A PROOF OF THE CONSISTENCY OF GEOMETRY

by

MICHAEL JAY JACOBS

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APPROVED:

Redacted for Privacy

Professor of Mathematics
In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Chairman of School Graduate Committee

Redacted for Privacy

Dean of Graduate School

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A PROOF OF THE CONSISTENCY OF GEOMETRY

INTRODUCTION

It is our purpose to prove the following Theorem:
If the axioms of arithmetic are consistent (i.e., don't lead to a contradiction) then so are the axioms of geometry.

For the axioms of arithmetic we take the well known axioms of Peano. For the axioms of geometry, we take the system of axioms given by Hilbert.

The proof of the theorem consists of exhibiting an arithmetic model for geometry. Once such a model is constructed, then any contradiction in geometry necessarily will involve a contradiction in arithmetic. Thus the theorem will follow.

The following notations will be used throughout the text:

Metadefinition -- A definition which assigns a geometric name to an arithmetic entity.

Definition -- A definition which assigns a name to a complex of geometric entities, or a definition which assigns a name to a complex of arithmetic entities.

Theorem -- A deductive result of arithmetic which has the same wording as an axiom of geometry.

Lemma -- All other deductive results.
I. AXIOMS OF CONNECTION

Metadef 1. A **point** is a triple \((x,y,z)\) of real numbers.

Metadef 2. A **line** is an equivalence class of pairs of equations equivalent to a pair of equations,
\[
A_1x + B_1y + C_1z + D_1 = 0 \\
A_2x + B_2y + C_2z + D_2 = 0
\]
where \(A_1, A_2, B_1, B_2, C_1, C_2, D_1\) and \(D_2\) are real constants and \(x, y\) and \(z\) are real variables and
\[
\text{rank } \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.
\]

Metadef 3. A **plane** is an equivalence class of equations equivalent to an equation,
\[
Ax + By + Cz + D = 0
\]
where \(A, B, C\) and \(D\) are real constants and \(x, y\) and \(z\) are real variables and
\[
A^2 + B^2 + C^2 \neq 0.
\]
Remark: We will sometimes say that 'a plane is an equation' or that 'a line is a pair of equations' when the equation or pair of equations in question are representatives of the plane or the line. The equations are used as name for the plane or line.

Metadef 4. A **line**,
The equation $A_1x + B_1y + C_1z + D_1 = 0$
and $A_2x + B_2y + C_2z + D_2 = 0$
is incident upon a pair of distinct points, $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ iff,

$$A_1x_1 + B_1y_1 + C_1z_1 + D_1 = 0$$
$$A_1x_2 + B_1y_2 + C_1z_2 + D_1 = 0$$
$$A_2x_1 + B_2y_1 + C_2z_1 + D_2 = 0$$
$$A_2x_2 + B_2y_2 + C_2z_2 + D_2 = 0$$

Lemma 1. Two points, $P_1 = (x_1y_1z_1)$ and $P_2 = (x_2y_2z_2)$, where $y_1 \neq y_2$, have a unique line incident upon them.

We will first show the uniqueness part of the lemma. Suppose therefore that a line represented by

(1) $A_1x + B_1y + C_1z + D_1 = 0$
and (2) $A_2x + B_2y + C_2z + D_2 = 0$
is incident upon $P_1$ and $P_2$.

Then we have,

(3) $A_1x_1 + B_1y_1 + C_1z_1 + D_1 = 0$
(4) $A_1x_2 + B_1y_2 + C_1z_2 + D_1 = 0$
(5) $A_2x_1 + B_2y_1 + C_2z_1 + D_2 = 0$
(6) $A_2x_2 + B_2y_2 + C_2z_2 + D_2 = 0$

From (3) and (4) and (5) and (6) respectively,

(7) $A_1(x_1-x_2) + B_1(y_1-y_2) + C_1(z_1-z_2) = 0$
(8) $A_2(x_1-x_2) + B_2(y_1-y_2) + C_2(z_1-z_2) = 0$
From (7) and (8),

(9) \( (A_1C_2 - A_2C_1)(x_1 - x_2) = (B_1C_2 - C_1B_2)(y_2 - y_1) \)

and

(10) \( (A_1C_2 - A_2C_1)(z_1 - z_2) = (A_2B_1 - A_1B_2)(y_2 - y_1) \)

From (3) and (5) and (4) and (6) respectively,

(11) \( (A_1C_2 - C_1A_2)x_1 + (C_2B_1 - C_1B_2)y_1 + (C_2D_1 - C_1D_2) = 0 \)

(12) \( (A_2B_1 - A_1B_2)y_2 + (C_1A_2 - C_2A_1)z_2 + (A_2D_1 - A_1D_2) = 0 \)

Since \( y_1 \neq y_2 \), from equations (9) and (10),

(13) \( (B_1C_2 - C_1B_2) = (A_1C_2 - A_2C_1)(x_1 - x_2)/(y_1 - y_2) \)

(14) \( (A_2B_1 - A_1B_2) = (A_1C_2 - A_2C_1)(z_1 - z_2)/(y_1 - y_2) \)

Now we notice using (13) and (14) that if

\( A_1C_2 - A_2C_1 = 0 \), the rank of \( \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \) is equal to zero.

Hence

(15) \( \ldots \ A_1C_2 - A_2C_1 \neq 0 \)

Hence from (13), (14), (15),

(16) \( \frac{B_1C_2 - C_1B_2}{A_1C_2 - A_2C_1} = \frac{x_1 - x_2}{y_1 - y_2} \)

(17) \( \frac{A_2B_1 - A_1B_2}{A_1C_2 - A_2C_1} = \frac{z_1 - z_2}{y_1 - y_2} \)

From (11) and (16) and (12) and (17)
We now write a pair of equations which are equivalent to (1) and (2),

\[
(A_1 C_2 - A_2 C_1)x + (B_1 C_2 - B_2 C_1)y + (D_1 C_2 - D_2 C_1) = 0
\]

\[
(A_2 B_1 - A_1 B_2)y + (A_2 C_1 - A_1 C_2)z + (D_1 A_2 - D_2 A_1) = 0
\]

From (16), (17), (18), (19), (20) and (21) we get another pair of equivalent equations,

\[
x + \frac{x_1 - x_2}{y_1 - y_2} y + \frac{x_2 - x_1}{y_1 - y_2} y_1 - x_1 = 0
\]

\[
\frac{z_1 - z_2}{y_1 - y_2} y + z + \frac{z_2 - z_1}{y_2 - y_1} y_2 - z_2 = 0
\]

This shows uniqueness. To show that such a line exists we write a pair of equations which are satisfied by the given triples.

\[
x(y_1 - y_2) + (x_1 - x_2)y + (x_2 - x_1)y_1 + x_1(y_2 - y_1) = 0
\]

\[
y(z_1 - z_2) + (y_1 - y_2)z + (z_2 - z_1)y_2 + z_2(y_2 - y_1) = 0
\]

Since \(y_1 \neq y_2\) the rank of the equations = 2 and thus (24) and (25) are a line incident upon \(P_1\) and \(P_2\).

