AN ABSTRACT OF THE THESIS OF

Sayooj Sunil Raj for the degree of Master of Science in Industrial Engineering presented on February 25, 2021.

Title: Measuring Instrument Precision Estimation in Destructive Testing.

Abstract approved: ____

David S Kim

Variability introduced into measurements by a measuring instrument is referred to as measurement instrument precision. Experimental procedures and analysis methods exist when measurements are repeatable and can be repeated on the same item. However, when the measurements are destructive and repeated measurements are not possible, estimating measuring instrument precision is difficult since measuring instrument precision is confounded with part variance. In this research formulas are developed for estimating measuring instrument precision and the measuring instrument precision estimate variance, from which confidence intervals can be obtained. The results are obtained by measuring two different part types, assuming the part measurement coefficient of variation is constant, the measurement instrument precision is constant, and that part measurements are normally distributed and independent. Equations are derived to estimate measuring instrument precision and its standard error when part type means are assumed known, and also when part type means are estimated from the measurement data. The results are validated using Monte Carlo simulation.

Measuring Instrument Precision Estimation in Destructive Testing

by

Sayooj Sunil Raj

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

Presented February 25, 2021
Commencement June 2021

Master of Science thesis of Sayooj Sunil Raj presented on February 25, 2021

APPROVED:

Major Professor, representing Industrial Engineering

Head of the School of Mechanical, Industrial and Manufacturing Engineering

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Sayooj Sunil Raj, Author

ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor Dr. David S. Kim, for allowing me to work with him on this research. This research would not have been possible without his persistent help and guidance throughout my time at Oregon State University. I would always be grateful to Dr. Kim for giving me the opportunity to work as a Graduate Assistant and for being my mentor and motivating me to push my limits, which helped me achieve this milestone.

I would like to thank Dr. Joseph Agor for serving as my committee member, Dr.Xue Lan for serving as my minor professor, and Dr. John Sessions for serving as my graduate council representative. Thank you all for taking the time to read this research document.

I would like to express my gratitude towards my father, Sunil Raj, my mother Sabitha, and my grandparents for always believing in my abilities and blessing me. I am also very thankful to my wife, Nivedha Satheesh, for always being there to support me during my masters. I would also like to thank my colleagues and my best friends Prashant Tiwari and Shardul Kandke for being nothing less than elder brothers at Oregon State University.

Lastly, I thank Oregon State University Graduate School and the Mechanical, Industrial, and Manufacturing Engineering department. I consider myself to be lucky to have come to Oregon State University and to have the opportunity to learn from Dr. David S Kim.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

The research in this thesis addresses measuring instrument precision. More specifically, estimating measuring instrument precision in destructive testing, where only a single measurement per item is possible. Destructive measurements are common and occur in many contexts. Measuring the sharpness of knife blades, the projectile velocity of rifle ammunition, and fracture tests are a few of many possible examples. Generally, a measuring instrument is considered capable and fit for use if the measuring instrument precision value is under 10 percent of the total variation (Senvar & Oktay Firat, 2010). Before discussing the difficulties of estimating measuring instrument precision for destructive testing, a more general discussion of measuring instrument precision and gauge capability testing is presented.

Measuring instrument precision is the variability in a set of numerical measurements that can be attributed to the measuring instrument. The ability to estimate measuring instrument precision is important so that a measuring instrument's ability to meet a specific application's needs can be assessed. For example, consider measuring the diameter of screws using a vernier caliper. The vernier caliper's precision needs to be analyzed to know if the measuring instrument has the ability to reliably identify screws that differ in diameter by some specified amount.

The measuring instrument itself is typically one component of a measurement system, and each component of a measurement system may contribute to the variation seen in the measurements realized through the system. A common example of a measurement system is one with a single measuring instrument, multiple operators that use the measuring instrument, and multiple samples of the same item that are measured. In this example, the measurement system components that contribute to measurement variability are the operators, the different item samples, and the measuring instrument. This example is the measurement system in the well-known gauge "reproducibility and repeatability" study.

Measuring instrument precision is analyzed as part of a measurement system capability study. A measurement system capability study focuses on quantifying the measurement process, which involves separating and estimating different variability sources. Measuring instrument precision is also commonly known as gauge repeatability, with a measuring instrument referred to as a gauge. Measuring instrument precision is typically estimated from repeated measurements of the same item obtained, with all other components of the measuring system being constant. For example, in gauge repeatability and reproducibility experiment, a items are randomly selected and measured by b randomly selected operators, and n repeated measurements are taken on the same item by each operator. The measurement data obtained from the experiment is assumed to be realizations of the following linear statistical model,

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \epsilon_{ijk} \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1,2, \dots, n \end{cases}$$

Where, y_{ijk} is the kth measurement on item i measured by the jth operator, μ is the overall mean of the measured items, and α_i, β_j and $\alpha\beta_j$ are the deviation from this overall mean. In this model, α_i is the random variable due to the item with $Normal(0, \sigma_\alpha^2)$ distribution, β is the random variable due to the operator with $Normal\left(0, \sigma_\beta^2\right)$ distribution, $\alpha\beta_{ij}$ is the random variable due to the interaction of the operator and the item with $Normal\left(0, \sigma_{\alpha\beta}^2\right)$, and ϵ_{ijk} is the measurement error with $Normal\ (0, \sigma^2)$. σ^2 is the measuring instrument precision. An estimate of σ^2 is obtained from the repeated measurements on the same item by the same operator. When a new measurement system component is introduced or removed from the study, the linear statistical model will change. For example, if the measurements in the experiment just described are taken using c gauges, then (excluding gauge interaction terms), the linear statistical model is now,

$$y_{ijlk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \gamma_l + \epsilon_{ijlk} \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ l = 1,2, \dots, c \\ k = 1,2, \dots, n \end{cases}$$

Where, y_{ijlk} is the kth measurement on item i measured by the jth operator using gauge l and γ_l is a random variable due to the gauge and follows a $Normal\left(0, \sigma_\gamma^2\right)$ distribution. One common trait discussed in the above models are the repeated measurements on the same item. Such testing, in which repeated measurements on the same item are possible, will be referred to as "<u>repeatable testing</u>". For example, radiography testing where gamma or X-rays are directed on an item to analyze its material property is repeatable testing because the measured item is not altered after a measurement. Thus, repeated measurements on the same item can be obtained.

To estimate measuring instrument precision in repeatable testing, the two common approaches used are ANOVA and maximum likelihood. ANOVA is a more common method available in most commercial statistical software. In the ANOVA the expected value of the mean squares is used to form estimates of the different variance components, including measuring instrument precision. However, in both ANOVA and maximum likelihood method, the measuring instrument precision estimation is possible only if repeated measurements on the same item are available. On the other hand, in "non-repeatable testing", repeated measurements on the same item are not possible.

Measuring Instrument Precision Estimation in Non-Repeatable Testing

In non-repeatable testing, measuring instrument precision estimation is not straightforward as in repeatable testing. Non-repeatable testing can occur because of different reasons. In some testing, the repeated measurements are not possible if the measurement value changes when some parameter in the study changes. For example, when a brake disc's temperature is measured during its operation, the measurement value varies with time or the measurement value is dependent on the time parameter. In this case repeated measurements are not possible since a particular measurement occurs at a single point of time. Similarly, the shrinkage of carpets varies with a carpet’s position, and the stress test value on the carpet depends on the extent of its shrinkage. During the stress test, the carpet at the position of measurement gets damaged, repeated measurements are then taken on a different position on the carpet. Thus, in this case, the measurement value varies with the parameter position. Such measurement testing in which measurements are dependent on a parameter will be referred to in this thesis as "parameter-dependent testing"

Another type of non-repeatable testing is "destructive testing". In destructive testing, the item measured is destroyed after a measurement is taken. For example, in a tensile test, the force required to break a plastic specimen is measured by destroying the specimen during each measurement. The ANOVA method cannot be used to estimate measuring instrument precision in non-repeatable testing because measuring instrument precision cannot be separated from item variance. For example, the linear statistical model for measurements on a items by a single operator with no repeated measurements of the same item is,

$$y_i = \mu + \alpha_i + \epsilon_i$$

Where, y_i is the measurement on item i, μ is the overall mean, and α_i is the deviation from this overall mean for item i. With repeated measurements, error term ϵ_i has a single subscript but is assumed to follow a $Normal\ (0, \sigma^2)$ distribution. Because if the lack of replicates, a measuring instrument precision estimate $\hat{\sigma}^2$ cannot be estimated based on the ANOVA expected mean squares. In this case, the measuring instrument precision is confounded with the other variance component in the linear statistical model.

Therefore, additional assumptions are needed in order to develop a method to estimate measuring instrument precision in destructive testing. For example, the homogeneity assumption considers different measured items homogeneous with negligible or zero-item variance. Under this assumption, measuring instrument precision is estimated considering measurements from different items as the repeated measurements. By disregarding item variance, total measurement variability is considered to be due to measuring instrument variability. However, this essentially ignores the underlying problem of no repeat measurements. Different items with no inter-item variance are rare and assuming homogenous items often leads to overestimated measuring instrument precision estimations. Moreover, other assumptions applied by other researchers are relevant only to some specific non-repeatable testing. Thus, there is a need for additional methods to estimate measuring instrument precision in non-repeatable testing.

Research Contribution

This research develops additional methods for estimating measuring instrument precision in destructive testing utilizing an assumption that is accurate in multiple situations. In this method items from multiple part types are measured. A part type is a distinct population of items with different mean measurements from which samples can be obtained. Using measurement data from these part types, a functional relationship between mean and variance that is assumed to hold is used as a basis for estimating measuring instrument precision. It is also assumed that measuring instrument precision is constant when measuring different part types.

In this research the simplest destructive measurement gauge capability study is considered. Items from two different part types are measured by a single operator using a single measuring instrument. The assumed functional relationship between part type means and variances is a constant coefficient of variation (linear functional relationship). A constant coefficient of variation implies constant relative variability so that parts types with larger mean measurements will also have larger variability. Using the constant coefficient of variation assumption, an estimator for measuring instrument precision computed from measurement data is derived. The variance of this estimator is also derived, which can then be used to calculate a confidence interval for measuring instrument precision.

Research Outline

The research is outlined as follows.

- In chapter 2, the existing literature already done on measuring instrument precision estimation in non-repeatable testing is discussed. Different assumptions and methods used in destructive testing and parameter-dependent testing are discussed in this chapter.
- In Chapter 3, the proposed assumption and the developed methodology is discussed to find an estimator for measuring instrument precision.
- In chapter 4, using statistical inference, the estimator's properties are derived and evaluated using simulations. Monte Carlo simulation is used to validate the derived equations.
- In chapter 5, conclusions and future work are discussed.

2. LITERATURE REVIEW

This chapter provides a review of the existing literature on estimating measuring instrument precision in non-repeatable testing. This chapter is arranged as follows. First, the different types of non-repeatable testing are explained. In the subsequent sections, assumptions and methods used in different types of non-repeatable testing are discussed.

The fundamental problem with non-repeatable testing is a lack of repeated measurements on the same item type. Repeated measurements are not possible due to several reasons. Destructive testing and parameter-dependent testing are two situations where repeated measurements are not possible. In destructive testing, after the item is measured, it is unfit for repeated measurements due to the destruction of the item. For example, in tensile strength testing, the force required to break a specimen is studied, which leads to specimen destruction after the measurement. *Parameter-dependent testing* was defined to describe the situation where a parameter that defines an item changes between measurements. When the parameter changes, the mean value of the measurements also changes. For example, an automobile disc brake during its operation generates heat. The temperature measurement taken on this disc-brake varies with the parameter time. Similarly, the shrinkage of carpets varies with a carpet's position, and the stress test value on the carpet depends on the extent of its shrinkage. During the stress test, the carpet at the position of measurement gets damaged, repeated measurements are then taken on a different position on the carpet. Thus, in this case, the measurement value varies with the parameter position. A summary of different approaches to measurement instrument precision estimation and existing research in non-repeatable testing is shown in Figure 1.

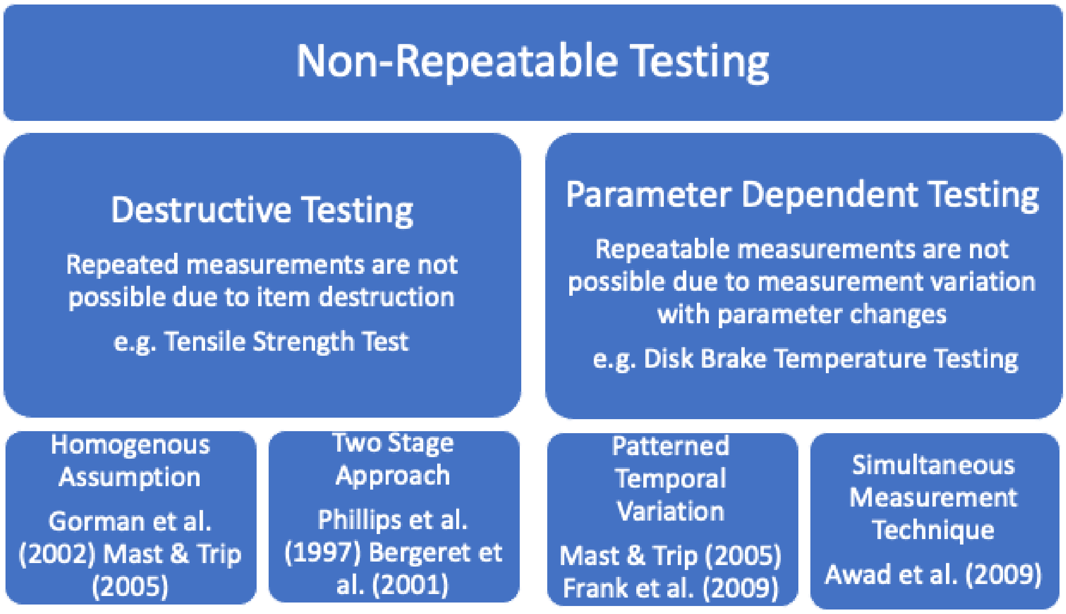


Figure 1: Classification of Non-Repeatable Testing

2.1. Destructive Testing

Gorman et al. (2002) estimate measurement instrument precision using a homogenous batch assumption. In the homogenous batch assumption, measurements are assumed to be taken on items sampled from a homogenous batch. Assuming items are homogenous, the measurements from different items are considered to have zero-item variance. Hence, assuming measurements from different items as replicates of the same item, measurement instrument precision can be estimated. For example, considering items cut from the same parent object as homogenous, measurements from different items are considered repeated measurements.

Mast and Trip (2005) provide a detailed discussion on the homogeneity assumption. Under this assumption, measurement instrument precision is estimated as the sample variance of the measurements. $\hat{\sigma}^2 = \frac{1}{k-1}\sum_{i=1}^{k}(Y(u_i) - \hat{u})^2$, where $u_1, u_2, u_3 \ldots . u_k$ are k samples from a homogenous batch and $\hat{u}$ is its sample mean. $Y(u_i)$ is the measured value for the item u_i, and $\hat{\sigma}^2$ is the measurement instrument precision estimate. The accuracy of the measurement instrument precision estimate using the homogeneity assumption depends on the extent of the measured items' homogeneity. But in reality, there are no completely homogenous items, and any variation from homogeneity results in overestimated measurement instrument precision.

When the homogeneity assumption does not hold, the presence of item variance σ_p^2 overestimates the measurement instrument precision as shown by the expected value of the measurement instrument precision estimate $E[\hat{\sigma}^2]$, where $E[\hat{\sigma}^2] = \sigma^2 + \sigma_p^2$. Thus, a high estimate of measurement instrument precision may be due to the contribution of item variance. Moreover, assuming zero-item variance and assigning the total variance to measurement instrument precision is essentially assuming the problem in destructive testing (separating measurement instrument precision from part variance) doesn't exist.

Mast and Trip (2005) also suggests a method to reduce the confounding of the item-to-item variance in homogenous assumption called the patterned object variation. Under this method, a function is fitted to the variation across items to make it more homogenous. The fitted function can be obtained from the historical data. For example, by fitting a linear function $\hat{f}(i) = \beta_0 + \beta_1 i$ to the variation between item i, the measurement instrument precision can be estimated as, $\hat{\sigma}^2 = \frac{1}{k-p}\sum_{i=1}^{k}\left(Y(u_i) - \hat{f}(i)\right)^2$. Where $\hat{\sigma}^2$ is the measurement instrument precision, k is the number of items, p is the number of parameters, and $Y(u_i)$ is the measured value.

Mast and Trip (2005) also suggests that when a homogenous batch of an item is not available, measurements can be taken on alternate items with similar properties. For example, in a destructive test where the pressure required to break an item is measured, and a homogenous item batch is not available. Measurements can be taken on an alternate homogenous batch of items that break at the same pressure as the item under study and consider as the repeated measurements. This approach is similar to the one used by Phillips et al. (1997) discussed later in this chapter. Suppose an item with a known true item value can be used in an experiment, measuring instrument precision can be estimated using this knowledge. For example, in destructive testing conducted by measuring plastic bars that break at a particular pressure, the measurement instrument precision can be estimated as, $\hat{\sigma}^2 = \frac{1}{k}\sum_{i=1}^{k}\left(Y(u_i) - T(u_i)\right)^2$, where $Y(u_i)$ is the measured value and $T(u_i)$ is the known true part value.

