Title: SIMULATION OF RANDOM, NONLINEAR WAVE FORCES ON A CIRCULAR CYLINDRICAL PILE

Abstract approved: Robert Turner Hudspeth

Two types of linear random wave simulation methods are investigated; viz., (1) Deterministic Spectral Amplitude (DSA) and (2) Nondeterministic Spectral Amplitude (NSA). These linear simulations are extended to second-order in a perturbation expansion in the frequency domain by utilizing a Fast Fourier Transform (FFT).

The nonlinear interaction matrix for waves in shallow-, intermediate-, and deep-water is also investigated. In shallow water, the nonlinear interaction matrix is not as well-behaved as it is in intermediate- and deep-water. The nonlinear interaction matrix is symmetrical with respect to its frequency indices. The value of this nonlinear interaction matrix may, therefore, be calculated only from two octants in the $\omega_m-\omega_n$ plane.

Water particle kinematics are computed from the simulated sea surface by three different Transfer Functions; viz., (1) Linear Frequency Domain (LFD), (2) Hybrid Frequency Domain (HFD), (3) Nonlinear Frequency Domain (NFD). Wave pressure forces on a circular cylindrical pile are
computed at approximately the mid-depth location in the time domain by the Morrison equation using the Dean and Aagaard coefficients. The statistics of these wave pressure forces are derived by the moment generating function approach by assuming that the initial sea surfaces are linear and Gaussian.

The statistics of these simulated sea surfaces and wave pressure forces are compared with those recorded during Hurricane Carla (1961) in the Gulf of Mexico. In the sea surface simulation, the NSA method gives better agreement with the statistics from the measured sea surfaces than the DSA method does. However, there is no conclusion that can be drawn about the wave forces which are simulated from only one record (NPII No. 6887).
Simulation of Random, Nonlinear Wave Forces on a Circular Cylindrical Pile

by

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1.0 INTRODUCTION

Dynamic analyses of pile-supported structures become increasingly important to offshore designers as more and more structures are being constructed in areas which are subjected to hazardous environmental conditions. A more common condition in many cases is one in which the structures are sited near to or in the storm generating area. In this case, the design wave forces should be predicted from design waves which are characterized by both randomness and nonlinearities.

1.1 Previous Studies in Random Waves and Random-Wave Forces Simulation

Random sea waves. - Rudnick (1951) and Birkhoff and Kotik (1952) suggested that a record of a sea surface taken at a fixed point might be best represented as a stationary Gaussian random process. Rice (1944) earlier presented mathematical models for noise currents which were normally distributed. These models have since been used for modeling linear Gaussian random ocean wave processes. The first time series model for linear ocean waves was presented as a pseudo-integral by Pierson (1955).

Longuet-Higgins (1952) has shown that if the wave spectrum is narrow banded and if the phase angles of the individual wave components are random, then the wave heights will be Rayleigh distributed.
The probability density function for the maximum of sea surface elevations was then presented by Cartwright and Longuet-Higgins (1956).

Expressions for a random nonlinear velocity potential and sea surface elevations in deep water was presented in a Fourier-Stieltjes integral by Tick (1959) who later extended them to the finite depth water (Tick, 1963). The continuous second-order spectral components of the nonlinear sea surface were then computed.

A discrete representation for a nonlinear wave energy spectrum was presented by Hasselman (1962, 1963). The perturbation expansion for the wave energy was carried to the sixth order. Longuet-Higgins (1963) developed a perturbation solution assuming that the solution was the sum and the difference of the first order waves having different frequencies and wave numbers but independent phase angles. The solution was used to examine the third cumulant or the skewness coefficients of sea surface elevations. Furthermore, the probability density function for a weakly nonlinear process which was initially Gaussian was approximated by successive sums of a Gram-Charlier series.

Simulation techniques for these models were first introduced by Borgman (1969). For unidirectional linear Gaussian sea surface elevations, Borgman proposed the following two simulation methods:

1. Wave superposition in which the random sea surface elevations at a fixed location are represented by the sum of sinusoidal waves with deterministic amplitudes, periods, but nondeterministic phase angles.

2. Linear filters in which white noise processes are filtered by the inverse Fourier transform of the square root of
the theoretical spectrum.

Both techniques produce a mean zero Gaussian stochastic process which has the specified spectral density. These techniques are equivalent to the mathematical models for random noise currents proposed by Rice (1944).

Hudspeth (1974) extended the wave superposition method of simulation to second-order perturbation correction. A Fast Fourier Transform (FFT) algorithm was utilized to obtain the time series from the complex-valued random amplitude spectrum.

Directional random ocean waves at a specified location were simulated by Sharma (1979) for several widths of spreading. The effects of the directionality on the skewness and other statistics of the random ocean waves were also investigated.

**Wave forces.** - One of the practical applications for the simulation of a random ocean wave is to obtain random wave forces for the dynamic analyses of pile-supported structures. The Morison equation proposed by Morison, O'Brien, Johnson, and Schaff (1950) has been used extensively for calculating wave forces on a vertical pile. Ideally, the water particle kinematics which are required to calculate the wave forces according to the Morison equation are determined by measurement. However, the instrumentation to accurately measure water particle velocities and accelerations have only been available recently and probably will not be available for the design wave area.

Reid (1958) developed a method for predicting the water particle kinematics from measured sea surface elevations. This method convolves
a sea surface realization with filters derived from linear wave theory. Wheeler (1969) used this technique but added a stretched coordinate in order to compute the pressure force coefficients from measured hurricane realizations. These methods were exercised by Borgman (1969) to simulate linear wave forces, and by Hudspeth (1974) to obtain the water particle kinematics from random simulated sea surface elevations. Hudspeth compared statistically the simulated wave forces with forces recorded during Hurricane Carla and found good agreement.

The so-called hybrid method was introduced by Dean (1977) for computing design wave forces and moment loadings on offshore structures. This method is a combination of both linear and nonlinear wave theories to determine wave forces and includes the nonlinearities, randomness, and directionality. Wave forces due to nonlinear waves with energy present over a continuum of frequencies and directions are represented by the product of a nonlinear wave force (calculated from one of the nonlinear wave theories) and a linear pseudo-transfer function.

Sharma (1979) developed a computer algorithm for simulating directional random water particle kinematics from which directional wave forces were simulated through the Morison equation. Significant reductions, depending on the width of spreading, in the magnitude of the resultant of wave forces were observed.

1.2 Thesis Objectives

The essential features of the thesis objectives and the computational scheme are described in the flow chart in Fig. 1.1.
Figure 1.1. Flow chart for computer program for simulating random sea surfaces and random wave forces
Wave realization simulations. - The following two mathematical models (Rice, 1944) are employed to simulate unidirectional seas:

1. Deterministic spectral amplitudes (DSA),
2. Nondeterministic spectral amplitudes (NSA).

These two linear seas are extended to include the second-order nonlinearities and are used to compare measured forces on a vertical cylinder.

Wave kinematic realization simulations. - Random horizontal water particle kinematic realizations required to simulate random wave forces by the Morison equation are simulated by the following three methods:

1. Linear transfer functions derived from linear wave theory. Simulations by these transfer functions are termed:
   a. LFD (Linear Frequency Domain) if the transfer functions operate on simulated linear random ocean waves,
   b. HFD (Hybrid Frequency Domain) if the transfer functions operate on simulated ocean waves correct to second-order.

2. Nonlinear transfer functions derived from the random velocity potential correct to second-order perturbation expansion. Simulations by these transfer functions are termed NFD (Nonlinear Frequency Domain).

The simulation of both the random sea surface elevations and the water particle kinematics is performed in the frequency domain using a Fast Fourier Transform (FFT).

Wave force simulations. - The unidirectional random wave forces at a fixed location on a vertical pile are simulated by the Morison
equation. The simulations are performed in the time domain and the pressure force coefficients are assumed to be constant.

**Statistical properties.** - Theoretical energy spectral densities for random sea surface realizations correct to second-order in the perturbation expansion parameter are derived by approaches which are different from those presented by Tick (1959) and Hudspeth (1974).

Theoretical moment generating function, probability density function, and autocovariance function for random wave forces realizations are also computed.

**Chapter by Chapter preview.** - This thesis is organized as follows:

(1) A brief introduction to theory of random processes is given in Chapter 2.

(2) In Chapter 3, the mathematical formulations of the wave problem are developed. General solutions to the boundary value problems are presented in the form of the perturbation expansion correct to second-order. The properties of the nonlinear interaction matrix for unidirectional waves are investigated.

(3) Simulation techniques for random sea surface realizations are discussed and the theoretical energy spectral density for sea surface realizations correct to second-order in the perturbation expansion parameter is derived in Chapter 4.

(4) In Chapter 5, three types of simulations for horizontal water particle kinematics are presented.
(5) In Chapter 6, the theoretical moment generating function, probability density function, and autocovariance function for linear random wave force realizations are derived. The possibility to utilize the linearized drag wave forces proposed by Borgman (1965) for simulating the random wave force realizations in the frequency domain is discussed.

(6) In Chapter 7, applications and results of simulation methods are presented. Methods for generating both uniformly distributed and normally distributed random numbers are discussed.

(7) Summary and conclusion of the simulation methods are given in Chapter 8.

(8) In the Appendices, methods for determining the force coefficients in the Morison equation are discussed. Some of the lengthy and cumbersome mathematical derivations are also given.
2.0 THE THEORY OF RANDOM PROCESSES

The classical theory of probability treats only those random variables whose values do not depend on time or any other parameter. However, the development of physics has also made it necessary to treat random variables which do depend on groups of parameters, most often including time. Consequently, in the classical theory of probability, the results of a series of experiments conducted under a definite set of conditions may be represented by a sequence

\[ z_1(\xi), z_2(\xi), \ldots, z_n(\xi) \]

or just by

\[ z_1, z_2, \ldots, z_n \]

which do not depend on time. The independent variable, \( \xi \), is not necessarily a number, but is rather an individual element in the event space \( S \) (Borgman, 1973, pp. 1-19). Now, let the subscripts of the random sequence be replaced by ascending time sequences, \( t_1, t_2, \ldots, t_n \). If experiments are performed in the very small time interval, i.e., \( t_{m+1} - t_m = 0 \), then the random sequence may be represented by a continuous random function \( z_t(\xi) \), which does depend on time.

The family of random functions, \( z_t(\xi) \), one for each \( \xi \), is called a random or stochastic process. Therefore, if a series of observations of a random process are obtained, they will be similar in their statistics but dissimilar in their physical detail. This observation
corresponds to every possible outcome, $\xi$, and is referred to as a realization of the random process $z_t(\xi)$ or $z(t)$ (cf Fig. 2.1).

![Figure 2.1. Realizations of a random function.](image)

The randomness manifests itself by the fact that the detailed form of each realization, $z_t(\xi)$, differs from one observation to the next in a totally random manner. Their statistics, however, do not differ from one observation to the next.

### 2.3. Probability Distribution

It has been shown that a stochastic process, $z_t(\xi)$, may be regarded as the transition from a random sequence $z_1(\xi), z_2(\xi), \ldots, z_n(\xi)$ to a discrete random process. Consequently, the probability concepts which are developed for the sequence of random variables are also applicable for random processes. It is, therefore, necessary to
introduce some probability laws which may be used as mathematical models to make predictions concerning random variables.

The probability distribution function (PDF) for the random sequence \( z_1(\xi), z_2(\xi), \ldots, z_n(\xi) \) is defined (Papoulis, 1965, p. 233) by

\[
F(z_1, z_2, \ldots, z_n) = P[z_1(\xi) \leq z_1, z_2(\xi) \leq z_2, \ldots, z_n(\xi) \leq z_n]
\] (2.1.1)

in which \( z_1, z_2, \ldots, z_n \) are fixed values.

The PDF, \( F(\cdot) \), is a nondecreasing function in the variables \( z_1, z_2, \ldots, z_n \). The probability density function (pdf), \( p(\cdot) \), may be given a mass interpretation. The total mass in a region \( M \) of the space equals the probability that the point \( z_1(\xi), z_2(\xi), \ldots, z_n(\xi) \) will be in \( M \); i.e.,

\[
P[z_1(\xi), z_2(\xi), \ldots, z_n(\xi); \text{in } M] = \int \int \cdots \int_{M} p(z_1, z_2, \ldots, z_n) \, dz_1 \, dz_2 \cdots dz_n
\] (2.1.2)

The pdf, \( p(\cdot) \), may be calculated from the PDF, \( F(\cdot) \), by the following nth order derivative:

\[
p(z_1, z_2, \ldots, z_n) = \frac{\partial^n F(z_1, z_2, \ldots, z_n)}{\partial z_1 \partial z_2 \cdots \partial z_n}
\] (2.1.3)

If certain variables in the PDF, \( F(z_1, z_2, \ldots, z_n) \), are replaced by very large values (e.g., by infinity) then the joint PDF of the remaining variables is called a marginal probability distribution.
function and is given by, for \( m < n \)

\[
F(z_1, z_2, \ldots, z_m) = F(z_1, z_2, \ldots, z_m, \infty, \infty, \ldots) \tag{2.1.4}
\]

This is equivalent to integrating the pdf with respect to certain variables over the interval \((-\infty, \infty)\) and the remaining variables over interval \((-\infty, z_i)\); i.e.,

\[
F(z_1, z_2, \ldots, z_m) = \int \cdots \int p(z_1, z_2, \ldots, z_n) \cdot dz_1 dz_2 \ldots dz_n \tag{2.1.5}
\]

From Eq. (2.1.5) the marginal pdf, \( p(z_1, z_2, \ldots, z_m) \), may be obtained by using Eq. (2.1.3).

The joint pdf of \( m \) random variables,

\[
y_1(\xi) = g_1[z_1(\xi), z_2(\xi), \ldots, z_n(\xi)]
\]

\[
y_m(\xi) = g_m[z_1(\xi), z_2(\xi), \ldots, z_n(\xi)] \tag{2.1.6}
\]

for \( m < n \), may be obtained from the following transformation:

\[
p[y_1, y_2, \ldots, y_m] = \int \cdots \int \frac{p(z_1, z_2, \ldots, z_n)}{J(z_1, z_2, \ldots, z_m)} \cdot \cdot \cdot dz_{m+1} \ldots dz_n \tag{2.1.7}
\]

in which the Jacobian of transformation is
For \( m = n \), the pdf of random variables \( y_1(\xi), y_2(\xi), \ldots, y_n(\xi) \) may be determined by

\[
p(y_1, y_2, \ldots, y_m) = \frac{p(z_1, z_2, \ldots, z_m)}{J(z_1, z_2, \ldots, z_m)}
\]  

(2.1.9)

For \( m > n \), the random variables \( y_1(\xi), y_2(\xi), \ldots, y_m(\xi) \) are not independent (i.e., one may be expressed in terms of the other), and the Jacobian of transformation matrix given by Eq. (2.1.8) is identically zero. In this case, only the pdf of the combination of any \( n \) random variables \( y_1(\xi), y_2(\xi), \ldots, y_n(\xi) \) may be computed from Eq. (2.1.9). We note also that this is sufficient to determine all statistics for the \( n \) random variables.

From equations presented above, it may be concluded that theoretically a finite sequence of random variables \( z_1(\xi), z_2(\xi), \ldots, z_n(\xi) \) is statistically determined if their joint PDF is known.

If the random variables \( z_1(\xi), z_2(\xi), \ldots, z_n(\xi) \) are independent, their joint PDF may be computed from the PDF of each random variable according to the following equation:
and their joint pdf according to

\[ p(z_1, z_2, \ldots, z_n) = p(z_1)p(z_2) \ldots p(z_n) \]  \hspace{1cm} (2.1.11)

Another important property of independent random variables is that the sequence of functions of random variables \( g_1[z_1(\xi)], g_2[z_2(\xi)], \ldots, g_n[z_n(\xi)] \) is also independent (Papoulis, 1965, p. 240).

