PURE TORSION OF A SLOTTED CHANNEL MEMBER

by

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LIST OF SYMBOLS

\( \alpha \) ................. Numerical coefficient

\( \theta \) .................. Angle of twist per unit length

\( \sigma \) .................. Unit normal stress

\( \tau \) .................. Unit shear stress

\( \tau_{yx}, \tau_{zx} \) .......... Unit shear stress on planes perpendicular to the \( y \) and \( z \) axes, and parallel to the \( x \) axis

\( \phi \) .................. Angle of twist

\( \phi', \phi'', \phi''' \) ....... First, second, and third derivatives of \( \phi \) with respect to \( x \)

\( A \) .................. Integration constant

\( a, b, d, g, j, s \) .......... Distances

\( C \) .................. Torsional rigidity

\( c \) .................. Subscript indicating channel

\( C_l \) .................. Warping rigidity

\( D \) .................. Diameter of circle, flexural rigidity

\( e \) .................. Distance from web to shear center

\( E \) .................. Modulus of elasticity

\( f \) .................. Subscript indicating flange

\( G \) .................. Modulus of elasticity in shear

\( h \) .................. Height of web

\( I \) .................. Moment of inertia of an area with respect to its neutral axis, subscript indicating channel section
II. ................ Subscript indicating slotted section
III. ............... Subscript indicating channel section
\( k^2 \) ............. Ratio of \( C/C_1 \)
K ..................... Torsion constant
L ...................... Length of channel beam
M ...................... Bending moment
Q ...................... Static moment of an area with respect to its neutral axis
r ...................... Radius, distance
S ...................... Axial force
s ...................... Subscript indicating slot
T ...................... Torque
t ...................... thickness
V ...................... Shearing force, end constants
w ...................... Stress function, subscript indicating web
x, y, z, .............. Rectangular coordinates
PURE TORSION OF A SLOTTED CHANNEL MEMBER

I. INTRODUCTION

Structural members of non-circular and open cross-section such as channels, wide flange and I beams are not usually designed to carry torsional loads. However, there are situations where these members are subjected to torsion. For instance, whenever the applied lateral loads do not pass through the shear center, the above members will twist. Consequently, the problem of torsion of beams with non-circular and open cross-section has been studied for many years.

In fact, the analysis of pure torsion* as applied to non-circular cylindrical members was first treated correctly by Saint Venant in 1855, and his general solution is applicable to any cross-section. In 1903 Prandtl showed that, if a thin membrane were stretched across a hole having the same shape as the cross-section in question and distended by being subjected to a slight difference of pressure on its two sides, the differential and boundary equations governing the deflection of its surface have the same form as the general differential and boundary equations involved in the torsion

* If a bar is twisted by couples applied at the ends and acting in planes normal to the axis of the bar, and if the ends of the bar are free to warp, we have the case of pure torsion (7, p. 212).
problem. Prandtl also showed that the torsional rigidity and shearing stress can be calculated directly from measurements of the volume and slopes of the displaced membrane. Prandtl's membrane analogy has been used in several torsion investigations (3, p. 271-277).

Although pure torsion of a slotted channel beam which has a discontinuous cross-section along the length of the beam may not be a common practical problem, it is conceivable that such a problem could arise. With this in mind, Mr. Mark Levinson, Assistant Professor of Mechanical Engineering, (on leave of absence), first suggested the problem, and the Engineering Experiment Station has sponsored this investigation.

This investigation was done to determine the effect on the torsional rigidity of a channel member loaded in pure torsion with different sizes of rectangular slots cut in the web. The width of the slot was equal to the height of the web; the other side varied in length.

Objectives of Investigation

The first objective was to develop an equation which gives the angle of twist of the slotted channel as a function of distance along the beam length and applied torque. To do this a strength of
materials approach was used. The stresses were not studied in this investigation; however, equations for the stresses are given, but they are not expected to be valid near the ends of the slot.

The second objective was to verify the analysis by applying torque to the slotted channel beam and comparing the measured values of the angles of twist along the length of the beam to the predicted values.

Finally, the results of the analytical and experimental work are intended to be used to evaluate the effect of the slot in the web on the torsional rigidity of the member.
II. THE TORSION THEORY

Part I. Uniform Torsion

The relation between torque and twist for any member subjected to torsion involves a constant, the value of which depends upon the material and the shape of the cross-section. The analysis required to determine this constant and to predict stresses will be outlined in this part.

General Problem

The analysis of torsion-induced stress in a bar of arbitrary cross-section consists primarily in determining the distribution of longitudinal displacements over the cross-section. The lateral shearing stress will be of uneven distribution, except in the case of the circular section, and as a result, the cross-section will be warped or displaced perpendicular to its plane during twisting.

It may be shown that the lateral displacements are proportional to the angular twist and to the distance from the twisting axis (as is the case in a circular section). The longitudinal displacement (warping) and the resulting distribution of shearing stress are described in terms of a stress function, \( w(y,z) \), \( y \) and \( z \) being
coordinates in the plane of the cross-section. This function must satisfy the differential equation:

\[ \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = -2G\theta, \]  

(1)

in which \( G \) equals the shearing modulus of elasticity, and \( \theta \) equals the unit angle of twist in radians per inch of length. It may be shown further that \( w \) must be a constant along the boundary of the cross-section and that the torque, \( T \), is related to \( \theta \) by the equation:

\[ T = KG\theta, \]  

(2)

in which \( K \) equals the torsion constant (6, p. 258-263).

The torsion constant, \( K \), may be determined from test results by observing the ratio of torsional moment to unit twist in radians per inch, at any load level below the yield point of the beam. \( K \) is equal to the polar moment of inertia for circular sections. Although for non-circular sections it is always less than the polar moment of inertia, there is no direct relation between the two factors.

If the boundary shape is such that \( w \) may be determined analytically from equation (1), it is possible to evaluate the torsion...
constant of the section and find the stresses at any point in the cross-section. Formulas for the torsion constant and maximum shearing stresses have been derived in this manner for such sections as the square, rectangle, ellipse, equilateral triangle, and sector of a circle.

Numerical results have been obtained in the past by application of Prandtl's membrane analogy, thereby overcoming the difficulties of the mathematical solution to equation (1).

In the experimental application of this analogy, a soap film is stretched across an opening having the same shape as the cross-section under consideration. The bubble is distended slightly by a variation in pressure. Prandtl showed that the following relations hold for this bubble: (1) The torsion constant, \( K \), is proportional to the total volume of the displaced bubble; (2) the shearing stress at any point is proportional to the maximum slope of the film at that point; and (3) the contour lines on the bubble give the direction of the shearing stress.