Phillips et al. (1997) suggest a different approach to separate measurement instrument precision from its item variance using a two-stage method. In the first stage, the item to be measured is replaced with an alternate item with negligible variance. Since the alternate item's variance is very close to zero in this stage, the measurement variability obtained is purely measurement instrument precision. In the next phase, the study is conducted using the item of interest. Using the measurement instrument precision value already estimated in phase one, the part variance is separated in phase two. For example, in destructive testing of fiberglass shingles, vinyl floor covering is used in the first phase. Vinyl floor covering has a low sample to sample variability. In the second phase, the test is conducted using the material of interest- fiberglass shingles. The measurement instrument precision estimated in the first phase is used in the second phase to separate the fiberglass shingles part variability. In the above example, $\hat{\sigma}_\beta^2$ is the measurement instrument precision obtained from the first phase using the vinyl floor covering and $\hat{\sigma}_\alpha^2$ is the measured variance in the second phase. The second phase variance includes measurement instrument precision $\hat{\sigma}^2$ and the fiberglass shingles variance $\hat{\sigma}_p^2$.

$$\hat{\sigma}^2 = \hat{\sigma}_\beta^2$$

$$\hat{\sigma}_\alpha^2 = \hat{\sigma}_p^2 + \hat{\sigma}^2$$

This method's main drawback is the difficulty in obtaining an alternate part that can be measured using the measuring instrument and with negligible part variance.

Bergeret et al. (2001) conduct a gauge capability study for destructive testing in which the item is not entirely destroyed, but measurements are possible on different locations on the item. This method involves a two-stage method. In the first stage, measurement instrument precision is estimated using a two factor nested design, in which locations are nested within items. An operator takes measurements at different locations on an item. In this stage, the measurement instrument precision is confounded with the locations on the item. From this stage, the independent item variance is obtained. In stage two, a two-factor nested design is used with items nested within operators. Operators measure one fixed location in different parts. In this case, the measurement instrument precision is confounded with part variance. Measurement instrument precision can then be estimated since the part variance was estimated in stage one. The results using this method show overestimation of measurement instrument precision. This method's application is also limited to items where measurements at different locations on the same item are possible.

Another variation of the two-stage method discussed above is to have two samples from the same batch (Mast and Trip, 2005). In this method, one sample is measured under a "perfect" destructive measurement system, and the other sample on the destructive measurement system under study. A perfect destructive measurement system is defined as a piece of an expensive lab equipment that has no measurement instrument precision error. The measurement given by this measuring instrument is assumed to be part variance. This part variance value from the perfect destructive measurement system is used when the second sample is tested using the destructive testing of interest. Again, this method's main constraint is the inaccessibility to the expensive, perfect destructive measurement system.

Finally, in some cases, destructive testing is preferred due to the high cost or immobility of non-destructive measurement system. In such cases, Mast and Trip (2005) propose testing the item first on non-destructive testing and then proceeding with the destructive testing. From the non-destructive testing, the part variance can be calculated, which can be used to separate measurement instrument precision from its confounded part variance during destructive testing. For example, to measure phosphor layers' thickness on displays, destructive and non-destructive testing are available. To obtain the measurement instrument precision in destructive testing, the test is first conducted using alternative non-destructive testing. The alternate non-destructive testing involves passing a ray of light through the object and analyzing the amount of light reaching the other end. This is a highly efficient digital radiography system but is immobile. Once the part variance from this method is obtained, measurement instrument precision in destructive testing can be estimated, which involves weighing a scrapped portion of the phosphor layered display. But this method may not be practical as access to non-destructive and destructive testing at the same time is very rare.

Kappele and Raffaldi (2010) suggest two ways to conduct gauge repeatability and reproducibility study in destructive testing. One way is to use alternate non-destructive testing that correlates and gives the same measurement result as the destructive testing. Another one by conducting the destructive study with substitute parts having identical measurements. Replacing the destructive test with the non-destructive test and conducting the measuring instrument precision estimation study doesn't make sense as we are interested in finding the measuring instrument precision of the destructive testing. Also, it fails to give more details on how the correlation between the two tests can be used to estimate measuring instrument precision.

Han and He (2007), Eva (2018), Sharma et al. (2019), and Gorman and Bower (2002) use the homogeneity assumption to estimate measuring instrument precision. The dissolution test conducted by Gao et al. (2007) is also a destructive test that uses tablets from a homogenous batch to analyze the amount of dissolution of tablets.

2.2. Parameter Dependent Testing

Mast and Trip (2005) suggests using a patterned temporal variation method to reduce the variation in measurements of an item whose measurement value varies with time. In the patterned temporal variation method, the variation in measurement over time of an object is represented by a function. Using this knowledge, the item measurement's linear statistical model is updated to form a fitted model, which reduces the effect of the item variance in measuring instrument precision estimations. For example, the measurement of pressure in a pipeline is different at different points in time. Suppose the fluctuation in the measured pressure value over time can be modeled. In that case, this knowledge can be used to reduce the measurement variations and reduce overestimation in measuring instrument precision estimation.

The patterned object variation and patterned temporal variation methods are further analyzed by Frank et al. (2009). It is shown that these methods can reduce the measuring instrument precision overestimation that happens when the homogenous assumption fails. That is when there is significant item-to-item variation. These methods fit a linear or non-linear model for the item-to-item variation or the variation in measurement taken during different points of time and correct these variations. Thus, any variation that gets added to the measuring instrument precision estimation can be reduced in this way. For example, in the temperature measurement of a number of food items, which is being cooled, the measurement value varies over different time points. A linear statistical model for n repeated measurements taken on a food item by each of the b operators at different points of time can be modeled as,

$$y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1,2, \dots, n \end{cases}$$

Where y_{ijk} is the measurement on food item i by the $j'th$ operator at time k, μ is the overall mean of the measured items, α_i is the random item effect, β_j is the random operator effect, and ε_{ijk} is the measurement error. The repeated measurements have high variations since the food item's temperature drops with each point of time. Assuming that the temperature measurement decreases linearly in time, the measurement value is corrected by updating the above model to $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma(t_k - 150) + \varepsilon_{ijk}$. The additional term $\gamma(t_k - 150)$ is the correction term to reduce the variation in measurement value with time, where γ is a fixed effect and t_k is the time during which measurements are taken. This method's drawback is that the measurement variation's function is derived from historical data, which may not be accurate. Thus, this method is used to reduce overestimations and does not provide accurate measurement instrument precision estimations.

Awad et al. (2009) propose a method using the assumption that measurements can be taken by two or more measuring instruments simultaneously during the experiment. The measuring instrument precision is estimated considering these simultaneous measurements as the repeated measurements. For example, two infrared guns are simultaneously used to measure a disc brake's temperature at each time t, and these measurements are considered as the repeated measurements. The drawback of this method is that more than one measuring instrument is used in this method. A single measurement instrument precision estimate has no meaning in this method.

Finally, Hamada and Borror (2012) conduct gauge repeatability and reproducibility study in non-repeatable testing when the replication is impractical. The replication is impractical as the item being tested was expensive that needs to be shipped out, leaving no time to conduct repeated tests or when the replicated measurements by the same operator are not taken if the operator can identify if it's the same part being measured. Using a Bayesian approach, the above scenario is analyzed to find the total measurement variability and showcase why without repeated measurements measuring instrument precision estimate can't be separated from other variance components

In this chapter, different assumptions and methods to estimate measuring instrument precision in non-repeatable testing are discussed. Mikulova et al. (2020) also provide an overview of the different approaches used in non-repeatable testing. But many of these assumptions have drawbacks and need further study. Therefore, this research proposes a new assumption and method that will add to the existing literature to estimate measuring instrument precision in destructive testing.

3. METHODOLOGY

This thesis addresses the research question- "*How can measuring instrument precision be estimated when the measurements are destructive?".* This chapter documents the development of a new methodology to estimate measurement instrument precision in destructive testing. The chapter explains why measurement instrument precision estimation is not possible in destructive testing. Then the need for an assumption to estimate measuring instrument precision in destructive testing is discussed. Finally, a reasonable assumption is proposed that enables estimation of measuring instrument precision in destructive testing.

3.1. Measurement Instrument Precision Estimation in Destructive Testing

In destructive testing, the item measured is destroyed after a measurement is taken. Therefore, repeated measurements are not possible in this type of testing. Example 1 demonstrates why measurement instrument precision estimation is not possible without repeated measurements.

Example – 1

Consider a destructive testing in which an operator takes a single measurement on a items with a single measuring instrument. The linear statistical model for the study is,

$$y_i = \mu + \alpha_i + \epsilon_i \tag{1}$$

Where y_i is the measurement on the item i, μ is the overall mean, and α_i is the deviations from this overall mean. In this model, α_i is the random variable for deviations of individual sample items and is assumed to be $Normal(0, \sigma_\alpha^2\ )$. ϵ_{ij} is the measurement error and is assumed to be $Normal\ (0, \sigma^2)$. The variance in the measurement error σ^2 is the measuring instrument precision. The linear statistical model determines the structure of the ANOVA. In the ANOVA, the expected value of the mean square is used to obtain estimators for the variance components $\sigma_\alpha^2\ and\ \sigma^2$. The expected mean square values for the model in equation (1) are shown in Table 1.

Table 1: Expected Mean Square ANOVA Table

Source of variability	**Sum of square (SS)**	**Degrees of freedom**	**Mean Square (MS)**	**Expected Mean Square E(MS)**
Parts	SS_P	$a-1$	$MS_p = \frac{SS_p}{a-1}$	$\sigma^2 + \sigma_\alpha^2$
Measurement Error	$SS_R = 0$	-	MS_R *cannot be estimated*	σ^2

Since there are no repeated measurements, there is no "Error" term estimate, which is an estimate of σ^2. The Table 1 results show that the measuring instrument precision, σ^2 is not independent and can't be separated from other variance components in the model. To sum up, in the above destructive testing example, the measuring instrument precision is confounded with the part variance. Therefore, to estimate measurement instrument precision in destructive testing some assumption needs to be made. The common assumption is the homogenous batch assumption. Under this assumption, measurements are assumed to be taken on a batch containing similar items. Assuming zero item variance, the measurements from different items are used as the repeated measurements in the study. However, the homogeneity assumption is very far from reality. Assuming zero item variance and conducting the study is like assuming the problem (separate measuring instrument precision value from item variance) away. Other assumptions mentioned in the literature review, like the two-stage method or conducting the measurement using a perfect destructive measurement instrument, are applicable only in special cases. Therefore, there is a need to develop a new methodology using a reliable assumption that is accurate and applicable in multiple situations to estimate measurement instrument precision in destructive testing.

3.2. Proposed Assumption to Estimate Measurement Instrument Precision in Destructive Testing

In this research, the method developed to estimate measurement instrument precision in destructive testing utilizes items from more than one part type. A part type is a distinct population of items with different mean measurements from which samples can be obtained. Additionally, it is assumed that there is a functional relationship between the mean and variance of different part types. Some possible functional relationships are linear (proportional), inverse, and quadratic (Figure 1).

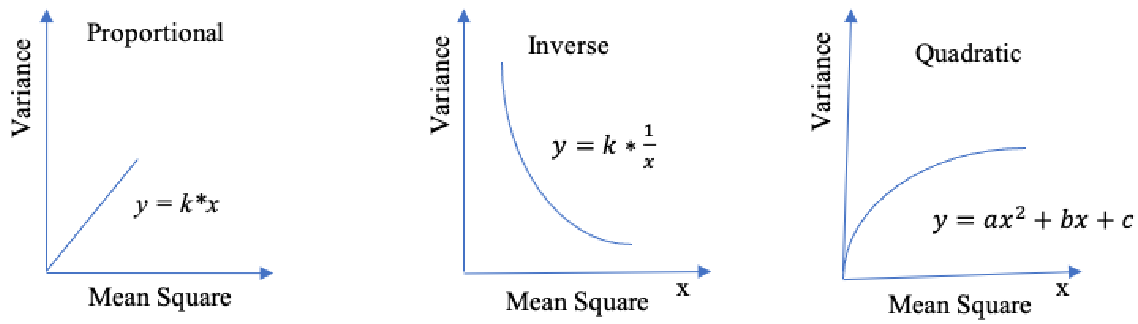


Figure 2: Functional Relationships

In this research, it is assumed that there is a linear relationship between the square of the part type means and the variances. A linear functional relationship between part type means and variances implies a constant coefficient of variation. The coefficient of variation, CV is a measure of relative variability and for a random variable is defined as the ratio of the standard deviation to the mean.

$$CV = \frac{\sigma}{\mu}$$

Here σ is the standard deviation, and μ is the mean of the random variable. Generally, when the mean of the part type increases the part type standard deviation also increases. Hence in the multiple part type system, to compare the variability of different part types, the coefficient of variation can be used. As a real example, consider the standard screw machine stock tolerance allowances from a commercial supplier of supplier of steel and aluminum bar, tubing and plate (EMJ Metals 2021) shown in Table 2. Assuming the standard screw machine stock tolerance is some constant multiple (k) of the standard deviation, the coefficients of variation (multiplied by k) for the various diameters of screw machine stock are calculated in Table 2.

Table 2: Standard Screw Machine Stock Tolerances Allowances

Diameter (Inches) (A)	**Tolerance (Inches) (B)**	***k**Coefficient of variation (B/A)**
0.25	±0.0015	0.006
0.5	±0.0020	0.004
1	±0.0025	0.0025
1.5	±0.0040	0.026
2	±0.0060	0.003
3	±0.0080	0.026

From Table 2 it can be seen that the parts with larger means have larger tolerance values. The relation between diameter and tolerance value of the standard screw machine stock is given in Figure 3.

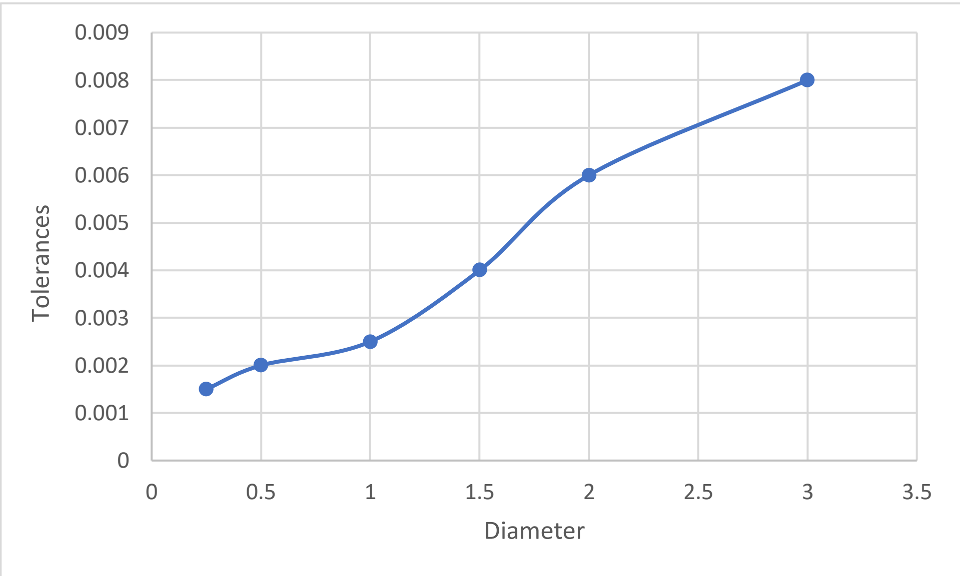


Figure 3: Standard Screw Machine Stock Tolerances Allowances Graph

From Figure 3 it can be seen that the standard screw machine stock tolerance allowance is a function of the stock diameter. Assuming the tolerance value as the *k**standard deviation, larger diameter machine stock has a higher standard deviation than smaller diameter stocks. Also, the stock diameter with a diameter of 1.5 inches and 3 inches has a constant coefficient of variation.

Here a constant coefficient of variation can be considered a reasonable assumption when multiple part types are used in a gauge capability study.

To model a linear relationship between the variance and squared means, data from two-part types is required. Hence, this research will utilize measurements from two-part types to estimate measuring instrument precision, assuming a constant coefficient of variation of part type measurements. Under the constant coefficient of variation assumption,

$$\frac{\sigma_1}{\mu_1} = \frac{\sigma_2}{\mu_2} = CV$$

Where μ_1 is the part type one mean with variance σ_1^2, and μ_2 is the part type two mean with variance σ_2^2. It is also assumed that the measuring instrument precision is constant over different part types. Using this assumption, the mathematics of the methodology to estimate measurement instrument precision in destructive testing is developed in the next chapter.

4. RESULTS

In this chapter, using the proposed part type constant coefficient of variation assumption, the mathematics for the methodology is developed. An estimator for measuring instrument precision is derived. Additionally, the estimator's properties- the expected mean, and the theoretical variance are derived so that a confidence interval for the measuring instrument precision estimate can be calculated.

The outline of the chapter is as follows. In section 4.1, the estimator for the measuring instrument precision is derived. In section 4.2, the estimator's properties- the expected mean, and the theoretical variance are derived. In section 4.3, the estimator's derived theoretical variance and the confidence interval coverage is validated using Monte Carlo simulation.

