Investigating Polymer Quantum Mechanics: Mathematical Formulation, the Particle on a Ring, and Time Evolution

By
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Loop quantum gravity is a theoretical framework which aims to quantize general relativity. One of the unique aspects of this theory is that it imposes a discrete structure on space-time. One toy model of loop quantum gravity is loop quantum cosmology, which maintains the idea of a discrete spatial structure while reducing the allowed degrees of freedom. The assumption of discrete space leads to a complication in quantization, because we are not able to take infinitesimal changes in order to define momentum on a discrete lattice. Thus the model of polymer quantum mechanics was formulated to test whether the results derived from using a discrete formulation of space could still generate verifiable results. Previous researchers have used polymer quantum mechanics on the free particle and harmonic oscillator toy problems in the momentum basis, and have found that in the continuum the results converge to those found in Schrodinger quantization which in turn confirmed the validity of polymer quantum mechanics in these situations. In this paper, I describe the construction and mathematical formulation of polymer quantum mechanics, as well as its application to the particle on a ring, and investigate time evolution of energy eigenstates. Using the derived energy eigenstates, I was able to verify that in the assumption that time is continuous, the Hamiltonian is still the generator of time evolution in polymer quantum mechanics. In addition, I constructed a dispersion relation of a localized superposition of energy eigenstates which corresponds to a single normalized position state. I then used this to calculate the expectation values of both position and momentum and found that position has an expectation value which depends on time. However we found instances which were inconsistent with Schrodinger quantum mechanics, including the normalization of distinct position eigenstates. These specific position eigenstates were formed as a linear superposition of energy eigenvalues, and serves as an interesting case example for where new and interesting results can emerge with different quantization schemes. Furthermore, when we introduce time evolution, it appears that the states transform the same way they would in the continuum, implying that there is some intrinsic connection between the discrete spatial structure and the preservation of time evolution.
Abstract approved:

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I understand that my project will become part of the permanent collection of Oregon State University Honors College. My signature below authorizes release of my project to any reader upon request.

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1 Introduction

1.1 Background

The search for a mathematically rigorous theory of quantum gravity started over a century ago with the emergence of theories such as quantum electrodynamics and particle physics. These theories created testable predictions of the way matter and energy interact at the quantum scale — the goal of quantum gravity is to do the same for gravity. Quantum gravity is an open problem in physics, as our current theories of quantum mechanics are not reconcilable with the theory of general relativity, which for now stands as our classical description of gravity [1]. The objective in quantum gravity research is to create agreement at the Planck scale which is approximately 1.6×10^{-35}m, as the physics of Einstein's general relativity breaks down at this scale.

Amongst the theories of quantum gravity the two most prominent are loop quantum gravity (LQG) — the quantum theory that accompanies loop quantum cosmology (LQC) — and string theory. String theory can be imagined as one dimensional objects moving in space and strives to provide a unification of our fundamental forces. What string theory does well is it provides a good description of perturbation theory, however the non-perturbative regime is not well understood as we do not know the background-independent formulation — meaning that our formulation must have a designated background manifold [2]. LQG can be thought of as one dimensional objects forming space, which strives not for a unification theory, but only a sufficient description of quantum gravity. In LQG, we have a mathematically rigorous, background-independent theory which works at the Planck scale, but it has an incomplete formulation of dynamics and provides no convincing derivation of general relativity in the classical limit [2]. This paper will focus on LQG formulations.

LQG is the theory of gravity that accompanies loop quantum cosmology. The goal of LQC, as described by Abhay Ashtekar, is to have a "theory that incorporates not only the dynamical nature of geometry but also the ramifications of quantum physics" [3]. One of the formulations of LQC does this by trying to apply loop quantum gravity to the universe at the large scale — assuming a spherically symmetric, homogeneous formulation of space which is isotropic — meaning measurements yield

the same result regardless of where they are measured or in what direction. For example, think about a chunk of the universe, containing large galaxy clusters — at the large scale this can appear like a continuous fluid in a cube as demonstrated in Figure 1. The importance of this, is with reduced degrees of complexity, we are able to treat the universe at large as simply a smooth, and continuous fluid.

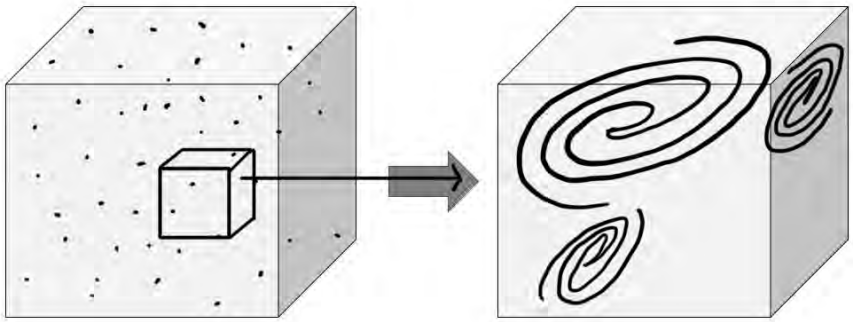


Figure 1: Left image: example of cosmological assumption that the universe is homogeneous. Right image: expanded section of left image to show Galaxy clusters, should we assume that the universe is no longer homogeneous.

This gives us a course-grained model of the universe, in which we can make hypotheses on what occurred at the start of the universe, and even possibly what may happen later in the universe at its collapse.

The main point of LQC as a whole, is to reduce degrees of freedom, so as to be able to perform calculations of a highly complicated and intricate system with some level of accuracy.

We can then, on this model of the universe, apply our quantum theory of gravity — loop quantum gravity. From the discrete nature of LQG came a result in LQC, that the founders Abhay Ashtekar and Parampreet Singh claim in the text [3]: loop quantum gravity is a fundamentally discrete theory both in its formulation and its results — meaning that the theory deals with a discrete spatial background instead of a continuous formulation generally assumed in canonical interpretations of space-time. This result of required discrete space was also mathematically confirmed [3,4].

Due to the requirement of discretization in LQC, those in the field decided to investigate how we can have discrete space and still have results which conquer with those from canonical quantum mechanics derived by Schrodinger and Heisenberg. This idea of discrete space led to the formulation of polymer quantum mechanics (PQM), a theory of quantum mechanics which assumes a discretized spacial structure. This discrete space can be thought of as assigning a fundamental minimal length of space, which physically is related to the Planck length.

However this discretization creates a problem; because momentum is classically defined as the derivative of space and the derivative cannot be well defined on a discrete space. Therefore we are required to find a way to represent these operators such that we can have fully defined conjugate operators in quantum mechanics — this was done via the Weyl algebra. The Weyl algebra is a vector space of exponentiated operators, which are called unitary operators which describe the translations and rotations of physical systems. Another problem which arrives from the discrete nature of space and the undefinedness of the momentum operator is a consequence of the Stone-von Neumann theorem. Stone-von Neumann theorem states, that given certain conditions, all representations of the canonical commutation relations are all equivalent, this restricts the physical interpretations we can make in quantum mechanics. In PQM, the conditions of the Stone-von Neumann theorem are violated as a consequence of our discretized space, and thus we have a quantization scheme which has the ability to yield distinct and interesting results separate from canonical quantization.

1.2 Motivation

In the past, there have been investigations into the application of PQM, in which certain systems have been investigated and compared to the classical quantization by taking a continuum limit — the limiting case in which PQM is approximately the same as the classical quantum mechanics [4]. However there has not been explicit pedagogical treatment of the underlying mathematics used, making these theories not as accessible to the general physicist.

In this paper, we will describe the required background mathematics and physical theory required to properly analyze PQM, in particular we will focus on the Stone

von-Neuman Theorem and the elements necessary to describe what the theorem states. Stone von-Neumann establishes a uniqueness of the classical commutation relationships (CCR), meaning that if you follow the theorem's conditions — also known as axioms — any way in which you represent the CCR will be unitarily equivalent. In violating any or all axioms of this theorem, one can generate new commutation relationships and open the door to potentially very physically interesting formulations of quantum mechanics which are distinct from the canonical quantum mechanics we are used to. Next in this paper we will take the particle on a ring example, apply our methods of polymer quantization to the ring, then develop a time evolution technique then apply it to a dispersion relation and generate expectation values and uncertainties related to the ring and our operators.

This is a project being done in collaboration with another member of the quantum foundations and cosmology group, Maxwell Siebersma, who is investigating the topological structure of PQM and the particle on a ring applications.

1.3 Course of the Thesis

This thesis will start with a description of linear algebra and vector spaces, including Hilbert spaces. The level of this discussion will be geared towards a student in the junior or senior level physics undergraduate. Next we will describe what quantization is, and the importance of phase space in the world of quantum mechanics; this discussion will occur at the level of a senior physics major or a junior undergraduate mathematics student along with the discussion on topological spaces and the Weyl Algebra. Using this mathematical foundation we will describe the Stone-von Neumann theorem in depth, which will lead us into the conversation on PQM.

For the discussion on PQM, we will describe exactly where the relationship between position and momentum comes from in LQC, as well as why the Stone-von Neumann theorem is violated. We will then follow the derivation from Corichi et al. in [5], as well as the tools used by Ashtekar et al. in [4] for the creation of the polymer Hilbert space.

The narrative will then shift to a discussion of the particle on a ring, how we construct the ring, and how we derive the Hamiltonian. As previously stated, this thesis is done in collaboration with Maxwell Siebersma, who derived the energy

eigenstates and the energy of the particle on a ring — we will continue assuming these results to the issue of time in PQM. From the subject of time, we will examine a dispersion relation of a position eigenstate on the ring, and from this wavefunction the expectation values of position as a function of time on the ring.

The original work done in this paper by myself and Max Siebersma is the derivation of the Hamiltonian of the particle on a ring, however the results follow the same formulation as that of the free particle described in [5]. Original work by myself is also in the construction of time evolution via the Hamiltonian, the dispersion relation of the position eigenstate, and the interpretation of the dispursion and position expectation value on the ring system.

