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Abstract approved: ______________________
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The approximation of a continuous function, in the maximum norm, by continuous splines in the Everett Interpolation Form is considered. The topics of characterization, uniqueness, and calculation of best approximations are investigated. Since uniqueness fails, a new vector-valued norm, which yields uniqueness, is introduced.

Approximation of Continuous Functions by Everett Splines

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APPROXIMATION OF CONTINUOUS FUNCTIONS BY EVERETT SPLINES

I. INTRODUCTION

1.1. Background Material

The standard approximation problem involves a fixed function space X, a norm $\|\cdot\|$ on X, and a set S of approximating functions which is a subset of X. For a fixed element x of X, $\rho(x, S)$ is defined to be the $\inf\{\|y - x\| \mid y \in S\}$. The set J(x, S) of best approximations to x from S is defined to be the set $\{y \in S \mid \|y - x\| = \rho(x, S)\}$.

Let S be a closed subset of a finite dimensional subspace of X. The general theory then yields that J(x, S) is nonempty and compact. If S is also convex, then J(x, S) is convex and extreme elements of J(x, S) can be found such that any element of S can be written as a convex sum of extreme elements. An extreme element y of a convex set is such that if y, P_1, and P_2 are elements of the convex set and $y = \frac{P_1 + P_2}{2}$, then $P_1 = P_2 = y$. This implies that an element y of a convex set is not an extreme element if and only if there exists a P, $P \not\equiv 0$, and a positive number β such that $y + \alpha P$ is an element of the convex set for any α where $|\alpha| < \beta$.

Let X be a closed subset of the set $C(\Omega)$ of real valued continuous functions on a compact subset Ω of the real line. Let X be normed by the maximum norm: $\|x\|_\Omega = \max_\Omega |x(t)|$. Let S be a closed convex subset of X. In this situation, the set J(x, S) can be characterized by the derivative of the norm. The maximum norm, a convex function, is finite at any element of X. Thus, if y is an element of S and P is such that $y + \alpha P$ for some α greater than zero is an element of S, the differential at y in the direction P exists. In fact, the differential is the limit as α goes to zero from the right of an increasing function ψy, where

$$\psi y(\alpha) = \frac{||y + \alpha P||_\Omega - ||y||_\Omega}{\alpha}$$

and

$$G(y, P, \Omega) = \lim_{\alpha \to 0^+} \psi y(\alpha) = \begin{cases} ||P|| & y \equiv 0 \\ \max\limits_{E(y)} \operatorname{sgn} y(t) P(t) & y \not\equiv 0 \end{cases}$$

E(y) is the set $\{t \in \Omega \mid |y(t)| = ||y||_\Omega\}$ and is referred to as the set of extreme points of y. Thus, if $y \not\equiv 0$, $\alpha > 0$, and $G(y, P, \Omega) > 0$, $||y||_\Omega < ||y + \alpha P||_\Omega$. If $y \not\equiv 0$ and $G(y, P, \Omega) < 0$, there exists a positive β such that $||y + \beta P||_\Omega < ||y||_\Omega$. The characterization of elements of J(x, S) then is as follows [3].

Theorem 1.1.1. Let X be a closed subset of $C(\Omega)$. Let S be a closed convex subset of X. Let $x \in X \setminus S$ and $y \in S$.

$$y \in J(x, S) \text{ iff } ||y - x||_\Omega \leq ||y_1 - x||_\Omega \;\; \forall y_1 \in S \text{ iff } G(y - x, P, \Omega) \geq 0 \;\forall P \text{ such that } \alpha P \in S - y \text{ for some } \alpha > 0$$

Furthermore, if $G(y - x, P, \Omega) > 0, \forall P \not\equiv 0$ such that $\alpha P \in S - y$ for some $\alpha > 0$, then $||y - x||_{\Omega} < ||y_1 - x||_{\Omega} \forall y_1 \in S \setminus y$. Thus, $J(x, S)$ is a singleton.

If Ω is replaced by a closed subset Ω' of Ω and S is replaced by a closed convex subset S' of S, the characterization of a best approximation to x on Ω' from S' remains basically the same; only the added restriction that $x|_{\Omega'} \notin S'|_{\Omega'}$ is needed. The statement concerning uniqueness is somewhat weaker: if $G(y - x, P, \Omega') > 0$, $\forall P|_{\Omega'} \not\equiv 0$ such that $\alpha P \in S' - y$ for some $\alpha > 0$, then $||y - x||_{\Omega'} < ||y_1 - x||_{\Omega'} \forall y_1 \in S'$ such that $y_1|_{\Omega'} \not\equiv y|_{\Omega'}$.

1.2. Statement of the Problem

The approximation of a real valued continuous function on a closed subset $[\bar{a}, \bar{b}]$ of the real line is considered in this paper. A finite subset of points having the properties that $z_0 = \bar{a}$, $z_N = \bar{b}$, and $z_I < z_{I+1}$ for $0 \leq I < N$ is assumed given. The norm on $C([\bar{a}, \bar{b}])$ is taken to be the maximum norm.

The set of approximating functions ES consists of all continuous functions H that reduce on every subinterval $[z_I, z_{I+1}]$ to a cubic polynomial and that have the additional property that $\lim_{t \to z_I^-} H''(t) = \lim_{t \to z_I^+} H''(t)$ for $1 \leq I \leq N - 1$. For ease of notation, if $f(t^*)$ is not defined, but the limit of f at t^* is defined, then $f(t^*)$ is taken to be the limit value. This convention will be used throughout this paper.

It is claimed that ES is the finite dimensional subspace spanned by the set of functions $\{A_I\}_{I=0}^{N} \cup \{B_I\}_{I=0}^{N}$. The definition of these functions is as follows:

$$\text{Let } h_I = 0 \qquad I < 0,\ I \geq N$$

$$h_I = z_{I+1} - z_I \qquad 0 \leq I < N$$

$$\text{Let } V_I(t) = 0 \qquad I < 0,\ I \geq N$$

$$V_I(t) = \begin{cases} 0 & t \in [z_0, z_I] \\ \dfrac{t - z_I}{h_I} & t \in [z_I, z_{I+1}] \\ 0 & t \in [z_{I+1}, z_N] \end{cases} \qquad 0 \leq I < N$$

$$\text{Let } W_I(t) = 0 \qquad I < 0,\ I \geq N$$

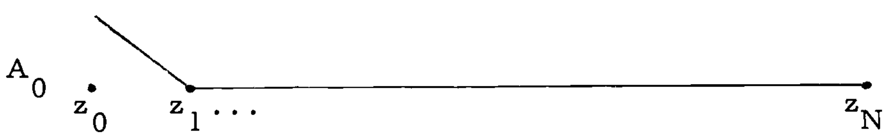
$$W_I(t) = \begin{cases} 0 & t \in [z_0, z_I] \\ 1 - V_I(t) = \dfrac{z_{I+1} - t}{h_I} & t \in [z_I, z_{I+1}] \\ 0 & t \in [z_{I+1}, z_N] \end{cases} \qquad 0 \leq I < N$$

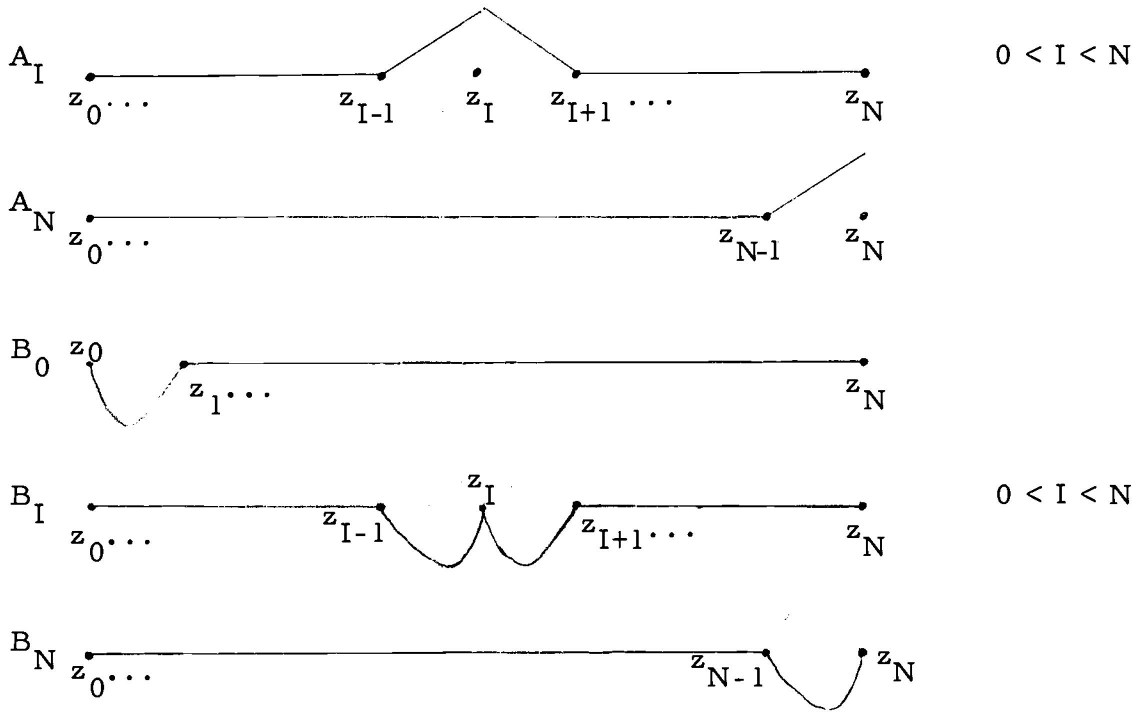
$$\text{Let } g(t) = \begin{cases} 0 & t \notin [0,1] \\ \dfrac{t(t^2 - 1)}{6} & t \in [0,1] \end{cases}$$

$$\text{Let } A_I = \max(V_{I-1}, W_I) \qquad 0 \leq I \leq N$$

$$\text{Let } B_I = \max(h_{I-1}^2\, g(V_{I-1}),\ h_I^2\, g(W_I)) \qquad 0 \leq I \leq N$$

The graphs of these functions would resemble the following graphs:





Since $V_{I-1}(z_{I-1}) = W_I(z_{I+1}) = 0$, $V_{I-1}(z_I) = W_I(z_I) = 1$, and $g(0) = g(1) = 0$, each of the functions A_I and B_I for $0 \leq I \leq N$ are continuous. Furthermore, the functions are second differentiable except at a finite number of points where the limit of the second derivative exists.

$$A_I''(t) = 0 \qquad t \in [z_0, z_N] \qquad 0 \leq I \leq N$$

$$B_0''(t) = \begin{cases} W_0(t) & t \in [z_0, z_1] \\ 0 & t \in [z_1, z_n] \end{cases}$$

$$B_I''(t) = \begin{cases} 0 & t \in [z_0, z_{I-1}] \\ V_{I-1}(t) & t \in [z_{I-1}, z_I] \\ W_I(t) & t \in [z_I, z_{I+1}] \\ 0 & t \in [z_{I+1}, z_N] \end{cases} \qquad 1 \leq I < N$$

$$B_N''(t) = \begin{cases} 0 & t \in [z_0, z_{N-1}] \\ V_{N-1}(t) & t \in [z_{N-1}, z_N] \end{cases}$$

Let M be $\{C(t) | C(t) = \sum_{I=0}^{N} (a_I A_I(t) + b_I B_I(t))\}$. Any element of M is a continuous real valued function. On any subinterval $[z_I, z_{I+1}]$, C(t) is a cubic polynomial.

(1.2.1)

$$C(t)\big|_{[z_I, z_{I+1}]} = C_I(t) = a_I A_I(t) + a_{I+1} A_{I+1}(t) + b_I B_I(t) + b_{I+1} B_{I+1}(t)$$

$$= \left[\frac{b_{I+1} - b_I}{6h_I}\right] t^3 + \left[\frac{b_I z_{I+1} - z_I b_{I+1}}{2h_I}\right] t^2$$

$$+ \left[\frac{a_{I+1} - a_I}{h_I} + \frac{b_I}{6h_I}(z_I^2 - 2z_{I+1} z_I - 2z_{I+1}^2) + \frac{b_{I+1}}{6h_I}(2z_I^2 + 2z_{I+1} z_I - z_{I+1}^2)\right] t$$

$$+ \left[\frac{a_I z_{I+1} - a_{I+1} z_I}{h_I} + \frac{b_I}{6h_I}(2z_I z_{I+1}^2 - z_I^2 z_{I+1}) + \frac{b_{I+1}}{6h_I}(z_{I+1}^2 z_I - 2z_I^2 z_{I+1})\right]$$

Thus, $C(z_I) = a_I$ for $0 \leq I \leq N$ and the limit of the second derivative of C at z_I for $1 \leq I \leq N - 1$ is b_I. Furthermore, if $C \equiv 0$, then $C_I \equiv 0$ for $0 \leq I \leq N - 1$ and so by (1.2.1), $a_I = b_I = 0$ for $0 \leq I \leq N$. The set M then is a subset of ES and is the space spanned by the set of linearly independent functions $\{A_I\}_{I=0}^{N} \bigcup \{B_I\}_{I=0}^{N}$.

It remains to show that ES is a subset of M. By definition, an element H of ES is a cubic polynomial on each subinterval $[z_I, z_{I+1}]$; it is continuous and

$$\lim_{t \to z_I^-} H''(t) = \lim_{t \to z_I^+} H''(t).$$

If $a_I = H(z_I)$ for $0 \leq I \leq N$, and $b_I = H''(z_I)$ for $0 \leq I \leq N$, it can be verified using (1.2.1) that

$$H = \sum_{I=0}^{N} (a_I A_I + b_I B_I).$$

Thus, ES is a subset of M and so ES is equal to M.

The set ES is referred to as the set of Everett Splines. Any element C of ES restricted to a subinterval corresponds in form to Everett's Interpolation Formula EIF [2] truncated after the second difference term:

$$C_I = a_I W_I + a_{I+1} V_I + b_I h_I^2 g(W_I) + b_{I+1} h_I^2 g(V_I)$$

$$EIF = f(z_I) W_I + f(z_{I+1}) V_I + \delta^2 f(z_I) g(W_I) + \delta^2 f(z_{I+1}) g(V_I)$$

Everett's Formula, since it involves only even order differences, truncated after the second difference term is equivalent to Bessel's Formula truncated after the third difference term. Thus, Everett's Formula is often used. In this investigation, it will be required that $a_I = f(z_I)$ for $0 \leq I \leq N$. Thus, only the constant b_I's will be sought to minimize the maximum error in absolute value on

$[\bar{a}, \bar{b}]$. Some work has been done in this area by Leslie Fox [1]. He considers minimizing the error on $[z_I, z_{I+1}]$ of the functions $f(z_I) W_I + d_I g(W_I)$ and $f(z_{I+1}) V_I + d_{I+1} g(V_I)$ separately by expanding in Tchebycheff series. Some comparison of results is given in Chapter VI for f equal to logarithm and exponential.

If f is a fixed element of $C([\bar{a}, \bar{b}])$, only elements C of ES having the property that $f(z_I) = C(z_I)$ for $0 \leq I \leq N$ are referred to as candidates for a best approximation to f from ES. If $C = \Sigma(a_I A_I + b_I B_I)$, then $C(z_I) = f(z_I)$ if and only if $a_I = f(z_I)$. Thus, if ES(f) is $\{C \in ES \mid C = \Sigma(f(z_I) A_I + b_I B_I)\}$, then ES(f) is the set of all candidates for a best approximation to f from ES. If $\rho(f, ES(f))$ is the $\inf\{\|C - f\| \mid C \in ES(f)\}$, then the set $J(f, ES(f))$ of all best approximations to f from ES is $\{C \in ES(f) \mid \|C - f\| = \rho(f, ES(f))\}$.

As stated above, this approximation problem differs from the standard approximation problem in that the set of candidates for best approximations is dependent upon the function to be approximated. This obstacle can be overcome by noting that each element of ES(f) is the sum of a fixed function dependent upon f, $\Sigma f(z_I) A_I$, and an independent function, $\Sigma b_I B_I$. Let X then be the closed subset of $C([\bar{a}, \bar{b}])$ $\{x \mid x = f - \Sigma f(z_I) A_I,\ f \in C([\bar{a}, \bar{b}])\}$. Let X be normed by the maximum norm. Let S be the set $\{P \mid P = \Sigma b_I B_I\}$. Since the B_I's are linearly independent and $\Sigma b_I B_I(z_J) = 0$ for $0 \leq J \leq N$, S is a closed finite dimensional subspace of X. Let $\rho(x, S)$ for x in X be the $\inf\{\|y - x\| \mid y \in S\}$. Let $J(x, S)$ be the set $\{y \in S \mid \|y - x\| = \rho(x, S)\}$. It is claimed that if $x = f - \Sigma f(z_I) A_I$, then y is an element of $J(x, S)$ if and only if $y + \Sigma f(z_I) A_I$ is an element of $J(f, ES(f))$. This is easily seen since the A_I's and B_I's are linearly independent and $\|\Sigma b_I B_I + \Sigma f(z_I) A_I - f\| = \|\Sigma b_I B_I - x\|$.

It is this equivalent approximation problem that is considered in the following chapters. It is assumed throughout that x is not an element of S. By the work of the previous section, $J(x, S)$ is a non-empty, convex, compact set. Furthermore,

(1.2.2)

$$G(y - x, P, [\bar{a}, \bar{b}]) \geq 0 \qquad \text{iff} \quad y \in J(x, S)$$
$$\forall P \text{ such that } \alpha P \in S - y$$
$$\text{for some } \alpha > 0$$

(1.2.3)

If $G(y - x, P, [\bar{a}, \bar{b}]) > 0 \ \forall P \not\equiv 0$ such that $\alpha P \in S - y$ for some $\alpha > 0$, then $J(x, S) = \{y\}$.

In Chapter II, condition (1.2.2) is investigated. Two conditions concerning the location of the extreme points of $y - x$ and the sign of $y - x$ at those points are determined. It is shown that satisfaction of one of those conditions is necessary and sufficient for y to be an element of $J(x, S)$.

In Chapter III, a characterization of a unique best approximation is found. However, the conditions would not be expected to be satisfied in most cases. In Chapter IV, a definition of a "best" best approximation to x from S is given. It is shown that the "best" best approximation to x from S is unique.

In Chapter V, characterization of the extreme elements of J(x, S) is investigated. Further assumptions on x must be made for straightforward characterization.

In Chapter VI, the "best" best approximations to logarithm and exponential are presented; some comments on how they were determined are included.

II. CHARACTERIZATION OF A BEST APPROXIMATION

2.1. Notation and Definitions

Since S is a subspace, the set S - y, for y an element of S, is the set S. Thus, for y to satisfy (1.2.2), the product of y - x and an arbitrary element of S must be positive or zero at one or more extreme points of y - x. By assumption, x is not an element of S. Therefore, y - x cannot be zero at any of its extreme points. Furthermore, - P is in S if and only if P is in S. Thus, if there exists a P in S that agrees in sign with y - x at every extreme point of y - x, y is not an element of J(x, S). If such a P does not exist, y is a best approximation.

