The relationship between monopole harmonics and spin-weighted spherical harmonics

Tevian Dray
Instituut voor Theoretische Fysica, Princetonplein 5, Postbus 80.006, 3508 TA Utrecht, The Netherlands[a] *and School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540*

(Received 1 October 1984; accepted for publication 28 December 1984)

We compare two independent generalizations of the usual spherical harmonics, namely monopole harmonics and spin-weighted spherical harmonics, and make precise the sense in which they can be considered to be the same. By analogy with the spin-gauge language, raising and lowering operators for the monopole index of the monopole harmonics can immediately be written down.

I. INTRODUCTION

Once again physicists in two completely different areas have independently developed the same mathematics. Wu and Yang[1] introduced[2] monopole harmonics as particular solutions of the Schrödinger equation for an electron in the field of a Dirac magnetic monopole. Newman and Penrose[3] introduced[2] spin-weighted spherical harmonics as a means to describe certain quantities exhibiting a particular "spin-gauge" behavior which occur naturally in the asymptotic expansion of the gravitational field in null directions.

In what follows we compare these two generalizations of the usual spherical harmonics and show that, for a particular choice of spin gauge, the spin-weighted spherical harmonics reduce to the monopole harmonics. As a simple application of this result, we note that the fundamental operators in the spin-gauge language raise or lower the spin weight by 1. Thus, writing these operators in the appropriate gauge immediately yields operators which raise or lower the monopole index of the monopole harmonics by 1. Going in the other direction, we adapt the angular momentum operators of the Schrödinger picture to the spin-gauge language and derive the corresponding operators there.

In Sec. II we first review monopole harmonics and in Sec. III we do the same for spin-weighted spherical harmonics. We compare the two in Sec. IV and then discuss our results in Sec. V.

II. MONOPOLE HARMONICS

The term "monopole harmonics" was first used by Wu and Yang[1] to describe solutions of the Schrödinger equation for an electron in the field of a magnetic monopole. However, the functions used in this description are almost as old as the relevant Schrödinger equation itself, which dates back to the original paper on monopoles by Dirac.[4]

These functions were first discussed by Tamm[5] and Fierz[6] and then by numerous other authors.[7]

The fundamental difference in the approach of Wu and Yang[1] is that elements of their Hilbert space are not functions at all, but rather sections of a particular fiber bundle. This eliminates the string singularity of the original description of the Dirac monopole. Although the presentation below follows Wu and Yang[1] we will deliberately deemphasize the underlying fiber bundle structure.

Define the regions R_a and R_b on the sphere by

$$R_a = \{0 \leqslant \theta < \pi\}, \quad R_b = \{0 < \theta \leqslant \pi\}. \tag{1}$$

The relevant Schrödinger equation is

$$[-(1/r^2)\partial_r(r^2\partial_r) + (1/r^2)[L^2 - q^2] + V - E]\psi = 0, \tag{2}$$

where $V(r)$ is the potential, E is the energy eigenvalue, L^2 is the total angular momentum operator, and $q = eg$ (see Ref. 8).

One makes the ansatz

$$\psi(r,\theta,\varphi) = R(r)Y_{qlm}(\theta,\varphi), \tag{3}$$

where the Y_{qlm} are characterized by their angular momentum eigenvalues

$$L^2Y_{qlm} = l(l+1)Y_{qlm}, \quad L_zY_{qlm} = mY_{qlm}. \tag{4a}$$

We also have

$$L_\pm Y_{qlm} = [(l \mp m)(l+1 \mp m)]^{1/2}Y_{qlm\pm 1}. \tag{4b}$$

The fiber bundle structure can be interpreted as follows: The angular momentum operators take different forms in regions R^a and R^b, leading to different functions Y^a_{qlm} and Y^b_{qlm} which *together* make up a monopole harmonic Y_{qlm}. In this paper, however, we will only be concerned with the *functions* Y^a_{qlm} and Y^b_{qlm}.