2 Mathematics

2.1 Linear Algebra — Vector Spaces

Linear algebra is a field of mathematics that focuses on vector spaces, matrices, and linear transformations. A vector space is a collection of vectors, whose axioms give us regulations for the vectors and the allowed operations. Given we let **v,w** be elements in a vector space V, and $\alpha \in \mathbb{C}$, then the axioms for a vector spcae are as follows [6]:

Commutivity	—	$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ for all **v,w** $\in V$.
Associativity	—	$(\mathbf{v} + \mathbf{u}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$ for all $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$.
Zero Vector	—	There exists $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Additive Inverse	—	For all $\mathbf{v} \in V$ there exists a vector $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. We denote $\mathbf{w}$ as $-\mathbf{v}$.
Multiplicative Identity	—	There exists an element $\mathbb{I} \in V$ such that $\mathbb{I} * \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Multiplicative Associativity	—	For all scalars $\alpha, \beta \in \mathbb{R}$, $(\alpha\beta) * \mathbf{v} = \alpha(\beta * \mathbf{v})$ for all $\mathbf{v} \in V$.
Distribution over vector addition	—	For all $\mathbf{v} \in V$ and for all $\alpha \in \mathbb{R}$, $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + a\mathbf{w}$.
Distribution of scalars	—	For all scalars $\alpha, \beta \in \mathbb{R}$, and for all $\mathbf{v} \in V$, $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Vectors can be the traditionally thought of single row matrix, or it can be anything that satisfies the above requirements, including functions. For example, when we are working in quantum mechanics we are dealing with wavefunctions in a vector space.

For example, we have our n-dimensional Euclidean space $\mathbb{R}^n$, composed of n dimensions in which we have for any two vectors $\psi = (x_1, x_2, ..., x_n)$ and $\phi =$

$(y_1, y_2, ..., y_n)$, we have that vector addition is component wise, giving that

$$\psi + \phi = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \tag{1}$$

and that our scalar multiplication is via distribution,

$$\alpha\psi = (\alpha x_1, \alpha x_2, ..., \alpha x_n). \tag{2}$$

If we were to say that in (2), α was a complex number of the form $\alpha = a + bi$ with $a, b \in \mathbb{R}$, then we would be in $\mathbb{C}^n$ — the n-dimensional complex space.

Now, the more applicable vector space for quantum mechanics are Hilbert spaces, as they are the "mathematical setting for quantum mechanics" [7].

2.1.1 Hilbert Spaces

Hilbert spaces are complex vector spaces equipped with an inner product which holds all possible states in whichever quantum mechanical system we are in. We have addressed complex spaces, but we have not addressed inner products. Inner products is an operation between two elements of the vector space, which is mapped to the complex numbers $\mathbb{C}$, or rather in the representation of bras and kets in our Hilbert space: $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. This would be represented as the inner product for elements $|a\rangle, |b\rangle \in \mathcal{H}$ a vector space of N dimensions as:

$$\langle a|b\rangle = \sum_{n=1}^{N} a_n^* b_n \tag{3}$$

If we are to represent our states as complex valued, continuous functions defined on a space X, and letting the complex conjugate of a function be represented by $f(x)^*$, the inner product would appear as

$$(f(x), g(x)) \coloneqq \int_X f(x)^* g(x) dx. \tag{4}$$

The most important distinctions for inner products is that they produce scalar quantities, and that they are the mathematical representation of taking the projection of one element onto another. In this interpretation, we can see that when we take an

inner product as seen in Equation (3), we are really saying 'how much of the vector a is in the direction of vector b.'

We now have an idea of what our elements could be and their inner product in quantum mechanics, we now wish to discuss operators in our context. A linear operator in mathematics is a function which maps elements of a vector space to elements in the same vector space — expressed as $A : X \to X$ for a vector space X— and which preserves vector addition and scalar multiplication. For example, taking a derivative is a linear operator, or simply multiplying the input by a scalar.

In quantum mechanics, we want a way to represent physical observables such as position, momentum, spin, angular momentum, energy, etc.. These observables are represented by what are called Hermitian operators — linear operators that are equivalent to their transpose and their complex conjugate. For example, if we let $\hat{H}$ be a Hermitian operator in our Hilbert space, then $\hat{H} = \hat{H}^\dagger = \hat{\bar{H}}^\star$, where we use $\hat{H}^\dagger$ to represent the hermitian adjoint of a matrix H, $\hat{H}^\star$ represents the complex conjugate of the matrix H, and $\hat{\bar{H}}$ represents the transpose of a matrix. We know from the study of linear algebra that the eigenvalues of Hermitian matrices are always real, and in quantum mechanics the eigenvalues of our operators which represent physical observables are the actual results of measurements. Thus, by requiring physical observables to be represented by Hermitian matrices, we are guaranteing that we will only have real results of measurement, as desired.

These Hermitian operators have associated eigenvectors and eigenvalues which are related via the eigenvalue-eigenvector equation. For example take an operator $\hat{H}$ with associated eigenvalues E_n and eigenvectors $|n\rangle$, then the eigenvalue-eigenvector equation is $\hat{H}\,|n\rangle = E_n\,|n\rangle$. These eigenvectors are expressed as sets of orthonormal functions — described below. In terms of notation, we will denote these as functions associated with the eigenvalues of the operator. For example, the energy eigenvalues of the harmonic oscillator are $E_n = (n + \frac{1}{2})\hbar\omega$, and the eigenfunctions can be represented by ψ_n, where $n \in \mathbb{N}$ is called the quantum number associated with energy. These functions, ψ_n, live inside the Hilbert space for all $n \in \mathbb{N}$

Some of the most important operators we will use in quantum mechanics are position, momentum, and the Hamiltonian which is the sum of the total kinetic and potential energy of the given system. These are invaluable operators in quantum

mechanics because of the way we choose to quantize a system — discussed in section 2.2 — and because in physics as a whole we analyze the dynamics of systems via the energy of the system, and the relations between kinetic and potential energy. It turns out that time and its effect on systems is very closely associated with the energy of the system itself — this is beautifully represented by the time dependent Schrodinger equation seen in section 4.2.

Another key definition to expand on is orthonormality. For a set of functions f_n to be orthonormal is such that

$$(f_n, f_m) = \int_a^b f_n^* f_m dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}. \tag{5}$$

An example of this would be the eigenstates of the infinite square well toy problem. These are expressed as the set of all $\sqrt{\frac{2}{L}} \sin(n\pi x/L)$ for all $n \in \mathbb{N}$.

We require this concept of normal because of quantum mechanics. Quantum mechanics is inherently probabilistic as opposed to classical mechanics being deterministic. Thus, the mathematics behind quantum mechanics is both linear algebra and probability. From this we can draw certain requirements for our eigenstates of operators and functions in general, such as the sum of all probabilities must equal 1, and that the only possible result of measurement comes from an eigenvalue of the operator associated with the corresponding quantity (position, momentum, energy, angular momentum, etc.) [8]. The requirement for our eigenvectors to be normal then comes from the probabilistic limitation for the sum of all probabilities to be equal to 1.

Another key concept in vector spaces is the idea of complete sets. A collection of vectors in a vector space is complete if they are linearly independent and that, from those vectors, you can construct any other vector in the space via a unique linear combination of the set. By definition then, the vectors in a complete set form a basis for the vector space. For example, take $\mathbb{R}^3$, then the set of vectors $\{(1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1)\}$ form a complete and orthonormal set.

What we want to focus on now is the L^2-space as our Hilbert space.

2.1.2 L^2 Space

The $L^2(\mathbb{R}, dq)$ space is a Hilbert space where dq is the Lebesgue measure, the way of assigning a concept of length, area, or volume, depending on the dimensions of the space. Because it is a Hilbert space, it has the same defined inner product shown in equation (5), however it only contains those functions which are square integrable. Square integrability is the idea that the square of a function is finite over a space X, and thus rigurously integrable, or in equations:

$$|f|^2 = f^* f \neq 0, \tag{6}$$

which gives

$$\int_X |f|^2 < \infty. \tag{7}$$

With this in mind, recal orthonormal functions — equation (6). If we want orthonormal bases for our operators, then we need to have that the norm square of our function is equal to one.

Now, explicitly how do we represent these operators and eigenvectors physically? That is dependent on our system. However, we can assert that in Schrodinger's quantum mechanics, if we let position $\hat{x}$ and momentum $\hat{p}$ be defined in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dq)$, we then have the following relations for a generic function of position $\psi(x)$:

$$\hat{x}\psi(x) = x\psi(x) \tag{8}$$

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x). \tag{9}$$

Let us understand what these operators are, and where they come from.

2.2 Quantization

Quantization can be thought of as forming a mathematical passageway from a classical system to a quantum system. In traditional quantum mechanics this is done via the classical commutation relations (CCR) using a phase space representation.

Phase space is a coordinate formulation which describes physics in terms of a generalized spacial coordinate, q, and the conjugate variable momentum, p. This

version of physics is called Hamiltonian mechanics, as the Hamiltonian in classical mechanics is the description of the total energy of our physical system in which all the equations of motion described by Newton can be derived.

The process of quantization starts with the classical Poisson bracket, also known as the classical algebraic bracket for n dimensions is defined in Equation (9) below.

$$\{f, g\}_{q,p} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}. \tag{10}$$

To quantize a system, We then define the CCR with the quantum mechanical position and momentum operators in terms of the commutator, seen as

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\mathbb{I}. \tag{11}$$

From this, we can see that the commutator takes on a similar structure as the Poisson bracket, however instead of our value of 1 when comparing the position and its conjugate momentum in the Poisson bracket, we get $i\hbar$.

$$[\hat{q}, \hat{q}] = [\hat{p}, \hat{p}] = 0 \tag{12}$$

The mathematical object shown This is a direct relationship from the Poisson bracket to the commutator — where

$$\frac{[\hat{x}, \hat{p}]}{i\hbar} \sim \{q, p\} \tag{13}$$

for a one dimensional phase space.

The CCR, represented in Equations (10, 11), can also be depicted in other ways by being derived in other representations. A representation is, in our context, a description of the way operators act of a given vectors space. So for us, let us recall our position and momentum operators, $\hat{p}, \hat{x}$. They act on vectors in $L^2(\mathbb{R}, dq)$ by $\hat{x}\psi(x) = x\psi(x)$ and $\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x)$. In an essence, these equalities are describing our transformations from operator and vector space relations to functions and vector space relations, and describing the way the operators act in the given vector space.