The ability of an element P of S to take on a specified sign at a designated point can be represented by a vector R(P) of relationships between the coefficients of P. Consider P restricted to subinterval $[z_I, z_{I+1}]$ where it reduces to the sum of two functions:

$$P|_{[z_I, z_{I+1}]} = P_I = h_I^2[b_I g(W_I) + b_{I+1} g(V_I)].$$

Since $V_I = 1 - W_I$, P_I can be expanded to the following form:

(2.1.1)

$$P_I = \frac{h_I^2}{6} V_I(1 - V_I)[(b_I - b_{I+1})V_I - (2b_I + b_{I+1})].$$

If t_1 and t_2 are elements of (z_I, z_{I+1}) and $t_1 < t_2$, then $0 < V_I(t_1) < V_I(t_2) < 1$. Thus, the sign of P_I at a designated point depends on the relationship between the coefficients b_I and b_{I+1}.

Since the form (2. 1. 1) would remain the same if P is restricted to any other subinterval with only the subscripts changing, a general polynomial of the same form is considered. The dependence of the sign of that polynomial on the relationship between its coefficients is determined in the following.

Lemma 2. 1. 1.

Let $Q(V) = K(V)(1 - V)[(C - D)V - (2C + D)]$, where K is a positive constant and V is an element of $[0, 1]$.

$Q(V) > 0 \quad 0 < V < 1$ iff $C = D < 0$ (r1)

or $C < 0,\ D > 0,\ C = -2D$ (r2)

or $C > 0,\ D < 0,\ D = -2C$ (r3)

or $C < 0,\ C < -2D < -2C$ (r4)

or $D < 0,\ D < -2C < -2D$ (r5)

$Q(V) < 0 \quad 0 < V < 1$ iff $C = D > 0$ (r6)

or $C > 0,\ D < 0,\ C = -2D$ (r7)

or $C < 0,\ D > 0,\ D = -2C$ (r8)

or $C > 0,\ -2C < -2D < C$ (r9)

or $D > 0,\ -2D < -2C < D$ (r10)

$$Q(V) > 0 \quad 0 < V < \frac{2C + D}{C - D} \qquad\qquad C < 0,\ D > 0$$
$$\text{and} \qquad\qquad \text{iff} \qquad\qquad \text{and} \qquad\qquad (r11)$$
$$Q(V) < 0 \quad \frac{2C + D}{C - D} < V < 1 \qquad\qquad 0 > 2C + D > C - D$$

$$Q(V) < 0 \quad 0 < V < \frac{2C + D}{C - D} \qquad\qquad C > 0,\ D < 0$$
$$\text{and} \qquad\qquad \text{iff} \qquad\qquad \text{and} \qquad\qquad (r12)$$
$$Q(V) > 0 \quad \frac{2C + D}{C - D} < V < 1 \qquad\qquad 0 < 2C + D < C - D$$

$$Q(V) = 0 \quad 0 \leq V \leq 1 \qquad\qquad \text{iff} \qquad\qquad C = D = 0 \qquad\qquad (r13)$$

<u>Proof</u>: If $C = D$, then $Q(V) = -3CKV(1 - V)$. Thus, if $C > 0$, $Q(V) < 0$; if $C < 0$, $Q(V) > 0$; if $C = 0$, $Q(V) = 0$.

If $C \neq D$, since $\operatorname{sgn} KV(1 - V) = +1$ for V in $(0, 1)$, $\operatorname{sgn} Q(V) = \operatorname{sgn}(C - D)\left[V - \left(\frac{2C + D}{C - D}\right)\right]$; Q is a cubic with roots at 0, 1, and $\frac{2C + D}{C - D}$.

Assume that $C > D$. $\frac{2C + D}{C - D} \leq 0$ implies that $Q(V) > 0$ and $2C \leq -D$. Thus, $2D < 2C \leq -D$. Therefore, $D < 0$. $\frac{2C + D}{C - D} \geq 1$ implies that $Q(V) < 0$ and $2C + D \geq C - D$. Thus, $C \geq -2D > -2C$. Therefore, $C > 0$. $0 < \frac{2C + D}{C - D} < 1$ implies that $Q(V) < 0$ for $0 < V < \frac{2C + D}{C - D}$ and $Q(V) > 0$ for $\frac{2C + D}{C - D} < V < 1$. Furthermore, $0 < 2C + D < C - D$. Thus, $D - 2C < 2D < -C < -D$ and $0 < 2C + D$. Therefore, $D < 0$ and $C > 0$.

The remaining results can be found by assuming that $D > C$.

Thus, given any P in S and any subinterval (z_I, z_{I+1}), the <u>coefficient pair</u> (b_I, b_{I+1}) of P_I must be related by one of the thirteen possible sets of inequalities. Each of the sets of inequalities is referred to as a relationship on the coefficient pair (b_I, b_{I+1}). The I component $R_I(P)$ of $R(P)$ is then an element of the set $\{rj \mid 1 \leq j \leq 13\}$ and is equal to the symbol corresponding to the relationship on (b_I, b_{I+1}). Thus, P_I is greater than zero on (z_I, z_{I+1}) if and only if $R_I(P)$ is rj for some j in $\{1, 2, 3, 4, 5\}$. P_I is less than zero on (z_I, z_{I+1}) if and only if $R_I(P)$ is rj for some j in $\{6, 7, 8, 9, 10\}$. For some t* in (z_I, z_{I+1}), P_I is greater than zero on (z_I, t^*) and less than zero on (t^*, z_{I+1}) if and only if $R_I(P)$ is r11. For some t* in (z_I, z_{I+1}), P_I is less than zero on (z_I, t^*) and greater than zero on (t^*, z_{I+1}) if and only if $R_I(P)$ is r12. P_I is identically zero on (z_I, z_{I+1}) if and only if $R_I(P)$ is r13.

Any vector R of N components from the set $\{rj \mid 1 \leq j \leq 13\}$ is referred to as a S vector if an element P of S can be constructed such that the coefficient pair (b_I, b_{I+1}) of each P_I is related by R_I. Not every vector R, however, is a S vector. Adjacent P_I's share a coefficient; in particular, the coefficient b_I of $g(W_I)$ in P_I appears as the coefficient of $g(V_{I-1})$ in P_{I-1}. The coefficient of $g(V_I)$ in P_I appears as the coefficient of $g(W_{I+1})$ in P_{I+1}. Thus, the relationship on (b_I, b_{I+1}) effects which relationships may be placed on (b_{I-1}, b_I) and (b_{I+1}, b_{I+2}).

If (b_I, b_{I+1}) is related by rj for j in $\{1, 3, 5, 7, 12\}$, then b_{I+1} must be negative. Thus, (b_{I+1}, b_{I+2}) can be related by rj only for j in {1, 2, 4, 5, 8, 10, 11}. It is to be noted that if (b_{I+1}, b_{I+2}) is to be related by r11 and b_{I+1} is a fixed negative constant, b_{I+2} can be chosen so that the interior root of P_{I+1} is at any specified point in (z_{I+1}, z_{I+2}).

If (b_I, b_{I+1}) is related by rj for j in {2, 6, 8, 10, 11}, then b_{I+1} must be positive. Thus, (b_{I+1}, b_{I+2}) can be related by rj only for j in {3, 5, 6, 7, 9, 10, 12}. If (b_{I+1}, b_{I+2}) is to be related by r12 and b_{I+1} is a fixed positive constant, b_{I+2} can be chosen so that the interior root of P_{I+1} is at any specified point in (z_{I+1}, z_{I+2}).

If (b_I, b_{I+1}) is related by rj for j in {4, 9}, then the sign of b_{I+1} is not specified. Thus, (b_{I+1}, b_{I+2}) can be related by rj for any j where $1 \leq j \leq 13$.

If (b_I, b_{I+1}) is related by r13, then b_{I+1} must be zero. Thus, (b_{I+1}, b_{I+2}) can be related by rj only for j in {5, 10, 13}.

These results are found in Table I. In a similar manner, the effects of the relationship on (b_I, b_{I+1}) on the sign of b_I and the possible relationships on (b_{I-1}, b_I) can be determined. The results are found in Table II.

Thus, by checking each pair of adjacent components for agreement with Table I or Table II, it can be determined whether or not a given vector R is a S vector. Furthermore, any S vector R, which does not consist entirely of components equal to r13, represents an infinite number of elements of S. Also, if the M component of a S

TABLE I

Relationship on (b_M, b_{M+1}) and Sign of P_M					Restriction on Sign of	Relationship on (b_{M+1}, b_{M+2}) Consistent with Restriction on b_{M+1} and Sign of P_{M+1}				
$P_M > 0$	$P_M < 0$	$P_M ><$	$P_M <>$	$P_M \equiv 0$	b_{M+1}	$P_{M+1} > 0$	$P_{M+1} < 0$	$P_{M+1} ><$	$P_{M+1} <>$	$P_{M+1} \equiv 0$
r1, r3, r5	r7	-	r12	-	$b_{M+1} < 0$	r1, r2, r4, r5	r8, r10	r11	-	-
r2	r6, r8, r10	r11	-	-	$b_{M+1} > 0$	r3, r5	r6, r7, r9, r10	-	r12	-
r4	r9	-	-	-	None	r1, r2, r3, r4, r5	r6, r7, r8, r9, r10	r11	r12	r13
-	-	-	-	r13	$b_{M+1} = 0$	r5	r10	-	-	r13

TABLE II

Relationship on (b_M, b_{M+1}) and Sign of P_M					Restriction on Sign of	Relationship on (b_{M-1}, b_M) Consistent with Restriction on b_M and Sign of P_{M-1}				
$P_M > 0$	$P_M < 0$	$P_M ><$	$P_M <>$	$P_M \equiv 0$	b_M	$P_{M-1} > 0$	$P_{M-1} < 0$	$P_{M-1} ><$	$P_{M-1} <>$	$P_{M-1} \equiv 0$
r1, r2, r4	r8	r11	-	-	$b_M < 0$	r1, r3, r4, r5	r7, r9	-	r12	-
r3	r6, r7, r9	-	r12	-	$b_M > 0$	r2, r4	r6, r8, r9, r10	r11	-	-
r5	r10	-	-	-	None	r1, r2, r3, r4, r5	r6, r7, r8, r9, r10	r11	r12	r13
-	-	-	-	r13	$b_M = 0$	r4	r9	-	-	r13

vector R is r11 or r12, a P in S exists such that the interior root of P_M is equal to any specified point in (z_M, z_{M+1}) and R(P) is equal to R.

Example 2.1.1. Let N = 3. The vector R = (r13, r11, r11) is not a S vector. First, if (b_0, b_1) is related by r13, $b_0 = b_1 = 0$. However, if (b_1, b_2) is related by r11, $b_1 < 0$. Second, if (b_1, b_2) is related by r11, $b_2 > 0$. However, if (b_2, b_3) is related by r11, $b_2 < 0$.

Example 2.1.2. Let N = 9. Let R = (r6, r3, r11, r12, r11, r3, r4, r9, r13). Let it be desired that P has roots at $1/2z_2 + 1/2\,z_3$, $1/3\,z_3 + 2/3\,z_4$, and $3/4\,z_4 + 1/4\,z_5$. If $b_0 = 1$, $b_1 = 1$, $b_2 = -2$, $b_3 = 2$, $b_4 = -8/5$, $b_5 = 56/25$, $b_6 = -112/56$, $b_7 = 1$, $b_8 = 0$, and $b_9 = 0$, then $P = \Sigma b_I B_I$ is an element of S having the desired roots and R(P) is equal to R. If $\alpha > 0$ and $P' = \alpha P$, then P' is also an element of S having the desired roots and R(P') is equal to R. In fact, if $\hat{b}_7$ is such that $0 < \hat{b}_7 < 56/25$, and $\hat{P} = \sum_{I \neq 7} b_I B_I + \hat{b}_7 B_7$, then $\hat{P}$ is an element of S having the desired roots and $R(\hat{P})$ is equal to R.

The sign of y - x at its extreme points can be represented by a vector SGN(y - x) where the I component $SGN_I(y - x)$ represents the sign of y - x at extreme points in $[z_I, z_{I+1}]$. Since y - x is zero at any z_I and nonzero at all of its extreme points, E(y - x) contains none of the z_I's. Thus, if $E_I(y - x)$ is the intersection of E(y - x) and (z_I, z_{I+1}), E(y - x) is the union of the $E_I(y - x)$ for $0 \leq I \leq N - 1$. To each set $E_I(y - x)$ is assigned a symbol from the set

$\{F, +, -, +-, -+, T\}$, by the rules given below and summarized in the first two columns of Table III, which represents the sign of $y - x$ on $E_I(y - x)$. $SGN_I(y - x)$ is the symbol assigned to $E_I(y - x)$.

If the set $E_I(y - x)$ is empty, the symbol F is assigned to $E_I(y - x)$. $SGN_I(y - x)$ is F then and is referred to as free.

If the set $E_I(y - x)$ is nonempty and $\operatorname{sgn}(y - x)(t) = +1$ for all t in $E_I(y - x)$, the symbol + is assigned to $E_I(y - x)$. $SGN_I(y - x)$ is + then and is referred to as a single.

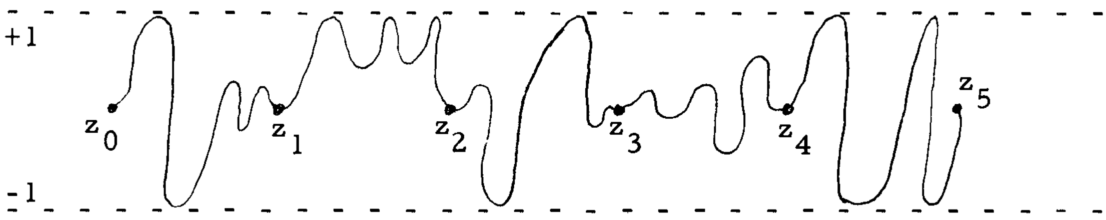
If the set $E_I(y - x)$ is nonempty and $\operatorname{sgn}(y - x)(t) = -1$ for all t in $E_I(y - x)$, the symbol - is assigned to $E_I(y - x)$. $SGN_I(y - x)$ is - then and is also referred to as a single.

If there exists a t* such that $E_I(y - x) \cap (z_I, t^*)$ is nonempty, $E_I(y - x) \cap (t^*, z_{I+1})$ is nonempty, $(y - x)(t^*)$ is zero, $\operatorname{sgn}(y - x)(t) = +1$ for all t in $E_I(y - x) \cap (z_I, t^*)$, and $\operatorname{sgn}(y - x)(t) = -1$ for all t in $E_I(y - x) \cap (t^*, z_{I+1})$, then the symbol +- is assigned to $E_I(y - x)$. $SGN_I(y - x)$ is +- then and is referred to as a double.

If there exists a t* such that $E_I(y - x) \cap (z_I, t^*)$ is nonempty, $E_I(y - x) \cap (t^*, z_{I+1})$ is nonempty, $(y - x)(t^*)$ is zero, $\operatorname{sgn}(y - x)(t) = -1$ for all t in $E_I(y - x) \cap (z_I, t^*)$, and $\operatorname{sgn}(y - x)(t) = +1$ for all t in $E_I(y - x) \cap (t^*, z_{I+1})$, then the symbol -+ is assigned to $E_I(y - x)$. $SGN_I(y - x)$ is -+ then and is referred to as a double.

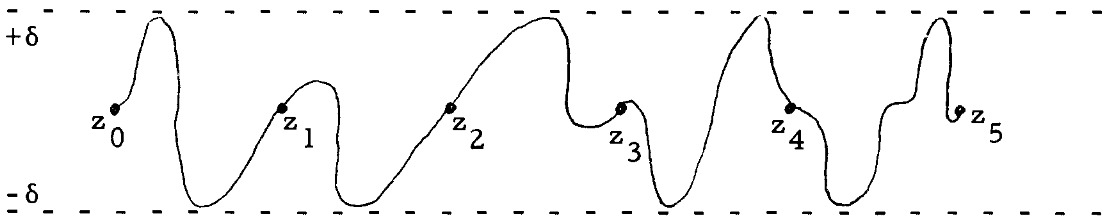
If there exists t_1, t_2, and t_3, elements of $E_I(y - x)$, such that $t_I < t_{I+1}$ and $\operatorname{sgn}(y - x)(t_I) = -\operatorname{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq 2$, then the symbol T is assigned to $E_I(y - x)$. $SGN_I(y - x)$ is T then and is referred to as a <u>triple</u>.

<u>Example 2.1.3.</u> Let N = 5. Let $||y - x|| = 1$. Let the graph of y - x be as follows:

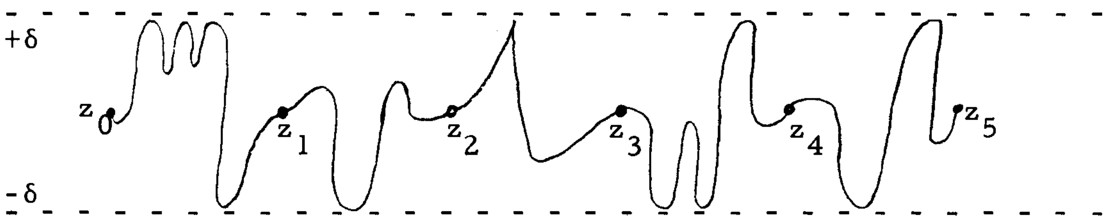


SGN(y - x) must be as follows: (+ -, +, - +, F, T).

<u>Example 2.1.4.</u> Let N = 5. Let SGN(y - x) be (+ -, -, +, - +, - +). The graph of y - x could be as follows:



where $\delta = ||y - x||$. The graph could also be:



where $\delta = ||y - x||$.

The existence of a P in S that agrees in sign with y - x at each point in E(y - x) can be thought of as the existence of a S vector R(P) that <u>matches</u> SGN(y - x). The I components "match" if the relationship $R_I(P)$ is such that P_I agrees in sign at appropriate points with the symbol $SGN_I(y - x)$. For example, if $SGN_I(y - x)$ is +, $R_I(P)$ can be rj only for j in {1, 2, 3, 4, 5, 11, 12}, if $R_I(P)$ is to "match" $SGN_I(y - x)$. The first five are obvious since P_I would be greater than zero on (z_I, z_{I+1}). Since y - x is continuous and is zero at each z_I and z_{I+1}, there exists a δ such that $E_I(y - x)$ is a subset of $(z_I + \delta, z_{I+1} - \delta)$. Thus, $R_I(P)$ can be r11; and, b_I and b_{I+1} can be chosen so that the interior root of P_I is greater than max $E_I(y - x)$. $R_I(P)$ can be r12; and, b_I and b_{I+1} can be chosen so that the interior root of P_I is less than the min $E_I(y - x)$. However, if $R_I(P)$ is rj for j in {6, 7, 8, 9, 10}, P_I is less than zero and has sign opposite $SGN_I(y - x)$. If $R_I(P)$ is r13, P_I is identically zero and so cannot "match" the sign in $SGN_I(y - x)$. The "matches" for other $SGN_I(y - x)$ can be found by similar arguments. The results are presented in column three of Table III.

The existence of a S vector to "match" a given SGN(y - x) vector is investigated further in the next section. Two other terms, however, will be needed throughout. A <u>segment</u> of a vector will refer to a group of consecutive components. A pair of adjacent nonfree components of a SGN(y - x) vector will be said to be <u>alternating</u> if the last sign in $SGN_I(y - x)$ is opposite the first sign in $SGN_{I+1}(y - x)$. A pair of components not satisfying this condition is said to be <u>non-alternating</u>.