Theorem 1. Two distinct points have a unique line incident upon them.
Proof. If \( y_1 = y_2 \) then either \( z_1 \neq z_2 \) or \( x_1 \neq x_2 \). Hence proceed with a parallel argument to that of the previous lemma.

Definition 1. A point \( P \), is a point of a line iff there exists a point \( Q \), such that the line is incident upon \( P \) and \( Q \). We sometimes say the point is on or in the line.

Theorem 2. Two distinct points \( P \) and \( Q \) of a line \( l \) have the line \( l \) incident upon them.
Proof. If \( P = (x', y', z') \), \( Q = (x'', y'', z'') \)
\[
\begin{align*}
1 &= Ax + By + Cz + D = 0 \\
A'x + B'y + C'z + D' &= 0
\end{align*}
\]
then we have that
\[
\begin{align*}
Ax' + By' + Cz' + D' &= 0 \\
Ax'' + By'' + Cz'' + D &= 0 \\
A'x' + B'y' + C'z' + D' &= 0 \\
A'x'' + B'y'' + C'z'' + D' &= 0
\end{align*}
\]

Theorem 3a. There exist two distinct points on any line.
Proof. Suppose we are given a line,
\[
\begin{align*}
A_1x + B_1y + C_1z + D_1 &= 0 \\
A_2x + B_2y + C_2z + D_2 &= 0
\end{align*}
\]
In order for the rank of \( \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \) to be 2 one of the following must be different from zero:
\[
\begin{align*}
A_1B_2 - A_2B_1 \\
A_1C_2 - A_2C_1 \\
B_1C_2 - B_2C_1
\end{align*}
\]
Suppose that $A_1B_2 - A_2B_1 \neq 0$. Then we define,

$$
\begin{align*}
x_0 &= B_1D_2 - B_2D_1/A_1B_2 - A_2B_1 \\
y_0 &= D_1A_2 - D_2A_1/A_1B_2 - A_2B_1 \\
l &= B_1C_2 - B_2C_1 \\
m &= C_1A_2 - C_2A_1 \\
n &= A_1B_2 - A_2B_1 \\
t &= \text{any non-zero real number.}
\end{align*}
$$

Then the points $(x_0, y_0, 0)$ and $(x_0 + lt, y_0 + mt, z_0 + nt)$ are distinct and are on the line.

A parallel proof is required for the assumption that either $l$ or $m$ is different from zero.

**Metadef 5.** A plane represented by $Ax + By + Cz + D = 0$ is incident upon three distinct points $(x_1, y_1, z_1)$ $(x_2, y_2, z_2)$ $(x_3, y_3, z_3)$ iff

$$
\begin{align*}
Ax_1 + By_1 + Cz_1 + D &= 0 \\
Ax_2 + By_2 + Cz_2 + D &= 0 \\
Ax_3 + By_3 + Cz_3 + D &= 0
\end{align*}
$$

**Definition 2.** A point is a point of a plane iff there exist 2 other distinct points such that the plane is incident upon the 3 points. We sometimes say that the point is on or in the plane. We say that a line is a line of or in a plane iff every point of the line is in the plane.
Theorem 3b. There exist three distinct points on every plane such that the three are not in any one line.

Proof. Given plane, \( Ax + By + Cz + D = 0 \), where \( A^2 + B^2 + C^2 \neq 0 \), then one of \( A \), \( B \) and \( C \) must be different from zero. Assume \( A \neq 0 \). Then

\[
Ax + By + Cz + D = 0 \\
y = 0
\]

and

\[
Ax + By + Cz + D = 0 \\
y + 1 = 0
\]

are lines. Since no triple with \( y = 0 \) is a solution to the second pair of equations, the lines are distinct. Call the lines \( l_1 \) and \( l_2 \). From theorem 3a we have that there exist two points \( P_1 \) and \( P_1' \) on the line \( l_1 \) and

\[
P_1 = (a, o, b), P_1' = (a', o, b')
\]

where \( a' \neq a \), or \( b' \neq b \).

There exists a point \( P_2 = (c, -1, d) \) on \( l_2 \).

\( P_1', P_1, P_2 \) are points of the plane since they must satisfy the equation of the plane.

Theorem 3c. There exists a plane.

Proof. \( x = 0 \) is a plane.

Theorem 4. Three points which are not in any one line have a unique plane incident upon them.

Proof. Let the three given points be: \( (x_1, y_1, z_1) \),
(x₂, y₂, z₂), (x₃, y₃, z₃)

Case 1.
Assume:

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
\end{vmatrix} = 0
\]

due to

\[
Ax_1 + By_1 + Cz_1 = 0 \\
Ax_2 + By_2 + Cz_2 = 0 \\
Ax_3 + By_3 + Cz_3 = 0
\]

for some values of A, B, C such that \(A^2 + B^2 + C^2 \neq 0\). (for if \(A^2 + B^2 + C^2 = 0\) then \(A = B = C = 0\))

Hence the points are on the plane \(Ax + By + Cz = 0\).

In order to show that this is the unique plane we must only show that the rank of \(\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}\) is 2.

The rank is less than 3 since the determinant is zero. The rank is not zero since then

\[
(x_1, y_1, z_1) = (x_2, y_2, z_2) = (x_3, y_3, z_3) \\
= (0, 0, 0)
\]

If the rank is 1 then we may write without loss of generality,

\[
(x_2, y_2, z_2) = k(x_1, y_1, z_1), \\
(x_3, y_3, z_3) = k'(x_1, y_1, z_1),
\]
in the usual vector notation.

Now suppose \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) satisfy an equation

\[
Ax + By + Cz + D = 0
\]

then

\[
Ax_1 + By_1 + Cz_1 + D = 0
\]

and

\[
Ax_1 + B'y_1 + C'z_1 + D = 0
\]

hence \(-D = Ax_1 + By_1 + Cz_1 = k(Ax_1 + By_1 + Cz_1)\).

If \((x_2, y_2, z_2) = (0, 0, 0)\) then \(D = 0\). If \((x_2, y_2, z_2) \neq (0, 0, 0)\) then \(k \neq 0\) and then \(Ax_1 + By_1 + Cz_1 = 0\) and hence \(D = 0\).

Therefore in any case, \(D = 0\).

Hence a line,

\[
Ax + By + Cz = 0
\]

\[
A'x + B'y + C'z = 0
\]

is incident upon \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\).

But since \((x_3, y_3, z_3) = k(x_1, y_1, z_1)\),

\[
A'kx_1 + B'ky_1 + C'kz_1 = 0
\]

\[
A'kx_1 + B'ky_1 + C'kz_1 = 0
\]

hence \((x_3, y_3, z_3)\) is on the line of \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) which is a contradiction.
Case 2.

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3
\end{vmatrix} \neq 0
\]

Then

\[
\begin{align*}
Ax_1 + By_1 + Cz_1 + D &= 0 \\
Ax_2 + By_2 + Cz_2 + D &= 0 \\
Ax_3 + By_3 + Cz_3 + D &= 0
\end{align*}
\]

has a solution for some \( D \neq 0 \).