4.1. Estimating Measuring Instrument Precision

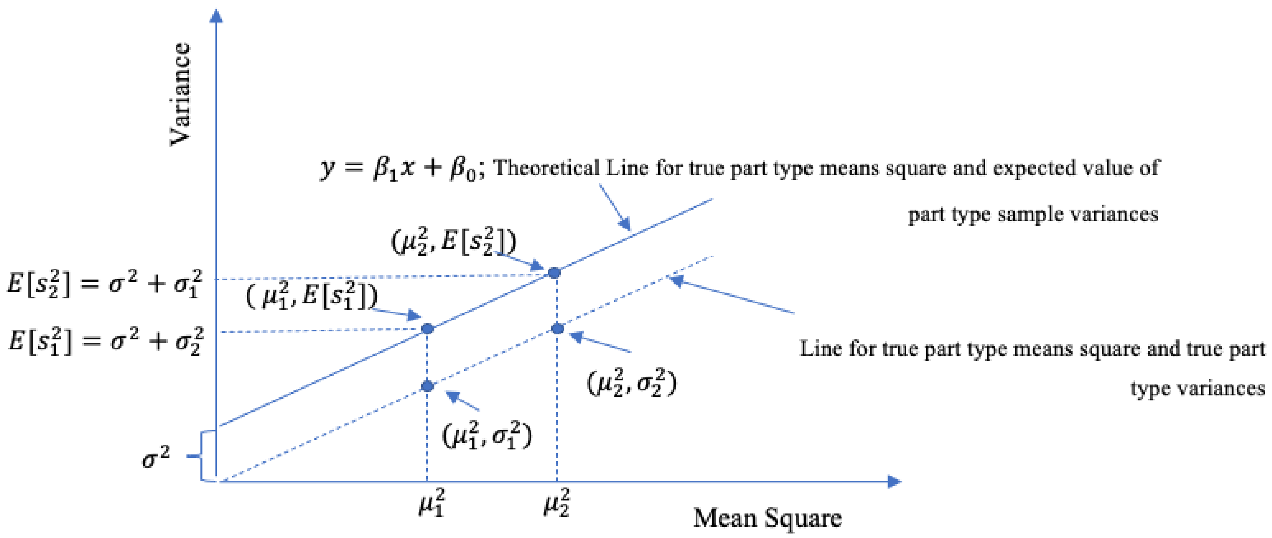
The developed methodology involves taking measurements on two-part types (part type one and part type two) by a single operator using a single measuring instrument (or a single automated measuring instrument). Let's assume that the true part type means are known. Once n samples from each part type are measured, the square of the true part type means, and the part type sample variances constitute the input data for estimating measuring instrument precision and a confidence interval for this estimate. The assumed model for measurements of part type i is

$$y_{ij} = \mu_i + \alpha_{ij} + \epsilon_{ij} \begin{cases} i = 1, 2 \\ j = 1, 2, \ldots, n \end{cases}$$

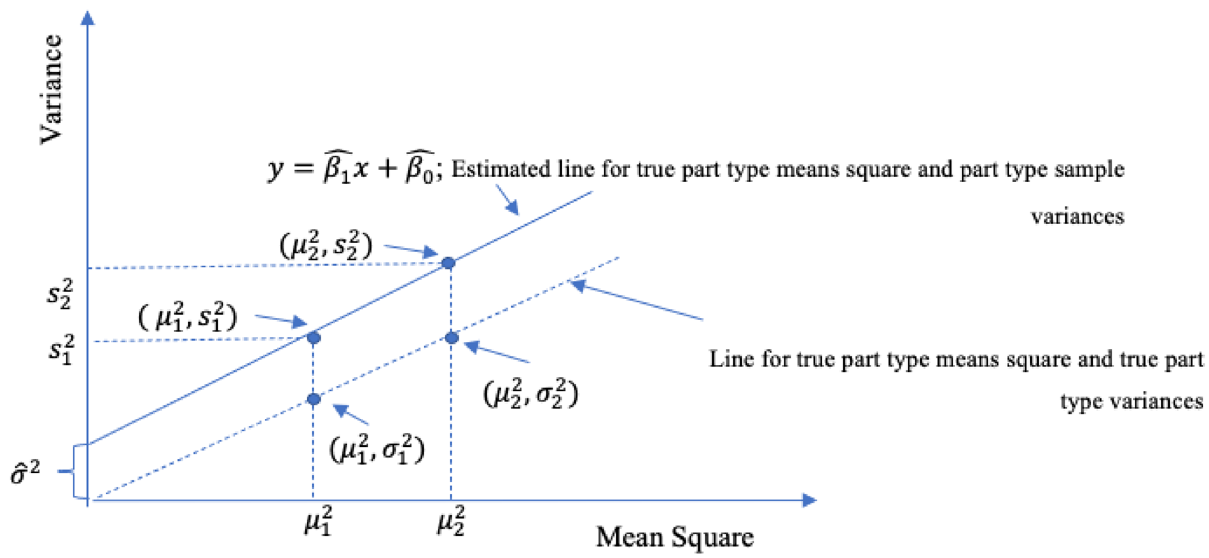
Where, y_{ij} is the measurement on the $j'th$ item of part type i and μ_i is the part type i mean measurement (a constant). α_{ij} is the random variable for deviations of individual items within a part type and is assumed to be $Normal(0, \sigma_\alpha^2)$. ϵ_{ij} is the measurement error and is assumed to be $Normal\ (0, \sigma^2)$. The variance in the measurement error σ^2 is the measuring instrument precision.

Before presenting the mathematical derivation of the measuring instrument precision estimate, a graphical presentation of the data will be presented. Let the square of part type i

mean, and the part type *i* sample variance constitutes an "*x*-*y*" pair. If the part type one and two "*x*-*y*" pairs are plotted, a line can be fit connecting the two points. This is shown in Figure 4. The means squares are on the "*x*-axis" and the sample part type variances are on the "y-axis".



(a)



(b)

Figure 4: Estimator for Measurement Instrument Precision

In Figure 4 (a) and (b), μ_1 and μ_2 are the true part type means and σ_1^2 and σ_2^2 are the true part type variances. When n measurements are taken from each of these two-part types, sample part type variances s_1^2 and s_2^2 are obtained. Assuming part type means are known, the true part type means square are plotted on the x-axis and the part type sample variances on the y-axis. A fitted line by the equation $y = \widehat{\beta_1}x + \widehat{\beta_0}$ with, $\widehat{\beta_1}$ as the slope and $\widehat{\beta_0}$ as the intercept is drawn connecting the plotted points of true part type mean square and part type sample variance as shown in Figure 4 (b). The line $y = \beta_1 x + \beta_0$ in Figure 4 (a) is the theoretical line drawn through the true part type means square and expected value of part type sample variances. The dotted lines are the conceptual lines through the points plotted by true part type means square and true part type variances, which passes through the origin. The sample part type variance s_1^2 and s_2^2 has two variance components, part type variance and measurement instrument precision. That is, $E[s_1^2] = \sigma^2 + \sigma_1^2$ and $E[s_2^2] = \sigma^2 + \sigma_2^2$. In the figures, when the square of the true part type mean is reduced, the part type sample variance also reduces. When the square of the true part type mean approaches zero, the true part type variance also approaches zero. The variance left behind, in this case, will be the measurement instrument precision and is given by the intercept of the line. Hence, it can be shown that the intercept of the line $\widehat{\beta_0}$ drawn through the plotted points is an estimator for the measurement instrument precision σ^2. The slope of the line β_1 is square of coefficient of variation.

4.2. Measurement Instrument Precision Estimator Properties

In this section, the estimator's properties, the expected value and the theoretical variance are derived, so that a confidence interval for the estimator can be generated. The estimator's properties are derived for two scenarios. In scenario one, the part type means are assumed known, and in scenario two, the part type means are estimated from the sample part type means. The notation in Table 3 will be used throughout the derivations.

Table 3: Notation Summary

Notation	Description
μ_1	Part type 1 mean
μ_2	Part type 2 mean
σ_1^2	Part type 1 variance
σ_2^2	Part type 2 variance
σ^2	Measuring instrument precision
$\overline{x_1}^2$	Part type 1 sample mean
$\overline{x_2}^2$	Part type 2 sample mean
s_1^2	Part type 1 sample variance
s_2^2	Part type 2 sample variance
n	No of samples
β_0	Intercept

4.2.1. Scenario 1: True Part-Type Means Known

Assuming the true part type means are known, consider two-part types- part type one with squared mean μ_1^2 and variance σ_1^2, and part type two with squared mean μ_2^2 and variance σ_2^2. A sample of n items are measured from each the part type to obtain part type one sample variance s_1^2 and part type two sample variance s_2^2. The constant measurement instrument precision is σ^2. The equation of a line through the points (μ_1^2, s_1^2) and (μ_2^2, s_2^2) is,

$$y = \frac{s_2^2 - s_1^2}{\mu_2^2 - \mu_1^2} x + \widehat{\beta_0}$$

Where, $\widehat{\beta_0}$ is the intercept of the line and $\frac{s_2^2 - s_1^2}{\mu_2^2 - \mu_1^2}$ is the slope of the line. Rearranging the above equation, a formula for the line's intercept is derived.

$$\widehat{\beta_0} = s_1^2 - \frac{s_2^2 - s_1^2}{\mu_2^2 - \mu_1^2} \mu_1^2$$

$$= \frac{s_1^2\mu_2^2 - s_1^2\mu_1^2 - s_2^2\mu_1^2 + s_1^2\mu_1^2}{\mu_2^2 - \mu_1^2}$$

$$= \frac{s_1^2\mu_2^2 - s_2^2\mu_1^2}{\mu_2^2 - \mu_1^2}$$

Thus, the formula for the line's intercept is

$$\widehat{\beta_0} = \frac{s_1^2\mu_2^2 - s_2^2\mu_1^2}{\mu_2^2 - \mu_1^2} \tag{2}$$

The above equation is the formula for the intercept estimate for a two-part type study with a constant coefficient of variation and constant measuring instrument precision. The estimator's expected mean and theoretical variance are derived next.

4.2.2. Scenario 1: Expected Value of the Estimator

The estimator's expected value is derived for scenario one.

$$\widehat{\beta_0} = \frac{s_1^2\mu_2^2 - s_2^2\mu_1^2}{\mu_2^2 - \mu_1^2}$$

The expected value of the estimate is,

$$E[\widehat{\beta_0}] = \frac{E[s_1^2]\mu_2^2 - E[s_2^2]\mu_1^2}{\mu_2^2 - \mu_1^2}$$

The expected value of the part type sample variances is,

$$E[s_1^2] = (\sigma_1^2 + \sigma^2)$$
$$E[s_2^2] = (\sigma_2^2 + \sigma^2)$$

Substituting the expected value of the part type sample variance to the expected value of the estimate equation,

$$E[\widehat{\beta_0}] = \frac{\mu_2^2 * (\sigma_1^2 + \sigma^2) - \mu_1^2 * (\sigma_2^2 + \sigma^2)}{\mu_2^2 - \mu_1^2}$$

$$= \frac{\mu_2^2\sigma_1^2 + \mu_2^2\sigma^2 - \mu_1^2\sigma_2^2 - \mu_1^2\sigma^2}{\mu_2^2 - \mu_1^2}$$
$$= \frac{\mu_2^2\sigma_1^2 - \mu_1^2\sigma_2^2 + (\mu_2^2 - \mu_1^2)\sigma^2}{\mu_2^2 - \mu_1^2}$$

From the constant coefficient of variation of part types assumption,

$$CV = \frac{\sigma_1}{\mu_1} = \frac{\sigma_2}{\mu_2}$$

Squaring the above equation,

$$CV^2 = \frac{\sigma_1^2}{\mu_1^2} = \frac{\sigma_2^2}{\mu_2^2}$$
$$CV^2 = \sigma_1^2\mu_2^2 = \sigma_2^2\mu_1^2$$

Substituting the above equation into the expected value of the estimate equation gives,

$$E[\widehat{\beta_0}] = \frac{\sigma^2(\mu_2^2 - \mu_1^2)}{\mu_2^2 - \mu_1^2}$$
$$= \sigma^2$$

The expected value of the estimator is measurement instrument precision. Hence, for scenario one, intercept of the line is an unbiased estimator for measurement instrument precision.

4.2.3. Scenario 1: Variance of the Estimator

The variance of the measurement instrument precision estimator is derived next. The distribution of the part type sample variance when computed from normal observations is utilized in the derivation.

Consider a random variable $X \sim N(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance. If s^2 is the sample variance computed from n items from this random variable then,

$$\frac{(n-1) * s^2}{\sigma^2} \sim \chi^2_{n-1}$$

Where χ^2_{n-1} is a random variable with a chi-square distribution having $n-1$ degrees of freedom. The above relationship implies that the sample variance s^2 follows a chi-square distribution multiplied by a constant.

$$s^2 \sim \frac{\chi^2_{n-1}\,\sigma^2}{n-1}$$

For a χ^2_{n-1} random variable, the expected value and the variance are $n-1$ and $2(n-1)$ respectively. Thus if, $X_i \sim N\left(\mu_i, (\sigma_i^2 + \sigma^2)\right)$ and sample variance, s_i^2 is computed from *n* observations then,

$$Var(s_i^2) = \frac{2(n-1)(\sigma^2 + \sigma_i^2)^2}{(n-1)^2}$$

Therefore, the variance of the sample variance is,

$$Var\left(s_i^2\right) = \frac{2(\sigma^2 + \sigma_i^2)^2}{(n-1)} \qquad (3)$$

The variance of the estimator is derived below. The estimator is,

$$\widehat{\beta_0} = \frac{s_1^2\mu_2^2 - s_2^2\mu_1^2}{\mu_2^2 - \mu_1^2}$$

$$Var\left[\widehat{\beta_0}\right] = Var\left(\frac{s_1^2\mu_2^2 - s_2^2\mu_1^2}{\mu_2^2 - \mu_1^2}\right)$$

Due to the independence of the part types,

$$Var\left[\widehat{\beta_0}\right] = \frac{var(s_1^2) * \mu_2^4 + var(s_2^2) * \mu_1^4}{(\mu_2^2 - \mu_1^2)^2}$$

Substituting equation (3) in the above variance equation,

$$Var[\widehat{\beta_0}] = \frac{\mu_2^4\left(\frac{2(\sigma^2+\sigma_1^2)^2}{(n-1)}\right)+\mu_1^4\left(\frac{2(\sigma^2+\sigma_2^2)^2}{(n-1)}\right)}{(\mu_2^2-\ \mu_1^2)^2}$$

$$= \frac{2}{(n-1)}\left(\frac{(\sigma^2+\sigma_1^2)^2*\mu_2^4+(\sigma^2+\sigma_2^2)^2*\mu_1^4}{(\mu_2^2-\ \mu_1^2)^2}\right)$$

Therefore, the derived equation for the variance of the estimator is given below.

$$Var[\widehat{\beta_0}] = \frac{2}{(n-1)}\left(\frac{(\sigma^2+\sigma_1^2)^2*\mu_2^4+(\sigma^2+\sigma_2^2)^2*\mu_1^4}{(\mu_2^2-\ \mu_1^2)^2}\right) \tag{4}$$

The above equation is the theoretical variance for the measurement instrument precision estimator. The theoretical variance of the estimator needs to be estimated from part type sample variances. Hence, the expected value of the intercept variance estimator will be discussed in the next section.

4.2.4. Scenario 1: Expected Value of the Intercept Variance Estimator

The variance derived for the measurement instrument precision estimator in section 4.2.3 is estimated from part type sample variances, since the part type variances and measuring instrument precision are not known. In this section, the expected value of the estimated intercept variance is derived.