2.3 Weyl Algebras

In the Heisenburg picture of quantum mechanics we imagine that the operators in our representation are changing with time as opposed to the wavefunctions. This time evolution of operators is represented by the exponentiated Hamiltonian operator — leading the Hamiltonian to be labeled the 'generator of time evolution' [15]. Similarly, as momentum is defined by an infinitesimal change in its conjugate spatial coordinate, the momentum operator is defined as the generator of infinitesimal translations. This can be seen more explicitly as viewing derivatives as translations of functions. Say we have a generic function of space which is shifted by a quantity $f(x+a)$, then, taking a Taylor expansion of this function we have that

$$f(x+a) = f(x) + a\frac{d}{dx}f(x) + \frac{a^2}{2}\frac{d^2}{dx^2}f(x) + \cdots$$

This can be simplified, by factoring our $f(x)$ as

$$f(x+a) = (1 + a\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2} + \cdots)f(x). \tag{14}$$

The first part of this equation then appears in the form of a Taylor expansion for e^x, thus we may simplify this expression even further as:

$$f(x+a) = e^{a\frac{d}{dx}}f(x). \tag{15}$$

From these concepts of operators as generators, we will investigate a representation in quantum mechanics of unitary operators which represent translations. A unitary operator is a Hermitian operator which preserves the inner product of the functions they act on.

For example, let us define unitary operators U, V with parameters $\lambda, \mu \in \mathbb{R}$ which are related to position and momentum operators as $U(\lambda)$ and $V(\mu)$ as:

$$U(\lambda) = e^{i\lambda\hat{x}/\hbar} \text{ and } V(\mu) = e^{i\mu\hat{p}/\hbar}. \tag{16}$$

From $U(\lambda)$ and $V(\mu)$ we can recover $\hat{x}$ and $\hat{p}$ by the power series below in Equation

(15) via [15]

$$e^{i\lambda\hat{x}/\hbar} = \sum_{n=0}^{\infty} \frac{(i\lambda\hat{x}/\hbar)^n}{n!}. \tag{17}$$

From the formulation of $U(\lambda)$ and $V(\mu)$ together, we also inherit the commutation relations from its exponents, which is then represented as

$$U(\lambda)V(\mu) = e^{-i(\lambda\mu)/\hbar}V(\mu)U(\lambda) \tag{18}$$

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2) \tag{19}$$

$$V(\mu_1)V(\mu_2) = V(\mu_1 + \mu_2), \tag{20}$$

as described in [5].

Another important concept in quantum mechanics is eigenvalues and eigenvectors. We would like to examine eigenvectors of the U(λ) operator, which we will say takes the form $|x_i\rangle$ as U(λ) is associated with position, these kets take the role of position eigenstates. When acted on our unitary operators we find that

$$U(\lambda)\,|x_i\rangle = e^{i\lambda x/\hbar}\,|x_i\rangle \text{ and } V(\mu)\,|x_i\rangle = |x_{i+\mu}\rangle \tag{21}$$

as shown in previous work [4].

From this we can see that $|x_i\rangle$ is unchanged by U(λ), as one would expect from an operator acting on its eigenket, but we see that V(μ) shifts the position by an index distance μ. We know that in quantum mechanics that we have complete, orthonormal sets of eigenvectors, so we assert that

$$\langle x_i|x_j\rangle = \delta_{ij} \tag{22}$$

where δ_{ij} is the Dirac delta.

Now, these operators U and V are defined using a variable parameter as said earlier, in this case λ and μ. These are the parameters of the operators and they dictate the step size of the translation when these are made analogous to our quantum mechanical systems. In quantum mechanics then our kets are represented by wavefunctions in $L^2(\mathbb{R}, dx)$, and from the relation with the kets shown in equation (10) we know that the operators, which as we said are unitary, do not take the eigen-

ket out of $L^2(\mathbb{R}, dx)$, only shift the original function and add a coefficient which is still in the scalar field $\mathbb{R}$.

In the context again of our operator-ket relation between V(μ) and $|x_i\rangle$, we can see that when we take the limit,

$$\lim_{\mu \to 0} \langle x_i | \, V(\mu) \, |x_i\rangle = \lim_{\mu \to 0} \langle x_i | x_{i+\mu} \rangle = 1 \tag{23}$$

because we defined μ as a continuous variable in $\mathbb{R}$.

These operators, when first described by Neumann in 1932, had specific conditions, such that the operator was continuous if the parameter was continuous [9]. This is demonstrated as

$$U(\sigma) \to U(\alpha) \text{ when } \sigma \to \alpha. \tag{24}$$

Thus if the parameter is defined as an element in a continuous space, then it is possible to take the limit as σ approaches α, and therefore we can conclude that continuity of the parameter is dependent on the structure of the space the parameter is in. This continuity, it turns out, is an important condition in the Stone-von Neumann Theorem.

2.4 Stone-von Neumann Theorem

The Stone-von Neumann Theorem states, that given that the Hilbert space is irreducible under the Weyl algebra, and the Weyl operators are weakly continuous, then those CCR in the Weyl reperesentation are unitarily equivalent to the CCR in the Schrodinger representation [4]. Let us break down this statement.

To start, I wish to define what a representation is. A representation is a mapping of elements in an algebraic space to linear operators in a vector space such that the mapping preserves the algebraic structure of the operators. For example, we said earlier that the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, this itself is representing an algebraic object which is coordinate independent ($\hat{p}$) to a linear operator in the position space which is coordinate dependent ($-i\hbar\frac{\partial}{\partial x}$). The algebraic structure is the commutation relation, so the operation in the algebraic space *is* the commutator, i.e $\hat{x}\cdot\hat{p} = [\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = [x, -i\hbar\frac{d}{dx}] = i\hbar$. We know what this commutator is by applying a generic

element of the vector space to the commutator as a whole — this could be either a matrix, a wavefunction in $L^2(\mathbb{R}, dx)$, or any element in any Hilbert space.

Irreducible representation of a vector space under an algebra implies that there cannot exist a smaller subspace of the algebra which could still produce a complete representation of the algebra [10, 16]. This idea of irreducibilty can be thought of simpler in terms of quadratics. For example, say there is a polynomial $p(x) = x^2 - 2x + 1$. Then $p(x)$ can be factored into $(x-1)^2$, and since $(x-1)$ cannot be factored any further, we know that $x-1$ is a unit, and irreducible. In the context of CCR then, we want to have an algebra which is the smallest it could be while still having a complete representation with respect to the vector space, in this case the Hilbert space. In the previous section, we introduced the $U(\lambda)$ and $V(\mu)$ operators and asserted that the CCR between the two was $U(\lambda)V(\mu) = e^{-i(\lambda\mu)/\hbar}V(\mu)U(\lambda)$. It is clear that if you have the union of all possible values for λ, μ in the operators, there would be no missing relations in the Hilbert space, and without every operator and value, it would not cover the entirety of the space.

Weakly continuous is connected to the parameters λ and μ of the Weyl operators. It means that the variables λ and μ we discussed earlier need to be continuous to have weakly continuous Weyl operators. The requirement for continuity needs be be true for all x_i shown below [11].

$$\lim_{\lambda \to 0} \langle x_i | \, U(\lambda) \, | x_i \rangle = \langle x_i | x_i \rangle \, . \tag{25}$$

This idea of continuity connects back to the idea of the algebra, as the parameters are part of the scalar field associated with the algebra, we need that field to be continuous. In physics, this relates to the topology of the field, i.e. what the topology of the spatial structure we are dealing with. This can also be related to the representation of matrices as our operators, and the indices of the matrix all being continuous. If this is the case then the translational parameter, λ or μ, is continuous. Relating back to the limit shown in Equation (23), taking the limit is in essence shortening the length of the translation until the translation is equal to 0.

The final aspect to investigate is the conclusion of our theorem, that the commutation relations derived using any representation which satisfies these aforementioned

conditions are unitarily equivalent. Unitary equivelence is the concept in which one thing can be transformed into another via a unitary transformation which preserves the relations of the algebra. Thus, if one were to perform a unitary transformation on a vector space, the transformation essentially forms as a homeomorphism, or a 1-to-1 and onto function, between the two spaces.

In the case of canonical quantum mechanics the criteria for weak continuity and irreducible representation are indeed satisfied, and thus we are able to conclude that the formulation of the CCR via the Weyl algebra expressed as $U(\lambda)V(\mu) = e^{-i(\lambda\mu)/\hbar}V(\mu)U(\lambda)$ is unitarily equivalent to $[\hat{q},\hat{p}] = i\hbar$, and $[\hat{q},\hat{q}] = [\hat{p},\hat{p}] = 0$.

2.5 Topology

Topology is a branch of mathematics that investigates the properties of open sets in order to investigate the structure of spaces. Open sets are the foundations of the study of topology, they are in fact the elements of the topology, and from them we derive all of our understanding of set structure — whether a point or a set is open gives us our closed sets, and our open sets define whether two objects or configurations may be the same even if they appear different. Continuity is an interesting concept applied to functions. In the most broad sense, continuity of functions is determined by the presence of what are called open sets.

A topological space (X, τ) is a set X with an associated topology τ— the collection of open sets. There are certain axioms of topology, they are:

(1) - The trivial sets $\emptyset$ (the empty set), and X (the entire set), are always in the topology.	—	$\{\emptyset, X\} \in \tau$
	–	
(2) - The finite intersection of open sets in the entire set X is in the topology.	—	$\bigcap_n^N \mathcal{O}_n \in \tau$ for
	–	$\mathcal{O}_i \in \tau$
(3) - The infinite union of open sets in X is open.	—	$\bigcup_n \mathcal{O}_n \in \tau$ for
	–	$\mathcal{O}_n \in \tau$

For our purposes of studying PQM, we would like to examine the idea of a discrete topology, in contrast to the standard topology of the reals, $\mathbb{R}^n$. The standard topology on the reals can be thought of most simply in the one dimensional case, $\mathbb{R}$, in which the open sets are open intervals $\{(a, b) \mid a, b \in \mathbb{R}\}$. These open intervals

when taken to higher dimensions take the form of open circles for two dimensions, and open balls for three dimensions. A discrete topological space is the idea that every single point in a set is separable from another by these open sets. The formal definition is that, a *discrete topology* is a topology in which every subset of our space X is open [12]. The discrete topology for this reason is refers to as the largest or maximal topology. Now we would like to focus on the idea of continuity. To be continuous is such that $f : X \to Y$ is continuous if and only if the inverse image of every open set in Y are open in X, i.e. if $f^{-1}(U) = V$ where $U \in Y$ and $V \in X$ are open [12]. This is visualized in Figure [2].