Hence it must be that

\[
\begin{align*}
\frac{A}{D}x_1 + \frac{B}{D}y_1 + \frac{C}{D}z_1 + 1 &= 0 \\
\frac{A}{D}x_2 + \frac{B}{D}y_2 + \frac{C}{D}z_2 + 1 &= 0 \\
\frac{A}{D}x_3 + \frac{B}{D}y_3 + \frac{C}{D}z_3 + 1 &= 0
\end{align*}
\]

But since

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3
\end{vmatrix} \neq 0
\]

this solution is unique for \( A/D, B/D, C/D \).

Theorem 5. Any 3 points of a plane \( \alpha \) which are not in the same straight line have \( \alpha \) incident upon them.

Proof. The 3 points \((x_1, y_1, z_1)\) \((x_2, y_2, z_2)\) \((x_3, y_3, z_3)\) satisfy
Theorem 6. If a plane contains two points of a line then it contains every point of the line.

Proof. Suppose a plane \( Ax + By + Cz + D = 0 \) contains two points, \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\), then
\[
Ax_1 + By_1 + Cz_1 + D = 0 \\
Ax_2 + By_2 + Cz_2 + D = 0.
\]

Now there exists a line,
\[
A'x + B'y + C'z + D' = 0 \\
A''x + B''y + C''z + D'' = 0
\]
incident upon \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) and
\[
\text{Rank } \begin{pmatrix} A' & B' & C' \\ A'' & B'' & C'' \end{pmatrix} = 2.
\]

Therefore
\[
\text{Rank } \begin{pmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{pmatrix} = 2
\]

It is impossible that the rank be 3 since the equations
A\,x + B\,y + C\,z + D = 0
A'x + B'y + C'z + D' = 0
A''x + B''y + C''z + D'' = 0

have more than one solution. Hence the rank is 2.
Therefore there exists a \( k \neq 0 \) such that either
\[
(kA, \, kB, \, kC) = (A', \, B', \, C')
\]
or
\[
(kA, \, kB, \, kC) = (A'', \, B'', \, C'').
\]
Suppose, without loss of generality, that it is the first of these conditions which is true. Then
\[
-D' = kAx_1 + kBy_1 + kCz_1 = -kD
\]
hence
\[
D' = kD
\]
and
\[
A'x + B'y + C'z + D' = 0
\]
and
\[
Ax + By + Cz + D = 0
\]
are equivalent. Hence every solution to the line is a solution to the plane.

Theorem 7. If two distinct planes have a point in common then there exists at least one other point in common with the planes.

Proof. Suppose planes
\[
Ax + By + Cz + D = 0
\]
and

\[ A'x + B'y + C'z + D' = 0 \]

have a point \((x_0, y_0, z_0)\) in common.

Since the planes have a point in common

\[
\text{Rank } \begin{pmatrix} A & B & C \\ A' & B' & C' \end{pmatrix} = 2
\]

And therefore there exists a line

\[
A'x + B'y + C'z + D' = 0 \]

such that each point of the line is a point of both planes. Then by theorem 3a, the present theorem is proved.

**Theorem 8.** There exist 4 points which are not in any one plane.

**Proof.** Consider the points, \(P_1 = (0, 0, 0)\), \(P_2 = (1, 0, 0)\), \(P_3 = (0, 1, 0)\), \(P_4 = (0, 0, 1)\).

If there were a plane

\[
Ax + By + Cz + D = 0
\]

which contained \(P_1\), then \(D = 0\). If it contained \(P_2\), then \(A = -D = 0\). If it contained \(P_3\), then \(B = -D = 0\). If it contained \(P_4\), then \(C = -D = 0\). But then \(A^2 + B^2 + C^2 = 0\).
II. AXIOMS OF ORDER

Lemma 2. The set of all points on a line
\[ Ax + By + Cz + D = 0 \]
\[ A'x + B'y + C'z + D' = 0 \]
is the same as the set of all points \((x, y, z)\) such that
\[ x = x_0 + lt \]
\[ y = y_0 + mt \]
\[ z = z_0 + nt, \]
where \((x_0, y_0, z_0)\) is any point of the line, \(l, m, n\) are real constants, \(l^2 + m^2 + n^2 = 1\) and \(t\) takes on all real values.

Proof. It was shown in the proof of theorem 3a that there exists a point \((x', y', z')\) on the line such that every point \((x, y, z)\) where
\[ x = x' + l't' \]
\[ y = y' + m't' \]
\[ z = z' + n't' \]
l', m', n' are real constants and \(t'\) assumes every real value, is a point of the line and that
\[ k = (l')^2 + (m')^2 + (n')^2 \neq 0 \]
Now let
\[ t'' = t' / \sqrt{k} \]
\[ l = l' / \sqrt{k} \]
\[ m = m' / \sqrt{k} \]
\[ n = n' / \sqrt{k} \]
then \[ x = x' + lt'' \]
\[ y = y' + mt'' \]
\[ z = z' + nt'' \]

where \( t'' \) assumes all real values, is an equivalent set of equations.

Also \[ l^2 + m^2 + n^2 = 1. \]

Now since \( (x_0, y_0, z_0) \) is on the line, there exists a \( t_o'' \) such that,
\[ x_0 = x' + lt_o'' \]
\[ y_0 = y' + mt_o'' \]
\[ z_0 = z' + nt_o'' \]

Now if
\[ x_1 = x' + lt_1'' \]
\[ y_1 = y' + mt_1'' \]
\[ z_1 = y' + nt_1'' \]

then
\[ x_1 = x_0 + l(t_1'' - t_o'') \]
\[ y_1 = y_0 + m(t_1'' - t_o'') \]
\[ z_1 = z_0 + n(t_1'' - t_o'') \]

Hence if we let \( T_1 = t_1'' - t_o'' \)
\[ x_1 = x_0 + lt_1 \]
\[ y_1 = y_0 + mt_1 \]
\[ z_1 = z_0 + nt_1 \]
Now assume that \((x_2, y_2, z_2)\) is on the line. Then
\[
A x_2 + B y_2 + C z_2 + D = 0
\]
\[
A' x_2 + B' y_2 + C' z_2 + D' = 0
\]
and since
\[
A x_0 + B y_0 + C z_0 + D = 0
\]
\[
A' x_0 + B' y_0 + C' z_0 + D' = 0
\]
we have
\[
A(x_0 - x_2) + B(y_0 - y_2) + C(z_0 - z_2) = 0
\]
\[
A'(x_0 - x_2) + B'(y_0 - y_2) + C'(z_0 - z_2) = 0
\]
Also
\[
A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0) = 0
\]
\[
A'(x_1 - x_0) + B'(y_1 - y_0) + C'(z_1 - z_0) = 0
\]
Hence
\[
Al + Bm + Cn = 0
\]
\[
A'l + B'm + C'n = 0
\]
Now since \(\begin{pmatrix} A & B & C \\ A' & B' & C' \end{pmatrix}\) has rank 2, there exists a \(t_2 \neq 0\) such that
\[
x_0 - x_2 = -t_2 l
\]
\[
y_0 - y_2 = -t_2 m
\]
\[
z_0 - z_2 = -t_2 n
\]
Hence
\[
x_2 = x_0 + t_2 l
\]
\[
y_2 = y_0 + t_2 m
\]
\[
z_2 = z_0 + t_2 n
\]
Since the steps can be reversed, the lemma is proved in both directions.