Let θ be the theoretical variance of the measurement instrument precision. The theoretical variance equation is,

$$Var[\widehat{\beta_0}] = \theta\ = \frac{2}{(n-1)}\left(\frac{(\sigma^2+\sigma_1^2)^2*\mu_2^4+(\sigma^2+\sigma_2^2)^2*\mu_1^4}{(\mu_2^2-\ \mu_1^2)^2}\right)$$

In the theoretical variance equation above, the measurement instrument precision σ^2 and part type variances, σ_1^2 and σ_2^2 are estimated from sample part type variances, s_1^2 and s_2^2. Replacing sample part type variances s_1^2 and s_2^2 in the place of $\sigma^2 + \sigma_1^2$ and $\sigma^2 + \sigma_2^2$ respectively gives

$$\widehat{Var\left[\widehat{\beta}_0\right]} = \hat{\theta} = \frac{2}{(n-1)}\left(\frac{\left(s_1^2\right)^2 * \mu_2^4 + \left(s_2^2\right)^2 * \mu_1^4}{\left(\mu_2^2 - \mu_1^2\right)^2}\right)$$

The expected value of this estimator is,

$$E\left[\hat{\theta}\right] = \frac{2}{(n-1)}\left(\frac{E[(s_1^2)^2] * \mu_2^4 + E[(s_2^2)^2] * \mu_1^4}{(\mu_2^2 - \mu_1^2)^2}\right)$$

The expected value of the part type sample variance squares, $E[(s_1^2)^2]$ and $E[(s_2^2)^2]$ is derived next.

$$Var(s_1^2) = E[(s_1^2)^2] - E[s_1^2]^2$$
$$E[(s_1^2)^2] = Var(s_1^2) + E[s_1^2]^2$$
$$E[(s_1^2)^2] = \frac{2(\sigma^2 + \sigma_1^2)^2}{(n-1)} + (\sigma^2 + \sigma_1^2)^2$$
$$E[(s_1^2)^2] = \frac{(n+1)(\sigma^2 + \sigma_1^2)^2}{(n-1)}$$

Similarly,

$$Var(s_2^2) = E[(s_2^2)^2] - E[s_2^2]^2$$
$$E[(s_2^2)^2] = Var(s_2^2) + E[s_2^2]^2$$
$$E[(s_2^2)^2] = \frac{2(\sigma^2 + \sigma_2^2)^2}{(n-1)} + (\sigma^2 + \sigma_2^2)^2$$
$$E[(s_2^2)^2] = \frac{(n+1)(\sigma^2 + \sigma_2^2)^2}{(n-1)}$$

Substituting the above derived expected value of the part type squared sample variances into the expected value of the estimated variance equation gives,

$$E[\hat{\theta}] = \frac{2}{(n-1)}\left(\frac{\left(\frac{(n+1)(\sigma^2+\sigma_1^2)^2}{(n-1)}\right)\mu_2^4 + \left(\frac{(n+1)(\sigma^2+\sigma_2^2)^2}{(n-1)}\right)\mu_1^4}{(\mu_2^2 - \mu_1^2)^2}\right)$$

$$E[\hat{\theta}] = \frac{2(n+1)}{(n-1)^2}\left(\frac{(\sigma^2+\sigma_1^2)^2\mu_2^4 + (\sigma^2+\sigma_2^2)^2\mu_1^4}{(\mu_2^2 - \mu_1^2)^2}\right)$$

Since there is a difference between $E[\hat{\theta}]$ and θ, there is bias in the estimated variance of the measurement instrument precision. The bias, $Bias[\hat{\theta}]$ is,

$$Bias[\hat{\theta}] = E[\hat{\theta}] - \theta$$

$$Bias[\hat{\theta}] = \frac{2(n+1)}{(n-1)^2}\left(\frac{(\sigma^2+\sigma_1^2)^2\mu_2^4 + (\sigma^2+\sigma_1^2)^2\mu_1^4}{(\mu_2^2 - \mu_1^2)^2}\right) - \frac{2}{(n-1)}\left(\frac{(\sigma^2+\sigma_1^2)^2 * \mu_2^4 + (\sigma^2+\sigma_2^2)^2 * \mu_1^4}{(\mu_2^2 - \mu_1^2)^2}\right)$$

$$\neq 0$$

Therefore, there is a bias associated with the estimated variance equation, and the bias correction factor is $\frac{(n-1)}{(n+1)}$.

$$E[\hat{\theta}] * \frac{(n-1)}{(n+1)} = \theta$$

Thus, in scenario one it has been shown that the intercept of the line through the square of the true part type means and sample part type variances is an estimator for the measurement instrument precision. It has been shown that this intercept is an unbiased estimator of measurement instrument precision, and the theoretical variance of the estimator has been derived. When the variance of the measurement instrument precision estimator is estimated, a bias correction factor has been derived that can be used to correct the bias associated with the variance estimate.

4.2.5. Scenario 2: True Part-Type Means Unknown

In this section, the instrument precision estimator properties, the expected value and theoretical variance are derived considering part type means are estimated from sample measurements.

4.2.6. Scenario 2: Expected Value of the Estimator

The expected value of the estimator for scenario two is derived below from the line's intercept equation (2),

$$\hat{\beta}_0 = \frac{\bar{x}_2^2 s_1^2 - \bar{x}_1^2 s_2^2}{\bar{x}_2^2 - \bar{x}_1^2}$$

Where $\bar{x}_i^2$ is the squared sample part type mean and s_i^2 is the sample part type variance of item type i measurements. Rearranging the above equation,

$$\hat{\beta}_0 = \frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2 - \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2$$

The expected value of the above estimate is,

$$E[\hat{\beta}_0] = E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2 - \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2\right]$$

From the independence of part type means and variances for normal observations, the above equation is modified to,

$$E[\hat{\beta}_0] = E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] E[s_1^2] - E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] E[s_2^2] \tag{5}$$

The terms, $\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ and $\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ in equation (5) are ratios of two correlated random variables. Each random variable is a non-central chi-square random variable. To make the derivations more straightforward, a partial fraction expansion is used to break up this ratio. In the partial fraction approach the ratio of two non-central chi-square random variables is converted to a ratio of correlated normally distributed random variables. The partial fraction expansion equation for $\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ and $\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ is given below.

$$\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} = 1 - \frac{1}{2}\left[\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}\right] + \frac{1}{2}\left[\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right]$$

$$\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} = -1 + \frac{1}{2}\left[\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}\right] + \frac{1}{2}\left[\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right]$$

The expected value of the above partial fraction expansion equation is,

$$E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] = 1 - \frac{1}{2}E\left[\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}\right] + \frac{1}{2}E\left[\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right] \tag{6}$$

$$E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] = -1 + \frac{1}{2}E\left[\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}\right] + \frac{1}{2}E\left[\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right] \tag{7}$$

The expected value of $\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ and $\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ is approximated using second-order Taylor series approximation. The second-order Taylor series approximation derivations are explained next.

The second-order Taylor series approximation equation for the expected value of correlated random variables X and Y is given below. This equation will be used to approximate equations (6) and (7).

$$E\left[\frac{X}{Y}\right] \approx \frac{E[X]}{E[Y]} - \frac{Cov(X,Y)}{(E[Y])^2} + \frac{Var(Y)E[X]}{(E[Y])^3} \tag{8}$$

Four covariance combination need to be obtained to apply the Taylor series approximation equation in equations (6) and (7). In the partial fraction expansion equations, there are four random variables. The distribution of these random variables is normal with mean x_i and variance σ_i^2 of part type i measurements with n the number of parts measured. The distribution of these random variables is given below.

$$\bar{x}_1 \sim N(\mu_1, (\sigma_1^2 + \sigma^2)/n)$$
$$\bar{x}_2 \sim N(\mu_2, (\sigma_2^2 + \sigma^2)/n)$$

Due to the independence of parts and the part types, we can define the distribution of other combinations of sample means as shown below.

$$\bar{x}_1 + \bar{x}_2 \sim N(\mu_1 + \mu_2, (\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)$$
$$\bar{x}_2 - \bar{x}_1 \sim N(\mu_2 - \mu_1, (\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)$$

Now that the distributions are defined, we can derive the covariance combination as shown below. These results are used in the Taylor series approximation to derive the approximate expected value.

$$Cov(\bar{x}_1, \bar{x}_1 + \bar{x}_2) = E[\bar{x}_1 * (\bar{x}_1 + \bar{x}_2)] - E[\bar{x}_1] * E[\bar{x}_1 + \bar{x}_2]$$
$$= E[\bar{x}_1^2] + E[\bar{x}_1\bar{x}_2] - E[\bar{x}_1]^2 - E[\bar{x}_1] * E[\bar{x}_2]$$
$$= E[\bar{x}_1^2] - E[\bar{x}_1]^2$$
$$= Var(\bar{x}_1^2) + E[\bar{x}_1]^2 - E[\bar{x}_1]^2$$
$$= Var(\bar{x}_1) + E[\bar{x}_1]^2 - E[\bar{x}_1]^2$$
$$= Var(\bar{x}_1)$$
$$= (\sigma_1^2 + \sigma^2)/n$$

$$Cov(\bar{x}_1, \bar{x}_2 - \bar{x}_1) = E[\bar{x}_1 * (\bar{x}_2 - \bar{x}_1)] - E[\bar{x}_1] * E[\bar{x}_2 - \bar{x}_1]$$
$$= -E[\bar{x}_1^2] + E[\bar{x}_1\bar{x}_2] + E[\bar{x}_1]^2 - E[\bar{x}_1] * E[\bar{x}_2]$$
$$= -E[\bar{x}_1^2] + E[\bar{x}_1]^2$$
$$= -Var(\bar{x}_1) - E[\bar{x}_1]^2 + E[\bar{x}_1]^2$$
$$= -Var(\bar{x}_1)$$
$$= -(\sigma_1^2 + \sigma^2)/n$$

$$Cov(\bar{x}_2, \bar{x}_1 + \bar{x}_2) = E[\bar{x}_2 * (\bar{x}_1 + \bar{x}_2)] - E[\bar{x}_2] * E[\bar{x}_1 + \bar{x}_2]$$
$$= E[\bar{x}_2^2] + E[\bar{x}_1\bar{x}_2] - E[\bar{x}_2]^2 - E[\bar{x}_1] * E[\bar{x}_2]$$
$$= E[\bar{x}_2^2] - E[\bar{x}_2]^2$$
$$= Var(\bar{x}_2) + E[\bar{x}_2]^2 - E[\bar{x}_2]^2$$

$= Var(\bar{x}_2)$

$= (\sigma_2^2 + \sigma^2)/n$

$Cov(\bar{x}_2, \bar{x}_2 - \bar{x}_1) = E[\bar{x}_2 * (\bar{x}_2 - \bar{x}_1)] - E[\bar{x}_2] * E[\bar{x}_2 - \bar{x}_1]$

$= E[\bar{x}_2^2] - E[\bar{x}_1 \bar{x}_2] - E[\bar{x}_2]^2 + E[\bar{x}_1] * E[\bar{x}_2]$

$= E[\bar{x}_2^2] - E[\bar{x}_2]^2$

$= Var(\bar{x}_2) + E[\bar{x}_2]^2 - E[\bar{x}_2]^2$

$= Var(\bar{x}_2)$

$= (\sigma_2^2 + \sigma^2)/n$

Using the above derived covariance equations using the second-order Taylor series approximation equation, the approximate expected value of equation (6) is derived as below,

$$E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] = 1 - \frac{1}{2} * E\left[\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}\right] + \frac{1}{2} * E\left[\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right]$$

$$\approx 1 - \frac{1}{2} * \left(\frac{\mu_1}{\mu_2 + \mu_1} - \frac{(\sigma_1^2 + \sigma^2)/n}{(\mu_2 + \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_1}{(\mu_2 + \mu_1)^3}\right) + \frac{1}{2} * \left(\frac{\mu_1}{\mu_2 - \mu_1} + \frac{(\sigma_1^2 + \sigma^2)/n}{(\mu_2 - \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_1}{(\mu_2 - \mu_1)^3}\right)$$

$$= 1 - \frac{1}{2} * \left(\frac{\mu_1}{\mu_1 + \mu_2} - \frac{\mu_1}{\mu_2 - \mu_1}\right) - \frac{1}{2} * \left[\frac{-(\sigma_1^2 + \sigma^2)/n}{(\mu_2 + \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_1}{(\mu_2 + \mu_1)^3} - \frac{(\sigma_1^2 + \sigma^2)/n}{(\mu_2 - \mu_1)^2} - \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_1}{(\mu_2 - \mu_1)^3}\right]$$

For the measurement instrument precision application, the quantity in the square brackets can be considered small (validated later) and will be assumed negligible giving,

$$E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] \approx 1 - \frac{1}{2} * \left(\frac{\mu_1}{\mu_1 + \mu_2} - \frac{\mu_1}{\mu_2 - \mu_1}\right) = 1 - \frac{1}{2} * \left(\frac{-2\mu_1^2}{\mu_2^2 - \mu_1^2}\right) = \frac{\mu_2^2}{\mu_2^2 - \mu_1^2} \quad (9)$$

Similarly, the expected value of the equation (7) using the Taylor series approximation equation,

$$E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] = -1 + \frac{1}{2} * E\left[\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}\right] + \frac{1}{2} * E\left[\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right]$$

$$\approx -1 + \frac{1}{2} * \left(\frac{\mu_2}{\mu_1 + \mu_2} - \frac{(\sigma_2^2 + \sigma^2)/n}{(\mu_2 + \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_2}{(\mu_2 + \mu_1)^3}\right) + \frac{1}{2} * \left(\frac{\mu_2}{\mu_2 - \mu_1} - \frac{(\sigma_2^2 + \sigma^2)/n}{(\mu_2 - \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_2}{(\mu_2 - \mu_1)^3}\right)$$

$$= -1 + \frac{1}{2} * \left(\frac{\mu_2}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_2 - \mu_1}\right) + \frac{1}{2} * \left[\frac{-(\sigma_2^2 + \sigma^2)/n}{(\mu_2 + \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_2}{(\mu_2 + \mu_1)^3} - \frac{(\sigma_2^2 + \sigma^2)/n}{(\mu_2 - \mu_1)^2} + \frac{((\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n)\mu_2}{(\mu_2 - \mu_1)^3}\right]$$

As before the quantity in the square brackets will be considered zero,

$$E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] \approx -1 + \frac{1}{2} * \left(\frac{\mu_2}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_2 - \mu_1}\right) = -1 + \frac{1}{2} * \left(\frac{-2{\mu_2}^2}{\mu_2^2 - \mu_1^2}\right) = \frac{\mu_1^2}{\mu_2^2 - \mu_1^2} \tag{10}$$

Substituting the expected value derived using the Taylor series approximation equations (9) and (10) in equation (5) and using $E[s_i^2] = \sigma_i^2 + \sigma^2$,

$$E[\hat{\beta}_0] = E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] E[s_1^2] - E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] E[s_2^2]$$

$$\approx \frac{\mu_2^2}{\mu_2^2 - \mu_1^2}(\sigma_1^2 + \sigma^2) - \frac{\mu_1^2}{\mu_2^2 - \mu_1^2}(\sigma_2^2 + \sigma^2)$$

$$= \frac{\mu_2^2\sigma_1^2 - \mu_1^2\sigma_2^2 + \sigma^2(\mu_2^2 - \mu_1^2)}{\mu_2^2 - \mu_1^2}$$

$$= \sigma^2$$

Where the last equality holds due to the constant coefficient of variation of part types assumption. Therefore, the approximate expected value of the line's intercept estimate is measurement instrument precision. Hence for scenario two, the line's intercept is an unbiased estimator for the measurement instrument precision ignoring small value terms and using Taylor series approximations.

4.2.7. Scenario 2: Variance of the Estimator

The theoretical variance of the estimator for scenario two is derived in this section. The theoretical variance is derived using a two-moment approximation for the ratio of variances, and partial fraction expansions. It will be shown that if part type sample means are used, then the theoretical variance of the estimator is close to the variance when the part means are assumed known. The line's intercept equation is,

$$\widehat{\beta_0} = \frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2 - \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2$$

$$Var\left[\widehat{\beta_0}\right] = Var\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2 - \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2\right]$$

$$= Var\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2\right] + Var\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2\right] - 2Cov\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2, \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2\right) \tag{11}$$

The covariance term is small relative to the variance terms (shown in Appendix 2). After neglecting the covariance term, the equation becomes,

$$Var\left[\widehat{\beta_0}\right] \approx Var\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2\right] + Var\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2} s_2^2\right] \tag{12}$$

The above variance term can be simplified using the partial fraction expansion equations (6) and (7). The first ratio term is simplified to,

$$Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) = Var\left(1 - \frac{1}{2}\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)} + \frac{1}{2}\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right)$$

$$= \frac{1}{4} Var\left(-\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)} + \frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right)$$

Using the independence of part type means, the above equation is simplified to,

$$Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) = \frac{1}{4}\left(Var\left(\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}\right) + Var\left(\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right) - 2Cov\left(\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right)\right)$$
$$= \frac{1}{4}Var\left(\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}\right) + \frac{1}{4}Var\left(\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right) - \frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right)$$

Thus, the first ratio term in equation (12) is

$$Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) = \frac{1}{4}Var\left(\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}\right) + \frac{1}{4}Var\left(\frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right) - \frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_1}{(\bar{x}_2 - \bar{x}_1)}\right) \tag{13}$$

Similarly, the second ratio term in the equation (12) can be simplified to,

$$Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) = Var\left(-1 + \frac{1}{2}\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)} + \frac{1}{2}\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right)$$
$$= \frac{1}{4}Var\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)} + \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right)$$
$$= \frac{1}{4}\left(Var\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)} + Var\left(\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right) + 2Cov\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right)\right)$$
$$= \frac{1}{4}\ Var\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}\right) + \frac{1}{4}\ Var\left(\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right) + \frac{1}{2}\ Cov\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right)$$

Thus, the second ratio term in equation (12) is,

$$\begin{aligned} Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) = \frac{1}{4}\ Var\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}\right) + \frac{1}{4}\ Var\left(\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right) \\ + \frac{1}{2}\ Cov\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right) \end{aligned} \tag{14}$$

Like the approximate expected value of the ratio of random variables, an approximate variance for the ratio of random variables can also be derived using second-order Taylor series approximation. The variance approximation using second-order Taylor series approximation for a ratio of random variables X and Y is,

$$Var\left(\frac{X}{Y}\right) \approx \frac{(\mu_X)^2}{(\mu_Y)^2}\left[\frac{Var(X)}{(\mu_X)^2} - 2\,\frac{Cov(X,Y)}{\mu_X\mu_Y} + \frac{Var(Y)}{(\mu_Y)^2}\right] \tag{15}$$

Using the above second-order Taylor series approximation, the variance terms of equations (13) and (14) are derived below.

