Predicting Critical Flow Velocity and Laminate Plate Collapse – Flat Plates

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Submitted to Nuclear Engineering and Design as a full length article
March 25, 2014

This manuscript has not been published elsewhere and has not been submitted simultaneously for publication elsewhere.
ABSTRACT

The Oregon State University (OSU), Hydro Mechanical Fuel test Facility (HMFTF) is designed to hydro-mechanically test prototypic plate type fuel. OSU’s fuel test program is a part of the Global Threat Reduction Initiative (GTRI), formerly known as the Reduced Enrichment for Research and Test Reactor program. One of the GTRI’s goals is to convert all civilian research, and test reactors in the United States from highly enriched uranium (HEU) to a low enriched uranium (LEU) fuel in an effort to reduce nuclear proliferation.

An analytical model has been developed and is described in detail which complements the experimental work being performed by the OSU HMFTF, and advances the science of hydro-mechanics. This study investigates two methods for determining the ‘critical flow velocity’ for a laminate plate. The objective is accomplished by incorporating a flexural rigidity term into the formulation of critical flow velocity originally derived by Donald R. Miller, and employing sandwich structure theory to determine the rigidity term. The final outcome of this study results in the developing of a single equation for each of three different edge boundary conditions which reliably and comprehensively predicts the onset of plate collapse. The two models developed and presented, are termed the monocoque analogy and the ideal laminate model. Of these two models, the ideal laminate model is the most resolved and comprehensive in its predictions.
1 INTRODUCTION

The Oregon State University (OSU), Hydro Mechanical Fuel test Facility (HMFTF) is designed to test prototypical plate type fuel (Marcum 2011). OSU’s fuel test program is a part of the Global Threat Reduction Initiative (GTRI), formerly known as the Reduced Enrichment for Research and Test Reactor program. One of the GTRI’s goals is to convert all civilian research, and test reactors in the United States from highly enriched uranium (HEU) to a low enriched uranium (LEU) fuel in an effort to reduce nuclear proliferation (Leventhal and Kuperman 1995; NNSA 2004; Loukianova and Hansell 2008). Although, numerous reactors have converted their fuel throughout the United States, several test reactors are designed in a manner which prevents them from converting to LEU fuel using previously qualified fuel forms (Glaser and VonHippel 2006; Staples 2008). This is because the reactors in questions operate with high power densities; they are referred to as High Performance Research Reactors (HPRR). The OSU HMFTF is in the process of experimentally evaluating a proposed LEU fuel under hydraulic loading. The proposed fuel is a laminated composite flat plate with Aluminum (Al) clad, and monolithic Uranium-Molybdenum (U-Mo) alloy core, which is in contrast to the uranium-aluminum dispersion fuel currently used (Wachs, Clark et al. 2008).

The HPRRs use a unique plate-type fuel geometry which promotes neutron efficiency and high power densities, however flat thin fuel plates tend to be mechanically weak under certain hydraulic conditions (1992). This weakness requires a design criterion that identifies conditions in which fuel plates of a certain form are susceptible to mechanical ‘failure’. Failure as defined by the OSU HMFTF test plan, is the ‘onset of plastic deformation’ (Marcum 2012), this is in contrast to ‘clad breach’ which is typically considered as fuel failure (Knief 2008). Prior work in plate fuel resulted in numerous flow metrics for predicting the hydro-mechanical stability of a plate under static and dynamic conditions (Miller 1958; Johansson 1960; Wambsganss 1967; Smith 1968; Marcum and Woods 2012). Specifically, flow metrics that predict the static stability limits for fuels plates have traditionally focused on predicting the maximum flow rate, known as critical fluid velocity (originally derived by D.R. Miller (Miller 1958)), across a plate which causes ‘plate collapse’ (Miller 1958),(Zabriskie 1958). Work has also been done which presents this critical flow velocity in terms of critical dynamic pressure (Smith 1968).
‘Plate collapse’ is considered mechanical failure in these studies, and is a term that is left without a specific definition. For the sake of this study it can also be thought of as failure, and implies that the plate has deflected to such an extent that a linearized beam deflection model is no longer acceptable to characterize the phenomena at hand (Wambsganss 1967). Such large deflections fall out of the applicability of the linearized method proposed in Miller’s original work, but also disallows the use of fundamental tools such as the Euler-Bernoulli beam equation which too requires small deflections in order to be credibly applied in a study (Bedford and Liechti 2000; Young and Budynas 2002; Goodwine 2011). The work presented herein utilizes the linearized method suggested by Miller along with the Euler-Bernoulli beam equation, therefore it is assumed to be valid up to, but not beyond the point of plate collapse.

Previous work has shown the effectiveness of flow metrics such as the critical flow velocity predicted by Miller’s model, for already qualified and currently deployed plate type fuel (Groninger and Kane 1963). However, at this time little theoretical work has been done toward the development of a critical flow velocity model for laminated plate-type fuel, or laminate fuel.

The laminate fuel plate in question, is constructed with three discrete layers (Wachs 2007; Wachs, Clark et al. 2008). The inner layer is the U-Mo fueled region, and the outer two layers are the aluminum cladding. Structures comprised of three discrete laminated regions encompass a class of structures studied in composites known as ‘sandwich structures’ (Avallone and Baumeister-III 1995; Vinson 1999; Carlsson and Kardomateas 2011). For this study, isotropic sandwich structure theory is adapted to critical flow velocity. This is because sandwich structure theory characterizes each region, and quantifies how the three regions interact and contribute to the overall stiffness of the structure (Reddy 1997; Vinson 1999; Carlsson and Kardomateas 2011).

A method must be employed that applies sandwich structure theory to critical flow velocity toward and application which is suitable for use in the safety analysis effort for reactors using plate type fuel with three discrete layers. Therefore, this analysis is intended to be transparent in
its derivation, and fundamental in use while providing a credible and reliable metric to supplement plate-type research reactor safety analyses.

This study develops a flow metric that predicts a fluid’s velocity, necessary to induce static mechanical instability of a plate undergoing hydraulic loading. This is referred to as critical flow velocity. It specifically caters to axial fluid flow over a flat laminate plate with two adjacent channels having fluid passing through them.

2 LITURATURE SURVEY

This survey of literature is broken into two sections. The first section surveys work on hydro-mechanical stability of plates under axial flow conditions. The second section surveys work on the mechanics of laminated plates (Jensen 2013).

2.1 Flow induced deflection of flat plates

Flow induced deformation and vibration of flat plates has been studied by numerous authors. This is primarily because the flat plate geometry is mechanically weaker than other fuel plate geometries (e.g. cylindrical, involute) (Marcum and Woods 2012).

One of the first studies concerning fuel plates and flow induced phenomena was done by Stromquist and Sisman in 1948 (Stromquist and Sisman 1948). They created one of the first experimental facilities dedicated to studying flow induced instability of reactor fuel plates, and tested curved and flat plates. They were able to measure pressure drop through their test element, frequency, and amplitude of the observed vibrations. Also noted was “buckling” of fuel plates, due to a pressure difference between flow channels. This reference to buckling was one of the first references to what has become known as plate collapse. Stomquist and Sissman concluded that the plates they tested were of “adequate strength” to withstand the vibrational phenomena measured. Also, they made very useful observations noting that channel dimensions, mechanical edge boundary conditions, and manufacturing defects (e.g. brazing defects) contributed to the occurrence of the buckling phenomena.
Following Stomquist and Sissman, Doan published a report on technical issues that arose during the initial operation of the Engineering Test Reactor in 1958 (Doan 1958). In this report Doan, noted that a pressure differential was formed between two adjacent flow channels which drove plate instability. Also listed were a number of proposed modifications to the reactor’s fuel in order to mitigate these effects.

In 1958 Donald R. Miller published his work which set about creating a formula to predict the critical flow velocity that causes “plate collapse” (Miller 1958). Miller’s work is paramount; it represents the first and most replicated attempt to quantify critical flow velocity. Miller studied both flat and curved plates, and devised a theoretical prediction for critical flow velocity by pairing fluid mechanics with solid mechanics. He approached the solid domain by noting that deflection of a plate is caused by a distributed force on its surface. This distributed force is analogues to the pressure difference that arises between two adjacent flow channels developed within the fluid domain. Knowing this, the plate deflection can be quantified using the Euler-Bernoulli beam equation in conjunction with the wide beam approximation. This deflection is then used to find the relative distortion of the flow channel. Bernoulli’s equation is then linearized, the velocity change between two adjacent channels due to plate deflection is noted, and the critical flow velocity is solved. This pairing of the fluid domain with the solid domain is important, because it showed that hydro-mechanical stability of plates is not exclusively dependent on the hydraulic conditions but also the solid mechanical conditions. This critical flow velocity is denoted as “Miller’s velocity” by many authors. Although Miller developed a new and useful model for predicting plate buckling under hydraulic loading, a significant number of assumptions were required. For flat plates, Miller assumed (Miller 1958):

- The plate is isotropic, homogenous, linearly elastic, initially un-deformed and perfectly flat, deforms symmetrical about their neutral axis, has perfect mechanical constants (i.e. Young’s Modulus, Poison’s Ratio), and its deflections are small enough to allow the use of the Euler-Bernoulli beam theorem.
- The fluid is incompressible and isothermal, and the flow is steady and uniform for all flow channels at any given point along the channel length.
- Shear in the plates is considered negligible, and the plane stress assumption is applied.
- The plate edge supports are represented by ideal edge boundary conditions.
Note that Miller’s use of the beam equation and the wide beam approximation, are only an estimation of the effects observed. This is because the fuel plates in question are inherently plates, and not beams. The use of the wide beam approximation also assumes that there is no lateral deformation in the $z$-direction as shown in Figure 1, which implies that the strain in the $z$ direction is zero ($\varepsilon_z \approx 0$), and that a plane stress state exist such that the strain in the $y$ direction is zero ($\varepsilon_y \approx 0$) (Budynas 1977).