The angular momentum operators are

$$L^a_z = -i\partial_\varphi - q, \tag{5a}$$

$$L^a_\pm = e^{\pm i\varphi}\left(\pm\partial_\theta + \frac{i\cos\theta}{\sin\theta}\partial_\varphi - \frac{q(1-\cos\theta)}{\sin\theta}\right), \tag{5b}$$

$$(L^2)^a = -\Delta + \frac{2iq}{\sin^2\theta}(1-\cos\theta)\partial_\varphi + \frac{2q^2}{\sin^2\theta}(1-\cos\theta)$$
$$= -\Delta + \frac{2q}{\sin^2\theta}(\cos\theta - 1)L^a_z, \tag{5c}$$

$$L^b_z = -i\partial_\varphi + q, \tag{5d}$$

$$L^b_\pm = e^{\pm i\varphi}\left(\pm\partial_\theta + \frac{i\cos\theta}{\sin\theta}\partial_\varphi - \frac{q(1+\cos\theta)}{\sin\theta}\right), \tag{5e}$$

$$(L^2)^b = -\Delta - \frac{2iq}{\sin^2\theta}(1+\cos\theta)\partial_\varphi + \frac{2q^2}{\sin^2\theta}(1+\cos\theta)$$
$$= -\Delta + \frac{2q}{\sin^2\theta}(\cos\theta + 1)L^b_z, \tag{5f}$$

[a] Permanent address.

 0022-2488/85/051030-04$02.50

where

$$\Delta = \partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi^2$$

is the Laplace operator on the two-sphere.

With appropriate normalization the Y_{qlm} satisfy

$$\int_S Y_{qlm}\overline{Y}_{ql'm'}\,dS = \delta_{ll'}\delta_{mm'}, \tag{6a}$$

where the integral is over the full two-sphere; we note that the integrand is the same in the regions R^a and R^b. We also have

$$Y_{0lm} = Y_{lm}, \tag{6b}$$

$$Y_{qlm} = 0, \quad \text{for } l < |q|, \tag{6c}$$

where the Y_{lm} denote the usual spherical harmonics. Finally, we note that $\{Y_{qlm}\}$ for given q is complete in the following sense: Given any section $f = (f^a, f^b)$, where f^a and f^b are functions on R^a and R^b, respectively, satisfying $f^a = e^{2qi\varphi}f^b$, then f can be expanded as a linear combination of the Y_{qlm}.

III. SPIN-WEIGHTED SPHERICAL HARMONICS

Newman and Penrose[3] introduced spin-weighted spherical harmonics based on ideas in Janis and Newman[9] in order to describe the asymptotic behavior of the gravitational field of isolated systems at large null distances from the source. Although they did this for a particular choice of spin gauge (the "standard" spin gauge) the concept can be immediately generalized to an arbitrary spin gauge. Except for this minor difference our presentation follows Newman and Penrose.[3]

Consider a two-sphere with the usual metric

$$g_{ab}\,dx^a\,dx^b = d\theta^2 + \sin^2\theta\,d\varphi^2. \tag{7}$$

Instead of the usual orthonormal basis $[\partial_\theta, (1/\sin\theta)\partial_\varphi]$, we introduce a complex null basis $(m^a, \overline{m}^a)$ via

$$g_{ab}m^am^b = 0, \quad g_{ab}m^a\overline{m}^b = 2, \tag{8}$$

where the bar denotes complex conjugation. The general m^a can thus be written

$$m^a(\partial_x)_a = e^{i\gamma}[\partial_\theta + (i/\sin\theta)\partial_\varphi]. \tag{9}$$

The choice of the function $\gamma(\theta,\varphi)$ will be called the choice of a spin gauge. We are thus led to consider transformations of the form

$$m^a \mapsto e^{i\Lambda}m^a. \tag{10}$$

A quantity Q whose behavior under this gauge transformation is

$$Q \mapsto e^{is\Lambda}Q \tag{11}$$

is said to have spin weight s [$\mathrm{sw}(Q) = s$]. The simplest example of this is

$$\mathrm{sw}(m^a) = +1, \quad \mathrm{sw}(\overline{m}^a) = -1. \tag{12}$$

Note that not all quantities have a well-defined spin weight. An example of this is

$$2\alpha = -\tfrac{1}{2}m^a\overline{m}^b\nabla_b\overline{m}_a, \tag{13}$$

where ∇_a denotes covariant differentiation on the two-sphere, which transforms under (10) as

$$2\alpha \mapsto e^{-i\Lambda}(2\alpha + i\overline{m}^a\partial_a\Lambda). \tag{14}$$