$$Var\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right) \approx \frac{\mu_1^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} - 2\,\frac{(\sigma_1^2+\sigma^2)/n}{(\mu_2+\mu_1)\mu_1} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]$$

$$Var\left(\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right) \approx \frac{\mu_1^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} + 2\,\frac{(\sigma_1^2+\sigma^2)/n}{(\mu_2+\mu_1)\mu_1} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]$$

$$Var\left(\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}\right) \approx \frac{\mu_2^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} - 2\,\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]$$

$$Var\left(\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right) \approx \frac{\mu_2^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} - 2\,\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]$$

Using the above second-order Taylor series approximation derivation, the equation (13) is solved as follows.

$$\begin{aligned}
Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2-\bar{x}_1^2}\right) &= \frac{1}{4}Var\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right) + \frac{1}{4}Var\left(\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right) - \frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)\\
&\approx \frac{1}{4}\frac{\mu_1^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} - 2\,\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]\\
&\quad + \frac{1}{4}\frac{\mu_1^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} + 2\,\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]\\
&\quad - \frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2-\bar{x}_1^2}\right) &\approx \frac{1}{4}\frac{\mu_1^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} - 2\,\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]\\
&\quad + \frac{1}{4}\frac{\mu_1^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} + 2\,\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]\\
&\quad - \frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)
\end{aligned} \tag{16}$$

Similarly, Using the above second-order Taylor series approximation derivation, the equation (14) is solved as follows.

$$Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right)$$
$$= \frac{1}{4} Var\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}\right) + \frac{1}{4} Var\left(\frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right) + \frac{1}{2} Cov\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right)$$
$$\approx \frac{1}{4}\frac{\mu_2^2}{(\mu_2 + \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 + \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 + \mu_1)^2}\right]$$
$$+ \frac{1}{4}\frac{\mu_2^2}{(\mu_2 - \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 - \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 - \mu_1)^2}\right]$$
$$+ \frac{1}{2} Cov\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right)$$

Therefore,

$$\begin{aligned} Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) &\approx \frac{1}{4}\frac{\mu_2^2}{(\mu_2 + \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 + \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 + \mu_1)^2}\right] \\ &+ \frac{1}{4}\frac{\mu_2^2}{(\mu_2 - \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 - \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 - \mu_1)^2}\right] \\ &+ \frac{1}{2} Cov\left(\frac{\bar{x}_2}{(\bar{x}_2 + \bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2 - \bar{x}_1)}\right) \end{aligned} \tag{17}$$

Due to the independence of part type means and variances, the theoretical variance equation (12) is solved below. The first term in equation (12) is derived below.

$$\begin{aligned} Var\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2} s_1^2\right] \\ &= \left(Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) + \left(E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]\right)^2\right)(Var(s_1^2) + (E[s_1^2])^2) \\ &- \left(\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}(\sigma_1^2 + \sigma^2)\right)^2 \end{aligned} \tag{18}$$

It is shown earlier in equation (9) and (10),

$$E\left[\frac{\bar{x}_i^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right]\approx\frac{\mu_i^2}{\mu_2^2-\mu_1^2}$$

Using this equality and substituting equation (16), each term in the equation (18) is solved below.

$$\begin{aligned}=\Bigg(&\frac{1}{4}\frac{\mu_1^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2}-\ 2\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2+\mu_1)}+\frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]\\&+\frac{1}{4}\frac{\mu_1^2}{\left(\mu_2-\mu_1\right)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2}+\ 2\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1\left(\mu_2-\mu_1\right)}+\frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{\left(\mu_2-\mu_1\right)^2}\right]\\&-\frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)+\frac{\mu_2^2}{\mu_2^2-\mu_1^2}^2\Bigg)\left(\frac{2(\sigma^2+\sigma_1^2)^2}{(n-1)}+(\sigma^2+\sigma_1^2)^2\right)\\&-\left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}(\sigma_1^2+\sigma^2)\right)^2\end{aligned}\tag{19}$$

Similarly, the second term in equation (12) is derived below.

$$\begin{aligned}&Var\left[\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}s_2^2\right]\\&\quad=\left(Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right)+E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right]^2\right)(Var(s_2^2)+E[s_2^2]^2)\\&\quad-\left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}(\sigma_1^2+\sigma^2)\right)^2\end{aligned}\tag{20}$$

It is shown earlier in equation (9) and (10),

$$E\left[\frac{\bar{x}_i^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right]\approx\frac{\mu_i^2}{\mu_2^2-\mu_1^2}$$

Using this equality and substituting equation (17), each term in the equation (20) is solved below.

$$= \left(\frac{1}{4}\frac{\mu_2^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]\right.$$
$$+ \frac{1}{4}\frac{\mu_2^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right.$$
$$\left. + \frac{1}{2}\,Cov\left(\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right) + \frac{\mu_1^2}{\mu_2^2-\mu_1^2}^{2}\right)\left(\frac{2(\sigma^2+\sigma_2^2)^2}{(n-1)}\right. \tag{21}$$
$$\left. + (\sigma^2+\sigma_2^2)^2\right) - \left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\,(\sigma_2^2+\sigma^2)\right)^2$$

Substituting the two equations (19) and (21) in theoretical variance equation (12),

$$Var[\hat{\beta}_0] = \left(\boldsymbol{\frac{1}{4}\frac{\mu_1^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} - 2\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]}\right.$$
$$+ \boldsymbol{\frac{1}{4}\frac{\mu_1^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1^2} + 2\frac{(\sigma_1^2+\sigma^2)/n}{\mu_1(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]}$$
$$\left. - \boldsymbol{\frac{1}{2}Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}, \frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)} + \frac{\mu_2^2}{\mu_2^2-\mu_1^2}^{2}\right)\left(\frac{2(\sigma^2+\sigma_1^2)^2}{(n-1)}\right.$$
$$\left. + (\sigma^2+\sigma_1^2)^2\right) - \left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}(\sigma_1^2+\sigma^2)\right)^2$$
$$+ \left(\boldsymbol{\frac{1}{4}\frac{\mu_2^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]}\right. \tag{22}$$
$$+ \boldsymbol{\frac{1}{4}\frac{\mu_2^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]}$$
$$\left. + \boldsymbol{\frac{1}{2}\,Cov\left(\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}, \frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right)} + \frac{\mu_1^2}{\mu_2^2-\mu_1^2}^{2}\right)\left(\frac{2(\sigma^2+\sigma_2^2)^2}{(n-1)}\right.$$
$$\left. + (\sigma^2+\sigma_2^2)^2\right) - (\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\,(\sigma_2^2+\sigma^2))^2$$

The bold terms in the above equation (22) are assumed small (validated later). Therefore, the equation is approximated as,

$$Var[\widehat{\beta_0}] = \left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}\right)^2\left(\frac{2(\sigma^2+\sigma_1^2)^2}{(n-1)} + (\sigma^2+\sigma_1^2)^2\right) - \left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}(\sigma_1^2+\sigma^2)\right)^2$$
$$+\left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\right)^2\left(\frac{2(\sigma^2+\sigma_2^2)^2}{(n-1)} + (\sigma^2+\sigma_2^2)^2\right) - \left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}(\sigma_2^2+\sigma^2)\right)^2$$

$$= \frac{2}{(n-1)}\frac{(\sigma^2+\sigma_1^2)^2\mu_2^4 + (\sigma^2+\sigma_2^2)^2\mu_1^4}{(\mu_2^2-\mu_1^2)^2} \quad (23)$$

The above equation is the theoretical variance of the measurement instrument precision estimator in scenario two. Assuming some terms are negligible and using Taylor series approximations, scenario two's theoretical variance equation is the same as in scenario one. The estimator's theoretical variance needs to be estimated from the part type sample means and part type sample variances. Hence, the expected value of the estimator variance is discussed in the next section.

4.2.8. Scenario 2: Expected Value of the Intercept Variance Estimator

The variance derived for the measurement instrument precision estimate in the above section is estimated from part type sample means and part type sample variances. In this section, the expected value of the estimated intercept variance is derived. Let θ be the theoretical variance of the measurement instrument precision in scenario two,

$$Var[\widehat{\beta_0}] = \theta = \frac{2}{(n-1)(\mu_2^2-\mu_1^2)^2}\left[(\sigma^2+\sigma_1^2)^2\mu_2^4 + (\sigma^2+\sigma_2^2)^2\mu_1^4\right]$$
$$\widehat{Var[\beta_0]} = \hat{\theta} = \frac{2}{(n-1)}\left(\frac{(s_1^2)^2 * \bar{x}_2^4 + (s_2^2)^2 * \bar{x}_1^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right)$$

$$E[\hat{\theta}] = \frac{2}{(n-1)}\left(E\left[(s_1^2)^2 * \frac{\bar{x}_2^4}{(\bar{x}_2^2-\bar{x}_1^2)^2}\right] + E\left[(s_2^2)^2 * \frac{\bar{x}_1^4}{(\bar{x}_2^2-\bar{x}_1^2)^2}\right]\right) \quad (24)$$

The above expected value of the equation is estimated using sample part type means and sample part type variances. Replacing with sample part type means and variances and solving the first term in the above expected value equation.

$$E\left[(s_1^2)^2 * \frac{\bar{x}_2^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] = E\left[(s_1^2)^2 \frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)}^2\right]$$

Since the part type means and variances are independent, the above equation can be solved as below.

$$E\left[(s_1^2)^2 * \frac{\bar{x}_2^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] = E[(s_1^2)^2] * E\left[\frac{\bar{x}_2^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right]$$

By making use of the variance of a random variable X, $Var(X) = E[X^2] - (E[X])^2$,

$$E\left[(s_1^2)^2 * \frac{\bar{x}_2^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right]$$
$$= (Var(s_1^2) + (E[s_1^2])^2)\left(Var\left(\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right) + \left(E\left[\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right]\right)^2\right)$$

By making use of $E[s_1^2] = \sigma^2 + \sigma_1^2$ and $Var(s_1^2) = \frac{2(\sigma^2+\sigma_1^2)^2}{(n-1)}$ derived in equation (3), and $Var\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right)$ derived in equation (16), and $E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]$ derived in equation (9), the above equation becomes,

$$E\left[(s_1^2)^2 * \frac{\bar{x}_2^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right]$$
$$= \left(\frac{2(\sigma^2 + \sigma_1^2)^2}{(n-1)} + (\sigma^2 + \sigma_1^2)^2\right)\left(\frac{\mathbf{1}}{\mathbf{4}}\frac{\boldsymbol{\mu_1^2}}{(\boldsymbol{\mu_2} + \boldsymbol{\mu_1})^2}\left[\frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_1^2}}\right.\right.$$
$$\left. - \ \mathbf{2}\frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_1}(\boldsymbol{\mu_2} + \boldsymbol{\mu_1})} + \frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma_2^2} + \mathbf{2}\boldsymbol{\sigma^2})/\boldsymbol{n}}{(\boldsymbol{\mu_2} + \boldsymbol{\mu_1})^2}\right]$$
$$+ \frac{\mathbf{1}}{\mathbf{4}}\frac{\boldsymbol{\mu_1^2}}{(\boldsymbol{\mu_2} - \boldsymbol{\mu_1})^2}\left[\frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_1^2}} + \ \mathbf{2}\frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_1}(\boldsymbol{\mu_2} - \boldsymbol{\mu_1})} + \frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma_2^2} + \mathbf{2}\boldsymbol{\sigma^2})/\boldsymbol{n}}{(\boldsymbol{\mu_2} - \boldsymbol{\mu_1})^2}\right]$$
$$\left. - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{Cov}\left(\frac{\bar{\boldsymbol{x}}_{\mathbf{1}}}{(\bar{\boldsymbol{x}}_{\mathbf{2}} + \bar{\boldsymbol{x}}_{\mathbf{1}})}, \frac{\bar{\boldsymbol{x}}_{\mathbf{1}}}{(\bar{\boldsymbol{x}}_{\mathbf{2}} - \bar{\boldsymbol{x}}_{\mathbf{1}})}\right) + \left(\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}\right)^2\right)$$

The quantity in bold is very small compared to $\left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}\right)^2$. Thus, approximating the bold terms to zero,

$$E\left[(s_1^2)^2 * \frac{\bar{x}_2^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] \approx \frac{(n+1)(\sigma^2+\sigma_1^2)^2}{n-1}\left(\frac{\mu_2^2}{\mu_2^2-\mu_1^2}\right)^2 \tag{25}$$

Similarly, the second term in the expected value equation using sample part type means and sample part type variances, the equation becomes,

$$E\left[(s_2^2)^2 * \frac{\bar{x}_1^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] = E\left[(s_2^2)^2 * \left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right)^2\right]$$

Since the part type means and variances are independent, the above equation can be derived as below.

$$E\left[(s_2^2)^2 * \frac{\bar{x}_1^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] = E[(s_2^2)^2] * E\left[\left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right)^2\right]$$

By making use of the variance of a random variable X, $Var(X) = E[X^2] - (E[X])^2$,

$$E\left[(s_2^2)^2 * \frac{\bar{x}_1^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] = \left(Var(s_2^2) + (E[s_2^2])^2\right)\left(Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right) + \left(E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]\right)^2\right)$$

By making use of $E[s_2^2] = \sigma^2 + \sigma_2^2$ and $Var(s_2^2) = \frac{2(\sigma^2+\sigma_2^2)^2}{(n-1)}$ derived in equation (3), and $Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2-\bar{x}_1^2}\right)$ derived in equation(17), and $E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2-\bar{x}_1^2}\right]$ derived in equation(10), the above equation becomes,

$$E\left[(s_2^2)^2 * \frac{\bar{x}_1^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right]$$

$$= \left(\frac{2(\sigma^2 + \sigma_2^2)^2}{(n-1)}\right.$$

$$+ (\sigma^2 + \sigma_2^2)^2\Bigg)\left(\frac{\mathbf{1}}{\mathbf{4}}\frac{\boldsymbol{\mu_2^2}}{(\boldsymbol{\mu_2} + \boldsymbol{\mu_1})^2}\left[\frac{(\boldsymbol{\sigma_2^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_2^2}} - \mathbf{2}\frac{(\boldsymbol{\sigma_2^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_2}(\boldsymbol{\mu_2} + \boldsymbol{\mu_1})}\right.\right.$$

$$\left. + \frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma_2^2} + \mathbf{2}\boldsymbol{\sigma^2})/\boldsymbol{n}}{(\boldsymbol{\mu_2} + \boldsymbol{\mu_1})^2}\right]$$

$$+ \frac{\mathbf{1}}{\mathbf{4}}\frac{\boldsymbol{\mu_2^2}}{(\boldsymbol{\mu_2} - \boldsymbol{\mu_1})^2}\left[\frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_1^2}} - \mathbf{2}\frac{(\boldsymbol{\sigma_2^2} + \boldsymbol{\sigma^2})/\boldsymbol{n}}{\boldsymbol{\mu_2}(\boldsymbol{\mu_2} - \boldsymbol{\mu_1})} + \frac{(\boldsymbol{\sigma_1^2} + \boldsymbol{\sigma_2^2} + \mathbf{2}\boldsymbol{\sigma^2})/\boldsymbol{n}}{(\boldsymbol{\mu_2} - \boldsymbol{\mu_1})^2}\right]$$

$$\left. + \frac{\mathbf{1}}{\mathbf{2}}\,\boldsymbol{Cov}\left(\frac{\bar{\boldsymbol{x}}_\mathbf{2}}{(\bar{\boldsymbol{x}}_\mathbf{2} + \bar{\boldsymbol{x}}_\mathbf{1})}, \frac{\bar{\boldsymbol{x}}_\mathbf{2}}{(\bar{\boldsymbol{x}}_\mathbf{2} - \bar{\boldsymbol{x}}_\mathbf{1})}\right) + \frac{\mu_1^2}{\mu_2^2 - \mu_1^2}^{2}\right)$$

The quantity in bold is very small compared to $\left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\right)^2$. Thus, approximating the bold term to zero,

$$E\left[(s_2^2)^2 * \frac{\bar{x}_1^4}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right] \approx \frac{(n+1)(\sigma^2 + \sigma_2^2)^2}{n-1}\left(\frac{\mu_1^2}{\mu_2^2 - \mu_1^2}\right)^2 \tag{26}$$

Therefore, by substituting the above equations (25) and (26) to the expected value equation (24),

$$E[\hat{\theta}] = \frac{2}{(n-1)}\left(\frac{(n+1)(\sigma^2 + \sigma_1^2)^2}{n-1}\left(\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}\right)^2 + \frac{(n+1)(\sigma^2 + \sigma_2^2)^2}{n-1}\left(\frac{\mu_1^2}{\mu_2^2 - \mu_1^2}\right)^2\right)$$

$$= \frac{2(n+1)}{(n-1)^2}\left(\frac{(\sigma^2 + \sigma_1^2)^2\mu_2^4 + (\sigma^2 + \sigma_2^2)^2\mu_1^4}{(\mu_2^2 - \mu_1^2)^2}\right)$$

There is a difference between $E[\hat{\theta}]$ and θ, hence, there is bias in the estimated variance of the measuring instrument precision. The bias, $Bias[\hat{\theta}]$ is,

$$Bias[\hat{\theta}] = E[\hat{\theta}] - \theta$$

$$Bias[\theta] = \frac{2(n+1)}{(n-1)^2}\left(\frac{(\sigma^2+\sigma_1^2)^2\mu_2^4+(\sigma^2+\sigma_2^2)^2\mu_1^4}{(\mu_2^2-\mu_1^2)^2}\right) - \frac{2}{(n-1)}\frac{(\sigma^2+\sigma_1^2)^2\mu_2^4+(\sigma^2+\sigma_2^2)^2\mu_1^4}{(\mu_2^2-\mu_1^2)^2}$$

$$\neq 0$$

Therefore, there is bias with the estimated variance equation, and the correction factor is $\frac{(n-1)}{(n+1)}$.

$$E[\hat{\theta}] * \frac{(n-1)}{(n+1)} = \theta$$

Thus, in scenario two also it has been shown that the intercept of the line through the square of the part type sample means and part type sample variances is an estimator for the measuring instrument precision. It has been shown that the estimator is approximately unbiased, and the theoretical variance of the estimator has been approximated. When the variance of the measurement instrument precision estimator is estimated, a bias correction factor has been derived that can be used to correct the bias associated with the estimate.