Remark: These equations, which are equivalent to $Ax + By + Cz + D = 0$, $A'x + B'y + C'z + D' = 0$ are referred to as the parametric equations of the line with respect to the point $(x_0, y_0, z_0)$.

**Metadef 6.** Let

\[
\begin{align*}
x &= x_0 + lt \\
y &= y_0 + mt \\
z &= z_0 + nt
\end{align*}
\]

be the parametric equations of a line with respect to $(x_0, y_0, z_0)$. We say that three distinct points on the line, $A$, $B$ and $C$, are such that $B$ is between $A$ and $C$ (written $ABC$) iff one of the following is true,

\[t_A > t_B > t_C \quad \text{or} \quad t_A < t_B < t_C,\]

where $t_A$, $t_B$, $t_C$ are the values of $t$ corresponding to $A$, $B$ and $C$ in the above equations.

Note that if $(x_1, y_1, z_1) \neq (x_0, y_0, z_0)$ is on the line, we will have

\[
\begin{align*}
x &= x_1 + lt' \\
y &= y_1 + mt' \\
z &= z_1 + nt'
\end{align*}
\]

is another set of parametric equations for the line.

If $t'_A$, $t'_B$, $t'_C$ are the values corresponding to $A$, $B$ and
C then it is obvious that if $t_B$ is between $t_A$ and $t_C$ in the real ordering then $t_B'$ is between $t_A'$ and $t_C'$ in the real ordering.

Theorem 9. If $A$, $B$ and $C$ are points, then $ABC$ implies $CBA$.
Proof. By symmetry.

Theorem 10. If $A$, $B$ and $C$ are points, then at most one of $ABC$, $BAC$ and $ACB$ hold.
Proof. At most one of three real numbers is between the other two in the ordering of the real numbers.

Theorem 11. If $A$ and $B$ are distinct points of a line, then there exists a point $C$ of the line such that $B$ is between $A$ and $C$.
Proof. By lemma 2, the parametric equations in the metadefinition of 'between' exist. Suppose $t_A$ corresponds to point $A$, $t_B$ corresponds to point $B$ (with respect to some point $(x_0, y_0, z_0)$), then

$$t_A < t_B$$

implies that there exists $t_C$ such that $t_A < t_B < t_C$

and

$$t_A > t_B$$

implies that there exists $t_C$ such that $t_A > t_B > t_C$. 
Definition 3. A **line segment** $AB$ is the set of all points between $A$ and $B$, where $A$ and $B$ are distinct points.

Definition 4. A **closed line segment** $\overline{AB}$ is the union of $AB$ and $A \cup B$, where $A$ and $B$ are distinct points.

Definition 5. A **triangle** $\triangle ABC$ is defined for 3 points $A, B, C$ such that they are not all in any one line and is the set $\overline{AB} \cup \overline{BC} \cup \overline{CA}$.

**Theorem 12.** If a line $l$, in the plane $\alpha$, of the points $P, Q, R$ ($P, Q$ and $R$ are not in the same straight line) has a point of $PQ \cup QR \cup RP$ on it, then it has another distinct point of the triangle $\triangle PQR$ on it.

**Proof.** Assume that the equation of the plane $\alpha$ is $A'x + B'y + C'z + D' = 0$. Since line $l$ has all its points in the plane $\alpha$, it may be represented by

$$A'x + B'y + C'z + D' = 0$$
$$Ax + By + Cz + D = 0$$

where rank $\begin{pmatrix} A' & B' & C' \\ A & B & C \end{pmatrix} = 2$, as was shown in the proof of theorem 6.
Now let

\[ P = (x_p, y_p, z_p) \]
\[ Q = (x_q, y_q, z_q) \]
\[ R = (x_r, y_r, z_r) \]

Then by lemma 2 there exists a representation of the line of \( P \) and \( Q \) as follows:

\[ x = x_p + lt \]
\[ y = y_p + mt \]
\[ z = z_p + nt \]

Since \( l \) has a point in common with the line of \( PQ \), the function

\[ f(t) = A(x_o + lt) + B(y_o + mt) + C(z_o + nt) + D \]

must have the value zero at some point of \( PQ \). We have two cases, since \( f(t) \) is a linear function of \( t \).

(1) \( f(t) \) is identically zero.

Then the theorem is proved since there must exist another point on \( PQ \) in common with \( l \).

(2) \( f(t_1) = 0 \) for exactly one value of \( t \). This value of \( t \) corresponds to a point of \( PQ \) and therefore

\[ 0 = t_p < t_1 < t_q \]

where \( t_q \) is the value of \( t \) for point \( Q \).

Therefore, since \( f \) is linear on \( t \), \( f(t_p) \) and \( f(t_q) \) have opposite sign.
Now in exactly symmetrical fashion we define \( g(t') \) and \( h(t'') \) to be the corresponding functions with respect to the lines of QR and RP respectively.

Then we have,

\[
h(t''_p) = f(t_p), \quad g(t'_{q}) = f(t_q)
\]

and

\[
h(t''_r) = g(t'_{r})
\]

Hence \( h(t''_p) \) and \( g(t'_{q}) \) have opposite signs. Now since \( h(t''_r) = g(t'_{r}) \) it must be that there is a zero of \( h(t'') \) over the segment RP or a zero of \( g(t') \) over the segment QR or that \( h(t''_r) = g(t'_{r}) = 0 \). In any case the theorem follows.
Remark: A transformation

\[
\begin{pmatrix}
    x' \\
    y' \\
    z'
\end{pmatrix} = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix}\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} + \begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{pmatrix}
\]

is denoted \{(aij), (ai)\}

where

\[
\begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

\{(aij)\}

\[
\begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{pmatrix}
\]

\{(ai)\}

The statement of equation (1) is written

\[
\begin{pmatrix}
    x' \\
    y' \\
    z'
\end{pmatrix} = \{(aij), (ai)\}\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
\]

The product of two transformations \{(aij), (ai)\} and \{(bij), (bi)\} is written

\[
\{(bij), (bi)\} \cdot \{(aij), (ai)\} = \{(bij)(aij), (bij)(ai) + (bi)\} 
\]
Lemma 3.

\[
\left[ \{ (bij), (bi) \} \cdot \{ (aij), (ai) \} \right] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \{ (bij), (bi) \} \begin{pmatrix} (aij), (ai) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

Proof. Follows from the above definition.

Remark: \( \{ (aij), (ai) \} \) is a \textit{rigid motion} iff

1. all the \( a \)'s in \( (aij) \) and \( (ai) \) are real.
2. \( (aij)(aij)^T = I \)

where \( (aij)^T = \) the transpose of \( (aij) \)

and \( I = \) the \( 3 \times 3 \) identity matrix.

3. \( |aij| = 1 \)

Lemma 4. The set of all rigid motions with the product mapping forms a group, \( G \).