TABLE III

Type of $E_M(y-x)$ and Sign of $y-x$ on $E_M(y-x)$	$SGN_M(y-x)$	Possible R_M to Match	Let $t_M = \dfrac{2b_M + b_{M+1}}{b_M - b_{M+1}}$ if $b_M \neq b_{M+1}$
$E_M(y-x) = \emptyset$	F	rj	$1 \leq j \leq 13$
$\forall t \in E_M(y-x) \neq \emptyset,\ sgn(y-x)(t) = +1$	+	rj	$1 \leq j \leq 5$
		r11	$t_M > \max E_M(y-x)$
		r12	$t_M < \min E_M(y-x)$
$\forall t \in E_M(y-x) \neq \emptyset,\ sgn(y-x)(t) = -1$	-	rj	$6 \leq j \leq 10$
		r11	$t_M < \min E_M(y-x)$
		r12	$t_M > \max E_M(y-x)$
$\forall t \in E_M(y-x) \cap (z_M, t^*) \neq \emptyset,\ sgn(y-x)(t) = +1$; $\forall t \in E_M(y-x) \cap (t^*, z_{M+1}) \neq \emptyset,\ sgn(y-x)(t) = -1$	+ -	r11	$\max(E_M(y-x) \cap (z_M, t^*)) < t_M < \min(E_M(y-x) \cap (t^*, z_{M+1}))$
$\forall t \in E_M(y-x) \cap (z_M, t^*) \neq \emptyset,\ sgn(y-x)(t) = -1$; $\forall t \in E_M(y-x) \cap (t^*, z_{M+1}) \neq \emptyset,\ sgn(y-x)(t) = +1$	- +	r12	$\max(E_M(y-x) \cap (z_M, t^*)) < t_M < \min(E_M(y-x) \cap (t^*, z_{M+1}))$
$\{t_I\}_{I=1}^{3} \subset E_M(y-x),\ t_I < t_{I+1},\ sgn(y-x)(t_I) = -sgn(y-x)(t_{I+1})$ for $1 \leq I \leq 2$	T	None	

Example 2.1.5. Each of the following pairs is alternating.

$SGN_I(y - x)$, $SGN_{I+1}(y - x)$		$SGN_I(y - x)$, $SGN_{I+1}(y - x)$	
+ -	+	- +	-
+ -	+ -	- +	- +
-	+	+	-
-	+ -	+	- +

Example 2.1.6. Each of the following pairs is nonalternating. There are others.

$SGN_I(y - x)$, $SGN_{I+1}(y - x)$		$SGN_I(y - x)$, $SGN_{I+1}(y - x)$	
F	F	+	+
F	+	+	+ -
F	-	-	-
F	+ -	-	- +
F	- +	+ -	- +

2.2. Characterization of the Vector SGN(y - x) for y in J(x, S)

From Table I and Table II, the characterization of a vector SGN(y - x) that cannot be matched is readily available.

Lemma 2.2.1.

Let $x \in X \setminus S$ and $y \in S$. One of the following conditions is satisfied if and only if y is in J(x, S).

(2.2.1, 1) Let SGN(y - x) contain a triple.

(2.2.1, 2) Let $0 \leq J < K \leq N - 1$. Let $SGN_J(y - x)$ and $SGN_K(y - x)$ be doubles. Let $SGN_I(y - x)$ for $J < I < K$ be a single. Let the pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ be alternating for $J \leq I < K$.

Proof:

Case I: Assume that (2.2.1, 1) is satisfied. Let $SGN_J(y - x) = T$. By definition, there must exist a subset $\{t_I\}_{I=1}^{3}$ of $E_J(y - x)$ such that $t_I < t_{I+1}$ and $sgn(y - x)(t_I) = -sgn(y - x)(t_{I+1})$ for $1 \leq I \leq 2$. If P_J is to have the same sign as $y - x$ at these points, there must exist t_1^* and t_2^* such that $z_J < t_1 < t_1^* < t_2 < t_2^* < t_3 < z_{J+1}$, and, $P_J(z_J) = P_J(t_1^*) = P_J(t_2^*) = P_J(z_{J+1}) = 0$. However, P_J is a cubic polynomial. Therefore, P_J must have sign opposite $y - x$ at some t_I; or, P_J must be identically zero. Thus, $\forall P \in S$, $G(y - x, P, [\bar{a}, \bar{b}]) \geq 0$.

Case II: Assume that (2.2.1, 2) is satisfied. Let $J + U = K$.

If $U = 1$, then the pair must be + -, + - or - +, - +. Since r11, r11 and r12, r12 are not segments of a S vector, $\forall P \in S$, $G(y - x, P, [\bar{a}, \bar{b}]) \geq 0$.

If $U > 1$, assume that $SGN_J(y - x) = + -$. The case where $SGN_J(y - x) = - +$ can be proven in a similar manner. Since each pair is alternating, $SGN_{J+I}(y - x) = +$, if I is odd; and, $SGN_{J+I}(y - x) = -$, if I is even. Furthermore, $SGN_K(y - x) = + -$, if U is odd; and, $SGN_K(y - x) = - +$, if U is even.

In a segment of a S vector, if $R_I = rj$ for j in {3, 5, 12}, and

R_{I+1} must match $SGN_{I+1}(y - x) = -$, then $R_{I+1} = rj$ for j in {8, 10, 11}. In a segment of a S vector, if $R_{I+1} = rj$ for j in {8, 10, 11} and R_{I+2} must match $SGN_{I+2}(y - x) = +$, then $R_{I+2} = rj$ for j in {3, 5, 12}.

Since $SGN_J(y - x) = +-$, $R_J = r11$. Thus, $R_{J+I} = rj$ for j in {3, 5, 12}, if I is odd; and, $R_{J+I} = rj$ for j in {8, 10, 11}, if I is even. Therefore, if U is odd, R_K cannot equal r11 in a segment of a S vector and so cannot match $SGN_K(y - x) = +-$. If U is even, R_K cannot equal r12 in a segment of a S vector and so cannot match $SGN_K(y - x) = -+$. Thus, $\forall P \in S$, $G(y - x, P, [\bar{a}, \bar{b}]) \geq 0$.

Case III: Assume that neither (2.2.1, 1) nor (2.2.1, 2) is satisfied. The vector SGN(y - x) must contain only frees, singles and doubles.

If SGN(y - x) contains only frees and singles, for $0 \leq I \leq N - 1$, let $R_I = r4$, if $SGN_I(y - x) = +$ or $SGN_I(y - x) = F$; let $R_I = r9$, if $SGN_I(y - x) = -$. The vector R is a S vector that matches SGN(y - x). Let P be in S and R(P) = R; then $G(y - x, -P, [\bar{a}, \bar{b}]) < 0$.

If SGN(y - x) contains one or more doubles, divide SGN(y - x) into overlapping segments of three types. The first type begins with the 0 component and ends with the first double. The second type begins with the last double and ends with the N - 1 component. The third type begins with a double, ends with the next double, and has at least one nonalternating pair. Since there is only one match for a fixed double, if a segment of a S vector can be found to match each segment type, the matching segments can be overlapped to form a matching S vector.

Type One Segments: Let $SGN_J(y - x)$ be the first double. For $0 \leq I < J$, let $R_I = r4$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r9$, if $SGN(y - x) = -$. If $SGN_J(y - x) = +-$, let $R_J = r11$. If $SGN_J(y - x) = -+$, let $R_J = r12$. This forms a matching segment of a S vector.

Type Two Segments: Let $SGN_K(y - x)$ be the last double. Let $R_K = r11$, if $SGN_K(y - x) = +-$. Let $R_K = r12$, if $SGN_K(y - x) = -+$. For $K < I \leq N - 1$, let $R_I = r5$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r10$, if $SGN_I(y - x) = -$. This forms a matching segment of a S vector.

Type Three Segments: Let $SGN_J(y - x)$ be the beginning double and $SGN_K(y - x)$ be the ending double. Let the pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ where $J \leq M < K$ be nonalternating. Let $J + U = K$.

If $M = J$, let $R_J = r11$ and $R_{J+1} = r9$, if $SGN_J(y - x) = +-$; let $R_J = r12$ and $R_{J+1} = r4$, if $SGN_J(y - x) = -+$. For $J + 2 \leq I < K$, let $R_I = r4$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r9$, if $SGN_I(y - x) = -$. Let $R_K = r11$, if $SGN_K(y - x) = +-$; let $R_K = r12$, if $SGN_K(y - x) = -+$. This forms a matching segment of a S vector.

If $M = K - 1$, let $R_J = r11$, if $SGN_J(y - x) = +-$; let $R_J = r12$, if $SGN_J(y - x) = -+$. For $J < I \leq K - 2$, let $R_I = r5$, if $SGN_I(y - x) = +$ or $SGN_I(y - x) = F$; let $R_I = r10$, if $SGN_I(y - x) = -$. Let $R_{K-1} = r5$

and R_K = r11, if $SGN_K(y - x) = +-$; let R_{K-1} = r10 and R_K = r12, if $SGN_K(y - x) = -+$. This forms a matching segment of a S vector.

If $J < M < K - 1$, let R_J = r11, if $SGN_J(y - x) = +-$; let R_J = r12, if $SGN_J(y - x) = -+$. For $J + 1 \leq I < M$, let R_I = r5, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let R_I = r10, if $SGN_I(y - x) = -$. If $SGN_M(y - x) = +$, or $SGN_{M+1}(y - x) = +$, then let R_M = r5 and R_{M+1} = r4; otherwise, let R_M = r10 and R_{M+1} = r9. For $M + 2 \leq I < K$, let R_I = r4, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let R_I = r9, if $SGN_I(y - x) = -$. Let R_K = r11, if $SGN_K(y - x) = +-$; let R_K = r12, if $SGN_K(y - x) = -+$. This forms a matching segment of a S vector.

Let R be the vector formed by overlapping appropriate matching segments as defined above; then, R is a S vector and R matches SGN(y - x). Thus, there must exist a P in S such that $R(P) = R$ and $G(y - x, -P, [\bar{a}, \bar{b}]) < 0$.

2.3. Summary

The characterization in the previous section is given in terms of conditions on SGN(y - x). For further use, it will be more convenient to refer to a characteristic pattern of extreme points as defined below.

Theorem 2.3.1. Let $x \in X \setminus S$ and $y \in S$. The following condition is satisfied if and only if y is in J(x, S).

(2. 3. 1, 1) Let $0 \leq J < J + U \leq N$. Let there exist points $\{t_I\}_{I=1}^{U+2}$ such that $t_1 \in E_J(y - x)$, $t_I \in E_{J+I-2}(y - x)$ for $2 \leq I \leq U + 1$, and $t_{U+2} \in E_{J+U-1}(y - x)$. Let $t_I < t_{I+1}$ and $\mathrm{sgn}(y - x)(t_I) = -\mathrm{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U + 1$.

Remark: If (2. 3. 1, 1) is satisfied, the pattern of extreme points from $E_J(y - x)$ to $E_{J+U-1}(y - x)$ is called a characteristic pattern of extreme points; every subinterval (z_I, z_{I+1}) for $J \leq I \leq J + U - 1$ is referred to as a characteristic subinterval of y; every component $SGN_I(y - x)$ for $J \leq I \leq J + U - 1$ is referred to as a characteristic component of y.

Proof: It is to be shown that: (2. 3. 1, 1) is satisfied if and only if SGN(y - x) satisfies (2. 2. 1, 1) or (2. 2. 1, 2).

If SGN(y - x) satisfies (2. 2. 1, 1) or (2. 2. 1, 2), then, by definition of SGN(y - x), (2. 3. 1, 1) is satisfied.

If there is a characteristic pattern from $E_J(y - x)$ to $E_{J+U-1}(y - x)$ and $U = 1$, then $SGN_J(y - x) = T$; and so, SGN(y - x) satisfies (2. 2. 1, 1).

If $U > 1$, let $U' = U - 1$. It is claimed that (2. 2. 1, 2) is satisfied by some segment between $SGN_J(y - x)$ and $SGN_{J+U'}(y - x)$.

If $U' = 1$, or $SGN_I(y - x)$ is a single for $J < I < J + U'$, then, by definition, SGN(y - x) satisfies (2. 2. 1, 2).

If $U' > 1$ and $SGN_M(y - x)$ is a double for some M where $J < M < J + U'$, let J(i) for $1 \leq i \leq b$ be such that $SGN_{J(i)}(y - x)$ is a double and $J = J(1) \leq J(i) < J(i + 1) \leq J(b) = J + U'$. Assume that the characteristic pattern from $E_{J(1)}(y - x)$ to $E_{J(b)}(y - x)$ demands that the point referred to in $E_{J(i)}(y - x)$, where $1 < i < b$, has the sign of $y - x$ equal to $+1$. If there exists a $t_1 < t_2$, elements of $E_{J(i)}(y - x)$, such that $\operatorname{sgn}(y - x)(t_1) = +1 = -\operatorname{sgn}(y - x)(t_2)$, then there is a characteristic pattern from $E_{J(1)}(y - x)$ to $E_{J(i)}(y - x)$. If such points do not exist, then there must be points $t_1 < t_2$, elements of $E_{J(i)}(y - x)$, such that $\operatorname{sgn}(y - x)(t_1) = -1 = -\operatorname{sgn}(y - x)(t_2)$. Thus, there is a characteristic pattern from $E_{J(i)}(y - x)$ to $E_{J(b)}(y - x)$. Therefore, there must exist a j, where $1 \leq j \leq b - 1$, such that there is a characteristic pattern from $E_{J(j)}(y - x)$ to $E_{J(j+1)}(y - x)$. Since $SGN_I(y - x)$ is a single for $J(j) < I < J(j+1)$, $SGN(y - x)$ must satisfy (2.2.1, 2).

It is important in the following material that every characteristic subinterval of y has been recognized; thus, all characteristic patterns, even those that overlap, must be determined.

III. CHARACTERIZATION OF A UNIQUE BEST APPROXIMATION

3.1. Statement of Characterization

It is desirable having found one element of J(x, S) to readily determine if there are other elements in the set. The general theory provides that if the element satisfies (1.2.3), then it is unique. In fact, it can be shown that satisfaction of (1.2.3) is both necessary and sufficient for uniqueness. The proof, which is presented in the remainder of this chapter, is both long and complicated. However, it not only yields the straightforward characterization of a unique best approximation as given below, but also forms the basis for the work of Chapter IV and Chapter V.

Theorem 3.1.1. Let $x \in X \setminus S$ and $y \in S$. Let CS be the union of all characteristic subintervals of y.

$\{y\} = J(x, S)$ iff (z_0, z_1) and (z_{N-1}, z_N) are subsets of CS; and, if (z_M, z_{M+1}) is not a subset of CS, then (z_{M+1}, z_{M+2}) is a subset of CS.

Furthermore, if y is in J(x, S), then $y|_{CS} \equiv y_1|_{CS}$ for all y_1 in J(x, S).

3.2. Notation and Preliminary Results

If y is in J(x, S) and does not satisfy condition (1.2.3), then there must exist a P in S, $P \not\equiv 0$, and $\alpha > 0$ such that αP is in S - y and $G(y - x, P, [\bar{a}, \bar{b}]) = 0$. Thus, the product of y - x and P must be zero at one or more points in E(y - x) and must be less than zero at all remaining extreme points. Since S is a linear subspace, S - y is S and -S is S. Thus, if y is in J(x, S) and condition (1.2.3) is not satisfied, there must exist a P in S, $P \not\equiv 0$, that is zero at one or more points in E(y - x) and has the same sign as y - x at all remaining extreme points. The conditions under which such a P exists must be determined.

Since P_I is a cubic polynomial with roots at z_I and z_{I+1}, and E(y - x) does not contain z_M for any M where $0 \leq M \leq N$, P_I can be zero at t in $E_I(y - x)$ only if either $P_I \equiv 0$, or P_I has a single interior root at t. Thus, if P is to be zero or agree in sign with y - x at points in E(y - x), and $(y - x)(t_1) > 0$ for some t_1 in $E_I(y - x)$, then either $P_I \equiv 0$, or $P_I(t_1) \geq 0$ and there exists a t_1^* in (z_I, z_{I+1}) such that $P_I(t_1^*) > 0$. If for some $t_1 < t_2$, elements of $E_I(y - x)$, $(y - x)(t_1) > 0$ and $(y - x)(t_2) < 0$, then if P is to be zero or agree in sign with y - x at points in E(y - x), either $P_I \equiv 0$, or $P_I(t_1) \geq 0$, $P_I(t_2) \leq 0$, and there exists points t_1^* and t_2^* where $z_I < t_1^* < t_1 < t_2 < t_2^* < z_{I+1}$ such that $P_I(t_1^*) > 0$ and $P_I(t_2^*) < 0$. The inequalities in statements involving y - x and P_I can be reversed and similar results would be obtained.

Thus, the existence of a P in S, $P \not\equiv 0$, that is zero at one or more points in E(y - x) and agrees in sign with y - x at remaining extreme points can be thought of as the existence of a S vector R(P), which does not consist entirely of components equal to r13, that zero-matches the vector SGN(y - x). The I components "zero-match" if the relationship $R_I(P)$ is such that $P_I \equiv 0$, or P_I agrees in sign at appropriate points with the symbol $SGN_I(y - x)$. For example, let $SGN_I(y - x)$ be + -; let E_1 be $\{t \in E_I(y - x) | (y - x)(t) > 0\}$; let E_2 be $\{t \in E_I(y - x) | (y - x)(t) < 0\}$. $R_I(P)$ can be rj only for j in {11, 13} if $R_I(P)$ is to "zero-match" $SGN_I(y - x)$. If $R_I(P)$ is r11, the interior root of P_I must be greater than or equal to max E_1 and less than or equal to min E_2. $R_I(P)$ equal to rj for j in {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12} could not "zero-match" $SGN_I(y - x)$. In the first ten cases, P_I would have constant sign on (z_I, z_{I+1}) while y - x changes sign on $E_I(y - x)$. If $R_I(P)$ is r12 and the interior root of P_I is less than or equal to min E_1, then P_I and y - x have opposite sign at points in E_2. If $R_I(P)$ is r12 and the interior root of P_I is greater than min E_1, then P_I and y - x have opposite sign at min E_1. The "zero-matches" for other $SGN_I(y - x)$ can be found by similar arguments and are presented in Table IV.

It is seen that the main difference between component matching

TABLE IV

$SGN_M(y - x)$		Possible R_M to Zero-Match Let $t_M = \dfrac{2b_M + b_{M+1}}{b_M - b_{M+1}}$ if $b_M \neq b_{M+1}$
F	rj	$1 \leq j \leq 13$
+	rj	$1 \leq j \leq 5$
	r11	$t_M \geq \max E_M(y - x)$
	r12	$t_M \leq \min E_M(y - x)$
	r13	
-	rj	$6 \leq j \leq 10$
	r11	$t_M \leq \min E_M(y - x)$
	r12	$t_M \geq \max E_M(y - x)$
	r13	
+ -	r11	$\max (E_M(y - x) \cap (z_M, t^*)) \leq t_M \leq \min (E_M(y - x) \cap (t^*, z_{M+1}))$
	r13	
- +	r12	$\max (E_M(y - x) \cap (z_M, t^*)) \leq t_M \leq \min (E_M(y - x) \cap (t^*, z_{M+1}))$
	r13	
T	r13	

of Chapter II and component zero-matching is that any component of SGN(y - x), even a component equal to T, can be zero-matched by $R_I(P)$ equal to r13. That the S vector that zero-matches SGN(y - x) must contain at least one component that is not r13 guarantees that the resulting P is not identically zero. It is to be noted that even if each component in one segment of SGN(y - x) must be zero-matched with components equal to r13, there may still exist a S vector that zero-matches SGN(y - x). However, if every segment must be zero-matched with a segment of a S vector consisting entirely of components equal to r13, then condition (1.2.3) must be satisfied.