In scenario two, the estimator's theoretical variance derivation uses approximations like the second-order Taylor series approximations and a few terms were assumed negligible. To check these approximations measurement instrument precision variance and confidence intervals are validated in the next section using Monte Carlo simulation.

4.3. Validation

The outline of the chapter is as follows. In section 4.3.1, the simulation set-up is explained. In section 4.3.2 the scenario one measuring instrument precision estimate confidence interval coverage is analyzed. In section 4.3.3, the scenario two estimator's derived theoretical variance equation accuracy and the measuring instrument precision estimate confidence interval coverage is tested.

4.3.1. Simulation Set-Up

In the simulation, part type one has $Normal(\mu_1, \sigma_1^2)$ distribution, part type two has $Normal(\mu_2, \sigma_2^2)$ distribution, and the measurement error has a distribution $Normal(0, \sigma^2)$, where σ^2 is the measurement instrument precision. 50 measurement samples are generated from each part type.

In each trial, part type sample means and part type sample variances can be obtained. Using the known part type means and part type sample variances measuring instrument precision estimator can be estimated. The simulation is repeated for 100,000 trials to obtain the sample variance of the estimator. The sample variance of the estimator is compared to the theoretical variance.

From the estimator (2) the probability distribution of the estimator is the difference of two scaled chi-square random variable with $n-1$ degrees of freedom, where n is the sample size. The chi square distribution is the distribution of the sum of squared standard normal distributions with degrees of freedom as its mean. The chi-square distribution is positively skewed with skewness decreasing with increasing degrees of freedom. Since a large sample size (= 50) is used in the simulation model, according to central limit theorem the chi-square distribution can be approximated to a normal distribution. Therefore, the distribution of the estimator can be approximated as the difference of two normal random variables. Also, when the histogram of the intercept estimate is generated, it shows approximate normality as shown in Figure 5. Hence, to form the measuring instrument precision confidence interval coverage, a 95 percent confidence interval is generated using the below formula.

$$Confidence\ interval = Intercept\ estimate \pm \alpha_{0.05} V_{theoretical}$$

Where, $V_{theoretical}$ is the derived variance and $\alpha_{0.05}$ = 1.96 is obtained from the standard normal distribution. Using the confidence interval generated in each trial, confidence interval coverage, whether the true measurement instrument precision is included in the confidence interval or not, is checked, and a percentage confidence interval coverage is calculated.

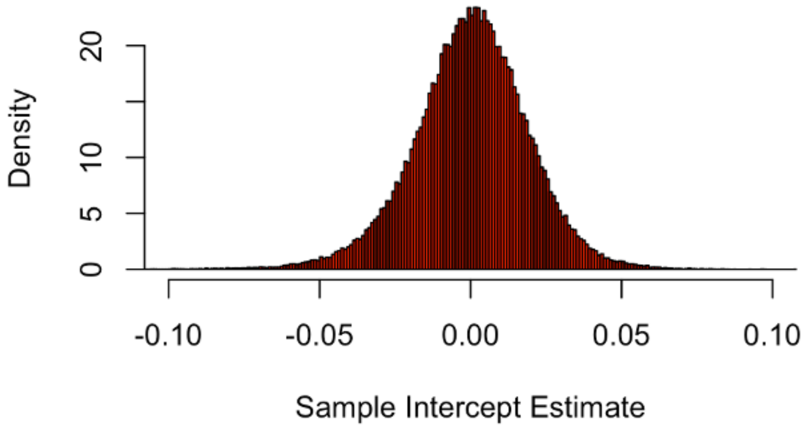


Figure 5: Sampling Distribution of the Estimator

Validation Space

The validation of the estimator's derived theoretical variance and confidence interval coverage is evaluated over a space defined by three factors at two levels each. The factors are part type one variance, part type two mean, and part type two variance. The part type one mean value is kept one throughout the simulation. Other variables, part type one standard deviation considered are 0.1 and 10 percent of part type one mean. The measurement instrument precision standard deviation considered are five and 25 percent of part type one standard deviation. The part type two means considered are 0.1 and ten-times of part type one mean. Part type two variance is chosen based on part type one mean, part type one standard deviation, and part type two mean so that the constant coefficient of variation assumption holds. Eight different treatment combinations are considered, as shown in Table 4. An additional case of part type one mean 10, standard deviation 0.5, measuring instrument precision standard deviation 0.2236, and part type two mean 20 is also analyzed.

Table 4: Sample Space Treatments

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)
-	10	20	0.5	0.2236
1	1	10	0.1	0.025
2	1	10	0.1	0.005
3	1	10	0.001	0.00025
4	1	10	0.001	0.00005
5	1	1.1	0.1	0.025
6	1	1.1	0.1	0.005
7	1	1.1	0.001	0.00025
8	1	1.1	0.001	0.00005

4.3.2. Scenario 1: Measurement Instrument Precision Confidence Interval Coverage

In scenario one, two test cases are considered. In Case 1, the theoretical variance is calculated using the known true part type means and known true part type variances. In Case 1.1, the theoretical variance is estimated using known true part type means and part type sample variances. The bias in the variance estimate in Case 1.1 is corrected using the bias correction factor. In both test cases, the intercept is estimated from true part type means and part type sample variances. A summary of the result obtained for scenario one is given in Table 5.

Table 5: Scenario One Test Cases

Test Case	Intercept Estimate (Simulation)	Theoretical Variance (Derived Equation)	Variance Difference (%)	Confidence Interval Coverage
1	Part Type True Means Part Type Sample Variances	Part Type True Means Part Type True Variances	Less than 1%	Nearly 95%
1.1		Part Type True Means Part Type Sample Variances	Less than 1%	More than 95%

<u>Case 1</u>

Case one results are shown in Table 6.

Table 6: Case1 Theoretical Variance and Confidence Interval Coverage

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
-	10	20	0.5	0.2236	5.0E-02	5.0E-02	1.2E-02	1.2E-02	0.8%	94.8%
1	1	10	0.1	0.025	6.3E-04	6.2E-04	8.9E-06	8.9E-06	0.9%	94.8%
2	1	10	0.1	0.005	2.5E-05	1.9E-05	8.4E-06	8.4E-06	0.9%	94.8%
3	1	10	0.001	0.00025	6.3E-08	6.2E-08	8.9E-14	8.9E-14	0.9%	94.8%
4	1	10	0.001	0.00005	2.5E-09	1.9E-09	8.4E-14	8.4E-14	0.9%	94.8%
5	1	1.1	0.1	0.025	6.3E-04	5.9E-04	3.1E-04	3.0E-04	0.8%	94.8%
6	1	1.1	0.1	0.005	2.5E-05	-6.9E-06	2.7E-04	2.7E-04	0.8%	94.8%
7	1	1.1	0.001	0.00025	6.3E-08	5.9E-08	3.1E-12	3.0E-12	0.8%	94.8%
8	1	1.1	0.001	0.00005	2.5E-09	-6.9E-10	2.7E-12	2.7E-12	0.8%	94.8%

The case one results show that the variance percentage difference is less than one percent for all the treatments. The confidence interval coverage is nearly 95 percent for all the treatments.

<u>Case 1.1</u>

Case 1.1 results are shown in Table 7.

Table 7: Case 1.1 Theoretical Variance and Confidence Interval Coverage

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Mean Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
-	10	20	0.5	0.2236	5.0E-02	5.0E-02	1.2E-02	1.2E-02	0.8%	95.6%
1	1	10	0.1	0.025	6.3E-04	6.2E-04	8.9E-06	8.9E-06	0.8%	95.6%
2	1	10	0.1	0.005	2.5E-05	1.9E-05	8.4E-06	8.4E-06	0.8%	95.7%
3	1	10	0.001	0.00025	6.3E-08	6.2E-08	8.9E-14	8.9E-14	0.8%	95.6%
4	1	10	0.001	0.00005	2.5E-09	1.9E-09	8.4E-14	8.4E-14	0.8%	95.7%
5	1	1.1	0.1	0.025	6.3E-04	5.9E-04	3.1E-04	3.0E-04	0.8%	95.7%
6	1	1.1	0.1	0.005	2.5E-05	-6.9E-06	2.7E-04	2.7E-04	0.8%	95.7%
7	1	1.1	0.001	0.00025	6.3E-08	5.9E-08	3.1E-12	3.0E-12	0.8%	95.7%
8	1	1.1	0.001	0.00005	2.5E-09	-6.9E-01	2.7E-12	2.7E-12	0.8%	95.7%

The Case 1.1 results show that the variance percentage difference is less than one percent for all the treatments and the confidence interval coverage is more than 95 percent for all the treatments.

Since the simulation results match with the derived equation values and there is 95 percent confidence interval coverage for the measuring instrument precision estimate, it can be concluded that the simulation model set up is correct.

4.3.3. Scenario 2: Measurement Instrument Precision Estimator Variance and Confidence Interval Coverage

In this section, the measurement instrument precision estimator's variance and the confidence interval coverage are validated. Since the theoretical variance in scenario two is derived using second-order Taylor series approximation and also by approximating a few negligible terms to zero, the validation is done to evaluate how well these approximations hold while estimating measurement instrument precision's variance and its confidence interval coverage.

Four test cases are presented in this section. Case 2 and Case 2.1 uses the estimator's derived theoretical variance equation. While Case 3 and Case 3.1 uses the estimator's derived theoretical variance equation, including all the left out small value terms. Case 2 and Case 3 uses true part type means and true part type variance while calculating estimator's theoretical variance. While in Case 2.1 and Case 3.1, the estimator's theoretical variance is estimated from part-type sample means and part type sample variances and then corrected the bias using the correction factor. In all these cases, the estimator is estimated from part type sample means and part type sample variances. A summary result of the test cases is given in Table 8.

Table 8: Scenario 2 Test Cases

Test Case	Intercept Estimate (Simulation)	Theoretical Variance (Derived Equation)	Variance Difference (%)	Confidence Interval Coverage
2	Part Type Sample Means Part Type Sample Variances	Part Type True Means Part Type True Variances	Low difference except for treatments 5 and 6	Nearly 95% except for treatments 5 and 6
2.1		Part Type Sample Means Part Type Sample Variances	Low difference	More than 95%
3		Part Type True Means Part Type True Variances	Low difference except for treatments 5 and 6	Nearly 95%, for treatment 5 and 6 < 95%
3.1		Part Type Sample Means Part Type Sample Variances	Low difference except for treatments 5 and 6	More than 95%

<u>Case 2</u>

The Case 2 results are shown in Table 9

Table 9: Case 2 Estimator Variance and Confidence Interval Coverage

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
-	10	20	0.5	0.2236	5.0E-02	5.0E-02	1.2E-02	1.2E-02	1.3%	94.8%
1	1	10	0.1	0.025	6.3E-04	6.1E-04	9.1E-06	8.9E-06	2.9%	94.6%
2	1	10	0.1	0.005	2.5E-05	1.2E-05	8.6E-06	8.4E-06	2.8%	94.6%
3	1	10	0.001	0.00025	6.3E-08	6.2E-08	8.9E-14	8.9E-14	0.9%	94.8%
4	1	10	0.001	0.00005	2.5E-09	1.9E-09	8.4E-14	8.4E-14	0.9%	94.8%
5	1	1.1	0.1	0.025	6.3E-04	-2.6E-05	3.7E-04	3.0E-04	19.0%	92.5%
6	1	1.1	0.1	0.005	2.5E-05	-5.8E-04	3.3E-04	2.7E-04	18.1%	92.6%
7	1	1.1	0.001	0.00025	6.3E-08	5.9E-08	3.1E-12	3.0E-12	0.8%	94.8%
8	1	1.1	0.001	0.00005	2.5E-09	-6.8E-10	2.7E-12	2.7E-12	0.8%	94.8%

The Case 2 results show that the percentage variance difference is less than three percent for all the treatments, and the confidence interval coverage is nearly 95 percent except for treatment five and six. In treatment five and six, the percentage variance difference is 19 percent and 18.1 percent respectively. The confidence interval is also slightly less for these treatments, with 92.5 percent and 92.6 percent, respectively.

The percentage variance difference in treatment five and six is because the second-order Taylor series approximation using which the theoretical variation equations are derived do not hold for these treatments. Second-order Taylor series approximations equations for the variance of random variables $\frac{X}{Y}$ estimates poorly when the mean of the denominator random variable is less than the standard deviation of the same denominator variable. This scenario becomes applicable for treatments five and six, thereby creating significant error in theoretical variance estimation. Additionally, the variance difference for treatment five and six can be neglected as the input value for these treatments are highly unlikely in a real measurement case. They are highly unlikely as the part type mean differences are very narrow and standard deviation of the part types are high.

Case 2.1:

Case 2.1 results are shown in Table 10.

Table 10: Case 2.1 Estimator Variance and Confidence Interval Coverage

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Mean Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
-	10	20	0.5	0.2236	5.0E-02	5.0E-02	1.2E-02	1.2E-02	1.2%	95.6%
1	1	10	0.1	0.025	6.3E-04	6.1E-04	9.1E-06	8.9E-06	2.7%	95.5%
2	1	10	0.1	0.005	2.5E-05	1.2E-05	8.6E-06	8.4E-06	2.6%	95.5%
3	1	10	0.001	0.00025	6.3E-08	6.2E-08	8.9E-14	8.9E-14	0.8%	95.6%
4	1	10	0.001	0.00005	2.5E-09	1.9E-09	8.4E-14	8.4E-14	0.8%	95.7%
5	1	1.1	0.1	0.025	6.3E-04	-2.6E-05	3.7E-04	3.6E-04	3.5%	95.5%
6	1	1.1	0.1	0.005	2.5E-05	-5.8E-04	3.3E-04	3.2E-04	3.5%	95.5%
7	1	1.1	0.001	0.00025	6.3E-08	5.9E-08	3.1E-12	3.0E-12	0.8%	95.7%
8	1	1.1	0.001	0.00005	2.5E-09	-6.8E-10	2.7E-12	2.7E-12	0.8%	95.7%

Case 2.1 results show that the percentage difference in variance for all the treatments is less than four percent. The confidence interval coverage is more than 95 percent. Compared to Case two, the confidence interval coverage is improved for all the treatments in Case 2.1. To further analyze Case 2.1 to understand if the improvement in percentage variance difference and confidence interval coverage is not by a random chance, the simulation is run with the number of trials increased to 500000 trials. Results are shown in Table 11.

Table 11: Case 2.1 Estimator Variance and Confidence Interval Coverage with 500000 trials

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Mean Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
1	1	10	0.1	0.025	6.3E-04	6.1E-04	9.1E-06	8.9E-06	2.0%	95.6%
2	1	10	0.1	0.005	2.5E-05	1.1E-05	8.5E-06	8.4E-06	2.0%	95.6%
3	1	10	0.001	0.00025	6.3E-08	-1.2E-09	8.3E-10	8.3E-10	0.1%	95.7%
4	1	10	0.001	0.00005	2.5E-09	-6.1E-08	8.3E-10	8.3E-10	0.1%	95.7%
5	1	1.1	0.1	0.025	6.3E-04	-3.6E-05	5.2E-04	3.1E-03	-501.1%	95.6%
6	1	1.1	0.1	0.005	2.5E-05	-5.9E-04	3.3E-04	3.3E-04	0.6%	95.6%
7	1	1.1	0.001	0.00025	6.3E-08	-3.4E-07	2.7E-08	2.7E-08	0.1%	95.7%
8	1	1.1	0.001	0.00005	2.5E-09	-4.0E-07	2.7E-08	2.7E-08	0.1%	95.7%

The Table 11 results show that there is no noticeable difference in the result compared to Table 10 results, except for treatment five. Treatment five performs poorly with a 501.1 percent difference in percentage variance difference. To analyze this case and to understand whether the high percentage variance difference is due to a random chance, the simulation model is run with the same number of trials but with a different seed value for treatment five. The result is shown in Table 12.

Table 12: Case 2.1 Estimator Variance and Confidence Interval Coverage with 500000 Trials with Different Seed Value

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Mean Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
5	1	1.1	0.1	0.025	6.3E-04	-4.7E-05	5.2E-04	3.1E-03	-503.5%	95.6%

The Table 12 result shows that a 503.5 percentage difference still exists for treatment five. During the analysis of the simulation intercept data, an extreme intercept value was noted, which caused the high percent variance difference. The extreme intercept value is the outlier in the sample of intercept values. When the extreme intercept estimate was removed and reanalyzed, the percentage difference is similar to the Table 10 result. Hence the small percentage variance difference and high confidence interval coverage is not due to a random chance. A summary of treatment five, Case 2.1 results are shown in Table 13.