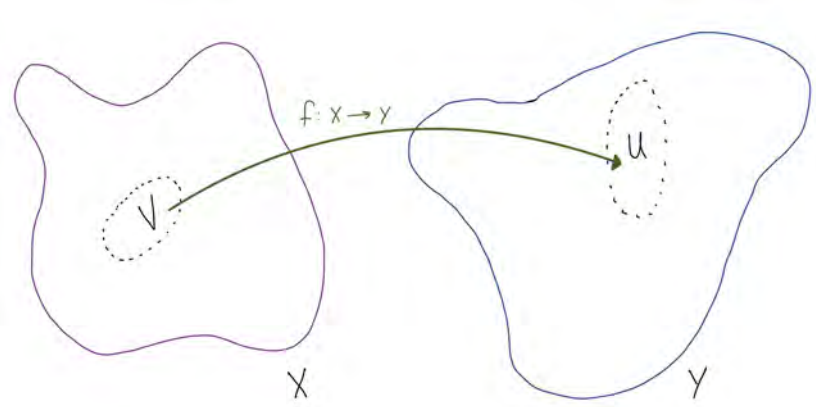


Figure 2: Figure representing two spaces with the same cardinality — number of points — with a defined function f from space X to space Y. The dashed objects represent open sets in both X and Y, and the inverse mapping of set U to set V is how a continuous function is defined.

Thus, as every subset is open in a discrete topology, a function from a discrete space mapped to anywhere will always be continuous. Furthermore, in the context of a linear operator in quantum mechanics working in a discrete space, every function is continuous as linear operators map elements to other elements in their own vector space.

3 Constructing PQM

In this section we will strive to make clear the differences, similarities, and the overall construction of polymer quantum mechanics as opposed to the canonical quantization scheme, as well as the motivation behind why we are use this particular formulation.

3.1 What is Different?

The essential component to polymer quantum mechanics is that we have a discrete spatial construction instead of the canonical continuous space. This in turn leads to an ill defined momentum operator. This ill-definedness derives from the canonical definition of momentum in quantum mechanics as $p = -i\hbar\frac{d}{dx}$, and as space is discrete there is no infinitesimal change in space as described by mathematics, thus the derivative operator cannot be defined as our generator of spatial translations.

Because of this, we will use the Weyl algebra to describe the operators in this quantization scheme, as even though momentum is ill-defined, the Weyl translational operator can be defined. Thus, we will define a specific lattice size for our momentum translational operator to shift our position wave function from spot to spot, and a specific position operator to act as the conjugate operator.

3.1.1 Discrete Space - Origins in Loop Quantum Cosmology

The requirement for a discrete space comes from the convention in loop quantum cosmology to foliate space time along the time-like dimension — seen in Figure (3). This foliation allows us to separate the spatial dimensions form the time-like dimension. In turn, the presence of now only spatial dimensions means that we can use the theory of quantized general relativity described in loop quantum gravity. However loop quantum cosmology requires the formulation of space into cells with fundamental minimum length, thus discretizing space. Around these cells in three dimensional space, are descriptions of the spatial geometry by triads, which impose a coordinate system for the cell. Because the cells are imposed, the consequence that the triads are discrete is required by the theory. More important in the construction of kinematics in LQC is the triad flux, which is the flux through

cells. Now, we have the idea of a spatial structure, but in physics we have a desire to know how things change. This information is typically given by a derivative, which in LQG is represented by the connection, A, which describes the curvature of the foliations. However on discrete space there cannot be a well defined curvature, but the exponentiated connection *is* well defined, and is called the holonomy which physically is the integral around a cell. Together, the triad flux and the holonomy form conjugate variables which behave exactly the same as position and momentum in canonical quantum mechanics and form a new representation of the CCR [11].

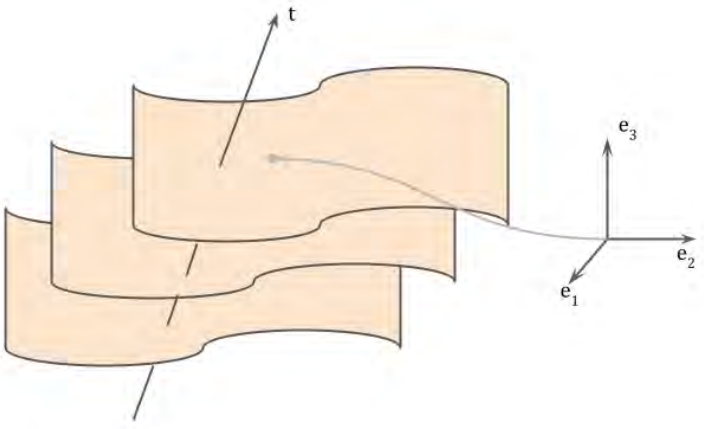


Figure 3: Example showing the foliations of space-time with the connection and associated triad, represented as the coordinate plane, at a point.

Recall again the idea of elementary cells. From the work by Freidel et al., we wish to think of these cells as the subdivision of space. We then wish, in an effort to simplify the system, "truncate" the allowed degrees of freedom [13] to better analyze dynamics of varying dimensions. For instance, in the particle on a ring problem, we are thinking of a one degree system, and thus the length of our cell μ_0 becomes the established lattice spacing of our real line.

In PQM, we use all of the ideas derived in LQC including the discrete space, except instead of dealing with the triad flux and holonomy, we wish to work with position and exponentiated momentum, which is exactly where the Weyl operators come into play.

3.1.2 Breaking Weak Continuity - New Classical Commutation Relations

Now that we have established the discrete nature of polymer quantum mechanics, we must discuss the implications of this discretization to our Weyl operators. While we know that classical momentum $p = -i\hbar\frac{d}{dx}$ is not well defined, we can also establish that the Weyl operator for momentum does not behave as it does in canonical quantum mechanics.

In continuous space, we were able to establish weak continuity of the variables λ, μ for both $U(\lambda)$ and $V(\mu)$. However, now that we have a distinct lattice from the elementary cells of length μ_0 imposed in our space, then our limit for weak continuity of μ appears as

$$\lim_{\mu\to 0} \langle x_i | \, V(\mu) \, | x_i \rangle = \langle x_i | x_{i+\mu_0} \rangle = 0 \tag{26}$$

as there is no way for $x_i = x_{i+\mu_0}$ no matter now small we make μ_0 it will still be distinctly different from 0. Therefore we do not have weak continuity of the Weyl operator associated with momentum on our discrete lattice.

While at first this may not seem to be a particularly significant result, we must think back to the Stone von-Neumann theorem, which requires weak continuity and irreducibility of our operators — thus μ not being weakly continuous means that we are not meeting the assumptions of the Stone-von Neumann theorem. Therefore we can conclude that in this construction of discrete space we have opened the door to a new representation of the CCR which is not equivalent to the CCR in the Schrodinger representation in canonical quantum mechanics, which has the potential to produce new and interesting physical predictions.

3.2 New Hilbert Space

There have been several techniques in defining new Hilbert spaces in PQM, we will follow the approach of Ashtekar et al. in their paper 4] and of Corichi et al. in their paper [5].

We have that in Schrodinger canonical quantum mechanics, our Hilbert space is $H = L^2(\mathbb{R}, dq)$ where L^2 is the space described in section 2.1.2. In PQM, we will use an entirely different formulation, where we will define what are called *cylindrical*

states, or Cyl.

In order to adequately define the Cyl group, we must first explore the geometric structure of polymer quantum mechanics.

3.2.1 Imposing a Different Geometry

We are now working in a space which is fundamentally discrete in its spatial coordinate, with a distinct lattice size. Let us call this lattice size μ_0.

We know that there are distinct values in which position can be, so we will follow the path led by Corichi et al., therefore let us imagine an abstract Hilbert space in which there are kets $|\chi\rangle$ which represent discrete position values in $\mathbb{R}_d$ with lattice size μ_0[5]. Symbolically this would appear as

$$\text{Let } \{|\chi\rangle \, | \chi \in \mathbb{R}_d\} \in H_{poly}. \tag{27}$$

Then, we know we have orthonormality for functions in Hilbert spaces as described in section 2.1.1, thus we may assert that the relation

$$\langle \chi_i | \chi_j \rangle = \delta_{ij} \tag{28}$$

holds. Let us now assert that this set of states constructs a basis, thus we know that the set spans our space such that we could express functions as linear superpositions of these kets with some associated probability amplitudes. This is demonstrated as

$$|\psi\rangle = \sum_{i}^{N} a_i \, |\chi_i\rangle \, . \tag{29}$$

Let us call the set of $|\psi\rangle$ as the Cyl set. Thus we know that $Cyl \subseteq H_{poly}$. From this we can say that there are operators $\hat{\epsilon}$ and $\hat{S}_\lambda$ such that

$$\hat{\epsilon} \, |\chi\rangle = \chi \, |\chi\rangle \tag{30}$$

$$\hat{S}_\lambda \, |\chi\rangle = |\chi + \lambda\rangle \, . \tag{31}$$

Let us call $\hat{\epsilon}$ the labeling operator, and $\hat{S}_\lambda$ the shifting operator, or translational operator. Now we know of physical operators which do the same operations, let

us first relate our momentum variable by 'momentum polarization' with the H_{poly} elements via a $|p\rangle$ which represents the variable value of momentum:

$$\varphi_\chi(p) = \langle p|\chi\rangle = e^{i\chi p/\hbar}. \tag{32}$$

We also know that when we act a Weyl operator on its eigenfunction it takes on the associated eigenvalue, thus the momentum operator acts as the labeling/multiplier of these momentum wavefunctions, and we can say

$$V(\lambda)\varphi_\chi(p) = e^{i\lambda p/\hbar}e^{i\chi p/\hbar} = \varphi_{\lambda+\chi}(p). \tag{33}$$

Thus we may conclude, since $V(\lambda)\varphi_\chi = \varphi_{\lambda+\chi}$, that $V(\lambda)$ functions as the translational operator for these cylindrical states.

Now let us still define position, $\hat{x} = i\hbar\frac{\partial}{\partial p}$, as the derivative of momentum, then when we act

$$\hat{x}\varphi_\chi(p) = -i\hbar\frac{\partial}{\partial p}e^{i\chi p/\hbar} \tag{34}$$

$$= -i\hbar(i/\hbar)\chi e^{i\chi p/\hbar} \tag{35}$$

$$= \chi e^{i\chi p/\hbar} \tag{36}$$

$$= \chi\varphi_\chi(p). \tag{37}$$

Therefore we can conclude that the position operator acts as the labeling operator for these cylindrical states.