Proof. (1) The operation is associative.

(2) To establish closure we must show

\[
(BA)(BA)^T = I
\]

where \( A = (aij) \) \( B = (bij) \)

and \( \{ (aij), (ai) \} \), \( \{ (bij), (bi) \} \) are rigid motions.

If we denote the right inverse of a matrix \( X \) by \( X^{-1} \), we have

\[
(BA)^{-1} = A^{-1}B^{-1} = A_T B_T = (BA)^T
\]
Also we note that
\[
\det (BA) = \det B \cdot \det A = 1 \cdot 1 = 1.
\]

(3) The identity is
\[
\left\{ \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}
\]

(4) \( \left\{ (aij), (ai) \right\}^{-1} = \left\{ (aij)^T - (aij)^T (ai) \right\} \)

where \(-1\) stands for the right inverse of a rigid motion with respect to the group operation and identity.

Lemma 5. A transformation \( \left\{ (aij), (ai) \right\} \)
is a member of the group G only if \( |aij| = 1 \) and

(1) \( \sum_{j=1}^{3} a_{ij}^2 = 1 \) for \( i = 1, 2, 3 \)

(2) \( \sum_{j=1}^{3} a_{ij} a_{i',j} = 0 \) for all \( i \neq i' \)

(3) \( \sum_{i=1}^{3} a_{ij}^2 = 1 \) for \( j = 1, 2, 3 \)

(4) \( \sum_{i=1}^{3} a_{ij} a_{i,j'} = 0 \) for all \( j \neq j' \)

all hold. Either conditions 1 and 2 or conditions 3 and 4 are sufficient for \( \left\{ (aij), (ai) \right\} \) to be in G if \( |aij| = 1 \).
Proof. \((aij)(aij)^T I \Rightarrow\) conditions 1 and 2
\(\{(aij), (ai)\} \in G \Rightarrow \{(aij)^T - (aij)^T(ai)\} \in G\)
\((aij)^T (aij)^T T = (aij)^T(aij) = I\)
\(\Rightarrow\) conditions 3 and 4.

The sufficiency statement is obtained by reversing the steps in the above arguments.

Metadef 7. An ordered pair of distinct points \((P_1, P_2)\) is congruent to an ordered pair of distinct points \((P_1', P_2')\), (written \(P_1, P_2) = (P_1', P_2')\)) where
\[P_1 = (x_1, y_1, z_1)\quad P_2 = (x_2, y_2, z_2)\]
\[P_1' = (x_1', y_1', z_1')\quad P_2' = (x_2', y_2', z_2')\]
iff that there exists a rigid motion \(\{(aij), (ai)\}\) such that
\[
\begin{pmatrix}
x_1'\\
y_1'\\z_1'
\end{pmatrix}
= \{(aij), (ai)\}
\begin{pmatrix}
x_1\\y_1\\z_1
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
x_2'\\y_2'\\z_2'
\end{pmatrix}
= \{(aij), (ai)\}
\begin{pmatrix}
x_2\\y_2\\z_2
\end{pmatrix}
\]

Remark. We will later change the above metadef.

Lemma 6.
1. \((P_1, P_2)\) is congruent to \((P_1', P_2')\) iff \((P_1', P_2')\)
is congruent to \((P_1, P_2)\).

2. \((P_1, P_2) \equiv (P_1'', P_2'')\)

and

\((P_1', P_2') \equiv (P_1'', P_2'')\)

implies \((P_1, P_2) \equiv (P_1', P_2')\)

3. \((P_1, P_2) \equiv (P_1, P_2)\)

Proof. Follows from the fact that \(G\) is a group.

Lemma 7. \((P_1, P_2) \equiv (P_1', P_2')\) iff

\[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2\]

Proof. Let \(u_x = x_1 - x_2\), \(r = u_x^2 + u_y^2\), \(u_y = y_1 - y_2\), \(u_z = z_1 - z_2\), then we have that at least one of \(u_x, u_y, u_z\) is not zero since the points are distinct. Assume either \(u_y \neq 0\) or \(u_x \neq 0\) and hence \(r \neq 0\) and \(\delta \neq 0\). We define a transformation, \(A\), as follows:

\[
A = \begin{pmatrix}
\begin{pmatrix}
\frac{r}{\delta} & 0 & \frac{u_z}{\delta} \\
0 & 1 & 0 \\
\frac{u_z}{\delta} & 0 & -\frac{r}{\delta}
\end{pmatrix}
& \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
\begin{pmatrix}
\frac{u_x}{\delta} & \frac{u_y}{\delta} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
& \begin{pmatrix}
u_y/\delta & u_x/\delta & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
-x_2 \\
-y_2 \\
-z_2
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
I
\end{pmatrix}
\]
It is easy to verify that $A$ is a rigid motion and

$$
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = A
\begin{pmatrix}
x_2 \\
y_2 \\
z_2
\end{pmatrix}
$$

$$
\begin{pmatrix}
\delta \\
0 \\
0
\end{pmatrix} = A
\begin{pmatrix}
x_1 \\
y_1 \\
z_1
\end{pmatrix}
$$

If both $x_1 = x_2$ and $y_1 = y_2$ then the same result is easily obtained by interchanging the second and third rows of each matrix in $A$, and writing $r = u_z^2 + u_x^2$ and interchanging $u_z$ and $u_y$ in each matrix.

Hence by repeated application of the previous lemma,

$$
(P_1, P_2) \equiv ((0, 0, 0), (\delta, 0, 0))
$$

Similarly we can show that

$$
(P'_1, P'_2) \equiv ((0, 0, 0), (\delta', 0, 0))
$$

where $\delta' = \sqrt{(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 + (z'_1 - z'_2)^2}$

But we have by hypothesis

$$
\delta' = \delta.
$$

Hence by the previous lemma,

$$(P_1, P_2) \equiv (P'_1, P'_2)$$

The steps are reversible so the implication holds in the other direction.
Lemma 8. \((P_1, P_2) \equiv (P_2, P_1)\) where \(P_1\) and \(P_2\) are arbitrary distinct points.

Proof. Follows from the previous lemma since

\[
\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}
\]

where \(P_1 = (x, y, z)\) and \(P_2 = (x_2, y_2, z_2)\)

Metadef 7'. The metadef 7 for congruence is now changed by striking the word 'ordered' wherever it appears.

Theorem 13. Congruence of pairs of points is transitive and symmetric.

Proof. Follows from the preceding lemmas.

Definition 6. A ray from a point \(P\) is a set containing a given point \(B\) and all points \(Y\) such that \(YPB\) is not true and \(Y\) is on the line of \(PB\).