We can, however, combine m^a and $\overline{\alpha}$ into operators which raise or lower the spin weight. For $\mathrm{sw}(Q) = s$ define[10]

$$ðQ = m^a\partial_aQ + 2\overline{\alpha}sQ, \quad \overline{ð} = \overline{m}^a\partial_aQ - 2\alpha sQ, \tag{15}$$

where ð is the Icelandic letter "edth"; note that $\overline{ðQ}$ is the complex conjugate of $ðQ$ since $\mathrm{sw}(\overline{Q}) = -\mathrm{sw}(Q)$. The fundamental property of these operators is

$$\mathrm{sw}(Q) = s \Rightarrow \begin{array}{l}\mathrm{sw}(ðQ) = s+1,\\ \mathrm{sw}(\overline{ð}Q) = s-1,\end{array} \tag{16}$$

i.e., $\mathrm{sw}(ð) = 1$, $\mathrm{sw}(\overline{ð}) = -1$. We also have

$$[ð,\overline{ð}]Q = -2sQ. \tag{17}$$

The standard gauge is given by choosing $\gamma = 0$ in (9), thus

$$\begin{aligned} ð_0 &= \partial_\theta + (i/\sin\theta)\partial_\varphi - s(\cos\theta/\sin\theta),\\ \overline{ð}_0 &= \partial_\theta - (i/\sin\theta)\partial_\varphi + s(\cos\theta/\sin\theta).\end{aligned} \tag{18}$$

In an arbitrary gauge we have

$$\begin{aligned} ð &= e^{i\gamma}\Big[\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi + s\Big(-\frac{\cos\theta}{\sin\theta} - i\gamma_{,\theta} + \frac{1}{\sin\theta}\gamma_{,\varphi}\Big)\Big],\\ \overline{ð} &= e^{-i\gamma}\Big[\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi - s\Big(-\frac{\cos\theta}{\sin\theta} + i\gamma_{,\theta} + \frac{1}{\sin\theta}\gamma_{,\varphi}\Big)\Big].\end{aligned} \tag{19}$$

We can now obtain the spin-weighted spherical harmonics (for integer spin) ${}_sY_{lm}$ by raising and lowering the spin weight of the usual spherical harmonics $Y_{lm}(\theta,\varphi)$ [$\mathrm{sw}(Y_{lm}) = 0$][11]

$${}_sY_{lm} = \begin{cases}\left[\dfrac{(l-s)!}{(l+s)!}\right]^{1/2} ð^sY_{lm}, & 0 \leqslant s \leqslant l,\\ \left[\dfrac{(l+s)!}{(l-s)!}\right]^{1/2}(-1)^s\overline{ð}^{-s}Y_{lm}, & -l \leqslant s \leqslant 0,\\ 0, \quad l < |s|.\end{cases} \tag{20}$$

We summarize the properties of the ${}_sY_{lm}$

$$\mathrm{sw}({}_sY_{lm}) = s, \tag{21a}$$

$$ð({}_sY_{lm}) = +[(l-s)(l+s+1)]^{1/2}\,{}_{s+1}Y_{lm}, \tag{21b}$$

$$\overline{ð}({}_sY_{lm}) = -[(l+s)(l-s+1)]^{1/2}\,{}_{s-1}Y_{lm}, \tag{21c}$$

$${}_0Y_{lm} = Y_{lm}, \tag{21d}$$

$$\int_S {}_sY_{lm}\,{}_s\overline{Y}_{l'm'}\,dS = \delta_{ll'}\delta_{mm'}. \tag{21e}$$

We can ask if there are generalizations of the usual angular momentum operators, i. e., operators L_z, $L_\pm$, L^2 satisfying [cf. (4)]

$$\begin{aligned} L_z\,{}_sY_{lm} &= m\,{}_sY_{lm},\\ L_\pm\,{}_sY_{lm} &= [(l\mp m)(l+1\pm m)]^{1/2}\,{}_sY_{lm\pm1},\\ L^2\,{}_sY_{lm} &= l(l+1)\,{}_sY_{lm}.\end{aligned} \tag{22}$$

