$$

where $K_2 = 1/(1-\alpha_1)$. Take $\delta < \sqrt{\frac{1}{1+K_2}\epsilon}$, then

$$E\{\|X(t)\|^2\} < \epsilon,$$

which proves 1.

Now, it is trivial to get 2. from (5.19).

Introducing $i = t - j$, (5.19) can be rewritten as

$$\begin{aligned}
\sum_{t=1}^{N} E\{\|X(t)\|^2\} &\leq \sum_{t=1}^{N} \alpha_1^t E\{\|X(0)\|^2\} + \sum_{t=1}^{N}\sum_{j=1}^{t} \alpha_1^{j-1} \|Q_w(i)\|^2 \\
&\leq K_1 E\{\|X(0)\|^2\} + \sum_{i=0}^{N-1} \sum_{t=i+1}^{N} \alpha_1^{t-i-1} \|Q_w(i)\|^2 \\
&\leq K_1 E\{\|X(0\|^2\} + K_2 \sum_{i=0}^{N-1} \|Q_w(i)\|^2,
\end{aligned}$$

where $K_1 = \alpha_1/(1-\alpha_1)$, $K_2 = \frac{1}{\alpha_1}$.

So the system (5.11) is mean-square finite-gain stable.

Remark 5.2: If $U(t)$ is 2nd-Order Stationary Process with mean zero, and has the ergodic property. Then, this theorem is held, and

$$\alpha(t) = \lambda_1(t) + \lambda_2(t) \sum_{i,j=1}^{m} R_{ij}(t),$$

and

$$R_{ij}(t) \approx \frac{1}{N}\sum_{t=1}^{N} u_i(t)u_j(t)$$

4. $U(t)$ is Generated by State Feedback, i.e. $U(t) = f(X(t))$

Now, we consider the system

$$X(t+1) = A(t)X(t) + \sum_{i=1}^{m} B_i(t)X(t)u_i(t) + W(t), \tag{5.20}$$

where $X \in R^n$, $U \in R^m$, and

$$U(t) = (u_1(t), u_2(t), \cdots, u_m(t))^T = f(X(t)), \tag{5.20a}$$

where $A(t)$, $B_i(t)$, $i = 1, \cdots, m$ are $n \times n$ matrices. $f : R^n \to R^m$ is defined in (2.1). $\{W(t), t \geq 0\}$ is white noise with zero mean and covariance matrix $Q_w(t)$.

The next theorem is based on the following lemma.

Lemma 5.2: Consider bilinear system (5.20) and $W(t)$ to be a white noise with zero mean and covariance matrix Q_w. Suppose there exist positive real numbers α_1, α_2 and α_3 such that

$$E\left\{\|X(t+1)\|^2\right\} \leq \alpha_1 E\left\{\|X(t)\|^2\right\} + \alpha_2 E\left\{\|X(t)\|^3\right\} + \alpha_3 E\left\{\|X(t)\|^4\right\} + D,$$

where $D = \sup_{t\geq 0} E\{W^T(t)W(t)\}$. Then the zero state for the system (5.20) is almost surely mean-square uniformly stable and asymptotically stable provided $\alpha_1 < 1 - r_2^2$, where $0 < r_1, r_2 < 1$ and $\|(X(0)\| < r_1\delta$, and $\sqrt{D} < r_2\delta$.

Proof: Let M denote the set of all $x \in \mathbf{X^n}$, and take $t = 0$; then we have

$$E\left\{\|X(1)\|^2\right\} \leq \alpha_1 E\left\{\|X(0)\|^2\right\} + \alpha_2 E\left\{\|X(0)\|^3\right\} + \alpha_3 E\left\{\|X(0)\|^4\right\} + D$$

Suppose $\|X(0)\| + \sqrt{D} < \delta$; then there exist r_1 and r_2, where $0 < r_1 < 1$, $0 < r_2 < 1$, such as $\|(X(0)\| < r_1\delta$, and $\sqrt{D} < r_2\delta$.

So $\|X(0)\|^j < r_1^j \delta^j, j = 2,3,4$. Let

$$\beta \triangleq \alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + r_2^2, \tag{*}$$

then

$$\begin{aligned} E\left\{\|X(1)\|^2\right\} &\le \delta^2(\alpha_1 r_1^2 + \alpha_2 r_1^3 \delta + \alpha_3 r_1^4 \delta^2 + r_2^2) \\ &< \delta^2(\alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + r_2^2) \\ &= \delta^2\beta. \end{aligned}$$

Now, we are going to find a condition which will ensure $\beta < 1$. Intuitively, from (*) if $\alpha_1 < 1 - r_2^2$, then $\beta < 1$, provided δ is small enough. In fact, from $\beta < 1$ the roots of the corresponding second-order polynomial equation regarding δ are

$$\delta_{1,2} = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_3(r_2^2 - 1 + \alpha_1)}}{2\alpha_3}.$$

There exist nonzero real roots for any positive real numbers α_1, α_2 and α_3 if and only if $\alpha_1 < (1 - r_2^2)$.

So, in this case, there exists a δ, $\delta_1 < \delta < \delta_2$ then $\beta < 1$, provided $\alpha_1 < 1 - r_2^2$. Assume $0 < \epsilon$, $\epsilon_1 < 1$, and take small enough such that $\beta < 1$, then

$$E\{\|X(1)\|^2\} < \delta^2\beta < \epsilon.$$

So, by the Tchebycheff inequality, for any given $\epsilon_1 > 0$, we find a $\delta(\delta = \epsilon_1^{3/2}\sqrt{\epsilon})$ such that

$$\begin{aligned} P\{\|X(1)\| > \epsilon_1\} &\le tr\{V_{x1}\}/\epsilon_1^2 \\ &\le E\left\{\|X(1)\|^2\right\}/\epsilon_1^2 \\ &= \delta^2\beta/\epsilon_1^2 < \epsilon\beta\epsilon_1 < \beta\epsilon_1 < \epsilon_1, \end{aligned} \tag{**}$$

where V_{x1} is the covariance matrix of $X(1)$.

Let

$$S_1^* = \{X \in \mathbf{X^n} : \|X(1)\| > \epsilon_1\},$$

then

$$P(S_1^*) < \epsilon_1,$$

by this above Tchebycheff inequality, for any $\epsilon_1 > 0$.

$$S_1 = \{X \in \mathbf{X^n} : \|X(1)\| > \delta\beta^{1/2}\},$$

then, we have $P(S_1) \leq \delta\sqrt{\beta}$. We have $S_1^* \subseteq S_1$, because

$$\|X(1)\| > \epsilon_1 > \epsilon_1^{3/2}\sqrt{\epsilon} \geq \delta > \delta\beta^{1/2}.$$

So, we have $P(S_1) < \delta\sqrt{\beta}$. If $X(t) \in M \setminus S_1$ (see section II. for the notation), then $\|X(1)\| < \delta^j\beta^{1/2}$, so $\|X(1)\|^j < \delta^j\beta^{j/2}$. Notice (*) and $\beta < 1$, we have

$$\begin{aligned} E\left\{\|X(2)\|^2\right\} &\leq \delta^2(\alpha_1\beta + \alpha_2\beta^{3/2}\delta + \alpha_3\beta^2\delta^2 + r_2^2) \\ &< \delta^2(\alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + r_2^2) \\ &= \delta^2\beta. \end{aligned}$$

Let

$$S_j = \{X \in \mathbf{X^n} : \|X(j)\| > \delta\beta^{j/2}\},$$

and

$$P(S_j) < \delta\beta^{j/2}.$$

Let

$$S_j^* = \{X \in \mathbf{X^n} : \|X(j)\| > \epsilon_1\},$$

then, $S_j^* \subseteq S_j$, for all j, so we obtain (see(**))

$$P(S_j^*) = P\{\|X(j)\| > \epsilon_1\} < \epsilon\beta^j\epsilon_1 < \epsilon.$$

Therefore, the zero state for the system (5.20) is almost surely asymptotically stable and uniformly stable if $\alpha_1 < 1 - r_2^2$.

Remark 5.3: Let $f(r_2) = 1 - r_2^2$, then $0 < f(r_2) < 1$. Notice the assumption $\|X(0)\| + \sqrt{D} < \delta$ (see section II-4), we may assume the variance of the noise is much small than $\|X(0)\|$, then $f(r_2) \approx 1$. Thus, in the Lemma 5.2, $\alpha_1 < 1 - r_2^2$ can be substituted by $\alpha_1 < 1$. If we take as $\|X(0)\| = \sqrt{D}$, then $f(r_2) = 1 - 0.5^2 = 0.75$.

Let

$$\lambda_1 \triangleq \sup_{t \geq 0} \lambda_{\max} \left[A^T(t)A(t)\right], \tag{5.21}$$

$$\lambda_2 \triangleq \sup_{t \geq 0} \max_{1 \leq i,j \leq m} \{\max |\lambda(B_i^T(t)B_j(t))|\}, \tag{5.22}$$

$$\lambda_3 \triangleq \sup_{t \geq 0} \max_{1 \leq i \leq m} \{\max |\lambda(B_i^T(t)A(t) + A^T(t)B_i(t))|\}. \tag{5.23}$$

Then, we can derive the following result.

Theorem 5.3: In the system (5.20) suppose $f:\ R^n \to R^m$ is defined as in (2.1). Assume $U(t) = f(X(t))$, and $W(t)$ is defined as above. Then the zero state for the system (5.20) is almost-surely uniformly stable and asymptotically stable if $\lambda_2 < \infty$, $\lambda_3 < \infty$, and $\lambda_1 < 1 - r_2^2$ where $0 < r_2 < 1$, and λ_1, λ_2, λ_3, r_2 are defined by (5.21)-(5.23) and Lemma 5.2 respectively.

Proof: By (5.20)

$$\begin{aligned} E\{X^T(t+1)X(t+1)\} = & E\Big\{X^T(t)\left[A^T(t)A(t)\right]X(t) \\ & + X^T(t)\sum_{i,j=1}^{m}[B_i^T(t)B_j(t)]X(t)u_i(t)u_j(t) \\ & + X^T(t)\sum_{i=1}^{m}[B_i^T(t)A(t) + A^T(t)B_i(t)]X(t)u_i(t) \\ & + W^T(t)W(t)\Big\}, \end{aligned} \tag{5.24}$$

where we used the hypothesis of $U(t)$ and the Independent Argument II.

$$I_1 = E\{X^T(t)\left[A^T(t)A(t)\right]X(t)\} \le \lambda_1 E\{\|X(t)\|^2\}.$$

$$\begin{aligned}
I_2 =& E\{X^T(t)\left[\sum_{i,j=1}^{m} B_i^T(t)B_j(t)\right]X(t)u_i(t)u_j(t)\} \\
=& E\left\{E\left\{X^T(t)\left[\sum_{i,j=1}^{m} B_i^T(t)B_j(t)\right]X(t)u_i(t)u_j(t)|X(t)\right\}\right\}. \qquad (5.25)
\end{aligned}$$

Equation 2.1 can be used to estimate $u(t)$ under the condition that $X(t)$ is given with

$$\begin{aligned}
E\{\sum_{i,j=1}^{m} u_i(t)u_j(t)|X(t)\} &\le E\left\{m[\sum_{i=1}^{m} u_i^2(t)]^{1/2}[\sum_{j=1}^{m} u_j^2(t)]^{1/2}|X(t)\right\} \\
&\le mE\{\|U(t)\|^2|X(t)\} \\
&\le mK_1^2E\{\|X(t)\|^2\}. \qquad (5.26)
\end{aligned}$$

Substitution of (5.26) into (5.25) yields

$$I_2 \le \lambda_2 mK_1^2E\left\{\|X(t)\|^4\right\}.$$

Similarly, we have

$$\begin{aligned}
I_3 =& E\{X^T(t)\left[\sum_{i=1}^{m} B_i^T(t)A(t) + A^T(t)B_i(t)\right]X(t)u_i(t)\} \\
=& E\left\{E\left\{X^T(t)\left[\sum_{i=1}^{m} B_i^T(t)A(t) + A^T(t)B_i(t)\right]X(t)u_i(t)|X(t)\}\right\}\right\} \\
\le& \lambda_3 E\left\{E\left\{\left[m\sum_{i=1}^{m} u_i^2(t)\right]^{1/2}\|X(t)\|^2|X(t)\right\}\right\} \\
\le& \lambda_3 mK_1\ E\{|X(t)\|^3\}.
\end{aligned}$$

So that (5.24) becomes

$$E\{\|X(t+1)\|^2\} \leq \widetilde{\lambda_1} E\{\|X(t)\|^2\} + \lambda_3 m K_1 E\{\|X(t)\|^3\} + \lambda_2 m K_1^2 E\{\|X(t)\|\}^4.$$

This theorem follows by application of Lemma 5.2.

Remark 5.4:

For the general bilinear system (5.20), and input $u(t)$ generated by state feedback (5.20a), that the zero state for the system (5.20) is almost-surely uniformly stable and asymptotically stable only depends on the eigenvalues of $A^T A$, does not depend the parameters $B_i, i = 1, 2, \cdots, n$. This conclusion is only when f satisfies $(a1)$, and $(a2)$ in which $p \geq 1$, and consider the zero state stability case. If $-1 < p < 0$, the conclusion will be opposite. We will discuss it in detail in next section. This theorem is easily applied. A good example is shown in section VII.

5. For the 2nd-Order Stationary Process

As in section IV, in the time-invariant bilinear systems with the 2nd-order stationary process, we have the simplified hypotheses for theses results.

Now, we consider the system

$$X(t+1) = AX(t) + \sum_{i=1}^{m} B_i X(t) u_i(t) + W(t), \tag{5.27}$$

where $X \in R^n$, $U \in R^m$, and

$$U(t) = (u_1(t), u_2(t), \cdots, u_m(t))^T = f(X(t)),$$

where A, B_i, $i = 1, \cdots, m$ are $n \times n$ matrices. $f : R^n \to R^m$ is defined in (2.1). $\{W(t), t \geq 0\}$ is the 2nd-order stationary white noise with zero mean and covariance matrix Q_w satisfy

$$E\{w_i w_j\} = \begin{cases} 0, & \text{if } i \neq j; \\ \sigma_j^2, & \text{if } i = j, \end{cases}$$

and we have

$$W = \{w_1, w_2, \cdots, w_p\}.$$

Let

$$\lambda_1 \triangleq \lambda_{\max}[A^T A], \tag{5.28}$$

we have

Theorem 5.4: In the system (5.27) suppose $f:\ R^n \rightarrow R^m$ is defined as in (2.1). Assume $U(t) = f(X(t))$, and $W(t)$ is defined as above. Then the zero state for the system (5.20) is almost-surely uniformly stable and asymptotically stable if $\lambda_1 < 1 - r_2^2$ where $0 < r_2 < 1$, and $\lambda_1,\ r_2$ is defined by (5.28) and Lemma 5.2 respectively.

VI. STABILITY OF BILINEAR SYSTEMS WITH MORE GERERAL FEEDBACK

In section III to Section V, we assume that the feedback function f satisfies (a1), (a2) (see section II): (1) $f(0) = 0$, (2) $\|f(Y(t))\| \leq K_1\|Y(t)\|^p$ where $p = 1$ (see (2.1)), and K_1 is a constant which may depend on Y. Please notice all results from section III to Section V can be developed to the $p \geq 2$. Specifically, a quadratic function satisfies (a1), (a2) if $p = 2$ (see Section II). Here we study the case in which the classes of feedback functions include the quadratic function, i.e. $p \geq 2$. Also, we will discuss when $-1 < p < 0$ case.

1. The Assumptions of Bilinear Systems

Now we consider the general form of bilinear system with output feedback as follows:

$$X(t+1) = A(t)X(t) + \sum_{i=1}^{m} B_i(t)X(t)u_i(t) + C(t)U(t), \tag{6.1}$$

$$Y(t) = H(t)X(t), \tag{6.2}$$

$$U(t) \triangleq (u_1(t)\cdots,\ u_m(t))^T = f(Y(t)), \tag{6.3}$$

where $X \in R^n$, $Y \in R^p$, $p \leq n$, $U \in R^m$. $A(t)$, $B_i(t)$, $i = 1,\cdots,m$ are $n \times n$ matrices, $C(t)$ is an $n \times m$ matrix, $H(t)$ is a $p \times n$ matrix, $f : R^p \to R^m$ is defined by (a1), (a2) (see section II-2), it means that f satisfies:

(a1): $f(0) = 0$,

(b1): For $p \geq 2$, $\|f(Y(t))\| \leq K_1\|Y(t)\|^p \leq K_1F_H^p\|X(t)\|^p$,

where $Y(t) = H(t)X(t)$, and K_1 is a constant which may depend on Y.

2. The Main Results and Proof

Now we first consider the simple form of bilinear system with output feedback.

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t), \tag{6.4}$$

$$Y(t) = H(t)X(t), \tag{6.5}$$

$$u(t) = f(Y(t)), \tag{6.6}$$

where $A(t), B(t), H(t)$ are $n \times n$ matrices, X and Y are n-vectors, $u(t)$ is scalar input. f satisfies (a1), (b1).

Let

$$\lambda_1 \triangleq \sup_{t \geq 0} \lambda_{\max}[A^T(t)A(t)], \tag{6.7}$$

$$\lambda_2 \triangleq \sup_{t \geq 0} \lambda_{\max}[B^T(t)B(t)], \tag{6.8}$$

$$\lambda_3 \triangleq \sup_{t \geq 0} \max \left|\lambda[B^T(t)A(t) + A^T(t)B(t)]\right|. \tag{6.9}$$

Theorem 6.1: In the system, (6.4) - (6.6), suppose $f : R^n \to R$ is defined as in (2.1), $H(t)$ is uniformly bounded on Z^+, and $\lambda_2 < \infty$, $\lambda_3 < \infty$. Then the zero state, for the system (6.4) - (6.6), is uniformly stable and asymptotically stable if $\lambda_1 < 1$.

Proof: Using (6.7) – (6.9), then (6.4) becomes

$$\|X(t+1)\|^2 \leq (\lambda_1 + \lambda_2 u^2(t))\|X(t)\|^2 + \lambda_3 \|X(t)\|^2 \, |u(t)| \, .$$

By (a1), (b1) and (6.3) we have

$$|u(t)| \leq K_1 F_H^p \|X(t)\|^p,$$

$$|u(t)|^2 \leq K_1^2 F_H^{2p} \|X(t)\|^{2p}.$$

Then,

$$\|X(t+1)\|^2 \leq \lambda_1 \|X(t)\|^2 + \lambda_3 K_1 F_H \|X(t)\|^{(2+p)} + \lambda_2 K_f^2 F_H^2 \|X(t)\|^{(2+2p)},$$

where $p \geq 2$. This Theorem follows by applying Lemma 3.2.