Figure 1 presents a general cantilever beam undergoing a load $P$, where the depth of the beam ($d$) is much greater than the thickness ($t$) ($d \gg t$) which indicates acceptable geometry for the application of wide beam theory.

(Please Insert Figure 1 Here)

The appropriateness of the application of the wide beam approximation may be debated. In most instances the deformation along the depth ($z$) of an actual fuel plate are in fact negligible. Also, the plane stress assumption remains valid along the depth. However, this is not true at the upstream or downstream edge of the plates (i.e. $z_{\text{max}}$ and $z_{\text{min}}$) (Smith 1968; Marcum and Woods 2012). It is known that the plane stress state is not valid at the edges (Seely and Smith 1952), and normal stress at the leading edge of the plate due to flow impingement causes a localized $z$-direction deformation. Regardless, most reactor fuel plates conform closely to the wide beam approximation because of their aspect ratios (i.e. $d \gg t$).

Following Miller’s study Zabriskie conducted experimental work to attempt to validate Miller’s critical flow velocity (Zabriskie 1958; Zabriskie 1959). Zabriskie’s experimental work addressed both measurements of critical flow velocity and the effects of varying length ($d$) and width ($L$) on critical flow velocity. He tested a number of assemblies with single plates, multiple plates, different channel dimensions, different lengths, and different widths. Most notably Zabriskie acknowledged that the critical flow velocity predicted by Miller’s method does not cause a “collapse” of the flow channel, rather a point of maximum plate deflection is reached. He also
noted the effects at the leading edges (i.e. assembly inlet) were very pronounced, and could be easily mitigated with the addition of an “inlet support comb”; this structural feature is designed to support and mechanical restrain the leading edge of such plates. One of the most ubiquitous trends started by Zabriskie was in comparing the measured critical velocity as a ratio to that predicted by Miller’s method. Subsequent authors have continued this trend by not only using the ratio of measured critical flow velocities to Miller’s method, but also the ratio of their own critical flow velocity models to Miller’s method.

Kane advanced the area of hydro-mechanics following Zabriskie by preforming a theoretical analysis that varied the inlet spacing conditions (Kane 1963). He concluded that small deviation in the inlet spacing conditions, could have profound effects on the observed deflections.

In 1963 Groninger and Kane tested three parallel plate assemblies (Groninger and Kane 1963). Their work is of significant relevance to this study because some of the plates that were tested were of heterogeneous construction. Groninger and Kane worked around this construction by empirically testing these plates in a manner that allowed them to create a hypothetical equivalent plate of homogenous zirconium with an equivalent thickness. This hypothetical plate with equivalent thickness was then applied to Miller’s method for predicting the critical flow velocity. The validity of such a method is difficult to gauge, because no details concerning the empirical test were provided by Groninger and Kane. The study presented had a number of interesting results which showed that the plates they tested never violently collapsed, but did show large deflections (agreeing with the findings of Zabriskie (Zabriskie 1958)). The point of maximum deflection was measured at approximately twelve to fifteen inches from the inlet, for a plate with a total length of approximately eighty six inches. They also noted that the onset of plate vibration occurred at approximately 1.9 times Miller’s critical flow velocity. Finally, they noted that adjacent plates consistently deform in opposite directions, with similar magnitudes.

Several authors have proposed modifications to Miller’s work; this has typically taken the form of a variety of multiplicative coefficients. Johansen was the first to do this, and included a number of terms that quantify flow redistribution and frictional effects within the flow channel adjacent to the plate of interest (Johansson 1960). Wambsganss did this as well; he attempted to
capture some of the information “lost in the linearization process” by approximating the deformation contour along the plate’s span-width direction (Wambsganss 1967).

Of these modifications the work of Smith, has been employed in the nuclear safety culture more than others (Smith 1968). Smith chose to redefine critical flow velocity as critical dynamic pressure. Smith’s model is also a semi-empirical model, based on a series of test preformed with gaseous fluid flows and several homogenous plate materials. He modified Miller’s theoretical analysis by including factors that quantified the increased deflection at the edges, and the angle of attack produced at the edge. These were termed the “area modification” and the “lift modification” factors.

In 1968 Smissaert did experimental and analytical work on flow induced deflection and vibration of flat plates (Smissaert 1968). Smissaert tested several sets of flat plates; most notably he performed tests with flow rates as high as three and a half times Miller’s critical flow velocity. He experimentally observed two critical flow velocities, the first being the critical flow velocity predicted by Miller which signified the onset of large static deflections. The second was termed “flutter velocity” and signified the onset of plate vibrations. This was observed at approximately two times Miller’s critical velocity (as observed by Groninger and Kane to be 1.9 (Groninger and Kane 1963)).

Kim and Scarton used various computational tools to better analyze flow induced deflections near the inlet of an assembly (Kim and Scarton 1977). They concluded that “viscous shear” played a large role in causing deflections at the plate’s leading edge. They also confirmed earlier work that recommended the use of plates with very small aspect ratios.

More recently, in 2007, a series of tests were performed by Ho, Hong, and Mack in support of the Australian Replacement Research Reactor (Ho, Hong et al. 2004). They observed that max deflection occurred at 75% of Millers critical flow velocity. It is important to note that their tests were performed without the use on an inlet support comb. This is significant as most authors have noted general agreement with the Miller’s critical flow velocity only when a support comb was used.
2.2 Sandwich Structure Theory

The mechanics of laminated structures have been studied by many researchers. A subclass of laminated structures is that of sandwich structures; sandwich structures have three discrete layers (i.e. two outer layers and one inner layer). Sandwich structures became the subject of intense study due to efforts to lighten aircraft structures without compromising the rigidity of the structure (Whitney 1987; Reddy 1997; Vinson 1999; Carlsson and Kardomateas 2011).

A very early study of sandwich structures was conducted by Gough, Elam, and DeBruyne in 1940 (Gough, Elam et al. 1940). The authors discussed in detail a number of laminated structures and physical arguments specific to certain structural configurations and why they are superior to others. This study contributed significantly to the body of knowledge in mechanics of laminate structures because it is explicit in stating the assumptions and reasoning behind sandwich structure design. They also, performed a number of experiments and presented results for early laminates used in aircraft production.

Another early study of sandwich construction was conducted in 1941 by William et al. for the United Kingdom’s Ministry of Aircraft Production (Williams, Leggett et al. 1941). This report is of note because it points out many distinctions between sandwich type construction, and other laminated constructions that may take a similar form. The primary difference between such structures centers on buckling of the sandwich plate; buckling is irreversible for plates with sandwich construction. This is because the inner and outer regions of a sandwich structure are bonded and move as one mechanical unit. In addition William et al. presents an analytical and approximate theoretical solution for buckling of a sandwich plate that is simply supported on all four sides.

In 1948 Libove and Batdorf developed a form of ‘small deflection theory’ for sandwich plates (Libove and Batdorf 1948), this work was performed under the National Advisory Committee for Aeronautics (NACA), an agency that later became known as the National Aeronautics and Space Administration (NASA). They developed both the energy expression and differential equation for displacement, for orthotropic and isotropic plates. In contrast to isotropic materials,
orthotropic material is one that has different mechanical properties in at least two of three mutually perpendicular directions (e.g. wood)

Also in 1948, Reissner preformed a study that focused on effects on the core material (i.e. inner region material) (Reissner 1948) of sandwich structures. He also developed quantitative methods for determining when solid mechanical analysis of a sandwich plate may no longer be analyzed linearly (i.e. small deflections), and noted that this effect becomes more pronounced as the core material became “softer”.

Hoff and Mautner presented results for beams of sandwich construction (Hoff and Mautner 1948). Their solutions are relatively simple, and they presented experimental work that closely matched their theoretical values.

An important study, which has been cited by many authors to follow, was done by Hoff in 1950 for NACA (Hoff 1950). Hoff solved the differential equation for bending, and buckling under compressive end loads for a sandwich plate with simply supported edge conditions.

Next Eringen presented the solution for a sandwich structure in which the face thicknesses were no longer considered to be thin, and “flattening” of the core material was taken into account (Eringen 1951). This is important because it acknowledges cases in which the plane stress assumption is no longer valid.

In 1958 Ericksen preformed an analysis on sandwich structures that had unequal face material (i.e. outer region) thicknesses (Ericksen and March 1958). A number of simplifications that greatly aid in the analysis of sandwich structures with equal face thicknesses no longer apply in the case where the faces are of unequal thickness. This results in a much more complex theoretical formulation for characterizing buckling of laminate plate structures.

A more recent paper by Yan and Dowell focused on vibrations in a sandwich structures, and cases in which the governing equations of motion may be acquired without directly solving the classical equations (Yan and Dowell 1974). These cases are typically specialized and involve
creating an equivalent beam with varying thickness, and may be thought of as a homogenization technique. Several other authors propose using techniques such as homogenization or the parallel axis theorem in order to find the flexural rigidity of a beam (Whitney 1987; Cavallaro and Jee 2008; Carlsson and Kardomeas 2011). It is important to note that these methods are only valid in certain circumstances.

Much of the work in sandwich structures that was done in the latter part of the twentieth century has been collected into printed books on the subject. Three such books of note are by Vinson (Vinson 1999), Carlsson and Kardomeas (Carlsson and Kardomeas 2011), and Reddy (Reddy 1997). The works of Vinson, and Carlsson and Kardomeas proved especially useful in the conducting this study.