In the literature on polymer quantum mechanics, which is outside the scope of the paper, is the conclusion that because of the structure of the Weyl operators$\mathcal{H}_{poly,p}$, is equivalent to $L^2(\mathbb{R}_b, d\mu_H)$, where $\mathbb{R}_b$ is what is called the 'Bohr compactification of the real line', and $d\mu_H$ is the Haar measure, which is an 'naturally probabilistic' measure [5].

The important description of the Bohr compactification, and that which is most relevent to this paper, is that there exists a Fourier transform between itself and to the real line with a discrete topology topology, $\mathbb{R}_d$, and the measure on $\mathbb{R}_d$ will be the 'counting measure', $d\mu_N$, where N are the total number of our finite set of points [5].

From this information we can extrapolate that $H_{poly,x} := L^2(\mathbb{R}_d, d\mu_c)$. Further, we can say that because of the relationship of the Weyl operators in the position space, their relationship to the wavefunctions in the position space is

$$V(\lambda)\varphi(q) = \varphi(q+\lambda), \tag{38}$$

$$\hat{x}\varphi_\chi(q) = \chi\varphi_\chi(q). \tag{39}$$

The full description of this process is seen in the paper by Corichi et al. [5].

Now, continuing to define exactly what functions are in this space, we would like to examine what is allowed. We know that, as this is a Hilbert space, we must have finite integration, and thus our functions must converge to some value less than infinity. We also know from the counting measure that our functions must exist on a strictly countable set of points. With this, let us construct a graph $\gamma = \{x_i\}$ such that $\{x_i\}$ is a countable set of points on the real line. We will now define our lattice construction for PQM from this graph.

We have from the construction of $\mathcal{H}_{poly}$ the technical condition that there exist no sequences of points with limit points in the real line when equipped with the standard metric. Thus, under this induced topology we are able to say that our graph γ has discrete topology.

With this construction we have an issue however, we know that we have a translational operator in which $V(\lambda)\varphi(q) = \varphi(q+\lambda)$, thus if we have a random scattering of points on the real line, this operation which we know is well defined in our standard topology would then create new points on our line. To correct this phenomenon we will create an isomorphism φ (a 1-1 and onto mapping/function) between our graph γ to a graph γ_{μ_0}, which we will say has a distance of exactly μ_0 between each discrete point. This idea is demonstrated in Figure 4.

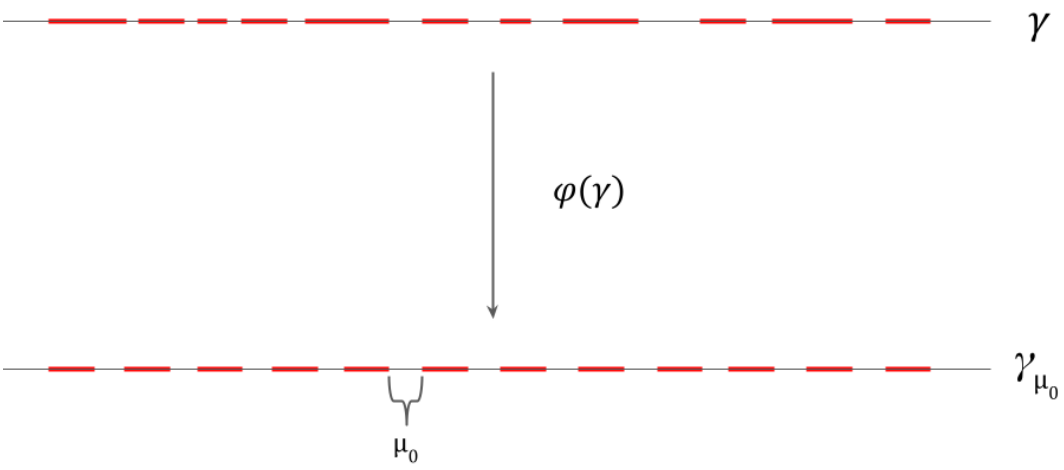


Figure 4: Visualization of the homomorphism between graph γ and γ_{μ_0}, φ

This graph γ_{μ_0} can then be defined as $\gamma_{\mu_0} := \{n\mu_0 | n \in \mathbf{Z}\}$ and our Hilbert space becomes specifically $H_{\gamma_{\mu_0}}$, which is the subspace of functions in $L^2(\mathbb{R}_d, d\mu_c)$ which satisfy the conditions of $\mathcal{H}_{poly}$ and the Weyl translational operator.

4 Particle on a Ring in PQM

4.1 Applying PQM

4.1.1 Defining the Hamiltonian

As we have been dealing with previously, we have the emergence of a discrete spacial structure, causing momentum to be undefined. In determining dynamics of our quantum mechanical system however, this becomes an issue as we need a way to express the kinetic energy, which is generically $\hat{P}^2/2m$.

For this paper, we have decided not to visualize the ring as a physical ring, but instead as a subset of of the real line with periodic boundary conditions using our graph γ_{μ_0} we described earlier. This is visualized in Figure [5]

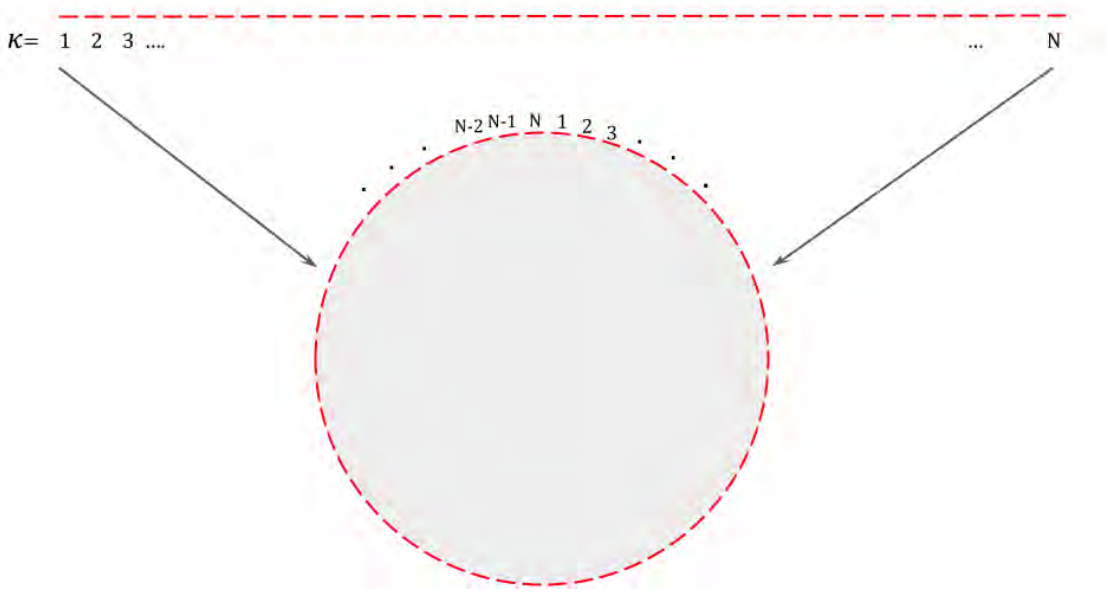


Figure 5: Visualization of the ring as a subspace of the real line with finite points N, where there is a periodic boundary condition such that the position values range from 1 to N, and such that for any position κ we have that $\kappa + N = \kappa$

.

Now, in polymer quantum mechanics it is standard to approximate $\hat{P}^2$ using $V(\mu)$, thus we will now generate an approximated momentum — labeled P_μ — to define our Hamiltonian [4]. It is important to note that when we are using p in the mathematical derivation below, we are really intending this to be interpreted as the variable p, or variable momentum. This can be thought of as the eigenvalue of

momentum when we try to take this system to the continuum.
Let

$$V(\mu_0) = e^{ip\mu_0/\hbar}, \tag{40}$$

now let us say that we have the Euler relation

$$\cos(\mu_0 p/h) = \frac{e^{ip\mu_0/\hbar} + e^{-ip\mu_0/\hbar}}{2}. \tag{41}$$

From this, if we impose that $\mu_0 p/\hbar$ is always real, then we can use a series expansion for cosine as

$$\cos(\mu_0 p/\hbar) \approx 1 - {\mu_0}^2 p^2/2\hbar^2. \tag{42}$$

It is important to note that because of the series expansion, this approximated momentum is well defined for values of $p << \hbar/\mu_0$. Essentially, a low energy approximation is required for this formulation.
Returning to the calculation, we will now combine equations to see

$$1 - {\mu_0}^2 p^2/2\hbar^2 \approx \frac{e^{ip\mu_0/\hbar} + e^{-ip\mu_0/\hbar}}{2} \tag{43}$$

and with a little algebraic rearranging we can find that

$$p^2 \approx \frac{\hbar^2}{{\mu_0}^2}(2\mathbb{I} - e^{ip\mu_0/\hbar} - e^{-ip\mu_0/\hbar}). \tag{44}$$

We will compact this even more to achieve our final expression

$$p^2 \approx \frac{\hbar^2}{{\mu_0}^2}(2\mathbb{I} - V(\mu_0) - V(-\mu_0)) = P_\mu^2. \tag{45}$$

If we then use this approximate momentum squared, P_μ^2, in our Hamiltonian and we find that

$$\hat{H} \approx \frac{P_\mu^2}{2m} = \frac{\hbar^2}{2m{\mu_0}^2}(2\mathbb{I} - V(\mu_0) - V(-\mu_0)). \tag{46}$$

4.1.2 Ansatz Formulation

Maxwell Siebersma was able to work through the process of deriving an ansatz for the eigenfunctions of the Hamiltonian through analyzing the behaviors of the V(μ_0) on the position eigenstates in matrix form, this relation can be thought of as a ring

of oscillators which is visualized below in Figure 6

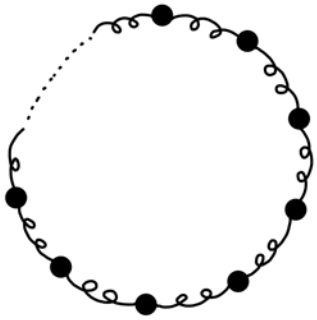


Figure 6: A visualization, where the dots are "masses" and the strings connecting then are springs, this is a classical physics problem solved in a course on periodic system behavior.