Now we consider the more general system (6.1) – (6.3) with multiple output feedback.

Theorem 6.2: In the system (6.1) – (6.3), suppose $C(t)$ and $H(t)$ are uniformly bounded on Z^+. If

$$\lambda_1 < 1,$$

then the zero state, for the system (6.1) - (6.3), is uniformly stable and asymptotically stable.

Proof: Let

$$X_1(t+1) = A(t)X(t) + \sum_{i=1}^{m} B_i(t)X(t)u_i(t)$$

$$X_1^T(t+1)X_1(t+1) \leq \lambda_1 \|X(t)\|^2 + \sum_{i=1}^{m} \lambda_3 \|X(t)\|^2 |u_i(t)| + \sum_i \sum_j \lambda_2 \|X(t)\|^2 |u_i(t)||u_j(t)|,$$

where λ_1 is defined in (6.7),

$$\lambda_2 \triangleq \sup_{t \geq 0} \max_{1 \leq i,j \leq m} \{\max |\lambda(B_i^T B_j)|\}, \tag{6.10}$$

$$\lambda_3 \triangleq \sup_{t \geq 0} \max_{1 \leq i \leq m} \{\max |\lambda(B_i^T A + A^T B_i)|\}. \tag{6.11}$$

By Hölder's inequality, we have

$$\sum_{i=1}^{m} |u_i(t)| \leq \sqrt{m} \|U(t)\| \leq \sqrt{m} K_1 F_H \|X(t)\|^p$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{m}|u_i(t)||u_j(t)| \le m\left(\sum_{i=1}^{m}u_i^2(t)\right)^{1/2}\left(\sum_{j=1}^{m}u_j^2(t)\right)^{1/2}$$
$$= m\|U(t)\|^2 \le mK_1^2F_H^2\|X(t)\|^{2p}.$$

So,

$$\|X_1(t+1)\|^2 \le \lambda_1\|X(t)\|^2 + \sqrt{m}\lambda_3K_1F_H\|X(t)\|^{(2+p)} + mK_1^2F_H^2\lambda_2\|X(t)\|^{(2+2p)}.$$

Hence,

$$\|X_1(t+1)\| \le \sqrt{\lambda_1}\|X(t)\| + m^{\frac{1}{4}}\sqrt{\lambda_3K_1F_H}\|X(t)\|^{1+\frac{p}{2}} + \sqrt{m\lambda_2}K_1F_H\|X(t)\|^{(1+p)}.$$

So,

$$\begin{aligned}\|X(t+1)\| &\le \|X_1(t+1)\| + F_C\|U(t)\| \le \|X_1(t+1)\| + K_1F_HF_C\|X(t)\|^p \\ &\le \sqrt{\lambda_1}\|X(t)\| + K_1F_HF_C\|X(t)\|^p \\ &\quad + m^{\frac{1}{4}}\sqrt{\lambda_3K_1F_H}\|X(t)\|^{1+\frac{p}{2}} + \sqrt{m\lambda_2}K_1F_H\|X(t)\|^{1+p}.\end{aligned}$$

Let $\|W(t)\|^2 \triangleq \|X(t)\|$. Substitute this into the above inequality and applying Lemma 3.2 to get the needed results.

Similarly, we can develop the all results of stability of section III-V to the more general feedback case.

3. The Discussions for Various Feedback Functions

From Theorem 6.1 and Theorem 3.3, we see that the stability of the bilinear systems only depends on $A(t)$, and does not depend on the f and B in homogeneous bilinear systems wherever the degree $p = 1$ or $p \ge 1$ in equation (6.2). In the non-homogeneous case, from the Theorem 6.2, we may find that the stability of the bilinear systems also only depends on $A(t)$, and does not depend on the f, nor even C. This is much different Theorem 3.4, in which the stability depends not only on $A(t)$, but also depends on the f, and C. This means that the feedback functions which satisfy (a1), (b1), degree of $p \geq 2$ can not be used to improve the local stability of the zero state for the feedback systems, because the stability results are independent with the feedback function, even in non-homogeneous case.

For the local stability of the zero state, to improve the stability of the bilinear systems, one useful way is use the linear feedback function, or a polynomial function with the linear part. The linear part can be used to improve the stability, and the part of degree ≥ 2 may increase the speed of the convergence, so an appropriate polynomial feedback can be used to improve the stability of the original systems.

Let us consider the following single input and single output example:

$$x(t+1) = a(t)x(t) + b(t)x(t)u(t) + c(t)u(t), \tag{6.12}$$

$$u(t) = f(x(t)) = a_1 x(t) + h(x(t)) = u_1(t) + u_2(t), \tag{6.13}$$

where $u_1 = a_1 x(t)$, and $u_2(t) = h(x(t))$; a_1 is a constant, and $h(.)$ is a polynomial with degree ≥ 2. Substitute (6.12) into (6.13), we have

$$x(t+1) = a^*(t)x(t) + \sum_{j=1}^{2} b(t)x(t)u_j(t) + c(t)u_2(t), \tag{6.14}$$

where $a^*(t) = a(t) + a_1 c(t)$. The same step as Theorem 6.2, we may obtain the conclusion: the stability at zero state for the bilinear systems (6.14) only depends on $a^*(t)$, and does not depend on the h, and b. So we may choose an appropriate a_1 such that $\lambda_1 < 1$, where

$$\lambda_1 \triangleq \sup_{t \geq 0} \lambda_{\max}[a^{*2}(t)].$$

This means that the linear feedback can improve the stability of the systems.

Assumption: the feedback function f satisfies:

(a1): $f(0) = 0$,

(b2): $u(t) = f(x(t)) = kx(t)^p$, where $-1/2 < p < 0$,

or $u(t) = f(x(t)) = kx(t)^{-q}$, where $0 < q = -p < 1/2$, and k is a constant.

Now we consider the following single input signal output example, where f satisfies (a1), (b2):

$$x(t+1) = a(t)x(t) + b(t)x(t)u(t), \tag{6.15}$$

$$u(t) = kx(t)^{-q}, \quad 0 < q < 1. \tag{6.16}$$

(6.15), (6.16) can be rewritten as

$$x(t+1) = a(t)x(t) + kb(t)x(t)^{1-q}.$$

So, we have

$$|x(t+1)|^2 \leq 2\lambda_1 x^2(t) + 2k\lambda_3 x^{2-q}(t) + k^2\lambda_2 |x(t)|^{2-2q},$$

where

$$\lambda_1 \triangleq \sup_{t\geq 0} \lambda_{\max}[a^2(t)] \tag{6.17}$$

$$\lambda_2 \triangleq \sup_{t\geq 0} \lambda_{\max}[b^2(t)b(t)] \tag{6.18}$$

$$\lambda_3 \triangleq \sup_{t\geq 0} \lambda_{\max}[a(t)b(t)] \qquad (6.19).$$

Notice $0 < 2 - 2q < 1$, and following the proof of Lemma 3.2, we may have the conclusion as the follows:

Theorem 6.3: In the system, (6.15)-(6.16), suppose $f : R \to R$ is defined as in (a1),(b2), and $\lambda_1 < \infty$, $\lambda_3 < \infty$. Then the zero state, for the system (6.15) - (6.1 6), is uniformly stable and asymptotically stable if $k^2\lambda_2 < 1$.

This theorem show that in this case: f satisfies (a1), (b2) in which $-1 < p < 0$, the stability of zero state only depends on B, not on A. Another way to improve the stability of zero state is use this feedback function which satisfies (a1) and (b2), then we may choose an appropriate k such that $k^2\lambda_2 < 1$, where λ_2 is defined by (6.18), and it only depends on B. Notice that this above feedback f can improve the homogeneous bilinear systems, but not the non-homogeneous bilinear systems, because add the non-homogeneous term $cu(t) = cx(t)^p \rightarrow \infty$, if $x(t) \rightarrow 0$.

An interesting example of a motor control problem is shown in section VII.

VII. THE APPLICATION OF STABILITY THEORY

1. The Humoral Immune Model

Here we provide an example for which the above stability theory of stochastic bilinear systems with additive noises is applied.

The humoral and the CMI (cell-mediated immune) dynamics may be divided into cellular and molecular subsystems which are coupled together by multipliers (see Mohler 1990). For the humoral system a conservation of cells leads to the following equations for concentration of immunocompetent cells (ICC) $x_1(t)$ and plasma cells $x_2(t)$:

$$\frac{dx_1}{dt} = \alpha u_1 x_1 - \frac{x_1}{\tau_1} + v_1, \tag{7.1}$$

$$\frac{dx_2}{dt} = 2\alpha u_2 x_1 - \frac{x_2}{\tau_2}. \tag{7.2}$$

The molecular(mass-action binding) behavior for free antibody $x_3(t)$, bound antibody-antigen complexes $x_4(t)$ and free antigen $x_5(t)$ become:

$$\frac{dx_3}{dt} = -\, c u_3 x_3 - \frac{x_3}{\tau_3} + \alpha_1 x_2 + c x_4 + \alpha_2 x_1, \tag{7.3}$$

$$\frac{dx_4}{dt} = c u_3 x_3 - (c + \frac{1}{\tau_4}) x_4, \tag{7.4}$$

$$\frac{dx_5}{dt} = v_2 - \frac{x_5}{\tau_5} - Nc(u_3 x_3 - x_4). \tag{7.5}$$

Here the ICC are sensitized lymphocyte cells with particular surface receptors for antigen according to a particular affinity. The plasma cells, are nonreproducing offspring of stimulated ICC. The free-antigen concentration triggers the response mechanism. u_1 is ICC multiplication; u_2 is plasma-cell multiplication;

u_3 is binding multiplication; v_1 is stem-cell source rate (from bone marrow), v_2 is inoculation rate of antigen.

The immune parameters are defined as follows: α is birthrate constant; N is a constant (to account for the total number of affinities); α_1 is plasma-cell antibody production rate; α_2 is ICC antibody production; τ_1 is the mean lifetime of immunocompetent cells; and τ_2, τ_3, τ_4, τ_5 are the appropriate lifetimes.

The additive signal v_1 is independent of the multiplicative control variables u_1, u_2, u_3 and can be significant in immunotherapy. Though this source of stem cells, v_1 is naturally distributed according to affinity (usually assumed to be Poisson or Gaussian), an average seems representative in most practical cases.

The other additive signal, rate of inoculation of antigen v_2 is independent of the other control variables. We assume $v_2(t)$ is independent of v_1. While $u_3 = kx_5$, u_1 and u_2 are dependent stochastic parameters which my be approximated by

$$\begin{aligned} u_1 =& p_s(1 - 2p_d), \\ u_2 =& p_s p_d, \end{aligned}$$

where p_s, p_d are coefficients or probabilities of stimulation and differentiation respectively. For convenience we assume

$$Eu_1(t) = Eu_2(t) = 0. \tag{$*$}$$

Equation (7.1) - (7.5) can be written as

$$\frac{dX(t)}{dt} = AX(t) + B_1X(t)u_1(t) + B_2X(t)u_2(t) + B_3X(t)u_3(t) + GV(t), \tag{7.6}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}, \tag{7.7}$$

$$A = \begin{bmatrix} -1/\tau_1 & 0 & 0 & 0 & 0 \\ 0 & -1/\tau_2 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & -1/\tau_3 & c & 0 \\ 0 & 0 & 0 & -(c+1/\tau_4) & 0 \\ 0 & 0 & 0 & Nc & -1/\tau_5 \end{bmatrix}, \tag{7.8}$$

$$B_1 = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & -Nc & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7.9}$$

$$V(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}. \tag{7.10}$$

Discretization of equations (7.1) by the use of a first-order Euler expansion to give

$$X(t+1) = X(t) + TAX(t) + TB_1X(t)u_1(t) + TB_2X(t)u_2(t)$$
$$+ TB_3X(t)u_3(t) + TGV(t), \tag{7.11}$$

where T is the sampling interval. Take $T = 1$ of (7.11) (if $T \neq 1$, then let $A^* = TA$; $N_i = TB_i, i = 1,2,3$; $G_1 = TG$). We have

$$X(t+1) = A_1X(t) + B_1X(t)u_1(t) + B_2X(t)u_2(t) + B_3X(t)u_3(t) + GV(t), \tag{7.12}$$

where $A_1 = A + I$, and $v_1(t), v_2(t)$ are additive signals, and assume they are independent of multiplicative control variables (u_1, u_2, u_3), and are independent of each other.

2. The Theoretical Results for this System

Let

$$\lambda_1 \triangleq \lambda_{\max}[A_1^T A_1], \tag{7.13}$$

$$\lambda_2 \triangleq \max_{1\le i,j\le 3}\{\max|\lambda(B_i^T B_j|\}, \tag{7.14}$$

$$\lambda_3 \triangleq \max_{1\le i\le 3}\{\max|\lambda(B_i^T A_1 + A_1^T B_i)|\}, \tag{7.15}$$

$$\widetilde{\lambda_2} \triangleq \max_{1\le i,j\le 2}\{\max|\lambda(B_i^T B_j|\}, \tag{7.16}$$

$$\widetilde{\lambda_3} \triangleq \max_{1\le i\le 2}\{\max|\lambda(B_i^T A_1 + A_1^T B_i)|\}, \tag{7.17}$$

$$\lambda_G \triangleq \lambda_{\max}\left[G^T G\right], \tag{7.18}$$

$$u_3 = kx_5. \tag{7.19}$$

Now we calculate $E\{\|X(t+1)\|^2\}$. First let

$$E\{\|X(t+1)\|^2\} = I_1 + I_2 + I_3 + I_4 + I_5, \tag{7.20}$$

where

$$I_1 = E\{X^T(t)\left[A_1^T A_1\right]X(t)\} \le \lambda_1 E\{\|X(t)\|^2\}. \tag{7.21}$$

Notice that for $u_1(t)$ and $u_2(t)$, the Independent Argument I can be used for calculation of I_2 and I_3.

$$\begin{aligned} I_2 =& E\{X^T(t)\sum_{i,j=1}^{2}[B_i^T B_j]X(t)u_i(t)u_j(t)\} \\ &\le 2\widetilde{\lambda_2}\sum_{i,j=1}^{2} R_{ij}(t)E\{\|X(t)\|^2\}, \end{aligned} \tag{7.22}$$

where $R_{ij}(t)$ is defined by (5.15a).

$$
\begin{aligned}
I_3 =& E\left\{ X^T(t) \sum_{i=1}^{2} [B_i^T A_1 + A_1^T B_i] X(t) u_i(t) \right\} \\
&\leq \widetilde{\lambda_3} E\left\{ \sum_{i=1}^{2} u_i(t) \|X(t)\|^2 \right\} \\
&\leq \widetilde{\lambda_3} [E\{u_1(t)\} + E\{u_2(t)\}] E\{|X(t)\|^2\}. \qquad (7.23)
\end{aligned}
$$

From the model (7.12) and the assumption as above, it is clear that $V(t)$ has an effect on $X(t+j), j > 0$, has no effect on $X(t+j), j \leq 0$, so from the Independent Argument I, we have

$$E\{x_5(t)X^T(t)X(t)V(t)\} = E\left\{x_5(t)X^T(t)X(t)E\{V(t)\}\right\} = 0,$$

$$E\{X^T(t)X(t)V(t)\} = E\left\{X^T(t)X(t)E\{V(t)\}\right\} = 0.$$

$$
\begin{aligned}
I_4 = & E\{kX^T(t)B_3^T x_5(t) A_1 X(t) + kX^T(t) B_3^T x_5(t) B_1 X(t) u_1(t) \\
& + kX^T(t) B_3^T x_5(t) B_2 X(t) u_2(t) + X^T(t) A_1^T k B_3 X(t) x_5(t) \\
& + k u_1(t) X^T(t) B_1^T B_3 X(t) x_5(t) + kX^T(t) B_2^T u_2(t) B_3 X(t) x_5(t)\} \\
& + k^2 X^T(t) B_3^T B_3 X(t) x_5^2(t) \\
\leq & k\lambda_3 E\{\|X(t)\|^2 |x_5(t)|\} + k\lambda_2 \sum_{j=1}^{2} E\{\|X(t)\|^2 |x_5(t)|\} E\{u_j(t)\} \\
& + k^2 \lambda_2 \sum_{j=1}^{2} E\{\|X(t)\|^2 |x_5(t)|^2\} \\
\leq & k\lambda_3 E\{\|X(t)\|^3\} + k\lambda_2 [E\{u_1(t)\} + E\{u_2(t)\}] E\{\|X(t)\|^3\} \\
& + k^2 \lambda_2 E\{\|X(t)\|^4\}. \qquad (7.24)
\end{aligned}
$$

$$I_5 = E\{V(t)^T G^T G V(t)\} \leq \lambda_G tr(Q_v(t)). \qquad (7.25)$$

where $Q_v(t)$is the covariance of $V(t)$. Substitute (7.21)-(7.25) into (7.20), and notice (*) yield

$$
\begin{aligned}
E\{\|X(t+1)\|^2\} \leq \lambda_1 E\{\|X(t)\|^2\} &+ \widetilde{\lambda_2} \sum_{i,j=1}^{2} R_{ij}(t) E\{\|X(t)\|^2\} \\
&+ \widetilde{\lambda_3}[E\{u_1(t)+u_2(t)]\}]E\{|X(t)\|^2\} + k\lambda_3 E\{\|X(t)\|^3\} \\
&+ k\lambda_2[E\{u_1(t)+u_2(t)]\}]E\{\|X(t)\|^3\} \\
&+ k^2\lambda_2 E\{\|X(t)\|^4\} + \lambda_G tr(Q_v(t)) \\
= \beta(t)E\{\|X(t)\|^2\} &+ \beta_1 E\{\|X(t)\|^3\} \\
&+ \beta_2 E\{\|X(t)\|^4\} + D, \qquad (7.26)
\end{aligned}
$$

where

$$\beta(t) = \lambda_1 + \widetilde{\lambda_2} \sum_{i,j=1}^{2} R_{ij}(t), \qquad (7.27)$$

$$\beta_1 = k\lambda_3,$$

$$\beta_2 = k^2\lambda_2.$$

Here, λ_1, $\widetilde{\lambda_2}$, $\widetilde{\lambda_3}$ are defined in (7.13), (7.16), (7.17) respectively. Thus, from lemma 5.2, we know that the system (7.12) is almost surely uniformly stable and asymptotically stable, if $\beta(t) < 1 - r_2$ for all t. So we have

Theorem 7.1 Consider system (7.12), if $\beta(t) < 1 - r_2$ for all t where $\beta(t)$ is defined in (7.27), and r_2 is defined by Lemma 5.2. Then the system (7.12) is almost surely uniformly stable and asymptotically stable.