The analogy is also useful as an aid in visualizing the rigidity and stress distribution in various sections loaded in uniform torsion.

**The Rectangular Cross-section**

In dealing with structural shapes, two principle types of
cross-section are often of interest, the rectangle, and the rectangle modified by sloping sides. In the case of a rectangle an accurate formula was derived originally by Saint Venant:

\[ K = \frac{bt^3}{3} - 2Vt^4, \quad (3) \]

in which \( t \) equals the thickness of the rectangular section; \( b \) equals the length of the rectangular section; and \( V \) equals a factor depending upon the ratio, \( \frac{b}{t} \), but is practically constant for \( \frac{b}{t} \geq 10 \) (8, p. 85-90).

Equation (3) finds a direct, qualitative interpretation in the soap film analogy. It is evident for long rectangular sections the bubble will be of constant cylindrical cross-section along the central part, but at the two ends it will be contracted and brought down to meet the small side. The quantity, \( -2Vt^4 \), then represents the "end loss", which for long sections is evidently a function of \( t \) only. That is, if the ends of the bubble were made discontinuous, as if they were parts of infinitely long rectangles, one might state without error:

\[ K = \frac{1}{3} bt^3. \quad (4) \]

The Section with Sloping Flanges

Equation (4) provides a basis for evaluating \( K \) for the sloping
flange section. Considering the section shown in Figure 1, let the thickness at any point be taken as \( r \). Then,

\[
dK = \frac{1}{3} r^3 \, dz,
\]

(5)

and since \( K \) represents the total volume enclosed under the de-flected membrane, \( K \) may be obtained by simply summing:

\[
K = \frac{1}{3} \int_{0}^{b} r^3 \, dz.
\]

(6)

Evaluating \( r \) in terms of \( t_1 \) and \( t_2 \), and integrating:

\[
K = \frac{b}{12} (t_2 + t_1)(t_2^2 + t_1^2),
\]

(7)

in which \( t_2 \) equals the major flange thickness and \( t_1 \) equals the minor flange thickness. A deduction must be made for end effects, as in the case of the simple rectangle, thus:

\[
K = \frac{b}{12} (t_2 + t_1)(t_2^2 + t_1^2) - V_2 t_2^4 - V_1 t_1^4,
\]

(8)

in which \( V_2 \) and \( V_1 \), are the end constants, for the large and small ends, respectively, of the flange (5, p. 240-241).
Figure 1. Sloping flange.

Figure 2. Cross-section of channel beam.

Figure 3. Cross-section of the channel beam at the slot.
The foregoing supplies a basis for evaluating the contributions to $K_c$ from the component parts which make up the cross-section of a channel beam.

Considering the section shown in Figure 2, the contribution to $K_c$ from the two flanges is expressed by:

$$K_f = 2 \left( \frac{1}{3} b t_f^3 - 2 V_f t_f^4 \right), \quad (9)$$

in which $K_f$ equals the contribution to $K_c$ from the flanges, and $V_f$ is the end constant for the flange. Consideration of the web in a similar manner gives:

$$K_w = \frac{1}{3} \left( d - 2t_w \right) t_w^3 - 2 V_w t_w^4, \quad (10)$$

in which $K_w$ equals the contribution to $K_c$ from the web; $d$ equals total depth of beam; and $t_w$ equals the thickness of the web (refer to Figure 2). There still remains a contribution to $K_c$ due to the connection of the flange and web and also due to the fillet at this point. It is evident that these will cause a considerable "hump" in the soap bubble.

The contribution of each of these "humps" to the torsion
constant is proportional to the fourth power of the diameter of the largest circle that can be inscribed at the juncture of the web and the flange (2, p. 178). This additional contribution from the two junctures is:

$$K_j = 2 \alpha D^4,$$  \hspace{1cm} (11)

in which \( D \) equals the diameter of the inscribed circle, and \( \alpha \) equals a factor that depends on two ratios, \( \frac{t_w}{t_f} \) and \( \frac{r}{t_f} \).

The various elements entering into the total \( K_c \) value can now be summarized as follows (refer to Figure 2):

$$K_c = K_f + K_w + K_j = 2 \left[ \frac{1}{3} bt_f^3 - 2V_{tf}t_f^4 + \frac{1}{6} (d - 2t_f)t_w^3 - V_w t_w^4 + \alpha D^4 \right].$$  \hspace{1cm} (12)

Simplifying equation (12) gives:

$$K_c = 2 \left[ \frac{1}{3} bt_f^3 - 2V_{tf}t_f^4 + \frac{1}{2} (d - t_f)t_w^3 - V_w t_w^4 + \alpha D^4 \right].$$  \hspace{1cm} (13)

Roark (2, p. 178) gives, for the channel shown in Figure 2, the following equation for \( K_c \):

$$K_c = 2 \left\{ \frac{b t_f^3}{3} \left[ \frac{1}{3} - 0.21 \frac{t_f}{b} \left( \frac{1 - t_f}{4} \frac{t_f}{b} \right) \right] + \left( \frac{d}{2} - t_f \right) t_w^3 \left[ \frac{1}{3} - 0.105 \frac{t_w}{(d - t_f)} \left( \frac{1}{4} \right) \right] + \alpha D^4 \right\},$$  \hspace{1cm} (14)
where \( \alpha \) is given by:

\[
\alpha = \frac{t_w}{t_f} \left( 0.07 + 0.076 \frac{\pi}{d} \right). \tag{15}
\]

When \( \frac{t_f}{b} < 1 \) and \( \frac{t_w}{\left( \frac{d - t_f}{2} \right)} < 1 \), the last term in each bracket in equation (14) is approximately zero. Using this approximation, equation (14) simplifies to:

\[
K_c = 2 \left( \frac{1}{3} b t_f^3 - 0.21 t_f^4 + \frac{1}{3} \frac{(d - t_f)}{t_f} t_w^3 - 0.105 t_w^4 + \alpha D^4 \right). \tag{16}
\]

In comparing equations (13) and (16) it is evident the equations are identical if \( 2V_f \) equals 0.21 and \( V_w \) equals 0.105. Equation (16) was used to calculate \( K_c \) for the channel section shown in Figure 2.

Where the slot is present the web of the channel has been removed, so that the cross-section consists of two equal rectangles, as indicated in Figure 3.

Roark (2, p. 174) gives for such a cross-section, the following equation for \( K_s \):

\[
K_s = 2 \left[ b t_f^3 \left( \frac{1}{3} - 0.21 \frac{t_f}{b} \right) \right]. \tag{17}
\]
The approximation used in obtaining equation (16) was also employed in equation (17). Equation (17) was used to calculate $K_s$ for the slotted section shown in Figure 3.