Vinson provides a significant amount of information on sandwich construction. This includes derivation of the governing equation, solutions for beams, columns, and rods, the application of energy methods to sandwich structures, the solutions for rectangular plates, dynamic effects and vibrations, and sandwich shells (Vinson 1999).

Carlsson and Kardomeas’ work provides similar information to that of Vinson’s. They include information on the derivation of the governing equations, first and higher order methods, global buckling, wrinkling and other localized instabilities, and information concerning de-bonding of layers (Carlsson and Kardomeas 2011).

2.3 Closing

The works presented herein represent a significant amount of research effort in the fields of hydro-mechanical instabilities and laminate structural analysis. While a number of works have analyzed hydro-mechanical instabilities of homogenous plates, very few have theoretically focused on plates of laminated construction. As noted this study aims to incorporate laminate structural analysis into the study of hydro-dynamical instability. Table 1 provides a summary of the literature reviewed organized by subject and study technique (i.e. experimental or theoretical).
3 MODEL AND METHODOLOGY

This section describes the derivation and physical construct of two unique models used to estimate the critical flow velocity necessary to cause plate collapse for a laminate plate having three discrete regions. These models follow a formal two step methodology. The first step defines the hydraulic domain which results in the net force on the plate of interest as a function of the plate’s flexural rigidity. The second step focuses exclusively on flexural rigidity, and the different perspectives one may take when constructing this term.

3.1 Critical Flow Velocity and Critical Dynamic Pressure

The critical flow velocity that predicts plate collapse for a flat rectangular plate was first formally derived by D.R. Miller (Miller 1958). Since then numerous studies have been performed on the subject of critical flow velocity, many of which propose new and innovative methods to quantify and explain this phenomena. All known studies which focus on Miller’s original method have only dealt with homogenous plates, as such these studies and subsequent models have incorporated the material and geometric properties of a homogenous plate. While the following derivation shares many similarities to Miller’s original work, it also expands on it and all known previous studies by creating three discrete mathematical regions for a rectangular flat laminate plate while maintaining continuity of stress throughout each cross-sectional region.

In order to determine the critical flow velocity for a laminated plate, a relation must be first formulated in terms of flexural rigidity. This is done through a simple modification to the Euler-Bernoulli beam equation (Boresi and Schmidt 2003). This modification takes place through the grouping of all material properties, and geometric information for a given cross-sectional region within the flexural rigidity \((D)\) term.

\[
D \frac{d^4 y}{dx^4} = P(x)
\]  

(1)

Equation (1) represents the out of plane deflection \((y)\) in relation to the imposed pressure on the beam \((P)\) as a function of span width \((x)\), and flexural rigidity \((D)\). This study’s motivation
centers on reactor fuel plates, as such there are three beam boundary conditions which are commonly used for reactor safety analysis. These three boundary conditions refer to the edges of the plate, including: (a) clamped on both edges (Figure 2a), (b) clamped on one edge and simply supported on the other (Figure 2b), and (c) simply supported on both edges (Figure 2a) (Luttrell 1995).

(Please Insert Figure 2 Here)

Mathematically, the boundary conditions presented in Figure 2 may be expressed as:

- Both edges clamped (Figure 2a):
  \[ y = \frac{\partial y}{\partial x} = 0, \quad x = 0, L \]  \hspace{1cm} (2)

- One edge clamped and one edge simply supported (Figure 2b):
  \[ y = \frac{\partial y}{\partial x} = 0, \quad x = 0 \]
  \[ y = M_x = 0, \quad x = L \] \hspace{1cm} (3) (4)

- Both edges simply supported (Figure 2c):
  \[ y = M_x = 0, \quad x = 0, L \] \hspace{1cm} (5)

The following outlines the complete derivation assuming both edges of the plate are clamped to provide context for the method employed. One may obtain the same results presented herein for all boundary condition sets considered if the same mathematical method is followed. Solving equation (1), assuming both edges clamped (utilizing the boundary conditions defined in equation (2)) yields the following analytical equation for out of plane deflection (Bedford and Liechti 2000).

\[ y(x) = \frac{P}{24D} \left( x^4 - 2Lx^3 + L^2x^2 \right) \] \hspace{1cm} (6)

It is assumed in this study that the fluid flow, or hydraulic load imposed on the plate(s), is steady at the instant in time the plate(s) begin to deform, resulting in a change in flow area. This change is found by doubling the integration of equation (6) across the entire span width of a plate \((x \in 0, L)\). The change in flow area impacts the hydraulic domain by yielding a net pressure
between adjacent channels. The net pressure is also the boundary condition imposed on the solid domain (i.e. the plate), which results in out of plain deflection. Assuming that the original cross-sectional area is of perfect rectangular geometry, the original flow channel may be represented as the product of channel height \((h)\) and span width \((L)\) yielding \(A_0 = Lh\). A graphical representation of the model geometry may be seen in Figure 3. The relative change in flow area may then be quantified by dividing the perturbed area found by integrating the beam deflection equation, by the original flow channel area as seen in (7).

\[
\frac{\Delta A}{A_0} = \frac{PL^4}{360Dh}
\]  

(7)

Equation (7) provides a relation between an imposed pressure \((P)\) on a ‘wide beam’, relative the beam’s change in out of plane position.

(Please Insert Figure 3 Here)

Leveraging the assumptions which relate conservation of energy to the Bernoulli equation, one may acquire the following form of conservation of energy considering inviscous flow, with no change of internal energy, along a streamline.

\[
\left(\frac{P}{\rho} + \frac{v^2}{2} + gz\right) = \left(\frac{P}{\rho} + \frac{v'^2}{2} + g'z\right) = 0
\]  

(8)

In equation (8), \(P\) represents the local domain pressure, \(\rho\) is fluid density, \(v\) is fluid velocity, \(g\) is the acceleration of gravity, and \(z\) is the height. It may be assumed that the gravitational terms are insignificant; equation (8) can be reformulated into the following. This represents the net pressure difference between two channels.

\[
P_1 - P_2 = \Delta P = \frac{1}{2} \rho \left(v'^2 - v^2\right)
\]  

(9)

In order to use equation (9), relations for each velocity term \((v)\) must developed. This is done by applying the conservation of mass, through a control volume shown by equation (10).

\[
\frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho (v \cdot \vec{n}) dA = 0
\]  

(10)
The volume integral represents the accumulation of mass in a control volume with respect to time, and the surface integral is taken to be the difference between the mass exiting the control volume to mass entering the control volume. Equation (10) is commonly referred to as the continuity equation, which is simply the conservation of mass on a control volume (White 2003). Using green's theorem, the surface integral in equation (10) can be converted to a volume integral shown in (11).

\[ \int_{CS} \rho (\mathbf{v} \cdot \mathbf{n}) dA \rightarrow \int_{CV} \nabla \cdot \rho \mathbf{v} dV \]  

(11)

Assuming steady flow of an incompressible fluid, equation (10) may be simplified. The steady flow assumption causes the time dependent terms to go to zero, and the incompressible assumption infers that there is no change in fluid density (\(\rho\)). The simplification is as follows.

\[ \nabla \cdot \mathbf{v} = 0 \]  

(12)

Integration of equation (12), for a one dimensional control volume assuming steady flow and an incompressible fluid yields the following.

\[ A_i v_i = A_j v_j \]  

(13)

(Please Insert Figure 4 Here)

The changes in flow area for two adjacent flow channels are now applied; this change is illustrated in Figure 4. It is assumed that two adjacent plates will deflect in opposite directions. Assuming the top plate was deflected upwards then the bottom plate would deflect downward with equal magnitude. This observation has been verified both theoretically and empirically (Miller 1958; Groninger and Kane 1963).

When it is assumed that each plate deflects in opposite direction, with equal magnitude, the increase in one flow channel is given by equation (14).

\[ (A_o + \Delta A) v_i = A_o v_o \]  

(14)

While the decrease in the neighboring flow channel is given by (15).

\[ (A_o - \Delta A) v_2 = A_o v_o \]  

(15)
The term \( v_0 \) is the initial flow velocity in the respective channel, and \( v_1 \) and \( v_2 \) are channel 1 and 2 velocities after a flow area perturbation of \( \Delta A(A_0)^{-1} \). Note that from equations (14) and (15) it is assumed that the initial velocities \( (v_0) \) of channel 1 and 2 are assumed to be equal. Reformulating equations (14) and (15) in respect to their initial velocities after a given change in flow area yields equation (16) for channel 1 and equation (17) for channel 2.

\[
v_1 = \frac{v_0}{1 + \Delta A(A_0)^{-1}} \tag{16}
\]
\[
v_2 = \frac{v_0}{1 - \Delta A(A_0)^{-1}} \tag{17}
\]

The net pressure may now be linearized by equating (9), (16), and (17) while taking the Taylor expansion as \( \Delta A(A_0)^{-1} \) goes to zero. The linearized pressure is a key assumption in formulating critical flow velocity and critical dynamic pressure. The linearization yields.

\[
\Delta P = 2\rho v_0^2 \Delta A(A_0)^{-1} \tag{18}
\]

Equations (7) and (18) are now combined, resulting in the critical flow velocity for a flat plate having both edge boundaries conditions clamped given by equation (19).

\[
V_{cr} = \left( \frac{h}{\rho L^4} \right)^{\frac{1}{2}} (180D)^{\frac{1}{2}} \tag{19}
\]

The critical flow velocity for a plate having one edge boundary condition clamped and the other simply supported is given by (20).

\[
V_{cr} = \left( \frac{h}{\rho L^4} \right)^{\frac{1}{2}} (80D)^{\frac{1}{2}} \tag{20}
\]

The critical flow velocity for a plate having both edge boundary conditions simply supported is given by (21).

\[
V_{cr} = \left( \frac{h}{\rho L^4} \right)^{\frac{1}{2}} (30D)^{\frac{1}{2}} \tag{21}
\]

Equations (19), (20), and (21) represent the most fundamental forms of critical flow velocity for a single flat plate for each of the three edge boundary conditions analyzed in this study. This
form is left in terms of flexural rigidity \((D)\), which will be discussed in subsequent sections of this chapter. It is useful to group the critical flow velocity terms such that all hydraulic components including channel height \((h)\), channel span width \((L)\), and fluid density \((\rho)\) together while solid mechanical components such as flexural rigidity \((D)\) are grouped separately.