Remark 7.1 If u_1, u_2 is uncorrelated, then $\beta(t) = \lambda_1$, this theorem is held if $\lambda_1 < 1 - r_2$.

3. The Simulation Results for this System

Example 7.1

Suppose for convenience (rather than immunological accuracy): $\tau_1 = 1.2$, $\tau_2 = 1.4$, $\tau_3 = 1.5$, $\tau_4 = 1.6$, $\tau_5 = 1.4$, where $\tau_j, j = 1, \cdots, 5$ is defined in (7.8), and also in (7.13) $N = 2, c = 2.2$, $\alpha_1 = 0.4$, $\alpha_2 = 0.5$, $\alpha = 0.25$. Thus, A_1, B_1, B_2, B_3 in (7.12) is given by

$$A1 = \begin{bmatrix} 0.167 & 0 & 0 & 0 & 0 \\ 0 & 0.286 & 0 & 0 & 0 \\ 0.4 & 0.5 & 0.333 & 0.2 & 0 \\ 0 & 0 & 0 & 0.175 & 0 \\ 0 & 0 & 0 & 0.4 & 0.286 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix},$$

G and $V(t)$ are the same as (7.9), (7.10). Here, $k = 0.5$, and $u_3(t) = 0.5x_5(t)$. The variances of random input u_1, u_2 and random noise v_1, v_2 are 0.4, 0.3, 0.001, and 0.002 respectively. From (7.13), (7.16) and (7.17), then

$$\lambda_1 = \lambda_{\max}[A_1^T A_1] \approx 0.45,$$

$$\widetilde{\lambda_2} = \max_{1 \le i,j \le 2} \{\max |\lambda(B_i^T B_j|\} \approx 0.31,$$

$$\widetilde{\lambda_3} = \max_{1 \le i \le 2} \{\max |\lambda(B_i^T A_1 + A_1^T B_i)|\} \approx 0.16.$$

From (7.27), then

$$\beta(t) = \lambda_1 + \widetilde{\lambda_2} \sum_{i,j=1}^{2} R_{ij}(t) \leq 0.932,$$

here $\beta < 1$, the computer simulations show that system (7.12) is almost surely uniformly stable and asymptotically stable. The states $x_1 - x_5$ are shown in Fig. 13 - Fig. 17 respectively.

4. The Model for a Motor Control Problem (Mohler, 1990)

Consider the DC motor and load such that

$$L_a \frac{di_a}{dt} = -R_a i_a - K_v^* w i_e + v_a,$$

$$J \frac{dw}{dt} = K_v i_e i_a - Dw,$$

where J : moment of inertia (including motor and load)

0.2 oz-in/rad/sec,

D : viscous damping ratio (including motor and load)

0.1 oz-in/rad/sec,

R_a : armature resistance, 1 ohm,

L_a : applied armature inductance, 0.05 henry,

K_v : motor const. 10 oz-in/A/A,

K_v^* : motor const. 70.6 mN.m/A/A,

i_a : armature current (A),

i_e : field current (A),

v_a : armature voltage (volts),

ω : angular velocity (rad/sec),

θ : angular position (rad).

Let $x_1 = i_a, x_2 = \theta, x_3 = \omega, u_1 = i_e, v = v_a$. Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_a/L_a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -D/J \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -K_v^*/L_a \\ 0 & 0 & 0 \\ K_v/J & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} u_1 + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} v, \tag{7.27a}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{7.27b}$$

(7.27) can be rewritten as the bilinear control system:

$$\dot{X} = AX + BXu_1 + CU, \tag{7.28}$$

$$Y = HX, \tag{7.29}$$

where

$$A = \begin{bmatrix} -R_a/L_a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -D/J \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & -K_v^*/L_a \\ 0 & 0 & 0 \\ K_v/J & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1/L_a \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$X = [x_1 \; x_2 \; x_3]^T,$$

$$U = [u_1 \; v]^T, \tag{7.30}$$

The motor control problem is to choose the functions f_1, f_2, such that the obtained feedback systems are stable. If possible, choose simple forms of f_1 and

f_2, such that $u(t) = f_1(X(t))$, $v(t) = f_2(X(t))$, and the obtained feedback system is uniformly stable and asymptotically stable.

5. The Stability for this Problem

Equations (7.28), (7.29) can be discretized by use of a first-order Euler expansion to give

$$X(t+1) \approx X(t) + TAX(t) + TBX(t)u_1(t) + TCU(t), \tag{7.31}$$

$$Y(t) = HX(t), \tag{7.32}$$

where T is the sampling interval. (7.31) can be rewritten as:

$$X(t+1) \approx A^*X(t) + B^*X(t)u_1(t) + C^*U(t), \tag{7.33}$$

where

$$A^* = I + TA = \begin{bmatrix} 1 - TR_a/L_a & 0 & 0 \\ 0 & 1 & T \\ 0 & 0 & 1 - TD/J \end{bmatrix},$$

$$B^* = TB = \begin{bmatrix} 0 & 0 & -K_v^*T/L_a \\ 0 & 0 & 0 \\ K_vT/J & 0 & 0 \end{bmatrix},$$

$$C^* = TC = \begin{bmatrix} 0 & T/L_a \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.33a}$$

For our particular example, these parameters are given as above : $R_a = 1$, $L_a = 0.05$, $K_v = 10$, $K_v^* = 70.6 \times 10^{-3}$, $J = 0.2$, $D = 0.1$. Here, there have

$$\lambda_1(A^*) = 1 - TR_a/L_a = 1 - T(1/0.05) = 1 - 20T,$$

$$\lambda_2(A^*) = 1,$$

$$\lambda_3(A^*) = 1 - TD/J = 1 - 0.5T.$$

Choose T such that $|1 - 20T| < 1$ and $|1 - 0.5T| < 1$, then the corresponding open-loop linear system will be stable from linear system theory.

a) $0 < T \leq 0.05$

In this case, $|\lambda_{\max} A^*| = \lambda_2(A^*) = 1$, then the corresponding open-loop linear system (U(t)=0) will be stable. Fig. 18- Fig. 20 show the corresponding stable open-loop linear system with $U(t) = 0$. Fig. 21-Fig. 23 show the corresponding bilinear system with input: $u_1 = 2$, $v = 3$. This system is bounded, but not asymptotically stable in this case (T=0.001).

b) $T > 0.05$

In this case, the corresponding stable open-loop linear system with $U(t) = 0$ is unstable, because $|\lambda_{\max} A^*| = \lambda_1(A^*) > 1$. Also, Fig. 24- Fig. 26 show the corresponding bilinear system with input: $u_1 = 2$, $v = 3$ and $T = 1$. Here this system is not stable and not bounded.

The sampling period is very important for the discrete-time systems. Faster sampling period can keep the same stability as in continuous case. In some case, sampling period is slow, the obtained discrete bilinear system will unstable even the original continuous-time system is stable. But it may appear that the minimum sampling period is restricted the time taken to update the parameters and output the control. Keeping the sampling period reasonably long has an advantage in some case (see Goodwin et al 1984). In this case, we may use the method in this thesis to improve the stability for the discrete-time systems.

From section VI-3, we know that the stability can be improved by using a feedback control containing the linear term. There are many ways to choose the feedback control. For simplicity, we choose the following type of feedback controls.

c) Using feedback control to improve the stability

Let $U(t) = SY(t)$, where S is a constant matrix.

Let

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}. \tag{7.34a}$$

Thus, the corresponding feedback system of (7.33), (7.32) is

$$X(t+1) = A^* X(t) + B^* X(t) u_1(t) + C^* U(t), \tag{7.34}$$

$$Y(t) = HX(t), \quad U(t) = SY(t), \tag{7.35}$$

where $X(t), A^*, B^*, C^*, U(t), S$ are defined by (7.33a).

Notice $U(t) = SHX(t)$, (7.34), (7.35) and can be represented as:

$$\begin{aligned} X(t+1) &= A^* X(t) + B^* X(t) u_1(t) + C^* SHX(t), \\ &= A^{**} X(t) + B^* X(t) u_1(t), \end{aligned}$$

$$A^{**} = A^* + C^* SH. \tag{7.36}$$

Let

$$\lambda_1 \triangleq \lambda_{\max}[A^{**T} A^{**}], \tag{7.37}$$

$$\lambda_2 \triangleq \lambda_{\max}[B^{*T} B^*], \tag{7.38}$$

$$\lambda_3 \triangleq \max |\lambda(B^{*T} A^{**} + A^{**T} B^*)|. \tag{7.39}$$

For our particular example:

$$\begin{aligned} C^* SH &= \begin{bmatrix} 0 & T/L_a \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \\ &\bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c_1 s_{21} & c_1 s_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{7.40}$$

where $c_1 = T/L_a$. So

$$A^{**} = \begin{bmatrix} 1 - TR_a/L_a + c_1 s_{21} & c_1 s_{22} & 0 \\ 0 & 1 & T \\ 0 & 0 & 1 - TD/J \end{bmatrix},$$

From (7.36)-(7.39) we have

$$|u_1(t)| \leq \|U(t)\| \leq \|S\| \|H\| \|X(t)\|,$$

$$|u_1(t)|^2 \leq \|U(t)\|^2 \leq \|S\|^2 \|H\|^2 \|X(t)\|^2,$$

it can be shown that

$$\begin{aligned} \|X(t+1)\|^2 \leq & \lambda_1 \|X(t)\|^2 + \lambda_3 \|X(t)\|^2 |u_1(t)| + \lambda_2 \|X(t)\|^2 u_1^2(t) \\ \leq & \lambda_1 \|X(t)\|^2 + \lambda_3 \|S\| \|H\| \|X(t)\|^3 + \lambda_2 \|S\|^2 \|X(t)\|^4. \end{aligned}$$

Then, by Lemma 3.2, we conclude that the system (7.34), (7.35) is uniformly stable and asymptotically stable when $\lambda_1 < 1$. So, we have

Theorem 7.2 Consider system (7.34), (7.35), if $\lambda_1 < 1$, where λ_1 is defined by (7.37), then the zero state of the system (7.34), (7.35) is uniformly stable and asymptotically stable.

The principle for choosing s_{ij} is to reduce the eigenvalues of $A^{**T} A^{**}$. There are a number of ways to do it. For convenience, we choose $s_{22} = 0$, and choose s_{12} such that

$$|1 - R_a/L_a + c_1 s_{21}| < 1,$$

it means

$$-1 < 1 - R_a/L_a + c_1 s_{21} < 1,$$

then,

$$s_{21} \in \left(\frac{-2}{c_1} + \frac{R_a}{c_1 L_a}, \frac{R_a}{c_1 L_a} \right).$$

For our particular example,

$$s_{21} \in (-2/20 + 1,\ 1) = (0.9, 1) \ \text{ if } T = 1,$$

$$s_{21} \in (-2/0.2 + 1,\ 1) = (-9, 1) \ \text{ if } T = 0.01,$$

$$s_{21} \in (-2/0.02 + 1,\ 1) = (-99, 1) \ \text{ if } T = 0.001.$$

Because the special C and H, such that the component s_{11} and s_{12} of feedback matrix S do not work (see *). And the component x_2 of X is a free with control: from the system model (7.33), we have

$$x_2(t+1) = x_2(t) + Tx_3(t),$$

nothing with control. This is the reason why we can not find a such feedback function $U = SY$ that the eigenvalues of $A^{**T}A^{**}$ less than 1. But we reduce the eigenvalues and improve the stability of this system. The maximum of eigenvalues of $A^{*T}A^{*}$for the original system (see (7.33)) is $19^2 = 361$ if $T = 1$. After this linear feedback, the maximum of eigenvalues of $A^{**T}A^{**}$ for this closed-loop system (see (7.34),(7.35)) is 2.1328. Although it does not satisfy the Theorem 7.2, but this system is stable, not asymptotical stable (see Fig. 24 - Fig. 26) at zero state. From the simulations, we see that $x_1(t)$ and $x_3(t)$ are asymptotical stable at zero state. But $x_2(t)$ is not asymptotical stable, only stable, because $x_1(t)$ and $x_3(t)$ are connected with feedback function u_1 and v respectively, but $x_2(t)$ is not. Simulations also show that the theorems still are strong. When $\lambda_1 = 2.1328$, the system is still stable at zero state. Fig. 27 - Fig. 29 show that this stability at zero state is improved by this feedback control, here, this system is stable and bounded at $T = 1$, $\lambda_1 = 2.1328$, $s_{21} = 9.5$ case. But the original, without this feedback control, this system is unstable and not bounded at zero state (see Fig.

24-Fig. 26). Fig. 30 - Fig. 32 show that the feedback control improve the stability also at $T = 0.001$ case, here $\lambda_1 = 1.0006$, $s_{21} = -98$. State variables x_1 and x_3 are asymptotical stable at zero state now. But the original, without this feedback control, x_1 and x_3 are bounded, not asymptotical stable at zero state (see Fig. 21-Fig. 23). Here λ_1 is the maximum of eigenvalues of $A^{**T}A^{**}$, defined by (7.37).

Remark 7.2 As mentioned above, we do not consider the case in which u_1 depends on v. If $v = Ku_1$, where K is a constant, the feedback control will be changed as the follows:

Let

$$u_1(t) = Z^T Y(t) = Z^T H X(t),$$

where $Z = [z_1,\ z_2]^T$. The system (7.33) becomes:

$$X(t+1) \approx A^* X(t) + B^* X(t) u_1(t) + C_1^* K u_1(t), \qquad (7.33')$$

where A^*, B^*, $X(t)$ are as above and

$$C_1^* = [T/L_a,\ 0,\ 0]^T.$$

The corresponding (7.36) becomes:

$$A^{**} = A^* + KC_1^* Z^T H. \qquad (7.36')$$

For our particular example:

$$\begin{aligned} KC_1^* Z^T H &= K \begin{bmatrix} T/L_a \\ 0 \\ 0 \end{bmatrix} \bullet [z_1\ z_2] \\ &\bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c_1 K z_1 & c_1 K z_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \qquad (7.41)$$

where $c_1 = T/L_a$. Comparing (7.40) with (7.41), we may find that we have $s_{21} = Kz_1$, $s_{22} = Kz_2$. It means that s_{21} and s_{22} in (7.40) can be replaced Kz_1, and Kz_2 in (7.41) respectively. So we obtain we choose $z_2 = 0$, and choose z_1 such that

$$|1 - R_a/L_a + c_1 K z_1| < 1,$$

it means

$$-1 < 1 - R_a/L_a + c_1 K z_1 < 1,$$

then,

$$z_1 \in \left(\frac{-2}{Kc_1} + \frac{R_a}{Kc_1 L_a}, \frac{R_a}{Kc_1 L_a} \right).$$

For our particular example,

$$s_{21} \in \frac{1}{K}(-2/20 + 1,\ 1) = \frac{1}{K}(-9, 1) \text{ if } T = 1,$$

$$s_{21} \in \frac{1}{K}(-2/0.2 + 1,\ 1) = \frac{1}{K}(-9, 1) \text{ if } T = 0.01,$$

$$s_{21} \in \frac{1}{K}(-2/0.02 + 1,\ 1) = \frac{1}{K}(-99, 1) \text{ if } T = 0.001.$$

So we have the similar results as above, also we get the same conclusion as u_1 with v independent case: the above feedback will improve the stability for the discrete bilinear systems.