In the statistical theory of random processes, emphasis is placed not on each individual realization \( z_t(\xi) \), but rather on the statistics of the set of all realizations. The PDF which characterizes the statistical behavior of the random process, may be represented by the following \( n \) dimensional PDF:

\[ F[z(t_1), z(t_2), \ldots, z(t_n); t_1, t_2, \ldots, t_n] = P[z_{t_1}(\xi) \leq z(t_1), z_{t_2}(\xi) \leq z(t_2), \ldots, z_{t_n}(\xi) \leq z(t_n)] \]  \hspace{1cm} (2.1.12)

2.2 Mathematical Expectations; Moments and Characteristic Functions

The expected value of the function

\[ g[z_1(\xi), z_2(\xi), \ldots, z_n(\xi)] \]

of the \( n \) random variables \( z_1(\xi), z_2(\xi), \ldots, z_n(\xi) \) is defined by
\[ E\{g[z_1(\xi), z_2(\xi), \ldots, z_n(\xi)] = \]
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(z_1, z_2, \ldots, z_n)p(z_1, z_2, \ldots, z_n) \cdot dz_1 dz_2 \ldots dz_n \quad (2.2.1) \]

From Eq. (2.2.1), it may be concluded that the expectation, \( E \), is a linear operator which may be defined by the following:

\[ E(a_1 g_1 + a_2 g_2 + \ldots + a_n g_n) = \]
\[ a_1 E(g_1) + a_2 E(g_2) + \ldots + a_n E(g_n) \quad (2.2.2) \]
in which the \( a_n \)'s are deterministic constants.

If the random variables, \( z_1(\xi), z_2(\xi), \ldots, z_n(\xi), \) are independent, then the expected value of the product of the function, \( g[z(\xi)], \) may be obtained from the following:

\[ E\{g_1[z_1(\xi)] g_2[z_2(\xi)] \ldots g_n[z_n(\xi)]\} = \]
\[ E[g_1[z_1(\xi)] E[g_2[z_2(\xi)] \ldots E[g_n[z_n(\xi)]\}] \quad (2.2.3) \]

It can be shown from Eq. (2.2.1) that if the sequence, \( z_1(\xi), z_2(\xi) \ldots, z_m(\xi), \) is independent of the sequence, \( z_{m+1}(\xi), \ldots, z_n(\xi), \) then

\[ E\{g[z_1(\xi), z_2(\xi), \ldots, z_m(\xi)] h[z_{m+1}(\xi), \ldots, z_n(\xi)]\} = \]
\[ E\{g[z_1(\xi), z_2(\xi), \ldots, z_m(\xi)] \cdot E[h[z_{m+1}(\xi), \ldots, z_n(\xi)]\} \quad (2.2.4) \]
The statistical moment of order \( k \) for random variables, \( z_1(\xi), \ldots, z_n(\xi) \), may be defined by

\[
m_{\ell_1 \ell_2 \ldots \ell_n} = \mathbb{E}[z_1^{\ell_1}(\xi) z_2^{\ell_2}(\xi) \ldots z_n^{\ell_n}(\xi)]
\] (2.2.5)

in which \( k = \ell_1 + \ell_2 + \ldots + \ell_n \). This statistical moment may be obtained from the joint characteristic function defined by

\[
\psi(s_1, s_2, \ldots, s_n) = \mathbb{E}\{\exp i[s_1 z_1(\xi) + s_2 z_2(\xi) + \ldots + s_n z_n(\xi)]\}
\] (2.2.6)

Expanding the exponential term in a MacLaurin series yields the following relationship (Papoulis, 1965, p. 244):

\[
m_{\ell_1 \ell_2 \ldots \ell_n} = (-i)^k \frac{\partial^k \psi(s_1, s_2, \ldots, s_n)}{\partial s_1^{\ell_1} \partial s_2^{\ell_2} \cdots \partial s_n^{\ell_n}} \bigg|_{s_j = 0, j=1,2,\ldots,n}
\] (2.2.7)

The joint pdf may also be determined from the joint characteristic function by the following Fourier transform (Papoulis, 1965, p. 213):

\[
p(z_1, z_2, \ldots, z_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp -i(s_1 z_1 + s_2 z_2 + \ldots + s_n z_n) \psi(s_1, s_2, \ldots, s_n) \cdot ds_1 ds_2 \ldots ds_n
\] (2.2.8)
2.3 Mean, the Autocorrelation and Autocovariance Function

As shown in the preceding section, the multidimensional law of probability distribution is sufficient to characterize random processes. In practice, however, there is no specific way to find this distribution function. Fortunately for many practical purposes, the statistical moments such as the mean, the autocorrelation function, and sometimes higher moments such as the trivariance function are adequate to describe the characteristics of random processes. Therefore, it is necessary to introduce some of these moments here briefly.

The mean, \( \mu_n \), of a random variable, \( z_n(\xi) \), is the expected value of the random variable \( z_n(\xi) \); i.e.,

\[
\mu_n = \mathbb{E}[z_n(\xi)] = \int_{-\infty}^{\infty} z_n p(z_n)dz_n \ldots
\]

(2.3.1.a)

Since \( n \) is arbitrary, it may be replaced by time variable \( t \), then the Eq. (2.3.1.a) may be rewritten as the following equation

\[
\mu_t = \mu(t) = \mathbb{E}[z_t(\xi)] = \int_{-\infty}^{\infty} z_t p(z_t; t)dz_t
\]

(2.3.1.b)

Equation (2.3.1.b) gives the mean for a random process \( z_t(\xi) \). The mean \( \mu(t) \) is in general, a function of time, \( t \). The correlation, \( R_{jk} \), between two random variables \( z_j(\xi) \) and \( z_k(\xi) \) is the expected value of the product of random variables \( z_j(\xi) \) and \( z_k^*(\xi) \); i.e.,

\[
R_{jk} = \mathbb{E}[z_j(\xi)z_k^*(\xi)]
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_j z_k^* p(z_j, z_k)dz_j dz_k
\]

(2.3.2.a)
The covariance, \( C_{jk} \), between two random variables \( z_j(\xi) \) and \( z_k^*(\xi) \) is the expected value of the product of the zero-mean random variables \( (z_j(\xi) - \mu_j) \) and \( (z_k^*(\xi) - \mu_k^*) \); i.e.,

\[
C_{jk} = E\{[z_j(\xi) - \mu_j][z_k^*(\xi) - \mu_k^*]\}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [z_j(\xi) - \mu_j][z_k^*(\xi) - \mu_k^*] p(z_j, z_k) dz_j dz_k \quad (2.3.2.b)
\]

in which the star, *, denotes the complex conjugate. The subscripts \( j \) and \( k \), may again be replaced by time \( t_1 \) and \( t_2 \). The correlation and the covariance functions for the same variables are often called the autocorrelation and the autocovariance respectively.

These functions may be expressed by the following equations:

\[
R(t_1, t_2) = E[z(t_1)z^*(t_2)]
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(t_1)z^*(t_2) p(z(t_1), z(t_2); t_1, t_2). dz(t_1)dz(t_2) \quad (2.3.3.a)
\]

\[
C(t_1, t_2) = E\{[z(t_1) - \mu(t_1)][z^*(t_2) - \mu^*(t_2)]\}
\]

\[
= \int_{-\infty}^{\infty} [z(t_1) - \mu(t_1)][z^*(t_2) - \mu^*(t_2)]. p(z(t_1), z(t_2); t_1, t_2) dz(t_1)dz(t_2) \quad (2.3.3.b)
\]

From Eq. (2.3.3.a), the autocovariance function, \( C(t_1, t_2) \), may be expressed in terms of the autocorrelation function; i.e.,
\[ C(t_1, t_2) = R(t_1, t_2) - \mu(t_1)\mu^*(t_2) \] (2.3.4)

For zero mean process the autocorrelation and the autocovariance are equivalent.

2.4 The Stationarity and Ergodicity of Random Processes

The special class of stationary random processes plays an important role in the theory of random processes. Random processes whose \( n \)-dimensional probability density functions (pdf) are invariant with respect to time, \( t \); i.e.,

\[ p[z(t_1), z(t_2), ..., z(t_n); t_1, t_2, ..., t_n] = \]

\[ p[z(t_1+\tau), z(t_2+\tau), ..., z(t_3+\tau); t_1+\tau, t_2+\tau, ...
\]

\[ t_n+\tau] \] (2.4.1)

are called stationary random process in strict sense (Papoulis, 1965, p. 301). Hence, we may obtain the following relationship:

\[ p(z(t); t) = p[z(t+\tau); t+\tau] \] (2.4.2)

Since this is true for every \( \tau \), it may be concluded that the pdf, \( p[z(t)] \), is independent of time, \( t \). A consequence of the above is that the mean, \( \mu(t) \), becomes

\[ E[z(t)] = \mu = \text{constant} \] (2.4.3)

The two-dimensional pdf may, after some thought, be represented by
in which \( \tau = t_2 - t_1 \), thus the pdf, \( p[z(t_1), z(t_2); \tau] \) is the joint pdf of random variables, \( z(t) \) and \( z(t+\tau) \), in which \( t \) is fixed.

Substituting Eq. (2.4.4) into Eqs. (2.3.2), it may be shown that the autocorrelation, \( R(t_1, t_2) \), is dependent only on the lag, \( \tau \). It may, therefore, be written in the form

\[
R(\tau) = E[(z(t)z^*(t+\tau)]
\]

\[
= E[z(t-\tau)z^*(t)] = R^*(-\tau)
\]

and the autocovariance

\[
C(\tau) = R(\tau) - \mu^2 = C^*(-\tau)
\]

It can be shown (Papoulis, 1965, p. 337) that

\[
|R(\tau)| \leq R(0)
\]

in which (from Eq. (2.4.5.a)) \( R(0) = E[|z(t)|^2] \geq 0 \).

The definition of stationarity given by Eq. (2.4.1) imposes a severe restriction that all moments be time invariant, a restriction which is usually not met in geophysical problems. Consequently, the correlation theory is limited in particular to only two-dimensional probability. Conditions, in which the ensemble mean is a constant (i.e., independent of time, \( t \)) and the autocorrelation function depends only on temporal lag, \( \tau \), are sufficient to define processes to be wide sense stationary. Strictly stationary random processes which satisfy Eq. (2.4.1) are also stationary in wide sense; the converse, however,
is not true. One special class of random processes (viz., Gaussian random processes), has statistics which are completely determined by the mean, \( \mu \), and the autocorrelation function, \( R(\tau) \) (Yaglom, 1962, pp. 14-16). Consequently, the concepts of stationarity in strict and in wide sense for Gaussian processes are identical.

Stationary processes also have a remarkable characteristic which is called ergodicity. Before further discussion about the properties of this characteristic, consider two different ways to average a stochastic process, \( z_t(\xi) \).

1. Ensemble average: for fixed \( t = t_1 \) the average over space \( S \), in which \( \xi \) is the member of \( S \), is given by

\[
E\{g[z_{t_1}(\xi)]\} \tag{2.4.6.a}
\]

2. Time average: for fixed \( \xi = \xi_1 \), the average over \( T \), in which \( t \) is the member of set \( T \); e.g., if \( T = [t_0, t_0 + T] \), is given by

\[
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0 + T} g[z_t(\xi_1)]dt \tag{2.4.6.b}
\]

The time average is frequently used to define the mean and the autocorrelation of a process \( z(t) \); e.g.,

\[
\bar{\mu} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} z(t)dt \tag{2.4.7.a}
\]

\[
\bar{R}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} z(t)z^*(t + \tau)dt \tag{2.4.7.b}
\]
However, they are obviously not constant, but are random variables. In order to equate \( \bar{\mu} \) and \( \bar{R}(\tau) \) to the ensemble average \( E[z(t)] \) and \( E[z(t)z^*(t+\tau)] \), respectively, the following expression must be proved; i.e.,

\[
E(\bar{\mu}) = E[z(t)] = \mu; \quad \sigma^2_{\mu} = 0 \quad (2.4.8a)
\]

\[
E[\bar{R}(\tau)] = E[z(t)z^*(t+\tau)] = R(\tau); \quad \sigma^2_R = 0 \quad (2.4.8b)
\]

in which \( \sigma^2 \) refers to the variance.

Fortunately, Eqs. (2.4.8.a,b) are true for many physical processes so that dealing with only a single realization will be sufficient to determine all statistics for second-order processes. A necessary condition that the random process \( z(t) \) be weakly ergodic is that \( z(t) \) is ergodic with respect to its mean and autocorrelation function (Bendat and Piersol, 1971, pp. 86-90) defined by Eqs. (2.4.8.a,b). A sufficient condition for the random process \( z(t) \) to be weakly ergodic is that it be weakly stationary (wide sense stationary) and that the following relationships be true (Papoulis, 1965, p. 328):

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} z(t) \, dt = E[z(t)] = \mu = \text{constant} \quad (2.4.9.a)
\]

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} R(\tau) \, d\tau = 0 \quad (2.4.9.b)
\]

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} z(t)z^*(t+\tau) \, dt = E[z(t)z^*(t+\tau)] = R(\tau) \quad (2.4.9.c)
\]
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (1 - \frac{T}{T_0} [R_{\phi}(\tau) - \mu^2]) d\tau = 0 \quad (2.4.9.d)
\]

in which \( \phi(t) = z(t)z^*(t+\tau) \).

Equation (2.4.9.d) demonstrates that it is necessary to compute the fourth-order moment in order to test the ergodicity of the autocorrelation function which is a requirement for a weakly ergodic process (Papoulis, 1965, p. 330).

### 2.5 Harmonic Analysis of Random Processes

If a random process \( z(t) \) is periodic in a mean-square sense, its autocorrelation \( R(\tau) \), is periodic; i.e., \( R(\tau+T) = R(\tau) \). This periodicity implies that it may be expanded in a Fourier series (Papoulis, 1965, p. 367) according to

\[
z(t) = \sum_{n=\infty}^{\infty} F(n) \exp(i\Delta \omega t) \quad (2.5.1)
\]

in which the fundamental frequency \( \Delta \omega \) is defined by \( \Delta \omega = \frac{2\pi}{T} \) (Wiley, 1975, p. 242). The complex-valued spectral amplitudes, \( F(n) \), are given by the Fourier transform,

\[
F(n) = \frac{1}{T} \int_{0}^{T} z(t) \exp(-i\Delta \omega t) dt \quad (2.5.2)
\]

and are orthogonal random variables such that (Papoulis, 1965, p. 461)

\[
E[z(t)] = E[F(n)] \delta_{n,0} = \mu \quad (2.5.3.a)
\]

\[
E[z(t)z^*(t)] = E[F(n)F^*(m)] = \sigma^2 \delta_{n,m} \quad (2.5.3.b)
\]
in which \( \delta_{n,m} \) is the Kronecker delta defined by the following expression:

\[
\delta_{n,m} = \begin{cases} 
1 & n = m \\ 
0 & n \neq m 
\end{cases} 
\tag{2.5.4}
\]

If the autocorrelation, \( R(\tau) \), is not periodic then the random complex-valued amplitudes, \( F(n) \), defined by Eq. (2.5.2) are no longer orthogonal, and the series expansion according to Eq. (2.5.1) is not true for every \( t \). However, it may be shown (Papoulis, 1965) that it will be true for \( |t| < \frac{T}{2} \).

If a random process \( z(t) \) can be expanded into a series given by Eq. (2.5.1), then the autocorrelation function, \( R(\tau) \), may be written in the form

\[
R(\tau) = \sum_{n=\infty}^{\infty} \sigma_n^2 \exp in\Delta \omega \tau 
\tag{2.5.5}
\]

The variance of the random process \( z(t) \) is obtained when \( \tau = 0 \); i.e.,

\[
R(0) = \sigma_z^2 = \sum_{n=\infty}^{\infty} \sigma_n^2 
\tag{2.5.6}
\]

The Fourier transform of Eq. (2.5.5) will give the variance distribution over the frequency, i.e.,

\[
S(n\Delta \omega) = \sigma_n^2 
\tag{2.5.7}
\]
The representation of \( S(\omega) \) for expressing the distribution density of variance or power of a random process \( z(t) \) over frequency and its relationship with the autocorrelation function, \( R(\tau) \), is given by the following Wiener-Khinchine's relationship:

\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) \exp(-i\omega \tau) d\tau \quad (2.5.8.a)
\]

\[
R(\tau) = \int_{-\infty}^{\infty} S(\omega) \exp(i\omega \tau) d\omega \quad (2.5.8.b)
\]

in which the discrete power spectrum \( S(n\Delta \omega) \) given by Eq. (2.5.7) may be related to the power density \( S(\omega) \) by the following:

\[
S(n\Delta \omega) = S(\omega_n) \Delta \omega
\]

in which \( S(\omega_n) \) = the value of power density at frequency \( \omega_n = n\Delta \omega \)

It is necessary to note that the power density \( S(\omega) \) of a stationary process is not directly related to the harmonic analysis, but rather is defined as the Fourier transform of the autocorrelation function, \( R(\tau) \), [cf. Eq. (2.5.8.a,b)].