Lemma 9. A set of points \(X\) is a ray from a point \((x_0, y_0, z_0)\) iff there exists a line \(l\) represented by

\[
\begin{align*}
x &= x_1 + lt \\
y &= y_1 + mt \\
z &= z_1 + nt
\end{align*}
\]

(where \(x_1, y_1, z_1\) is any point of the line)

and
\[x_0 = x_1 + lt_o\]
\[y_0 = y_1 + mt_o\]
\[z_0 = z_1 + nt\]

and \((x_A, y_A, z_A), (x_B, y_B, z_B)\) are members of \(X\) imply that either

\[t_A > t_o \text{ and } t_B > t_o\]

or

\[t_A < t_o \text{ and } t_B < t_o\]

where \(x_A = x_1 + lt_A\) and \(x_B = x_1 + lt_B\)
\[y_A = y_1 + mt_A\] \(y_B = y_1 + mt_B\)
\[z_A = z_1 + nt_A\] \(z_B = z_1 + nt_B\)

Proof. Follows from lemma 2, metadef 6 and definition 6.

**Theorem 14.** In a ray from a point \(P\) there exists a unique point \(Q\) such that \(PQ \equiv RS\) where \(R\) and \(S\) are arbitrary distinct points.

Proof. Let \(P = (x_P, y_P, z_P), Q = (x_Q, y_Q, z_Q), R = (x_R, y_R, z_R), S = (x_S, y_S, z_S)\). We have for any point \((x_1, y_1, z_1)\) on a ray from \(P\) that

\[x_1 = x_p + lt_1\]
\[y_1 = y_p + mt_1\]
\[z_1 = z_p + nt_1\]

Then if \((x_2, y_2, z_2)\) is another point on this ray
\[ x_2 = x_p + lt_2 \]
\[ y_2 = y_p + mt_2 \]
\[ z_2 = z_p + nt_2 \]

and it follows from lemma 9 that \( t_1 \) and \( t_2 \) have the same non-zero sign. Hence

\[
\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2} = |t_1|
\]
\[
\sqrt{(x_2-x_0)^2 + (y_2-y_0)^2 + (z_2-z_0)^2} = |t_2|
\]

since \( l^2 + m^2 + n^2 = 1 \). Hence if

\[
\sqrt{(x_r-x_s)^2 + (y_r-y_s)^2 + (z_r-z_s)^2} = |t_x|
\]

it follows that there exists a unique point on the ray such that

\[
x = x_p + l|t_x|
\]
\[
y = y_p + m|t_x|
\]
\[
z = z_p + n|t_x|
\]

where either the + or the - sign applies according to whether the ray is characterized by positive or negative values of \( t \).

Theorem 15. If \( ABC \) on a line \( l \) and if \( A'B'C' \) on a line \( l' \) and \( AB \equiv A'B' \), \( BC \equiv B'C' \) then \( AC \equiv A'C' \).

Proof. Let
\[ A = (a_x, a_y, a_z) \quad A' = (a'_x, a'_y, a'_z) \]
\[ B = (b_x, b_y, b_z) \quad B' = (b'_x, b'_y, b'_z) \]
\[ C = (c_x, c_y, c_z) \quad C' = (c'_x, c'_y, c'_z) \]

Then the line \( l \) is represented by
\[ x = b_x + lt \]
\[ y = b_y + mt \]
\[ z = b_z + nt \]

and line \( l' \) by
\[ x' = b'_x + l't' \]
\[ y' = b'_y + m't' \]
\[ z' = b'_z + n't' \]

Let \( t_a, t_b, t_c \) and \( t'_a, t'_b, t'_c \) represent the values of \( t \) and \( t' \) corresponding to \( A, B, C \) and \( A', B', C' \) on lines \( l \) and \( l' \) respectively. Then
\[ t_b = t'_b = 0 \]

and we may suppose that \( t_a \geq 0 \) and hence \( t_c \leq 0 \). Now, in this case, if \( t'_a = 0 \) we let \( m'' = -m', n'' = -n' \) and \( l'' = -l' \). Evidently this new set of equations is equivalent to the old set. So in any case we have
\[ t_a > 0, \quad t'_a > 0 \]
\[ t_c < 0, \quad t'_c < 0 \]

Also, it is evident that
Then
\[ t'_{a} = t_{a} \]
\[ t'_{c} = t_{c} \]

\[
\sqrt{(a_{x} - c_{x})^{2} + (a_{y} - c_{y})^{2} + (a_{z} - c_{z})^{2}} = t_{a} - t_{c}
\]
\[ = t'_{a} - t'_{c} = \sqrt{(a'_{x} - c'_{x})^{2} + (a'_{y} - c'_{y})^{2} + (a'_{z} - c'_{z})^{2}} \]

**Definition 7.** If we have two distinct rays \( r_{1} \) and \( r_{2} \) from a point \( A \), we define an **angle** \( \angle (r_{1}, A, r_{2}) \) as the ordered set \( (r_{1}, A, r_{2}) \).

**Remark.** We will later change the definition of angle.

**Definition 8.** A **straight angle** is an angle \( \angle (r_{1}, A, r_{2}) \) such that the points of \( r_{1} \) and \( r_{2} \) are on the same line.

**Metadef 8.** \( \angle (r_{1}, A, r_{2}) \equiv \angle (r_{1}', A', r_{2}') \) (read **congruent**) iff there exists a rigid motion which takes points of \( r_{1} \) to points of \( r_{1}' \) and points of \( r_{2} \) to points of \( r_{2}' \) and \( A \) to \( A' \).

**Lemma 10.** A rigid motion takes points of lines into points of lines. Therefore if a rigid motion takes two distinct points of a line into two image points, the rigid motion takes the line of the points into the line
of the image points.

Proof. A rigid motion \((a_{ij}), (ai)\) takes points of the line

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}
\]

into points of the line

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \begin{pmatrix} (a_{ij}), (ai) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}
\]

The second part of the lemma follows from theorem 2.

Lemma 11. \(X \in (r_1 P_1 r_2) \equiv X \in (r_2 P_1 r_1)\)

Proof. Let \(P_1 = (x_1, y_1, z_1)\) and \(P_2\) be a point of \(r_1\) and let \(P_2 = (x_2, y_2, z_2)\).

Now the rigid motion 'A' in the proof of lemma 7 takes \(P_1\) into \((0, 0, 0)\) and \(P_2\) into \((0, 0, 0)\). Hence by the preceding lemma, it takes \(r_1\) into a ray \(r_1'\) from \((0, 0, 0)\) and containing all points of the form \((x, 0, 0)\) where \(x > 0\) as is obviously verifiable.

Now on \(r_2\) there is a point \(Z\) such that

\[(P_1, Z) \equiv (0, 0, 0), (0, 0, 1)\]

hence the transformation \(A\) takes \(Z\) into a point \(Z' = (1, m, n)\) such that \(1^2 + m^2 + n^2 = 1\). Let the image of \(r_2\) under \(A\) be \(r_2'\). Then \((1, m, n) \in r_2'\).
If we can find a rigid motion $B$, such that $B$ takes $r_1'$ into $r_2'$ and vice versa, the present lemma would follow since $A^{-1}BA$ will take $r_2$ into $r_1$ and vice versa.

We want to find a rigid motion $B$ such that

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
= B
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
m \\
n
\end{pmatrix}
= B
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
= B
\begin{pmatrix}
m \\
n
\end{pmatrix}
\]

Such a rigid motion is,

\[
B = \begin{pmatrix}
1 & m & n \\
m & \frac{-1-n^2}{1+1} & \frac{mn}{1+1} \\
n & \frac{mn}{1+1} & \frac{-1-m^2}{1+1}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Definition 7'. Henceforth definition 7 should omit the word 'ordered'.
Theorem 16. An angle is congruent to itself.