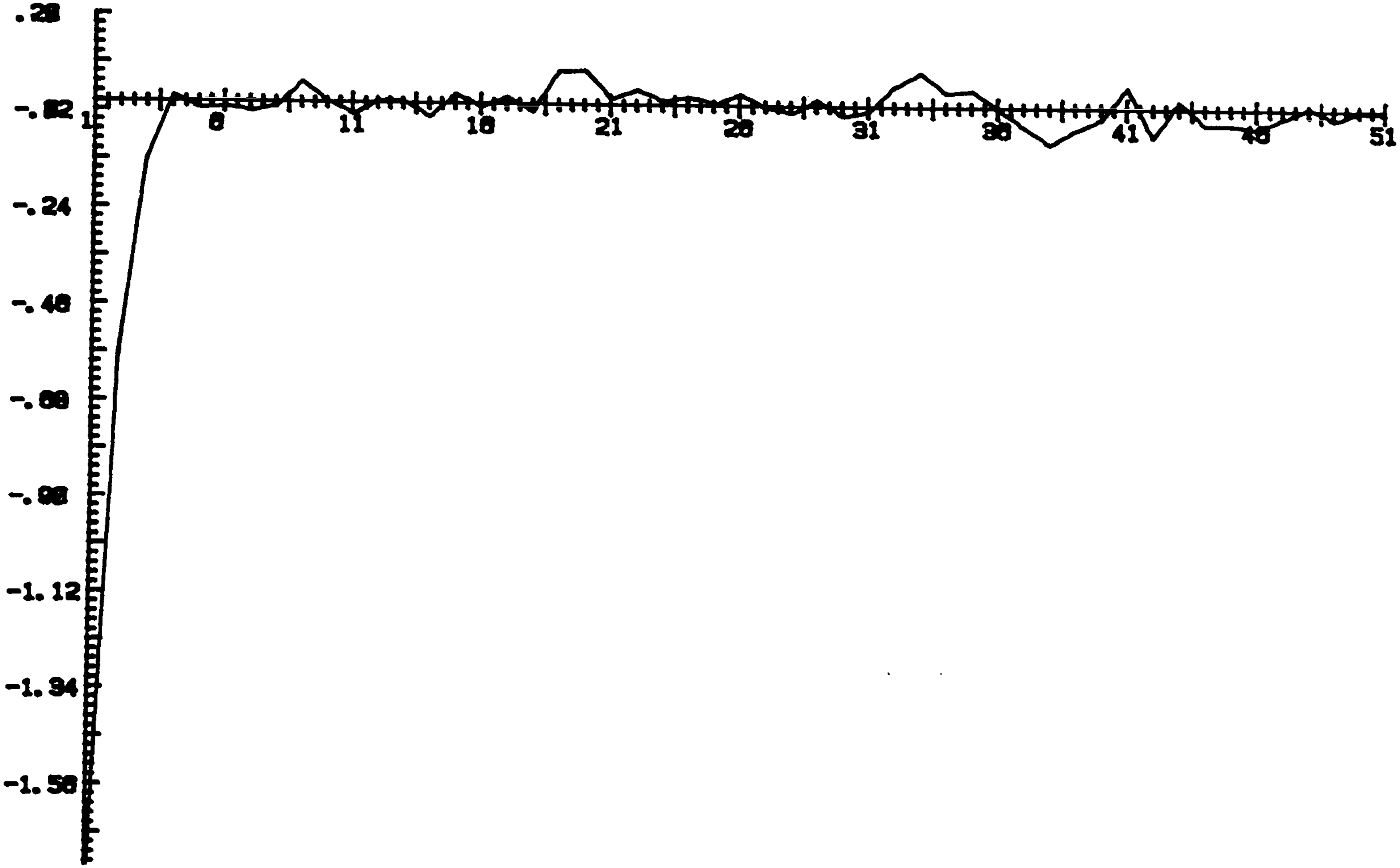

Fig. 13. The State $x_1(t)$ of this Immunologic System

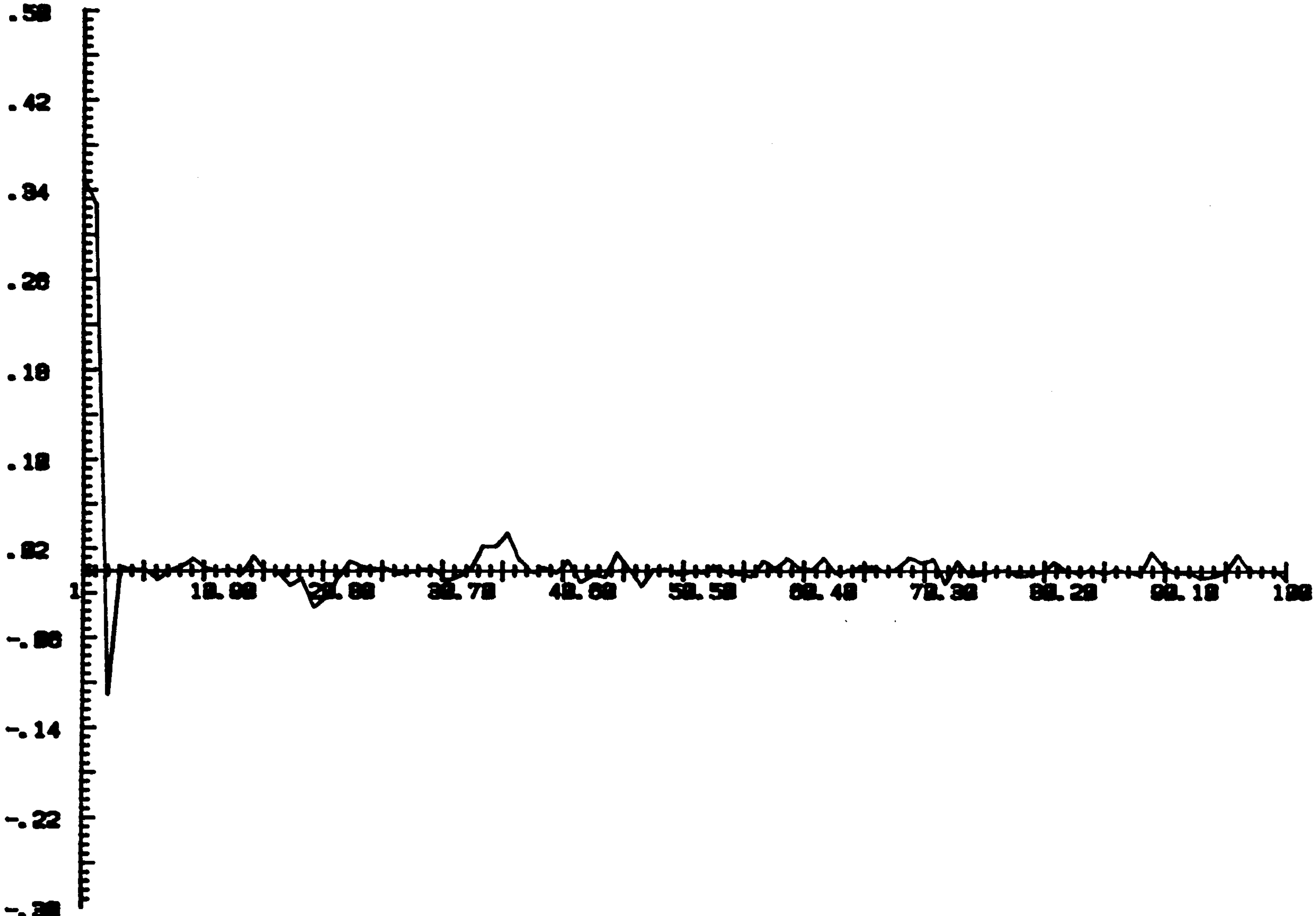

Fig. 14. The State $x_2(t)$ of this Immunologic System

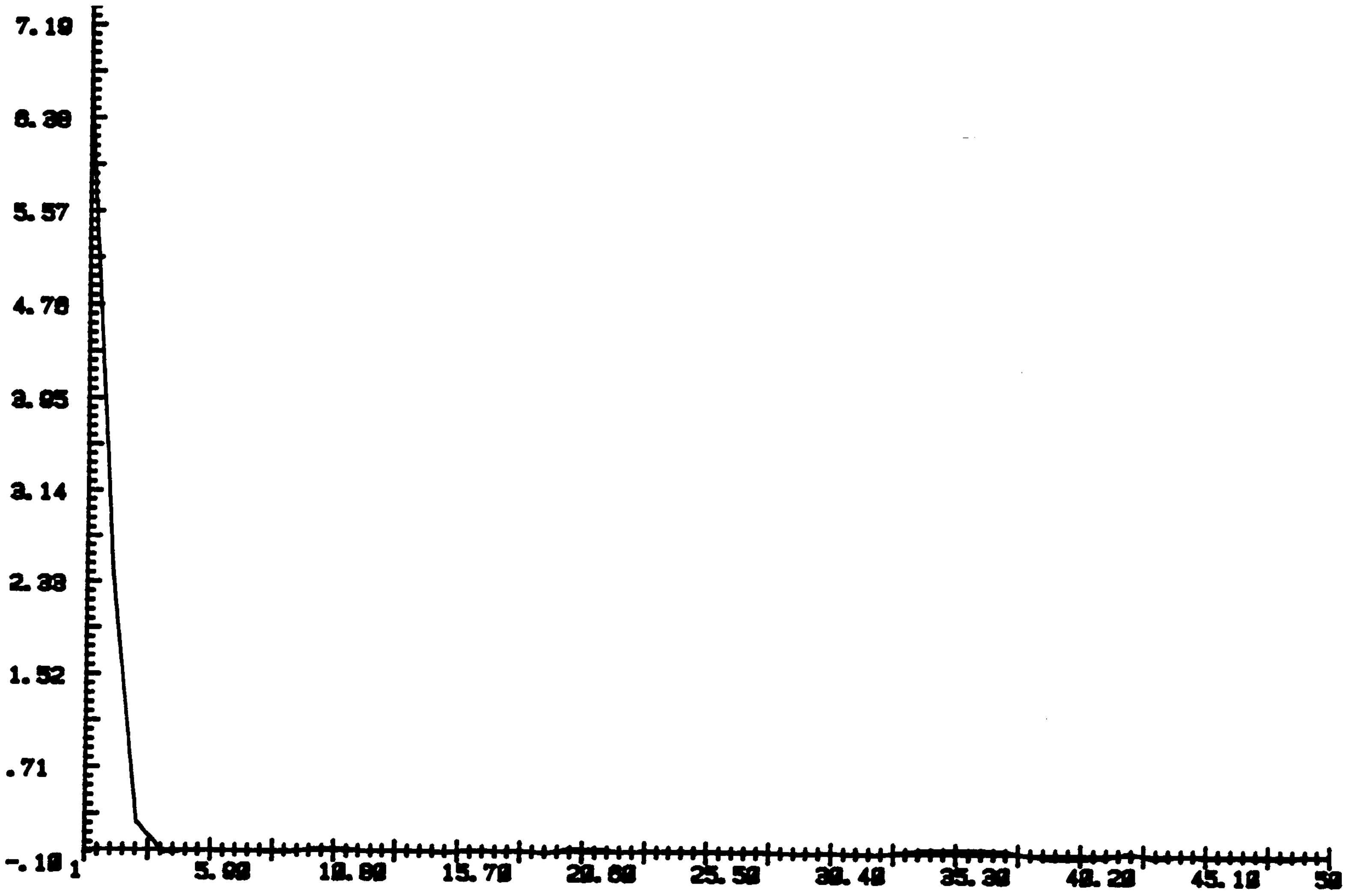

Fig. 15. The State $x_3(t)$ of this Immunologic System

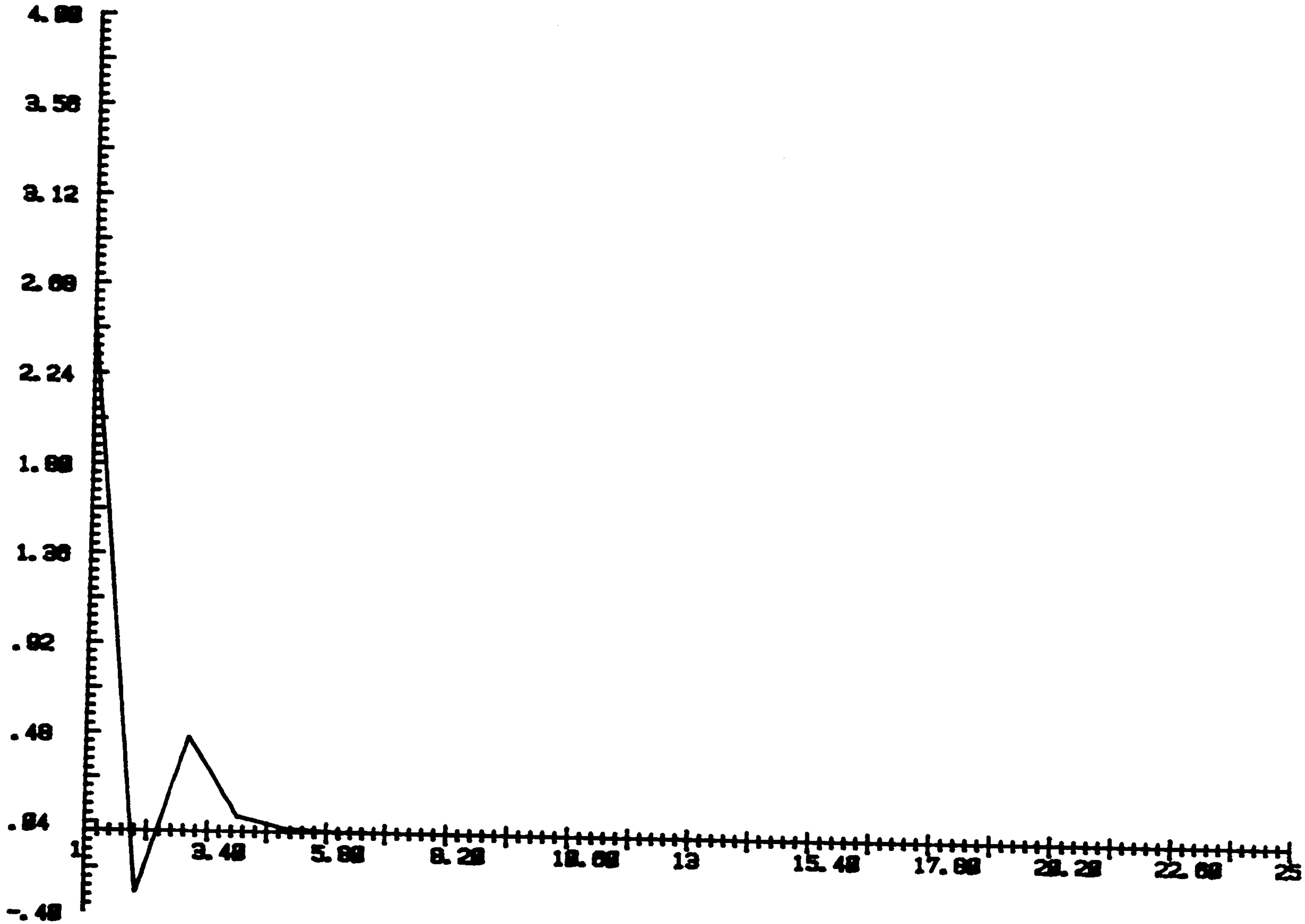

Fig. 16. The State $x_4(t)$ of this Immunologic System

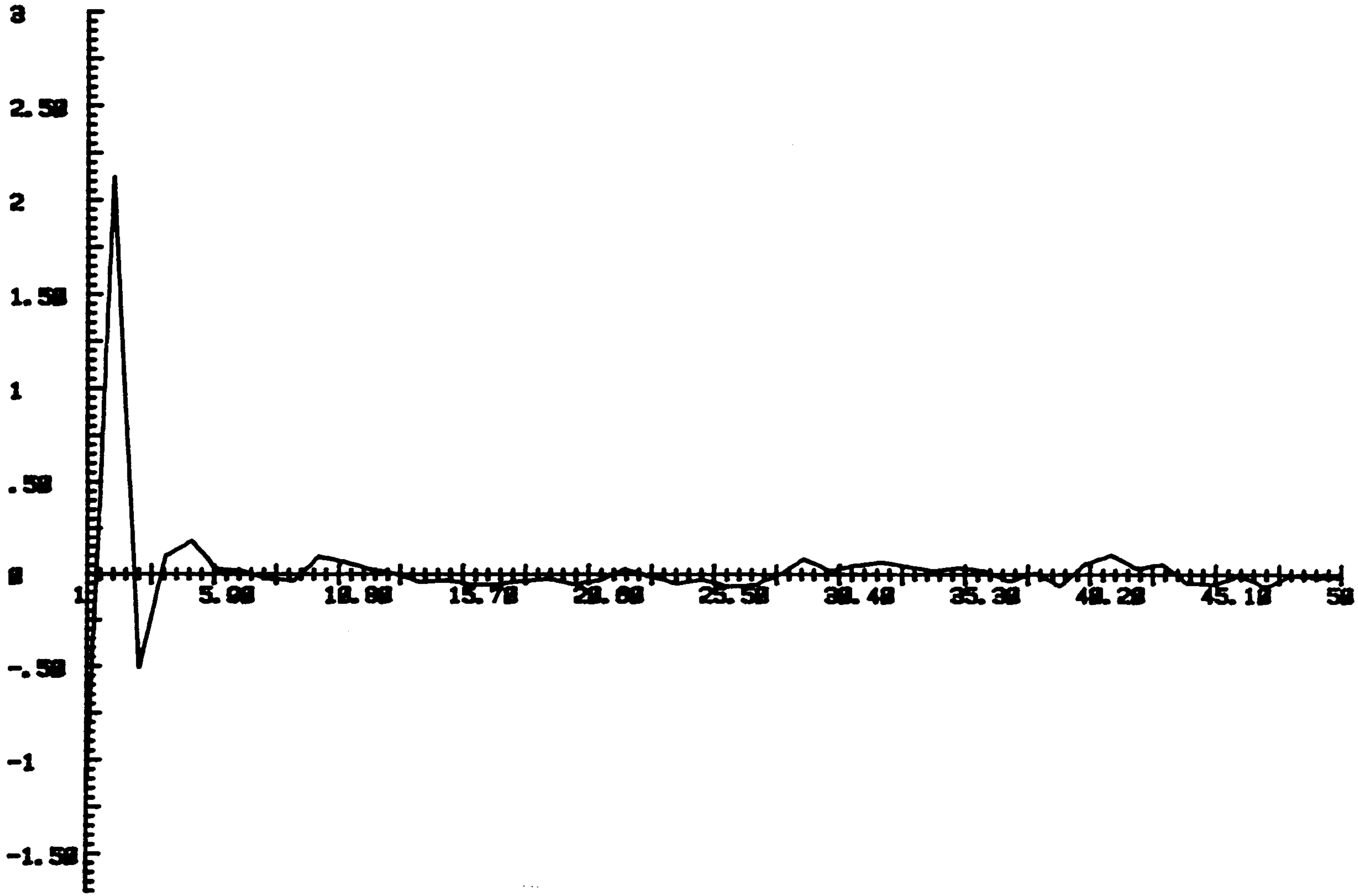

Fig. 17. The State $x_5(t)$ of this Immunologic System