Given our Hamiltonian $\hat{H} \approx \frac{P_\mu^2}{2m} = \frac{\hbar^2}{2m{\mu_0}^2}(2\mathbb{I} - V(\mu_0) - V(-\mu_0))$, the eigenstate that Siebersma found were then

$$|\varphi_n\rangle = \frac{1}{\sqrt{N}} \sum_{\kappa=1}^{N} e^{2\pi i \kappa n/N} |x_\kappa\rangle \,, \tag{47}$$

where N is the total number of lattice positions possible using the step size μ_0 as visualized in Figure 4, κ is the label given for the discrete labeling of the position of the particle on a ring, where again seen in Figure 5 we have a periodic structure to κ [14].

Thus eigenstates for our Hamiltonian then have the associated eigenvalues:

$$E_n = \frac{\hbar^2}{m\mu_0^2}(1 - \cos(2\pi n/N)). \tag{48}$$

For the full proof of this see [14]. It is important to note that due to the nature of the derivation of the eigenstates, there are only as many energy eigenstates are there are positions on a ring, and thus there are only as many energy eigenvalues as there are positions on the ring.

By examining this eigenvalue, we can see that we have a maximal energy in two cases. When N is even, then $N/2$ is an integer and $\cos(2\pi(n = N/2)/N) = \cos(2\pi(N/2)/N) = \cos(\pi) = 0$. When N is odd, we have a degeneracy of maximal energy at for values of $\frac{N-1}{2}$ and $\frac{N+1}{2}$. In fact because of the cosine in our equation for energy, we will have degeneracy in our energy for all values of n which are symmetric

about N/2 for even, and $\frac{N-1}{2}$ and $\frac{N+1}{2}$ for odd. More explicitly, for even value of N, we know that the energies $E_1 = E_{N-1}$, $E_2 = E_{N-2}$, ... , $E_{\frac{N}{2}-1} = E_{\frac{N}{2}+1}$, and $E_{\frac{N}{2}} = E_{\frac{N}{2}}$. For the odd case we have the same process except at the middle point: $E_1 = E_{N-1}$, $E_2 = E_{N-2}$, ... , $E_{\frac{N-1}{2}-1} = E_{\frac{N+1}{2}+1}$, and finally $E_{\frac{N-1}{2}} = E_{\frac{N+1}{2}}$. Thus, for the even case, even though there are N energy levels described, there are only $N/2$ distinct energy levels for the ring, and for the odd case there are only $\frac{N-1}{2}$ distinct energy levels for the ring.

From this point on, what I am interested in is the behavior of these states as a function of time.

4.2 Time

Time is a tricky concept to talk about in quantum gravity. If you are talking about quantum gravity, there is no way to have continuous time and have that relate to a development of a general covariance — that we have background independence, that the foundations of our theorem do not rely on an inlay of coordinates. It is also the case that general relativity is a constrained theory, in which we essentially freeze the system, and determine the variables which are invariant under the symmetries we impose in the theory. Thus it is generally impossible to have a continuous time-like variable as those symmetries would then evolve with time as well. There is also the issue with talking the foliations of space time *along* the time-like dimension. This in itself is a discretization of time in our formulation of the theory of loop quantum cosmology.

However, in the context of polymer quantum mechanics, there are exceptions in which we desire to know how a particular toy model may change in time, or evolve. Because PQM is in its most general form a tool to investigate the ramifications of conceptualizing space as discrete, it would be in the continuum limit of space *and* time in which we could correlate those results to that of the tested theory in canonical quantum mechanics. Thus, even if our Hamiltonian which canonically describes time evolution is discrete, time itself may be considered continuous. Therefore for our purposes, we will consider time to be continuous to observe the behavior of our states as they evolve through time.

First we would like to confirm that with our approximate Hamiltonian, we are

still allowed to use the Hamiltonian as a generator for time evolution. We will do this by solving the time-dependent Schrodinger Equation.

The Schrodinger equation for a time independent Hamiltonian takes the form

$$\hat{H}\left|\varphi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\varphi(t)\right\rangle. \tag{49}$$

Since our position eigenfunctions are discrete, we will use the state $|\varphi_n\rangle$ as our state here to find time evolution. The equation then becomes:

$$\hat{H}\left|\varphi_n(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\varphi_n(t)\right\rangle \tag{50}$$

which can be expressed as

$$\hat{H}\sum_{\kappa=1}^{N} f(t)e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle = i\hbar\frac{\partial}{\partial t}\sum_{\kappa=1}^{N} f(t)e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle. \tag{51}$$

We are allowed to make this assumption that, essentially, our functions for energy and time are distinct, by means of a separation of variables technique where some $f(x,t) = X(x)T(t)$. Let us now apply the eigenvalue-eigenvector relation $\hat{H}\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle$:

$$E_n\sum_{\kappa=1}^{N} f(t)e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle = i\hbar\frac{\partial}{\partial t}\sum_{\kappa=1}^{N} f(t)e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle \tag{52}$$

We know by calculus that for continuous functions of t we have that

$$\frac{\partial}{\partial t}\sum f(t) = \sum\frac{\partial}{\partial t}f(t). \tag{53}$$

Therefore, if we are to go off of our assumption that time is continuous in this case, we have that

$$E_n\sum_{\kappa=1}^{N} f(t)e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle = i\hbar\sum_{\kappa=1}^{N}\frac{\partial}{\partial t}f(t)e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle. \tag{54}$$

Let us now compress the equation slightly, to more clearly see the relationship we have created with $\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N}\left|x_\kappa\right\rangle = \left|\varphi_n\right\rangle$:

$$E_n f(t)\left|\varphi_n\right\rangle = i\hbar\frac{\partial}{\partial t}f(t)\left|\varphi_n\right\rangle. \tag{55}$$

$$\frac{E_n}{i\hbar} f(t) \left|\varphi_n\right\rangle = \frac{\partial}{\partial t} f(t) \left|\varphi_n\right\rangle \tag{56}$$

From inspection we can see that this is a first order homogeneous ordinary differential equation, and thus

$$\frac{\partial}{\partial t} f(t) = \frac{E_n}{i\hbar} f(t) \tag{57}$$

and we can use the ansatz $f(t) = e^{E_n t/i\hbar} = e^{-iE_n t/\hbar}$. This concludes in the relation that

$$\left|\varphi_n(t)\right\rangle = \sum_{\kappa=1}^{N} e^{-iE_n t/\hbar} e^{2\pi i \kappa n/N} \left|x_\kappa\right\rangle \tag{58}$$

$$= \sum_{\kappa=1}^{N} e^{i\{2\pi\kappa n/N - E_n t/\hbar\}} \left|x_\kappa\right\rangle \tag{59}$$

This result implies that the Hamiltonian is still the generator of time evolution, and thus the time translation operator can be expressed as $e^{-i\hat{H}t/\hbar}$ where when acted on an eigenstate $\left|\varphi_n\right\rangle$ of the Hamiltonian, $\hat{H}$, we get the relation

$$e^{-i\hat{H}t/\hbar} \left|\varphi_n\right\rangle = e^{-iE_n t/\hbar} \left|\varphi_n\right\rangle \tag{60}$$

where E_n is the energy eigenvalue associated with the eigenstate.

4.2.1 Dispersion of a Position Eigenstate

What we are now going to investigate is what is called the dispersion for a state with a particular initial energy. What this dispersion is, is the examination of what happens to that particular state over time. We will construct a dispersion relation of a state in a singular position eigenstate, thus with associated probability of being at that position with a probability of 1. With the time evolution factor, we will then be able to examine how the state disperses over time.

One way to perform time evolution is to use the translational operator associated

with time, generated by the Hamiltonian.

$$|\varphi_m(t)\rangle = \mathcal{U}(t)\,|\varphi_m\rangle = e^{-i\hat{H}t/\hbar}\,|\varphi_m\rangle \tag{61}$$

$$= e^{-iE_m t/\hbar}\,|\varphi_m\rangle \tag{62}$$

$$= e^{-iE_m t/\hbar}\frac{1}{\sqrt{N}}\sum_{\kappa=1}^{N} e^{2\pi i\kappa m/N}\,|x_\kappa\rangle \tag{63}$$

Now in creating a dispersion relation we need to examine a state which is a particular eigenstate of position. We will for the moment concider the simplest case, and start at the position $\kappa\mu = N\mu$. This we can call $|x_N\rangle$. Now we know that because the the Hilbert space is a vector space, and the Hamiltonian is a Hermitian matrix, we can say that the eigenstates of the Hamiltonian form a basis, and thus we can represent the position eigenstate $|x_N\rangle$ as a superposition of energy eigenstates as described in section via a completeness relation in the energy basis, $\mathbb{I}_E$. We wish to do this in order to time evolve the state, as we must be in the energy eigenbasis to time evolve as it is the generator of time evolution. This is seen as:

$$|x_N\rangle = \mathbb{I}_E\,|x_N\rangle = \sum_{n=1}^{N}|\varphi_n\rangle\,\langle\varphi_n|x_N\rangle \tag{64}$$

$$= \sum_{n=1}^{N}\frac{1}{\sqrt{N}}\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N}\,|x_\kappa\rangle\,\frac{1}{\sqrt{N}}\sum_{l=1}^{N} e^{2\pi iln/N}\,\langle x_l|x_N\rangle \tag{65}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N}\,|x_\kappa\rangle\sum_{l=1}^{N} e^{2\pi iln/N}\delta_{l,N} \tag{66}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N}\,|x_\kappa\rangle\,e^{2\pi iNn/N} \tag{67}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N}\cdot 1\cdot|x_\kappa\rangle \tag{68}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N}\,|x_\kappa\rangle \tag{69}$$

We know that the eigenkets of position act as Kronecker deltas, thus we know that $\langle x_\kappa|x_N\rangle = 1$ if and only if $\kappa = N$, which because we are on the ring is equivalent to $\kappa = N$. Thus we may collapse the sum, and say that the exponential goes to one, as the exponent goes to $i2\pi n$. Thus our position eigenstate in the energy basis appears as is shown in Equation (67).

Now we may time evolve, which as seen earlier will produce the function

$$|x_N(t)\rangle = \frac{1}{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N} e^{2\pi i\kappa n/N} e^{-iE_n t/\hbar} |x_\kappa\rangle \tag{70}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N} e^{i(2\pi\kappa n/N - E_n t/\hbar)} |x_\kappa\rangle \,. \tag{71}$$

Here are some examples of the real component of this dispersion as a relation of time.