Note that in \((19), (20),\) and \((21)\) the coefficient tied to the flexural rigidity term is the only component of the equation that changes as the edge boundary condition changes. This demonstrates, that the critical flow velocity required for a plate with both edges clamped as seen in \((19)\) is much larger in magnitude from that of a plate having both edges simply supported \((21)\). This is expected, and agrees with basic solid mechanical (Bedford and Liechti 2000).

### 3.2 Flexural Rigidity

If left as currently defined, equations \((19), (20),\) and \((21)\) may be solved given a homogenous flat plate by applying the flexural rigidity shown in equation \((22)\), yielding critical flow velocity as defined by Miller in 1958 (Miller 1958).

\[
D = \frac{t^3 E}{12 (1-\nu^2)}
\]

The work described herein extends Miller’s method for predicting critical flow velocity, by quantifying this phenomenon for a heterogeneous laminated plate having three discrete layers. This is done, by formulating flexural rigidity for a laminated plate.

The flexural rigidity contains all of the information pertaining to material properties and geometry, for a given cross-section. The flexural rigidity may be thought of as “stiffness”. Stiffness is simply a cross-sections resistance to deflection, as seen in \((22)\).

A case where a plate has three discrete regions, with both outer regions having equal thickness and being symmetric about the centerline is known as a sandwich structure (Vinson 1999; Carlsson and Kardomateas 2011). Sandwich structures are a subclass of structures studied in composites. The motivation for developing a critical flow velocity relation, which incorporates the characteristics of a sandwich structure, is driven by the desire to analyze reactor fuel plates.
Reactor fuel plates, comprised of three discrete layers typically have two outer cladding regions and one inner fueled region. The outer two layers of the LEU fuel introduced in this study are made of Aluminum while the inner region is comprised of Uranium-Molybdenum alloy.

Previous authors have done significant work in analyzing the mechanics of sandwich structures (Reddy 1997; Vinson 1999; Carlsson and Kardomateas 2011). The aim of this work is to adapt and extend previous work in a manner that is useful to studying the mechanics of reactor fuels, with a hydraulic boundary.

Prior to defining and expanding on flexural rigidity, it is important to note that although a traditional reactor fuel plate is clad all around with an outer layer, it is assumed during this study that this outer layer is defined only on the top and bottom of the inner region; this is shown in Figure 5.

The actual fuel plate is shown on the left in Figure 5, while the simplified geometry is shown on the right. This geometric simplification is made in order to better characterize each discrete region’s flexural rigidity. This simplification does not significantly impact the solution, based on the fact that flexural rigidity is highly dependent on thickness rather than width. This is true for all beams having a large length to thickness ratios, and is shown mathematically by the second area moment of inertia for a cross-section in which the thickness term is raised to the third power. Thus variations across the width of a sandwich structure are of little importance, while variations across the thickness are highly important.

Furthermore, it is common practice to analyze flexural rigidity in terms of unit width (Ugural and Fenster 2003), as is done throughout this study. Note that early literary references commonly refer to flexural rigidity in terms of unit width as ‘specific flexural rigidity’.

The dependence of stiffness on thickness indicates that the outermost fiber of a beam contributes the most to the flexural rigidity, and the overall stiffness. This is the fundamental driving force behind the structural mechanics of sandwich structure design. The sandwich’s outermost layers
are considered to contribute greatly to the stiffness of the structure, while the inner region contributes much less.

An implication of the sandwich structure model is that no slip may occur at the interface of each layer. If such slip occurs the structure is no longer a sandwich structure and cannot be analyzed using such techniques.

Figure 5 shows the general sandwich structure analyzed throughout this study. The body in Figure 5 is symmetric about the centerline of the plate, and each region is isotropic and homogenous.

(Please Insert Figure 5 Here)

3.2.1 The Monocoque Analogy

A monocoque is a structure that is supported by its skin (Vinson 1999). The sandwich structure presented in Figure 5 may be thought of as a monocoque, because the outer region contributes the most to flexural rigidity. The flexural rigidity of a sandwich structure may be approximated, by assuming the outer region of the sandwich structure bares the entire load on the plate. This section derives and defines a version of flexural rigidity, based on this and terms it the monocoque analogy. Note that the monocoque analogy is a simplified estimation of a plate’s flexural rigidity, as all loading is assumed to be taken by the outer regions (regions 1 and 3 in Figure 5), because of this the applicability of this method will be discussed later.

A monocoque form of the flexural rigidity term, was first derived by Carlson and Kardomateas (Carlsson and Kardomateas 2011). Carlson and Kardomateas derived this relation for the application to structures used in the aerospace industry. It is important to note that Carlson and Kardomateas defined each region at its centroid in reference to the sandwich centerline. This is in contrast to other work, which defined each region at its interface (Vinson 1999).
In order to arrive at the monocoque analogy form of flexural rigidity the general stress states must be simplified into a one dimensional form. This form is necessary because the critical flow velocity and critical dynamic pressure functions have been developed assuming a one-dimensional flexural rigidity.

Applying Hooke’s Law to the elemental body in Figure 6, yields the general stress state for the $k$-th region of a sandwich structure shown by equation (23). The six by six coefficient matrix is most commonly referred to as the stiffness matrix having discrete stiffness parameters $Q_{ij}$.

$$
\begin{bmatrix}
\sigma_{yj} \\
\sigma_{yi} \\
\sigma_{xy}
\end{bmatrix}_k = \begin{bmatrix}
Q_{yi} \\
Q_{yi} \\
Q_{xy}
\end{bmatrix}_k 
\begin{bmatrix}
\varepsilon_{yi} \\
\varepsilon_{yi} \\
\varepsilon_{xy}
\end{bmatrix}_k
$$

Equation (23) may be simplified by assuming that each layer is isotropic in material and stress load characteristics. In the application to reactor fuel plates this assumption is valid, because the Aluminum outer regions and Uranium-Molybdenum inner regions are isotropic (Avallone and Baumeister-III 1995; Rest, Kim et al. 2009). The isotropic assumption reduces the stiffness matrix in equation (23) to a diagonally symmetric form due to the homogeneity of material composition and comprises three of the original thirty six stiffness parameters. This stress versus strain relation may be further simplified by assuming a plane stress state. A plane stress state exists when $\sigma_z = \sigma_{yz} = \sigma_{xz} = 0$, requiring out of plane normal stress and shear terms to be null.

As a result of this simplification, the forcing function is now in the form of strain, the coefficient matrix relates a layers effective extensional rigidity in each cardinal direction and the solution matrix is in the form of strain. In this form, the coefficient matrix is most commonly referred to as the compliance matrix. Inverting the simplified version Fick’s law around the compliance matrix yields (24)

$$
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix}_k = \frac{E_k}{(1-\nu_k^2)} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1}{2}(1-\nu)
\end{bmatrix}_k 
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
2\varepsilon_{xy}
\end{bmatrix}_k
$$

where $\nu$ represents Poisons Ratio for the $k$-th layer and $E$ represents Young’s Modulus for the $k$-th layer. Equation (24) is the plane stress matrix for the $k$-th layer of a sandwich structure.
For direct integration into the critical flow velocity the one dimensional plane stress case must be taken. This further simplifies (24), yielding the plane stress for the $k$-th layer of the sandwich structure, as seen in (25).

$$\sigma_k = Q_k \varepsilon_k = \frac{E_k}{(1-\nu_k^2)} \varepsilon_k$$  \hspace{1cm} (25)

Equation (25) may be reformulated for the top and bottom layers of the sandwich structure shown as (26) and (27), respectively.

$$\sigma_1 = \frac{E_1}{(1-\nu_1^2)} \varepsilon_1$$ \hspace{1cm} (26)

$$\sigma_3 = \frac{E_3}{(1-\nu_3^2)} \varepsilon_3$$ \hspace{1cm} (27)

The bending strain for each layer must now be considered. To do this it is useful to define the bending displacement of the sandwich structure in terms of its curvature. This is shown in equation (28) where $u_0(x)$ is the displacement of the inner region; $z$ is term that defines the distance from the center of the sandwich, and $\kappa$ is the curvature of the structure.

$$u(x) = u_0(x) + z\kappa$$ \hspace{1cm} (28)

The derivative of displacement with respect to $x$ is taken yielding the bending strain on the sandwich structure.

$$\varepsilon = \frac{\partial u_0}{\partial x} + z\kappa$$ \hspace{1cm} (29)

Equation (29), may be reworked in the form of the inner region’s strain. This is the bending strain on the sandwich structure, in terms of the $k$-th layer.

$$\varepsilon_k = \varepsilon_x^0 + z_k\kappa$$ \hspace{1cm} (30)

The bending strain for the bottom region is shown in equation (31), and the top region is shown in equation (32). These relations are acquired by inputting the appropriate values of $z$, which is calculated from the centerline of the sandwich to the centroid of each layer.

$$\varepsilon_1 = \varepsilon_x^0 - \frac{(t_1 + t_2)}{2} \kappa$$ \hspace{1cm} (31)

$$\varepsilon_3 = \varepsilon_x^0 + \frac{(t_2 + t_3)}{2} \kappa$$ \hspace{1cm} (32)
The force resultant $N$ and moment couple $M$ for the $k$-th layer is generated by creating a force balance on the generic body shown in Figure 6.

\[ \begin{align*}
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}_k &= \int_{-t_2/2}^{t_2/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} \mathrm{d}z + \int_{t_2/2}^{t_2+t_1} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} \mathrm{d}z \\
\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}_k &= \int_{-t_2/2}^{t_2/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} z \mathrm{d}z + \int_{t_2/2}^{t_2+t_1} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} z \mathrm{d}z
\end{align*} \]

(Equation 33)

Note the limits of integration, are from the center line of the sandwich panel to the centroid of the $k$-th layer.