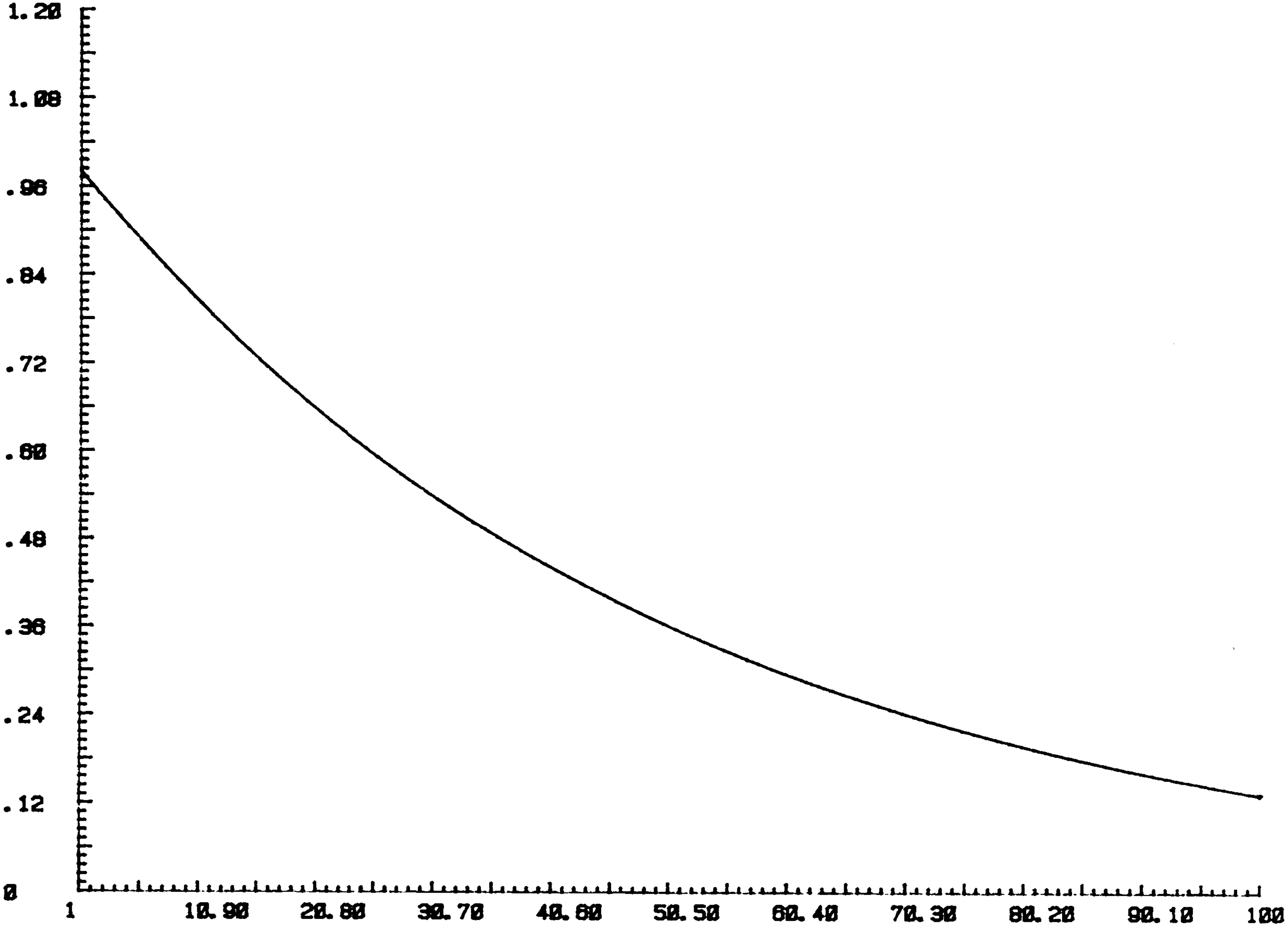

Fig. 18. The State $x_1(t)$ of Eq. (7.33),(7.32) with $u(t) = 0$, $T = 0.001$

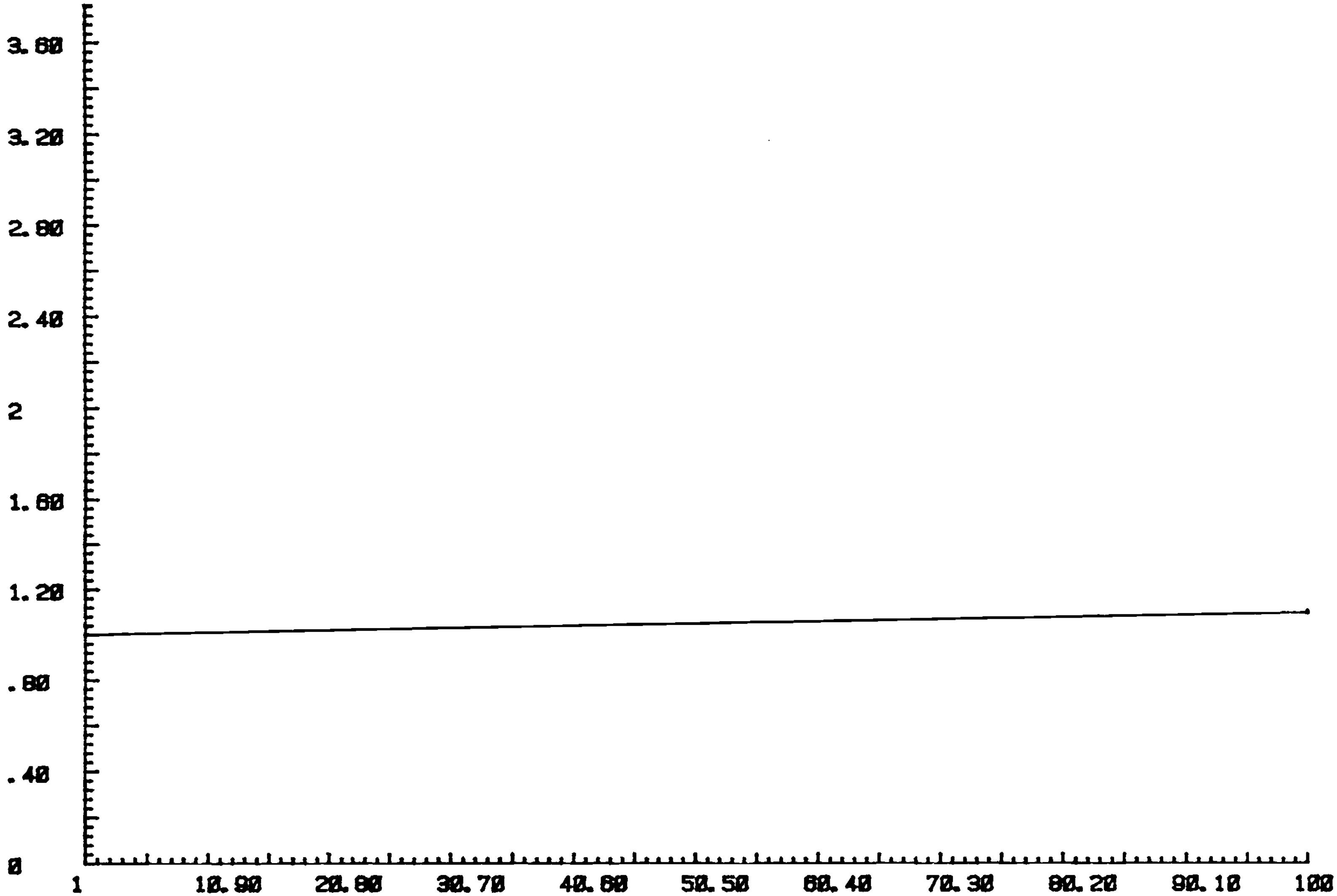

Fig. 19. The State $x_2(t)$ of Eq. (7.33),(7.32) with $u(t) = 0$, $T = 0.001$

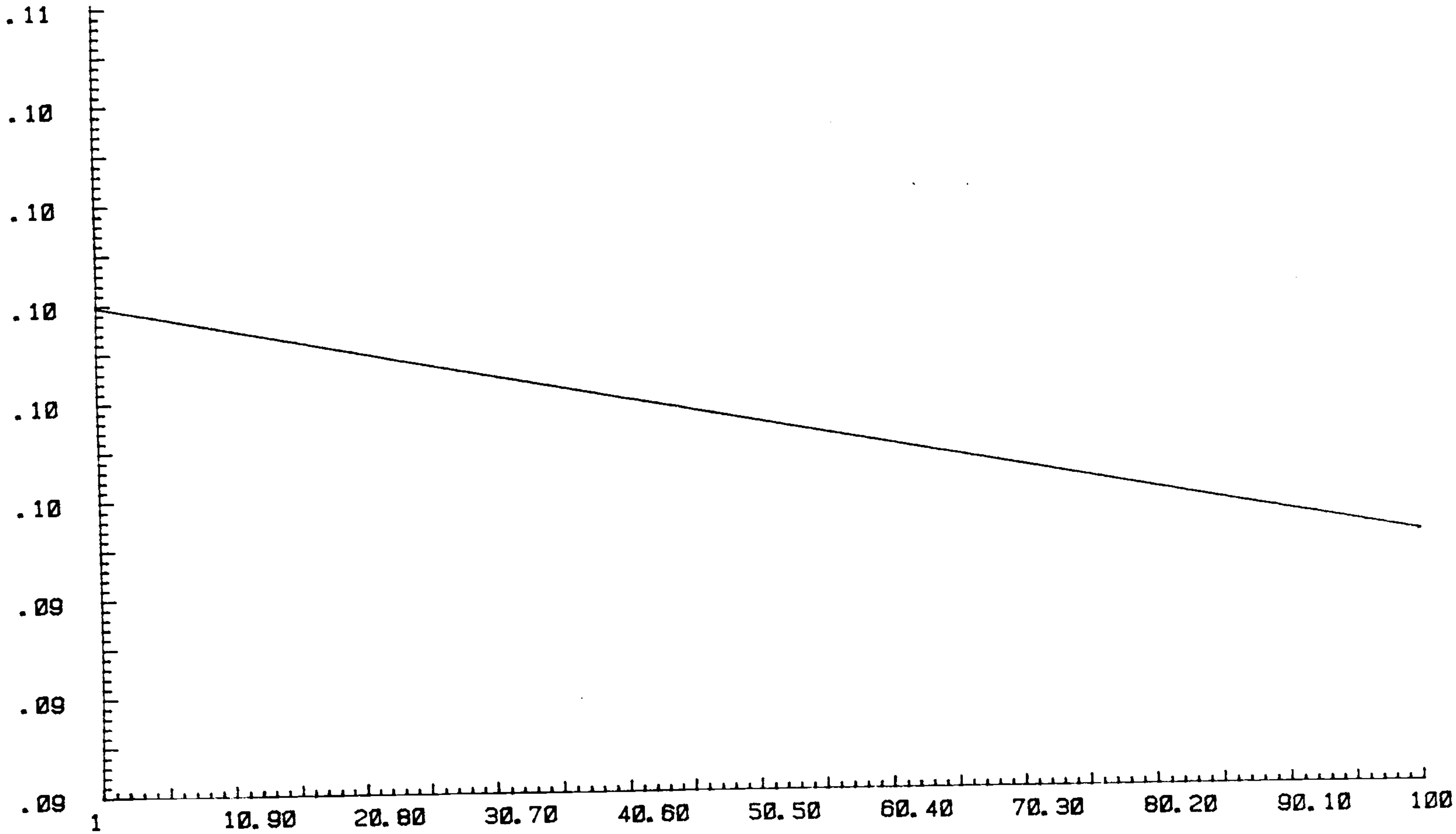

Fig. 20. The State $x_3(t)$ of Eq. (7.33),(7.32) with $u(t) = 0$, $T = 0.001$

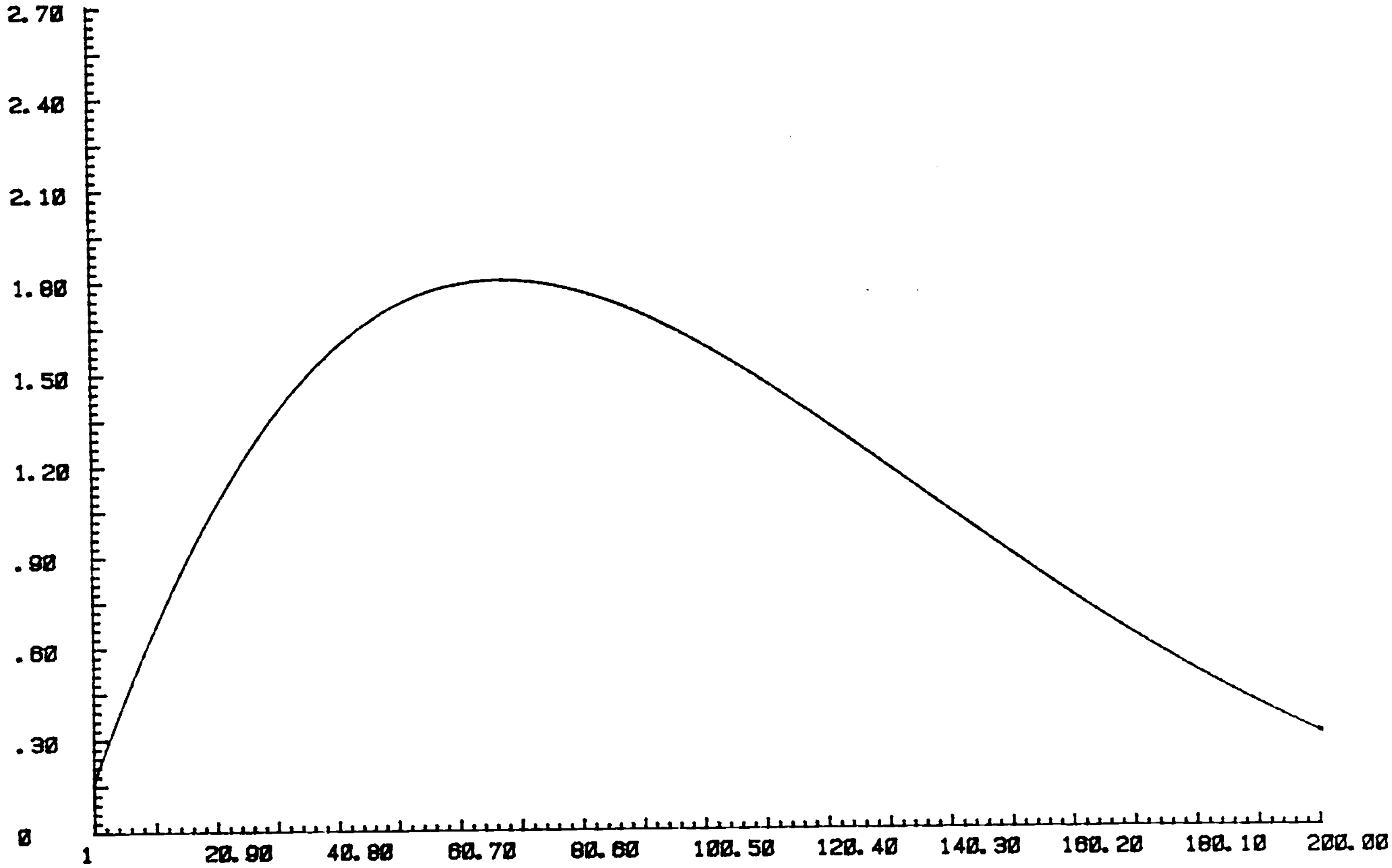

Fig. 21. The State $x_1(t)$ of Eq. (7.33),(7.32) with $u_1 = 0.2, v = 3$, $T = 0.001$