Proof. Follows from the previous lemma and the fact that \( \begin{bmatrix} I, & 0 \\ 0, & 0 \end{bmatrix} \) transforms rays and points into themselves.

Definition 9. A side of a line in a plane is a set of points containing a given point \( A \) not on the line, and all points \( X \) in the plane of \( A \) and the line, such that there does not exist a point \( Q \) of the line such that \( XQA \).

Remark. We will use the expression "A rigid motion takes planes into planes" to mean that all the points of a plane are taken into all the points of an image plane by the rigid motion. Similarly we will say that 'lines are taken into lines' to mean that points of lines are taken into points of lines.

Lemma 11. A rigid motion takes a plane to an image plane.

Proof. If \( A_x + B_y + C_z + D = 0 \) is a plane, and \( (aij)(ai) \) is a rigid motion and

\[
\begin{pmatrix}
x_1 \\
y_1 \\
z_1
\end{pmatrix} = \begin{bmatrix} (aij)(ai) \end{bmatrix} \begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}
\]

and

\[
Ax_o + By_o + Cz_o + D = 0
\]
then we have
\[ A'x_1 + B'y_1 + C'z_1 + D' = 0 \]
where
\[
\begin{pmatrix}
  A \\
  B \\
  C
\end{pmatrix}
\begin{pmatrix}
  a_{ij}
\end{pmatrix}
\begin{pmatrix}
  A' \\
  B' \\
  C'
\end{pmatrix}
\]
and \[ D' = -A'a_1 - B'a_2 - C'a_3 + D. \]

Lemma 12. A set of points \( X \) is a side of a line
\[ A'x + B'y + C'z + D' = 0 \]
\[ Ax + By + Cz + D = 0 \]
in a plane
\[ A'x + B'y + C'z + D' = 0 \]
iff
\[ (x_1, y_1, z_1) \in X \text{ and } (x_2, y_2, z_2) \in X \]
implies
\[ Ax_1 + By_1 + Cz_1 + D \]
has the same non-zero sign as does
\[ Ax_2 + By_2 + Cz_2 + D. \]

Proof. If the points \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \) have no point of 1 between them and the line of \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \) is represented by
\[
\begin{align*}
x &= x_0 + lt \\
y &= y_0 + mt \\
z &= z_0 + nt
\end{align*}
\]
we define
\[ f(t) = A(x_0 + lt) + B(y_0 + mt) + C(z_0 + nt) + D \]
then in the range \( t_1 \leq t \leq t_2 \) or \( t_2 \leq t \leq t_1 \) (which ever applies), \( f(t) \) has no zero values. Since \( f(t) \) is a linear function of \( t \), \( f(t_1) \) and \( f(t_2) \) must have the same sign. Reversing the steps we see that the converse follows.

Lemma 13. A rigid motion takes a side of a plane with respect to a line into a side of the image plane with respect to the image of the line.

Proof. The proof follows from the fact that lines are transformed into lines and hence the line determined by two points on a given side of the plane goes into an image line. This image line does not intersect the image of the line defining the side of the plane because 'distances' are preserved and hence a non-zero 'distance' cannot go into a zero 'distance' as would be the case if the image lines had a common point.

Theorem 17. If \( l \) is a line in a plane \( \alpha \), \( P \) is a point of \( l \), \( r_1 \) is a ray from \( P \) on \( l \), and \( X \) is a side of \( l \) in \( \alpha \), there exists a ray \( r_2 \) in \( X \) such that the angle \( \angle (r_1, A, r_2) \) is congruent to a given angle \( \angle (r_1', B, r_2') \) in plane \( \beta \).
Proof. Let the equation of \( \alpha \) be

\[ Ax + By + Cz + D = 0 \]

where

\[ A^2 + B^2 + C^2 \neq 0. \]

Now consider the matrix,

\[
A = \begin{pmatrix}
0 & 0 & A/k \\
1 & 0 & B/k \\
0 & 1 & C/k \\
\end{pmatrix}
\]

where \( k = A^2 + B^2 + C^2 \neq 0. \)

Obviously the columns of \( A \) are linearly independent and

\[ (A/k)^2 + (B/k)^2 + (C/k)^2 = 1. \]

Hence by the Gram-Schmidt process we can find a matrix

\[
B = \begin{pmatrix}
a_{11} & a_{12} & A/k \\
a_{21} & a_{22} & B/k \\
a_{31} & a_{32} & C/k \\
\end{pmatrix}
\]

where the columns are mutually orthogonal and are normal.

We notice that

\[
B \begin{pmatrix}
0 \\
0 \\
1 \\
\end{pmatrix} = \frac{1}{k} \begin{pmatrix}
A \\
B \\
C \\
\end{pmatrix}
\]

Further the determinant of \( B \) is either \(+1\) and (if it is \(-1\)) can be made \(+1\) by multiplying one of the first two columns by \(-1\). Hence by lemma 11, since
\[ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{k} B^{-1} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \]

the rigid motion \( \{ B^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \) takes \( Ax + By + Cz + D \)

into the plane \( z = K \)

where \( K \) is some constant. Further, the rigid motion \( \{ I, \begin{pmatrix} -K \\ -K \\ -K \end{pmatrix} \} \) will take \( z = K \) into the plane \( z = 0 \).

We also note that sides of the plane are taken into sides of the image plane, \( z = 0 \), by lemma 13. Let \( l' \) be the image of the line \( l \) in the plane \( z = 0 \) and let the image of the point \( P \) be \( (x', y', 0) \). Then the rigid motion \( \{ I, \begin{pmatrix} -x' \\ -y' \\ 0 \end{pmatrix} \} \)

takes points of \( z = 0 \) into points of \( z = 0 \) and the point \( (x', y', 0) \) into \( (0, 0, 0) \). Now the image of \( l' \) under this last transformation is a line \( l'' \). There exists a point \( (x'', y'', 0) \) on \( l'' \) such that \( \sqrt{x''^2 + y''^2} = 1 \).

Now the rigid motion \( \{ \begin{pmatrix} x' & y' & 0 \\ -y' & x' & 0 \\ 0 & 0 & 1 \end{pmatrix} \} \)
takes points of \( z = 0 \) to points of \( z = 0 \), \((0, 0, 0)\) into itself and \((x'', y'', 0)\) into \((1, 0, 0)\).

Hence in the proof of the theorem we will assume that these transformations have all been applied to the original plane and that the image of line 1 is \( y = 0 \) \( z = 0 \) (by lemma 10) and that the image of \( P \) is \((0, 0, 0)\). Then if we can prove the theorem for this particular line and this particular point on the line and in the plane \((z = 0)\), we can then apply the inverse transformations one at a time and prove the theorem in general.