Equations (35) and (36) show the force resultant and moment couple of a one dimensional sandwich plate, with only the outer regions accounted for.

\[ \begin{align*}
N_x &= \int_{-t_2/2}^{t_2/2} \sigma_x \mathrm{d}z + \int_{t_2/2}^{t_2+t_1} \sigma_x \mathrm{d}z \\
M_x &= \int_{-t_2/2}^{t_2/2} \sigma_x z \mathrm{d}z + \int_{t_2/2}^{t_2+t_1} \sigma_x z \mathrm{d}z
\end{align*} \]

(Equation 35)

Equations (37) and (38) are the general relations for the force resultants $N$ and moment couples $M$, for an isotropic sandwich plate in a plane stress state. Where $A$ is the extensional stiffness, $D$ is the flexural rigidity, and $B$ and $C$ are elastic coefficients.
The one dimensional form of (37) and (38) is presented by equations (39) and (40).

\begin{align*}
N &= A\varepsilon_x^0 + Bk \\
M &= C\varepsilon_x^0 + Dk
\end{align*}

(39)  (40)

It must now be stated that the sandwich plate in question is symmetric about the centerline. In this case, then \([B] = 0 = [C]\). It is also observed that the thicknesses of the outer regions are of equal thickness (i.e. \(t_1 = t_3 = t_\omega\)). These assumptions allow for equations (37) and (38) as well as equations (39) and (40), to be uncoupled. This yields equations (41) and (42) for the general case.

\begin{align*}
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ 2\varepsilon_{xy}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \\
M_x &= C_{11} C_{12} C_{16} \varepsilon_x^0 + D_{11} D_{12} D_{16} \kappa_x \\
M_y &= C_{12} C_{22} C_{26} \varepsilon_y^0 + D_{12} D_{22} D_{26} \kappa_y \\
M_{xy} &= C_{16} C_{26} C_{66} 2\varepsilon_{xy}^0 + D_{16} D_{26} D_{66} \kappa_{xy} \\
\end{align*}

(37)  (38)

Equations (43) and (44) represent (41) and (42) in a one dimensional state.

\begin{align*}
N &= A\varepsilon_x^0 \\
M &= D\kappa_x \\
\end{align*}

(43)  (44)

It may be seen by inspection that the form of the extensional stiffness matrix \(A\) in equation (37), and the stiffness parameter matrix \(C\) in equation (38), are nearly identical in form to the stiffness matrix \(Q\) in equation (24). In fact they only differ by a factor of \(t\), that is \(A = tQ\). Furthermore, the grouping of terms during the integration of equation (36), noting that the force resultant and moment couple are now independent, allows the bending stiffness matrix (i.e. flexural rigidity) \(D\) to be filled independently of \(\kappa\). This yields a simple relation for the one-dimensional case of \(D\),
shown in equation (45) where $A=tQ$. Relations for the coefficient matrices $A$ and $D$ are independent of strain $\varepsilon^0_x$ and $\kappa$, allowing them to be found by grouping terms after integration.

$$D = \frac{1}{4} \left( (t_2 + t_1)^2 A_1 + (t_2 + t_3)^2 A_3 \right) \quad (45)$$

From the assumption of symmetry and equal thickness for the outer region, it is assumed that extensional stiffness for each region is equal (i.e. $A_i = A_3 = A_{or}$). The subscript “or” refers to the outer region of the sandwich plate. This implies that the extensional stiffness, for the sandwich structure, is the following.

$$A = A_i + A_3 = 2A_{or} \quad (46)$$

Combining equations (45) and (46) yields the following relation for flexural rigidity.

$$D = \left( \frac{t_r + t_{or}}{2} \right)^2 A_{or} \quad (47)$$

Recall that the outer regions are of equal thickness ($t_{or} = t_1 = t_3$), with equivalent material properties ($E_{or} = E_i = E_3$ and $\nu_{or} = \nu_1 = \nu_3$). The interior region of the sandwich structure (region 2 in Figure 5) will be denoted with the subscript “ir” indicating the inner region.

Combining and reformulating equation (46), yields the extensional stiffness for the monocoque analogy (48).

$$A_{mon} = \frac{2E_{or}t_{or}}{1-\nu_{or}^2} \quad (48)$$

Inserting the extensional stiffness given by (48) into the previously defined flexural rigidity term seen in equation (47), yields the flexural rigidity for the monocoque analogy.

$$D_{mon} = \frac{\left( t_r + t_{or} \right)^2 t_{or}}{2} \frac{E_{or}}{\left( 1-\nu_{or}^2 \right)} \quad (49)$$

Hereinafter results referring to the monocoque analogy will use the flexural rigidity form presented in equation (49).

It is commonly known that the outermost fiber in a beam contributes the most to stiffness of the beam as theoretically and experimentally demonstrated through numerous studies (Budynas...
Sandwich structures take advantage of this by moving material away from the inner region. Historically, this has been done to lighten structures while maintaining mechanical integrity. For instance, early designers often choose a material such as aluminum for the outer region and a very light material such as balsa wood for the inner region (Hoff 1950). Modern sandwich structures typically use a stiff fiber composite (i.e., carbon fiber) as the outer material, and a very light Honey Comb matrix for the inner region (Reddy 1997). The single most important design parameter with regard to the sandwich structure is the distance between the outer regions. The greater this distance, the stiffer the structure will be. So, the ratio of the outer region thickness to the inner region thickness is optimized by the designer to provide as much stiffness as possible for a desired mass and total plate thickness. This kind of structure had been used in the aerospace industry for many decades (Reddy 1997; Vinson 1999; Carlsson and Kardomateas 2011).

The monocoque analogy takes advantage of this philosophy. It allows the analysis of rigidity without consideration of the inner region material. The only information about the inner region that impacts the overall stiffness of the sandwich structure is the distance it moves the outer regions away from each other. This allows flexural rigidity to take a very simple and useful form.

Ultimately, the monocoque analogy is an estimation-based calculation, as such it has limitations. Specifically, the outer region material cannot have material properties that are drastically ‘weaker’ than the inner region material, and the thickness of the inner region in relation to the thickness of outer region must fall into a certain usability criteria for a given application. It is important to note that the application of this sort of rigidity to reactor fuel plates may fall outside of the bounds of similar criteria provided in literature. This discrepancy can be seen in Carlsson and Kardomateas (Carlsson and Kardomateas 2011) and Vinson’s work (Vinson 1999). This is because these authors focused on design applications that are not necessarily applicable to reactor fuel plates. Specifically, attention is directed towards optimizing the ‘strength’ of a sandwich structure for a case where the design variables and thicknesses are not fixed. These authors’ design criteria work best in a case where the designer may choose from many different outer region materials. In applications directed towards reactor fuel plates, this is not the case,
typically the choice of inner region and outer region material has already been determined (i.e. aluminum and uranium-molybdenum).

This does not mean that the monocoque analogy is not applicable. It may still be used, but its applicability is dependent on the designers engineering judgment. Several test cases demonstrating the applicability of the monocoque analogy are provided in the results chapter.

3.2.2 Ideal Laminate Model

The ideal laminate model represents the full analytical solution to the flexural rigidity of a sandwich structure. Unlike the monocoque analogy, it takes into account information from the outer region as well as the inner region. It does not suffer from the same usability problems as the monocoque analogy, because it considers all portions of the sandwich structure.

A form of the original rigidity term considered within the ideal laminate model was initially presented by Vinson; he focused on lightweight structures for the aerospace industry (Vinson 1999). It is important to note that Carlsson and Kardomeas’ work is very similar to Vinson’s work; a large portion of both derivations are identical. The primary difference is that Vinson’s work defines the regions of the sandwich at the interface between regions. This is in contrast to Calsson and Kardomeas who define their regions from the centroid.

Note that Vinson concerned himself with fiber reinforced structures. As such he included terms to quantify thermoelasticity (i.e. thermal expansion) and hygrothermal effects (i.e. moisture absorption). This study is not concerned with such structures, so these terms have been omitted. Isotropic homogenous sandwich structures are the focus of this study, so the work below reflects this.

As mentioned the work below shares many similarities with the derivation of the monocoque analogy. In fact all of the steps from (23) to (30) are identical. These steps have been omitted from this section.
The derivation of the ideal laminate model diverges from that of the monocoque analogy, when the force resultants \( N \) and moment couples \( M \) are generated from the body in Figure 6. This is because the limits of integration with respect to \( N \) and \( M \) are at the surface of the \( k \)-th region. The force resultants are shown mathematically by equation (50), and the moment couples are given by equation (51).