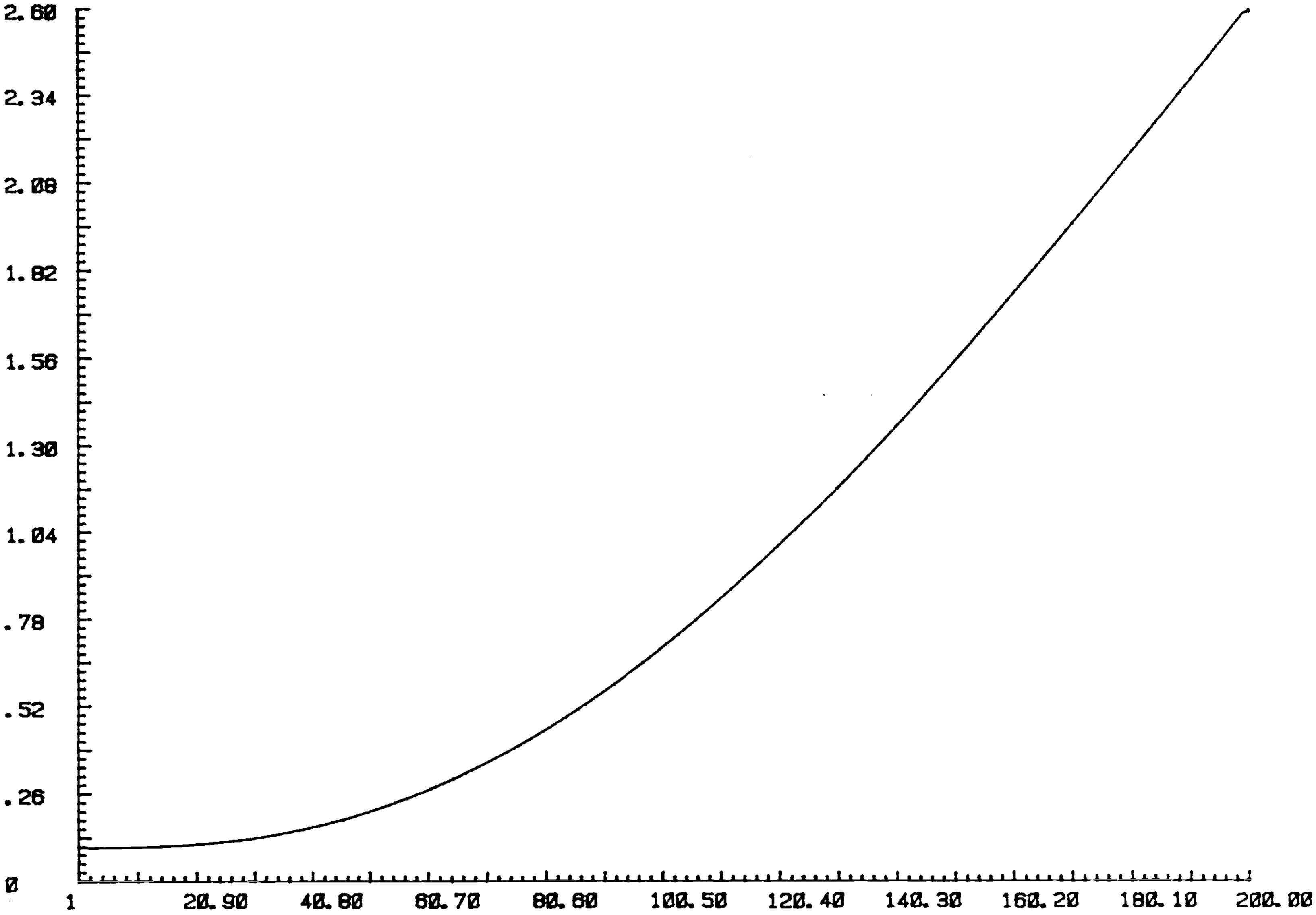

Fig. 22. The State $x_2(t)$ of Eq.(7.33),(7.32) with $u_1 = 2, v = 3, T = 0.001$

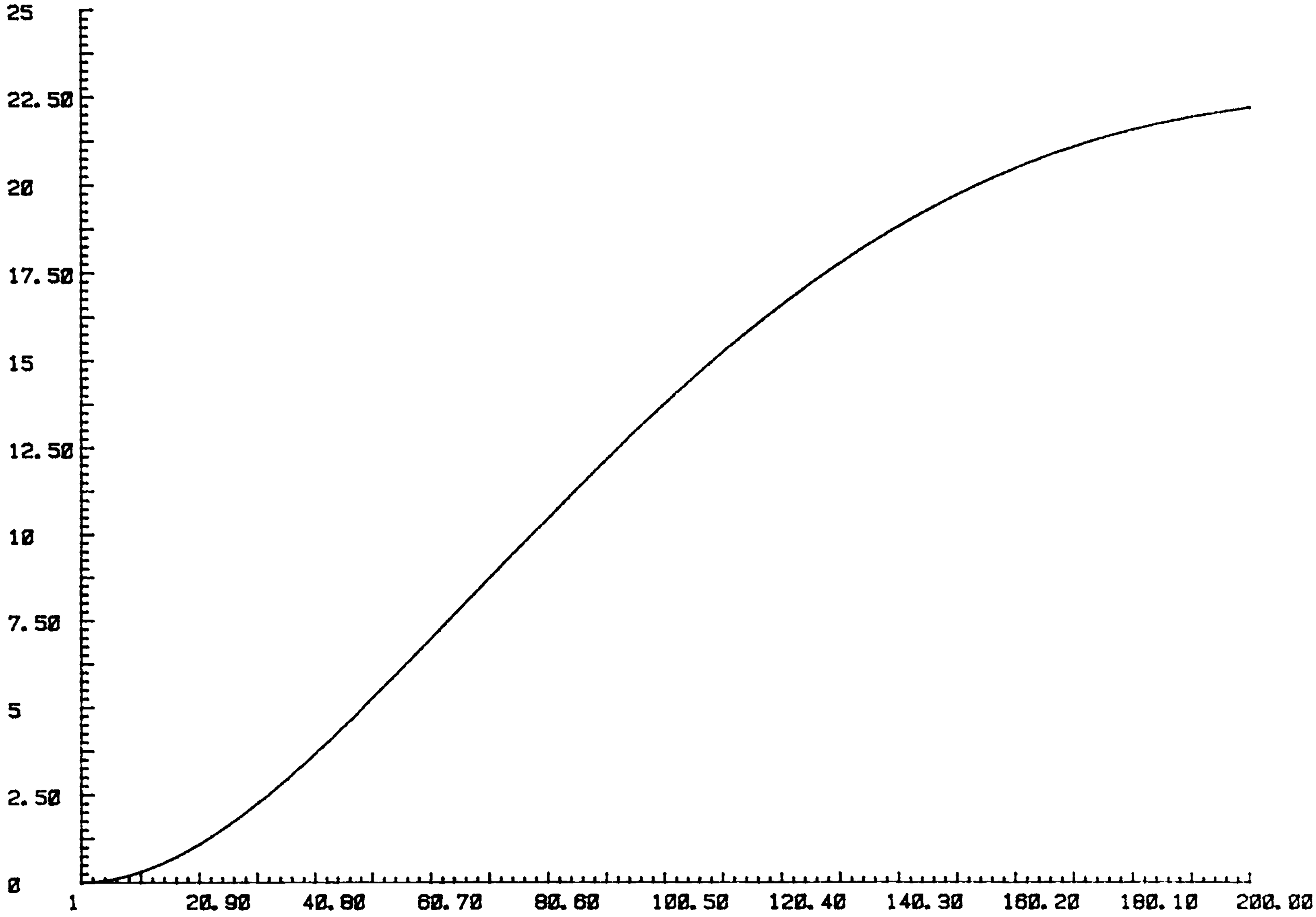

Fig. 23. The State $x_3(t)$ of Eq.(7.33),(7.32) with $u_1 = 2, v = 3$, $T = 0.001$