As stated earlier, if a stationary process \( z(t) \) is not periodic in mean square sense, then it is not possible to expand it according to Eq. (2.5.1). However, the approximation may be obtained by applying a data window on the process, i.e.,

\[
\hat{z}(t) = \sum_{n=-\infty}^{\infty} F(n) \exp(in\Delta \omega t) \quad (2.5.9.a)
\]

\[
F(n) = \int_{-\infty}^{\infty} z(t) h(t) \exp(-in\Delta \omega t) dt \quad (2.5.9.b)
\]
or in discrete form

\[ F(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=-N/2}^{N/2} Z(m \Delta t) h(m \Delta t) \exp(-inm/N) \]  

(2.5.9.c)

in which \( h(t) \) is given by

\[
h(t) = \begin{cases} 
1 & \text{for } |t| < \frac{T}{2} \\
0 & \text{otherwise}
\end{cases}
\]

(2.5.10.a)

or

\[
h(t) = \frac{\sin(\Delta \omega t/2)}{\pi t} \text{ for } |t| < \infty
\]

(2.5.10.b)

Data windows given by Eq. (2.5.10.a) are equivalent to filtering the power density spectrum \( S(\omega) \) by

\[
T_n(\omega) = \frac{\sin \frac{\omega T}{2}}{\omega / 2}
\]

(2.5.11.a)

while for Eq. (2.5.10.b) by

\[
T_n(\omega) = \begin{cases} 
1 & \text{for } (n-\frac{1}{2}) \Delta \omega < \omega < (n+\frac{1}{2}) \Delta \omega \\
0 & \text{otherwise}
\end{cases}
\]

(2.4.11.b)

For more about data windows, reference is made to (Brigham, 1971), (Jenkins and Watts, 1968, pp. 48-53), and (Papoulis, 1965).
3.0 MATHEMATICAL EQUATIONS AND FORMULATION OF A RANDOM SURFACE GRAVITY WAVE PROBLEM

The fluid motion is defined with respect to three axes in Cartesian coordinates. Any point is defined by coordinates $x, y, z$. The $x, y$ axes lie in a horizontal plane at the still water level, with the $z$ axis positive upwards (cf., Fig. 3.1.1). For notational compactness, the horizontal plane will be denoted by

$$\mathbf{s} = s = x\mathbf{i}_x + y\mathbf{i}_y$$

in which $\mathbf{i}_s = \text{unit vector in horizontal } s \text{ direction}$

$\mathbf{i}_x = \text{unit vector in } x \text{ direction};$ and

$\mathbf{i}_y = \text{unit vector in } y \text{ direction}$

3.1 Formulation of Boundary Value Problem

The fluid is assumed to be inviscid and incompressible. The fluid motion is assumed to be irrotational. A random velocity potential, $\phi$, is defined by

$$\mathbf{u}(x, y, z; t) = -\nabla \phi$$

The governing partial differential field equation for the random velocity potential, $\phi$, is the equation of continuity for an irrotational fluid flow;

$$\nabla^2 \phi = 0; \quad |\mathbf{s}| < \infty, \quad -h \leq z \leq \eta(\mathbf{s}, t), \quad t \geq 0$$
Figure 3.1.1. Definition sketch for wave terminology.
in which the three-dimensional gradient operator $\nabla(\cdot)$, is defined by

$$\nabla(\cdot) = \frac{\partial (\cdot)}{\partial x} \hat{x} + \frac{\partial (\cdot)}{\partial y} \hat{y} + \frac{\partial (\cdot)}{\partial z} \hat{z}$$

(3.1.3)

and $\eta(s,t) = \text{the water surface elevation at position, } s, \text{ and time, } t.$

The finite depth fluid domain is assumed to be unbounded in the horizontal dimensions so that the boundary conditions required at the vertical boundaries located at infinity are those which preclude partial standing waves. The boundary conditions required to quantify either the unknown multiplicative coefficients or the eigenvalues are prescribed at the free surface and horizontal bottom boundaries. The fluid is assumed to occupy a finite depth, $h$, in the lower half plane and to satisfy the following bottom boundary condition (BBC) requiring no flow across the impermeable bottom:

$$\text{BBC : } \frac{\partial \phi}{\partial z} = 0 ; \quad z = -h, \quad |\hat{s}| < \infty, \quad t \geq 0$$

(3.1.4)

The fluid free surface, $\eta(s,t)$, is unconstrained and a kinematic free surface boundary condition (KFSBC) is required to prescribe mathematically the continuity of fluid motion at the interface and to ensure physically that no fluid particles are convected across the interface;

$$\text{KFSBC : } \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial s} \frac{\partial \eta}{\partial s} + \frac{\partial \phi}{\partial z} = 0 ; \quad z = \eta(s,t), \quad |\hat{s}| < \infty, \quad t \geq 0$$

(3.1.5)
in which the coordinate, $s$, is given by Eq. (3.1).

An additional dynamic free surface boundary constraint is required to ensure that all stresses acting along the free surface are continuous. In the absence of surface tension and for an atmospheric pressure assumed to be zero, the dynamic free surface boundary condition (CFSBC) may be obtained from the Bernoulli equation; i.e.,

$$\frac{\partial \Phi}{\partial t} + g \frac{\partial \eta}{\partial t} + \frac{1}{2} \left| \nabla \Phi \right|^2 = Q(t); \quad z = \eta(s, t),
$$

$$\left| \frac{\partial}{\partial s} \right| < \infty, \quad t \geq 0 \quad (3.1.6)$$

The total material derivative of Eq. (3.1.6) combined with KFSBC yields the following combined free surface boundary condition (CFSBC):

$$\left( \frac{\partial - \nabla \cdot \nabla}{\partial t} \right) \left( -\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left| \nabla \Phi \right|^2 - Q(t) \right) - g \frac{\partial \Phi}{\partial z} = 0$$

which may be expanded to the following form:

$$\text{CFSBC} : \frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \frac{\partial Q}{\partial t} = \frac{3}{2} \left| \nabla \Phi \right|^2 - \frac{1}{2} \nabla \cdot \nabla \left| \nabla \Phi \right|^2 ;$$

$$z = \eta(s, t), \quad \left| \frac{\partial}{\partial s} \right| < \infty, \quad t \geq 0 \quad (3.1.7)$$

With given initial conditions, these equations suffice to determine the wave motion. The free surface boundary conditions are nonlinear and, more seriously, they are specified on the unknown free surface $\eta(s, t)$. Some approximation techniques are usually required to solve these equations.
3.2 Solution to Boundary Value Problem
(Perturbation Method)

In order to avoid evaluating the CFSBC at the unknown free surface, \( \eta(s, t) \), and in order to utilize a separable coordinate system to solve the governing field equation [Eq. (3.1.1)], the CFSBC is expanded in a MacLaurin series about the still water level, \( z = 0 \); i.e.,

\[
\text{CFSBC: } \sum_{n=0}^{\infty} \frac{n^n}{n!} \frac{3^n}{3z^n} \left( \frac{2\phi}{\partial t^2} + g \frac{3\phi}{3t} + \frac{3\phi}{3t} \right) = \sum_{n=0}^{\infty} \frac{n^n}{n!} \frac{3^n}{3z^n} \left( \frac{3}{3t} \left| \vec{\psi} \right|^2 - \frac{1}{2} \vec{\nabla} \vec{\phi} \cdot \vec{\nabla} \vec{\phi} \right),
\]

\[ z = 0, \quad |s| < \infty, \quad t \geq 0 \quad (3.2.1) \]

In order to obtain an approximate solution to Eq. (3.2.1) subject to the nonlinear CFSBC, the nonlinearities are assumed to impose small perturbations on an initial linear approximation (Van Dyke, 1975, pp. 45-76). The measure of the perturbation is assumed to be proportional to wave slope and will be used as the measure of the perturbation ordering parameter.

The random velocity potential, random sea surface realization, and the Bernoulli constant are assumed to be expandable in the following power series with the perturbation ordering parameter implicitly absorbed into the functional notation:

\[
\phi(s, z, t) = \sum_{n=1}^{\infty} \phi^{(n)}(s, z, t) \quad (3.2.2.a)
\]
Substituting Eqs. (3.2.2.a,b,c) into the equations for the boundary value problem and equating equal orders of \( n \) leads to a set of linear boundary value problems.

3.2.a First-order Perturbation Solution

The first-order solution is obtained by solving the linear boundary value problem for \( n = 1 \); i.e.,

\[
\nabla^2 \phi^{(1)} = 0; \quad |\vec{s}| < \infty, \quad -h < z < 0, \quad t > 0 \tag{3.2.a.1.a}
\]

\[
\frac{\partial \phi^{(1)}}{\partial z} = 0; \quad |\vec{s}| < \infty, \quad z = -h, \quad t > 0 \tag{3.2.a.1.b}
\]

\[
\frac{\partial^2 \phi^{(1)}}{\partial t^2} + g \frac{\partial \phi^{(1)}}{\partial z} + \frac{\partial Q^{(1)}}{\partial t} = 0; \quad |\vec{s}| < \infty, \quad z = 0, \quad t > 0 \tag{3.2.a.1.c}
\]

\[
\eta^{(1)} = \frac{1}{g} \left( \frac{\partial \phi^{(1)}}{\partial t} + Q^{(1)} \right); \quad |\vec{s}| < \infty, \quad z = 0, \quad t > 0 \tag{3.2.a.1.d}
\]

A solution which satisfies exactly Eqs. (3.2.a.1.a,b) is given by the following series:

\[
\phi^{(1)}(\vec{s}, z, t) = -i \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{g}{\omega_m} \cosh[k_n(h+z)] \frac{\cosh(k_nh)}{\cosh(k_nh)} \cdot \vec{f}(\vec{k}_n, \omega_m) \exp(-i(k_n \cdot \vec{s} - \omega_m t)) \tag{3.2.a.2}
\]
in which $\vec{F}(\vec{k}_n, \omega_m)$ is the complex-valued random amplitude of the velocity potential, and $\vec{k}_n$ is the directional wave number which can be expressed by the following:

$$\vec{k}_n = k_n \hat{x} + k_n \hat{y}$$

$$k_n = |\vec{k}_n|$$  \hspace{1cm} (3.2.a.3)

Absorbing the Bernoulli constant, $Q^{(1)}$, in Eq. (3.2.a.1.d) into the value of $\frac{\partial \phi^{(1)}}{\partial t}$ by requiring that the random sea surface realization maintain a zero mean, yields the following expression for the random sea surface realization:

$$\eta^{(1)}(\vec{s}, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \vec{F}(\vec{k}_n, \omega_m) \exp(-i(\vec{k}_n \cdot \vec{s} - \omega_m t))$$

(3.2.a.4)

Substituting Eq. (3.2.a.2) into Eq. (3.2.a.1.c), and assuming that the temporal derivative of the Bernoulli constant, $\frac{\partial Q^{(1)}}{\partial t}$, may be included in the value of $\frac{\partial^2 \phi^{(1)}}{\partial t^2}$, we obtain

$$i \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [\omega_m^2 - g k_n \tanh(k_n h)] \vec{F}(\vec{k}_n, \omega_m) \cdot \exp(-i(\vec{k}_n \cdot \vec{s} - \omega_m t)) = 0$$

(3.2.a.5)

which implies the following:

$$[\omega_m^2 - g k_n \tanh(k_n h)] \cdot \vec{F}(\vec{k}_n, \omega_m) = 0$$

(3.2.a.6)
Equation (3.2.a.b) is satisfied either by the trivial solution

\[ \tilde{F}(\vec{k}_n, \omega_m) = 0 \]  \hspace{1cm} (3.2.a.7)

or by the zeroes of

\[ \omega_m - g k_n \tanh(k_n h) = 0 \]  \hspace{1cm} (3.2.a.8)

Equation (3.2.a.6) requires that the complex-valued random amplitude, \( \tilde{F}(\vec{k}_n, \omega_m) \), vanish except when \( m = n \), in which case Eq. (3.2.a.8) then must be satisfied. This restriction on the amplitude, \( \tilde{F}(\vec{k}_n, \omega_m) \), may be given by the following:

\[ F(\omega_n) = F(n) = \tilde{F}(\vec{k}_n, \omega_m) \delta_{n,m} \]  \hspace{1cm} (3.2.a.9)

in which the Kronecker delta, \( \delta_{m,n} \), is defined by Eq. (2.5.4). As a consequence of the Kronecker delta function, the subscript, \( m \), in Eq. (3.2.a.8) may be replaced by \( n \); i.e.,

\[ \omega_n^2 = g k_n \tanh(k_n h) \]  \hspace{1cm} (3.2.a.10)

Substituting Eq. (3.2.a.9) into the general solution given by Eq. (3.2.a.2), the following random scalar velocity potential is obtained:

\[ \phi^{(1)}(\vec{s}, z, t) = -i \sum_{n=-\infty}^{\infty} \frac{g}{\omega_n} \frac{\cosh[k_n (h+z)]}{\cosh(k_n h)} \cdot \tilde{F}(n) \exp(-i(k_n \cdot \vec{s} - \omega_n t)) \]  \hspace{1cm} (3.2.a.11)

in which the reality of the random velocity potential requires the following complex conjugate relationship:
The random sea surface realization is determined from Eq. (3.2.a.4), i.e.,

\[ \eta_1(\hat{s}, t) = \sum_{n=-\infty}^{\infty} f(n) \exp -i(k_n \cdot \hat{s} - \omega_n t) \]  

(3.2.a.13)

3.2.b Second-order Perturbation Solution

The functional form of the second-order solution is obtained by solving the linear boundary value problem in \( \phi^{(2)} \) which is given by Eq. (3.1.6), (3.1.7) and (3.2.2.a,b,c) for \( n = 2 \); viz.,

\[ \nabla^2 \phi^{(2)} = 0; \quad |\hat{s}| < \infty, -h \leq z \leq 0, \quad t \geq 0 \]  

(3.2.b.1.a)

\[ \frac{\partial \phi^{(2)}}{\partial z} = 0; \quad |\hat{s}| < \infty, z = -h, \quad t \geq 0 \]  

(3.2.b.1.b)

\[ \frac{\partial^2 \phi^{(2)}}{\partial t^2} + g \frac{\partial \phi^{(2)}}{\partial z} + \frac{\partial \phi^{(1)}}{\partial t} = \tilde{a} \left[ \tilde{b} \right] \]  

(3.2.b.1.c)

The general solution for \( \phi^{(2)} \) is assumed to be similar to the first order solution given by Eq. (3.2.a.2). Denoting the frequency, \( \Omega_k \), and the eigenvalues, \( k_\ell \), in a general form which are to be evaluated through the boundary conditions, a solution which satisfies exactly
Eqs. (3.2.b.1.a,b) is given by the following:

\[
\phi^{(2)}(\mathbf{s}, z, t) = -i \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\cosh[\kappa (h+z) \ell]}{\cosh(\kappa h)} \cdot B(\mathbf{s}, \Omega_{\ell}) \exp -i(\frac{k}{\kappa} \cdot \mathbf{s} - \Omega_{\ell} t)
\]

Substituting Eq. (3.2.b.2) into Eq. (3.2.b.1.c), requiring that the random sea surface relation maintain a zero mean leads to the following equation:

\[
i \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left[ \Omega_{\ell}^2 - g \kappa \tanh(\kappa h) \right] B(\mathbf{s}, \Omega_{\ell}) \exp -i(\frac{k}{\kappa} \cdot \mathbf{s} - \Omega_{\ell} t)
\]

\[
= i \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{2 \omega_n \omega_m} \left[ 2(\omega_n + \omega_m) (g k_{2n} \cdot k_{2m} -
- \omega_n^2 - \omega_m^2 + \omega_n \omega_m (\omega_n^2 + \omega_m^2) + g^2 (k_{2n}^2 \omega_m^2 + k_{2m}^2 \omega_n^2)) \right] -
\cdot F(n) F(m) \exp -i[(\frac{k_{2n}}{\kappa} + \frac{k_{2m}}{\kappa}) \cdot \mathbf{s} - (\omega_n + \omega_m) t]
\]