**Shearing Stresses**

Neglecting stress concentrations at the corners, the maximum shearing stresses which occur in the flanges and the web of the channel section shown in Figure 2 are given by the following equations:

\[
\tau_{\text{max}}^{\text{flange}} = \frac{T t_f}{K_c}, \quad (18)
\]

\[
\tau_{\text{max}}^{\text{web}} = \frac{T t_w}{K_c}. \quad (19)
\]

These stresses act at the surface, along the middle of the sides of the flanges and the web respectively (5, p. 240). Similarly, the maximum shearing stress which occurs in the flanges of the slotted section shown in Figure 3 is given by the following equation:

\[
\tau_{\text{max}}^{\text{flange}} = \frac{T t_f}{K_s}. \quad (20)
\]
Part II. Non-Uniform Torsion

General Behavior

If all the cross-sections in a bar subjected to twisting are free to warp (as has been assumed in Part I) the longitudinal elements (lines) of the surface of the twisted bar remain practically straight lines with negligible change in their lengths unless the angle of twist per unit length is very large and the cross-sections are unusually extended; likewise, longitudinal stresses which accompany the small changes in length in rolled shapes may usually be neglected (3, p. 284).

If however, a bar of thin wall, open section, such as channel, subjected to a twisting moment, has some restraint against warping, then the longitudinal elements of the surface become decidedly curved with marked changes in their lengths, and the accompanying longitudinal stresses in the outer elements of the flanges and the web are not negligible. This prevention of warping during twist (non-uniform torsion) may also have considerable effect on the angle of twist (3, p. 284).

It is evident from Figure 4 that the torsional rigidity of the slotted section is less than that of the channel section. Therefore, a torque applied to the beam will tend to twist the slotted section more than the channel section. This will result in restrained warping of the cross-sections of the beam and will cause the flanges and
Figure 4. Slotted channel beam.
If a transverse force is applied at the shear center of a channel beam, the beam will bend without twisting and if a torque is applied in the plane of the cross-section, the beam will twist without bending the shear center axis. Therefore, the shear center axis of a channel beam remains straight during torsion, and the cross-sections of the beam rotate with respect to that axis. It is evident then, that the simplest way to represent the displacements in the plane of the cross-section is by a rotation about its shear center axis.

For the same reason, the simplest way to represent the displacements in the plane of the slotted section is by a rotation about its shear center axis, which does not coincide with that of the channel section. These displacements will not be the displacements which take place relative to the frame of reference chosen for the unslotted sections. However, they differ only by a rigid body displacement and a rigid body rotation normal to the axis of the channel; hence, the resulting expressions for stresses and relative rotations in the plane of the cross-section will be correct.

Derivation of the Differential Equation for Non-Uniform Torsion

The following assumptions have been made in the derivation
of the differential equation for non-uniform torsion of the slotted channel beam.

1. The material of the beam is continuous, homogenous and isotropic.

2. Hookes Law is obeyed.

3. Displacements of the cross-sections in their own planes consist only of rotations.

4. The lines, in the middle plane of the cross-section (planes mid-way between the flat sides of the flanges and the web) which are parallel and perpendicular to the axis of the beam before twisting remain mutually perpendicular after twisting. This can be shown to be true for any prismatical member of arbitrary, open cross-section loaded in uniform torsion, and it is assumed to be true for non-uniform torsion as well.

5. The unit angle of twist, \( \theta \), is small.

6. The distribution of shearing stresses, due to bending, across the thickness of the flanges and the web are uniform.

7. Bending moments in the flanges and the web, about lines along their respective mid-thicknesses and in the plane of the cross-section, are negligible, since the bending rigidities of these parts in the directions perpendicular to their respective planes is small in comparison to the bending rigidities of these parts in their own planes.

8. The two ends of the beam are twisted between mutually parallel planes and are free to warp in the longitudinal direction of the beam as twisting takes place.

9. The influence of shearing forces on the curvature in the planes of the flanges is negligible.

10. The weight of the beam is neglected.

The twisting moment, \( T \), is transmitted along the member
near the free end mainly by torsional shearing stresses, but near the restrained sections the twisting moment is transmitted by both torsional shearing stresses and lateral shearing stresses which accompany bending of the flanges and the web.

Consider the equilibrium of any section of the channel, such as shown in Figure 5. The external twisting moment is restrained in two ways. First, a twisting moment, $T_1$, results from the torsional shearing stresses which produce the deformations described as uniform torsion. This twisting moment is proportional to the rate of change of the angle of twist along the length of the beam. The value of $T_1$ is found from equation (2):

$$T_1 = K_c G \frac{d\phi}{dx},$$

in which $d\phi/dx$ equals the unit angle of twist. Secondly, a twisting moment, $T_2$, is produced by the lateral shearing forces which accompany bending in the flanges:

$$T_2 = Vh.$$

The shear force, $V$, is related, through equilibrium relationships, to the changes in bending moments in the flanges and web. These bending moments are in turn related to the angular twist $\phi$, since
Figure 5. Channel section of slotted channel beam.
the curvatures in the flanges and web can be expressed in terms of \( \phi \).

Since the flanges bend, a shearing force, \( V_1 \), is set up in each flange. The shearing stresses which make up this shearing force have a parabolic distribution over the cross-section of each flange. \( V_1 \) is given by the following equation:

\[
V_1 = \frac{dM_f}{dx} = -EI_f \frac{d^3 z}{dx^3}, \quad (23)
\]

in which \( E \) equals the modulus of elasticity and \( I_f \) equals the moment of inertia of one flange about its neutral axis. It is evident from Figure 5 that:

\[
z = \frac{h}{2} \phi. \quad (24)
\]

Substitution of the relation (24) into equation (23) gives:

\[
V_1 = \frac{dM_f}{dx} = -EI_f \frac{h}{2} \frac{d^3 \phi}{dx^3}. \quad (25)
\]

Since the web also bends, this causes an additional shearing force, \( V_2 \), to be set up in each flange. The shearing stresses which make up this shearing force have a linear distribution over the cross-section of each flange. \( V_2 \) is given by the following
The total shear force $V$ is the sum of the two effects $V_1$ and $V_2$ mentioned above:

$$V = V_1 + V_2$$

Substitution of the expressions (30), (29), (28), and (25) into
equation (22) gives the torque due to non-uniform torsion:

\[ T_2 = - \frac{D_t h^2}{2} \left[ 1 + \frac{t_w h^3}{4I_z} \right] \frac{d^3 \phi}{dx^3}, \]  

(31)

in which \( D_t \) equals \( \frac{E t_f b^3}{12} \).