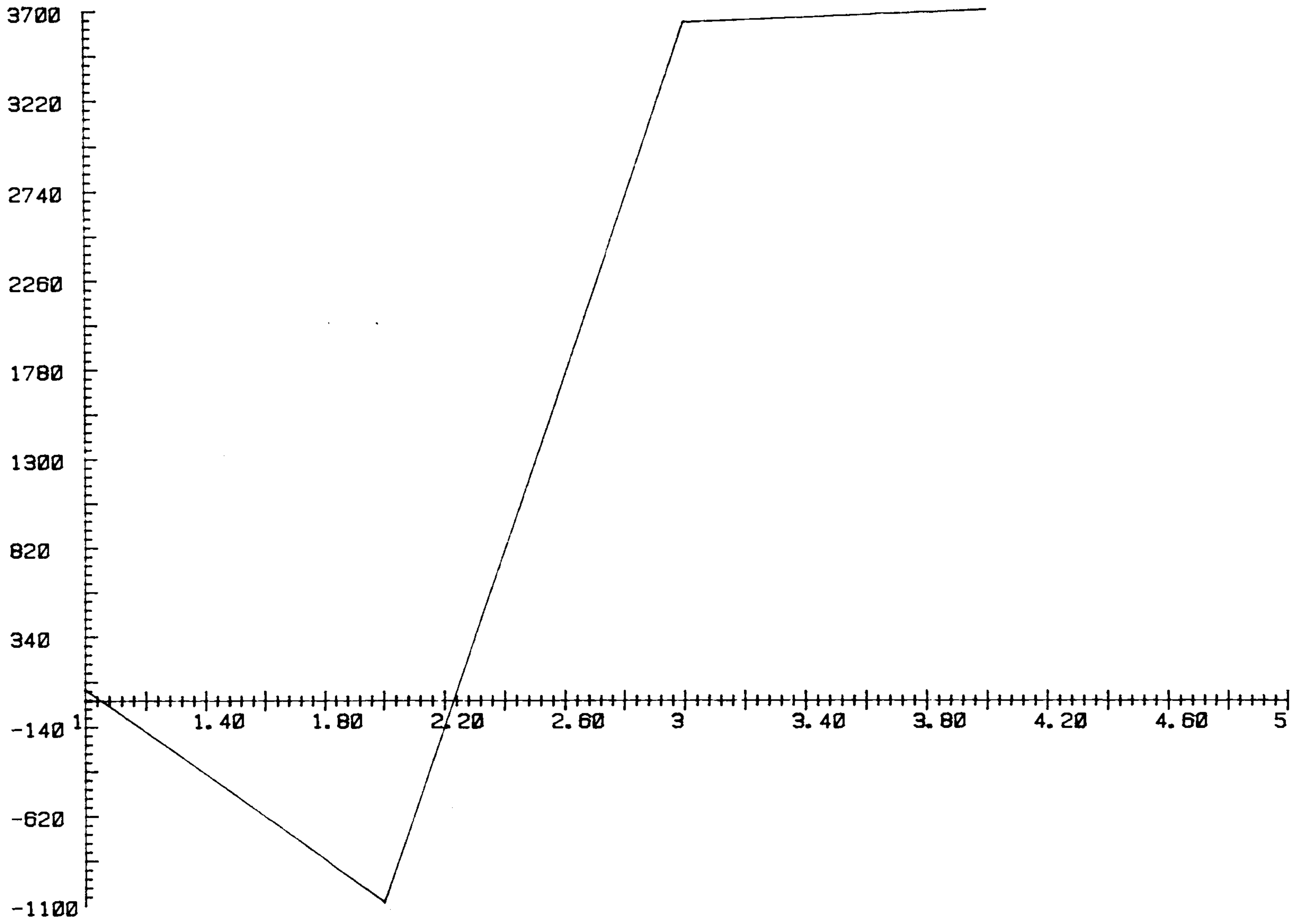

Fig. 24. The State $x_1(t)$ of Eq.(7.33),(7.32) with $u_1 = 2, v = 3, T = 1$

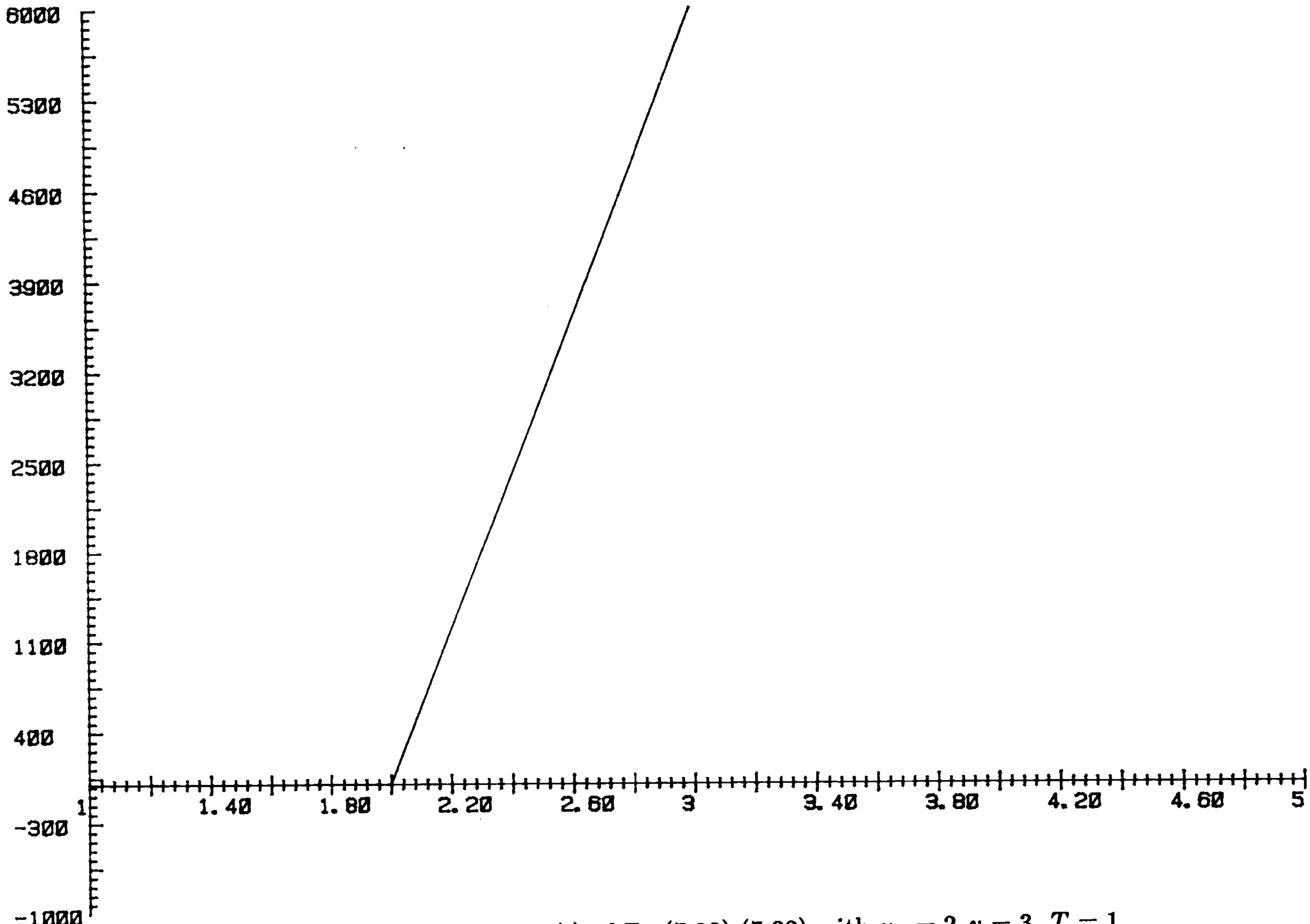

Fig. 25. The State $x_2(t)$ of Eq.(7.33),(7.32) with $u_1 = 2, v = 3, T = 1$

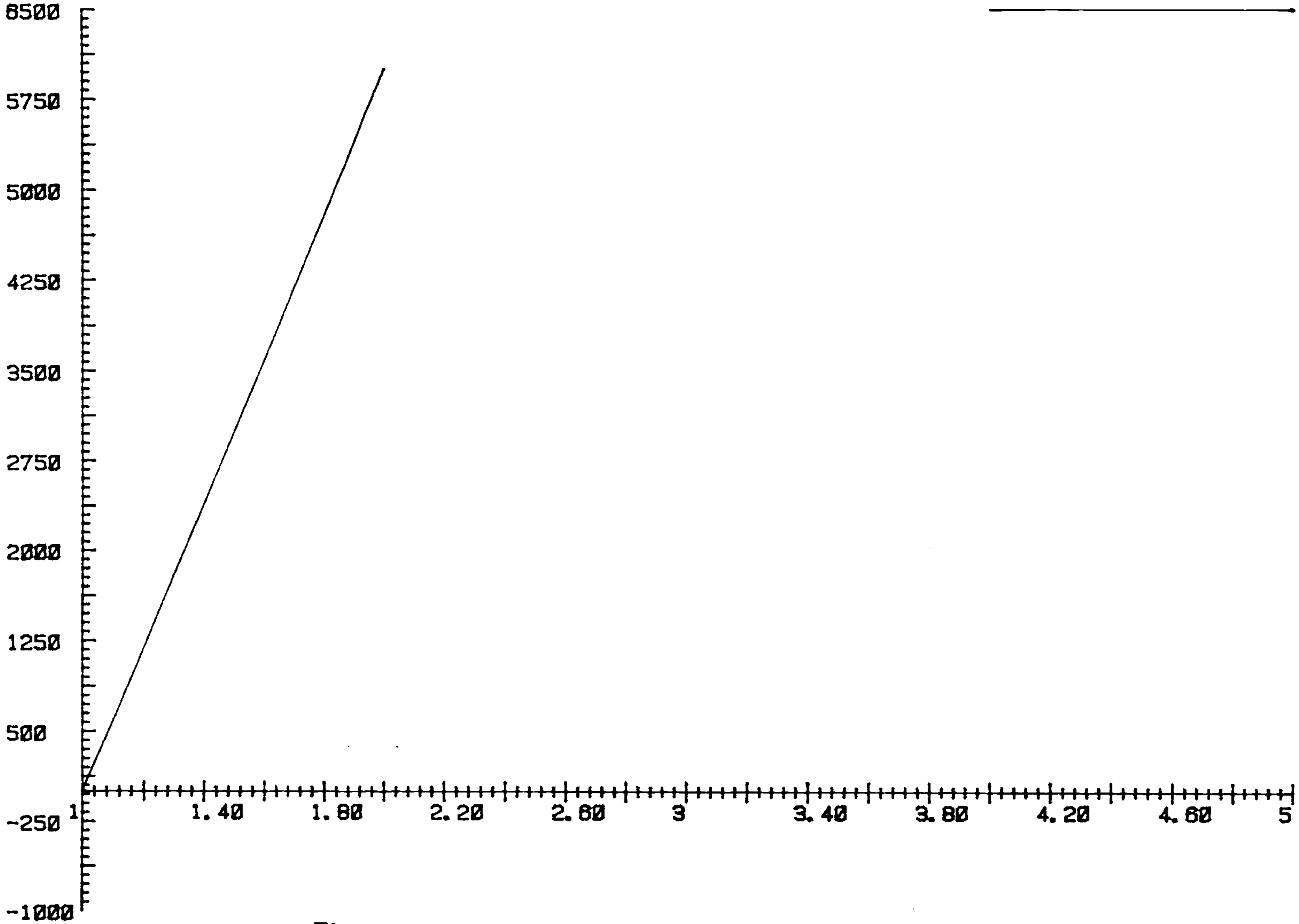

Fig. 26. The State $x_3(t)$ of Eq.(7.33),(7.32) with $u_1 = 2, v = 3$, $T = 1$

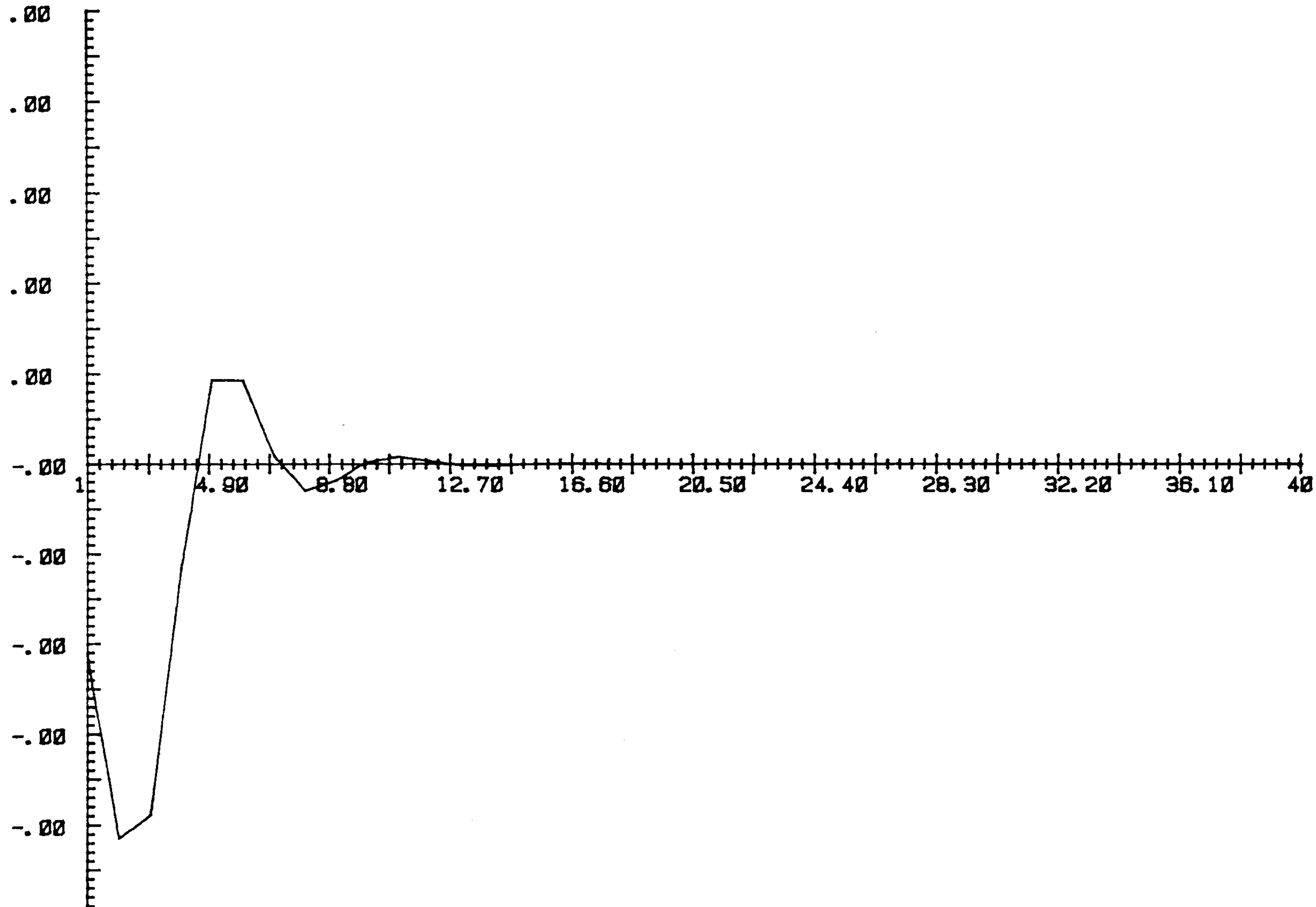

Fig. 27. The State $x_1(t)$ of Eq.(7.33),(7.32) with linear feedback at $T = 1$

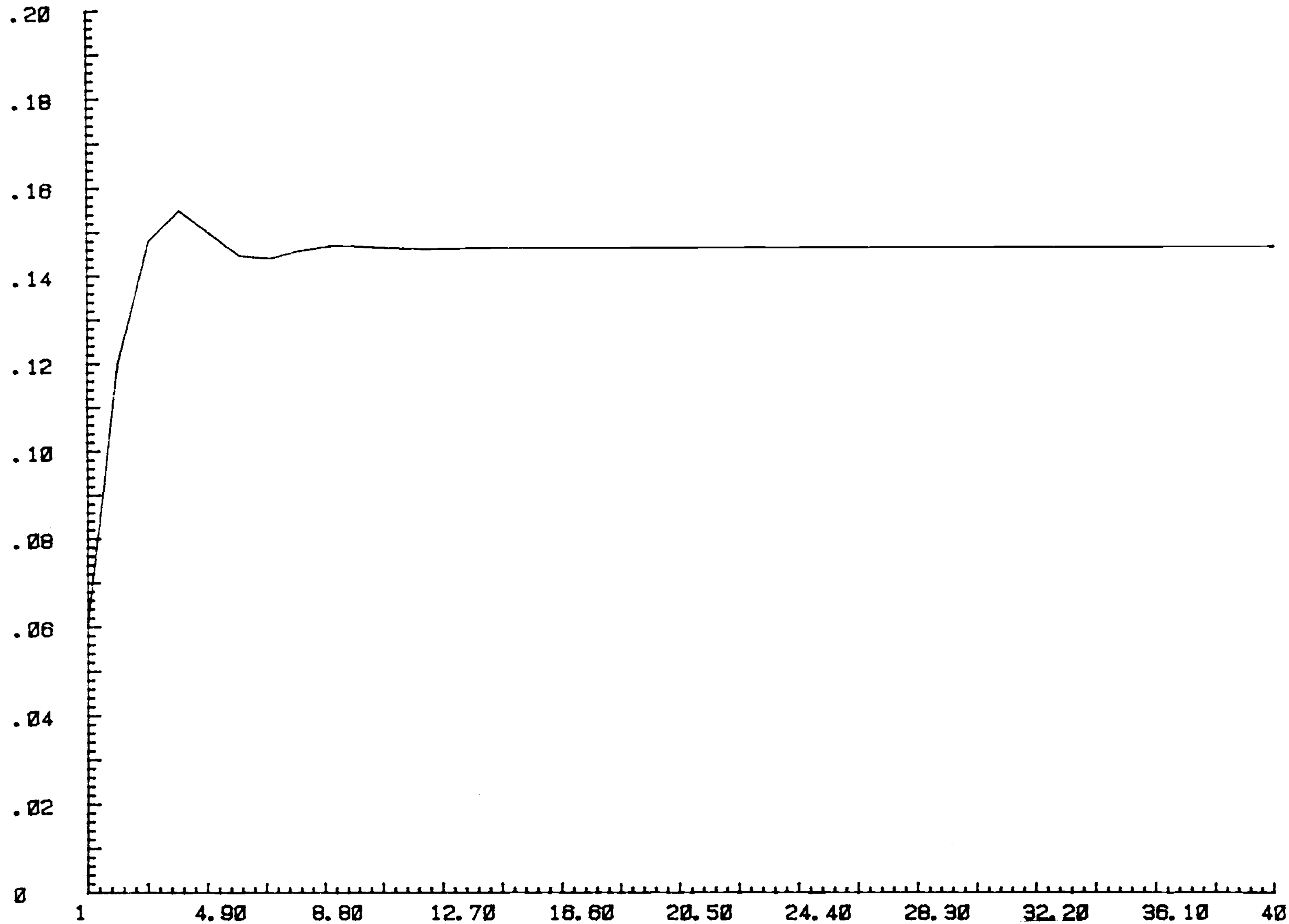

Fig. 28. The State $x_2(t)$ of Eq.(7.33),(7.32) with linear feedback at $T = 1$

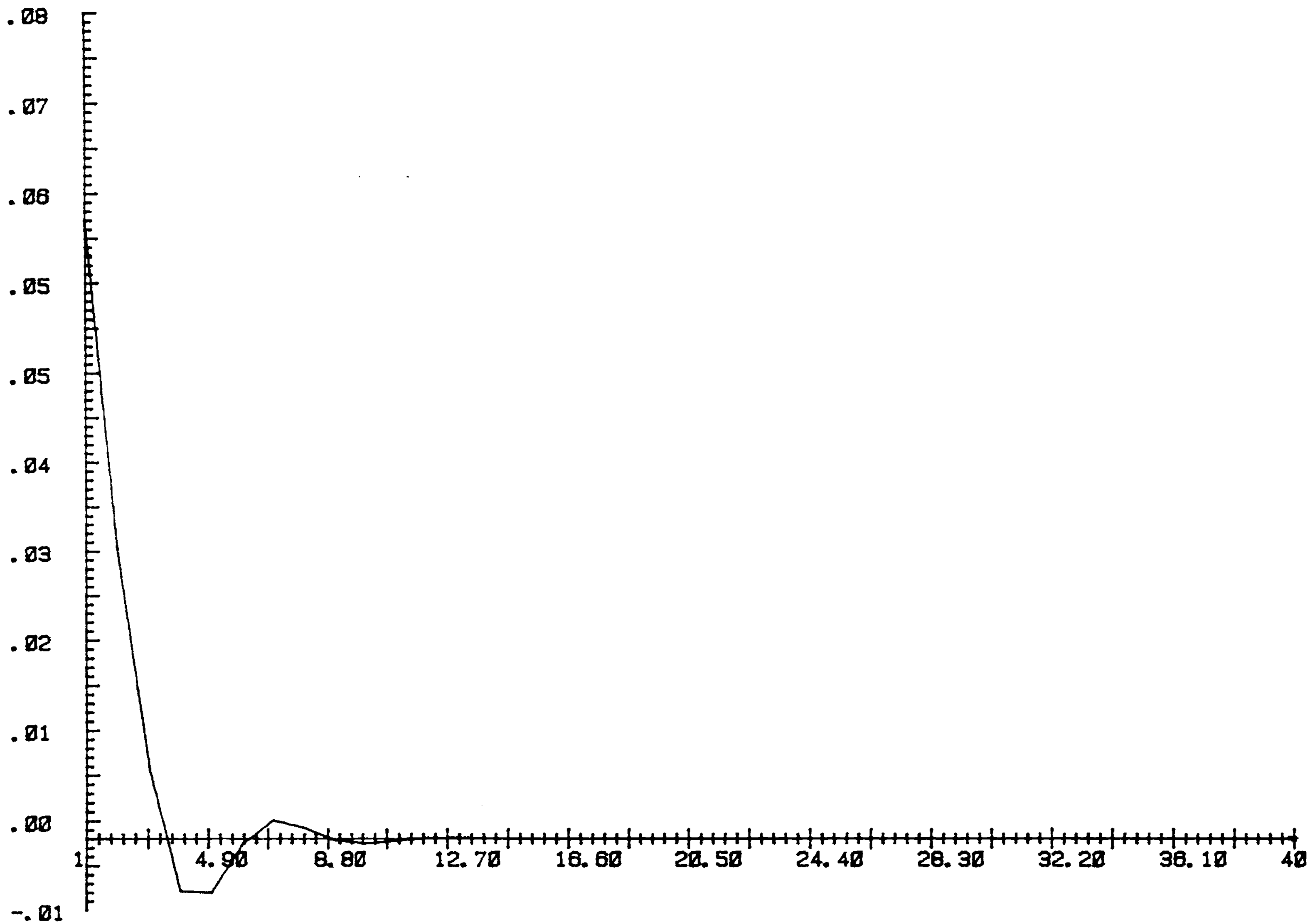

Fig. 29. The State $x_3(t)$ of Eq.(7.33), (7.32) with linear feedback at $T = 1$

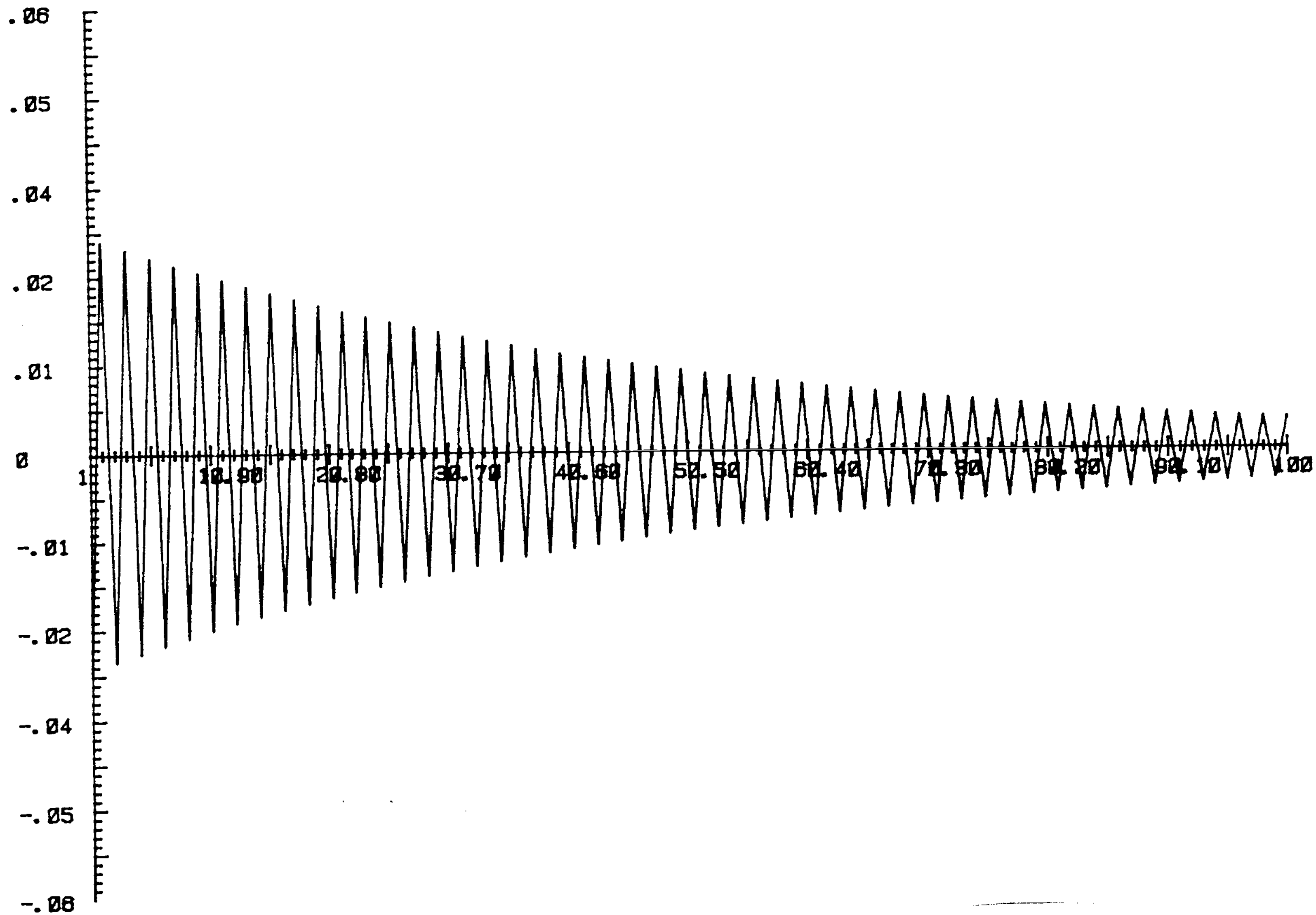

Fig. 30. The State $x_1(t)$ of Eq.(7.33),(7.32) with linear feedback at $T = 0.001$

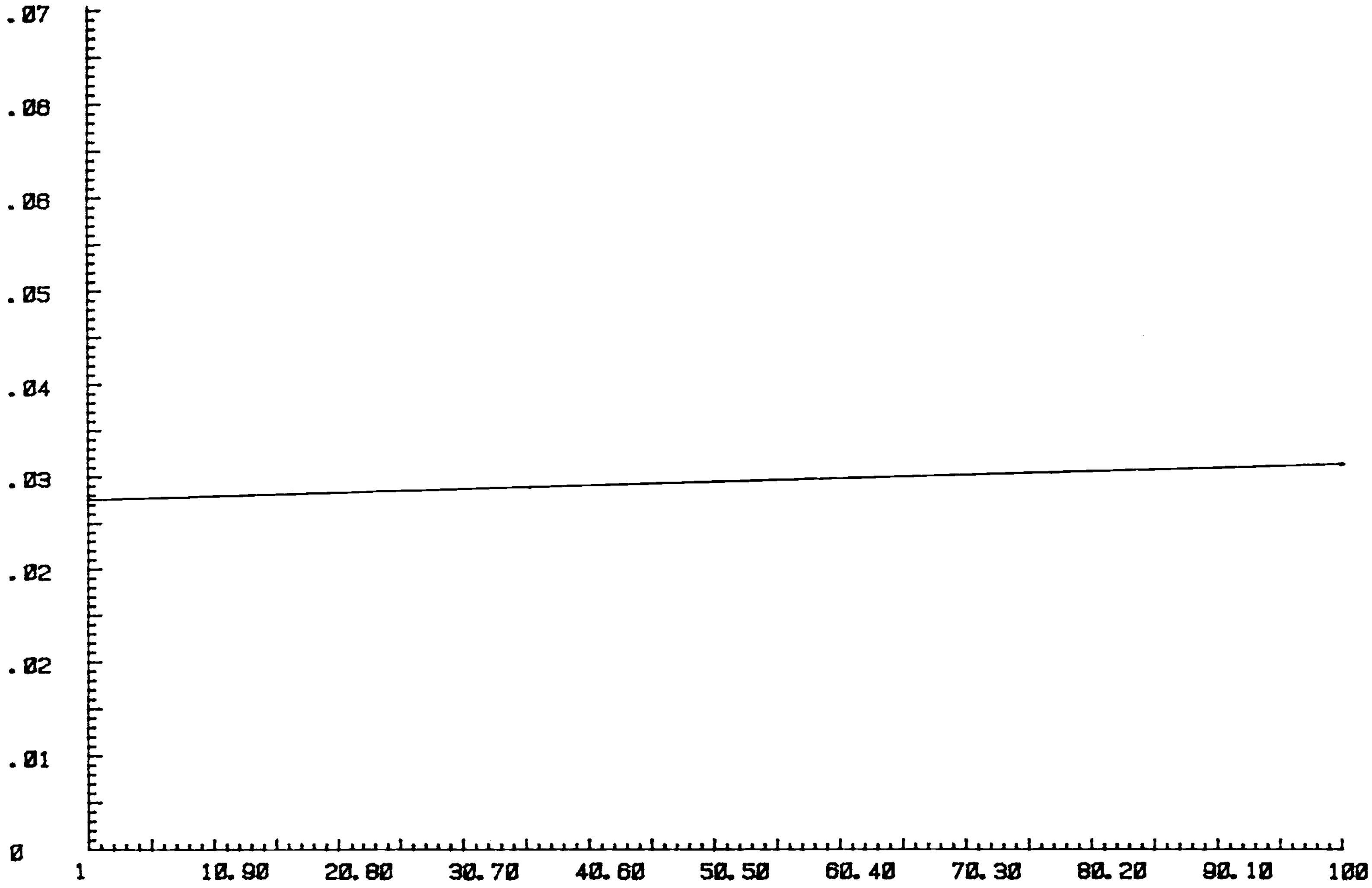

Fig. 31. The State $x_2(t)$ of Eq.(7.33),(7.32) with linear feedback at $T = 0.001$