Table 13: Case 2.1 Estimator Variance and Confidence Interval Coverage Treatment 5 Comparisons

Trials	Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (MIP) (SD)	MIP (Var)	Intercept		Mean Theoretical Variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
							Sample Mean	Sample Variance (A)			
100k	5	1	1.1	0.1	0.025	6.3E-04	-2.6E-05	3.7E-04	3.6E-04	3.5%	95.5%
200k	5	1	1.1	0.1	0.025	6.3E-04	-1.6E-05	3.8E-04	3.6E-04	3.8%	95.5%
500k	5	1	1.1	0.1	0.025	6.3E-04	-3.6E-05	5.2E-04	3.1E-03	-501.1%	95.6%
500k (Extreme cases removed)	5	1	1.1	0.1	0.025	6.3E-04	-2.4E-05	3.7E-04	3.6E-04	2.8%	95.6%

Overall, for Case 2.1, with low percentage variance difference and high confidence interval coverage, the theoretical variance derived using second-order Taylor series approximation and ignored terms are reasonable. Also, the improvement in percentage variance difference in Case 2.1 compared to Case two is not due to a random chance since the difference remained the same even when the number of trials is increased.

Case 3:

The results for Case three are shown in Table 14.

Table 14: Case 3 Estimator Variance and Confidence Interval Coverage

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
-	10	20	0.5	0.2236	5.0E-02	5.0E-02	1.2E-02	1.2E-02	0.2%	94.9%
1	1	10	0.1	0.025	6.3E-04	6.1E-04	9.1E-06	9.2E-06	-1.2%	95.0%
2	1	10	0.1	0.005	2.5E-05	1.2E-05	8.6E-06	8.7E-06	-1.2%	95.0%
3	1	10	0.001	0.00025	6.3E-08	6.2E-08	8.9E-14	8.9E-14	0.9%	94.8%
4	1	10	0.001	0.00005	2.5E-09	1.9E-09	8.4E-14	8.4E-14	0.9%	94.8%
5	1	1.1	0.1	0.025	6.3E-04	-2.6E-05	3.7E-04	7.1E-04	-90.1%	98.6%
6	1	1.1	0.1	0.005	2.5E-05	-5.8E-04	3.3E-04	6.2E-04	-86.4%	98.6%
7	1	1.1	0.001	0.00025	6.3E-08	5.9E-08	3.1E-12	3.0E-12	0.8%	94.9%
8	1	1.1	0.001	0.00005	2.5E-09	-6.8E-10	2.7E-12	2.7E-12	0.8%	94.8%

The Case three results show that the percentage variance difference is less than two percent for all the treatments except for treatment five and six. The confidence interval coverage is also as expected (95 percent) for all the treatments. The exceptions for treatment five and six are due to the same reason discussed in Case two concerning the second-order Taylor series approximation not holding true for these treatments.

Case 3.1:

The Case 3.1 results are shown in Table 15.

Table 15: Case 3.1 Estimator Variance and Confidence Interval Coverage

Treatments	Part Type 1 Mean	Part Type 2 Mean	Part Type 1 SD	Measurement Instrument Precision (SD)	Measurement Instrument Precision (Var)	Intercept		Mean Theoretical variance (B)	% Variance Difference (A-B)/B	Confidence Interval Coverage
						Sample Mean	Sample Variance (A)			
-	10	20	0.5	0.2236	5.0E-02	5.0E-02	1.2E-02	1.2E-02	-0.3%	95.8%
1	1	10	0.1	0.025	6.3E-04	6.1E-04	9.1E-06	9.3E-06	-1.5%	95.9%
2	1	10	0.1	0.005	2.5E-05	1.2E-05	8.6E-06	9.3E-06	-7.8%	95.9%
3	1	10	0.001	0.00025	6.3E-08	6.2E-08	8.9E-14	8.9E-14	0.8%	95.6%
4	1	10	0.001	0.00005	2.5E-09	1.9E-09	8.4E-14	8.4E-14	0.8%	95.7%
5	1	1.1	0.1	0.025	6.3E-04	-2.6E-05	3.7E-04	1.3E-03	-242.0%	99.8%
6	1	1.1	0.1	0.005	2.5E-05	-5.8E-04	3.3E-04	1.0E-03	-214.0%	99.7%
7	1	1.1	0.001	0.00025	6.3E-08	5.9E-08	3.1E-12	3.0E-12	0.8%	95.7%
8	1	1.1	0.001	0.00005	2.5E-09	-6.8E-10	2.7E-12	2.7E-12	0.8%	95.7%

The results are similar to Case 3, but with more percentage variance difference for treatment five and six. This is expected and due to the reason discussed in Case two as the second-order Taylor series approximation does not hold for these treatments. For the rest of the treatments, the percentage variance difference is less, and the confidence interval coverage is more than 95 percent for all the treatments.

Considering all the cases in scenario two, it can be concluded that the theoretical variance equation derived using the second-order Taylor series approximation and by approximating terms to zero with negligible value are reasonable. The measuring instrument precision confidence interval also has an expected 95 percent coverage.

5. CONCLUSION AND FUTURE WORK

5.1 Conclusions

In this research, a new assumption to estimate measuring instrument precision in destructive testing is proposed. The assumption is to sample data from two-part types with a constant coefficient of variation. Using the constant coefficient of variation assumption, a new methodology is developed. Using this methodology, the intercept of the line through the points plotted by part type means squares on the x-axis, and part type sample variances on the y-axis is an estimator for measuring instrument precision. Through the mathematics of the new method, the estimator's properties- expected value, theoretical variance, and a bias for the variance estimate are derived. Through the derivation, it is shown that the intercept is an unbiased estimator for measuring instrument precision. The theoretical variance derived can be used to calculate the measuring instrument precision estimate's confidence interval. The estimator's properties are derived assuming part type means are known, and also when part type means are estimated from the sample. The derived equations are validated against Monte Carlo simulation. A sample space is defined, and the validation is tested for different test cases. The simulation results show that the derived estimate has high confidence interval coverage and is a good estimate for measuring instrument precision in destructive testing.

5.1 Future Research Scope

The following are the recommendations that can be considered to extend this research:

1. The current research focuses on estimating measuring instrument precision when a single operator takes measurements using a single measuring instrument. One extension of this research is to analyze using the assumption proposed in the research, how measuring instrument precision can be estimated when multiple operators are involved in the study.
2. The assumption used in this research is a constant coefficient variation of part types. This assumption might be extended to part types with a non-constant coefficient of variation. That is to analyze how the measuring instrument precision can be estimated when the part types do not follow a constant coefficient of variation but follow other non-linear relationships.

REFERENCES

Awad, M., Erdmann, T. P., Shanshal, Y., & Barth, B. (2009). A Measurement System Analysis Approach for Hard-to-Repeat Events. *Quality Engineering*, *21*(3), 300–305.

Bergeret, F., Maubert, S., Sourd, P., and Puel, F. (2002). Improving and Applying Destructive Gauge Capability. *Quality Engineering*, *14*(1), 59–66.

De Mast, J., and Trip, A. (2005). Gauge R&R Studies for Destructive Measurements. *Journal of Quality Technology*, *37*(1), 40–49.

Gao, Z., Moore, T., Smith, A. P., Doub, W., Westenberger, B., & Buhse, L. (2007). Gauge repeatability and reproducibility for accessing variability during dissolution testing: A technical note. *AAPS PharmSciTech*, *8*(4), 11–15.

Gorman, D., and Bower, K. M. (2002). Measurement system analysis and destructive testing. Six Sigma Forum Magazine, 1(4), 16–19.

Hamada, M. S., & Borror, C. M. (2012). Analyzing Unreplicated Gauge R&R Studies. *Quality Engineering*, *24*(4), 543–551.

Han, Y., He, Z., 2007. An Applied Study of Destructive Measurement System Analysis, *2nd IEEE Conference on Industrial Electronics and Applications*, Tianjin, China, 30-34.

Jarošová, E. (2018). Destructive R&R study – evaluation problems. Paper presented at the Proceedings of the 17th Conference on Applied Mathematics APLIMAT 2018, Bratislava, Slovakia.

Kang, C. W., Lee, M. S., Seong, Y. J., & Hawkins, D. M., (2007). Using Repeatability and Reproducibility Studies to Evaluate a Destructive Test Method. *Journal of Quality Technology,* 39(2), 151

Kang, C. W., Lee, M. S., Seong, Y. J., & Hawkins, D. M., (2007). A Control Chart for the Coefficient of variation. *Journal of Quality Technology,* 39(2), 151

Kappele W, Raffaldi J (2005) An introduction to gauge R&R. Quality 44(13):24–25

Mikulová, P., Plura, J. ř., & Knop, K. (2020). Repeatability and Reproducibility Studies for Non-Replicable Tests. *System Safety: Human - Technical Facility - Environment*, *2*(1), 275–284.

Phillips, A. R., Jeffries, R., Schneider, J., and Frankoski, S. P. (1997). Using Repeatability and Reproducibility Studies to Evaluate a Destructive Test Method. *Quality Engineering*, *10*(2), 283–290.

Senvar, O., & Oktay Firat, S. U. (2010). An overview of capability evaluation of Measurement Systems and Gauge Repeatability and Reproducibility Studies. *International Journal of Metrology and Quality Engineering*, *1*(2), 121–127.

Sharma, M., Sahni, S. P., & Sharma, S. (2019). Validating a destructive measurement system using Gauge R&R — a case study. *Engineering Management in Production and Services*, *11*(4), 34–42.

Van Der Meulen, F., De Koning, H., & De Mast, J. (2009). Nonrepeatable Gauge R&R Studies Assuming Temporal or Patterned Object Variation. *Journal of Quality Technology*, *41*(4), 426–439.

Z. Gao, T. Moore, A.P. Smith, W. Doub, B. Westenberger, L. Buhse, Gauge repeatability and reproducibility for accessing variability during dissolution testing: a technical note, AAPS PharmSciTech 8 (4) (2007) (article 82).

"Earle M. Jorgensen Company Reference Book." Edited by EMJ Metals, *Earle M. Jorgensem Company,* EMJ Metals, 2007, www.emjmetals.com/pdf_indexer/pdfs/Tolerances_and_Machining_Allowances.pdf.

APPENDICES

Appendix 1: The variance of the product of two independent random variables X and Y

$$\boldsymbol{Var}(\boldsymbol{XY}) = \boldsymbol{E}[(\boldsymbol{XY})^{\boldsymbol{2}}] - (\boldsymbol{E}[\boldsymbol{XY}])^{\boldsymbol{2}}$$
$$= \boldsymbol{E}[\boldsymbol{X}^{\boldsymbol{2}}]\boldsymbol{E}[\boldsymbol{Y}^{\boldsymbol{2}}] - (\boldsymbol{E}[\boldsymbol{XY}])^{\boldsymbol{2}}$$
$$= (\boldsymbol{Var}(\boldsymbol{X}) + (\boldsymbol{E}[\boldsymbol{X}])^{\boldsymbol{2}})\,(\boldsymbol{Var}(\boldsymbol{Y}) + (\boldsymbol{E}[\boldsymbol{Y}])^{\boldsymbol{2}}) - (\boldsymbol{E}[\boldsymbol{XY}])^{\boldsymbol{2}}$$

Appendix 2: Covariance in equation (11)

The covariance term in equation (11) is derived and shown to be small below.

$$Cov\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}s_1^2, \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}s_2^2\right)$$
$$= E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}s_1^2 * \frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}s_2^2\right] - E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}s_1^2\right] * E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}s_2^2\right]$$

Using the independence of part type sample means and part type sample variances, the above equation is solved as below.

$$= E\left[\frac{\bar{x}_2^2\,\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)^2}s_1^2 s_2^2\right] - E\left[\frac{\bar{x}_2^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]E[s_1^2] * E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2 - \bar{x}_1^2}\right]E[s_2^2]$$

This last expression is an approximation, again using $E\left[\frac{\bar{x}_i^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] \approx \frac{\mu_i^2}{\mu_2^2 - \mu_1^2}$. By splitting each term in the above equation and due to the independence of the sample mean and sample variance, the first term in the equation becomes,

$$\approx E\left[\frac{\bar{x}_2^2\,\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)^2}\right]E[s_1^2 s_2^2] - \frac{\mu_2^2}{\mu_2^2 - \mu_1^2}(\sigma_1^2 + \sigma^2) * \frac{\mu_1^2}{\mu_2^2 - \mu_1^2} * (\sigma_2^2 + \sigma^2)$$

The terms within the first bracket are rearranged.

$$\approx E\left[\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)} / \frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right] E[s_1^2 s_2^2] - \frac{\mu_2^2}{\mu_2^2 - \mu_1^2}(\sigma_1^2 + \sigma^2) * \frac{\mu_1^2}{\mu_2^2 - \mu_1^2} * (\sigma_2^2 + \sigma^2) \tag{27}$$

The second-order Taylor series approximation in equation (8) is used to simplify the equation (27). Before the derivation, first, the $E\left[\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right]$ is solved.

$$E\left[\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right] = E\left[\frac{1}{\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}}\right]$$

$$E\left[\frac{1}{\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}}\right] \approx \frac{E[1]}{E[\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}]} - \frac{Cov\left(1, \frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right)}{\left(E\left[\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right]\right)^2} + \frac{Var\left(\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right) E[1]}{\left(E\left[\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right]\right)^3}$$

$$\begin{aligned}\approx \frac{1}{\frac{\mu_1^2}{\mu_2^2 - \mu_1^2}} - 0 \\ &+ \left(\frac{1}{4}\frac{\mu_2^2}{(\mu_2 + \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 + \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 + \mu_1)^2}\right]\right. \\ &\left. + \frac{1}{4}\frac{\mu_2^2}{(\mu_2 - \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 - \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 - \mu_1)^2}\right]\right) \\ &/ \left(\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}\right)^3\end{aligned}$$

$$\begin{aligned}= \frac{\mu_2^2 - \mu_1^2}{\mu_1^2} &+ \left(\frac{1}{4}\frac{\mu_2^2}{(\mu_2 + \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 + \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 + \mu_1)^2}\right]\right. \\ &\left. + \frac{1}{4}\frac{\mu_2^2}{(\mu_2 - \mu_1)^2}\left[\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} - 2\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2(\mu_2 - \mu_1)} + \frac{(\sigma_1^2 + \sigma_2^2 + 2\sigma^2)/n}{(\mu_2 - \mu_1)^2}\right]\right) \\ &/ \left(\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}\right)^3\end{aligned}$$

The second term in the above equation is very small compared to the first term as shown empirically in a structured manner for realistic data for the application considered. Approximating this term to zero we obtain,

$$E\left[\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right] \approx \frac{\mu_2^2 - \mu_1^2}{\mu_1^2} \tag{28}$$

The equation (27) is solved below using Taylor series approximation in equation (8).

$$E\left[\frac{\bar{x}_2^2\ \bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)^2} s_1^2 s_2^2\right] \approx \left(\frac{\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}}{\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}} - \frac{Cov\left(\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)}, \frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right)}{\left(\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}\right)^2} + \frac{Var\left(\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right) * \frac{\mu_2^2}{\mu_2^2 - \mu_1^2}}{\left(\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}\right)^3}\right) E[s_1^2 s_2^2] \tag{29}$$

Assuming $E\left[\frac{\bar{x}_i^2}{\bar{x}_2^2 - \bar{x}_1^2}\right] \approx \frac{\mu_i^2}{\mu_2^2 - \mu_1^2}$ and $E\left[\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right] \approx \frac{\mu_2^2 - \mu_1^2}{\mu_1^2}$, the covariance term in the equation (29) is approximately zero as shown below.

$$Cov\left(\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)}, \frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right) = E\left[\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)} * \frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right] - E\left[\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)}\right] E\left[\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2}\right]$$

$$= E\left[\frac{\bar{x}_2^2}{\bar{x}_1^2}\right] - \left(\frac{\mu_2^2}{\mu_2^2 - \mu_1^2} * \frac{\mu_2^2 - \mu_1^2}{\mu_1^2}\right)$$

$$= Var\left(\frac{\bar{x}_2}{\bar{x}_1}\right) + \left(E\left[\frac{\bar{x}_2}{\bar{x}_1}\right]\right)^2 - \frac{\mu_2^2}{\mu_1^2}$$