In the following lemmas, the conditions necessary and sufficient for the existence of a S vector that zero-matches SGN(y - x) are determined.

Lemma 3.2.1.

Let $x \in X \setminus S$. If y is in J(x, S), P is in S, and $G(y - x, P, [\bar{a}, \bar{b}]) = 0$, then $P_I \equiv 0$ for every I such that (z_I, z_{I+1}) is a characteristic subinterval of y.

Proof: It is to be shown that every characteristic component $SGN_K(y - x)$ must be zero-matched with $R_K = r13$.

Step I: In the following, the effects of a zero-match using r13 on the zero-matching of surrounding components is determined.

Ia: Let $0 \leq J < K \leq N - 1$. Let $J + 2 \geq K$; or, let $SGN_I(y - x)$

for $J < I < K$ be a single and each pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J + 1 \leq I < K - 1$ be alternating. If R_J and R_K are r13, and R is to be a S vector that zero-matches SGN(y - x), then R_I must be r13 for $J \leq I \leq K$.

Proof of Ia: If $J + 1 = K$, the result is true by assumption.

If $J + 2 = K$, since $R_J = r13$, R_{J+1} can be rj only for j in {5, 10, 13}. Of these, only $R_{J+1} = r13$ can be followed by $R_{J+2} = r13$ in a segment of a S vector.

Assume the result has been shown for $J + U = K$. Let $J + U + 1 = K$. Assume that $SGN_{J+1}(y - x) = +$; the case where $SGN_{J+1}(y - x) = -$ can be shown in a similar manner. Since each pair is alternating, $SGN_{J+I}(y - x) = +$, if I is odd, and $SGN_{J+I}(y - x) = -$, if I is even.

Since $R_J = r13$, R_{J+1} can be rj only for j in {5, 13}, if R_{J+1} is to zero-match $SGN_{J+1}(y - x) = +$. If $R_{J+1} = r5$, for R to be a S vector that zero-matches SGN(y - x), R_{J+I} can be rj only for j in {3, 5, 12}, if I is odd, and only for j in {8, 10, 11}, if I is even. Thus, $R_{K-1} = rj$ for j in {3, 5, 8, 10, 11, 12}. None of these can be followed in a S vector by $R_K = r13$. Thus, $R_{J+1} = r13$. Since $(J + 1) + U = K$, by the induction hypothesis, $R_I = r13$ for $J + 1 \leq I \leq K$.

Ib: Let $0 \leq J < K \leq N - 1$. Let $SGN_J(y - x)$ be a double. Let $J + 1 = K$; or, let $SGN_I(y - x)$ for $J < I < K$ be a single, and each pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J \leq I < K - 1$ be alternating.

If R_K is r13, and R is to be a S vector that zero-matches SGN(y - x), then R_I must be r13 for $J \leq I \leq K$.

<u>Proof of Ib:</u> If J + 1 = K, since R_K = r13, R_{K-1} can be rj only for j in {4, 9, 13}. Of these, only R_{K-1} = r13 zero-matches a double.

Assume the result has been shown for J + U = K. Let J + U + 1 = K. Assume that $SGN_{K-1}(y - x) = +$; the case where $SGN_{K-1}(y - x) = -$ can be shown in a similar manner. Since each pair is alternating, $SGN_{K-I}(y - x) = +$, if I is odd, and $SGN_{K-I}(y - x) = -$, if I is even. If U is odd, $SGN_J(y - x) = + -$. If U is even, $SGN_J(y - x) = - +$.

Since R_K = r13, R_{K-1} can be rj only for j in {4, 13}, if R_{K-1} is to zero-match $SGN_{K-1}(y - x) = +$. If R_{K-1} = r4, for R to be a S vector that zero-matches SGN(y - x), R_{K-I} can be rj only for j in {2, 4, 11}, if I is odd, and only for j in {7, 9, 12}, if I is even. If U is odd, $SGN_{K-U-1}(y - x) = + -$ and R_{K-U} must be rj for j in {2, 4, 11}. None of these can be preceded by R_{K-U-1} = r13 or R_{K-U-1} = r11. If U is even, $SGN_{K-U-1}(y - x) = - +$ and R_{K-U} must be rj for j in {7, 9, 12}. None of these can be preceded by R_{K-U-1} = r13 or R_{K-U-1} = r12. Thus, R_{K-1} = r13. Since J + U = (K - 1), by the induction hypothesis, R_I = r13 for $J \leq I \leq K - 1$.

<u>Ic:</u> Let $0 \leq J < K \leq N - 1$. Let $SGN_K(y - x)$ be a double. Let J + 1 = K; or, let $SGN_I(y - x)$ for $J < I < K$ be a single, and each

pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J + 1 \leq I < K$ be alternating. If R_J is r13, and R is to be a S vector that zero-matches SGN(y - x), then R_I must be r13 for $J \leq I \leq K$.

Proof of Ic: If J + 1 = K, since R_J = r13, R_{J+1} can be rj only for j in {5, 10, 13}. Of these, only R_{J+1} = r13 zero-matches a double.

Assume the result has been shown for J + U = K. Let J + U + 1 = K. Assume that $SGN_{J+1}(y - x)$ = +; the case where $SGN_{J+1}(y - x)$ = - can be shown in a similar manner. Since each pair is alternating, $SGN_{J+I}(y - x)$ = +, if I is odd, and $SGN_{J+I}(y - x)$ = -, if I is even. If U is odd, $SGN_K(y - x)$ = - +. If U is even, $SGN_K(y - x)$ = + -.

Since R_J = r13, R_{J+1} can be rj only for j in {5, 13}, if R_{J+1} is to zero-match $SGN_{J+1}(y - x)$ = +. If R_{J+1} = r5, for R to be a S vector that zero-matches SGN(y - x), R_{J+I} can be rj only for j in {3, 5, 12}, if I is odd, and only for j in {8, 10, 11}, if I is even. If U is odd, $SGN_{J+U+1}(y - x)$ = - + and R_{J+U} must be rj for j in {3, 5, 12}. None of these can be followed by R_{J+U+1} = r13 or R_{J+U+1} = r12. If U is even, $SGN_{J+U+1}(y - x)$ = + - and R_{J+U} must be rj for j in {8, 10, 11}. None of these can be followed by R_{J+U+1} = r13 or R_{J+U+1} = r11. Thus, R_{J+1} = r13. Since (J + 1) + U = K, by the induction hypothesis, R_I = r13 for $J + 1 \leq I \leq K$.

Step II: If $SGN_J(y - x) = T$, the only zero-match is $R_J = r13$.

If $SGN(y - x)$ satisfies (2.2.1, 2) from $SGN_J(y - x)$ to $SGN_K(y - x)$, if $R_I \neq r13$ for $J \leq I \leq K$, by Lemma 2.2.1, R cannot zero-match $SGN(y - x)$. Thus, at least one R_M where $J \leq M \leq K$ must be r13 if R is to be a S vector that zero-matches $SGN(y - x)$. By Ib, $R_I = r13$ for $J \leq I \leq M$. By Ic, $R_I = r13$ for $M \leq I \leq K$.

If $SGN_K(y - x)$ is a characteristic component, then there must be a characteristic pattern from $E_J(y - x)$ to $E_{J+U}(y - x)$ where $J \leq K \leq J + U$. In the proof of Theorem 2.3.1, it was shown that there must exist a segment between $SGN_J(y - x)$ and $SGN_{J+U}(y - x)$ that satisfies (2.2.1, 1) or (2.2.1, 2). Let M(j) for $1 \leq j \leq a$ be such that $SGN_{M(j)}(y - x)$ is in a segment satisfying (2.2.1, 1) or (2.2.1, 2) and $J \leq M(j) < M(j + 1) \leq J + U$. By the work above, if R is to be a S vector that zero-matches $SGN(y - x)$, $R_{M(j)} = r13$ for $1 \leq j \leq a$.

Case I: If $J < M(1)$, let J(i) for $1 \leq i \leq b$ be such that $SGN_{J(i)}(y - x)$ is a double and $J = J(1) \leq J(i) < J(i + 1) < M(1)$. Since (2.3.1, 1) is satisfied, and (2.2.1, 1) and (2.2.1, 2) are not satisfied, for each i, $1 \leq i < b$, the pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J(i) \leq I < J(i + 1) - 1$ is alternating, and the pair $SGN_{J(i+1)-1}(y - x)$ and $SGN_{J(i+1)}(y - x)$ is nonalternating. Furthermore, the pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J(i) \leq I < M(1) - 1$ is alternating. Thus, by Ib, every $SGN_I(y - x)$ for $J \leq I \leq M(1)$ must be zero-matched with $R_I = r13$.

Case II: If $M(a) < J + U$, let $J(i)$ for $1 \leq i \leq b$ be such that $SGN_{J(i)}(y - x)$ is a double and $M(a) < J(i) < J(i + 1) \leq J(b) = J + U$. Since (2. 3. 1, 1) is satisfied, and (2. 2. 1, 1) and (2. 2. 1, 2) are not satisfied, for each i, $1 \leq i < b$, the pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J(i) < I < J(i + 1)$ is alternating and the pair $SGN_{J(i)}(y - x)$ and $SGN_{J(i)+1}(y - x)$ is nonalternating. Furthermore, the pair $SGN_I(y-x)$ and $SGN_{I+1}(y - x)$ for $M(a) < I < J(1)$ is alternating. Thus, by Ic, every $SGN_I(y - x)$ for $M(a) \leq I \leq J + U$ must be zero-matched with $R_I = r13$.

Case III: If $M(j) + 1 \neq M(j + 1)$ and $SGN_I(y - x)$ for $M(j) < I < M(j+1)$ is a single, since there is a characteristic pattern from $E_J(y - x)$ to $E_{J+U}(y - x)$, the pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $M(j) < I < M(j + 1) - 1$ is alternating. By Ia, every $SGN_I(y - x)$ for $M(j) \leq I \leq M(j + 1)$ must be zero-matched with $R_I = r13$.

If there are doubles, let $J(i)$ for $1 \leq i \leq b$, be such that $SGN_{J(i)}(y - x)$ is a double and $M(j) < J(i) < J(i + 1) < M(j + 1)$. In the proof of Theorem 2. 3. 1, it was shown that since there is a characteristic pattern from $E_J(y - x)$ to $E_{J+U}(y - x)$, and $J < J(i) < J + U$ for all i, for each i, there is a characteristic pattern either from $E_J(y - x)$ to $E_{J(i)}(y - x)$, or from $E_{J(i)}(y - x)$ to $E_{J+U}(y - x)$.

If there is a characteristic pattern from $E_J(y - x)$ to $E_{J(b)}(y - x)$, as in Case II, every $SGN_I(y - x)$ for $M(j) \leq I \leq J(b)$ must be zero-matched with R_I = r13. Since there is a characteristic pattern from $E_J(y - x)$ to $E_{J+U}(y - x)$, each pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J(b) < I < M(j + 1) - 1$ is alternating. By Ia, every $SGN_I(y - x)$ for $J(b) \leq I \leq M(j + 1)$ must be zero-matched with R_I = r13. Similarly, by Case I and Ia, if there is a characteristic pattern from $E_{J(1)}(y - x)$ to $E_{J+U}(y - x)$, every $SGN_I(y - x)$ for $M(j) \leq I \leq M(j + 1)$ must be zero-matched with R_I = r13.

If the above does not apply, let i_1 be the maximum of the integers such that there is a characteristic pattern from $E_J(y - x)$ to $E_{J(i)}(y - x)$. Let i_2 be the minimum of the integers such that there is a characteristic pattern from $E_{J(i)}(y - x)$ to $E_{J+U}(y - x)$. Notice that either $J(i_2) \leq J(i_1)$, or, $J(i_1 + 1) = J(i_2)$. By Case II, every $SGN_I(y - x)$ for $M(j) \leq I \leq J(i_1)$ must be zero-matched with R_I = r13. By Case I, every $SGN_I(y - x)$ for $J(i_2) \leq I \leq M(j + 1)$ must be zero-matched with R_I = r13. If $J(i_1) < J(i_2)$, since there is a characteristic pattern from $E_J(y - x)$ to $E_{J+U}(y - x)$, every pair $SGN_I(y - x)$ and $SGN_{I+1}(y - x)$ for $J(i_1) + 1 \leq I < J(i_2) - 1$ is alternating. By Ia, every $SGN_I(y - x)$ for $J(i_1) \leq I \leq J(i_2)$ must be zero-matched with R_I = r13.

<u>Step III:</u> Thus, if (z_K, z_{K+1}) is a characteristic subinterval, and $G(y - x, P, [\bar{a}, \bar{b}]) = 0$, (b_K, b_{K+1}) must be related by r13. Therefore, $P_K \equiv 0$.

Lemma 3.2.2.

Let $x \in X \backslash S$ and $y \in J(x, S)$. The following three statements are equivalent.

(3.2.2, 1) There exists a P in S, $P \not\equiv 0$, such that $G(y-x, P, [\bar{a}, \bar{b}])=0$.

(3.2.2, 2) (z_0, z_1) or (z_{N-1}, z_N) is not a characteristic subinterval of y; or, there exists a M where $0 \leq M < N - 1$ such that (z_M, z_{M+1}) and (z_{M+1}, z_{M+2}) are not characteristic subintervals of y.

(3.2.2, 3) There exists a P in S, $P \not\equiv 0$, such that $P_I \equiv 0$ if (z_I, z_{I+1}) is a characteristic subinterval of y; and, P_I and $y - x$ have opposite sign at each point in $E_I(y - x)$ if (z_I, z_{I+1}) is not a characteristic subinterval of y.

Proof: That (3.2.2, 3) implies (3.2.2, 1) is immediate. It is to be shown that (3.2.2, 1) implies (3.2.2, 2) which implies (3.2.2, 3).

Step I: It is to be shown that not (3.2.2, 2) implies not (3.2.2, 1). Let L(j) for $1 \leq j \leq a$ be such that $SGN_{L(j)}(y - x)$ is a characteristic component and $L(j) < L(j + 1)$. Let P be such that P is in S, $P \not\equiv 0$, and $G(y - x, P, [\bar{a}, \bar{b}]) = 0$. By Lemma 3.2.1, $R_{L(j)}(P) = r13$ for $1 \leq j \leq a$. By not (3.2.2, 2), $L(1) = 0$, $L(a) = N - 1$, and $L(j) + 2 \geq L(j + 1)$. Thus, by Lemma 3.2.1, Step I, Ia, $R_{L(j)+1}(P) = r13$ for $1 \leq j \leq a - 1$. Therefore, $R_I(P) = r13$ for $0 \leq I \leq N - 1$. This contradicts the assumption that $P \not\equiv 0$.

Step II: It is to be shown that (3.2.2, 2) implies (3.2.2, 3). Let L(j) for $1 \leq j \leq a$ be such that $SGN_{L(j)}(y - x)$ is a characteristic component of y and $L(j) < L(j + 1)$.

Case I: Assume that $0 < L(1)$.

Step I, I: Since the components before $SGN_{L(1)}(y - x)$ are not characteristic, there is not a segment satisfying (2.2.1, 1) or (2.2.1, 2). Furthermore, if there are doubles before $SGN_{L(1)}(y - x)$, and $SGN_{J(b)}(y - x)$ is the last double before $SGN_{L(1)}(y - x)$, there exists a pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ where $J(b) \leq M < L(1) - 1$ that is nonalternating.

If this is not true, then it is claimed that there is a characteristic pattern starting at $E_{J(b)}(y - x)$. This contradicts the definition of the L's.

Assume that each pair is alternating and that a characteristic pattern starting at $E_{J(b)}(y - x)$ demands that the first point referred to in $E_{L(1)}(y - x)$ has the sign of $y - x$ equal to +1. If there exists a $t_1 < t_2$, elements of $E_{L(1)}(y - x)$, such that $\text{sgn}(y - x)(t_1) = +1 = - \text{sgn}(y - x)(t_2)$, then there is a characteristic pattern from $E_{J(b)}(y - x)$ to $E_{L(1)}(y - x)$. If there does not exist such points, then for some $U \geq 1$, there is a characteristic pattern from $E_{L(1)}(y - x)$ to $E_{L(1)+U}(y - x)$; and, the pattern must demand that the last point referred to in $E_{L(1)}(y - x)$ has sign of $y - x$ equal to +1. Thus, there is a characteristic pattern from $E_{J(b)}(y - x)$ to $E_{L(1)+U}(y - x)$.

Step I, II: If there are not any doubles, for $0 \leq I < L(1)$, let $R_I = r4$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r9$, if $SGN_I(y - x) = -$. Let $L(1) = r13$. This forms a zero-matching segment of a S vector.

If there are doubles, the segment from $SGN_0(y - x)$ to $SGN_{J(b)}(y - x)$ can be matched as in Chapter II.

If $M = J(b)$, let $R_M = r11$ and $R_{M+1} = r9$, if $SGN_{J(b)}(y-x) = +-$; let $R_M = r12$ and $R_{M+1} = r4$, if $SGN_{J(b)}(y - x) = -+$.

If $M \neq J(b)$, let $R_{J(b)} = r11$, if $SGN_{J(b)}(y - x) = +-$; let $R_{J(b)} = r12$, if $SGN_{J(b)}(y - x) = -+$. For $J(b) < I < M$, let $R_I = r5$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r10$, if $SGN_I(y - x) = -$. If $SGN_M(y - x) = +$, or $SGN_{M+1}(y - x) = +$, then let $R_M = r5$ and $R_{M+1} = r4$; otherwise, let $R_M = r10$ and $R_{M+1} = r9$.

In either case, for $M + 2 \leq I < L(1)$, let $R_I = r4$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r9$, if $SGN_I(y - x) = -$. Let $L(1) = r13$. This forms a zero-matching segment of a S vector.

Case II: Assume that $L(a) < N - 1$.

Step II, I: Since the components after $SGN_{L(a)}(y - x)$ are not characteristic, there is not a segment satisfying (2. 2. 1, 1) or (2. 2. 1, 2). Furthermore, if there are doubles after $SGN_{L(a)}(y - x)$, and $SGN_{J(1)}(y - x)$ is the first double after $SGN_{L(a)}(y - x)$, there exists a pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ where $L(a) < M < J(1)$ that is nonalternating.

If this is not true, then it is claimed that there is a characteristic pattern ending at $E_{J(1)}(y - x)$. This contradicts the definition of the L's.

Assume that each pair is alternating and that a characteristic pattern ending at $E_{J(1)}(y - x)$ demands that the last point referred to in $E_{L(a)}(y - x)$ has the sign of $y - x$ equal to +1. If there exists $t_1 < t_2$, elements of $E_{L(a)}(y - x)$, such that $\text{sgn}(y - x)(t_1) = -1 = -\text{sgn}(y - x)(t_2)$, then there is a characteristic pattern from $E_{L(a)}(y-x)$ to $E_{J(1)}(y - x)$. If there does not exist such points, then for some $U \geq 1$, there is a characteristic pattern from $E_{L(a)-U}(y - x)$ to $E_{L(a)}(y - x)$; and, the pattern must demand that the first point referred to in $E_{L(a)}(y - x)$ has the sign of $y - x$ equal to +1. Thus, there is a characteristic pattern from $E_{L(a)-U}(y - x)$ to $E_{J(1)}(y - x)$.