Equations (21) and (31) express separate torques implied by the twist \( \phi(x) \), the St. Venant torque being related to the first derivative of \( \phi \), and the torque due to bending of the flanges and the web being related to the third derivative of \( \phi \). It follows from the principle of superposition that the total torque at any section is simply the sum of these two torques:

\[ T = T_1 + T_2 = C_1 \frac{d\phi}{dx} - C_{11} \frac{d^3 \phi}{dx^3}, \]  

(32)

in which \( C_1 \) equals \( K_c G \) and \( C_{11} \) equals \( D_t h^2 \left[ 1 + \frac{t_w h^3}{4I_z} \right] \).

The differential equation for non-uniform torsion of any thin-walled, open section will have the same form as equation (32), with the constants \( C_1 \) and \( C_{11} \) being different expressions depending on the geometry of the cross-section. For instance, for the slotted section, \( C_1 \) reduces to \( K_s G \) and \( C_{11} \) reduces to \( D_t h^2 \), since \( t_w \) equals zero. Then in general, the differential
equation for non-uniform torsion of thin-walled, open sections may be expressed:

\[ T = C \frac{d\phi}{dx} - C_1 \frac{d\phi}{dx}^3, \quad \text{(33)} \]

in which \( C \) and \( C_1 \) depend upon the material and geometry of the cross-section.

Solution of equation (33) will give the angle of twist, \( \phi \), as a function of beam length and applied torque, and permit evaluation of the torsional rigidity of the slotted channel member.

Equations for the Stresses (5, p. 260-264)

When the slotted channel beam is loaded in pure torsion, there are shearing stresses and bending stresses in the flanges and the web.

Shearing Stresses

The shearing stresses may be divided into two types. Those of the first type are torsional shearing stresses resulting from uniform torsion. Neglecting stress concentrations, the maximum torsional shearing stresses which occur in the flanges and the web are given by the following equations:

\[ \tau_{\text{max}} \] _\text{flange} = Gt \frac{d\phi}{dx}, \quad \text{(34)} \]
\[ \tau_{\text{max}}_{\text{web}} = Gt \frac{d\sigma}{dx} \]  

Those of the second type are shearing stresses resulting from a change in bending moment in the flanges and the web. These shearing stresses are shown in Figure 6, acting at the junctions of the flanges and the web. First, consider the shearing stresses, \( \tau_{yx} \), in the web of the channel section shown in Figure 4.

Considering the equilibrium of the shaded element between two adjacent cross-sections \( mn \) and \( m_1n_1 \), Figure 7, gives:

\[ \tau_{yx} t_w dx - (\tau_{yx})_o t_w dx - \frac{dM_w}{dx} \frac{Q_w}{T_w} dx = 0, \tag{36} \]

in which \( Q_w \) is the static moment, with respect to the z axis, of the shaded portion of the cross-section of the web. Defining \( S \) by the following equation,

\[ \frac{dS}{dx} = (\tau_{yx})_o t_w = - (\tau_{zx})_o t_f, \tag{37} \]

noting that

\[ \frac{dM_w}{dx} = - (\tau_{yx})_o t_w h = - \frac{dS}{dx} h, \tag{38} \]

and substituting equation (37) and (38) into equation (36) gives
Figure 6. Flanges and web of channel section.

Figure 7. Web of channel section.
the following equation for the shearing stresses in the web:

\[
\tau_{yx} = \frac{1}{t_w} \frac{dS}{dx} \left[ 1 - \frac{h}{i_w} Q_w \right].
\]  

(39)

By integrating equation (39) over the area of the web, one can conclude that there is no net vertical shearing force in the web. This is reasonable since there are no external vertical forces applied to the beam.

The axial force, \( S \), in the flange may be found by substituting equations (28) and (29) into equation (38) and integrating. This results in the following equation for \( S \):

\[
S = \frac{Et_w t_f h^4 b^2}{4I_z} \frac{d^2 \phi}{dx^2}.
\]  

(40)

Next, consider the shearing stresses, \( \tau_{zx} \), in the flange of the channel section. Considering the equilibrium of the shaded elements between the cross-sections \( mn \) and \( m_1 n_1 \), Figure 8, gives:

\[
\tau_{zx} t_f dx - (\tau_{zx})_0 \left( \frac{b-z}{b} \right)_f t_f dx - \frac{dM_f}{dx} Q_f \frac{dx}{I_f} = 0,
\]  

(41)

in which \( Q_f \) is calculated for the flange in the same manner as \( Q_w \) is calculated for the web. Substituting equations (25) and (37) into
Figure 8. Flange of channel section.

Figure 9. Flange of slotted section.
equation (41) gives the following equation for the shearing stresses in the flange:

\[ \tau_{zx} = -\frac{1}{t_f}\frac{dS}{dx}\left(\frac{b-z}{b}\right) - \frac{Eh}{t_f^2}Q_f\frac{d^3\phi}{dx^3} \]  

Now consider the shearing stresses, \( \tau_{zx} \), in the flange of the slotted section in Figure 4. Considering the equilibrium of the shaded element shown in Figure 9 gives:

\[ \tau_{zx f}\frac{dM_f}{dx} - \frac{Q_f}{t_f}dx = 0. \]  

Substituting equation (25) into equation (43) gives the following equation for the shearing stresses in the flange:

\[ \tau_{zx} = -\frac{Eh}{t_f^2}Q_f\frac{d^3\phi}{dx^3} \]  

Bending Stresses

To calculate the bending stresses in the flanges and the web respectively, the following equation may be used:

\[ \sigma = \frac{Ms}{I} \]  

When applying equation (45) to the flange, $M$ may be found from equation (25); $s$ is measured from the neutral axis of the flange to the point under consideration and $I$ equals $I_f$. Similarly, for the web, $M$ may be found from equation (28); $s$ is measured from the neutral axis of the web and $I$ equals $I_w$.

It should be noted that the bending stresses in the web, as implied by equation (45) and the fourth assumption, can not exist at the ends of the slot, since this surface is stress-free. However, this discrepancy will effect the validity of the predicted stresses and deformations only locally and will have little effect on the overall torsional rigidity of the member.

Since a complicated stress distribution exists near the ends of the slotted section, the above equations for the stresses are not expected to be valid in this region.

Solution of the Differential Equation for Non-Uniform Torsion

Equation (33) will now be solved for the angle of twist, $\phi$.

Equation (33) is a linear, third order, non-homogeneous differential equation; its general solution is given by the following equation:

$$\phi(x) = A_1 + A_2 \sinh kx + A_3 \cosh kx + \frac{Tx}{C}, \quad (46)$$
in which the A's are constants of integration and $k^2 = \frac{C}{C_1}$.