By making use of Taylor series approximation (equation (15)) in the above equation to derive $Var\left(\frac{\bar{x}_2}{\bar{x}_1}\right)$ and expected value applied to the prior expression gives,

$$\approx \left(\frac{\mu_2^2}{\mu_1^2}\left(\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} + \frac{(\sigma_1^2 + \sigma^2)/n}{\mu_1^2}\right) + \left(\frac{\mu_2}{\mu_1} + \frac{(\sigma_1^2 + \sigma^2)/n * \mu_2}{{\mu_1}^3}\right)^2\right) - \frac{\mu_2^2}{\mu_1^2}$$

$$= \left(\frac{\mu_2^2}{\mu_1^2} \left(\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} + \frac{(\sigma_1^2 + \sigma^2)/n}{\mu_1^2} \right) + \left(\frac{\mu_2}{\mu_1} \right)^2 + \left(\frac{(\sigma_1^2 + \sigma^2)/n * \mu_2}{{\mu_1}^3} \right)^2 + \left(2 * \frac{\mu_2}{\mu_1} * \frac{(\sigma_1^2 + \sigma^2)/n * \mu_2}{{\mu_1}^3} \right) \right) - \frac{\mu_2^2}{\mu_1^2}$$

$$= \left(\frac{\mu_2^2}{\mu_1^2} \left(\frac{(\sigma_2^2 + \sigma^2)/n}{\mu_2^2} + \frac{(\sigma_1^2 + \sigma^2)/n}{\mu_1^2} \right) + \left(\frac{\mu_2}{\mu_1} \right)^2 + \left(\frac{(\sigma_1^2 + \sigma^2)/n * \mu_2}{{\mu_1}^3} \right)^2 + \left(2 * \frac{\mu_2^2}{\mu_1^4} * (\sigma_1^2 + \sigma^2)/n \right) \right) - \frac{\mu_2^2}{\mu_1^2}$$

$$= \left(\frac{\mu_2^2}{\mu_1^2} \left(\frac{(\sigma_2^2 + \sigma^2)}{n * \mu_2^2} + \frac{(\sigma_1^2 + \sigma^2)}{n * \mu_1^2} \right) + \frac{\mu_2^2}{\mu_1^2} \left(1 + \frac{(\sigma_1^2 + \sigma^2)^2}{(n * \mu_1^2)^2} + \frac{2 * (\sigma_1^2 + \sigma^2)}{\mu_1^2 * n} \right) \right) - \frac{\mu_2^2}{\mu_1^2}$$

$$= \frac{\mu_2^2}{\mu_1^2} \left[\frac{(\sigma_2^2 + \sigma^2)}{n * \mu_2^2} + \frac{(\sigma_1^2 + \sigma^2)}{n * \mu_1^2} + 1 + \frac{(\sigma_1^2 + \sigma^2)^2}{(n * \mu_1^2)^2} + \frac{2 * (\sigma_1^2 + \sigma^2)}{\mu_1^2 * n} \right] - \frac{\mu_2^2}{\mu_1^2}$$

All of the ratios in the square brackets are small relative to one. Hence, the above equation is simplified as below.

$$\approx \frac{\mu_2^2}{\mu_1^2} - \frac{\mu_2^2}{\mu_1^2}$$

$$= 0$$

Now that the covariance term is zero, equation (29) is simplified as below.

$$E\left[\frac{\bar{x}_2^2}{(\bar{x}_2^2 - \bar{x}_1^2)} / \frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2} \right] E[s_1^2 s_2^2] \approx \left(\frac{\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}}{\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}} + \frac{Var\left(\frac{(\bar{x}_2^2 - \bar{x}_1^2)}{\bar{x}_1^2} \right) * \frac{\mu_2^2}{\mu_2^2 - \mu_1^2}}{\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}^3} \right) E[s_1^2 s_2^2]$$

$$= \left(\frac{\frac{\mu_2^2}{\mu_2^2 - \mu_1^2}}{\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}} + \frac{Var\left(\frac{1}{\frac{\bar{x}_1^2}{(\bar{x}_2^2 - \bar{x}_1^2)}} \right) * \frac{\mu_2^2}{\mu_2^2 - \mu_1^2}}{\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}^3} \right) E[s_1^2 s_2^2]$$

$$= \left(\frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2} + \frac{\mu_2^2(\mu_1^2)^3}{(\mu_2^2-\mu_1^2)^4} Var\left(\frac{1}{\frac{\bar{x}_1^2}{(\bar{x}_2^2-\ \bar{x}_1^2)}} \right) \right) E[s_1^2 s_2^2] \tag{30}$$

By making use of Taylor series approximation (equation (15)) to derive the variance term in the above equation (30), and using equation (17)

$$Var\left(\frac{1}{\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}} \right) \approx \frac{(E[1])^2}{\left(E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right]\right)^2} \left[\frac{var(1)}{E[1]^2} - 2\frac{Cov\left(1, \frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right)}{E[1]E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right]} + \frac{Var\left(\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right)}{E\left[\frac{\bar{x}_1^2}{\bar{x}_2^2-\ \bar{x}_1^2}\right]^2} \right]$$

$$= \frac{1}{\left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\right)^2} \left[\frac{\frac{1}{4}\frac{\mu_2^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} -\ 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right]}{\left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\right)^2} + \frac{\frac{1}{4}\frac{\mu_2^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} -\ 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right]}{\left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\right)^2} \right]$$

$$= \frac{1}{\left(\frac{\mu_1^2}{\mu_2^2-\mu_1^2}\right)^4} \left[\frac{1}{4}\frac{\mu_2^2}{(\mu_2+\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} -\ 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2+\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2+\mu_1)^2}\right] + \frac{1}{4}\frac{\mu_2^2}{(\mu_2-\mu_1)^2}\left[\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2^2} -\ 2\frac{(\sigma_2^2+\sigma^2)/n}{\mu_2(\mu_2-\mu_1)} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)/n}{(\mu_2-\mu_1)^2}\right] \right]$$

$$= \left(\frac{\mu_2^2-\mu_1^2}{\mu_1^2}\right)^4 * \frac{1}{4}\left(\frac{(\sigma_2^2+\sigma^2)}{n(\mu_2+\mu_1)^2} -\ 2\frac{(\sigma_2^2+\sigma^2)\mu_2}{n(\mu_2+\mu_1)^3} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)\mu_2^2}{n(\mu_2+\mu_1)^4} + \frac{(\sigma_2^2+\sigma^2)}{n(\mu_2-\mu_1)^2} -\ 2\frac{(\sigma_2^2+\sigma^2)\mu_2}{n(\mu_2-\mu_1)^3} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)\mu_2^2}{n(\mu_2-\mu_1)^4} \right) \tag{31}$$

Substituting the above variance value in equation (30),

$$E\left[\frac{\bar{x}_2^2\ \bar{x}_1^2}{(\bar{x}_2^2 - \ \bar{x}_1^2)^2} s_1^2 s_2^2\right]$$

$$= \left(\frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2} + \frac{\mu_2^2(\mu_1^2)^3}{(\mu_2^2-\mu_1^2)^4}\left(\frac{\mu_2^2-\mu_1^2}{\mu_1^2}\right)^4 * \frac{1}{4}\ \frac{(\sigma_2^2+\sigma^2)}{n(\mu_2+\mu_1)^2} - \ 2\frac{(\sigma_2^2+\sigma^2)\mu_2}{n(\mu_2+\mu_1)^3} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)\mu_2^2}{n(\mu_2+\mu_1)^4} + \frac{(\sigma_2^2+\sigma^2)}{n(\mu_2-\mu_1)^2} - \ 2\frac{(\sigma_2^2+\sigma^2)\mu_2}{n(\mu_2-\mu_1)^3} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)\mu_2^2}{n(\mu_2-\mu_1)^4}\right) E[s_1^2 s_2^2]$$

$$= \left(\frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2} + \boldsymbol{\frac{\mu_2^2}{\mu_1^2} * \frac{1}{4}\left(\ \frac{(\sigma_2^2+\sigma^2)}{n(\mu_2+\mu_1)^2} - \ 2\frac{(\sigma_2^2+\sigma^2)\mu_2}{n(\mu_2+\mu_1)^3} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)\mu_2^2}{n(\mu_2+\mu_1)^4} + \frac{(\sigma_2^2+\sigma^2)}{n(\mu_2-\mu_1)^2} - \ 2\frac{(\sigma_2^2+\sigma^2)\mu_2}{n(\mu_2-\mu_1)^3} + \frac{(\sigma_1^2+\sigma_2^2+2\sigma^2)\mu_2^2}{n(\mu_2-\mu_1)^4}\right)}\right) E[s_1^2 s_2^2]$$

The bold terms in the above equation are small relative to the first term $\frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2}$. Approximating these terms to zero,

$$E\left[\frac{\bar{x}_2^2\ \bar{x}_1^2}{(\bar{x}_2^2 - \ \bar{x}_1^2)^2} s_1^2 s_2^2\right] \approx \frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2}\ E[s_1^2 s_2^2]$$

$$= \frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2}(\sigma_1^2+\sigma^2)(\sigma_2^2+\sigma^2) \tag{32}$$

Finally, substituting equation (32) to equation (27), the covariance term becomes,

$$Cov\left(\frac{\bar{x}_2^2}{\bar{x}_2^2 - \ \bar{x}_1^2} s_1^2, \frac{\bar{x}_1^2}{\bar{x}_2^2 - \ \bar{x}_1^2} s_2^2\right) = \frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2}(\sigma_1^2+\sigma^2)(\sigma_2^2+\sigma^2) - \frac{\mu_2^2\mu_1^2}{(\mu_2^2-\mu_1^2)^2}(\sigma_1^2+\sigma^2)(\sigma_2^2+\sigma^2)$$

$$= 0$$

Hence, based on the above approximation, the covariance term in the variance equation (11) can be approximated to zero.

Appendix 3: Covariances in equation (22)

The covariance in equation (22) can be simplified as shown below.

$$Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)=E\left[\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}*\ \frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right]-E\left[\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right]*E\left[\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right]$$

$$=E\left[\frac{x_1^2}{x_2^2-x_1^2}\right]-E\left[\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right]*E\left[\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right]$$

The first term in the above equation can be simplified as shown below.

$$E\left[\frac{x_1^2}{x_2^2-x_1^2}\right]=-1+\frac{1}{2}*E\left[\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}\right]+\frac{1}{2}*E\left[\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right]$$

$$=-1+\frac{1}{2}*\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\boldsymbol{(\sigma_2^2+\sigma^2)/n}}{\boldsymbol{(\mu_2+\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_2}}{\boldsymbol{(\mu_2+\mu_1)^3}}\right)$$

$$+\frac{1}{2}*\left(\frac{\mu_2}{\mu_2-\mu_1}-\frac{\boldsymbol{(\sigma_2^2+\sigma^2)/n}}{\boldsymbol{(\mu_2-\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_2}}{\boldsymbol{(\mu_2-\mu_1)^3}}\right)$$

The bold terms are small and hence approximating to zero,

$$\approx -1+\frac{1}{2}\frac{\mu_2}{\mu_1+\mu_2}+\frac{1}{2}\frac{\mu_2}{\mu_2-\mu_1}$$

$$=\frac{\mu_1^2}{\mu_2^2-\mu_1^2}$$

The second term in the covariance equation can be simplified as shown below.

$$E\left[\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right]*E\left[\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right]$$
$$=\left(\frac{\mu_1}{\mu_2+\mu_1}-\frac{\boldsymbol{(\sigma_1^2+\sigma^2)/n}}{\boldsymbol{(\mu_2+\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_1}}{\boldsymbol{(\mu_2+\mu_1)^3}}\right)$$
$$*\left(\frac{\mu_1}{\mu_2-\mu_1}+\frac{\boldsymbol{(\sigma_1^2+\sigma^2)/n}}{\boldsymbol{(\mu_2-\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_1}}{\boldsymbol{(\mu_2-\mu_1)^3}}\right)$$

The bold terms are small. Approximating to zero,

$$\approx\frac{\mu_1}{\mu_2+\mu_1}\left(\frac{\mu_1}{\mu_2-\mu_1}\right)$$
$$=\frac{\mu_1^2}{\mu_2^2-\mu_1^2}$$

Substituting the simplified first and second term in the covariance term,

$$Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)=\frac{\mu_1^2}{\mu_2^2-\mu_1^2}-\frac{\mu_1^2}{\mu_2^2-\mu_1^2}$$
$$=0$$

The covariance in equation (22) can be simplified as shown below.

$$Cov\left(\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right)=E\left[\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}*\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right]-E\left[\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}\right]*E\left[\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right]$$
$$=E\left[\frac{x_2^2}{x_2^2-x_1^2}\right]-E\left[\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}\right]*E\left[\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right]$$

The first term in the above equation can be simplified as shown below.

$$E\left[\frac{x_2^2}{x_2^2-x_1^2}\right]=1-\frac{1}{2}*E\left[\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right]+\frac{1}{2}*E\left[\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right]$$

Applying Taylor series approximation for $E\left[\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)}\right]$ and $E\left[\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right]$

$$\approx 1-\frac{1}{2}*\left(\frac{\mu_1}{\mu_2+\mu_1}-\frac{\boldsymbol{(\sigma_1^2+\sigma^2)/n}}{\boldsymbol{(\mu_2+\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_1}}{\boldsymbol{(\mu_2+\mu_1)^3}}\right)+\frac{1}{2}$$
$$*\left(\frac{\mu_1}{\mu_2-\mu_1}+\frac{\boldsymbol{(\sigma_1^2+\sigma^2)/n}}{\boldsymbol{(\mu_2-\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_1}}{\boldsymbol{(\mu_2-\mu_1)^3}}\right)$$

The bold terms are small and hence approximating to zero,

$$\approx 1-\frac{1}{2}*\left(\frac{\mu_1}{\mu_2+\mu_1}\right)+\frac{1}{2}*\left(\frac{\mu_1}{\mu_2-\mu_1}\right)$$
$$=\frac{\mu_2^2}{\mu_2^2-\mu_1^2}$$

The second term in the covariance equation can be simplified as shown below.

$$E\left[\frac{\bar{x}_2}{(\bar{x}_2+\bar{x}_1)}\right]*E\left[\frac{\bar{x}_2}{(\bar{x}_2-\bar{x}_1)}\right]$$
$$=\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\boldsymbol{(\sigma_2^2+\sigma^2)/n}}{\boldsymbol{(\mu_2+\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_2}}{\boldsymbol{(\mu_2+\mu_1)^3}}\right)$$
$$*\left(\frac{\mu_2}{\mu_2-\mu_1}-\frac{\boldsymbol{(\sigma_2^2+\sigma^2)/n}}{\boldsymbol{(\mu_2-\mu_1)^2}}+\frac{\boldsymbol{((\sigma_1^2+\sigma_2^2+2\sigma^2)/n)\mu_2}}{\boldsymbol{(\mu_2-\mu_1)^3}}\right)$$

The bold terms are small. Approximating to zero,

$$\approx\frac{\mu_2}{\mu_1+\mu_2}\left(\frac{\mu_2}{\mu_1+\mu_2}\right)$$
$$=\frac{\mu_2^2}{\mu_2^2-\mu_1^2}$$

Substituting the simplified first and second term in the covariance term,

$$Cov\left(\frac{\bar{x}_1}{(\bar{x}_2+\bar{x}_1)},\frac{\bar{x}_1}{(\bar{x}_2-\bar{x}_1)}\right)=\frac{\mu_2^2}{\mu_2^2-\mu_1^2}-\frac{\mu_2^2}{\mu_2^2-\mu_1^2}$$
$$=0$$

Appendix 4: Simulation R Code

The R code for Scenario two, Case 2.1 with Treatment one as inputs is given below. The same R code can be modified to run for the rest of the validations.

```
```{r}
start.time <- Sys.time()
set.seed(1)
#Inputs
n = 50
trials = 100000

mean1 = 1
mean2 = 10
sd1 = 0.1
var1 = sd1^2
repeatability = 0.025
rep = repeatability^2

CV = sd1/mean1
sd2 = CV*mean2
var2 = sd2^2
mean1_2 = mean1^2
mean2_2 = mean2^2

#Output DataFrame
df <- data.frame()

#Trial Loop
for (ii in 1:trials)
{
  #Sample1
```
```

```
    measerror_df1 <- data.frame(no=1, mean = rnorm(n = n, mean = mean1, sd = sd1),
meas_var = rnorm(n = n, mean = 0, sd = repeatability))
    measerror_df1$meas_val <- ( measerror_df1$mean + measerror_df1$meas_var )
    smean1 = mean(measerror_df1$meas_val)
    smean1_2 = smean1^2
    svar1 = var(measerror_df1$meas_val)

    #Sample2
    measerror_df2 <- data.frame(no=2, mean = rnorm(n = n, mean = mean2, sd = sd2),
meas_var = rnorm(n = n, mean = 0, sd = repeatability))
    measerror_df2$meas_val <- ( measerror_df2$mean + measerror_df2$meas_var )
    smean2 = mean(measerror_df2$meas_val)
    smean2_2 = smean2^2
    svar2 = var(measerror_df2$meas_val)

    #InputDataFrame
    measerror_df <- data.frame(rbind(measerror_df1, measerror_df2))
   #Intercept Estimate
   intercept_e <- ((smean2_2*svar1)-(smean1_2*svar2))/(smean2_2-smean1_2)



    #Estimated Estimator Variance
    var_theory = ( 2*( ((svar1)^2 * smean2^4) + ((svar2)^2 * smean1^4) ) )/ ( ((n-1) *
(smean2^2 - smean1^2)^2) )

    #Variance Bias Correction
    var_theory = var_theory*(n-1)/(n+1)

    #Confidence Interval
    e_CI_start = intercept_e - (1.96*sqrt(var_theory))
    e_CI_end   = intercept_e + (1.96*sqrt(var_theory))

   #Confidence Interval Coverage
```

```
 if (rep>=e_CI_start & rep<=e_CI_end){
    e_coverage = 1
  } else { e_coverage = 0
}

  #Output data frame
  Result <- data.frame(
  intercept_e=intercept_e, e_coverage=e_coverage, var_theory=var_theory)

  #Result
  df <- rbind(df, Result)

  }
end.time <- Sys.time()
print(end.time-start.time)

#OUTPUT
apply(df, 2, mean)
apply(df, 2, var)
```
```
```