Step II, II: If there are not any doubles, for $L(a) < I \leq N - 1$, let $R_I = r5$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r10$, if $SGN_I(y - x) = -$. Let $R_{L(a)} = r13$. This forms a zero-matching segment of a S vector.

If there are doubles, the segment from $SGN_{J(1)}(y - x)$ to $SGN_{N-1}(y - x)$ can be matched as in Chapter II.

If $M = J(1) - 1$, let $R_M = r5$ and $R_{M+1} = r11$, if $SGN_{J(1)}(y - x) = +-$; let $R_M = r10$ and $R_{M+1} = r12$, if $SGN_{J(1)}(y - x) = -+$.

If $M \neq J(1) - 1$, let $R_{J(1)} = r11$, if $SGN_{J(1)}(y - x) = +-$; let $R_{J(1)} = r12$, if $SGN_{J(1)}(y - x) = -+$. For $M + 2 \leq I < J(1)$, let

R_I = r4, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let R_I = r9, if $SGN_I(y - x) = -$. If $SGN_M(y - x) = +$, or $SGN_{M+1}(y - x) = +$, let R_M = r5 and R_{M+1} = r4; otherwise, let R_M = r10 and R_{M+1} = r9.

In either case, for $L(a) < I < M$, let R_I = r5, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let R_I = r10, if $SGN_I(y - x) = -$. Let $R_{L(a)}$ = r13. This forms a zero-matching segment of a S vector.

Case III: Assume that $L(j) + 1 \neq L(j + 1)$.

Step III, I: Since the components between $SGN_{L(j)}(y - x)$ and $SGN_{L(j+1)}(y - x)$ are not characteristic, there is not a segment satisfying (2. 2. 1, 1) or (2. 2. 1, 2). If there are doubles, let $SGN_{J(1)}(y-x)$ be the first double after $SGN_{L(j)}(y - x)$, and let $SGN_{J(b)}(y - x)$ be the last double before $SGN_{L(j+1)}(y - x)$. As in Step I, I, there must exist a pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ where $J(b) \leq M < L(j + 1) - 1$ that is nonalternating. As in Step II, I, there must exist a pair $SGN_{M_1}(y - x)$ and $SGN_{M_1+1}(y - x)$ where $L(j) < M_1 < J(1)$ that is nonalternating. If there are not any doubles, then either $L(j) + 2 = L(j + 1)$ and $SGN_{L(j)+1}(y - x) = F$; or, there exists a pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ where $L(j) < M < L(j + 1) - 1$ that is nonalternating.

If the latter is not true, then it is claimed that there is a characteristic pattern from $E_J(y - x)$ to $E_K(y - x)$ where $J \leq L(j) < L(j + 1) \leq K$. This contradicts the definition of the L's.

If $L(j) + 2 = L(j + 1)$, assume that $SGN_{L(j)+1}(y - x)$ is a single.

If $L(j) + 2 < L(j + 1)$, assume that each pair is alternating. Assume that a characteristic pattern involving those subintervals demands that the last point referred to in $E_{L(j)}(y - x)$ has sign of $y - x$ equal to +1; and, that the first point referred to in $E_{L(j+1)}(y - x)$ has sign of $y - x$ equal to +1.

If there exists a $t_1 < t_2$, elements of $E_{L(j)}(y - x)$, such that $\operatorname{sgn}(y - x)(t_1) = -1 = -\operatorname{sgn}(y - x)(t_2)$, then as in Step I, I, either there is a characteristic pattern from $E_{L(j)}(y - x)$ to $E_{L(j+1)}(y - x)$, or, there exists a $U \geq 1$, such that there is a characteristic pattern from $E_{L(j+1)}(y - x)$ to $E_{L(j+1)+U}(y - x)$ and from $E_{L(j)}(y - x)$ to $E_{L(j+1)+U}(y - x)$.

If such points do not exist, then there is a $U_1 \geq 1$ such that there is a characteristic pattern from $E_{L(j)-U_1}(y - x)$ to $E_{L(j)}(y - x)$; and, the pattern demands that the first point referred to in $E_{L(j)}(y - x)$ has the sign of $y - x$ equal to +1. If there exists points $t_1 < t_2$, elements of $E_{L(j+1)}(y - x)$, such that $\operatorname{sgn}(y - x)(t_1) = +1 = -\operatorname{sgn}(y - x)(t_2)$, then there is a characteristic pattern from $E_{L(j)-U_1}(y - x)$ to $E_{L(j+1)}(y - x)$. If such points do not exist, then there is a $U \geq 1$ such that there is a characteristic pattern from $E_{L(j+1)}(y - x)$ to $E_{L(j+1)+U}(y - x)$; and, the pattern demands that the last point referred to in $E_{L(j+1)}(y - x)$ has the sign of $y - x$ equal to +1. Thus, there is a characteristic pattern from $E_{L(j)-U_1}(y - x)$ to $E_{L(j+1)+U}(y - x)$.

Step III, II: If $L(j) + 2 = L(j + 1)$, then $SGN_{L(j)+1}(y - x) = F$; let $R_{L(j)} = R_{L(j)+1} = R_{L(j+1)} = r13$. This is a zero-matching segment of a S vector.

If $L(j) + 2 < L(j + 1)$, and there are not any doubles for $L(j) < I < M$, let $R_I = r5$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r10$, if $SGN_I(y - x) = -$. If $SGN_M(y - x) = +$ or $SGN_{M+1}(y-x) = +$, let $R_M = r5$ and $R_{M+1} = r4$; otherwise, let $R_M = r10$ and $R_{M+1} = r9$. For $M + 2 \leq I < L(j + 1)$, let $R_I = r4$, if $SGN_I(y - x) = +$, or $SGN_I(y - x) = F$; let $R_I = r9$, if $SGN_I(y - x) = -$. Let $R_{L(j)} = R_{L(j+1)} = r13$. This forms a zero-matching segment of a S vector.

If there are doubles, zero-match $SGN_{L(j)}(y - x)$ through $SGN_{J(1)}(y - x)$ as in Step II, II. Since none of the components between $SGN_{J(1)}(y - x)$ and $SGN_{J(b)}(y - x)$ are characteristic, the segment can be matched as in Chapter II. Zero-match $SGN_{J(b)}(y - x)$ through $SGN_{L(j+1)}(y - x)$ as in Step I, II. This forms a zero-matching segment of a S vector.

Case IV: Let $R_{L(j)} = r13$ for $1 \leq j \leq a$.

Let R be the vector formed by overlapping appropriate zero-matching segments as defined above; since $0 < L(1)$, or $L(a) < N - 1$, or $L(j) + 2 < L(j + 1)$ for some j, R does not consist entirely of components equal to r13; also, R is a S vector. Thus, R zero-matches $SGN(y - x)$. Since it was not necessary to specify the location at an extreme point of any interior roots, there exists a $\hat{P}$ in S such that

$R(\hat{P}) = R$, $\hat{P}_I \equiv 0$ if (z_I, z_{I+1}) is a characteristic subinterval of y, and if (z_I, z_{I+1}) is not a characteristic subinterval, $\hat{P}_I$ and y - x agree in sign at each point in $E_I(y - x)$. P equal to $-\hat{P}$ is the element of S that is sought.

3. 3. Proof of Theorem 3. 1. 1

The proof follows almost directly from the lemmas of Section 3. 2 and the properties of continuous functions on a compact set.

Proof of Theorem 3. 1. 1.

Step I: Assume that (z_0, z_1) and (z_{N-1}, z_N) are subsets of CS; assume that if (z_M, z_{M+1}) is not a subset of CS, then (z_{M+1}, z_{M+2}) is a subset of CS. This implies that $CS \neq \emptyset$; therefore, y is an element of J(x, s) and $G(y - x, P, [\bar{a}, \bar{b}]) \geq 0, \forall P \in S$. Since (3. 2. 2, 2) is not true, (3. 2. 2, 1) is not true. Therefore, if $P \not\equiv 0$, $G(y - x, P, [\bar{a}, \bar{b}]) > 0$. By the general theory, y is a unique best approximation to x from S.

Step II: Assume that (z_0, z_1) or (z_{N-1}, z_N) is not a subset of CS; or that (z_M, z_{M+1}) and (z_{M+1}, z_{M+2}) where $0 \leq M < N - 1$ are not subsets of CS. This implies that either $CS = \emptyset$, or y is in J(x, S) and (3. 2. 2, 2) is true. If CS is empty, $y \notin J(x, S)$ and so $\{y\} \neq J(x, S)$. If (3. 2. 2, 2) is true, it is to be shown that there exists a $\beta > 0$ such that $|(y - x + \beta P)(t)| \leq ||y - x||$ for all t in $[\bar{a}, \bar{b}]$. Thus,

$||y - x + \beta P|| \leq ||y - x||$. Since y is in J(x, S), $||y - x|| \leq ||y - x + \beta P||$. Therefore, $y - x + \beta P$ is in J(x, S) and $\{y\} \neq J(x, S)$.

Since $P \not\equiv 0$, without loss of generality, assume that $||P|| = 1$. If $P_I \equiv 0$, let $\beta_I = ||y - x||$. For all t in $[z_I, z_{I+1}]$ and all α where $0 \leq \alpha \leq \beta_I$, $|(y - x + \alpha P)(t)| = |(y - x)(t)| \leq ||y - x||$. If $P_I \not\equiv 0$, then let θ_1 be $\{t \in [z_I, z_{I+1}] \mid P(t) \cdot (y - x)(t) < 0\}$. Since P and y - x are continuous, θ_1 is an open set. Let θ_2 be $[z_I, z_{I+1}] \setminus \theta_1$; then, θ_2 is a compact set, and, since P_I and y - x have opposite sign at each point in $E_I(y - x)$, $\theta_2 \cap E_I(y - x) = \emptyset$. Thus, for some $\beta_I > 0$, $\max_{\theta_2} |(y - x)(t)| + \beta_I = ||y - x||$. For all α where $0 \leq \alpha \leq \beta_I$, and for all t in θ_2, $|(y - x + \alpha P)(t)| \leq |(y - x)(t)| + \alpha |P(t)| \leq \max_{\theta_2} |(y - x)(t)| + \beta_I = ||y - x||$. For t in θ_1, sgn P(t) = -sgn(y - x)(t); assume that $(y - x)(t) > 0$. For all α where $0 \leq \alpha \leq \beta_I$, if $|(y - x + \alpha P)(t)| = (y - x + \alpha P)(t)$, then $|(y - x + \alpha P)(t)| \leq |(y - x)(t)| \leq ||y - x||$; if $|(y - x + \alpha P)(t)| = -(y - x + \alpha P)(t)$, then $|(y - x + \alpha P)(t)| \leq \alpha |P(t)| \leq \beta_I \leq ||y - x||$.

Therefore, if β is the $\min_{0 \leq I \leq N-1} \{\beta_I\}$, for all t in $[\bar{a}, \bar{b}]$, $|(y - x + \beta P)(t)| \leq ||y - x||$.

<u>Step III:</u> Since J(x, S) is convex, if y and y_1 are elements of J(x, S), $y + \alpha(y_1 - y)$ is an element of J(x, S) for all α where $0 \leq \alpha \leq 1$; thus, $G(y - x, y_1 - y, [\bar{a}, \bar{b}]) = 0$. By Lemma 3.2.1, $(y_1 - y)|_{[z_I, z_{I+1}]} \equiv 0$ for all I such that (z_I, z_{I+1}) is a subset of CS. Therefore, $y_1|_{CS} \equiv y|_{CS}$.

There are several simple conditions which indicate that an element of $J(x, S)$ is not a unique best approximation.

Lemma 3.3.1.

Let $x \in X \setminus S$ and $y \in J(x, S)$. If $SGN(y - x)$ satisfies one of the following, then y is not a unique best approximation.

(3.3.1, 1) Let $SGN_0(y - x)$ or $SGN_{N-1}(y - x)$ be a single or free.

(3.3.1, 2) Let $0 \leq M < N - 1$. Let the pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ be nonalternating. Let $SGN_I(y - x)$ for $M \leq I \leq M + 1$ be a single or free.

Proof: Since (z_0, z_1) is a characteristic subinterval of y only if there is a characteristic pattern starting at $E_0(y - x)$, if $SGN_0(y - x)$ is a single or free, (z_0, z_1) is not a characteristic subinterval of y. Since (z_{N-1}, z_N) is a characteristic subinterval of y only if there is a characteristic pattern ending at $E_{N-1}(y - x)$, if $SGN_{N-1}(y - x)$ is a single or free, (z_{N-1}, z_N) is not a characteristic subinterval of y. If $SGN_I(y - x)$ for $M \leq I \leq M + 1$ is a single or free, (z_M, z_{M+1}) and (z_{M+1}, z_{M+2}) are characteristic subintervals of y only if there is a characteristic pattern from $E_J(y - x)$ to $E_K(y - x)$ where $J < M < M + 1 < K$. However, if the pair is also nonalternating, there cannot exist a t_1 in $E_M(y - x)$ and t_2 in $E_{M+1}(y - x)$ such that $sgn(y - x)(t_1) = - sgn(y - x)(t_2)$. Therefore, if the pair $SGN_M(y - x)$ and $SGN_{M+1}(y - x)$ is nonalternating, and $SGN_I(y - x)$

for $M \leq I \leq M + 1$ is a single or free, (z_M, z_{M+1}) and (z_{M+1}, z_{M+2}) are not characteristic subintervals of y.

IV. A QUASINORM ON X YIELDING UNIQUE "BEST" BEST APPROXIMATIONS

4.1. Definition of the Quasinorm and "Best" Best Approximations

It is unsatisfactory that a unique best approximation is rare; with a new norm, a unique solution is obtained. A quasinorm M on X, a map from X into R^N which, if R^N is ordered by the dictionary order, has the three main properties of a norm, is defined below. The "distance" of an element of S from a fixed element of X is then "measured" by the quasinorm.

Let the finite set of points $\{z_I\}_{I=0}^{N}$ be given as before. Let $||x||_\Omega$ be $\max_\Omega |x(t)|$ where Ω is a closed subset of $[\bar{a}, \bar{b}]$. Let the components $M_k(x)$ of the vector $M(x)$ be defined as follows:

Let $\Omega_1(x) = [\bar{a}, \bar{b}]$, and $j = 1$.

Step I:

Let $C_j(x) = \{(z_I, z_{I+1}) \mid \max_{[z_I, z_{I+1}]} |x(t)| = ||x||_{\Omega_j(x)}\}$.

Let $m_j(x)$ be the cardinality of $C_j(x)$.

Let $M_k(x) = ||x||_{\Omega_j(x)}$ for

$$\sum_{i=1}^{j-1} m_i(x) + 1 \leq k \leq \sum_{i=1}^{j} m_i(x).$$

Let $\Omega_{j+1}(x) = \Omega_j(x) \setminus \bigcup_{(z_I, z_{I+1}) \in C_j(x)} (z_I, z_{I+1})$

Step II:

If $\Omega_{j+1}(x) \neq \{z_I\}_{I=0}^{N}$, repeat from Step I with j equal to j + 1.

If $\Omega_{j+1}(x) = \{z_I\}_{I=0}^{N}$, then

$$\sum_{i=1}^{j} m_i(x) = N$$

and the process is completed.

The map $M: X \to R^N$ is well defined since every element of X is zero at every z_I. Let R^N be ordered by the dictionary order; then, the three basic properties of a norm are satisfied by M.

Lemma 4.1.1.

Let x and x_1 be in X. Let λ be in R. The map $M: X \to R^N$ has the following properties:

(1) $M(x) > \emptyset$ unless $x \equiv 0$

(2) $M(\lambda x) = |\lambda| M(x)$

(3) $M(x + x_1) \leq M(x) + M(x_1)$

Proof:

(1) Since $M_k(x) = ||x||_{\Omega_j(x)}$ for some j, $M_k \geq 0$; and so, $M(x) \geq \emptyset$. If $M(x) = \emptyset$, then $M_1(x) = ||x|| = 0$; thus, $x \equiv 0$ and $m_1(x) = N$; thus, $M(x) = \emptyset$.

(2) Since $||\lambda x||_\Omega = |\lambda| \; ||x||_\Omega$, for any closed set Ω in $[\bar{a}, \bar{b}]$, if $\lambda \neq 0$, $\Omega_j(x) = \Omega_j(\lambda x)$, $M_k(\lambda x) = |\lambda| M_k(x)$, and $M(\lambda x) = |\lambda| M(x)$. If $\lambda = 0$, then $\lambda x \equiv 0$, $M(\lambda x) = \emptyset$, and $|\lambda| M(x) = \emptyset$.

(3) For any closed set Ω in $[\bar{a},\bar{b}]$, either $\|x + x_1\|_\Omega < \|x\|_\Omega + \|x_1\|_\Omega$; or, there exists a t in Ω such that $|(x+x_1)(t)| = \|x + x_1\|_\Omega$, $|x(t)| = \|x\|_\Omega$, and $|x_1(t)| = \|x_1\|_\Omega$. Thus, either $M_1(x + x_1) < M_1(x) + M_1(x_1)$; or there exists a K such that

$$\max_{[z_K, z_{K+1}]} |(x + x_1)(t)| = M_1(x + x_1),$$

$$\max_{[z_K, z_{K+1}]} |x(t)| = M_1(x),$$

and

$$\max_{[z_K, z_{K+1}]} |x_1(t)| = M_1(x_1);$$

and, if $\Omega_2 = [\bar{a},\bar{b}] \setminus (z_K, z_{K+1})$, $\|x + x_1\|_{\Omega_2} = M_2(x + x_1)$, $\|x\|_{\Omega_2} = M_2(x)$, and $\|x_1\|_{\Omega_2} = M_2(x_1)$. The argument can be repeated and it is found that $M(x + x_1) \leq M(x) + M(x_1)$.

The set of "best" best approximations to x from S is defined to be the set of all elements of S at a minimum "distance" from x where "distance" is "measured" by the quasinorm M. Thus, the set of "best" best approximations $J^*(x, S)$ is $\{y \in S \mid M(y - x) \leq M(y_1 - x)\ \forall y_1 \in S\}$.

Theorem 4.1.1. For any x in X, there exists a unique "best" best approximation.

The proof of this theorem depends upon the properties of an approximation problem involving a closed subset of $[\bar{a},\bar{b}]$ and a constrained subset of S; it is delayed until the last section of this chapter. The assertions in the following section concerning the constrained approximation problem mentioned above will not be proven. The necessary basic facts were actually proven in the proofs of lemmas in Chapter II and Chapter III. The style of proof would parallel that of previous lemmas.