It is convenient to write these constants of integration in terms of quantities defining the conditions at one end of the bar. With the designations $\phi(0) = \phi_0$, $\phi'(0) = \phi'_0$, and $\phi''(0) = \phi''_0$, equation (46) becomes:

$$\frac{\phi(x)}{T} = \frac{\phi_0}{T} + \frac{\phi'_0}{T} \frac{1}{k} \sinh kx + \phi''_0 \frac{1}{T} \frac{1}{k^2} \left[ \cosh kx - 1 \right] + \frac{1}{C} \left[ \frac{x-1}{k} \sinh kx \right].$$

(47)

Applying equation (47) to the first, second, and third sections, respectively, shown in Figure 4 gives:

$$\frac{\phi_{\Pi}(x_1)}{T} = \frac{\phi_0}{T} + \frac{\phi'_0}{T} \frac{1}{k_1} \sinh k_1 x_1 + \phi''_0 \frac{1}{T} \frac{1}{k_1^2} \left[ \cosh k_1 x_1 - 1 \right] + \frac{1}{C_1} \left[ \frac{x_1}{k_1} \sinh k_1 x_1 \right],$$

(48)

$$\frac{\phi_{\Pi}(x_2)}{T} = \frac{\phi_a}{T} + \frac{\phi'_a}{T} \frac{1}{k_2} \sinh k_2 x_2 + \frac{\phi''_a}{T} \frac{1}{k_2^2} \left[ \cosh k_2 x_2 - 1 \right] + \frac{1}{C_2} \left[ \frac{x_2}{k_2} \sinh k_2 x_2 \right],$$

(49)
\[
\frac{\phi_{III}(x_3)}{T} = \frac{\phi_{g}}{T} + \frac{\phi_{g}'}{T} \frac{1}{k_1} \sinh k_1 x_3 + \frac{\phi_{g}''}{T} \frac{1}{k_1^2} \left[ \cosh k_1 x_3 - \frac{1}{k_1} \right] \left[ x_3 - \frac{1}{k_1} \sinh k_1 x_3 \right].
\]

Taking \( x_1 = 0 \) as a reference point from which relative rotations are measured, \( \phi_o \) is set equal to zero at that point. Since there is no restraint against warping at that point, \( \phi_{o''} \) is also zero in equation (48).

At \( x = a \) and \( x = a + g \), \( \phi(x) \) must of course be continuous, and compatibility of the flanges demands that \( \phi'(x) \) also be continuous. The in-plane bending moment in each flange is given by:

\[
M_f = -EI_f \frac{d^2 \phi}{dx^2} = -EI_f h \frac{d^2 \phi}{dx^2}.
\]

If these are to be equal, \( \phi'' \) must also be continuous at these points. The fact that the function \( \phi(x) \) satisfies the corresponding differential equation for non-uniform torsion for each section (see equations (32) and (33)), with a common value of \( T \) for all sections, guarantees that the total torque derived from the resulting function \( \phi(x) \) will be the same everywhere.
Since $\phi$, $\phi'$, and $\phi''$ have been shown to be continuous, $\frac{\phi_a}{T}$, $\frac{\phi'_a}{T}$, and $\frac{\phi''_a}{T}$ in equation (49) may be found in terms of $\frac{\phi'_o}{T}$ by evaluating $\phi_1$ and its derivatives at $x = a$, from equation (48). This results in the following equations for $\frac{\phi_a}{T}$, $\frac{\phi'_a}{T}$, and $\frac{\phi''_a}{T}$:

\[
\frac{\phi_a}{T} = \frac{\phi'_o}{T} \sinh k_1 a + \frac{1}{C_1} \left[ a - \frac{1}{k_1} \sinh k_1 a \right], \tag{52}
\]
\[
\frac{\phi'_a}{T} = \frac{\phi'_o}{T} \cosh k_1 a + \frac{1}{C_1} \left[ 1 - \cosh k_1 a \right], \tag{53}
\]
\[
\frac{\phi''_a}{T} = \frac{\phi'_o}{T} k_1 \sinh k_1 a - \frac{1}{C_1} k_1 \sinh k_1 a. \tag{54}
\]

By using equation (49) in a similar manner, the constants $\frac{\phi_0}{T}$, $\frac{\phi'_0}{T}$, and $\frac{\phi''_0}{T}$ may be evaluated in terms of $\frac{\phi'_o}{T}$.

This results in the following equations for $\frac{\phi_g}{T}$, $\frac{\phi'_g}{T}$, and $\frac{\phi''_g}{T}$:

\[
\frac{\phi_g}{T} = \frac{\phi_a}{T} + \frac{\phi'_a}{T} \left[ \frac{1}{k_2} \sinh k_2 g + \frac{1}{k_2} \frac{1}{C_2} \left[ \cosh k_2 g - 1 \right] + \frac{1}{k_2} \frac{1}{C_2} \left[ \frac{1}{k_2} \sinh k_2 g \right] \right]. \tag{55}
\]
\[
\frac{\phi''}{g} = \frac{\phi'}{T} \cosh k_2 g + \frac{\phi''}{T} \frac{1}{k_2} \sinh k_2 g + \frac{1}{C_{II}} \left[ 1 - \cosh k_2 g \right], \quad (56)
\]

\[
\frac{\phi'''}{T} = \frac{\phi'}{T} k_2 \sinh k_2 g + \frac{\phi''}{T} \cosh k_2 g - \frac{1}{C_{II}} k_2 \sinh k_2 g. \quad (57)
\]

Substituting equations (52), (53), (54), (55), (56), and (57) into equation (50) and applying the third and last boundary condition, \( \phi'' = 0 \) at \( x_3 = j \), the following equation for the constant, \( \phi'_0 \), is obtained:

\[
\frac{\phi'_0}{T} = \frac{k_1 \sinh k_1 j}{C_I} \left\{ \cosh k_1 a \cosh k_2 g + \frac{k_1}{k_2} \sinh k_1 a \sinh k_2 g \right\} + k_2 \cosh k_1 j \left\{ \cosh k_1 a \sinh k_2 g + \frac{k_1}{k_2} \sinh k_1 a \cosh k_2 g \right\} =
\]

\[
- \frac{1 - \cosh k_1 a}{C_I} \cosh k_2 g - \frac{k_1}{k_2} \frac{1}{C_I} \sinh k_1 a \sinh k_2 g +
\]

\[
- \frac{1 - \cosh k_2 g}{C_{II}} \frac{-1}{C_I} \frac{k_1}{C_I} \sinh k_1 j - \frac{1 - \cosh k_1 a}{C_I} \sinh k_2 g
\]

\[
- \frac{k_1}{k_2} \frac{1}{C_I} \cosh k_2 g - \frac{1}{C_{II}} \frac{k_2}{C_I} \sinh k_2 g \frac{k_2}{k_1} \cosh k_1 j. \quad (58)
\]
Since slots symmetrical with respect to the beam length were considered in the investigation, it is necessary to use only equations (48) and (49) to calculate the angles of twist at any distance along the length of the beam. However, for the case of asymmetry, equations (48), (49), and (50) must be used.