We first notice that there exist exactly two sides of the plane with respect to the line \( y = 0 \).

\[
\begin{align*}
\text{The one side is given by the points } (x, y, 0) \text{ where } y > 0 \text{ and the other by the points } (x, y, 0) \text{ where } y < 0.
\end{align*}
\]

That these sets of points each constitutes a side of the plane is obvious. To show that there cannot exist another side of the plane with respect to the given line we note that two distinct sides of a plane with respect to a given line cannot have a non-empty intersection. This is obvious when we consider the definitions involved.

Hence let us suppose that the side of the plane in question is given by the set \( \{(x, y, 0) : y > 0\} \).
We may also suppose that the ray in question is the ray \((x, 0, 0): x > 0\) since the arguments will be symmetric for all four cases.

Now it is obvious that the plane \(\beta\) may be transformed by rigid motions into the plane \(z = 0\) and the ray \(r_1'\) into the ray \((x, 0, 0): x > 0\). Now if the image of \(r_2'\) is in the side \((x, y, 0): y > 0\) we can pause for a moment. If the image ray of \(r_2'\) is in the other side of the plane, the rigid motion

\[
\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{array} \right) \left( \begin{array}{c}
x \\
y \\
0 \\
\end{array} \right) = \left( \begin{array}{c}
x' \\
y' \\
0 \\
\end{array} \right)
\]

will take it into \((x, y, 0): y > 0\) and leave the other ray and the plane of the two rays unchanged. Hence the existence part of the theorem is proved since an angle is congruent to itself.

The uniqueness part is next proved. Suppose there is another distinct ray \(r_3\) from \((0, 0, 0)\) in the given side of the plane such that the two angles are congruent. Then there exists a point \((x, y, 0)\) where \(x^2 + y^2 = 1\) on this ray. It is obvious that any rigid motion which leaves the ray \((x, 0, 0): x > 0\) fixed must leave

\[
\sqrt{(x'-1)^2 + y'^2}
\]

fixed where \(x'\) and \(y'\) are the image points of \(x\) and \(y\).
and also we have $x'^2 + y'^2 = 1$. It is apparent that these two conditions completely determine the point $(x', y', 0)$ and further that $(x', y', 0) = (x, y, 0)$. This proves the theorem since the line of $r_3$ must be the line of $r_2$.

**Definition.** In a triangle ABC we define $\mathcal{A} \triangle ABC$ as the angle $\mathcal{A} \triangle r_1B r_2$ where $r_1$ contains A and $r_2$ contains B.

**Theorem 18.** If in \( \triangle ABC \) and \( \triangle A'B'C' \) we have
\[
\begin{align*}
\mathcal{A} \triangle ABC & \equiv \mathcal{A} \triangle A'B'C' \\
AB & \equiv A'B' \\
BC & \equiv B'C'
\end{align*}
\]
then 
\[
\mathcal{A} \triangle BAC \equiv \mathcal{A} \triangle B'A'C'.
\]

**Proof.** By the same series of transformations used in theorem 17, we can transform \( \mathcal{A} \triangle \)'s ABC and A'B'C' into the angle \( \mathcal{A} \triangle (r_1, P, r_2) \) where \( P = (0, 0, 0) \), \( r_1 = \{ (x, 0, 0) | x > 0 \} \) and \( r_2 \) is a ray from \( P \) in the plane \( z = 0 \). Under this set of transformations it is obvious that the image points of A and A', of B and B' will coincide. Hence \( \mathcal{A} \triangle BAC \) and \( \mathcal{A} \triangle B'A'C' \) coincide.
IV. PLAYFAIR'S AXIOM

Theorem 19. If in a plane $\alpha$ we have a line $l$ and a point $P$ not on $l$ then there exists a unique line containing $P$ such that it is in $\alpha$ and contains no point of $l$. (Playfair's Axiom)

Proof. Let $Ax + By + Cz + D = 0$ represent $\alpha$ and let

$$Ax + By + Cz + D = 0$$

$$A'x + B'y + C'z + D' = 0$$

represent $l$. Any line $l'$ which is in $\alpha$ can be represented by

$$Ax + By + Cz + D = 0$$

$$A''x + B''y + C''z + D'' = 0.$$ 

Now if $l'$ does not have a point in common with $l$, then

$$\begin{vmatrix}
A & B & C \\
A' & B' & C' \\
A'' & B'' & C''
\end{vmatrix} < 2.$$ 

Hence since

$$\begin{vmatrix}
A & B & C \\
A' & B' & C'
\end{vmatrix} = 2,$$

the rank of the first matrix must be exactly 2. Therefore (since

$$\begin{vmatrix}
A & B & C \\
A'' & B'' & C''
\end{vmatrix} = 2$$)

we have that $(A'', B'', C'')$ and

$(A', B', C')$ must be linearly dependent. Hence

$$(A'', B'', C'') = k(A', B', C')$$

where $k \neq 0$. Now if $l'$ is to contain $(x_0, y_0, z_0)$ then
\[ D'' = -A''x_o - B''y_o - C''z_o = -k(A'x_o + B'y_o + C'z_o). \]

Therefore a unique line \( l' \) exists which satisfies the stated conditions.
V. ARCHIMEDES' AXIOM

Definition. A set of points $B_1, B_2, \ldots$ on a line is **linearly ordered** iff $i < j < k$ is equivalent to $B_i B_j B_k$.

Theorem 20. For any segment $XY$ there exists a sequence $B_1, B_2, B_3, \ldots$ of points on a line $l$, such that the sequence is linearly ordered and $B_i B_{i+1} \equiv XY$ for all $i$, and such that for any point $Z$ on $l$ there exists a $B_n$ such that $B_1 Z B_n$.

Proof. For any line $l$ with parametric equations

$$x = x_0 + lt$$
$$y = y_0 + mt$$
$$z = z_0 + nt,$$

let $B_1 = (x_0, y_0, z_0)$. Then there exists a $t_1$ such that the points $(x_0 \pm lt_1, y_0 \pm mt_1, z_0 \pm nt_1)$ and $(x_0, y_0, z_0)$ are congruent to $X$ and $Y$. It follows that the sequences

$$+B_1 = x_0 \pm lit_1, y_0 \pm lit_1, z_0 \pm lit_1$$

are each linearly ordered. Now if the point $Z$ is $(x_z, y_z, z_z)$ then there exists at $Z$ such that
\[ x_z = x_o + lt_z \]
\[ y_z = y_o + mt_z \]
\[ z_z = y_o + nt_z \]

If \( t_z \geq 0 \) then the sequence \( \{ +B_i \} \) contains a point \( B_k = (x_o + k_1t, y_o + m_kt, z_o + n_kt) \) where \( k_1 > t_z \geq 0 \). If \( t_z < 0 \) a symmetric proof is required.
VI. THE COMPLETENESS AXIOM

The completeness axiom of geometry reads as follows:

If we have a system of points, lines and planes which satisfy the previous axioms, then it is impossible to add additional points to the set of points on a line and still have a system which satisfies the axioms.

It then follows as a theorem of geometry that the set of points on a line is the set of real numbers. Since the set of points on a line in the present model is the set of real numbers, it follows that we cannot add points to this set without contradicting the above theorem of geometry. Hence the completeness axiom is satisfied.
BIBLIOGRAPHY