Since these imply that

$$[L,ð] = 0 = [L,\overline{ð}], \tag{23}$$

where L represents any of the angular momentum operators, one can easily solve for these operators. The result is [12]

$$L_z = -i\,\partial_\varphi - s\gamma_{,\varphi},$$

$$L_\pm = e^{\pm i\varphi}\Big[\pm\partial_\theta + \frac{i\cos\theta}{\sin\theta}\partial_\varphi + s\Big(-\frac{1}{\sin\theta} \mp i\gamma_{,\theta} + \frac{\cos\theta}{\sin\theta}\gamma_{,\varphi}\Big)\Big],$$

$$L^2 = -\Delta' + \frac{2s\cos\theta}{\sin^2\theta}L_z + \frac{s^2}{\sin^2\theta}$$
$$= -\Delta + is(\Delta\gamma) - 2s\gamma_{,\theta}(L_y\cos_\varphi - L_x\sin_\varphi) + \frac{2s}{\sin^2\theta}(\cos\theta - \gamma_{,\varphi})L_z - s^2\Big(\gamma_{,\theta}^2 + \frac{\gamma_{,\varphi}^2 - 1}{\sin^2\theta}\Big),$$

where

$$\Delta' = \Delta - is(\Delta\gamma) - 2is(\gamma_{,\theta}\partial_\theta + \gamma_{,\varphi}\partial_\varphi/\sin^2\theta) - s^2(\gamma_{,\theta}^2 + \gamma_{,\varphi}^2/\sin^2\theta),$$

and

$$L_\pm = L_x \pm iL_y. \quad (24)$$

Here, Δ' is just the operator obtained from Δ by the substitutions

$$\partial_\varphi \mapsto \partial_\varphi - is\gamma_{,\varphi}, \quad \partial_\theta \mapsto \partial_\theta - is\gamma_{,\theta}.$$

Note that in the standard gauge, denoted "0" the $L^0_\pm$ are just the angular momentum operators $\hat{J}_\pm$ given in Landau and Lifschitz[13] for the symmetric top (with k there identified with $-s$ here). The similarity between the symmetric top operators and the Y_{qlm} has already been pointed out, e. g., in Ref. 6.

IV. COMPARISON OF MONOPOLE AND SPIN-WEIGHTED SPHERICAL HARMONICS

Comparing (24) with (5) we see that if we introduce the gauges A, defined by $\gamma = +\varphi$, and B, defined by $\gamma = -\varphi$ [in (9)], and if we make the identification $q = s$, then

$$L^a \equiv L^A, \qquad L^b \equiv L^B, \quad (25)$$

where L again represents any of the angular momentum operators. But since the Y_{qlm} are fully determined *up to a constant phase factor for each* q by specifying q, the behavior of the angular momentum operators [Eq. (4)], and the normalization condition (6a), and since the ${}_sY_{lm}$ have the same behavior with respect to angular momentum [Eq. (22)] and the same normalization [Eq. (21e)], we see that Y_{qlm} and ${}_qY_{lm}$ differ at most by a constant (q-dependent) phase factor. With our ${}_sY_{lm}$ as defined in (20) we have

$$Y^a_{qlm} = {}_qY^A_{lm}, \quad Y^b_{qlm} = {}_qY^B_{lm}. \quad (26)$$

This is our main result.

Note that we can now immediately give raising and lowering operators for the monopole index of the monopole harmonics; these are just $\eth$ and $\bar{\eth}$ in the appropriate gauge:

$$\eth^A = e^{+i\varphi}\Big(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi + q\frac{(1-\cos\theta)}{\sin\theta}\Big),$$
$$\eth^B = e^{-i\varphi}\Big(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi - q\frac{(1+\cos\theta)}{\sin\theta}\Big), \quad (27)$$
$$\bar{\eth}^A = e^{-i\varphi}\Big(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi - q\frac{(1-\cos\theta)}{\sin\theta}\Big),$$
$$\bar{\eth}^B = e^{+i\varphi}\Big(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi + q\frac{(1+\cos\theta)}{\sin\theta}\Big).$$

[To obtain the correct normalization merely divide these by the constant on the right side of (21b) or (21c) with $s = q$.]