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}_k = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix}_k dz
\]

(50)

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}_k = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix}_k zdz
\]

(51)

Equations (50) and (51) may be written in the form shown by equations (52) and (53), by inputting the values for stress and applying the one-dimensional case.

\[
N = \sum_{k=1}^{N} \left( \int_{t_{k-1}}^{t_k} Q_k \varepsilon_x^0 dz + \int_{t_{k-1}}^{t_k} Q_k \kappa_x dz \right)
\]

(52)

\[
M = \sum_{k=1}^{N} \left( \int_{t_{k-1}}^{t_k} Q_k \varepsilon_x^0 zdz + \int_{t_{k-1}}^{t_k} Q_k \kappa_x zdz \right)
\]

(53)

Because the material of each region is isotropic, \( Q_k \) must be a constant. In addition, the displacements \( u \) are not a function of \( z \), and neither are their derivatives (i.e. strain \( \varepsilon_x^0 \)). Finally, the curvature \( \kappa \) is not a function of \( z \). These simplifications to the force resultants and moment couples allow a number of terms to be grouped outside of the integrals, yielding equation (54) and (55).

\[
N = \sum_{k=1}^{N} \left( Q_k \varepsilon_x^0 \int_{t_{k-1}}^{t_k} dz + Q_k \kappa_x \int_{t_{k-1}}^{t_k} dz \right)
\]

(54)

\[
M = \sum_{k=1}^{N} \left( Q_k \varepsilon_x^0 \int_{t_{k-1}}^{t_k} zdz + Q_k \kappa_x \int_{t_{k-1}}^{t_k} zdz \right)
\]

(55)

Recall that the sandwich plate in question is symmetric about the sandwich centerline of the plate thickness, and the outer regions are of equal thickness. This causes equations (54) and (55) to
become uncoupled, where \( [B] = 0 = [C] \). Also, equations (54) and (55) may be restated in their one-dimensional forms. As seen in equations (56) and (57).

\[
N = A \varepsilon_x^0 + B \kappa 
\]

\[
M = B \varepsilon_x^0 + D \kappa 
\]

Integrating equations (56) and (57), factoring out \( \varepsilon_x^0 \) and \( \kappa \), and grouping relevant term yields the following very useful equation for \( A, B, \) and \( D \). Where \( A \) is the extensional stiffness given by equation (58).

\[
A = \sum_{k=1}^{N} Q_k \left( t_k - t_{k-1} \right) 
\]

\( B \) is the elastic coefficient given by equation (59).

\[
B = \sum_{k=1}^{N} Q_k \left( t_k^2 - t_{k-1}^2 \right) 
\]

\( D \) is the flexural rigidity given by equation (60).

\[
D = \frac{1}{3} \sum_{k=1}^{N} Q_k \left( t_k^3 - t_{k-1}^3 \right) 
\]

The relations for extensional stiffness and flexural rigidity, equations (59) and (60), are very useful in this study. They allow for calculation of extensional stiffness and flexural rigidity in one-dimension for \( k \) layers. If this is extended for more than three layers, the indices \( k \) are only valid in factors of 2 (i.e. \( k = 1, 2, 4, 6, \) etc.). Refer to Vinson for further information concerning this (Vinson 1999) This assumes that each layer in the sandwich structure is symmetric with respect to its partner, isotropic, homogenous, the outer regions are identical in thickness and material properties, and that there is no slip at the interface. Equations (58) and (60) are now solved for a one-dimensional sandwich structure having three discrete layers that satisfies the assumptions listed above.

The stiffness term for the top and bottom outer regions are given by equation (61).

\[
Q_{or} = Q_1 = Q_3 = \frac{E_{or}}{\left(1 - \nu_{or}^2\right)} 
\]

The stiffness term for the inner region is given by equation (62).
\[ Q_{ir} = Q_2 = \frac{E_{ir}}{1 - v_{ir}^2} \]  

(62)

Applying equation (58) for a sandwich plate with three discrete layers yields the extensional stiffness (63).

\[ A = Q_{ir} \left( -\frac{t_{ir}}{2} - \left( -\frac{t_{ir}}{2} - t_{or} \right) \right) + Q_{or} \left( \frac{t_{ir}}{2} - \left( -\frac{t_{ir}}{2} \right) \right) + Q_{or} \left( \frac{t_{ir} + t_{or}}{2} - \frac{t_{ir}}{2} \right) \]  

(63)

Combining equations (61), (62), and (63) yields the extensional stiffness for the ideal laminate model (64).

\[ A_{ij} = 2Q_{or}t_{or} + Q_{or}t_{ir} = \frac{2E_{or}}{1 - v_{or}^2} t_{or} + \frac{E_{ir}}{1 - v_{ir}^2} t_{ir} \]  

(64)

Applying equation (60) for a sandwich plate with three discrete regions results in the following relation for flexural rigidity (65).

\[ D = \frac{1}{3} Q_{or} \left( \frac{t_{ir}}{2} \right)^3 - \left( -\frac{t_{ir}}{2} - t_{or} \right)^3 + \frac{1}{3} Q_{or} \left( \frac{t_{ir}}{2} \right)^3 - \left( -\frac{t_{ir}}{2} \right)^3 \]  

\[ + \frac{1}{3} Q_{or} \left( \frac{t_{ir} + t_{or}}{2} \right)^3 - \left( -\frac{t_{ir}}{2} \right)^3 \]  

(65)

Combining equations (61), (62), and (65) yields the flexural rigidity for the ideal laminate case.

\[ D_{ij} = \frac{1}{3} \left( \frac{2E_{or}}{1 - v_{or}^2} \left( \frac{3}{4} t_{ir}^2 t_{or} + \frac{3}{2} t_{ir} t_{or}^2 + t_{or}^3 \right) + \frac{E_{ir}}{1 - v_{ir}^2} t_{ir}^3 \right) \]  

(66)

Equation (66) is the complete analytical solutions for the flexural rigidity of a sandwich structure, derived from a general stress state (i.e. first principles). Throughout this work, it will be referred to as the ideal laminate model. It does not have the same usability and applicability limitations as the monocoque analogy. It will be shown, in subsequent sections that the ideal laminate model will consistently predict the flexural rigidity of a sandwich structure regardless of the dimensions of the inner or outer region.

3.2.3 Model Closure

This monocoque analogy, acknowledges that the inner region of the laminate plate provides insignificant mechanical stiffness to the laminate, and subsequently ignores this region. These equations may be simply reformulated into one equation utilizing the constant \( k \) which shall be
termed the edge boundary constant. This constant assumes a value of 90 when both edges are clamped, 40 when one edge is clamped and one is simply supported, and 15 when both edges are simply supported. The final form of the monocoque analogy is shown in equation (67).

\[
V_{cr-mon} = \left( \frac{h}{\rho L^2} \right)^{\frac{1}{2}} \left( k \left( t_{ir} + t_{or} \right)^2 t_{or} \left( \frac{E_{or}}{\left(1 - v_{or}^2\right)} \right) \right)^{\frac{1}{2}}
\]

(67)

The ideal laminate model, makes fewer assumptions about the mechanical contribution of each region, and subsequently includes the mechanical contribution for all regions. It is shown to be robust in comparison to the monocoque analogy, by Test Cases 1, 2, and 3. These equations may also be simplified by including an edge boundary constant \( k \). The edge boundary constant assumes the value of 60 when both edges are clamped, 80/3 when one edge is clamped and one is simply supported, and 10 if both edges are simply supported. The final form of the ideal laminate model is shown in equation (68).

\[
V_{cr-il} = \left( \frac{h}{\rho L^2} \right)^{\frac{1}{2}} \left( k \frac{2E_{or}}{\left(1 - v_{or}^2\right)} \left( \frac{3}{4} t_{ir}^2 t_{or} + \frac{3}{2} t_{ir} t_{or}^2 + t_{or}^3 \right) + \frac{E_{ir}}{\left(1 - v_{ir}^2\right)} \left( t_{ir}^3 \right) \right)^{\frac{1}{2}}
\]

(68)

4 RESULTS AND DISCUSSION

This section presents and describes several test cases, which highlight the robustness and applicability of the ideal laminate model along with the monocoque analogy. The boundary conditions for each test case are detailed, and discussions of each test case’s results are presented. In addition, a comparison to previous experimental work is provided and demonstrates relatively good agreement throughout.

4.1 Test Cases

Three ‘Test Cases’ are described and performed in comprehensive detail. These test cases are intended to qualitatively and quantitatively demonstrate the robustness and applicability of the models developed herein. For all test cases the clamped-clamped edge boundary condition was employed.
4.1.1 Test Case 1- Sensitivity Due to Region Thicknesses

As the thickness of the inner and/or outer region is varied, the relative percent of load carried by each respective region will change. This presents a liability for the monocoque analogy. The monocoque analogy, assumes that the outer region carries a significant amount of the load; so much so that the overall stiffness of the plate is approximated by the outer region only. Thus the mechanical contribution of the outer region is used solely to approximate the plate’s flexural rigidity, and the inner region is ignored. This is generally a safe assumption under most cases where the sandwich structure is sufficiently thin that it may be viewed as a laminated shell. Note that a shell is a thin three dimensional body in space with large aspect ratio. Yet even for a thin sandwich plate, certain cases exist in which the applicability of the monocoque analogy may be questioned. These cases arise, as the thickness of the inner and outer regions is varied.