In calculating the angles of twist along the beam length for the particular beam used in this investigation, equation (58) was evaluated for the constant \( \frac{\phi'}{Q} \). Then \( \phi(x) \) was evaluated from equations (48) and (49). This was done for each slot size. The results of these calculations are shown in Figure 10 and Tables 2, 3, and 4 in the appendix.

The torsional rigidity of the slotted channel beam, \( C_{sc} \), may be expressed by:

\[
C_{sc} = \frac{TL}{\phi(L) - \phi(0)}. \tag{59}
\]

Equation (49), evaluated at \( x = g/2 \), was used to calculate the theoretical values of the torsional rigidity. The results of the calculations are shown in Figure 11 and Table 5 in the appendix.
Figure 10. Predicted curves of $\phi/T$ vs. $x/L$ for slotted channel beam, 44" long, used in torsion tests.
Figure 11. Plot of torsional rigidity, $C_{sc}$, vs. $k_1 a^2/L$ for slotted channel beam, $sc$ 44" long, used in torsion tests.
Apparatus

A structural steel channel beam, with a test section length of 44" and average cross-sectional dimensions of 5.9" x 3" x 0.388", was made by welding together two equal angles. The channel was thoroughly annealed after welding. Rectangular slots cut in the web, symmetrical with respect to the center of the beam, had one side equal to the height of the web; the other side varied in length from 4" to 13".

Torque was applied to the beam through end brackets clamped in the jaws of a Tinius Olsen torsion testing machine (60,000 in.-lb. capacity), Figure 12. Each bracket, Figure 13, was made from a 10" x 10" x 3/4" steel plate, and a 1.5" diameter solid steel bar 6" long. A slot to fit the cross-section of the channel was cut out of the plate, and the bar welded to the plate at the shear center of the cut out cross-section, which was made such that the channel beam was essentially simply supported, that is, such that the ends of the beam could not rotate about a longitudinal axis but were free to warp.

With the ends of the channel beam inserted into the slots in the
Figure 12. Tinius Olsen torsion testing machine.
Figure 13. End bracket.
plate, torque was applied normal to the plane of the cross-section of the channel, while restraint against displacements in the direction of the axis of the channel was minimized.

A gunner's quadrant, Figure 14, was used to measure the angles of twist at various positions along the beam. The total range of the instrument was $\pm 45^\circ$ or 800 mils from the original level position. A vernier permitted reading to 0.1 mil (1 mil = 0.05625°), and the bubble was sensitive to about 0.1 mil. Before the gunner's quadrant was used it was checked against known angles, which were set up on a sine bar by the use of gage blocks. Readings with the gunner's quadrant differed from the angles indicated by the sine bar by 0.2% and 0.04% for angles of 3° and 17° respectively.

The measuring system on the torsion machine, Figure 15, permitted torque reading to 10 in.-lb. and was sensitive to about 5 in.-lb.

Procedure

Before each torsion test, positions along the length of the beam where the angles of twist would be measured were scribed to $\pm 0.02"$.

In order to compare the predicted values of the angles of twist to the measured values, it was necessary to know the numerical
Figure 14. Gunner's quadrant.
Figure 15. Measuring system on torsion machine.
values of $E$ and $G$. $G$ was calculated from equation (2) by assuming equation (16), for the torsion constant, $K_c$, to be correct and measuring values of torque and angular twist for the solid channel beam (no slot). An initial torque of 100 in.-lb. was applied to the beam and increased in 100 in.-lb. increments to 8000 in.-lb. The angles of twist were measured along the length of the beam in 4" increments for each value of torque. The average calculated value of $G$ was $12.8 \times 10^6$ lb./in.$^2$; the maximum variation was 1.14%. $G$ for steel is usually greater than $12 \times 10^6$ lb./in.$^2$, and a value of $12.4 \times 10^6$ lb./in.$^2$ is often used. However, the measured value of $12.8 \times 10^6$ lb./in.$^2$ was used in all calculations.

The values of $E$ for structural steel may vary from $28.5 \times 10^6$ lb./in.$^2$ to $31 \times 10^6$ lb./in.$^2$, and the value of $30 \times 10^6$ lb./in.$^2$ is commonly used. The ratio of $E/G$ for $E = 30 \times 10^6$ lb./in.$^2$ and $G = 12.4 \times 10^6$ lb./in.$^2$ is 2.42; the ratio of $E/G$ for $E = 31 \times 10^6$ lb./in.$^2$ and $G = 12.8 \times 10^6$ lb./in.$^2$ is also 2.42. $E$ was assumed equal to $31 \times 10^6$ lb./in.$^2$, and this value was used in the calculations for predicting the angles of twist and the torsional rigidity of the slotted channel member.

The modulus $E$ has a relatively unimportant role in determining the torsional rigidity of the slotted channel member as supported in this investigation, and has no effect on the torsional rigidity of
the un-slotted channel. Calculations for the channel with an 8" slot, using a value of 28.5 x 10^6 lb./in.^2 for E, gave a value of rigidity, C_{sc}, which was only 0.44% lower than the value calculated with E taken equal to 31 x 10^6 lb./in.^2. It may be noted that a lower value of E will, for most points, result in an even better comparison between the theoretical and experimental results. (See Tables 2, 3, 4, and 5 in the appendix)

The torsion tests for the 4", 8", and 13" slots were performed by applying an initial torque of 500 in.-lb. to the beam and increasing the torque in 500 in.-lb. increments to 5000 in.-lb. The angles of twist were measured along the length of the beam for each value of torque.

The results of these tests were normalized and averaged, and are given in Tables 2, 3, and 4 in the appendix.

The torsional rigidity was calculated for each slot size in the beam from the torsion test data by using the following equation:

$$ C_{sc} = \frac{TL}{2\phi} \left( x = \frac{L}{2} \right) $$

(60)

in which C_{sc} equals the torsional rigidity of the slotted channel.

Normalized values of $\frac{\phi}{T}$ at the center of the beam were used. The results are shown in Figure 11 and in Table 5 in the appendix.
Comparison of Experimental and Theoretical Results

The experimental results, obtained from the torsion tests of the 4", 8", and 13" slotted channel beam, agreed well with the predicted values (See Figures 10 and 11 and Tables 2, 3, 4, and 5 in the appendix.)