V. DISCUSSION

Our result (26) should not be surprising. The monopole harmonics are analytic, whereas the operator $\eth_0$ has a direction-dependent limit at $\theta = 0$ and $\theta = \pi$. Going to the gauge A or B is necessary in order to turn $\eth$ into an analytic operator on the region R^a or R^b![14]

Futhermore, since the ${}_sY_{lm}$ of course have spin weight s, our result can be interpreted as follows: Remove the explicit q dependence (i. e., $e^{\pm iq\varphi}$) from the ${}_qY^{a,b}_{lm}$. The result is precisely the spin-weighted spherical harmonics ${}_qY^0_{lm}$ *in standard gauge*.

We have only explicitly treated the spin-weighted spherical harmonics for integer spin. However, the argument used in Sec. III to introduce the angular momentum operators L can be inverted: we could equally well *define* the spin-weighted spherical harmonics as eigenfunctions of L. It is then obvious that the results of Sec. IV are also valid for half-integer spin.

Note added: In fact, if we let ${}_sY^\gamma_{lm}$ denote the spin-weighted spherical harmonics in *spin gauge* γ [Eq. (9)] then[11]

$${}_sY^\gamma_{lm} \equiv e^{is\gamma}\,{}_sY^0_{lm}(\theta,\varphi) \quad (28)$$
$$\equiv \frac{(-1)^s(2l+1)^{1/2}}{(4\pi)^{1/2}}D^l_{-sm}(\varphi,\theta,\gamma),$$

where the D^l_{-sm} are the Wigner D functions as given by Goldberg *et al.*[15] Thus, choosing a gauge γ in the sense of this paper corresponds to fixing a Euler angle $(-\gamma)$ in the argument of the Wigner D functions. As pointed out by the referee, the spin-weighted spherical harmonics in standard gauge ${}_sY^0_{lm}$ and the monopole harmonics $Y^{a,b}_{qlm}$ merely correspond to different choices of this Euler angle.

ACKNOWLEDGMENTS

I would like to thank Professor Chen Ning Yang for providing Ref. 7, Peter Batenburg for bringing the symmetric top (Ref. 13) to my attention and for pointing out some mistakes in the original manuscript, Malcolm Perry and Annti Niemi for discussions on the Wigner D functions, and the referee for suggesting Ref. 15.

This work was supported in part by the Stichting voor Fundamenteel Onderzoek der Materie.

[1]T. T. Wu and C. N. Yang, Nucl. Phys. B **107**, 365 (1976).

[2]In both cases the original ideas can be traced to earlier work. This is discussed in Secs. II and III below.

[3]E. T. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966).

[4]P. A. M. Dirac, Proc. R. Soc. London Ser. A **133**, 60 (1931).

[5]Ig. Tamm, Z. Phys. **71**, 141 (1931).

[6]M. Fierz, Helv. Phys. Acta **17**, 27 (1944).

[7]P. P. Banderet, Helv. Phys. Acta **19**, 503 (1946); Harish-Chandra, Phys. Rev. **74**, 883 (1948); A. S. Goldhaber, Phys. Rev. **140**, B1407 (1965).

[8]We have set $\hbar = c = 1$; e is the electron charge and g the strength of the monopole.

[9]A. I. Janis and E. T. Newman, J. Math. Phys. **6**, 902 (1965).

[10]These differ by a factor of (-1) from the operators defined in Ref. 3.

[11]These $_sY_{lm}$ differ from those in Ref. 3 by a factor $(-1)^s$. The reason for this choice will become clear in Sec. IV.

[12]These operators reduce to the standard ones for $s = 0$. Note that strictly speaking "s" must be interpreted as an operator.

[13]L. D. Landau and E. M. Lifschitz, *Quantum Mechanics* (Pergamon, New York, 1977), 3rd ed., p. 417; see also F. Reich and H. Rademacher, Z. Phys. **39**, 444 (1926).

[14]This idea is implicit in Ref. 3.

[15]J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudershan, J. Math. Phys. **8**, 2155 (1967).