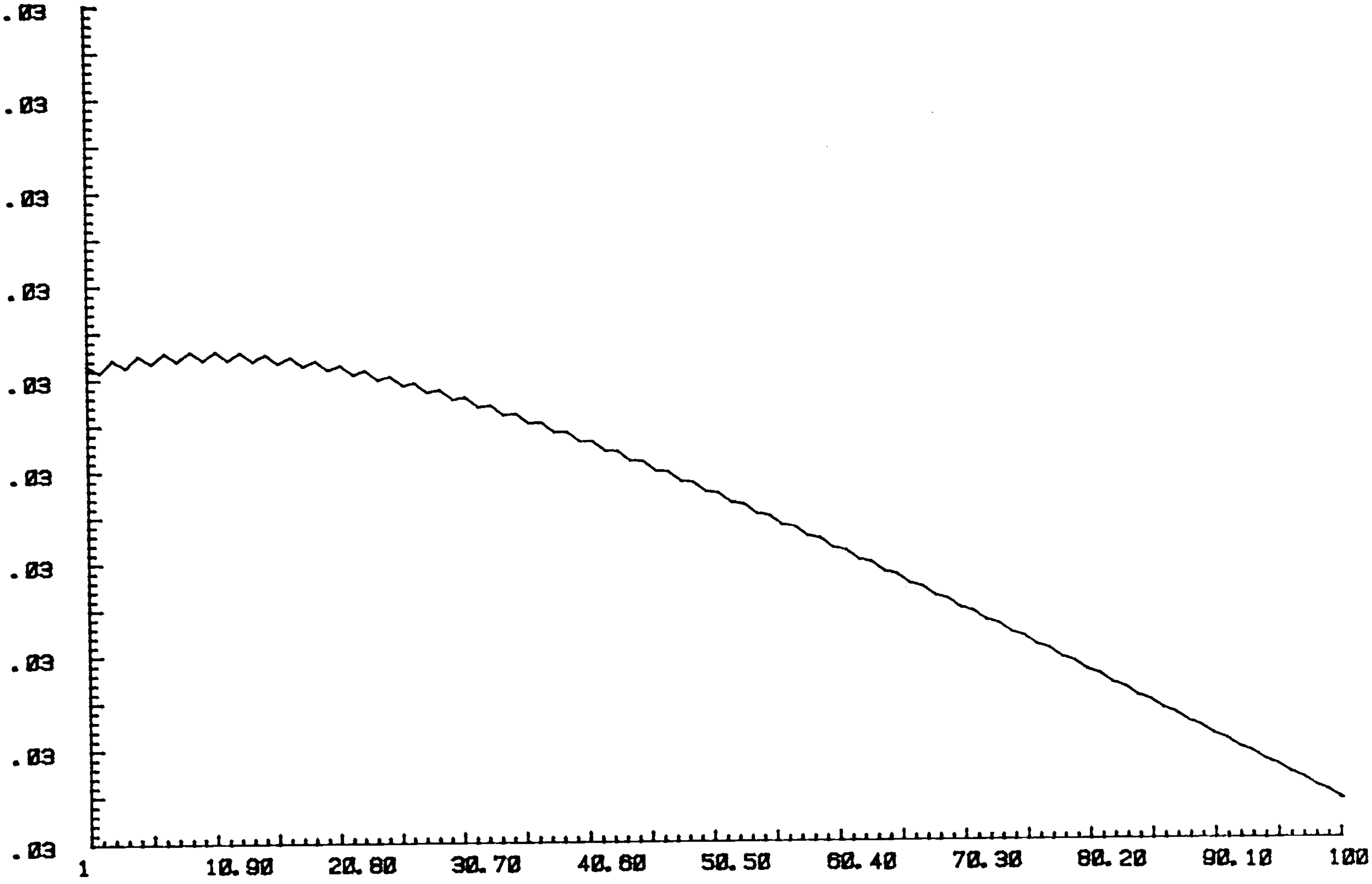

Fig. 32. The State $x_3(t)$ of Eq.(7.33),(7.32) with linear feedback at $T = 0.001$

VIII. CONCLUSIONS

In this thesis, stability of deterministic and stochastic discrete time-varying bilinear systems is studied. We consider the input, $u(t)$, as not only a signal but also a feedback function depending on the present and the previous output, it is $u(t) = f(Y(t))$. Also the feedback function f, we assume, is a wider class than most papers which have been published. The f can be linear, a function satisfying the Lipschitz condition or a quadratic function or a high-order polynomial function. Very few papers consider such cases as in this thesis. The other contribution in the thesis is that all given hypotheses for stability are simplified. Those hypotheses depend on the coefficient matrices of the systems and are already given in most existing models. So, these results are very easy to check and to apply in engineering problems. Computer simulations illustrate the utility of the theorems.

Because of the random nature of the phenomena involved stochastic system models have been suggested. Here we study stability of the bilinear systems with random parameters, also stability of bilinear systems with additive noises. We give mean-square, stability conditions for the stochastic models without the stationarity. Also all derived conditions which assure stability for the corresponding bilinear systems are convenient to check as in deterministic case.

Two practical examples (one is the deterministic bilinear model, another is the stochastic bilinear system model) are introduced in this thesis. The examples show that these results of stability for the bilinear systems are useful because they can be applied easily. Also, the stability analysis in this thesis will be helpful for the system design. It shows the way how to improve the stability by using the feedback function: the linear term of the feedback function is an important part for improving the zero state stability for bilinear systems. Also, we know that we may choose an appropriate feedback function including linear term to stabilize the system which is unstable originally. The appropriate quadratic term may increase the speed of convergence. In other cases, we may choose $u(t) = kx^{-p}(\ 0 < p < 1)$, to improve the zero state stability in homogeneous bilinear systems.

Most of the theorems in this thesis are local stability. And some results are restrictive. So the study of stability in large and improvement some results are remained in the further research.

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X. APPENDICES

1. Stability Analysis in term of Norms

General discrete bilinear systems can be described by difference equations of the form

$$X(t+1) = A(t)X(t) + \sum_{i=1}^{m} B_i(t)X(t)u_i(t) + C(t)U(t),$$
$$Y(t) = H(t)X(t), \quad u_i(t) = f_i(Y(t)), \quad U(t) = (u_1(t), \cdots, u_m(t))^T, \quad (A.1)$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$, $U(t) \in R^m$, $Y(t) \in R^p$, $p \leq n$, $A(t)$, $B_i(t)$ $i = 1, \cdots, m$ are $n \times n$ matrices, $C(t)$ is $n \times m$ matrix, $H(t)$ is a $p \times n$ matrix, f_i, $i = 1, \cdots m$, are arbitrarily bounded measurable functions from R^p to R.

Here, first, we consider bilinear systems with scalar input as follows:

$$X(t+1) = A(t)X(t) + B(t)X(t)u(t) + C(t)u(t),$$
$$Y(t) = H(t)X(t), \qquad u(t) = f(Y(t)), \qquad (A.2)$$

where $X(t), Y(t)$ are n-dimension vectors and A, B, H are $n \times n$ matrices, and C is $n \times 1$ matrix. A more general extension can be found at the end of this Appendix.

We define the norm of vector X in R^n, $X = (x_1, \cdots, x_n)^T$, and norm of matrix $A \in R^{n \times n}$, $A = (a_{ij})$, $i, j = 1, \cdots, n$, to be

$$\|X\|_\infty = \sup_j |x_j| \quad \text{and} \quad \|A\|_\infty = \sup_{ij} |a_{ij}|. \qquad (A.3a)$$

Here, we should remark that the results in following theorems hold if the norms are replaced by

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \text{and} \quad \|A\|_{q,p} = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |a_{ij}|^q\right)^{p/q}\right)^{1/p}, \qquad (A.3b)$$

where $1/p + 1/q = 1$.

If f be a linear functional from R^n to R. The norm of f is defined by

$$\|f\|_\infty = \sup_{\|x\|_\infty \leq 1} \{|f(x)| \; x \in R^n\}. \tag{A.4}$$

The same assumption as of f in Section II, but the norm defined in Section II will be replaced by (A.3a) or (A.3b).

Suppose f is a bounded measurable function (defined from R^n to R) and satisfies (2.1).

Let us denote

$$\sup_{t\in Z^+} \|A(t)\|_\infty = F_A, \qquad \sup_{t\in Z^+} \|A(t)\|_{q,p} = F_A^*, \tag{A.5}$$

$$\sup_{t\in Z^+} \|B(t)\|_\infty = F_B, \qquad \sup_{t\in Z^+} \|B(t)\|_{q,p} = F_B^*, \tag{A.6}$$

$$\sup_{t\in Z^+} \|B_i(t)\|_\infty = F_{B_i}, \qquad \sup_{t\in Z^+} \|B_i(t)\|_{q,p} = F_{B_i}^*, \tag{A.7}$$

$$\sup_{t\in Z^+} \|H(t)\|_\infty = F_H, \qquad \sup_{t\in Z^+} \|H(t)\|_{q,p} = F_H^*. \tag{A.8}$$

The following lemma will be useful.

Lemma A.1: In systems (A.2), assume $\{X(k), k \in Z^+\}$ to be a sequence in R^n, and the norms of $A(t)$, $B(t)$, $C(t)$, $H(t)$ be uniformly bounded on Z^+. Suppose $f : R^n \to R$ is defined as in (A.4). Then

(i) $\|X(k+1)\|_\infty \leq (nF_A + 4nK_1F_CF_H)\|X(k)\|_\infty + 4n^2K_1F_BF_H\|X(k)\|_\infty^2$

for all $k \in Z^+$. Furthermore, if the norm, $\| \ \|$ is replaced by the norm $\| \ \|_p$, $\| \ \|_{q,p}$, (see (3)), then

(ii) $\|X(k+1)\|_p \leq (F_A^* + 4K_1F_C^*F_H^*)\|X(k)\|_p + 4K_1F_B^*F_H^*\|X(k)\|_p^2$

for all $k \in Z^+$.

Proof: Let us fix k. Let $A(k) = (a_{ij}(k))$, $X(k) = (x_1(k), \cdots, x_n(k))^T$. So,

$$A(k)X(k) = \left(\sum_{j=1}^{n} a_{1j}(k)x_j(k), \cdots, \sum_{j=1}^{n} a_{nj}(k)x_j(k) \right)^T . \tag{A.9}$$

Hence,

$$\begin{aligned} \|A(k)X(k)\|_\infty &\le \max_{1 \le i \le n} \left\{ \left| \sum_{j=1}^{n} a_{ij}(k)x_j(k) \right| \right\} \\ &\le \sum_{j=1}^{n} \|A\| \|X(k)\|_\infty \le nF_A \|X(k)\|_\infty . \end{aligned}$$

Following the same way as above, we have

$$\|H(k)X(k)\|_\infty \le n\|H(k)\|_\infty \|X(k)\|_\infty \le nF_H \|X(k)\|_\infty .$$

Applying Minkowski's inequality (Rudin, 1987) and the above two estimates, we have

$$\begin{aligned} \|X(k+1)\|_\infty &\le n\|A(k)\|_\infty \|X(k)\|_\infty + n\|B(k)\|_\infty \|X(k)\|_\infty |u(k)| + \|C(k)\|_\infty |u(k)| \\ &\le nF_A \|X(k)\|_\infty + (nF_B \|X(k)\|_\infty + F_C)|u(k)|, \end{aligned} \tag{A.10}$$

for all $k \in Z^+$. Let us write

$$\begin{aligned} |u(k)| &= |f(H(k)X(k))| \\ &\le \sum_{j=0}^{\infty} \sup_{\|H(k)X(k)\|_\infty \le 2^{-j} nF_H \|X(k)\|_\infty} |f(H(k)X(k))| \\ &= \sum_{j=0}^{\infty} \omega(2^{-j} nF_H \|X(k)\|_\infty) \\ &= 2\sum_{j=0}^{\infty} \left\{ \omega(2^{-j} nF_H \|X(k)\|_\infty)/2^{-j+1} \right\} \left\{ 2^{-j+1} - 2^{-j} \right\}. \end{aligned}$$

It is clear to see $\omega(t)$ is a bounded increasing function. Therefore, the term in the first pair of parentheses is not bigger than the minimum value of the function $\omega(tnF_H\|X(k)\|_\infty)/t$ on the interval $t \in [2^{-j},\ 2^{-j+1}]$. So, the last sum is less than

$$2\int_0^2 \frac{\omega(tnF_H\|X(k)\|_\infty)}{t}dt = 2\int_0^{2nF_H\|X(k)\|_\infty} \frac{\omega(t)}{t}dt.$$

From (A.4), we conclude that

$$|u(k)| \leq 4nK_1F_H\|X(k)\|_\infty. \tag{A.11}$$

Part (i) follows from substituting (A.11) into (A.10). To prove part (ii), we need to estimate $\|A(k)X(k)\|_p$ and $\|H(k)X(k)\|_p$. From (A.9), we have

$$\|A(k)X(k)\|_p = \left(\sum_{i=1}^n\left|\sum_{j=1}^n a_{ij}(k)x_j(k)\right|^p\right)^{1/p}.$$

By Hölder's inequality (Rudin, 1987), the last term is bounded by

$$\left(\sum_{i=1}^n\left(\sum_{j=1}^n|a_{ij}(k)|^q\right)^{p/q}\right)^{1/p}\left(\sum_{j=1}^n|x_j(k)|^p\right)^{1/p} = \|A\|_{q,p}\|X(k)\|_p.$$

As before, we have the same conclusion,

$$\|H(k)X(k)\|_p \leq \|H\|_{q,p}\|X(k)\|_p.$$

Hence, part (ii) follows by repeating the same proof of part (i).

Lemma A.2: In systems (A.2), assume $\{X(k), k \in Z^+\}$ to be a sequence in R^n, and if there exist nonzero positive numbers α_1 and α_2 such that

$$\|X(t+1)\|_\infty \leq \alpha_1\|X(t)\|_\infty + \alpha_2\|X(t)\|_\infty^2.$$

Then the zero state for the system (A.2) is stable and asymptotically stable, if $\alpha_1 < 1$.

Proof: Let

$$\beta \triangleq \alpha_1 + \alpha_2\delta.$$

For every $\epsilon > 0$, take δ such that

$$\delta\beta < \epsilon \qquad \text{and} \qquad \beta < 1.$$

It is easy to prove that the δ exists if $\alpha_1 < 1$, i.e. taking

$$\delta = \min\{\epsilon,\ (1-\alpha_1)/\alpha_2\}.$$

So, if $\|X(0)\|_\infty < \delta$, then

$$\|X(1)\|_\infty < \delta\beta < \epsilon,$$

$$\|X(2)\|_\infty < \delta\beta[\alpha_1 + \alpha_2\delta\beta] < \delta\beta^2.$$

Without difficulty by mathematical induction, one can show that

$$\|X(k)\|_\infty \leq \delta\beta^k.$$

It implies that the zero state for the systems (A.2) is stable and asymptotically stable, if $\beta \leq 1$ or $\beta < 1$, respectively.

Remark A.2: This lemma A.2 does not depend systems model (A.2), so this result can be developed to the general nonlinear systems.

Theorem A.1: Suppose $f : R^n \to R$ is defined as in (A.4) and the norms of $A(t), B(t)$, $C(t)$, $H(t)$ are uniformly bounded on Z^+. Let a denote either

$$nF_A + 4nK_1F_CF_H$$

or

$$F_A^* + 4K_1F_C^*F_H^*,$$

where F_A, F_C, F_H and F_A^*, F_C^*, F_H^* are defined as (A.5), (A.3), (A.8) respectively. Then the zero state for the systems (A.2) is stable if $a \leq 1$.

Proof: From Lemma A.1, we have

$$\text{(i)} \quad \|X(k+1)\|_\infty \leq (nF_A + 4nK_1F_CF_H)\|_\infty X(k)\|_\infty + 4n^2K_1F_BF_H\|X(k)\|_\infty^2$$

for all $k \in Z^+$. Furthermore, if the norm, $\| \ \|$ is replaced by the norm $\| \ \|_p$, $\| \ \|_{q,p}$, (see (A.3)), then

$$\text{(ii)} \qquad \|X(k+1)\|_p \leq (F_A^* + 4K_1F_C^*F_H^*)\|X(k)\|_p + 4K_1F_B^*F_H^*\|X(k)\|_p^2$$

for all $k \in Z^+$. Let a denote either

$$nF_A + 4nK_1F_CF_H$$

or

$$F_A^* + 4K_1F_C^*F_H^*,$$

where F_A, F_C, F_H and F_A^*, F_C^*, F_H^* are defined as (A.5), (A.3), (A.8) respectively.

The result is clear when following the lemma A.2.

Corollary A.1: Suppose $f : R^n \to R$ is a linear function and the norms of $A(t), B(t), C(t), H(t)$ are uniformly bounded on Z^+, Then

(i) The zero state for the system (A.2) is stable and asymptotically stable if

$$\|f\| \leq (1 - nF_A)/nF_CF_H,$$

or

$$\|f\| \leq (1 - F_A^*)/F_C^*F_H^*;$$

(ii) The zero state for the system (A.2) is asymptotically stable if

$$\|f\| < (1 - nF_A)/nF_CF_H,$$

or $$\|f\| < (1 - F_A^*)/F_C^* F_H^*.$$

Proof: Notice that (A.11) can be changed to:

$$|u(k)| \le n\|f\| F_H \|X(k)\|_\infty,$$

if f is a linear function. This result is clear by following the Theorem A.1.

The homogeneous bilinear systems with scalar input is as follows:

$$\begin{aligned} X(t+1) &= A(t)X(t) + B(t)X(t)u(t) \\ Y(t) &= H(t)X(t), \qquad u(t) = f(Y(t)), \end{aligned} \tag{A.12}$$

where $X(t), Y(t)$ are n-dimension vectors and A, B, H are $n \times n$ matrices, we have

Corollary A.2: Suppose $f : R^n \to R$ is defined as in (A.4) and the norms of $A(t), B(t)$, $H(t)$ are uniformly bounded on Z^+. Let a denote either

$$nF_A \text{ or } F_A^*,$$

where F_A, F_B, F_H and F_A^*, F_B^*, F_H^* are defined as (A.5), (A.6), (A.8) respectively. Then the zero state for the systems (A.12) is stable if $a \le 1$.

Proof: This corollary can be obtained when $C = 0$ of Theorem 1.

Following the same procedure as in Theorem A.1, we have the next theorem.

Theorem A.2: (i) The zero state for the system (A.1) is stable if

$$F_A^* + 4\left(\sum_{i=1}^{m} |K_{f_i}|^p\right)^{1/p} F_C^* F_H^* \le 1;$$

(ii) The zero state for the system (A.1) is asymptotically stable if

$$F_A^* + 4\left(\sum_{i=1}^{m} |K_{f_i}|^p\right)^{1/p} F_C^* F_H^* < 1;$$

where F_A^*, F_C^*, $F_{B_i}^*$ and F_H^* are defined as (A.5) – (A.8) respectively, K_1 is defined by (2.1).

2. An Example

let

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} -0.3\sin t & 0 \\ -0.2e^{-t} & -0.3\cos t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.3e^{-t} & 0.2\cos 2t \\ -0.5\sin t & 0.4e^{-3t} \end{bmatrix} u(t) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} u(t)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.2e^{-t} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$u(t) = f(Y(t)) = a_1 y_1(t) + a_2 y_2(t).$$

Let us set $p = q = 2$, since $F_A^* = \|A\| = 0.361$, $F_B^* = \|B\| = 0.735$, $F_C^* = \|C\| = 0.5$, $F_H^* = \|H\| = 0.375$, and $\|f\| = \sup |a_1 y_1 + a_2 y_2| = (a_1^2 + a_2^2)^{1/2}$ where the supremum is over $(y_1, y_2) \in R^2$ and $(y_1^2 + y_2^2)^{1/2} \leq 1$, by the corollary, the equilibrium at the origin for this system is stable if $(a_1^2 + a_2^2)^{1/2} \leq 3.5$ and the equilibrium at the origin for this system is asymptotically stable if $(a_1^2 + a_2^2)^{1/2} < 3.5$.