The eigenvalue, \(\kappa\), and frequency, \(\Omega\), may now be quantified by requiring equivalence of the temporal and spatial functions in Eq. (3.2.b.3). This equivalence results in the following relationship:

\[
\kappa = k_n + k_m
\]

\[
\Omega = \omega_n + \omega_m
\]

and

\[
B(\mathbf{s}, \Omega) = D(\omega_n, \omega_m; \kappa, \Omega) F(n) F(m)
\]
in which the nonlinear interaction matrix

\[
D(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) = \\
\frac{2(\omega_n + \omega_m)(g^2 \vec{k}_n \cdot \vec{k}_m - \omega_n^2 \omega_m^2 - \omega_n \omega_m (\omega_n^2 + \omega_m^2) + g^2 (\omega_n^2 + \omega_m^2))}{2\omega_n \omega_m [(\omega_n + \omega_m)^2 - g |\vec{k}_n + \vec{k}_m|^2 \tanh(|\vec{k}_n + \vec{k}_m|h)]}
\]

(3.2.b.6)

The wave number, \( \vec{k}_n \), and frequency, \( \omega_n \), satisfy Eq. (3.2.a.10). Equation (3.2.b.2) may now be written by the following:

\[
\phi^{(2)}(s, z, t) = -i \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\cosh[|\vec{k}_n + \vec{k}_m|(h+z)]}{\cosh[|\vec{k}_n + \vec{k}_m|h]} \\
\cdot D(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) F(m) F(n) \exp[-i((\vec{k}_n + \vec{k}_m) \cdot \hat{s} - (\omega_n + \omega_m)t)]
\]

(3.2.b.7)

Substituting the appropriate derivatives into Eq. (3.2.b.1.c), and requiring that the random sea surface relation maintain a zero mean yields the following expression for the second-order correction for water surface elevation:

\[
\eta^{(2)}(s, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F(n) F(m) H(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) \\
\cdot \exp[-i((\vec{k}_n + \vec{k}_m) \cdot \hat{s} - (\omega_n + \omega_m)t)]
\]

(3.2.b.8)

in which the nonlinear interaction matrix, \( H(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) \), is given by

\[
H(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) = \frac{1}{2g} \frac{[2(\omega_n + \omega_m) \cdot D(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m)]}{g^2 \frac{\vec{k}_n \cdot \vec{k}_m}{\omega_n \omega_m} + \omega_n + \omega_m^2}
\]

(3.2.b.9)
The important point to note about Eq. (3.2.b.7) and Eq. (3.2.b.8) is that for a fixed set of horizontal spatial coordinates, the second-order contributions occur at the same discrete frequencies as those of the linear superposition realization. This permits the application of the finite Fourier transform (FFT) algorithm to compute the nonlinear random realization.

3.3 Nonlinear Interaction Matrix

A second-order nonlinear solution for both the velocity potential, \( \phi(s,z,t) \), and the water surface elevation, \( \eta(s,t) \), may be expressed in terms of the first-order frequencies, \( \omega \), and wave numbers, \( k \), which are determined from the frequency dispersion relationship given by Eq. (3.2.a.10)

\[
\omega_n^2 = g k_n \tanh k_n h
\]  

(3.2.a.10)

in which \( \omega_n = -\omega_n \) and \( k_n = -k_n \).

The properties of the nonlinear interaction matrix for the velocity potential may be determined from Eq. (3.2.b.6). For waves moving in the same direction (colinear), Eq. (3.2.b.6) becomes

\[
D(\omega_n, \omega_m; k_n, k_m) = \frac{2(\omega_n + \omega_m)(g^2 k_n^2 k_m^2 - \omega_n^2 \omega_m^2 - \omega_n \omega_m (\omega_n^3 + \omega_m^3) + g^2 (k_n^2 + k_m^2) \Delta \omega_n \Delta \omega_m)}{2 \omega_n \omega_m ((\omega_n + \omega_m)^2 - g(k_n + k_m) \tanh[(k_n + k_m)h])}
\]  

(3.3.1)
If \( \omega_n \) and \( \omega_m \) are taken as horizontal and vertical axes of plane Cartesian coordinates, it is convenient to divide this plane into 45° sectors and to label them \( R_1 \) through \( R_8 \) (cf., Fig. 3.3.1).

It is easily seen from Eq. (3.3.1) that the nonlinear interaction matrix is symmetrical with respect to the indices \( n \) and \( m \); i.e.,

\[
D(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) = D(\omega_m, \omega_n; \vec{k}_m, \vec{k}_n)
\]

(3.3.2.a)

and anti symmetrical with respect to indices \( -n \) and \( m \); i.e.

\[
D(\omega_{-n}, \omega_m; \vec{k}_{-n}, \vec{k}_m) = -D(\omega_m, \omega_{-n}; \vec{k}_m, \vec{k}_{-n})
\]

(3.3.2.b)

From this fact, it is necessary only to calculate the value of \( D \) on the regions \( R_1, R_8 \) which represent the value of \( D \) on regions \( R_1, R_2, R_5, R_6 \) and \( R_3, R_4, R_7, R_8 \), respectively.
Let us next examine the behavior of this nonlinear interaction matrix in shallow-, intermediate-, and deep-water. Definitions for shallow-, intermediate-, and deep-water are given according to the magnitude of the ratio of water depth to wave length, i.e.,

\[ \frac{h}{L_n} \geq \frac{1}{2} \] deep water

\[ \frac{1}{25} < \frac{h}{L_n} < \frac{1}{2} \] intermediate \hspace{1cm} (3.3.3)

\[ \frac{h}{L_n} \leq \frac{1}{25} \] shallow water

in which \( L_n \) = wave length for the
\( T_n \) = wave period

**Shallow water**

The frequency dispersion relationship becomes

\[ \frac{2}{\omega_n^2} = g k_n^2 h \]

\[ k_n = \frac{\omega_n}{\sqrt{gh}} \] \hspace{1cm} (3.3.4)

Substituting these approximation values into Eq. (3.2.6), noting that for small values of \( k_n h, k_m h \)

\[ \tanh[(k_n + k_m)h] = (k_n + k_m)h[1 - (k_n h)(k_m h) + \]

\[ O((k_n h)^2(k_m h)^2)] \] \hspace{1cm} (3.3.5)

gives the following:

\[ \delta(\omega_n, \omega_m; k_n^+, k_m^+) = \frac{3(g/h)^2 - (g/h)(\omega_n^2 + \omega_m^2 + \omega_m^2)}{2(\omega_n^+ + \omega_m^+) \omega_n \omega_m - \frac{1}{(g/h)} O(\omega_n^2 \omega_m^2)} \] \hspace{1cm} (3.3.6)
The denominator decreases faster than the numerator as \( \omega_n \) and \( \omega_m \) decrease, therefore \( D(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) \) is not well-behaved in shallow water.

**Intermediate Water**

The nonlinear interaction matrix, \( D \), behaves quite well in this region except when \( \omega_n = -\omega_m \). This condition will produce a wave with zero frequency and will cause a change in the mean level of the ocean. In section 3.2.b, it was required that the random sea surface maintain a zero mean by an appropriate Bernoulli's constant. The condition when \( \omega_n = -\omega_m \), therefore, may be included in Bernoulli's constant.

**Deep Water**

The frequency dispersion relationship in this region becomes approximately

\[
\frac{\omega_n^2}{g} = \frac{k_n}{n} \text{sgn}(n) \tag{3.3.7.a}
\]

in which the signum function, \( \text{sgn}(n) \), is defined by

\[
\text{sgn}(n) = \begin{cases} 
1 & n > 0 \\
0 & n = 0 \\
-1 & n < 0 
\end{cases} \tag{3.3.7.b}
\]

Substituting this approximation into Eq. (3.3.1) gives the approximate value for \( D \) in deep water as
or

\[ D(w_n, w_m; k_n, k_m) = \begin{cases} 0 & \text{in } R_1, R_2, R_5, R_6 \\ -\omega_m & \text{in } R_3 \\ \omega_n & \text{in } R_4 \\ \omega_m & \text{in } R_7 \\ -\omega_n & \text{in } R_8 \end{cases} \]  \tag{3.3.8.a}

Unlike the nonlinear interaction matrix for velocity potential, the nonlinear interaction matrix for water surface elevation, \( H(w_n, w_m; k_n, k_m) \), is symmetrical with respect to indices \((n,m)\) and \((-n,-m)\), i.e.,

\[ H(w_n, w_m; k_n, k_m) = H(w_m, w_n; k_m, k_n) = H(-w_n, -w_m; -k_n, -k_m) = H(-w_m, -w_n; -k_m, -k_n) \]  \tag{3.3.9}

In shallow water and intermediate water regions, this function behaves in the same manner as that of the velocity potential. In deep water its value may be observed from Eqs. (3.2.b.9) and (3.3.8.a); i.e.,

\[ H(w_n, w_m; k_n, k_m) = \begin{cases} \frac{\omega_n + \omega_m}{2g} & \text{in } R_1, R_2, R_5, R_6 \\ \frac{\omega_n - \omega_m}{2g} & \text{in } R_3, R_7 \\ \frac{\omega_m - \omega_n}{2g} & \text{in } R_4, R_8 \end{cases} \]  \tag{3.3.10}
This expression is similar to that presented by Longuet-Higgins (1963) but is different from that presented by Tick (1959). The difference is because Tick did not symmetrize this interaction matrix.

The self interaction occurs when \( n = m \) and Eq. (3.3.10), for this case, is the Stoke's second-order waves (Ippen, 1966, p. 105).

The calculation for the two nonlinear interaction matrices, \( D \) and \( H \), were made for 0.0077 rad/sec increment of frequency and with water depth \( h = 99 \) feet. The results of these calculations are plotted in Figs. (3.3.2.a,b).
$\Delta \omega = 0.0077 \text{ rad/sec}$
$h = 99 \text{ feet}$

$D(\omega_n, \omega_m)$

Figure 3.3.2.a. Nonlinear interaction matrix for velocity potential.
\[ \Delta \omega = 0.0077 \text{ rad/sec} \]
\[ h = 99 \text{ feet} \]

Figure 3.3.2.b. Nonlinear interaction matrix for water surface elevation.
4.0 SIMULATIONS OF A RANDOM SEA SURFACE ELEVATION

An objective of our simulation study is the analyses of a stochastic process by mathematical methods. This requires the formulation of a mathematical model for the process being investigated. The model may be a mathematical expression in which the dependent variables behave in a fashion similar to the actual process. It may also contain the element of chance which is governed by a probability law.

Basically, simulation techniques are either numerical or analog. Artificial data are generated which have specified properties. Since the simulation of a stochastic process is theoretically equivalent to observing or sampling a real process, the realization obtained may be considered as sample observations with specified statistics.

Following development, the model must be calibrated with real data to see how well it approximates the physical process simulated. This calibration usually requires correlations with measured physical data. Only if and when the simulated processes are statistically similar to the real data can the model be used for predicting the behavior of the physical process. The simulation of random waves for engineering purposes was introduced by Borgman (1969), and was later extended to weakly nonlinear and slightly non-Gaussian second-order waves utilizing a Fast Fourier Transform algorithm by Hudspeth (1974) and Sharma (1979).

A mathematical model for random ocean wave simulation was developed in section 3.2, and was extended to second-order approximation. Since there is a similarity in the statistical properties of the
representation of noise current (cf, Rice, 1944, pp. 328-330) to that of random sea waves, it is appropriate to express linear random sea waves at a fixed location in discrete form by the second of the two representations given by Rice; viz.,

\[
\eta(n) = \sum_{m=0}^{NP-1} A_m \cos \left(\frac{2\pi mn}{NP} \right) \tag{4.1}
\]

in which \( n \) and \( m \) are positive integers such that

\[
t = n \Delta t
\]

\[
f_m = \omega / 2\pi = m\Delta f, \text{ and}
\]

\[
NP = \text{number of points}
\]

\[
A_m = \left( S_{\eta\eta}(m) \right)^{\frac{1}{2}}; S_{\eta\eta}(m) \text{ is the discrete two-sided deterministic wave spectral density which is given by the following relationship}
\]

\[
S_{\eta\eta}(m) = S_{\eta\eta}(m\Delta\omega)
\]

in which \( S_{\eta\eta}(m\Delta\omega) \) = the continuous spectral density evaluated at

\[
\omega_m = m\Delta\omega
\]

The random phase angles, \( \theta_1, \theta_2, \ldots, \theta_m \) are uniformly distributed over the range \((-\pi, \pi)\).

Another representation which is statistically equivalent to that given by Eq. (4.1) is also given by Rice as the first of the two representation for noise current; viz.,

\[
\eta(n) = \sum_{m=0}^{NP-1} \left[ \frac{S_{\eta\eta}(m)}{2} \right]^{\frac{1}{2}} (A_m \cos (2\pi mn/\alpha) - \beta_m \sin (2\pi mn/\alpha)) \tag{4.3}
\]
The nondeterministic coefficients \( \alpha_m \) and \( \beta_m \) are independent random variables which are normally distributed with zero mean and unit variance. Equation (4.3) may be written alternatively by the following:

\[
\eta(n) = \sum_{m=0}^{NP-1} R_m \cos(2\pi mn/NP - \theta_m) \tag{4.4}
\]

in which

\[
R_m = \frac{S_{\eta\eta}(m)}{2} \left( \alpha_m^2 + \beta_m^2 \right)^{1/2}
\]

\[
\theta_m = \arctan(-\beta_m/\alpha_m)
\]

Both \( R_m \) and \( \theta_m \) are nondeterministic. For this reason, we shall call the first model given by Eq. (4.1) a deterministic spectral amplitude (DSA) model and the second model given by Eq. (4.4) a nondeterministic spectral amplitude (NSA) model.

4.1 Deterministic Spectral Amplitude Simulation (DSA)

Equation (4.1) may be written in complex Fourier series form (Wylie, 1975, pp. 229-233) as

\[
\eta(n) = \sum_{m=-N}^{N} F(m) \exp(i2\pi mn/NP) \tag{4.1.1.a}
\]

in which the complex-valued Fourier coefficients are

\[
F(m) = [S_{\eta\eta}(m)]^{1/2} \exp - i\theta_m \tag{4.1.1.b}
\]

with

\[
S_{\eta\eta}(-m) = S_{\eta\eta}(m) \quad \text{and} \quad \theta_{-m} = -\theta_m
\]
The random spectral measure in this case is given by the phase of the complex-valued coefficient, $F(m)$.

The simulation problem now rests with the determination of the complex values for $F(m)$. This involves the numerical generation of discrete random phase angles which are uniformly distributed between $(-\pi, \pi)$.

Discrete simulated values of $\eta(n)$ are obtained by first initializing the array for $F(m)$ in accordance with Eq. (4.1.1.b) and then by taking the inverse Fourier transform of $F(m)$. The second-order correction for the random sea waves may be calculated from Eq. (3.2.b.8).

In section 3.2.b, it was shown that for a fixed location, the weakly second-order nonlinear interactions between two frequencies produce many harmonic waves determined by the sums and differences between interacting frequencies. These harmonic waves will add linearly to the first order waves having equal frequency.

The result of the nonlinear wave-wave interaction will be the addition of energy to each component of the first order wave spectrum, and the magnitude of the wave amplitude spectrum will increase at each discrete frequency. The DSA model for nonlinear waves correct to second-order may be expressed in discrete form as the sum of individual collinear waves

$$\eta(n) = \sum_{m=-N}^{N} F(m) \exp(i2\pi mn/N) \tag{4.1.2}$$

in which

$$F(m) = F(m) + \sum_{k=-N}^{N} F(k) F(m-k) H(\vec{k}, \vec{k}-\vec{m}-\vec{k})$$
Discrete nonlinear random waves correct to second-order may be obtained by synthesizing the Fourier coefficients in Eq. (4.1.2) into a time sequence.

4.2 Nondeterministic Spectral Amplitude Simulation (NSA)

The NSA method is equivalent to simulating by the linear filtering techniques which have been introduced by Borgman (1969). The basic problem becomes one of determining values of the random coefficients \( \alpha_m \) and \( \beta_m \) in Eq. (4.3).