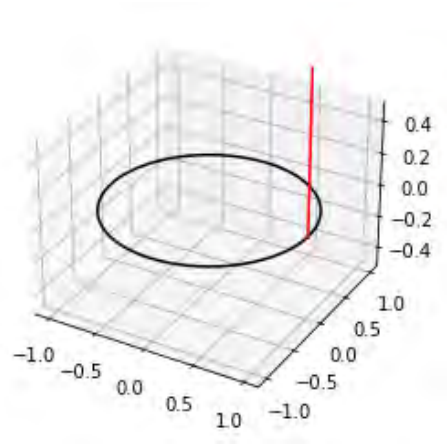
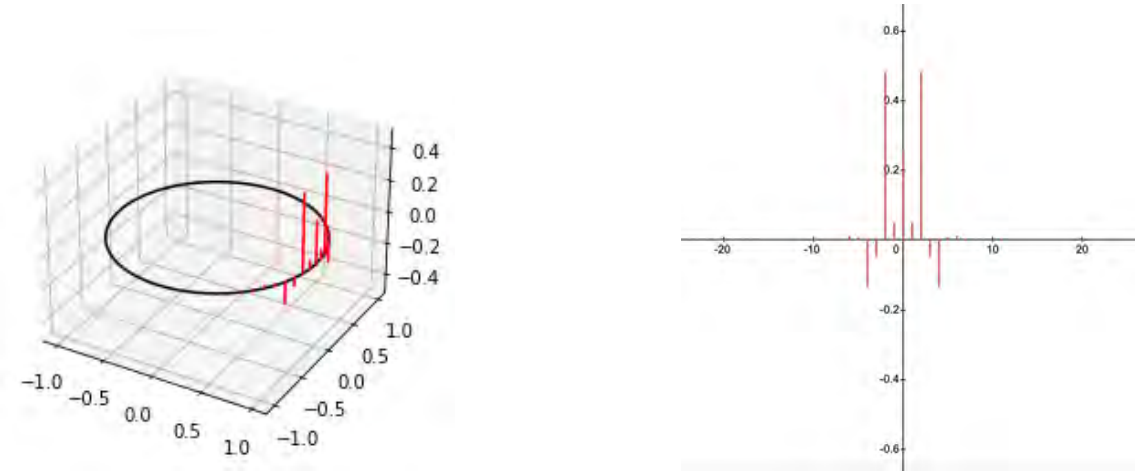
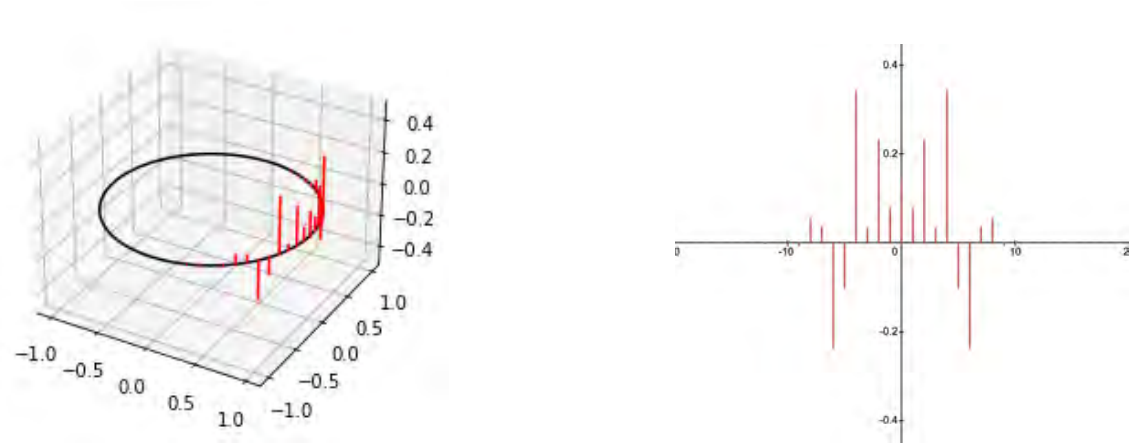


Figure 7: A visualization of the particle on a ring with an initial state $|x_N\rangle$ at t=0, real part only.



(a) (b)

Figure 8: A representation of the dispersion of an initial state at $\kappa = N$ at t=3 on the ring (a) and on a flat plane with periodic boundary conditions (b), real part only.



(a) (b)

Figure 9: A representation of the dispersion of an initial state at $\kappa = N$ at t=6 on the ring (a) and on a flat plane (b), real part only.

We will now take a small aside and examine the commutator of position and the Hamiltonian, our $V(\mu_0)$ translational operator and the Hamiltonian in order to see if the eigenfunctions of these operators will change in time. We will also see what the commutator between the position operator and the translational operator are, and see if there is a connection to the canonical quantization results in regards to the uncertainty relation.

We know from the previous literature that the commutator between $\hat{x}$ and $V(\mu_0)$ is $[\hat{x}, V(\mu_0)] = -\mu_0 V(\mu_0)$ [4]. From this, we can examine the relation between position and our Hamiltonian for the particle on a ring solution previously defined.

$$[\hat{x}, \hat{H}] = [\hat{x}, \frac{\hbar^2}{2m{\mu_0}^2}(2\mathbb{I} - V(\mu_0) - V(-\mu_0))] \tag{72}$$

$$= \frac{\hbar^2}{2m{\mu_0}^2}\{[\hat{x}, 2\mathbb{I}] - [\hat{x}, V(\mu_0)] - [\hat{x}, V(-\mu_0))]\}. \tag{73}$$

We know that the identity operator commutes with all operators, and thus the commutator $[\hat{x}, 2\mathbb{I}] = 0$. For the other components we can use our result from [4]:

$$[\hat{x}, \hat{H}] = \frac{\hbar^2}{2m{\mu_0}^2}\{\mu_0 V(\mu_0) - \mu_0 V(-\mu_0)\} \tag{74}$$

We know that $V(\mu_0) = e^{i\mu_0 p/\hbar}$, thus we can see that we have the Euler relation for

$\sin \mu_0 p/\hbar$, thus our commutator becomes:

$$[\hat{x}, \hat{H}] = \frac{\hbar^2 \mu_0}{2m{\mu_0}^2}(2i \sin(\mu_0 p/\hbar)) \tag{75}$$

$$= \frac{\hbar^2 i}{m\mu_0} \sin(\mu_0 p/\hbar). \tag{76}$$

When either μ_0 or p is equal to 0, then this commutator will equal 0, however, as we know μ_0 does not equal 0 in PQM, and 'momentum' p is simply a variable akin to an eigenvalue, this will not equal 0 in the continuum or in PQM. Therefore this commutator does not equal 0 and we can expect the expectation value of position in time to in fact depend on time. However, we can see by inspection that the commutator between our translational operator and the Hamiltonian will be zero, as the Hamiltonian is simply composed of linear combinations of the translational operator itself. We would therefore expect the expectation values of the translational operator — i.e. the momentum — to not depend on time.

Let us now pick up where we left off with our dispersion relation. We want to examine how the position expectation value, what we expect the value of position to be, changes as a function of time after first representing it in the energy eigenbasis.

$$\langle x_N(t)| \hat{x} |x_N(t)\rangle = \langle x_N(t)| \hat{x} \frac{1}{N} \sum_{n=1}^{N} \sum_{\kappa=1}^{N} e^{i(2\pi\kappa n/N - E_n t/\hbar)} |x_\kappa\rangle \tag{77}$$

$$= \langle x_N(t)| \frac{1}{N} \sum_{n=1}^{N} \sum_{\kappa=1}^{N} \kappa\mu_0 e^{i(2\pi\kappa n/N - E_n t/\hbar)} |x_\kappa\rangle . \tag{78}$$

Now, since we want the most general case, we need to assume that $\langle x_N(t)|$ and $|x_N(t)\rangle$ are not aligned in their energies and positions, thus for $\langle x_N(t)|$ we will use l as the index for position, and b as the index for energy. We do know that the total number of allowed positions remains N, so this will remain as our upper bound. This then gives

$$\langle x_N(t)| \hat{x} |x_N(t)\rangle = \frac{1}{N} \sum_{b=1}^{N} \sum_{l=1}^{N} e^{-i(2\pi lb/L - E_b t/\hbar)} \langle x_l| \frac{1}{N} \sum_{n=1}^{N} \sum_{\kappa=1}^{N} \kappa\mu_0 e^{i(2\pi\kappa n/N - E_n t/\hbar)} |x_\kappa\rangle . \tag{79}$$

We can then compress this relation as

$$\langle x_N(t)|\,\hat{x}\,|x_N(t)\rangle = \frac{1}{N^2}\sum_{b=1}^{N}\sum_{l=1}^{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N}\kappa\mu_0 e^{-i(2\pi lb/L - E_b t/\hbar)}e^{i(2\pi\kappa n/N - E_n t/\hbar)}\langle x_l|x_\kappa\rangle\,. \tag{80}$$

From this we can clearly see that the terms in this expectation value are zero when $l \neq \kappa$ for all values of l and κ. For nonzero terms in the expectation value, we compact the sums such that $l = \kappa$ for all values of $x_l, x_\kappa \in \gamma_{\mu_0}$.

$$\langle x_N(t)|\,\hat{x}\,|x_N(t)\rangle = \frac{1}{N^2}\sum_{b=1}^{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N}\kappa\mu_0 e^{-i(2\pi\kappa b/N - E_b t/\hbar)}e^{i(2\pi\kappa n/N - E_n t/\hbar)} \tag{81}$$

$$= \frac{1}{N^2}\sum_{b=1}^{N}\sum_{n=1}^{N}\sum_{\kappa=1}^{N}\kappa\mu_0 e^{i(2\pi\kappa(n-b)/N - (E_n - E_b)t/\hbar)}. \tag{82}$$

When we set $t = 0$ we are able to evaluate the summations to achieve $\langle x_N(t)|\,\hat{x}\,|x_N(t)\rangle = \frac{1}{N^2}N^3\mu_0$, so we end up with an expectation value of simply $N\mu_0$, which is equivalent to our initial state being $\kappa = N\mu_0$. However as t increases, or is nonzero, we see that the expectation value varies about the ring — as seen in Figure [10].