4.2. The Constrained Approximation Problem

A set of integers $\{L(j)\}_{j=1}^{a}$ where $0 \leq L(j) \leq N - 1$ and if $a > 1$, $L(j) < L(j + 1)$ is assumed given. Let Ω_1 be the union of $(z_{L(j)}, z_{L(j)+1})$ for $1 \leq j \leq a$. Let Ω_2 be $[\bar{a}, \bar{b}] \setminus \Omega_1$. Assume an element $\bar{y}$ of S is given such that $||\bar{y} - x||_{\Omega_2} \leq \delta$, where δ is

$$\min_{1 \leq j \leq a} \{ \max_{[z_{L(j)}, z_{L(j)+1}]} |(\bar{y} - x)(t)| \}.$$

Let $\bar{S}$ be the closed convex subset of S such that y is in $\bar{S}$ if and only if y is in S, $y|_{\Omega_1} \equiv \bar{y}|_{\Omega_1}$, and $||y - x||_{\Omega_2} \leq \delta$. Let $\rho(x, \bar{S})$ be $\inf \{ ||y - x||_{\Omega_2} | y \in \bar{S} \}$. Let $J(x, \bar{S})$ be $\{ y \in \bar{S} \mid ||y - x||_{\Omega_2} = \rho(x, \bar{S}) \}$. Let $E(y - x, \bar{S})$ be $\{ t \in \Omega_2 \mid |(y - x)(t)| = ||y - x||_{\Omega_2} \}$.

The general theory yields that $J(x, \bar{S})$ is a nonempty, closed convex subset of $\bar{S}$. If $x|_{\Omega_2}$ is not in $\bar{S}|_{\Omega_2}$ and y is in $\bar{S}$, then y is in $J(x, \bar{S})$ if and only if $G(y - x, P, \Omega_2) \geq 0$ for all P such that $\alpha P \in \bar{S} - y$ for some $\alpha > 0$. Furthermore, if $G(y - x, P, \Omega_2) > 0$

for all $P|_{\Omega_2} \not\equiv 0$ such that $\alpha P \in \bar{S} - y$ for some $\alpha > 0$, then $||y - x||_{\Omega_2} < ||y_1 - x||_{\Omega_2}$ for all y_1 in $\bar{S}$ such that $y_1|_{\Omega_2} \not\equiv y|_{\Omega_2}$.

Let $Z\bar{S}$ be $\{P \in S | P|_{\Omega_1} \equiv 0\}$. By definition, $\bar{S} - y$ for any y in $\bar{S}$ is a subset of $Z\bar{S}$. Therefore, if $G(y - x, P, \Omega_2) \geq 0$ for all P in $Z\bar{S}$, then $G(y - x, P, \Omega_2) \geq 0$ for all P such that $\alpha P \in \bar{S} - y$ for some $\alpha > 0$. If there exists a P in $Z\bar{S}$ such that $G(y - x, P, \Omega_2) < 0$, then it is claimed that if β is the minimum of the β_I defined below, then $\beta > 0$ and βP is in $\bar{S} - y$. If $P_I \equiv 0$, let β_I be $||y - x||_{\Omega_2}$. If $P_I \not\equiv 0$, let θ_1 be $\{t \in [z_I, z_{I+1}] | P(t) \cdot (y - x)(t) < 0\}$; and, let θ_2 be the compact set $[z_I, z_{I+1}] \setminus \theta_1$. Since $G(y - x, P, \Omega_2) < 0$ and $P \not\equiv 0$, let β_I be the positive number such that $\max_{\theta_2} |(y - x)(t)| + \beta_I ||P|| = ||y - x||_{\Omega_2}$. Thus, if $x|_{\Omega_2}$ is not in $\bar{S}|_{\Omega_2}$ and y is in $\bar{S}$,

$$G(y - x, P, \Omega_2) \geq 0 \ \forall P \text{ such that } \alpha P \in \bar{S} - y \text{ for some } \alpha > 0 \quad \text{iff} \quad G(y - x, P, \Omega_2) \geq 0 \ \forall P \in Z\bar{S} \quad \text{iff} \quad y \in J(x, \bar{S})$$

<u>Lemma 4.2.1.</u>

Let $x|_{\Omega_2} \notin \bar{S}|_{\Omega_2}$ and $y \in \bar{S}$. One of the following conditions is satisfied if and only if y is in $J(x, \bar{S})$.

(4.2.1, 1) Let $0 \leq J < J + U \leq N$. Let there exist points $\{t_I\}_{I=1}^{U+2}$ such that $t_1 \in E_J(y - x, \bar{S})$, $t_I \in E_{J+I-2}(y - x, \bar{S})$ for $2 \leq I \leq U + 1$, and $t_{U+2} \in E_{J+U-1}(y - x, \bar{S})$. Let $t_I < t_{I+1}$ and $\operatorname{sgn}(y - x)(t_I) = -\operatorname{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U + 1$.

(4.2.1, 2) Let $L(j) < L(j) + U = L(j+1) - 1$. Let there exist points $\{t_I\}_{I=1}^{U}$ such that $t_I \in E_{L(j)+I}(y - x, \bar{S})$ for $1 \leq I \leq U$. Let $\operatorname{sgn}(y - x)(t_I) = -\operatorname{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U - 1$.

(4.2.1, 3) Let $L(j) < L(j) + U$. Let there exist points $\{t_I\}_{I=1}^{U+1}$ such that $t_I \in E_{L(j)+I}(y - x, \bar{S})$ for $1 \leq I \leq U$ and $t_{U+1} \in E_{L(j)+U}(y - x, \bar{S})$. Let $t_I < t_{I+1}$ and $\operatorname{sgn}(y - x)(t_I) = -\operatorname{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U$.

(4.2.1, 4) Let $L(j) - U < L(j)$. Let there exist points $\{t_I\}_{I=1}^{U+1}$ such that $t_1 \in E_{L(j)-U}(y - x, \bar{S})$, $t_I \in E_{L(j)-U+I-2}(y - x, \bar{S})$ for $2 \leq I \leq U + 1$. Let $t_I < t_{I+1}$ and $\operatorname{sgn}(y - x)(t_I) = -\operatorname{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U$.

Remark: The subintervals (z_I, z_{I+1}) for $J \leq I \leq J + U - 1$, if (4.2.1, 1) is satisfied; for $L(j) + 1 \leq I \leq L(j) + U$, if (4.2.1, 2) is satisfied; for $L(j) + 1 \leq I \leq L(j) + U$, if (4,2,1, 3) is satisfied; and, for $L(j) - U \leq I \leq L(j) - 1$, if (4.2.1, 4) is satisfied, are referred to as $\bar{S}$ - characteristic subintervals of y.

Lemma 4.2.2.

Let $x|_{\Omega_2} \notin \bar{S}|_{\Omega_2}$. If y is in $J(x, \bar{S})$, P is in $Z\bar{S}$, and $G(y - x, P, \Omega_2) = 0$, then $P_I \equiv 0$ for every I such that (z_I, z_{I+1}) is a $\bar{S}$ - characteristic subinterval of y.

Lemma 4.2.3.

Let $x|_{\Omega_2} \notin \bar{S}|_{\Omega_2}$, and $y \in J(x, \bar{S})$. Let $C\bar{S}$ be the union of Ω_1 and all $\bar{S}$ - characteristic subintervals of y. The following three statements are equivalent.

(4.2.3, 1) There exists a P, $P \not\equiv 0$, $\alpha P \in \bar{S} - y$ for some $\alpha > 0$ such that $G(y - x, P, \Omega_2) = 0$.

(4.2.3, 2) (z_0, z_1) or (z_{N-1}, z_N) is not a subset of $C\bar{S}$; or, there exists a M where $0 \leq M < N - 1$ such that (z_M, z_{M+1}) and (z_{M+1}, z_{M+2}) are not subsets of $C\bar{S}$.

(4.2.3, 3) There exists a P in $Z\bar{S}$, $P \not\equiv 0$, such that $P_I \equiv 0$ if (z_I, z_{I+1}) is a subset of $C\bar{S}$; and, P_I and $y - x$ have opposite sign at each point in $E_I(y - x, \bar{S})$ if (z_I, z_{I+1}) is not a subset of $C\bar{S}$.

Lemma 4.2.4.

Let $x|_{\Omega_2} \notin \bar{S}|_{\Omega_2}$, and $y \in \bar{S}$. Let $C\bar{S}$ be as in Lemma 4.2.3.

$\{y\} = J(x, \bar{S})$ iff (z_0, z_1) and (z_{N-1}, z_N) are subsets of $C\bar{S}$; and, if (z_M, z_{M+1}) is not a subset of $C\bar{S}$, then (z_{M+1}, z_{M+2}) is a subset of $C\bar{S}$.

Furthermore, if y is in $J(x, \bar{S})$, then $y|_{C\bar{S}} \equiv y_1|_{C\bar{S}}$ for all y_1 in $J(x, \bar{S})$.

4.3. Proof of Theorem 4.1.1

A process which constructs a sequence of approximation problems of the type introduced in Section 4.2 and terminates with a problem whose set of best approximations J^D is a singleton, is considered. It is then shown that $J^*(x, S)$ is J^D.

Proof of Theorem 4.1.1.

Step I: Let the original problem be the first problem in the sequence. By the general theory, $J(x, S)$ is a nonempty, closed, convex set. Let J^1 be $J(x, S)$ and ρ^1 be $\rho(x, S)$.

If J^1 is a singleton, the process is finished. If J^1 is not a singleton, let $\overline{y}_1$ be an arbitrary element of J^1; let $L^1(j)$ for $1 \leq j \leq a_1$ be such that $(z_{L^1(j)}, z_{L^1(j)+1})$ is a characteristic subinterval of $\overline{y}_1$ and $L^1(j) < L^1(j+1)$; let CJ^1 be the union of all characteristic subintervals of $\overline{y}_1$; let θ_1 be $[\overline{a}, \overline{b}] \setminus CJ^1$; let ZJ^1 be $\{P \in S \mid P|_{CJ^1} \equiv 0\}$.

By Theorem 3.1.1, CJ^1 is independent of the choice of $\overline{y}_1$, since every characteristic subinterval of any element of J^1 is a characteristic subinterval of every element of J^1. In fact, for any y_1 in J^1, y is in J^1 if and only if y is in S, $y|_{CJ^1} \equiv y_1|_{CJ^1}$ and

$$||y-x||_{\theta_1} \leq \rho^1 = \min_{1 \leq j \leq a_1} \{ \max_{[z_{L^1(j)}, z_{L^1(j)+1}]} |(y_1 - x)(t)| \}.$$

By the results of Section 4.2, $J^2 = J(x, J^1)$ is a nonempty, closed, convex subset of J^1. If there exists a y in J^1 such that $||y - x||_{\theta_1} = \rho^1$, then $E(y - x) \cap \theta_1 = E(y - x, J^1)$. Since J^1 is not a singleton, by Lemma 3.2.2, there exists a P in ZJ^1 such that $G(y - x, P, \theta_1) < 0$. Thus, y could not be in J^2 and so $\rho^2 = \rho(x, J^1) < \rho^1$.

The remainder of the process for $i \geq 2$ is given iteratively:

If J^i is a singleton, as must be if $x|_{\theta_{i-1}}$ is in $J^{i-1}|_{\theta_{i-1}}$, the process is finished. If J^i is not a singleton, let $\bar{y}_i$ be an arbitrary element of J^i; let $L^i(j)$ for $1 \leq j \leq a_i$ be such that $(z_{L^i(j)}, z_{L^i(j)+1})$ is a subinterval which is in CJ^{i-1} or is a J^{i-1}-characteristic subinterval of $\bar{y}_i$, and $L^i(j) < L^i(j + 1)$; let CJ^i be the union of $(z_{L^i(j)}, z_{L^i(j)+1})$ for $1 \leq j \leq a_i$; let θ_i be $[\bar{a}, \bar{b}] \setminus CJ^i$; let ZJ^i be $\{P \in S | P|_{CJ^i} \equiv 0\}$.

By Lemma 4.2.4, CJ^i is independent of the choice of $\bar{y}_i$. In fact for any y_1 in J^i, y is in J^i if and only if y is in J^{i-1}, $y|_{CJ^i} \equiv y_1|_{CJ^i}$ and

$$||y - x||_{\theta_i} \leq \rho^i = \min_{1 \leq j \leq a_i} \{ \max_{[z_{L^i(j)}, z_{L^i(j)+1}]} |(y_1 - x)(t)| \}.$$

By the results of Section 4.2, $J^{i+1} = J(x, J^i)$ is a nonempty, closed, convex subset of J^i. If there exists a y in J^i such that $||y - x||_{\theta_i} = \rho^i$, then $E(y - x, J^{i-1}) \cap \theta_i = E(y - x, J^i)$. Since J^i is not a singleton, by Lemma 4.2.3, there exists a P in ZJ^i such that $G(y - x, P, \theta_i) < 0$. Thus, y is not in J^{i+1} and $\rho^{i+1} = \rho(x, J^i) < \rho^i$.

It is to be noted that there is at least one characteristic subinterval of $\bar{y}_1$ and at least one J^{i-1} - characteristic subinterval of $\bar{y}_i$ for any $i \geq 2$. Thus, for some i less than or equal to N, J^i is a singleton and the process would terminate.

Step II: Let D be the number of approximation problems considered in Step I; let $a_0 = 0$ and J^0 be S. By the definition of ρ^{i+1}, for $0 \leq i < D - 1$, $M_k(y - x) = \rho^{i+1}$ for all y in J^{i+1}, and for all k where $a_i + 1 \leq k \leq a_{i+1}$; also, for $0 \leq i < D$, $M_{a_i+1}(y_1 - x) > M_{a_i+1}(y - x)$ for all y_1 in $J^i \setminus J^{i+1}$ and all y in J^{i+1}. Thus, if $\bar{y}$ is the element of J^D, $M(y_1) > M(\bar{y})$ for all y_1 in $S \setminus \bar{y}$. Therefore, $\{\bar{y}\} = J^*(x, S)$.

V. EXTREME ELEMENTS OF THE SET J(x, S)

5.1. Sufficient Conditions for Extreme and Nonextreme Elements of J(x, S)

The set J(x, S) is known to be convex and a characterization of the extreme elements of the set might be useful. The general theory yields that an element y of J(x, S) is not an extreme element if and only if there exists a P in S, $P \not\equiv 0$, and a $\beta > 0$ such that $y \pm \alpha P$ are elements of J(x, S) for all α where $0 \leq \alpha \leq \beta$. If $y \pm \alpha P$ are elements of J(x, S), then $||y + \alpha P - x||$, $||y - \alpha P - x||$, and $||y - x||$ are all equal; therefore, $G(y - x, P, [\bar{a}, \bar{b}]) = 0$ and $G(y - x, -P, [\bar{a}, \bar{b}]) = 0$. Thus, P must be zero at each extreme point of y - x.

The existence of a P in S, $P \not\equiv 0$, that is zero at all extreme points of y - x is then necessary for y not to be an extreme element of J(x, S). The conditions under which such a P exists are to be determined. An element P of S is represented by the R(P) vector as in Chapter II. The number and placement of extreme points of y - x is represented by a vector NUM (y - x) of N components from the set {0, 1, 2}. If $E_I(y - x)$ is empty, $NUM_I(y - x)$ is 0. If $E_I(y - x)$ is a singleton, $NUM_I(y - x)$ is 1. If $E_I(y - x)$ contains two or more points, $NUM_I(y - x)$ is 2.

The existence of a P in S, $P \not\equiv 0$, that is zero at each point in

$E(y - x)$ can be thought of as the existence of a S vector $R(P)$, not consisting entirely of components equal to r13, that <u>zeroes</u> $NUM(y - x)$. $R_I(P)$ "zeroes" $NUM_I(y - x)$ if the relationship $R_I(P)$ is such that P_I has at least $NUM_I(y - x)$ zeroes in (z_I, z_{I+1}). For example, if $NUM_I(y - x)$ is 1, $R_I(P)$ can be rj only for j in $\{11, 12, 13\}$, if $R_I(P)$ is to "zero" $NUM_I(y - x)$. The interior root of P_I can be located at the element of $E_I(y - x)$ if $R_I(P)$ is r11 or r12. Since P_I is a cubic polynomial with roots at z_I and z_{I+1}, if $NUM_I(y - x)$ is 2, $R_I(P)$ must be r13 to "zero" $NUM_I(y - x)$. However, if $NUM_I(y - x)$ is 0, $R_I(P)$ can be rj for any j where $1 \leq j \leq 13$ and still "zero" $NUM_I(y - x)$.

Lemma 5.1.1.

Let $y \in J(x, S)$. There exists a P in S, $P \not\equiv 0$, such that $P(t) = 0$ for all t in $E(y - x)$ if and only if one of the following conditions is satisfied.

(5.1.1, 1) Let $0 \leq J \leq N - 1$. Let $NUM_J(y - x) = 0$ and $NUM_I(y - x) \leq 1$ for $0 \leq I < J$.

(5.1.1, 2) Let $0 \leq J \leq N - 1$. Let $NUM_J(y - x) = 0$ and $NUM_I(y - x) \leq 1$ for $J < I \leq N - 1$.

(5.1.3, 3) Let $0 \leq J < J + U \leq N - 1$. Let $NUM_J(y - x) = NUM_{J+U}(y - x) = 0$ and $NUM_I(y - x) \leq 1$ for $J < I < J + U$.

Proof:

Case I: Assume that (5.1.1, 1) is satisfied. Since $NUM_I(y - x) \leq 1$ for $0 \leq I < J$, $R_I = rj$ for j in $\{11, 12\}$ zeroes $NUM_I(y - x)$. For $0 \leq J - I < J$, let $R_{J-I} = r12$, if I is odd; let $R_{J-I} = r11$, if I is even. Let $R_J = r4$. Let $R_I = r13$ for $J < I \leq N - 1$. R is a S vector; and there exists a P in S such that $R(P) = R$ and $P(t) = 0$ for all t in $E(y - x)$.

Case II: Assume that (5.1.1, 2) is satisfied. Since $NUM_I(y - x) \leq 1$ for $J < I \leq N - 1$, $R_I = rj$ for j in $\{11, 12\}$ zeroes $NUM_I(y - x)$. For $0 \leq I < J$, let $R_I = r13$. Let $R_J = r5$. For $J < J + I \leq N - 1$, let $R_{J+I} = r11$, if I is odd; let $R_{J+I} = r12$, if I is even. R is a S vector; and, there exists a P in S such that $R(P) = R$ and $P(t) = 0$ for all t in $E(y - x)$.

Case III: Assume that (5.1.1, 3) is satisfied. Since $NUM_I(y - x) \leq 1$ for $J < I < J + U$, $R_I = rj$ for j in $\{11, 12\}$ zeroes $NUM_I(y - x)$. For $0 \leq I < J$, let $R_I = r13$. Let $R_J = r5$. For $1 \leq I < U$, let $R_{J+I} = r11$, if I is odd; let $R_{J+I} = r12$, if I is even. Let $R_{J+U} = r4$, if U is odd; let $R_{J+U} = r9$, if U is even. Let $R_I = r13$ for $J + U < I \leq N - 1$. R is a S vector; and, there exists a P in S such that $R(P) = R$ and $P(t) = 0$ for all t in $E(y - x)$.

Case IV: Assume that none of the conditions are satisfied. It is to be shown that $R_I = r13$ for $0 \leq I \leq N - 1$, if R_I is to zero $NUM_I(y - x)$.

Since $y \in J(x, S)$, there is at least one component of $NUM(y - x)$ equal to 2. Let $L(j)$ for $1 \leq j \leq a$ be such that $NUM_{L(j)}(y - x) = 2$ and $L(j) < L(j + 1)$. If $R_{L(j)}$ is to zero $NUM_{L(j)}(y - x)$, then $R_{L(j)} =$ r13.