In contrast to the DSA method, the random measure in the NSA method is composed of two independent random variables which are normally distributed with zero mean and unit variance. The complex-valued Fourier coefficient, \( F(m) \), in Eq. (4.1.1.a) implicitly depends on the random variables \( \alpha_m \) and \( \beta_m \) through the nondeterministic amplitude, \( R_m \), and phase angle, \( \theta_m \), given by Eq. (4.4); i.e.,

\[
F(m) = \begin{cases} 
R_m \exp(-i\theta_m) & m > 0 \\
|R_m| \exp(i\theta_m) & m < 0 
\end{cases}
\]  

(4.2.1)

It can be shown (Appendix D) that the nondeterministic amplitude, \( R_m \), is distributed according to the Rayleigh distribution and the phase angle, \( \theta_m \), is uniformly distributed between \((-\pi, \pi)\).

The simulation procedure for this method is the following:

1. Generate a sequence of paired random numbers which are normally distributed with zero mean and unit variance.
(2) Compute $R_m$ and $\sigma_m$ according to Eq. (4.4).

(3) Compute $F(m)$ according to Eq. (4.2.1).

(4) Inverse Fourier transform the sequence of $F(m)$ to obtain a sequence of discrete time series values for $\eta(n)$.

The nonlinear random sea waves correct to second-order may be obtained by the same manner as in the DSA method.

Both simulation methods will give a Gaussian distribution for the random sea if the simulation is carried only to the first-order approximation. In the second-order approximations, it has been shown by Longuet-Higgins (1963) that the distribution of a random sea which is initially linear and Gaussian is given by successive terms of a Gram-Charlier series.

Both linear simulation methods are equivalent to filtering the white noise (Rice, 1945, pp. 329; Borgman, 1969). In the DSA method, the white noise spectral density is employed, while in NSA method a sample spectrum of a Gaussian white noise realization with zero mean and unit variance is filtered in the frequency domain by the target wave amplitude spectrum. The sample spectral estimate of the white noise realization obtained in the NSA method is not, in fact, a constant unit value; but rather, the spectral estimate fluctuates randomly about its mean value (Jenkins and Watts, 1968, pp. 420).

The expected results from these two amplitude simulation methods are schematically illustrated in Fig. 4.1. Each method has certain advantages and disadvantages. In the DSA method the target spectrum will be exactly obtained for every realization simulated. In other words, the sample spectrum estimated from every simulation by DSA will
Figure 4.1. Schematic representation of deterministic spectral amplitude (DSA) and nondeterministic spectral amplitude (NSA) simulation methods.
be exactly the same as the target spectrum. In contrast, the sample spectrum estimated from NSA simulations will fluctuate about the target spectrum. However, the statistical behavior of the simulation of the NSA method correlates better with the statistics of measured random sea waves.

4.3 Fast Fourier Transform (FFT)

Since the concept of a spectral representation of a stochastic process plays a very important role in studying stationary random processes, the development of the Fast Fourier Transform (FFT) algorithm is regarded as revolutionizing time series analysis. So important has it become that several FORTRAN computer programs for this algorithm may be found in several recent publications (Borgman, 1973, p. 7-21; Bloomfield, 1976, pp. 69-76; Brigham, 1971), and some computer operating system may have FFT programs intrinsic to their FORTRAN library.

The mathematical basis for the FFT is a matter of bookkeeping in which the repetitious computations are eliminated. This is done by representing the indices with binary numbers and permutation is used to trace the correct indices.

Consider now the complex-valued Fourier series

\[ z(t) = \sum_{m=-\infty}^{\infty} F(m) \exp(i \frac{2\pi mt}{T}) \]  

(4.3.1.a)

with the coefficients given by

\[ F(m) = \frac{1}{T} \int_{0}^{T} z(t) \exp\left(-i \frac{2\pi mt}{T}\right) \, dt \]  

(4.2.1.b)
in which $T$ = the period of $z(t)$.

Let the positive integer, $NP$, and the positive quantities, $\Delta t$, $\Delta f$, and $T$ satisfy the relations

$$\begin{align*}
t &= n\Delta t \\
\Delta t &= T/NP \\
\Delta f &= 1/T
\end{align*} \quad (4.3.2)$$

Substituting Eqs. (4.3.2) into Eq. (4.3.1.a) gives the following discrete presentation for $z(t)$:

$$z(n) = \sum_{m=-\infty}^{\infty} F(m) \exp(i2\pi mn/NP)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=0}^{(k+1)NP-1} F(l) \exp(i2\pi ln/NP) \quad (4.3.3)$$

in which $z(n) = z(n\Delta t)$.

That is, the summation over all values of $m$ is subdivided into smaller blocks of summations each of which contains $NP$ terms. Substituting the variable $m = l - kNP$ in Eq. (4.3.3) and interchanging the order of summation gives

$$z(n) = \sum_{m=0}^{NP-1} \sum_{k=-\infty}^{\infty} F(m+kNP) \exp(i2\pi mn/NP)$$

$$= \sum_{m=0}^{NP-1} \sum_{k=-\infty}^{\infty} F(m+kNP) \exp(i2\pi mn/NP) \exp(i2\pi nk)$$

$$= \sum_{m=0}^{NP-1} \sum_{k=-\infty}^{\infty} F(m+kNP) \exp(i2\pi mn/NP) \quad (4.3.4)$$
We may choose the number of points, $N_P$, so that the value of complex-valued amplitude, $F(k)$, is zero for $|k| > 0$, then Eq. (4.3.4) becomes

$$z(n) = \sum_{m=0}^{N_P-1} F(m) \exp(i2\pi mn/N_P) \quad (4.3.4.a)$$

and its inverse is given by

$$F(m) = \frac{1}{N_P} \sum_{n=0}^{N_P-1} z(n) \exp(-i2\pi mn/N_P) \quad (4.3.4.b)$$

To prove this relationship, we consider the following pair of equations given by

$$z(n) = r_1 \sum_{m=0}^{N_P-1} Z(m) \exp(i2\pi mn/N_P) \quad (4.3.5.a)$$

$$Z(m) = r_2 \sum_{n=0}^{N_P-1} z(n) \exp(-i2\pi mn/N_P) \quad (4.3.5.b)$$

Equations (4.3.4.a,b) are the special case of Eqs. (4.3.5.a,b) when $r_1 = 1$ and $r_2 = \frac{1}{N_P}$. Substitution of Eq. (4.3.5.a) into the right hand side of Eq. (4.3.5.b) gives

$$Z(m) = r_1 r_2 \sum_{n=0}^{N_P-1} \sum_{\ell=0}^{N_P-1} Z(\ell) \exp(i2\pi \ell n/N_P) \cdot \exp(-i2\pi mn/N_P)$$

$$= r_1 r_2 \sum_{\ell=0}^{N_P-1} Z(\ell) \sum_{n=0}^{N_P-1} \exp[i2\pi (\ell-m)n/N_P] \quad (4.3.6)$$
The second summation may be calculated by using the following geometric series (Borgman, 1973, p. 6-2)

\[
\sum_{n=0}^{NP-1} \exp[i2\pi(\ell-m)n/NP] = \exp[i2\pi(\ell-m)/NP] \cdot \\
\cdot \left\{ \frac{1 - \exp[i2\pi(\ell-m)]}{1 - \exp[i2\pi(\ell-m)/NP]} \right\} = NP \delta_{\ell,m}
\]

(4.3.7)

in which \(\delta_{\ell,m}\) is the Kronecker delta defined by Eq. (2.5.4). Substituting Eq. (4.3.7) into Eq. (4.3.6) yields

\[
Z(m) = r_1 r_2 NP Z(m)
\]

This equation will be identical if only when

\[
\frac{r_1 r_2}{NP} = 1
\]

(4.3.8)

The random sea surface realization given by Eq. (3.2.a.l3) may be written as a discrete Fourier transform pair; viz.,

\[
\eta(n) = \sum_{m=0}^{NP-1} F(m) \exp(i2\pi mn/NP) ~~~~~~~~~~~~~~~~~~~ (4.3.9.a)
\]

\[
F(m) = \frac{1}{NP} \sum_{n=0}^{NP-1} \eta(n) \exp(-i2\pi mn/NP) ~~~~~~~~~~~~~~~~~~~ (4.3.9.b)
\]

in which \(F(m)\) = the complex-valued wave amplitude.

Since most FORTRAN computer languages do not allow zero indices, Eqs. (4.3.9.a,b) are rewritten to comply with this requirement as

\[
\eta(n) = \sum_{m=1}^{NP} F(m) \exp[i2\pi(m-1)(n-1)/NP] ~~~~~~~~~~~~~~~~~~~ (4.3.10.a)
\]
\[ F(m) = \frac{1}{NP} \sum_{n=1}^{NP} \eta(n) \exp\left[-i2\pi(m-1)(n-1)/NP\right] \quad (4.3.10.b) \]

The mean value of the time sequence, \( \eta(n) \), is represented by the first array element \( \text{[F(1)]} \). The time sequence, \( \eta(n) \), is real and, therefore, the following complex conjugate relationship for the complex-valued spectral amplitude holds

\[ F(NP + 2 - m) = F^*(m) \quad \frac{NP}{2} > m > 1 \quad (4.3.11) \]

This equality may be demonstrated by substituting \( m = N + 2 - m \) into Eq. (4.3.10.b) to obtain

\[
F(NP + 2 - m) = \frac{1}{NP} \sum_{n=1}^{NP} \eta(n) \exp\left[-i2\pi(NP+1-m)(n-1)/NP\right] \\
= \frac{1}{NP} \sum_{n=1}^{NP} \eta(n) \exp\left[i2\pi(m-1)(n-1)/NP\right] \cdot \\
\cdot \exp(-i2\pi) \\
= \frac{1}{NP} \sum_{n=1}^{NP} \eta(n) \exp\left[i2\pi(m-1)(n-1)/NP\right] \\
= F^*(m)
\]

4.4 Wave Spectrum

Basically, there are three types of energy (power) density spectra for the random wave process:

1. One-dimensional frequency spectrum
(2) Two-dimensional wave-number spectrum

(3) Three-dimensional spectrum.

The frequency dispersion relationship between the wave number, $k_n$, and frequency, $\omega_n$, for small-amplitude waves is given by Eq. (3.2.a.10)

$$\omega_n^2 = g k_n \tanh k_n h$$  \hspace{1cm} (3.2.a.10)

The one-dimensional frequency spectrum may be derived for the two-dimensional wave-number spectrum by the following:

$$S_{\eta \eta}(\omega_n) = \int_0^\pi S_{\eta \eta}(k_n, \theta) |J| d\theta$$  \hspace{1cm} (4.4.1)

in which $S_{\eta \eta}(k_n, \theta)$ is a two-dimensional wave-number spectrum, and the Jacobian of transformation, $|J|$, is given by

$$|J| = \left| \begin{array}{c}
k_n, \theta \\
\partial(\omega_n, \theta)
\end{array} \right| = \left| \begin{array}{cc}
\frac{\partial k_n}{\partial \omega_n} & \frac{\partial \theta}{\partial \omega_n} \\
\frac{\partial k_n}{\partial \theta} & \frac{\partial \theta}{\partial \theta}
\end{array} \right|
$$

in which $k_n = |\vec{k}_n|$.

$\theta$ = angular direction between $\vec{k}_n$ and x axis.

From the dispersion relationship given by Eq. (3.2.a.10), the expression for $|J|$ then becomes
\[ |J| = \frac{1}{U_g} \]  \hspace{1cm} (4.4.2)

in which \( U_g \) = wave group velocity = \( \omega/2k_n \left(1 + 2k_n h/\sinh 2k_n h\right) \).

In engineering practice, Eq. (4.4.1) is very rarely applied because of the difficulties in measuring two-dimensional wave records. The most common engineering application is the transformation of a one-dimensional frequency spectrum into a two-dimensional wave-number spectrum. Several formulae have been presented and used for practical engineering applications. The variables in the general two-dimensional spectrum, \( S_{\eta\eta}(k_n, \theta) \), may be separated and the spectrum may be expressed by the product of the one-dimensional frequency spectrum, \( S_{\eta\eta}(\omega_n) \), and a dimensionless directional spreading function, \( M(\theta) \), according to

\[ S_{\eta\eta}(k_n, \theta) = |J|^{-1} S_{\eta\eta}(\omega_n) M(\theta) \]  \hspace{1cm} (4.4.3)

in which \( |J| \) is given by Eq. (4.4.2) and

\[ \int_{-\pi/2}^{\pi/2} M(\theta) d\theta = 1 \]  \hspace{1cm} (4.4.4.a)

The most commonly used expression for \( M(\theta) \) is (Price and Bishop, 1974, p. 137)

\[ M(\theta) = c(p) \cos^2 P(\theta/2) \]  \hspace{1cm} (4.4.4.b)

c(p) and \( p \) are determined such that Eq. (4.4.4.a) is satisfied.

A three-dimensional spectrum, theoretically, gives a general expression for a wave spectrum in which the wave-number and frequency are considered to be independent. The three-dimensional spectrum may be obtained from measurements made at sea; e.g., from several stereo
photos which are taken continuously. The wavelength and the angle of spreading may be obtained from each photo by two-dimensional Fourier analysis (e.g., two-dimensional analysis from grids formed on each photo), and the wave frequencies may be determined for a fixed location from the successive photos (temporal). If this three-dimensional spectrum could be constructed, by assuming that the small-amplitude wave theory is applicable, it may be transformed to a two-dimensional wave-number spectrum or to a one-dimensional frequency spectrum (Kinsman, 1965, pp. 341).

Since the one-dimensional frequency spectrum is the easiest spectrum of the three types to measure, most engineering applications have employed this type of spectrum.

4.4.a Theoretical Spectral Representations

The wave frequency spectrum $S_{\eta\eta}(\omega_n)$ of an ocean surface profile, $\eta(x,t)$, at a fixed point on the ocean surface may be calculated from the Wiener-Khinchine's relationship (Eq. 2.5.8.a), for zero mean process, according to

$$S_{\eta\eta}(\omega_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{\eta\eta}(\tau) \exp(-i\omega_n \tau) d\tau$$  \hspace{1cm} (2.5.8.a)

in which the autocovariance function $C_{\eta\eta}(\tau)$ is given by Eq. (2.4.5.b).

Statistically, the wave spectrum, $S_{\eta\eta}(\omega_n)$, may be determined by averaging a number of individual energy spectra sampled from distinct time series of the water surface recorded at a particular location.
The wave records are selected such that they are stable (i.e., the ocean is exposed to the same condition).

The interaction between wind and waves is not completely understood, but wave generation theories have been presented by Phillips (1957), Miles (1957, 1959a, 1959b, 1960) and Hasselmann (1962, 1963) and have provided a good explanation of the history of the wave growth. Hasselmann (1962, 1963) has determined the theoretical rate of transfer of energy among the different wave components. He found that most of energy transfer is among pairs of almost identical wave numbers. This energy transfer is usually not sufficient to maintain a statistical equilibrium by this means alone since it is not capable of transferring as rapidly as it is supplied by wind (Phillips, 1977, pp. 107-119).

The growth of waves due to the wind will be limited by the ability of the wave to support further growth of its height. The surface eventually becomes unstable; and once the wave breaks and energy is released, the stability is restored. From these arguments, Phillips (1977) was able to determine the parameters that govern the occurrence of sharp wave crest which becomes the equilibrium subrange of the wave spectrum. The waves should not grow beyond this range. By similarity laws, it may be shown (Phillips, 1977) that this equilibrium subrange may be written as

\[ S_{n\eta}(\omega) = \beta g^2 \omega^{-5} \]  

(4.4.a.1)

in which \( \beta \) = a constant which may be determined from experiment. Equation (4.4.a.1) is valid for frequencies which are much larger
than the peak spectral frequency, but much smaller than the frequencies of capillary waves.

Kitaigorodski (1962) proposed the similarity hypothesis in which, for fully developed seas, the wave spectrum is only a function of wave frequency, $\omega$, acceleration of gravity, $g$, wind shear velocity, $U_*$, and the length of fetch, $X$. He showed that the wave spectrum, in general, may be expressed as

$$S(\omega) = g^2 \omega^{-5} f(\omega/\omega_p)$$  \hspace{1cm} (4.4.a.2)

in which $\omega_p$ = the peak frequency.

Equation (4.4.a.2) will approach the equilibrium spectrum represented by Eq. (4.4.a.1) when $\omega/\omega_p >> 1$.