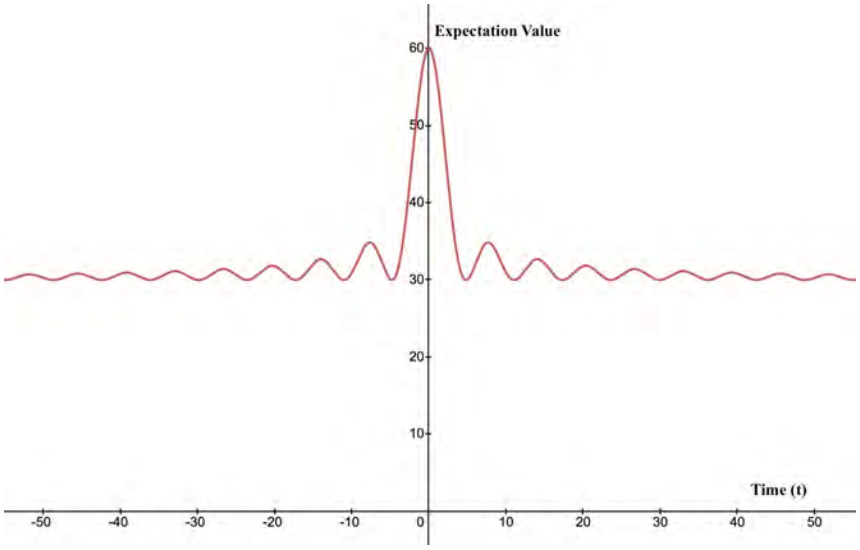


Figure 10: A graphical representation of the expectation value of position as a function of time, horizontal axis is the time axis, and the vertical axis is the position expectation value. Plotted for a ring with $N = 60$, for sake of computing power.

It seems that the expectation value levels out to approximately $N/2$ as time goes on, in this case $N = 60$. However we know that the wave propagates symmetrically about the point it originates as seen in the previous Figures [7,8, 9]. One would then

think that the expectation value should not change in time until the two propagating waves collide at $\frac{N}{2}$ for even or $\frac{N-1}{2}$ for odd.

To investigate this further, let us change our initial point of our position eigenstate to instead be at any point on the ring in the set $\{\mu_0, 2\mu_0, 3\mu_0, ..., N\mu_0\}$. The equation for our dispersion relation changes, instead of what occurs in Equation 81 we find that the inner product for some $c \in \{\mu_0, 2\mu_0, 3\mu_0, ..., N\mu_0\}$ is

$$\langle \varphi_n | x_c \rangle = \frac{1}{\sqrt{N}} \sum_{\kappa=1}^{N} e^{-2\pi i \kappa n/N} \langle x_\kappa | x_c \rangle . \tag{83}$$

We know that these positions act as Dirac deltas, so this relation is equivalent to

$$\langle \varphi_n | x_c \rangle = \frac{1}{\sqrt{N}} \sum_{\kappa=1}^{N} e^{-2\pi i \kappa n/N} \delta_{\kappa,c}. \tag{84}$$

We know that this is only non-zero when $\kappa = c$, thus for a nontrivial result we know that κ is indeed equal to c. This makes the position eigenstate equal to

$$|x_c\rangle = |\varphi_n\rangle \langle \varphi_n | x_c \rangle \tag{85}$$

$$= \frac{1}{\sqrt{N}} \sum_{\kappa=1}^{N} e^{2\pi i \kappa n/N} |x_\kappa\rangle \frac{1}{\sqrt{N}} e^{2\pi i c n/N} \tag{86}$$

$$= \frac{1}{N} \sum_{\kappa=1}^{N} e^{-2\pi i (\kappa - c) n/N} |x_\kappa\rangle . \tag{87}$$

When time evolved following the same process as above we find the state is then,

$$|x_c(t)\rangle = \frac{1}{N} \sum_{n=1}^{N} \sum_{\kappa=1}^{N} e^{i(2\pi(\kappa-c)n/N - E_n t/\hbar)} |x_\kappa\rangle . \tag{88}$$

This, when examined as a dispersion evolves in time exactly the same for all c's, and is seen as a symmetric distribution about the starting point on the ring.

Let us examine the expectation value of the dispersion now. Similarly derived as in the energy eigenstate, we find the time evolved position eigenstate to be:

$$\langle x_c(t) | \hat{x} | x_c(t) \rangle = \frac{1}{N^2} \sum_{b=1}^{N} \sum_{n=1}^{N} \sum_{\kappa=1}^{N} \kappa \mu_0 e^{i(2\pi(\kappa-c)(n-b)/N - (E_n - E_b)t/\hbar)} \tag{89}$$

The below figure is a representation of the expectation values of the system as a function of time, where each line is representing an expectation value with respect to the initial starting point on the ring and of time. We can see that, even when the initial point is not N, the expectation values still vary greatly.

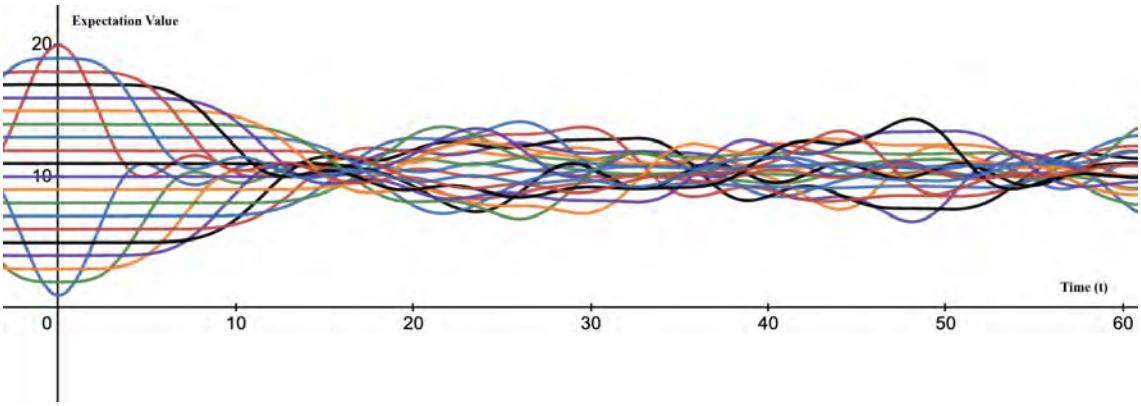


Figure 11: Representation of the different expectation values as a function of time, starting at different points on the ring for an even number of points on the ring, in this case N=20.

From this graph we can see that these results suggest our chosen orientation and our initial starting position greatly influences what we would expect to measure with respect to position. This is contrary to what we would expect the expectation value be, as the ring is symmetric we would think that the there would be no change in the expectation value from the point of origin until the wave propagated entirely around the ring. In fact, by observation it would seem the point in which all the waves propagate about the ring occurs at about 18t on the graph, and this is the point in which the system entirely looses all information regarding the starting places on the ring. This could be calculated analytically in relation to the frequencies in the exponent of the expression. It is also good to note that, here I have pictured the real part of the expectation value shown, but in fact the imaginary component does always go to zero. This is seen by the fact that the expectation value is a symmetric function, and thus will only have contributions from cosine, and none from sine. Thus, we have a real expectation value as one would expect.

We can also note from the graph that it appears all the expectation values are attempting to converge to the value $N/2$, which in the case of this visualization is the value 10. However this is not necessarily a bad sign. Let us recall the classical ring problem, where we have an operator $\hat{\phi}$, and eigenstates $|m\rangle \doteq \frac{1}{\sqrt{2\pi}} e^{im\phi}$. When

we take the expectation value of the angle operator ϕ as a function of time, we have that

$$\begin{aligned}\langle m|\,\hat{\phi}\,|m\rangle &= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{-im\phi}\phi \frac{1}{\sqrt{2\pi}} e^{im\phi}\, d\phi \\ &= \int_0^{2\pi} \frac{1}{2\pi}\phi\, d\phi \\ &= \frac{1}{2\pi}(2\pi^2) \\ &= \pi\end{aligned}$$

Since this was the case for a generic eigenstate $|m\rangle$, it is evident that the value π emerges from the integral regardless of the initial function.

This can be thought of qualitatively as when, after enough time has passed, there is an even probability distribution across the ring. The question of expectation value is then 'what is the average value of the circle', or in other words, the expectation value is simply asking what the average value of the position label is.

5 Conclusion

A mathematical understanding for quantum gravity still evades the top researchers in theoretical physics. One of the leading theories is loop quantum gravity, a framework which asserts a discretized structure on space-time. This is used most in loop quantum cosmology is a toy model itself which is used to make predictions of what occurred in the early universe, with reduced degrees of freedom. With its discrete structure however, we need new mathematical representations of quantum mechanics as our standard commutation relations don't hold with a discrete construction for space.

This led to the idea of polymer quantum mechanics, which assumes a discrete spatial structure, and makes the momentum operator undefined in our traditional sense. We were then led to interpreting quantum mechanics using the Weyl algebra, which can be done in Schrodinger's quantum mechanics, but can also account for discrete movements along our space-like formulation. The Weyl algebra, instead of using the classically defined operators, uses a representation of unitary translational operators instead. In this development of new operators for position and momentum in PQM, it is known that the commutation relation between our conjugate operators are now distinctly different from that of Schrodinger's quantum mechanics, as the operators themselves do not satisfy the requirement of weak continuity of the Stone von-Neuman theorem.

The construction of polymer quantum mechanics is well studied and has been used to investigate different toy models such as the free particle and the harmonic oscillator, but not on the particle on a ring which is what we have investigated here. Once establishing the Hilbert space in which we operate in PQM, we then construct the Hamiltonian which can be used for the particle on a ring. We decided to look solely at the ring as a finite subset of the real line equiped with a discrete topology. From this assumption, our Hamiltonian appears as that of the free particle, with an approximated momentum standing in for our P^2 term. From this, Max Siebersma who I am doing this project with, discovered an eigenvector which satisfies the time independent Schrodinger equation with the Hamiltonian. He also found that in the continuum limit, the results we found approach the results in Schrodinger's quantum mechanics.

After this, the question went to time evolution, where we realized that in quantum gravity time is, in its essence, not continuous. However to understand the mechanics of how polymer quantum mechanics functions we assumed continuity of our time-like variable. Using this assumption, I solved the time dependent Schrodinger equation, and found that the Hamiltonian is still the generator for time evolution. Next, I found that, as in Schrodinger quantum mechanics, the position operator does not commute with the Hamiltonian. I also derived an expression for a dispersive relation of a singular position eigenstate. Using this relation and the position operators, I calculated the expectation value and discovered that it does in fact change in time. However, I also found that the expectation value is reliant on the initial starting point on the ring, even though we have an imbedded cyclic nature to our structure.

Overall, from this research we are able to investigate the mathematical foundations for loop quantum cosmology and polymer quantum mechanics, and in understanding these toy models we are able to chip away at the possible framework for a mathematical understanding of quantum gravity.

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