<u>Step I</u>: If $0 < L(1)$, since (5. 1. 1, 1) is not satisfied, $NUM_I(y - x) = 1$ for $0 \leq I < L(1)$. In a segment of a S vector, if $R_M =$ r13 and R_{M-1} is to zero $NUM_{M-1}(y - x) = 1$, then $R_{M-1} =$ r13. Thus, if R is to be a S vector and R_I is to zero $NUM_I(y - x)$, for $0 \leq I \leq L(1)$, $R_I =$ r13.

<u>Step II</u>: If $L(a) < N - 1$, since (5. 1. 1, 2) is not satisfied, $NUM_I(y - x) = 1$ for $L(a) < I \leq N - 1$. In a segment of a S vector, if $R_M =$ r13 and R_{M+1} is to zero $NUM_{M+1}(y - x) = 1$, then $R_{M+1} =$ r13. Thus, if R is to be a S vector and R_I is to zero $NUM_I(y - x)$, for $L(a) \leq I \leq N - 1$, $R_I =$ r13.

<u>Step III</u>: If $L(j) + 1 < L(j + 1)$, since (5. 1. 1, 3) is not satisfied, $NUM_I(y - x) = 1$ for $L(j) < I < L(j + 1)$, or there exists at most one M such that $L(j) < M < L(j + 1)$ and $NUM_M(y - x) = 0$. In the first case, as in Step I, $R_I =$ r13 for $L(j) \leq I \leq L(j + 1)$. In the second case, $R_I =$ r13 for $L(j) \leq I < M$ and $M < I \leq L(j + 1)$ as in Step I and Step II. If R is to be a S vector, since $R_{M-1} =$ r13 $= R_{M+1}$, $R_M =$ r13. Thus, $R_I =$ r13 for $L(j) \leq I \leq L(j + 1)$, if R is to be a S vector and R_I is to zero $NUM_I(y - x)$.

Therefore, if $P(t) = 0$ for all t in $E(y - x)$, $P \equiv 0$. Thus, there does not exist a P in S, $P \not\equiv 0$, such that $P(t) = 0$ for all t in $E(y - x)$.

Since the existence of a P in S, $P \not\equiv 0$, that is zero at every extreme point in $E(y - x)$ is a necessary condition for y not to be an extreme element, a sufficient condition for y to be an extreme element of $J(x, S)$ has been found.

Lemma 5.1.2.

Let $y \in J(x, S)$. If both (5.1.2, 1) and (5.1.2, 2) are satisfied, then y is an extreme element of $J(x, S)$.

(5.1.2, 1) If $NUM_J(y - x) = 0$, then there exists a M_1 and M_2 where $0 \leq M_1 < J < M_2 \leq N - 1$ such that $NUM_{M_1}(y - x) = NUM_{M_2}(y - x) = 2$.

(5.1.2, 2) If $NUM_J(y - x) = NUM_K(y - x) = 0$ where $J < K$, then there exists a M where $0 \leq J < M < K \leq N - 1$ such that $NUM_M(y - x) = 2$.

For y in $J(x, S)$ not to be an extreme element of $J(x, S)$, besides the existence of a P in S, $P \not\equiv 0$, that is zero at each point in $E(y - x)$, there must exist a $\beta > 0$ such that $y \pm \beta P$ are elements of $J(x, S)$. Three subconditions of Lemma 5.1.1 are sufficient to show that such a P and such a β exist.

Lemma 5.1.3.

Let $y \in J(x, S)$. If one of the following conditions is satisfied, then y is a nonextreme element of J(x, S).

(5.1.3, 1) Let $NUM_0(y - x) = 0$.

(5.1.3, 2) Let $NUM_{N-1}(y - x) = 0$.

(5.1.3, 3) Let $0 \leq J < N - 1$. Let $NUM_J(y - x) = NUM_{J+1}(y - x) = 0$.

Proof: It is to be noted that if $NUM_I(y - x) = 0$, then

$$\max_{[z_I, z_{I+1}]} < ||y - x||.$$

Case I: Assume that (5.1.3, 1) is satisfied. Let $R_0 = r4$ and $R_I = r13$ for $0 < I \leq N - 1$. Let P be such that R(P) = R and $||P|| = 1$. Let β be such that $\max_{[z_0, z_1]} |(y - x)(t)| + \beta = ||y - x||$. $y \pm \beta P$ are elements of J(x, S).

Case II: Assume that (5.1.3, 2) is satisfied. Let $R_I = r13$ for $0 \leq I < N - 1$ and $R_{N-1} = r5$. Let P be such that R(P) = R and $||P|| = 1$. Let β be such that $\max_{[z_{N-1}, z_N]} |(y-x)(t)| + \beta = ||y-x||$. $y \pm \beta P$ are elements of J(x, S).

Case III: Assume that (5.1.3, 3) is satisfied. Let $R_I = r13$ for $0 \leq I < J$. Let $R_J = r5$ and $R_{J+1} = r4$. Let $R_I = r13$ for $J + 1 < I \leq N - 1$. Let P be such that R(P) = P and $||P|| = 1$. Let β be such that $\max_{[z_J, z_{J+2}]} |(y - x)(t)| + \beta = ||y - x||$. $y \pm \beta P$ are elements of J(x, S).

When J(x, S) is not a singleton, Lemma 5.1.3 is sufficient to prove that the "best" best approximation to x is not an extreme element of J(x, S).

Lemma 5.1.4.

If J(x, S) is not a singleton, the "best" best approximation to x from S is a nonextreme element of J(x, S).

Proof: Let $\overline{y}$ be the "best" best approximation to x. Let CS be the union of characteristic subintervals of $\overline{y}$. By the characterization of $\overline{y}$ given in the proof of Theorem 4.1.1, $E(\overline{y} - x)$ is a subset of CS. By Theorem 3.1.1, since J(x, S) is not a singleton, (z_0, z_1), (z_{N-1}, z_N), or for some M where $0 \leq M < N - 1$, $(z_M, z_{M+1}) \cup (z_{M+1}, z_{M+2})$ is not a subset of CS. Thus, (5.1.3, 1), (5.1.3, 2), or (5.1.3, 3) is satisfied.

5.2. Necessary and Sufficient Conditions When x is Continuously Differentiable

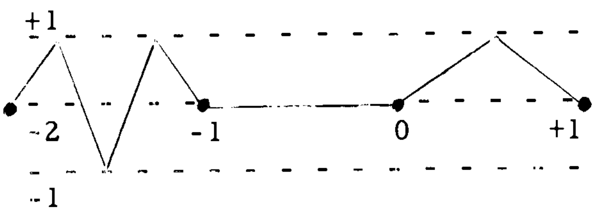
With the added restriction that x be continuously differentiable, satisfaction of one of the conditions of Lemma 5.1.3 is found to be both necessary and sufficient for y to be a nonextreme element of J(x, S). That the restriction on x is necessary for this result is shown in the following example.

Example 5.2.1. Let $N = 3$, $z_0 = -2$, $z_1 = -1$, $z_2 = 0$ and $z_3 = 1$.

$$\text{Let } x(t) = \begin{cases} -4t - 8 & t \in [-2, -7/4] \\ 8t + 13 & t \in [-7/4, -6/4] \\ -8t - 11 & t \in [-6/4, -5/4] \\ 4t + 4 & t \in [-5/4, -1] \\ 0 & t \in [-1, 0] \\ -2t & t \in [0, 1/2] \\ 2t - 2 & t \in [1/2, 1] \end{cases}$$

x is an element of $X \setminus S$, $\rho(x, S)$ is 1, and 0 is an element of $J(x, S)$.

Graph of $0 - x$:



$E(0 - x) = \{-7/4, -6/4, -5/4, 1/2\}$

$\rho(x, S) = ||0 - x|| = 1$

$SGN(0 - x) = (T, F, +)$

$NUM(0 - x) = (2, 0, 1)$

$NUM(0 - x)$ satisfies none of the conditions of Lemma 5.1.3; however, if $b_0 = 0$, $b_1 = 0$, $b_2 = 1$, $b_3 = -1$, and

$$P = \sum_{I=0}^{3} b_I B_I,$$

then P is in S and $||\pm P - x|| \leq 1$. Thus, $0 \pm P$ are elements of $J(x, S)$ and $0 = 1/2(P) + 1/2(-P)$ is not an extreme element of $J(x, S)$.

The characterization of an extreme element of $J(x, S)$ is straightforward when x is continuously differentiable.

Theorem 5.2.1.

Let $x \in C^1([\bar{a},\bar{b}]) \cap X$, and $y \in J(x, S)$.

y is an extreme element of J(x, S)	iff	$E_0(y - x) \neq \emptyset$, $E_{N-1}(y - x) \neq \emptyset$, and if $E_M(y - x) = \emptyset$, then $E_{M+1}(y - x) \neq \emptyset$.

Proof:

Step I: If $E_0(y - x) = \emptyset$, $E_{N-1}(y - x) = \emptyset$, or $E_M(y-x) = E_{M+1}(y-x) = \emptyset$, then $NUM_0(y - x) = 0$, $NUM_{N-1}(y - x) = 0$, or $NUM_M(y - x) = NUM_{M+1}(y - x) = 0$. Thus, by Lemma 5.1.3, y is not an extreme element of J(x, S).

Step II: Assume that $E_0(y - x) \neq \emptyset$, $E_{N-1}(y - x) \neq \emptyset$, and if $E_M(y - x) = \emptyset$, then $E_{M+1}(y - x) \neq \emptyset$.

It is claimed that if $y \pm \beta P \in J(x, S)$ for some $\beta > 0$, then $P_I \equiv 0$ for all I such that $NUM_I(y - x) \geq 1$. Since $y \pm \beta P \in J(x, S)$, $P(t) = 0$ for all t in $E(y - x)$ and $|(y - x)(t) \pm \beta P(t)| \leq ||y - x||$. If $NUM_I(y - x) = 2$, then $P_I \equiv 0$. If $NUM_I(y - x) = 1$, then $P_I \equiv 0$, or $P_I \not\equiv 0$, $P_I(t_I) = 0$, and $E_I(y - x) = \{t_I\}$ for some t_I in (z_I, z_{I+1}). Assume that $P_I \not\equiv 0$ and that $(y - x)(t_I) = ||y - x||$; then,

$$(y-x)(t) + \beta |P_I(t)| \leq \max \{|(y-x)(t) + \beta P_I(t)|, |(y-x)(t) - \beta P_I(t)|\} \leq ||y - x||.$$

Since y - x is continuously differentiable on (z_I, z_{I+1}), for some ε sufficiently small, for all t in $(t_I, t_I + \varepsilon)$, and for some d_t in

$(t_I, t_I + \varepsilon)$, $0 < (y - x)(t) = (y - x)(t_I) + \frac{d}{dt}(y - x)(d_t)(t - t_I)$

$$= ||y - x|| + \frac{d}{dt}(y - x)(d_t)(t - t_I).$$

Thus, $||y - x|| + \frac{d}{dt}(y - x)(d_t)(t - t_I) + \beta\,|P_I(t)| \leq ||y - x||$ and,

$$\frac{d}{dt}(y - x)(d_t) + \beta\,(\hat{P}_I(t)) \leq 0, \text{ where } \hat{P}_I(t) = \frac{|P_I(t)|}{(t - t_I)}.$$

Since t_I is a simple root of P_I, $\hat{P}_I(t) \to d > 0$ as $t \to t_I^+$. However, since t_I is a relative maximum of $y - x$, $\frac{d}{dt}(y - x)(d_t) \to 0$ as $t \to t_I^+$. This is a contradiction. Similar results would occur if it is assumed that $P_I \not\equiv 0$ and $(y - x)(t_I) = -||y - x||$. Thus, if $NUM_I(y - x) \geq 1$, $P_I \equiv 0$.

It is claimed that if $y \pm \beta P \in J(x, S)$ for some $\beta > 0$, then $P \equiv 0$. Let $R(P)$ be the vector of P. By the previous claim, $R_I(P) = r13$ if $NUM_I(y - x) \geq 1$. Since $y \in J(x, S)$, there is at least one component of $NUM(y - x)$ that is 2. Let $L(j)$ for $1 \leq j \leq a$ be such that $NUM_{L(j)}(y - x) \geq 1$ and $L(j) < L(j + 1)$. Since $E_0(y - x) \neq \emptyset$, $NUM_0(y - x) \geq 1$ and $L(1) = 0$. Since $E_{N-1}(y - x) \neq \emptyset$, $NUM_{N-1}(y - x) \geq 1$ and $L(a) = N - 1$. Since if $E_M(y - x) = \emptyset$, then $E_{M+1}(y - x) \neq \emptyset$, $L(j) + 2 \geq L(j + 1)$ for $1 \leq j < a$. Thus, since $R(P)$ is a S vector, $R_I(P) = r13$ for $0 \leq I \leq N - 1$. This implies that $P \equiv 0$.

Therefore, there does not exist a P in S, $P \not\equiv 0$, such that $y \pm \beta P \in J(x, S)$ for some $\beta > 0$; thus, y is an extreme element of $J(x, S)$.

VI. THE "BEST" BEST APPROXIMATION TO EXPONENITAL AND LOGARITHM ON [1,2]

6.1. Comments on the Method Used

The approximation to x from S has many similarities to the classical Chebycheff approximation to a continuous function by a finite dimensional Haar space; thus, the Remes ascent method, applicable in the classical case, was considered. Four main conditions would make the Remes ascent method feasible: (1) easily recognizable lower bounds on $\rho(x, S)$; (2) easily computable lower bounds on $\rho(x, S)$; (3) an easily obtainable montone increasing sequence of lower bounds on $\rho(x, S)$; (4) the ability to show that the limit of such sequences is $\rho(x, S)$. The first two conditions are satisfied in this problem. The remaining two are very complicated claims in this situation; the method actually used was a cross between a modified Remes method and guess work.

A result similar to the de La Vallée Poussin Theorem yields lower bounds on $\rho(x, S)$. Let FP(J, K) be the set $\{\{t_I\}_{I=1}^{U+2} \mid K - J = U,$ $t_1 \in (z_J, z_{J+1})$, $t_I \in (z_{J+I-2}, z_{J+I-1})$ for $2 \leq I \leq U+1$, and $t_{U+2} \in (z_{J+U-1}, z_{J+U})$ where $t_I < t_{I+1}$ for $1 \leq I \leq U+1\}$.

<u>Theorem 5.1.1.</u> Let $x \in X$ and $y \in S$. Let $C \in FP(J, K)$ where $0 \leq J < K \leq N$; let $C = \{t_I\}_{I=1}^{U+2}$ where $K - J = U$. Let $(y-x)(t_I) \neq 0$ for $1 \leq I \leq U+2$. Let $\text{sgn}(y-x)(t_I) = -\text{sgn}(y-x)(t_{I+1})$ for

$1 \leq I \leq U + 1$. Let

$$\text{Min} = \min_{1 \leq I \leq U+2} |(y - x)(t_I)|.$$

Let

$$\text{Max} = \max_{1 \leq I \leq U+2} |(y - x)(t_I)|.$$

If the above conditions are satisfied, then Min $\leq \rho(x, S)$; furthermore, if Max > Min, then Min $< \rho(x, S)$.

Proof: Assume that $\rho(x, S) \leq$ Min, and without loss of generality, that $(y - x)(t_1) > 0$. Since $J(x, S)$ is nonempty, there exists an element y_1 of S such that $\rho(x, S) = ||y_1 - x|| \leq$ Min. Thus, $\pm(y_1 - x)(t_I) \leq ||y_1 - x|| \leq \text{Min} \leq |(y - x)(t_I)|$ for $1 \leq I \leq U + 2$. Therefore, if P is $y_1 - y$, P is in S,

$$\begin{aligned} &P(t_I) \geq 0 \quad \text{if } I \text{ is even} \\ (5.1.1) \qquad &\text{and} \\ &P(t_I) \leq 0 \quad \text{if } I \text{ is odd.} \end{aligned}$$

Since P_I is a cubic polynomial with roots at z_I and z_{I+1}, $P_I(\bar{t}) \geq 0$ for $\bar{t}$ in (z_I, z_{I+1}), only if $P_I \equiv 0$, or $P_I(\bar{t}) \geq 0$ and there exists a t^* in (z_I, z_{I+1}) such that $P_I(t^*) > 0$. If $P_I(\bar{t}_1) \geq 0$ and $P_I(\bar{t}_2) \leq 0$ where $z_I < \bar{t}_1 < \bar{t}_2 < z_{I+1}$, then either $P_I \equiv 0$, or $P_I(t) > 0$ for t in $(z_I, \bar{t}_1)$, $P_I(t) < 0$ for t in $(\bar{t}_2, z_{I+1})$, and the interior root of P_I is greater than or equal to $\bar{t}_1$ and less than or equal to $\bar{t}_2$. The inequalities in statements involving P_I can be reversed and similar results would be obtained.

Thus, condition (5. 1. 1) can be represented as a segment of a SGN vector expressing the sign demands on P just as the sign demands of y - x were expressed by SGN(y - x). If U = 1, $SGN_J = T$. If U > 1, $SGN_J = -+$, and if U is even, $SGN_{J+U-1} = -+$; if U is odd, $SGN_{J+U-1} = +-$. For $1 \leq I < U - 1$, $SGN_{J+I} = -$, if I is odd, and $SGN_{J+I} = +$, if I is even. The vector R(P) must be a S vector and $R_I(P)$ must zero-match SGN_I for $J \leq I \leq J + U - 1$. It is shown in Lemma 3. 2. 1, that $R_I(P)$ must be r13 for $J \leq I \leq J + U - 1$. Thus, $P_I \equiv 0$ for $J \leq I \leq J + U - 1$, and $y|_{[z_J, z_{J+U}]} \equiv y_1|_{[z_J, z_{J+U}]}$.

This implies that

$$\rho(x, S) \leq \text{Min} \leq \text{Max} \leq ||y - x||_{[z_J, z_{J+U}]} = ||y_1 - x||_{[z_J, z_{J+U}]} \leq ||y_1 - x||$$

$$= \rho(x, S).$$

Therefore, Min $\leq \rho(x, S)$; and if Max > Min, Min $< \rho(x, S)$.

It can be shown that for any C in FP(J, K) where $0 \leq J < K \leq N$, there exists a unique λ^C and set of coefficients $\{b_I^C\}_{I=J}^K$ such that if

$$y^C = \sum_{I=J}^{K} b_I^C B_I,$$

then $(-1)^{I+1} \lambda^C + y^C(t) = x(t)$ for all t in C. By definition, $|\lambda^C|$ is $||y^C - x||_C$. By Theorem 5. 1. 1, $||y^C - x||_C$ is a lower bound on $\rho(x, S)$. Thus, lower bounds on $\rho(x, S)$ are easily computable.

If $||y^C - x||_C$ is strictly less than $||y^C - x||_{[z_J, z_K]}$, it can be shown that by inspection of the curve $y^C - x$ on $[z_J, z_K]$, a J^1, K^1, and a C^1 in $FP(J^1, K^1)$, where $J \le J^1 < K^1 \le K$ and C^1 is $\{t_I^1\}_{I=1}^{U^1}$ can be chosen so that for some L, $|(y^C - x)(t_L^1)|$ is $||y^C - x||_{[z_J, z_K]}$, and for I not equal to L, $|(y^C - x)(t_I^1)|$ is $||y^C - x||_C$. It can be shown that $||y^{C_1} - x||_{C_1}$ is strictly greater than $||y^C - x||_C$.