To illustrate this consider a sandwich plate having three discrete regions (as presented in Figure 5right) with constant total thickness. Assume that this plate has three discrete regions with equal materials properties. Also, consider a homogenous plate (one region) with a thickness equal to the total thickness of the sandwich plate. Assuming that each layer in the sandwich is perfectly bound to one another, logic dictates that a valid model to predict critical flow velocity for a sandwich plate would predict the same values as the homogenous case. Test case 1 compares the laminate models’ ability to predict the critical flow velocity of a homogenous plate as compared against Miller’s model.

In this test case, the relative thickness of the inner and outer region is varied while the total thickness is held constant. The boundary conditions used are presented in Table 2.

(Please Insert Table 2 Here)

(Please Insert Figure 7 Here)
In Figure 7 the x-axis represents the relative percent that the inner region occupies, while the y-axis presents the ratio of critical flow velocity predicted by each model to that predicted by Miller’s method.

When Compared against Miller’s method, the ideal laminate model precisely predicts the critical flow velocity through the entire range of varied thicknesses. This is expected as the mathematical formulation, although including a number of assumptions, is derived on the basis that the stress loading imposed by the hydraulic domain is carried in all the regions of the plate regardless of each region’s thickness.

In contrast, the monocoque analogy provides a best estimate between 20 and 50 percent, with the most accurate prediction occurring when each layer is equal (i.e. the inner layer occupies approximately one third of the total thickness). The variance in the monocoque analogy’s ability to predict critical flow velocity is due to the load being shifted from the outer region material to the inner region material as the thickness of each region is changed. The monocoque analogy does not consider the rigidity of the core material. No mechanical properties or geometric properties are taken into the monocoque analogy for the inner region material, as seen in equation (67). This causes the monocoque analogy to significantly under predict the critical flow velocity as the inner region becomes thick. In addition, as the outer regions become thick (i.e. inner plate region thickness is less than one third), the monocoque analogy also under predicts the critical flow velocity. This is driven by the fact that the geometric properties of the outer region do not properly scale as the laminate plate approaches the geometry of the homogenous plate. This may be observed by comparing the flexural rigidities for a homogenous plate given by equation (22) and the flexural rigidity given by the monocoque analogy shown in equation (68).

Additionally an alternative perspective may be taken, which demonstrate the monocoque analogy’s inability to predict the critical flow velocity as the inner regions thickness approaches zero. This is demonstrated by utilizing the superposition method to derive the flexural rigidity of the monocoque analogy. In which the parallel axis theorem is modified, by assuming that the second area moment of inertia for each region is negligible. For this assumption to hold true,
each region must have some thickness which is not true as the inner region percentage approaches zero. A complete derivation of this is shown in the appendix.

4.1.2 Test Case 2- Sensitivity Due to Total Thickness

As the total thickness of a sandwich plate increase, more load is carried by the inner region. The transfer of load results in the monocoque analogy and the ideal laminate model gradually diverging as the total thickness increases. To be specific the monocoque analogy will under predict the critical flow velocity as the total thickness increases.

As with the previous test case, the critical flow velocity for a plate with three discrete layers of equal material properties was calculated using both models and compared to the critical flow velocity for a homogenous plate predicted using Miller’s method. The boundary conditions used for Test Case 2 are shown in Table 3.

(Please Insert Table 3 Here)

Figure 8 demonstrates that with a constant ratio of inner-to-outer region, the monocoque analogy gradually becomes less accurate as the total thickness of the plate is increased. As shown in Figure 7, there will always be a small discrepancy between the monocoque analogy and ideal laminate model. This discrepancy increases in magnitude as the total thickness of the plate increases, due to the monocoque analogy’s inability to quantify effects occurring in the inner region of the plate.

(Please Insert Figure 8 Here)

4.1.3 Test Case 3- Sensitivity Due to Material Composition

The material composition of each region may also adversely affect each model’s ability to accurately predict critical flow velocity. An actual fuel plate will have cladding material and fueled material; the material properties of each have profound effects on the plate’s resistance to
flow induced instabilities. In the case of high performance research reactor fuel plates the Young’s Modulus of the fuelled region is larger than that of the clad [51]. In such a case, the inner region will contribute more to the overall rigidity of the plate than if it were comprised of a ‘weaker’ material. Each model’s sensitivity to a change in the material composition of each region is compared in this test case. The boundary conditions used for Test Case 3 are given in Table 4.

(Please Insert Table 4 Here)

Figure 9 shows the sensitivity of each model to changes in the material composition of the inner region. As before the critical flow velocity for each model is compared to Miller’s method. Miller’s method was calculated by using the material properties for the outer region, and the total thickness of the laminated plate. The thickness of each region in the laminated plate was assumed to be one-third the total thickness of the plate. As shown in Figure 9, the ratio of critical flow velocities is compared to the ratio of inner and outer region Young’s Moduli.

(Please Insert Figure 9 Here)

Figure 9 demonstrates that as the inner regions Young’s Moduli increases the accuracy of the monocoque analogy relative to that of the ideal laminate case is diminished. In addition, simply using the material information for the clad material in Miller’s method is also unacceptable in this case. This is because neither method considers the contribution of the inner region to the rigidity of the structure. Notice that all models show close agreement until the ratio of Young’s Moduli reaches unity. Beyond unity the ideal laminate model performs well, while the monocoque analogy and Miller’s method becomes less accurate. Clearly, the ideal laminate model predicts the most representative critical flow velocity.
4.2 Model Comparison with Experimental Data

The models presented herein, are compared to previous experimental work. Currently, there exists no relevant available experimental data on critical flow velocity of laminated plates, however, numerous studies have been performed on this subject while employing homogeneous plates. While it is not ideal to compare a laminate model to homogenous model, it has been shown in the previous Test Cases that such a comparison is valid under certain conditions.

Several authors provide critical flow velocity data for homogenous plates. Most applicable to this study is the work of Zabriskie (Zabriskie 1958) and Smith (Smith 1968); these authors included test results for failure of plates under static conditions. Of the two authors, Smith’s data was determined to be most applicable. This is because Smith conducted tests for four separate materials, and presented the experimental variability of results for each test. In contrast, Zabriskie did not provide any information concerning the variability of his measured results.

There are some undesirable aspects to Smith’s work, however: (1) the error associated with some of his experimental test cases is very large, and (2) the method he choose in conveying his results made it very difficult to accurately extract usable information from his work. Relatively large error and difficulty in reproducibility is not uncommon among critical flow velocity and critical dynamic pressure studies (Stromquist and Sisman 1948; Groninger and Kane 1963), and only highlights the need for modern experimental facilities such as the Oregon State University Hydro Mechanical Fuel Test Facility.

Smith’s work focused on small beams of lead, aluminum, copper, and steel. Of the materials tested lead was the least stiff material and steel was the stiffest. The test facility Smith used was designed to circulate helium or air as the working fluid. The data set chosen for comparison was taken from figure 6, in Smith’s study (Smith 1968). This data set compared the critical dynamic pressure at beam failure, against a material parameter for each beam. The critical dynamic pressure \( q_{cr} \) for the both the monocoque analogy and ideal laminate model for appropriate edge boundary conditions may be obtained by inserting the evaluated critical flow velocity obtained in either (67) or (68) into

\[
q_{cr} = \frac{1}{2} \rho v_{cr}^2
\]  

(69)
As with the previous homogenous examples, the laminate models were compared to Miller’s method for a homogenous plate. Table 5 outlines the boundary conditions for each model which reflects those tested by Smith.

(Please Insert Table 5 Here)

Figure 10 compares the predicted critical dynamic pressure, to Smith’s observed critical dynamic pressure at plate failure. It should be noted that as in Test Case 1 the results predicted by Miller’s method and the ideal laminate model agree perfectly due to the homogenous nature of the plates under consideration, while the monocoque analogy presents results that are slightly less than that of the other two models.

Figure 10, shows the results compared to Smith’s data for a plate with both edge boundaries clamped. This is the most mechanically rigid case. As is seen, there is great variability in Smith’s data, and the critical dynamic pressure predicted by the models is slightly greater than the average presented by Smith for each material.

(Please Insert Figure 10 Here)

Although each model’s ability to predict critical dynamic pressure trends with that observed by Smith, the validity of this comparison is hard to gauge, because there is significant variability in Smith’s data. Smith chose to quantify plate failure when a plate’s deflection is greater than or equal to 0.254 mm. This definition of plate failure can be debated. Recall, the point of failure when using critical flow velocity does not refer to a specific deflection. Plate collapse, as previously defined above, is the point at which the plate has deflected to such an extent that a linearized beam deflection model is no longer acceptable to characterize the phenomena at hand.
4.3 General Observations

The scope of results presented in Test Case 3 was expanded upon by exploring how changes in each layers material composition effects the Ideal Laminate model as the thickness of the inner region relative to the outer region is changed.

Figure 11 compares the ratio of critical velocities to the ratio of Young’s Moduli as the inner region thickness is varied from 1 percent to 100 percent of the total sandwich thickness. Young’s modulus is varied from 0.1 to 10, all other inputs are the same as those presented in Test Case 3. From Figure 11 it may be seen that when inner region effectively occupies a larger percent of the total thickness as compared to the outer region (i.e. IL > 33%) and the inner region’s Young’s Modulus is less than that of the outer region, the predicted critical flow velocity predicted by the Ideal Laminate model is less than that of Miller’s model. In contrast, when Young’s Modulus for the inner region is greater than that of the outer region, and for cases where the inner region occupies a greater percent of the total thickness, the Ideal Laminate model yields critical flow velocity values which are greater than that of Miller’s model. For all cases considered, if each region’s material composition is unity with respect to its neighboring region, the Ideal Laminate model’s results match that of Test Case 3 and Miller’s model. This figure demonstrates the robustness of the Ideal Laminate model over all ranges of region thicknesses and Young’s Moduli while also as well as reliably returning back to Miller’s solution, when the appropriate boundary conditions are input. From this observation one may conclude that the Ideal Laminate model may be utilized reliably to predict the critical flow velocity for laminate flat plates, but may also be used in any relevant setting as a direct substitute for Miller’s model.