Experimental data may be used to determine the best fit for the function, $f(\omega/\omega_p)$. Based on Eq. (4.4.a.2), many theoretical two-parameter spectra have been developed. One of the most widely used for engineering purposes is the Pierson-Moskowitz spectrum (Pierson, and Moskowitz, 1963) represented by

$$S(\omega) = \left(\frac{\omega g^2}{\omega_p}\right) \exp \left[-\beta \left(\frac{h}{y/\omega}\right)^4\right]$$  \hspace{1cm} (4.4.a.3)

in which $\alpha = 8.10 \times 10^{-3}$

$\beta = 0.74$

$U$ = wind speed measured at 10 m above the ocean surface

$\alpha$ and $\beta$ are dimensionless parameters.

An analogous two-parameter sea state energy density spectrum was introduced by Bretschneider (Bretschneider, 1966). The function,
\( f(\omega/w_p) \), in Eq. (4.4.a.2) was determined from analysis of data collected in the Gulf of Mexico. The Bretschneider frequency spectrum is given by

\[
S_{\eta\eta}(\omega) = \left(\frac{\gamma^2}{\omega^5}\right) \exp\left[-0.675(\frac{\gamma}{\omega B_2})^4\right]
\]  

(4.4.a.4)

in which \( \gamma \) and \( B_2 \) are dimensionless quantities determined by

\[
\gamma = 3.437 \frac{B_2^4}{B_1^4}
\]

\[
B_1 = \frac{g H_{1/3}}{\gamma^2}
\]

\[
B_2 = \frac{g T_{1/3}}{2\pi U}
\]

in which \( H_{1/3} \) and \( T_{1/3} \) are the significant wave height and period, respectively.

Unlike the development of the Pierson-Mostowitz and Bretschneider spectrum, the Scott spectrum was not developed according to the similarity hypothesis proposed by Kitaigorodski but rather was the modification of Darbyshire spectrum (Scott, 1965).

Given the significant wave height and the average period of a wave train, the wave energy density distribution may be expressed by the following:

\[
S_{\eta\eta}(\omega) = 0.214 \left(\frac{H_{1/3}}{w_p}\right)^2 \exp\left[-\frac{(\omega - w_p)^2}{0.065(w - w_p + 0.26)}\right]^{1/8}
\]

for \(-0.26 < \omega - w_p < 1.65\)

\[= 0 \quad \text{otherwise} \quad (4.4.a.5)\]
in which \( \omega_p = \frac{3.15}{T} + \frac{8.98}{T^2} \)

\[ \bar{T} = 0.085 \frac{H_{1/3}}{T} + 7.1 \]

in which the average wave period, \( \bar{T} \), is in seconds and the significant wave height, \( H_{1/3} \), is in feet.

Many other wave spectra which provide a best fit to the wave records for a particular area have been proposed. A careful judgement is required to choose one of them before it is applied to design.

4.4.b Comparison of Theoretical to Measured Wave Spectra

The unknown parameters required for the various theoretical spectra vary from spectrum to spectrum; however, two easily measured parameters for design purposes are the total energy content, \( m_0 \), and the peak spectral frequency, \( \omega_p \). The total energy content of the measured spectrum is assumed to be equal to that of a theoretical spectrum. The peak frequency of a theoretical spectrum then may be computed in a best least square sense from the measured data. These two parameters, \( m_0 \) and \( \omega_p \), are enough to generate any two-parameter theoretical spectrum. Table 4.4.1 summarizes the results of this method for WPII field data.

The Pierson-Moskowitz and Bretschneider spectra are equivalent when determined from the two parameters given by the total energy content and the peak frequency. Since the Bretschneider spectrum was developed from analysis of wave data gathered from the Gulf of Mexico, and since the measured data used to calculate \( m_0 \) and \( \omega_p \) in Table
Table 4.4.1. Theoretical two-parameter spectral representation with transformation to parameters \( m_0 \) and \( \omega_p \) (from Hudspeth and Chen, 1979).

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>Spectral density</th>
<th>( m_0 )</th>
<th>( \omega_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pierson-Moskowitz</td>
<td>( \frac{m_0}{\omega_p} \left( \frac{-\omega}{\omega} \right)^5 \exp\left[ 1.25 \left( \frac{-\omega}{\omega} \right)^4 \right] )</td>
<td>( \frac{a q^2}{5 \omega_p} )</td>
<td>( \frac{q}{\eta} \left( \frac{8}{1.25} \right)^\frac{1}{4} )</td>
</tr>
<tr>
<td>Bretschneider</td>
<td>( \frac{m_0}{\omega_p} \left( \frac{-\omega}{\omega} \right)^5 \exp\left[ 1.25 \left( \frac{-\omega}{\omega} \right)^4 \right] )</td>
<td>( \frac{y q^2}{\omega_p^4} )</td>
<td>( \frac{2 \pi}{T_{1/3}} \left( \frac{0.675}{1.25} \right)^\frac{1}{4} )</td>
</tr>
<tr>
<td>Scott</td>
<td>3.424 ( m_0 ) ( (\omega-\omega_p)^2 ) ( \exp\left[ \frac{(\omega-\omega_p)^2}{0.065(\omega-\omega_p-0.25)} \right] )</td>
<td>( \frac{H_{1/3}}{4} )</td>
<td>( \frac{3.15}{T} + \frac{8.98}{T^2} )</td>
</tr>
</tbody>
</table>

4.4.1 were also obtained from the data during the Hurricane Carla in the Gulf of Mexico, only the Bretschneider spectrum will be referred to in the remaining presentation.

4.5 Second-order Spectral Density

The autocovariance function for a stationary random sea correct to second perturbation order at a fixed point is defined by

\[
C_{\eta \eta}(\tau) = E\{[\eta^{(1)}(t) + \eta^{(2)}(t)] [\eta^{(1)*}(t + \tau) + \eta^{(2)*}(t + \tau)]\}
= E[\eta^{(1)}(t)\eta^{(1)*}(t + \tau)] + E[\eta^{(1)}(t)\eta^{(2)*}(t + \tau)]
+ E[\eta^{(2)}(t)\eta^{(1)*}(t + \tau)] + E[\eta^{(2)}(t)\eta^{(2)*}(t + \tau)]
\]

(4.5.1)
in which the superscript asterisk * denotes a complex-conjugate value.

Substituting the expressions for \( \eta^{(1)}(t) \) and \( \eta^{(2)}(t) \) given by Eq. (3.2.a.13) and Eq. (3.2.b.8) respectively into Eq. (4.5.1) yields

\[
C_{\eta\eta}(\tau) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} \exp\left[ i(\omega_m - \omega_n) t - \omega_n \tau \right] E[F(m)F^*(n)]
\]

\[+ \sum_{k=-N}^{N} \sum_{m=-N}^{N} \sum_{n=-N}^{N} \sum_{\tilde{m}=-N}^{N} \sum_{\tilde{n}=-N}^{N} H(\omega_m, \omega_n; \omega_{\tilde{m}}, \omega_{\tilde{n}}) \exp\left[ i(\omega_{\tilde{m}} - \omega_{\tilde{n}}) t - \omega_{\tilde{n}} \tau \right] E[F(\tilde{m})F^*(\tilde{n})]
\]

\[+ \sum_{k=-N}^{N} \sum_{\tilde{m}=-N}^{N} \sum_{\tilde{n}=-N}^{N} \sum_{\tilde{k}=-N}^{N} \sum_{\tilde{\tilde{n}}=-N}^{N} H(\omega_k, \omega_{\tilde{m}}; \omega_{\tilde{k}}, \omega_{\tilde{\tilde{n}}}) \cdot H(\omega_m, \omega_n; \omega_{\tilde{m}}, \omega_{\tilde{n}}) \exp\left[ i(\omega_k + \omega_{\tilde{k}} - \omega_m - \omega_n) t - (\omega_m + \omega_n) \tau \right] \cdot
\]

\[\cdot E[F(k)F(\tilde{k})F^*(m)F^*(n)] \quad (4.5.2)
\]

It can be shown (Appendix F) that any expectation of the triple product of the random variables \( F(n) \) for both the DSA and NSA methods is identically zero. Therefore, the second and the third terms in Eq. (4.5.2) vanish. The autocovariance function for a random sea surface correct to second-order may be expressed by the following (Appendix F);
\[ C_{\eta \eta}(\tau) = \sum_{n=-N}^{N} \left| F(n) \right|^2 \exp \left( -i\omega_n \tau \right) \]
\[ + \sum_{m=-N}^{N} \sum_{n=-N}^{N} \left( 2 - \delta_{m,n} \right) \hat{a}^2(\omega_n, \omega_m, \vec{k}_n, \vec{k}_m) \cdot \left| F(m) \right|^2 \left| F(n) \right|^2 \exp \left( -i(\omega_m + \omega_n) \tau \right) \]  \hspace{1cm} (4.5.3)

in which \( \delta_{m,n} = \) Kronecker delta defined by Eq. (2.5.4).

The first term of Eq. (4.5.3) is the autocovariance function for the first-order (linear) waves; while the second term is the nonlinear contribution which is due to the nonlinear interaction among pairs of linear waves.

Tick (1959) has derived a continuous autocovariance function for deep water waves which has been extended to waves in intermediate water depth by Hasselmann (1962, 1963), Longuet-Higgins (1963), and Tick (1963).

From the Wiener-Khinchine's relationship given by Eq. (2.5.8.a), the discrete random wave spectrum correct to second-order may be calculated according to

\[ S_{\eta \eta}(\lambda) = \frac{1}{NP} \left[ \sum_{p=0}^{NP-1} C_{\eta \eta}(p) \exp \left( i2\pi \frac{p\lambda}{NP} \right) \right] \hspace{1cm} (4.5.4) \]

in which \( S_{\eta \eta}(\lambda) = S_{\eta \eta}(i\Delta\omega)\Delta\omega \)

\( C_{\eta \eta}(p) = C_{\eta \eta}(p\Delta\tau) \)

Substituting Eq. (4.5.3) into Eq. (4.5.4) and noting that

\( \omega_n = 2\pi n/NP\Delta t \) leads to the following:
\[ S_{\eta\eta}(\ell) = \frac{1}{NP} \left\{ \sum_{n=-N}^{N} |F(n)|^2 \sum_{p=0}^{NP-1} \exp[i2\pi p(\ell-n)/NP] \right. \]
\[ + \sum_{m=-N}^{N} \sum_{n=-N}^{N} (2 - \delta_{mn}) H^2(\omega_m, \omega_n; \mathbf{k}_m, \mathbf{k}_n) \cdot \]
\[ \left. \cdot |F(m)|^2 |F(n)|^2 \sum_{p=0}^{NP-1} \exp[i2\pi p(\ell-m-n)/NP] \right\} \quad (4.5.5) \]

From Eq. (4.3.7), the sum of exponential terms are equal to NP only when its argument is zero and vanishes when its argument is not zero. Eq. (4.5.5) then may be expressed as follows:

\[ S_{\eta\eta}(\ell) = \sum_{n=-N}^{N} |F(n)|^2 \delta_{\ell,n} + \sum_{m=-N}^{N} \sum_{n=-N}^{N} (2 - \delta_{mn}) \cdot \]
\[ \cdot H^2(\omega_m, \omega_n; \mathbf{k}_m, \mathbf{k}_n) |F(m)|^2 |F(n)|^2 \delta_{\ell-m,n} \quad (4.5.6) \]

The Kronecker delta reduces the number of summations by one. Then the representation of discrete two-sided wave spectrum correct to second-order may be given by the following:

\[ S_{\eta\eta}(\ell) = |F(\ell)|^2 + \sum_{m=-N}^{N} (2 - \delta_{m,\ell-m}) \cdot \]
\[ \cdot H^2(\omega_m, \omega_{\ell-m}; \mathbf{k}_m, \mathbf{k}_{\ell-m}) |F(m)|^2 |F(\ell-m)|^2 \quad (4.5.7) \]

Equation (4.5.7) may now be expressed in terms of discrete first-order spectral density values by

\[ S_{\eta\eta}(\ell) = S^{(1)}_{\eta\eta}(\ell) + \sum_{m=-N}^{N} (2 - \delta_{m,\ell-m}) H^2(\omega_m, \omega_{\ell-m}; \mathbf{k}_m, \mathbf{k}_{\ell-m}) \]
\[ \cdot S^{(1)}(m) S^{(1)}(\ell-m) \quad (4.5.8) \]
The second term in Eq. (4.5.8) represents the second perturbation order spectrum that was first introduced by Tick (1959) for a continuous spectrum of deep water and was later extended for intermediate depth waves (Tick, 1963). Hasselmann (1962) represented this second perturbation order spectrum in discrete form.
5.0 SIMULATION OF RANDOM WAVE KINEMATICS
IN THE HORIZONTAL DIRECTION

The mathematical model for water particle velocity may be obtained from Eq. (3.1.1)
\[
\dot{\mathbf{u}}(s,z,t) = -\nabla \phi
\]  
(3.1.1)

and the acceleration by the total derivative of the velocity, i.e.,
\[
\dot{\mathbf{a}}(s,z,t) = \frac{\partial \dot{\mathbf{u}}}{\partial t} + (\dot{\mathbf{u}} \cdot \nabla) \mathbf{u}
\]  
(5.1)

in which the gradient operator, \( \nabla \), is defined in section 3.1 by Eq. (3.1.3).

The expansion of the velocity and acceleration may be written in similar form to the expansion given by Eqs. (3.2.2.a,b,c); i.e.,
\[
\mathbf{\ddot{u}}(s,z,t) = \sum_{n=1}^{\infty} \mathbf{\ddot{u}}^{(n)}(s,z,t)
\]  
(5.2.a)
\[
\mathbf{\ddot{a}}(s,z,t) = \sum_{n=1}^{\infty} \mathbf{\ddot{a}}^{(n)}(s,z,t)
\]  
(5.2.b)

The perturbation expansion is carried to second-order and two types of transfer functions are employed. The first type is the linear transfer function which was introduced by Reid (1958) and was later applied by Wheeler (1969). The second type is the nonlinear transfer function which is obtained directly from formulae for wave kinematics given by Eqs. (3.1.1), (5.2.4) and (5.2.10).
5.1 Linear Transfer Function

The expression for the wave kinematics originates from small-amplitude waves, in which the expression for horizontal components of water particle velocity and acceleration are carried to first-order in the perturbation expansion, i.e.,

\[
\begin{align*}
\frac{u_x}{x} &= -\frac{\partial \phi}{\partial x} \\
&= \sum_{m=-\infty}^{\infty} \frac{g k_m}{\omega_m} \frac{\cosh[k_m (h+z)]}{\cosh(k_m h)} F(m) \cdot \\
&\cdot \exp \left(-i(k_m^x \cdot \hat{s} - \omega_m t) \right) \quad (5.1.1.a)
\end{align*}
\]

\[
\begin{align*}
\frac{a_x}{x} &= \frac{\partial u_x}{\partial t} \\
&= i \sum_{m=-\infty}^{\infty} \frac{g k_m}{\omega_m} \frac{\cosh[k_m (h+z)]}{\cosh(k_m h)} F(m) \cdot \\
&\cdot \exp \left(-i(k_m^x \cdot \hat{s} - \omega_m t) \right) \quad (5.1.1.b)
\end{align*}
\]

in which \( k_m = \|k_m^x\| \)

\[
k_m^x = k_m \cdot \hat{x}_m
\]

\( \hat{x}_m \) = unit vector in \( x \) direction

To improve the solution, the complex spectral amplitude correct to second-order is used in place of \( F(m) \); viz.,

\[
F(m) = F(m) + \sum_{n=-\infty}^{\infty} F(n)F(m-n)H(\omega_n, \omega_m; k_n^x, k_m^x) \quad (5.1.2)
\]
The linear transfer function, \( T_{x \ell} \), in which \( x \) denotes the variable and \( \ell \) denotes the order, is the quantity which premultiplies \( F(m) \).