The average differences between the measured and predicted values for the angles of twist along the length of the beam for the 4", 8", and 13" slots were 1.07%, 1.06% and 2.27% respectively; the maximum differences were 6.0%, 3.40%, and 3.63%. The results are given in Tables 2, 3, and 4 in the appendix. The above value of 6.0% occurred at x equal to 4" where the angles of twist are small. Consequently, a misreading in the angle of twist by a few tenths of a mil would account for this difference.

The minimum and maximum differences between the calculated and predicted values for the torsional rigidities of the slotted channel beam were 0.0% and 3.09%. The results are shown in Figure 11 and Table 5 in the appendix.
IV. DISCUSSION

The fourth assumption made in deriving the non-uniform torsion equation (see p. 17) implies that the flanges and the web of the channel are equally restrained against warping at the ends of the slotted section. It is this assumption that leads to inconsistencies in the equations for the stresses, because this restraint against warping would require stresses on the free surface of the web at \( x = a \) and \( x = a+g \), which is actually stress-free. Equations (35) and (39) do, in fact, give a shearing stress distribution of the free surface of the web, and equation (45) gives a bending stress distribution. The inconsistency is probably not as pronounced in the flanges, since there is material there which can be expected to provide restraint, which is not too different from that implied by assumption 4.

When the shearing stress given by equation (39) is integrated over the area of the web, the resultant shearing force is zero. Therefore, this force does not contribute anything to balancing the external torque.

Equation (35) gives a uniform (St. Venant) torsional shearing stress acting on the free surface of the web, and these non-existent shearing stresses do contribute to balancing the external torque.
Also, equation (45) gives a bending stress distribution on the free surface of the web which, in itself, does not contribute to balancing the external torque. However, it is the change in this bending stress which results in a linearly distributed shearing stress distribution in the flanges of the channel. This linearly distributed shearing stress, when integrated over the area of the flanges, gives a shearing force which contributes to balancing the external torque.

When considering the slotted section (Section II, Figure 4), the above inconsistencies in the stresses are not apparent since the equation for $\phi(x)$ indicates, as it should, that the internal torques and the bending moments at both ends of section II equal the internal torques and bending moments of their adjacent sections. However, if equations (34), (44), and (45) were applied to evaluate the stresses at the ends of section II, they would be invalid.

It is evident that assumption 4 will give erroneous values for the stresses at the ends of the slotted section but what effect the inconsistencies in the stresses have on the angle of twist, $\phi$, at the ends of the slotted section is not apparent. However, the percent differences between the measured and predicted values of the angle of twist, $\phi$, at the ends of the slotted section for the 4", 8" and 13" slots are shown in the following table:
Table 1. Differences in percent between measured and predicted values for the angles of twist at the ends of the slotted section. (Refer to Tables 2, 3, and 4 in the appendix)

<table>
<thead>
<tr>
<th>Slot size</th>
<th>( x_2 = 0 )</th>
<th>( x_2 = g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4&quot;</td>
<td>0.866</td>
<td>-0.513</td>
</tr>
<tr>
<td>8&quot;</td>
<td>0.131</td>
<td>-1.07</td>
</tr>
<tr>
<td>13&quot;</td>
<td>-1.61</td>
<td>-3.14</td>
</tr>
</tbody>
</table>

From the values in the above table it may be concluded that while the equations developed for the angles of twist imply shearing and bending stresses which are not actually present near the ends of the slot, these equations provide an accurate means for predicting the rotations of the member due to torsion.
V. SUMMARY AND CONCLUSIONS

Summary

The material presented in this paper may be summarized as follows:

1. The essential features of the torsion analysis are presented.

2. Formulas for calculating the torsion constants and maximum shearing stresses in channel and rectangular sections are given.

3. The differential equation for non-uniform torsion is developed.

4. Neglecting stress concentrations, equations for the stresses in the flanges and the web of the slotted channel beam are given. These equations are not expected to be valid near the ends of the slotted section.

5. The differential equation for non-uniform torsion is solved for the angle of twist as a function of beam length and applied torque. The equations for the angles of twist are developed for the case of a slot not necessarily symmetrical with respect to the center of the beam.

6. An equation is given for the torsional rigidity of the slotted channel beam.

7. For the particular beam used in the torsion tests, in which the slots were symmetrical with respect to the center of the beam, the numerical values of the angles of twist at different locations along the length of the beam were calculated using the equations developed for the angles of twist. These values are given in Tables 2, 3, and 4 in the appendix, and they are also presented in Figure 10. Using equation (59) the numerical values of
the torsional rigidity of the slotted channel were also calculated. These values are presented in Figure 11 and Table 5 in the appendix.

8. Torsion tests were performed on the slotted channel beam to verify the equations developed for the angles of twist as a function of beam length and applied torque, and the equation developed for torsional rigidity.

Conclusions

From the results of the torsion tests the following conclusions can be made for a slotted channel beam of thin wall cross-section:

1. The equations developed for the angles of twist as a function of beam length and applied torque may be used to calculate the angle of twist at any position along the length of the beam.

2. Equation (59) may be used to calculate the torsional rigidity of the slotted channel beam.

3. The torsional rigidity of a channel beam is reduced significantly (see Table 6 in the appendix) when a slot is cut in the web; the torsional rigidity decreases as the slot size increases.
BIBLIOGRAPHY


The measured values of $\phi/T$ in Tables 2, 3, and 4 are average normalized values and were obtained from torsion tests of a slotted channel beam. Measurements were taken in 500 in.-lb. increments of torque from 500 in.-lb. to 5000 in.-lb. The predicted values of $\phi/T$ were calculated by using equations (48) and (49).

Table 2. Comparison of the measured and predicted values for the angles of twist along the length of the 4" slotted channel beam.