(Please Insert Figure 11 Here)

Figure 12 extends the results presented in Figure 11 to provide a trend of both the relevant range of Young’s Modulus and ratio of each region that occupies the entire plate thickness in a continuum. Note that with significant increases in the inner region’s Young’s Modulus along with its occupied thickness, one experiences corresponding increases in predicted critical flow
velocities. However, these large increased predictions in critical flow velocity reduce quickly as the inner regions material properties approach that of the outer region.

(Please Insert Figure 12 Here)

5 CONCLUSIONS

This study investigates two methods for determining the critical flow velocity for a pair of laminate plates. The objective is accomplished by incorporating a flexural rigidity term into the formulation of critical flow velocity originally derived by Miller, and employing sandwich structure theory to determine the rigidity term.

The flexural rigidity term is derived in two ways. The first is termed the monocoque analogy, and only considers the mechanical contribution of the outer regions material to a plate’s flexural rigidity. The second considers every layer’s contribution to flexural rigidity. Both are derived from a general stress state.

The methods presented herein, allow for a single simple calculation to be performed in order to predict the flow velocity or dynamic pressure that causes ‘plate collapse’. The simplicity of these methods is intended to provide designers with a useful tool which compliments other forms of analysis.

The final outcome of this study results in the developing of a single equation for each of three different edge boundary conditions which reliably and comprehensively predicts the onset of plate collapse.

This monocoque analogy, acknowledges that the inner region of the laminate plate provides insignificant mechanical stiffness to the laminate, and subsequently ignores this region. This result is presented in (67).
The ideal laminate model, makes fewer assumptions about the mechanical contribution of each region, and subsequently includes the mechanical contribution for all regions. It is shown to be robust in comparison to the monocoque analogy, by Test Cases 1, 2, and 3. This model is presented in (68).

The formulation of critical flow velocity presented herein includes several assumptions, listed below:

- The plate is initially un-deformed and perfectly flat, deforms symmetrical about its neutral axis, and its deflections are small enough to allow the use of the Euler-Bernoulli beam theorem.

- The fluid is incompressible and isothermal, and the flow is steady and uniform for the flow channel at any given point along the channel length.

- Internal shear in the plate is considered negligible, and the plane stress assumption is applied.

- The plate edge supports are perfectly rigid.

- The plate behaves as a wide beam.

- The plate is a sandwich structure, which is symmetric with one inner region and two outer regions.

- The regions of the sandwich plate are isotropic, linearly elastic, have perfect mechanical constants (i.e. Young’s Modulus, Poison’s Ratio), and no slip occurs at the interface of the regions.

The applicability of each model presented herein, has also been thoroughly explored in previous chapters. The ideal laminate model presented in equation (68), was shown to be robust, and consistently produced valid results. The monocoque analogy presented in equation (67), showed some applicability issues when the percentage of the inner region increased beyond 50 percent or
decreased below 20 percent with ideal range equal to 33 percent, this is illustrated in Figure 7. Also, the monocoque analogy may lose some fidelity if the sandwich plate becomes very thick. Finally, its applicability is also poor in cases where the inner regions Young’s modulus or material parameter \((E/(1-\nu^2))\) becomes of the order or greater than outer regions, this is shown in Figure 8.
Appendix

It is interesting to note, that the monocoque flexural rigidity can be simultaneously obtained by applying the parallel axis theorem to the simplified sandwich beam shown in Figure 5. This section is only meant to aid the users understanding of the monocoque analogy, while section 3.2.1 above derives the monocoque analogy from a general stress state (i.e. first principles).

Application of the parallel axis theorem to sandwich structures has been referred to as a “good starting point” or “back of the envelope” method by many authors (Cavallaro and Jee 2008; Carlsson and Kardomateas 2011). As above the contribution of the outer regions is considered, while the contribution of the inner region is ignored. This is shown below.

\[
D = \frac{EI}{(1-v^2)}
\]  
(70)

\[
I_x = \int_A y^2 da
\]  
(71)

\[
I_x = \overline{I}_x + Ad^2
\]  
(72)

Equation (70) is the general form that the flexural rigidity will take in this case. Equation (71) is the rote definition of the parallel axis theorem (Bedford and Liechti 2000), for a Cartesian system. Equation (72) is the parallel axis theorem expanded for a rectangular beam. The \(\overline{I}_x\) term is the second area moment of inertia, of an individual component of the sandwich beam with respect to the centroid of the beam. As discussed the sandwich structure derives most of its stiffness from the outer regions, by increasing the distance between corresponding outer regions. Thus the single component \(\overline{I}_x\) term for the outer regions is unimportant, and can be ignored. The inner region is also ignored in the monocoque analogy, so the \(\overline{I}_x\) for the inner region is also ignored. The sandwich structure, is also analyzed as having unit width, this reduces the \(A\) term to the thickness of the outer region. This is shown below, each \(I\) shown is the second area moment of inertia for either the top or bottom outer region with the simplifications mentioned above, and the \(d\) term is the distance from the centroid of the entire sandwich structure to the centroid of the respective outer region. Equations (73) and (74) below imply that the stiffness of a sandwich structure is most highly dependent on the term \(d\).
\[ I_1 = t_1d_1^2 \] 
\[ I_3 = t_3d_2^2 \] 
\[ d_1 = \frac{1}{2}(t_1 + t_2) \] 
\[ d_3 = \frac{1}{2}(t_2 + t_3) \] 

Equation (73)-(76) can now be combined by simply summing the inertia terms. This will yield \( \bar{I} \).

\[ \bar{I} = I_1 + I_2 \] 
\[ \bar{I} = \frac{t_1}{2}(t_1 + t_2)^2 + \frac{t_3}{2}(t_2 + t_3)^2 \] 

Assuming the sandwich is symmetric, isotropic, and the outer regions are of equal thickness will yield the monocoque analogy. Equations (79)-(82) restate this information and the constants used in this derivation.

\[ t_{or} = t_1 = t_3 \] 
\[ t_r = t_2 \] 
\[ E_{or} = E_1 = E_3 \] 
\[ v_{or} = v_1 = v_3 \] 

Equation (83), represents the form second area moment of inertia takes when inputting the information from equations (79) through (82).

\[ \bar{I} = \frac{t_{or}}{2}(t_r + t_{or})^2 \] 

Combing equation (70) and (83), yields the monocoque analogy derived from using the parallel axis theorem.

\[ D_{mon} = \frac{E}{(1-v^2)} \bar{I} \] 
\[ D_{mon} = \frac{(t_r + t_{or})^2 t_{or}}{2} \frac{E_{or}}{(1-v_{or}^2)} \]
### NOMENCLATURE

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<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$A$</td>
<td>Area, Extensional stiffness</td>
</tr>
<tr>
<td>$B$</td>
<td>Stiffness parameter</td>
</tr>
<tr>
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<td>Stiffness parameter</td>
</tr>
<tr>
<td>$D$</td>
<td>Flexural rigidity</td>
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<td>$D_{mon}$</td>
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<td>$E$</td>
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<td>$E_{ir}$</td>
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<td>$E_{or}$</td>
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<td>$g$</td>
<td>Gravitational acceleration</td>
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<td>$h$</td>
<td>Flow channel height</td>
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<tr>
<td>$I$</td>
<td>Second area moment of inertia</td>
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<td>$k$</td>
<td>$k_i$th substrate layer, Edge boundary constant</td>
</tr>
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<td>$L$</td>
<td>Span width</td>
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<td>$M$</td>
<td>Moment couple</td>
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<td>$N$</td>
<td>Force resultant</td>
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<td>$\nu_{ir}$</td>
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<td>$P$</td>
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Table 2: Test Case 1 boundary conditions

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<td>Fluid density [kg/m³]</td>
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<tr>
<td>Channel height to total thickness ratio [#]</td>
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<tr>
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Table 5: Comparison against experimental work boundary conditions

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<td>Plate span width [cm]</td>
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<tr>
<td>Flow channel height [mm]</td>
<td>1.62</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Lead</th>
<th>Aluminum</th>
<th>Copper</th>
<th>Steel</th>
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</thead>
<tbody>
<tr>
<td>Young’s Modulus [GPa]</td>
<td>15.9</td>
<td>69.0</td>
<td>117</td>
<td>210</td>
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<tr>
<td>Poisson’s Ratio [#]</td>
<td>0.44</td>
<td>0.33</td>
<td>0.33</td>
<td>0.27</td>
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Figure 1: Basic fixed beam, illustrating the wide beam approximation
Figure 2: Three edge boundary conditions considered in study.
Figure 3: Plate and flow channel geometry
Figure 4: Sectional view of flow channel, and deflected plates
Figure 5: Fuel plate (left) and model geometry (right)
Figure 6: An element of flat plate geometry (top) force resultants and (bottom) moments
Figure 7: Critical flow velocity ratio versus percent inner region thickness
Figure 8: Critical flow velocity versus total thickness
Figure 9: Critical velocity ratio vs. Young’s Moduli ratio
Figure 10: Clamped-Clamped edge boundary condition
Figure 11: Critical velocity ratio vs. Young’s Moduli ratio and thickness
Figure 12: Ratio critical velocities, Young’s Moduli, & region thickness