Thus, an initial guess for J^0, K^0 and C^0 in $FP(J^0, K^0)$ is made where $0 \le J^0 < K^0 \le N$, and, in the manner above, a monotone increasing sequence of lower bounds on $\rho(x, S)$ can be obtained. For some k, for all i greater than k, J^i is J^k and K^i is K^k where $J_0 \le J^k < K^k \le K_0$. Let $\underline{J}$ be J^k and $\underline{K}$ be K^k. It can be shown that the sequence $\{y^{C_i}\}$ of elements of S converges on $[z_{\underline{J}}, z_{\underline{K}}]$ to some

$$\bar{y} = \sum_{I=\underline{J}}^{\underline{K}} \bar{b}_I B_I.$$

For some $\underline{C}$ in $FP(z_{\underline{J}}, z_{\underline{K}})$, $\bar{y}$ is $y^{\underline{C}}$, and $||\bar{y} - x||_{[z_{\underline{J}}, z_{\underline{K}}]}$ is $||\bar{y} - x||_{\underline{C}}$. In fact, $||\bar{y} - x||_{\underline{C}}$ is the inf $\{||y - x||_{[z_{\underline{J}}, z_{\underline{K}}]} | y \in S\}$.

It is here that the difficulty arises. By Lemma 5.1.1, it is known that $||\bar{y} - x||_C$ is a lower bound on $\rho(x, S)$. By the characterization theorem, it is also known that for some C^* in $FP(J^*, K^*)$ where $0 \le J^* < K^* \le N$, $||y^{C^*} - x||_{C^*}$ is equal to $\rho(x, S)$. However,

$\underline{J}$ and $\underline{K}$ need not be J^* and K^*. Thus, $||\bar{y} - x||_{\underline{C}}$ may be strictly less than $\rho(x, S)$.

The next step in the process is to assume that $||\bar{y} - x||_{\underline{C}}$ is equal to $\rho(x, S)$. If this assumption is correct, then by Lemma 3.2.1, $y_1|_{[z_{\underline{J}}, z_{\underline{K}}]} \equiv \bar{y}|_{[z_{\underline{J}}, z_{\underline{K}}]}$, for all y_1 in $J(x, S)$. For some choice of

$$\{b_I\}_{I=0}^{\underline{J}-1} \bigcup \{b_I\}_{I=\underline{K}+1}^{N},$$

if

$$y = \sum_{I=0}^{\underline{J}-1} b_I B_I + \sum_{I=\underline{J}}^{\underline{K}} \bar{b}_I B_I + \sum_{I=\underline{K}+1}^{N} b_I B_I,$$

$||y - x||$ will equal $\rho(x, S)$.

If Ω_1 is $(z_{\underline{J}}, z_{\underline{K}})$, Ω_2 is $[\bar{a}, \bar{b}] \setminus \Omega_1$, and $\bar{S}$ is $\{P \in \bar{S} \,|\, P|_{\Omega_1} \equiv \bar{y}|_{\Omega_1}\}$, this problem corresponds to the constrained problem of Section 4.2. The only difference is that the norm on Ω_2 of the difference of an element of $\bar{S}$ and x is not bounded by a fixed constant. This does not change the characterization of elements of $J(x, \bar{S})$. It does, however, allow that $\rho(x, \bar{S})$ be greater than $||\bar{y} - x||_{\Omega_1}$. In fact, if a lower bound on $\rho(x, \bar{S})$ is found that is greater than $||\bar{y} - x||_{\Omega_1}$, then it is known that the wrong assumption has been made, and that $||\bar{y} - x||_{\Omega_1}$ is strictly less than $\rho(x, S)$. If an element y_1 of $\bar{S}$ can be found such that $||y_1 - x||_{\Omega_2}$ is less than or equal to $||\bar{y} - x||_{\Omega_1}$, then $\rho(x, S)$ is equal to $||\bar{y} - x||_{\Omega_1}$ and y_1

must be an element of $J(x, S)$.

The following theorem yields lower bounds on $\rho(x, \bar{S})$. Let $\overline{FP}(J,K)$ be the set $\{\{t_I\}_{I=1}^{U+1} \mid K - J = U,\ t_I \in (z_{J+I-1}, z_{J+I})$ for $1 \le I \le U$, and $t_{U+1} \in (z_{J+U-1}, z_{J+U})$ where $t_I < t_{I+1}$ for $1 \le I \le U\}$. Let $\underline{FP}(J, K)$ be the set $\{\{t_I\}_{I=1}^{U+1} \mid K - J = U,\ t_1 \in (z_J, z_{J+1})$, and $t_I \in (z_{J+I-2}, z_{J+I-1})$ for $2 \le I \le U + 1$ where $t_I < t_{I+1}$ for $1 \le I \le U\}$. Let $\overline{\underline{FP}}(J,K)$ be the set $\{\{t_I\}_{I=1}^{U} \mid K - J = U$ and $t_I \in (z_{J+I-1}, z_{J+I})$ for $1 \le I \le U\}$.

Theorem 5.1.2. Let $L(j)$ for $1 \le j \le a$ be given integers such that $0 \le L(j) \le N - 1$ and $L(j) < L(j + 1)$ if $a > 1$. Let Ω_1 be the union of $(z_{L(j)}, z_{L(j)+1})$ for $1 \le j \le a$. Let Ω_2 be $[\bar{a}, \bar{b}] \setminus \Omega_1$. Let $\bar{S}$ be $\{P \in S \mid P|_{\Omega_1} \equiv \bar{y}|_{\Omega_1}\}$ where $\bar{y}|_{\Omega_1}$ is a given element of $S|_{\Omega_1}$. Let $x \in X$ and $y \in \bar{S}$.

(5.1.2, 1) Let $J < K$ and $[z_J, z_K]$ be a subset of Ω_2. Let $C = \{t_I\}_{I=1}^{U+2} \in FP(J,K)$ where $K - J = U$. Let $(y - x)(t_I) \neq 0$ for t_I in C. Let $\operatorname{sgn}(y - x)(t_I) = -\operatorname{sgn}(y - x)(t_{I+1})$ for $1 \le I \le U + 1$. Let

$$\text{Min} = \min_{1 \le I \le U+2} |(y - x)(t_I)|$$

and

$$\text{Max} = \max_{1 \le I \le U+2} |(y - x)(t_I)|.$$

(5.1.2, 2) Let $L(j) + 1 < L(j + 1)$. Let $C = \{t_I\}_{I=1}^{U} \in \overline{\underline{FP}}(L(j)+1, L(j+1))$ where $L(j + 1) - L(j) - 1 = U$. Let $(y - x)(t_I) \neq 0$ for every t_I in C.

Let $\mathrm{sgn}(y - x)(t_I) = -\mathrm{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U - 1$. Let

$$\mathrm{Min} = \min_{1 \leq I \leq U} |(y - x)(t_I)|$$

and

$$\mathrm{Max} = \max_{1 \leq I \leq U} |(y - x)(t_I)|.$$

(5.1.2, 3) Let $L(j) + 1 < K$ and $[z_{L(j)+1}, z_K]$ be a subset of Ω_2. Let $C = \{t_I\}_{I=1}^{U+1} \in \overline{FP}(L(j) + 1, K)$ where $K - L(j) - 1 = U$. Let $(y - x)(t_I) \neq 0$ for every t_I in C. Let $\mathrm{sgn}(y - x)(t_I) = -\mathrm{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U$. Let

$$\mathrm{Min} = \min_{1 \leq I \leq U + 1} |(y - x)(t_I)|$$

and

$$\mathrm{Max} = \max_{1 \leq I \leq U + 1} |(y - x)(t_I)|.$$

(5.1.2, 4) Let $J < L(j)$ and $[z_J, z_{L(j)}]$ be a subset of Ω_2. Let $C = \{t_I\}_{I=1}^{U+1} \in \underline{FP}(J, L(j))$ where $L(j) - J = U$. Let $(y - x)(t_I) \neq 0$ for every t_I in C. Let $\mathrm{sgn}(y - x)(t_I) = -\mathrm{sgn}(y - x)(t_{I+1})$ for $1 \leq I \leq U$. Let

$$\mathrm{Min} = \min_{1 \leq I \leq U + 1} |(y - x)(t_I)|$$

and

$$\mathrm{Max} = \max_{1 \leq I \leq U + 1} |(y - x)(t_I)|.$$

If (5.1.2, 1), (5.1.2, 2), (5.1.2, 3), or (5.1.2, 4) is satisfied, then $\mathrm{Min} \leq \rho(x, \overline{S})$; furthermore, if Max > Min, then $\mathrm{Min} < \rho(x, \overline{S})$.

Thus, a Remes ascent method on Ω_2 is attempted; all necessary systems of equations are uniquely solvable and a monotone increasing sequence of lower bounds on $\rho(x, \bar{S})$ can be obtained. As before, the limit of the sequence need not equal $\rho(x, \bar{S})$. However, for some $\underline{\underline{C}}$ in FP(J, K) where $\underline{J} < J < K \leq N$ or $0 \leq J < K < \underline{K}$ and θ is $[z_J, z_K]$; or for some $\underline{\underline{C}}$ in $\overline{FP}(\underline{K}, K)$ where $\underline{K} < K \leq N$ and θ is $[z_{\underline{K}}, z_K]$; or some $\underline{\underline{C}}$ in $\underline{FP}(J, \underline{J})$ where $0 \leq J < \underline{J}$ and θ is $[z_J, z_{\underline{J}}]$, if $y^{\underline{\underline{C}}}$ is $\bar{\bar{y}}$, the limit of the sequence will be $||\bar{\bar{y}} - x||_{\underline{\underline{C}}}$ which will equal $||\bar{\bar{y}} - x||_\theta$.

If $||\bar{\bar{y}} - x||_\theta$ is greater than $||\bar{y} - x||_{\Omega_1}$, then by inspection of the curve $\bar{y} - x$ on Ω_1 and $\bar{\bar{y}} - x$ on θ, a new initial guess J^0, K^0 and C^0 in $R(J^0, K^0)$ can be chosen so that $||y^{C_0} - x||_{C_0}$ is strictly greater than $||\bar{y} - x||_{\Omega_1}$. The process is then begun again.

If $||\bar{\bar{y}} - x||_\theta$ is less than or equal to $||\bar{y} - x||_{\Omega_1}$, it is assumed that $\rho(x, \bar{S})$ is equal to $||\bar{\bar{y}} - x||_\theta$. Another constrained approximation problem where Ω_1^1 is the union of Ω_1 and the interior of θ, Ω_2^1 is $[\bar{a}, \bar{b}] \setminus \Omega_1^1$, and $\bar{\bar{S}}$ is $\{P \in S | P|_{\Omega_1^1} \equiv \bar{y}|_{\Omega_1}, P|_\theta \equiv \bar{\bar{y}}|_\theta\}$ is considered. The Remes ascent method on Ω_2^1 is then attempted.

In finding the "best" best approximations to logarithm and exponential, the coefficients $\{d_I\}$ suggested by the work of Leslie Fox were computed. The curve of $d_I B_I + d_{I+1} B_{I+1} - x$ on each subinterval was inspected. From this information, an initial guess for J^0, K^0 and C^0 in $R(J^0, K^0)$ was made. Only one other guess at J^0 and K^0

was needed before an element of J(x, S) was found. In two subsequent attempts, the "best" best approximation was determined. That both functions were more difficult to approximate at one end than the other gave direction to the process. For less smooth functions, it is believed that the process would be highly difficult to program.

6.2. The "Best" Best Approximations

Points equal to $1 + .004(J)$ for $0 \leq J \leq 250$ were chosen to discretize $[1, 2]$. The interpolation points were chosen to be $z_I = 1 + I(.2)$ for $0 \leq I \leq 5$. An initial guess $\{d_I\}_{I=0}^{5}$ for the coefficients $\{b_I\}_{I=0}^{5}$ of the "best" best approximation was determined using the equations suggested by Leslie Fox [1]. The equation for d_I is a linear combination of even order differences at z_I. The terms through the tenth order difference were used to determine d_I where possible. However, in computing d_0 for approximating the logarithm on $[1, 2]$, only terms through the eighth order difference were used since, using spacing of .2, the tenth order difference at 1 is undefined. The dependence of Fox's coefficients on function values at points outside the interval of approximation is a drawback to the method; it is to be noted that only function values at points in the interval of approximation are needed for the determination of the coefficients of a "best" best approximation. Both sets of coefficients, FER which is

$||\Sigma d_I B_I + \Sigma f(z_I) A_I - f||$, and BBA which is $||\Sigma b_I B_I + \Sigma f(z_I) A_I - f||$ are given.

LOGARITHM

$FER = 3.08 \times 10^{-4}$

$BBA = 3.63 \times 10^{-6}$

Fox's Coefficients

$d_0 = -.8559959015$

$d_1 = -.6844518120$

$d_2 = -.5042471145$

$d_3 = -.3870616893$

$d_4 = -.3063979794$

$d_5 = -.2485200537$

"Best" Best Coefficients

$b_0 = -.9802853799$

$b_1 = -.6809059179$

$b_2 = -.5070605068$

$b_3 = -.3845308080$

$b_4 = -.3090930670$

$b_5 = -.2459355112$

$f_1 = \Sigma d_I B_I + \Sigma \text{Log}(z_I) A_I - \text{Log}$

$f_2 = \Sigma b_I B_I + \Sigma \text{Log}(z_I) A_I - \text{Log}$

$\max_{[1.0,1.2]} |f_1(t)| = 308 \times 10^{-6}$

$\max_{[1.2,1.4]} |f_1(t)| = 4.44 \times 10^{-6}$

$\max_{[1.4,1.6]} |f_1(t)| = 1.24 \times 10^{-6}$

$\max_{[1.6,1.8]} |f_1(t)| = .776 \times 10^{-6}$

$\max_{[1.8,2.0]} |f_1(t)| = .488 \times 10^{-6}$

$\max_{[1.0,1.2]} |f_2(t)| = 3.63 \times 10^{-6}$

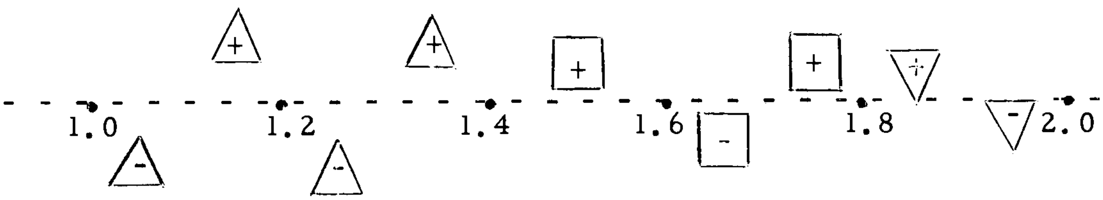
$\max_{[1.2,1.4]} |f_2(t)| = 3.63 \times 10^{-6}$

$\max_{[1.4,1.6]} |f_2(t)| = 2.13 \times 10^{-6}$

$\max_{[1.6,1.8]} |f_2(t)| = 2.13 \times 10^{-6}$

$\max_{[1.8,2.0]} |f_2(t)| = 1.59 \times 10^{-6}$

Pattern of Extreme Points of "Best" Best Approximation



△ - $||f_2|| = 3.63 \times 10^{-6}$

□ - $||f_2|| = 2.13 \times 10^{-6}$

▽ - $||f_2|| = 1.59 \times 10^{-6}$

EXPONENTIAL

FER = 5.94×10^{-6}

Fox's Coefficients

$d_0 = 2.707380370$

$d_1 = 3.306801833$

$d_2 = 4.038936920$

$d_3 = 4.933168668$

$d_4 = 6.025385810$

$d_5 = 7.359422843$

$$f_1 = \Sigma d_I B_I + \Sigma \mathrm{Exp}(z_I) A_I - \mathrm{Exp}$$

$\max\limits_{[1.0,\,1.2]} |f_1(t)| = 2.67 \times 10^{-6}$

$\max\limits_{[1.2,\,1.4]} |f_1(t)| = 3.26 \times 10^{-6}$

$\max\limits_{[1.4,\,1.6]} |f_1(t)| = 3.98 \times 10^{-6}$

$\max\limits_{[1.6,\,1.8]} |f_1(t)| = 4.87 \times 10^{-6}$

$\max\limits_{[1.8,\,2.0]} |f_1(t)| = 5.94 \times 10^{-6}$

BBA = 5.12×10^{-6}

"Best" Best Coefficients

$b_0 = 2.705817454$

$b_1 = 3.308476580$

$b_2 = 4.037616263$

$b_3 = 4.934709961$

$b_4 = 6.024276760$

$b_5 = 7.360707027$

$$f_2 = \Sigma b_I B_I + \Sigma \mathrm{Exp}(z_I) A_I - \mathrm{Exp}$$

$\max\limits_{[1.0,\,1.2]} |f_2(t)| = 2.41 \times 10^{-6}$

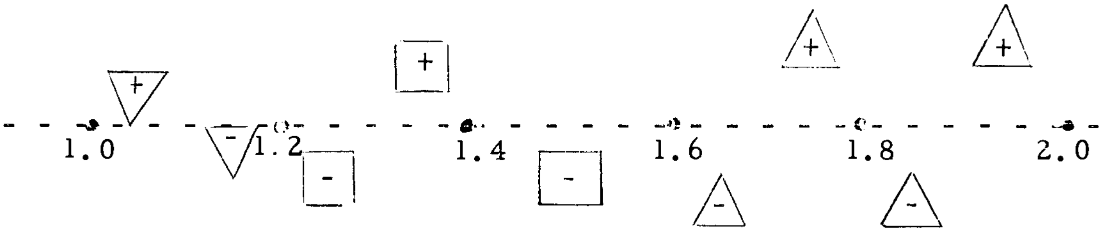
$\max\limits_{[1.2,\,1.4]} |f_2(t)| = 3.69 \times 10^{-6}$

$\max\limits_{[1.4,\,1.6]} |f_2(t)| = 3.69 \times 10^{-6}$

$\max\limits_{[1.6,\,1.8]} |f_2(t)| = 5.12 \times 10^{-6}$

$\max\limits_{[1.8,\,2.0]} |f_2(t)| = 5.12 \times 10^{-6}$

Pattern of Extreme Points of "Best" Best Approximation



△ - $||f_2|| = 5.12 \times 10^{-6}$

□ - $||f_2|| = 3.69 \times 10^{-6}$

▽ - $||f_2|| = 2.41 \times 10^{-6}$

VII. SUMMARY

It has been shown that J(x, S) is a nonempty set for any choice of x in X. An element y of J (x, S) is characterized by the condition of Theorem 2. 3.1 which concerns the location of the extreme points of y-x and the sign of y-x at those points. When X is normed by the maximum norm, J(x, S), in general, is not a singleton. However, when the quasinorm of Chapter IV is used, a unique element of S, the "best" best approximation, at a minimum distance from x is obtained. The "best" best approximation is always an element of J(x, S). Unfortunately, an easily programmable method for calculating "best" best approximations has not been found. However, a method, modelled on the Remes algorithm and outlined in Chapter VI, can be used to calculate the "best" best approximation in some cases.

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