<table>
<thead>
<tr>
<th>x in inches</th>
<th>$\phi \times 10^{-2}$ mils/in.-lb.</th>
<th>Difference in %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Measured</td>
<td>Predicted</td>
</tr>
<tr>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>0.150</td>
<td>0.160</td>
</tr>
<tr>
<td>8</td>
<td>0.309</td>
<td>0.320</td>
</tr>
<tr>
<td>12</td>
<td>0.470</td>
<td>0.481</td>
</tr>
<tr>
<td>16</td>
<td>0.640</td>
<td>0.644</td>
</tr>
<tr>
<td>17</td>
<td>0.683</td>
<td>0.685</td>
</tr>
<tr>
<td>18</td>
<td>0.716</td>
<td>0.725</td>
</tr>
<tr>
<td>19</td>
<td>0.763</td>
<td>0.767</td>
</tr>
<tr>
<td>20</td>
<td>0.802</td>
<td>0.809</td>
</tr>
<tr>
<td>21</td>
<td>0.843</td>
<td>0.851</td>
</tr>
<tr>
<td>22 (center)</td>
<td>0.888</td>
<td>0.892</td>
</tr>
<tr>
<td>23</td>
<td>0.932</td>
<td>0.934</td>
</tr>
<tr>
<td>24</td>
<td>0.981</td>
<td>0.976</td>
</tr>
<tr>
<td>25</td>
<td>1.018</td>
<td>1.018</td>
</tr>
<tr>
<td>26</td>
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<td>1.060</td>
</tr>
<tr>
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<td>1.100</td>
</tr>
<tr>
<td>28</td>
<td>1.146</td>
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<td>1.304</td>
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<td>1.465</td>
</tr>
<tr>
<td>40</td>
<td>1.643</td>
<td>1.625</td>
</tr>
<tr>
<td>44</td>
<td>1.814</td>
<td>1.785</td>
</tr>
</tbody>
</table>

Average difference = 1.072%
Table 3. Comparison of the measured and predicted values for the angles of twist along the length of the 8" slotted channel beam.

<table>
<thead>
<tr>
<th>x in inches</th>
<th>$\theta \times 10^{-2}$ mils/in. -lb.</th>
<th>Difference in %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Measured</td>
<td>Predicted</td>
</tr>
<tr>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>0.163</td>
<td>0.165</td>
</tr>
<tr>
<td>8</td>
<td>0.330</td>
<td>0.322</td>
</tr>
<tr>
<td>12</td>
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<td>0.501</td>
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<tr>
<td>15</td>
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<td>0.630</td>
</tr>
<tr>
<td>16</td>
<td>0.674</td>
<td>0.675</td>
</tr>
<tr>
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<td>0.718</td>
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<td>18</td>
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<td>0.806</td>
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<td>0.851</td>
</tr>
<tr>
<td>21</td>
<td>0.905</td>
<td>0.895</td>
</tr>
<tr>
<td>22(center)</td>
<td>0.950</td>
<td>0.941</td>
</tr>
<tr>
<td>23</td>
<td>0.998</td>
<td>0.987</td>
</tr>
<tr>
<td>24</td>
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<td>1.032</td>
</tr>
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<td>1.077</td>
</tr>
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<td>1.122</td>
</tr>
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</tr>
<tr>
<td>28</td>
<td>1.224</td>
<td>1.207</td>
</tr>
<tr>
<td>29</td>
<td>1.270</td>
<td>1.253</td>
</tr>
<tr>
<td>32</td>
<td>1.407</td>
<td>1.381</td>
</tr>
<tr>
<td>36</td>
<td>1.578</td>
<td>1.550</td>
</tr>
<tr>
<td>40</td>
<td>1.755</td>
<td>1.717</td>
</tr>
<tr>
<td>44</td>
<td>1.944</td>
<td>1.880</td>
</tr>
</tbody>
</table>

Average difference = 1.063%
Table 4. Comparison of the measured and predicted values for the angles of twist along the length of the 13" slotted channel beam.

<table>
<thead>
<tr>
<th>x in inches</th>
<th>$\phi \times 10^{-2}$ mils/in.-lb.</th>
<th>Difference in %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T Measured</td>
<td>Predicted</td>
</tr>
<tr>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>0.172</td>
<td>0.172</td>
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<tr>
<td>8</td>
<td>0.349</td>
<td>0.346</td>
</tr>
<tr>
<td>12</td>
<td>0.532</td>
<td>0.524</td>
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<tr>
<td>13</td>
<td>0.577</td>
<td>0.569</td>
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<tr>
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<td>0.626</td>
<td>0.615</td>
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<tr>
<td>15</td>
<td>0.675</td>
<td>0.661</td>
</tr>
<tr>
<td>15.5</td>
<td>0.695</td>
<td>0.684</td>
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<tr>
<td>17</td>
<td>0.772</td>
<td>0.755</td>
</tr>
<tr>
<td>18</td>
<td>0.810</td>
<td>0.802</td>
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<tr>
<td>19</td>
<td>0.868</td>
<td>0.850</td>
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<tr>
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<tr>
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<tr>
<td>22(center)</td>
<td>1.024</td>
<td>0.995</td>
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<tr>
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<td>1.075</td>
<td>1.044</td>
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<td>1.369</td>
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<tr>
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<td>1.407</td>
<td>1.376</td>
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<tr>
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<td>1.463</td>
<td>1.422</td>
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<tr>
<td>32</td>
<td>1.515</td>
<td>1.467</td>
</tr>
<tr>
<td>36</td>
<td>1.695</td>
<td>1.645</td>
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<tr>
<td>40</td>
<td>1.884</td>
<td>1.818</td>
</tr>
<tr>
<td>44</td>
<td>2.059</td>
<td>1.991</td>
</tr>
</tbody>
</table>

Average difference = 2.27%
For the measured values of the torsional rigidity of the slotted channel beam, $C_{sc}$, in Table 5, the angles of twist were measured at the center of the beam. The range of torques are the same as those given on the first page of the appendix. The predicted values of $C_{sc}$ were calculated by using equation (59).

Table 5. Comparison of the measured and predicted values for the torsional rigidity of the slotted channel beam.

<table>
<thead>
<tr>
<th>Slot size in inches</th>
<th>$C_{sc} \times 10^6$ lb.-in.$^2$</th>
<th>Difference in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (no slot)</td>
<td>2.62</td>
<td>----</td>
</tr>
<tr>
<td>4</td>
<td>2.52</td>
<td>0.0</td>
</tr>
<tr>
<td>8</td>
<td>2.36</td>
<td>0.84</td>
</tr>
<tr>
<td>13</td>
<td>2.19</td>
<td>3.09</td>
</tr>
<tr>
<td>44 (no web)</td>
<td>----</td>
<td>1.75</td>
</tr>
</tbody>
</table>

Table 6. Percent difference between the measured torsional rigidities of the slotted channel members compared to the un-slotted member.

<table>
<thead>
<tr>
<th>Slot size in inches</th>
<th>$C_{sc} \times 10^6$ lb.-in.$^2$</th>
<th>Difference in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (no slot)</td>
<td>2.62</td>
<td>----</td>
</tr>
<tr>
<td>4</td>
<td>2.52</td>
<td>3.82</td>
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<td>8</td>
<td>2.36</td>
<td>9.93</td>
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<td>13</td>
<td>2.19</td>
<td>16.4</td>
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<tr>
<td>44 (no web)</td>
<td>----</td>
<td>----</td>
</tr>
</tbody>
</table>