The linear transfer function for horizontal water particle velocity is given by

\[
T_{ul}(\omega_m; k_m, z) = \frac{g k_m \cosh[k_m(h+z)]}{\omega_m \cosh(k_m h)} \tag{5.1.3}
\]

and the linear transfer function for horizontal water particle acceleration is given by

\[
T_{al}(\omega_m; k_m, z) = ig k_m \cosh[k_m(h+z)] \frac{\cosh(k_m h)}{\omega_m \cosh(k_m h)}
\]

\[
= i \omega_m T_{ul}(\omega_m; k_m, z) \tag{5.1.4}
\]

Equations (5.1.1.a,b) may be expressed in terms of linear transfer functions, \( T_{ul} \) and \( T_{al} \), by the following:

\[
u_x(z', z, t) = \sum_{m=-\infty}^{\infty} T_{ul}(\omega_m; k_m, z) F(m) \exp(-i k_m \cdot z' - \omega_m t) \tag{5.1.5.a}
\]

\[
a_x(z', z, t) = \sum_{m=-\infty}^{\infty} T_{al}(\omega_m; k_m, z) F(m) \exp(-i k_m \cdot z' - \omega_m t) \tag{5.1.5.b}
\]

Wheeler (1969) improved the method for predicting random wave forces from the Morison equation by introducing a stretched coordinate defined as

\[
z' = \frac{z - \eta(t)}{1 + \eta(t)/h} \tag{5.1.6}
\]

in the argument of the hyperbolic function. Since pressures at only approximately mid-depth will be simulated and since the stretched
coordinate correction is small at mid-depth, no stretched coordinate will be used in this presentation.

### 5.2 Nonlinear Transfer Function

The horizontal component of water particle velocity correct to second-order may be written as (cf., Eq. (5.2.a))

\[ u_x(s,z,t) = u_x^{(1)} + u_x^{(2)} \quad (5.2.1) \]

in which \( u_x^{(1)} \) is given by Eq. (5.1.1.a) and \( u_x^{(2)} \) is given by the following expression:

\[
\begin{align*}
\frac{\partial u_x^{(2)}}{\partial x} (s,z,t) &= -\frac{\partial \phi^{(2)}}{\partial x} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} T_{u2} (\omega_n, \omega_m; \vec{k}_n, \vec{k}_m, z) \cdot \\
&\cdot F(n)F(m)\exp[-i(\vec{k}_n + \vec{k}_m) \cdot \vec{s} - (\omega_n + \omega_m)t] \quad (5.2.2)
\end{align*}
\]

in which the nonlinear transfer function, \( T_{u2} \),

\[
T_{u2} (\omega_n, \omega_m; \vec{k}_n, \vec{k}_m, z) = (k_{nx} + k_{mx}) \cdot \\
\cdot \frac{\cosh[(|\vec{k}_n + \vec{k}_m|)(h+z)]}{\cosh(|k_{nx} + k_{mx}|h)} \cdot D(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m) \quad (5.2.3)
\]

Substituting Eq. (5.1.1.a) and (5.2.2) for \( u_x^{(1)} \) and \( u_x^{(2)} \) in Eq. (5.2.1) yields the following expression for the horizontal component of water particle velocity:
\[ u_x(s, z, t) = \sum_{n=-\infty}^{\infty} T_1(\omega_n; \vec{k}_n, z) \exp(-i(\vec{k}_n \cdot \hat{s} - \omega_n t)) \]
\[ + \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} T_2(\omega_n, \omega_m; \vec{k}_n, \vec{k}_m, z) F(n) F(m) \cdot \exp(-i[(\vec{k}_n + \vec{k}_m) \cdot \hat{s} - (\omega_n + \omega_m) t]) \] (5.2.4)

The horizontal component of water particle acceleration correct to second-order may be expressed according to Eq. (5.2.b) by the following:

\[ a_x(s, z, t) = a_x^{(1)} + a_x^{(2)} \] (5.2.5)

in which \( a_x^{(1)} \) is given by Eq. (5.1.1.b) and \( a_x^{(2)} \) is obtained from Eq. (5.1); i.e.,

\[ a_x^{(2)} = \frac{\partial u^{(2)}_x}{\partial t} + u_x^{(1)} \frac{\partial u_x^{(1)}}{\partial x} + u_y^{(1)} \frac{\partial u_x^{(1)}}{\partial y} + u_z^{(1)} \frac{\partial u_x^{(1)}}{\partial z} \] (5.2.6)

in which \( u_y^{(1)} \) = the first-order perturbation expansion for the water particle velocity in the lateral \( y \) axis direction, \( u_x^{(1)} \) = the first-order perturbation expansion for the vertical water particle velocity.

\( u_y^{(1)} \) and \( u_z^{(1)} \) may be obtained from the velocity potential, \( \phi \), given by Eq. (3.2.a.11) according to:

\[ u_y^{(1)} = -\frac{\partial \phi^{(1)}}{\partial y} \] (5.2.7.a)

\[ u_z^{(1)} = -\frac{\partial \phi^{(1)}}{\partial z} \] (5.2.7.b)
The last three terms on the right hand side of Eq. (5.2.6) are the convective terms which do not appear in the linear Eq. (5.1.1 b). Substituting the appropriate value for \( u_x^{(2)}, u_x^{(1)}, u_y^{(1)}, u_z^{(1)} \) and their appropriate derivatives in Eq. (5.2.6) yields an expression for the second-order horizontal component of water particle acceleration:

\[
a_x^{(2)} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{T}_{a2}^{(n,m)} (\omega_n, \omega_m; \vec{k}_n, \vec{k}_m, z) F(n) F(m) \cdot \\
\cdot \exp[-i((\vec{k}_n+\vec{k}_m) \cdot \vec{s}-(\omega_n+\omega_m)t)]
\]

in which the nonlinear transfer function, \( \mathcal{T}_{a2} \), is given by

\[
\mathcal{T}_{a2}^{(n,m)} (\omega_n, \omega_m; \vec{k}_n, \vec{k}_m, z) = i((\omega_n+\omega_m) T_{u2}^{(n,m)} (\omega_n, \omega_m; \vec{k}_n, \vec{k}_m, z)
\]

\[
+ \frac{g^2 (k_n x + k_m x)}{2 \omega_n \omega_m \cosh(k_n h) \cosh(k_m h)} \left[ -\left( k_n k_n x + k_m k_m x \right) \cdot \\
\cdot \cosh(k_n (h+z)) \cosh(k_m (h+z)) + k_n k_m \sinh(k_n (h+z)) \right]
\]

\[
\cdot \sinh(k_m (h+z)) \right] \}
\]

in which \( T_{u2} \) = the nonlinear transfer function for horizontal water particle velocity given by Eq. (5.2.3)

\( k_n y \) = the component of the wave number in the \( y \) direction

The horizontal component of water particle acceleration,

\[ a_x (s, z, t), \] may now be expressed by
\[ a_x(s, z, t) = \sum_{n = -\infty}^{\infty} T(\omega_n; k_n, z)F(n)\exp(-i(k_n \cdot s - \omega_n t)) \]
\[ + \sum_{n = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} T(\omega_n, \omega_m; k_n, k_m, z)F(n)F(m) \cdot \exp(-i((k_n + k_m) \cdot s - (\omega_n + \omega_m) t)) \]  

Equations (5.2.4) and (5.2.10) are the final forms required for simulating wave kinematics in the frequency domain. The random measures for the stochastic wave process are implicitly included in the complex-valued wave amplitude spectrum \( F(m) \) through the random phase angle, \( \theta_m \) (cf. Eq. (4.1.1.b)).
6.0 SIMULATION OF RANDOM WAVE FORCES

The hydrodynamic forces produced on offshore structures by surf-
face waves vary both in time and space. The hydrodynamic force at a
specific location on a circular pile may be calculated by the Morison
equation if the motion of the water particle is not sensibly affected
by the presence of the structure. This may be assumed to be the case
if the diameter of the circular member of the structure is small com-
pared to the incident wave length.

For larger objects, the scattering of the waves by the structure
should be included in force calculations and a diffraction wave theory
should be used. The validity and application for either of these
methods may be determined from the ratio of the diameter of the member
to the wave length.

Theoretically, the forces due to irregular waves which are com-
posed of many single waves ranging from long waves to short waves may
be calculated from either method depending upon the value of the
diameter-wave length ratio, D/L. Since the spectral decomposition of
random sea waves results in spectral amplitude values that are small at
the higher frequencies (short waves), these high frequency waves may be
assumed to make negligible contributions to the total wave forces.
By this assumption, we shall use the Morison equation to calculate the
random wave forces.
6.1 Morison Equation

When a cylinder is immersed in an accelerating stream of inviscid fluid, it experiences a hydrodynamic force associated with the fluid acceleration that acts in the direction of the acceleration force. This is an inertia force and may be derived theoretically by solving the boundary value problem for a wave velocity potential. For the two-dimensional case of a circular cylindrical section, the inertia force per unit length is given by

\[ F_I = C_M \rho A \hat{a} \]  

(6.1.1)

in which \( \rho \) = mass density of the fluid
\( A \) = cross section area of cylinder parallel to the flow
\( \hat{a} \) = fluid acceleration vector
\( C_M \) = inertial coefficient which is theoretically equal to 2.0

The inertia force may be considered as the sum of two terms. The first is the force that would act on the cylinder due to the pressure field of the accelerating fluid (i.e., as if the cylinder itself were absent). The second is the additional force resulting from the pressure field due to the presence of the body. The term added mass or induced mass is often used to describe the equivalent mass associated with this latter component. In the corresponding case where the fixed circular cylinder, rather than fluid, is motionless, these two components are equal (Le Méhauté, 1976, pp. 177-178).

If a real fluid rather than an inviscid fluid is considered, the determination of the hydrodynamic forces acting is less straightforward
due to the effects of viscosity. Additional complexities arise due to the separation of the viscous boundary layer that results in vortex shedding, in wave formations, and in a complex flow pattern around the body contour which is difficult to determine analytically. Moreover, the forces at any instant depends upon the instantaneous state of flow and will, therefore, be influenced not only by instantaneous velocity and its temporal derivatives but also by the time history of the flow. The Morison equation, first introduced by Morison, O'Brien, Johnson, and Schaaf (1950) is a semi-empirical formulation for the in-line force in accelerated motion and has, in particular, been extensively applied to oscillatory flows. It is based on the important assumption that the total in-line wave forces may be obtained by adding two component forces, each of which may be determined separately. The first such force is analogous to the inertia force discussed previously. Note that $C_M$ is now no longer restricted to the theoretical value of 2.0 corresponding to inviscid motion but rather is given an empirical value. The second term is analogous to a steady flow drag force and is given in terms of an empirical drag coefficient by:

$$F_D = \frac{1}{2} C_D \rho D \|\mathbf{u}\| \mathbf{u}$$

(6.1.2)

in which $C_D = \text{empirical drag coefficient}$, $D = \text{cylinder diameter}$, $\mathbf{u} = \text{water particle velocity vector}$.

Here, $\|\mathbf{u}\|$ is used rather than $\mathbf{u}^2$ in order to include the case of the reversal of the vector direction in oscillatory flow. $C_D$ is given
Figure 6.1.1. Definition sketch.
an empirical value which depends upon the characteristic Reynolds number and Keulegan-Carpenter parameter which may differ from steady flow drag coefficients at the corresponding Reynolds number. The Morison equation for the hydrodynamic force per unit length is given by:

\[ F_T = C_M \rho \frac{\pi D^2}{4} \ddot{a} + C_D \rho \frac{D}{2} \dot{u} |\dot{u}| \quad (6.1.3) \]

The Morison equation was developed from the Froude-Kriloff hypothesis which states that the structure does not measurably affect the dynamic field of the surrounding wave system. The kinematics used in the Morison equation may then be determined at the center of the circular cylinder as if the cylinder were not present.

The accuracy of the Morison equation depends on the choice of the empirical inertia and drag coefficients \( C_M \) and \( C_D \) and the wave theory selected to represent the wave field kinematics. A number of laboratory and field tests have been conducted to determine these coefficients. However, the scatter found in most of the empirical force coefficients are typical of these experiments. One of the reasons for this scatter is that in most studies reported, the velocities and accelerations were not measured but were determined from the measured sea surface elevations using a theoretical wave theory (Dean, 1976). These results then become dependent on the accuracy of the wave theory selected for predicting the kinematics of the waves. Sarpkaya (1976) has found that, except for a limited range in which the wake effects are rather erratic, the Morison equation represents the oscillating forces on a cylinder in oscillating flow with good accuracy. This is
because the flow kinematics in his oscillating U tube experiment were measured rather accurately. There are two basic types of empirical evaluations used to determine the force coefficients. The first is determined in the laboratory. The second is determined from the oscillatory ocean waves. In Appendix E, an outline of the various methods used to determine the empirical force coefficients for the Morison equations is given.

6.2 Statistical Theory of Wave Forces

A statistical model for sea surface elevations implies a corresponding model for wave forces, although mathematical complexities may make it difficult to demonstrate the complete details of the model. This is particularly true if the waves are of finite amplitude so that the nonlinearities become significant. Therefore, we do not attempt to derive the probability function for the pressure force distribution obtained from the kinematics of the nonlinear random waves since this is a very complex problem due to the nonlinearities involved.

The statistics of a wave force represented by the Morison equation using linear wave theory which is distributed according to Gaussian distribution was introduced by Pierson (1965). Later Borgman (1965) derived the probability density function for this wave force, and Hino (1969) applied the method of characteristic function to derive the statistics of the wave force in which the current was included. Tung (1975) included the effects of the fluctuations of free surface profile on those points of a cylinder which are above the wave troughs and are not continuously submerged.
To obtain the probability density function for wave forces given by the Morison equation, the following assumptions are involved:

1. The water surface elevation, \( \eta(t) \), is a Gaussian stochastic process with a zero mean. This implies the linearity of the waves.

2. The horizontal water particle velocity \( u_x(s,z,t) \) and acceleration \( a_x(s,z,t) \) are also Gaussian stochastic processes.

3. The random variables \( u_x(s,z,t) \) and \( a_x(s,z,t) \) are uncorrelated with zero means, and variances \( \sigma_u^2 \) and \( \sigma_a^2 \), respectively.

The last two assumptions are the consequences of assumption (1), in which the wave realization may be expressed as the sum of individual waves with distinct frequencies.

The dimensionless wave force variable may be defined by

\[
Y = \frac{F}{(\sigma_a C_M) \rho \frac{\pi D^2}{4}} = c_u |u_x| + c_a a_x
\]  

in which \( c_u = 2/\pi (C_D/(\sigma_a C_M)) \); and

\( c_a = 1/\sigma_a \)

The moment generating function of the random variable \( Y \) may be calculated from

\[
\psi_Y(s) = E[\exp(isY)] = E[\exp is(c_u |u_x| + c_a a_x)]
\]

\[
= \int \int \exp is(c_u |u_x| + c_a a_x) p(u_x, a_x) \, du_x \, da_x
\]  

(6.2.3)
in which the joint probability density function for $u_x$ and $a_x$ given by

$$p(u_x, a_x) = \frac{1}{2\pi \sigma_u \sigma_a} \exp \left(-\frac{1}{2} \left(\frac{u_x^2}{\sigma_u^2} + \frac{a_x^2}{\sigma_a^2}\right)\right) \tag{6.2.4}$$

Substituting Eq. (6.2.4) into Eq. (6.2.3) and using the dimensionless variables $u_x/\sigma_u = t^{\frac{3}{2}}$ and $a_x/\sigma_a = q^{\frac{3}{2}}$ yields

$$\psi(s) = \frac{1}{2\pi} \left[ \int_0^\infty t^{-\frac{3}{2}} \cos(c_a \sigma_a s \sqrt{t}) \exp(-t/2) \, dt \right] \cdot \left[ \int_0^\infty q^{-\frac{3}{2}} \cos(c_a \sigma_a q^{\frac{3}{2}}) \exp(-q/2) \, dq \right]$$

$$= \frac{1}{2\pi} [L_1(\frac{3}{2}) L_2(\frac{3}{2})] \tag{6.2.5.a}$$

in which $L_1(\cdot)$ and $L_2(\cdot)$ are the Laplace transforms of the functions $t^{-\frac{3}{2}} \cos(c_a \sigma_a s \sqrt{t})$ and $q^{-\frac{3}{2}} \cos(c_a \sigma_a q^{\frac{3}{2}})$ respectively (Oberhettinger, 1973) and are defined by

$$L_1(\frac{3}{2}) = (2\pi)^{\frac{3}{2}} [1 + (s/\xi)^2]^{-\frac{3}{4}} \cos\{\frac{3}{4} \arctan(s/\xi)\}$$

$$L_2(\frac{3}{2}) = (2\pi)^{\frac{3}{2}} \exp - \frac{3}{8} (c_a \sigma_a s)^2 \tag{6.2.5.b}$$

in which $\xi = 1/(2c_a \sigma_a^2)$.

Substituting Eq. (6.2.5.b) into Eq. (6.2.5.a) yields

$$\psi(s) = [1 + (s/\xi)^2]^{-\frac{3}{4}} \cos\{\frac{3}{4} \arctan(s/\xi)\} \cdot \exp - \frac{3}{8} (c_a \sigma_a s)^2 \tag{6.2.6}$$

Substituting Eq. (6.2.2) into Eq. (6.2.6) yields
The probability density function for the random variable, $Y$, is the Fourier transform of the moment generating function, $\psi_Y(s)$, (cf., section 2.2). Since $\psi_Y(s)$ is an even function, its Fourier transform may be expressed as a cosine transform according to

$$p(Y) = \frac{1}{\pi} \int_{0}^{\infty} \psi_Y(s) \cos Ys \, ds \quad (6.2.8)$$

Substituting Eq. (6.2.7) into Eq. (6.2.8) and making a change of variable from $s/\xi$ to $t^{\frac{1}{2}}$ leads to

$$p(Y) = \xi/2\pi \int_{0}^{\infty} t^{\frac{1}{2}}(1+t)^{-\frac{3}{4}} \cos(\xi Yt^{\frac{3}{2}}) \cos(\arctan t^{\frac{1}{2}}) \cdot$$

$$\cdot \exp(-\frac{\xi^2}{2}t) \, dt$$

$$= \xi/2\pi \quad L(\xi^2/2) \quad (6.2.9)$$

in which $L(\xi^2/2)$ is the Laplace transform of the function which pre-multiplies $\exp(-\xi^2/2t)$ and is given by:

$$L(\xi^2/2) = (\pi/2\xi)^{\frac{3}{4}} \exp(-\xi^2/2) \exp\left\{\frac{(\xi+Y)^2}{4}\right\} D_{-\frac{3}{2}}(\xi+Y)$$

$$+ \exp\left\{\frac{(\xi-Y)^2}{4}\right\} D_{-\frac{3}{2}}(\xi-Y)\}$$

Hence, the probability function of random variable, $Y$, is given by:

$$p(Y) = (\xi/8\pi)^{\frac{3}{4}} \exp(-\xi^2) \left\{\exp\left\{\frac{(\xi+Y)^2}{4}\right\} D_{-\frac{3}{2}}(\xi+Y) \right.$$\n
$$+ \exp\left\{\frac{(\xi-Y)^2}{4}\right\} D_{-\frac{3}{2}}(\xi-Y)\} \quad (6.2.10)$$
in which $D_{-\frac{1}{2}}(z)$ is the parabolic cylindrical function which may be expressed in terms of Bessel functions (Miller, 1964):

$$D_{-\frac{1}{2}}(t) = (t/2\pi)^{\frac{1}{2}}K_{\frac{1}{2}}(t^2/4)$$

$$= \frac{1}{\pi} (\frac{1}{2} t)^{\frac{1}{2}} [I_{-\frac{1}{2}}(t^2/4) - I_{\frac{1}{2}}(t^2/4)], \text{ for } t > 0$$

$$D_{-\frac{1}{2}}(z) = \frac{1}{\pi} (|t|)^{\frac{1}{2}} [I_{-\frac{1}{2}}(t^2/4) + I_{\frac{1}{2}}(t^2/4)], \text{ for } t < 0 \quad (6.2.11)$$

in which $K_\nu$ and $I_\nu$ are modified Bessel functions of the second kind and first kind of order $\nu$ respectively. These Bessel functions may be expanded by the following polynomial series (Luke, 1975):

$$I_{\frac{1}{4}}(t) = t^{\frac{1}{4}} \sum_{n=0}^{\infty} b_n \cos n[\arccos(t/8)]$$

$$I_{-\frac{1}{4}}(t) = t^{-\frac{1}{4}} \sum_{n=0}^{\infty} c_n \cos n[\arccos(t/8)], \text{ for } t < 8$$

$$I_{\frac{1}{4}}(t) = (2\pi t)^{-\frac{1}{2}} \exp(t) \sum_{n=0}^{\infty} d_n \cos n[\arccos(16/t-1)]$$

$$I_{-\frac{1}{4}}(t) = (\pi t)^{-\frac{1}{2}} \exp(-t) \sum_{n=0}^{\infty} r_n \cos n[\arccos (10/t-1)]$$

$$+ I_{\frac{1}{4}}(t), \text{ for } t > 8 \quad (6.2.12)$$

These series converge very rapidly and the value of coefficients $b_n$, $c_n$, $d_n$ and $r_n$ are tabulated in Appendix G.

Equation (6.2.10) generates a curve that is symmetric to the vertical axis with a sharper peak than normal curve. This is demonstrated in Figure 6.2.1.

Statistical moments of a random variable, $Y$, may be expressed in terms of derivatives of characteristic function $\psi_Y(s)$ by:
Figure 6.2.1. Comparison between normal probability density function and parabolic probability density function.
\[ E(Y^n) = \left( i^n \frac{d^n}{ds^n} \psi_Y(s) \right) \bigg|_{s=0} \quad (6.2.13.a) \]

Since \( p(Y) \) is an odd function in \( Y \) \( (p(Y) = p(-Y)) \), the odd moments are zero while the even moments are given by

\[ E(Y^{2m}) = (-1)^m \frac{d^{2m}}{ds^{2m}} \psi_Y(s) \bigg|_{s=0} \quad (6.2.13.b) \]

Mean and skewness measures are zero and the standard deviation and the excess of kurtosis are given by

\[ \sigma_Y = (3 \frac{2}{u' u} + 1)^{\frac{1}{2}} \]
\[ \lambda_{4Y} = 105 \frac{4}{u' u} + 18 \frac{2}{a^2} \quad (6.2.14) \]

The expression for moment generating function given by Eq. (6.2.7) is different from that derived by Borgman (1965). However, the probability density function given by Eq. (6.2.10) which is the Fourier transform of the moment generating function is exactly similar to that of given by Borgman. Both Borgman (1965) and Hino (1969) also presented the expression for standard deviation for dimensionless wave forces (Eq. 6.2.1) as a function of the variance of water particle velocity. Although the mathematical expressions given by Hino and that given by Eq. (6.2.14) and by Borgman are different from each other, the approximate forms of them coincide with each other.

6.3 The Autocovariance Function and the Energy Spectrum for Wave Forces

The autocovariance function of the dimensionless stochastic wave forces, \( Y(t) \), is obtained from
\[ E[Y(t)Y^*(t+\tau)] = E\{[c_{ux}Y(t) + c_{ax}a_x(t)] \}
\]
\[
\begin{align*}
&= c_{ux}^2 E[u_x(t)u_x(t+\tau) | u_x(t)| u_x(t+\tau)] \\
&+ c_{ux} c_{ax} E[u_x(t) | a_x(t+\tau)] \\
&+ c_{ux} c_{ax} E[a_x(t)u_x(t+\tau) | u_x(t+\tau)] \\
&+ c_{ax}^2 E[a_x(t)a_x(t+\tau)]
\end{align*}
\]
\[(6.3.1)\]

Before evaluating each term in Eq. (6.3.1), the joint probability function for \( u_x(t) \) and \( u_x(t+\tau) \), \( u_x(t) \) and \( a_x(t+\tau) \), \( a_x(t) \) and \( u_x(t+\tau) \), \( a_x(t) \) and \( a_x(t+\tau) \) must be determined. Let subscripts 1 and 2 represent the value at \( t \) and complex-conjugate value at \( t+\tau \), respectively; then, we may write (Rice, 1945, pp. 50)

\[
p(u_1,u_2) = (2\pi)^{-1}(\sigma_u^2)^{-\frac{1}{2}} \exp\left\{-\frac{\sigma_u^{-2} + \sigma_a^{-2} - 2\sigma_{ua}^{-1}u_1u_2}{2(\sigma_u^{-2} - \sigma_{ua}^{-2})}\right\}
\]
\[(6.3.2)\]

in which \( \sigma_{uu} \) = the autocovariance function for random variable \( u_x \)

\[
p(u_1,a_2) = (2\pi)^{-1}(\sigma_u^2\sigma_a^2 - \sigma_{ua}^2)^{-\frac{1}{2}} \exp\left\{-\frac{\sigma_{ua}^{-2} + \sigma_a^{-2} - 2\sigma_{ua}^{-1}a_1a_2}{2(\sigma_u^2\sigma_a^2 - \sigma_{ua}^2)}\right\}
\]
\[(6.3.3)\]

in which \( \sigma_{ua} \) = the covariance function for random variables \( u_x \) and \( a_x \)

\[
p(a_1,u_2) = (2\pi)^{-1}(\sigma_u^2\sigma_a^2 - \sigma_{au}^2)^{-\frac{1}{2}} \exp\left\{-\frac{\sigma_{au}^{-2} + \sigma_u^{-2} - 2\sigma_{au}^{-1}a_1u_2}{2(\sigma_u^2\sigma_a^2 - \sigma_{au}^2)}\right\}
\]
\[(6.3.4)\]
in which \( C_{au} \) = the covariance function for random variables \( a_x \) and \( u_x \).

\[
p(a_1, a_2) = (2\pi)^{-1}(\sigma_a^4 - C_{aa}^2)^{-\frac{3}{2}} \exp \left[ -\frac{\sigma_{a1}^2 + \sigma_{a2}^2 - 2C_{aa}a_1a_2}{2(\sigma_a^4 - C_{aa}^2)} \right]
\]  

(6.3.5)

in which \( C_{aa} \) = the autocovariance function for random variable \( a_x \).

The first term in Eq. (6.3.1) may be evaluated by the following

\[
c^2 E(u_1u_2 | u_1 | u_2 |) = c^2_u \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_1u_2 | u_1 | u_2 | p(u_1, u_2) \, du_1 \, du_2
\]  

(6.3.6)

Substituting Eq. (6.3.2) into Eq. (6.3.6) and making a change of variables according to:

\[
u_1 = c_u (1-r^2)^{\frac{1}{2}} x_1, \quad u_2 = c_u (1-r^2)^{\frac{1}{2}} x_2
\]  
in which \( r = C_{uu}/\sigma_u^2 \) yields:

\[
(u_1u_2 | u_1 | u_2 |) = c^2_u \sigma_u^4 (1-r^2)^{2} E(x_1x_2 | x_1 | x_2 |)
\]

\[
= (2\pi)^{-1} c^2_u \sigma_u^4 (1-r^2)^{5/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1x_2 | x_1 | x_2 |) \cdot \exp \left[ -\frac{1}{2}(x_1^2 + x_2^2 - 2rx_1x_2) \right] dx_1 dx_2
\]

(6.3.7.a)

Equation (6.3.7.a) may be shown to be

\[
2(\pi)^{-1} c^2_u \sigma_u^4 (1-r^2)^{5/2} \int_{0}^{\infty} x^2 \exp (-x^2/2) \left\{ \int_{0}^{\infty} x^2 \sinh(rx_2x_1) \cdot \exp(-x^2/2) dx_1 \right\} dx_2
\]

(6.3.7.b)

Using a table of Laplace transforms (Oberhettinger, 1973) Eq. (6.3.7.b) may be shown to integrate to

\[
c^2 E(u_1u_2 | u_1 | u_2 |) = c^2_u \sigma_u^2 r
\]

(6.3.7.c)
in which \( \psi(r) \) is defined by

\[
\psi(r) = 2r^2 + 1 - \frac{32\sqrt{2}}{15\pi} (1-r)\frac{5/2}{F_1} \frac{F_1}{2} (\frac{1}{2}, \frac{3}{2}; \frac{7}{2}; \frac{1-r}{2}) \quad (6.3.8.a)
\]

The hypergeometric function, \( _2F_1 \), may be calculated from the series

(Miller, 1964)

\[
_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)\ n!} \ z^n; \quad |n| < 1 \quad (6.2.8.b)
\]

in which the gamma function, \( \Gamma(v) \) is defined by indefinite integral

\[
\Gamma(v) = \int_0^\infty t^{v-1} \exp(-t)dt \quad (6.2.8.c)
\]

The second and third terms in Eq. (6.3.1) may be evaluated simultaneously, i.e.,

\[
c_{u}c_{a}[E(u_1|u_1|a_2) + E(a_1u_2|u_2|)] = \\
c_{u}c_{a} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u_1|u_1|a_2)p(u_1,a_2)du_1da_2 + \right. \\
\left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (a_1u_2|u_2)p(a_1,u_2)da_1du_2 \right\} \quad (6.3.9)
\]

Substituting Eq. (6.3.3) and (6.3.4) into Eq. (6.3.9) and noting that \( C_{u}a = -C_{a}u \) yields:

\[
c_{u}c_{a}[E(u_1|u_1|a_2) + E(a_1u_2|u_2|)] = \\
(\pi)^{-1}c_{u}c_{a}(\sigma_{u}^2a_{2}^2 - C_{a}u^2) \frac{1}{2} \int_{-\infty}^{\infty} u_2|u_2| \exp\left[-\frac{\sigma_{u}^2a_{2}^2}{2(\sigma_{u}^2-C_{u}u^2)}\right] \\
\left( \int_{-\infty}^{\infty} \frac{a_1\exp\left[-\frac{\sigma_{a}^2a_{1}^2}{2(\sigma_{a}^2-C_{a}u^2)}\right]\cosh\left(\frac{C_{u}u_{2}a_{1}}{2\sigma_{u}^2-C_{u}u^2}\right)da_1\right)du_2 \quad (6.3.10)
\]
Noting that the integrand under the integral in the bracket \( \{ \} \) is an odd function in \( a_1 \), the integral over \((-\infty, \infty)\) is equal to zero and we have:

\[
c_u c_a [E(u_1|u_1|a_2) + E(a_1u_2|u_2|)] = 0 \quad (6.3.11)
\]

Finally, the last term in Eq. (6.3.1) is the auto covariance function for random variable \( a_x \) which may be written as follows:

\[
c_a^2 E[a_1a_2] = c_a^2 C_{aa} \quad (6.3.12)
\]

Eq. (6.3.1) may now he expressed in terms of only autocovariance functions for horizontal velocity \( u_x \) and acceleration \( a_x \); i.e.,

\[
E[Y(t)Y^*(t+\tau)] = c_u^2 \sigma^2 [\frac{uu}{2}] + c_a^2 C_{aa}(\tau) \quad (6.3.13)
\]

Recalling that \( c_u = 2/\pi(C_D/\sigma_a M D) \) and \( c_a = 1/\sigma_a \), we may write the autocovariance function for random wave forces, \( F_T \), as follows

\[
C_{FTFT}(\tau) = \frac{\rho C_D}{2} \sigma_u [\frac{uu}{2}] + \frac{\rho C_M}{4} \sigma_u^2 C_{aa}(\tau) \quad (6.3.14)
\]

The spectral density for \( F_T \) may be calculated from the Wiener-Khinchine's relationship given by Eqs. (2.5.8.a)

\[
S_{FTFT}(\omega) = \int_{-\infty}^{\infty} C_{FTFT}(\tau) \exp(-i\omega \tau) d\tau \quad (2.5.8.a)
\]

This integral may be difficult to evaluate in closed form and will be calculated numerically for the force spectral density. Knowing the value of \( C_{uu}(\Delta \tau) \) and \( C_{aa}(\Delta \tau) \), we may calculate the wave force auto-covariance function by Eq. (6.3.14) and determine the wave force.
spectral density from Eq. (6.3.5) by using a Fast Fourier Transform algorithm (FFT).

Borgman (1965) derived a representation for the autocovariance function for wave forces by a different approach. The solution to the autocovariance of the drag term in the Morison equation given by the first term of Eq. (6.3.1) was obtained by solving a third-order differential equation in $C_{uu}(t)$ with given boundary conditions. The differential equation was developed based on a theorem due to Price (1958). The solution to this differential equation was presented in a simple form of the function of the autocovariance of the horizontal water particle velocity.

Hino (1969) employed the Laplace transformation method to avoid the complication caused by the nonlinearity of the drag term in the Morison equation. In contrast to the derivation by Borgman (1965) and by the author, Hino obtained the expression for the autocovariance of the wave forces by a time averaging approach. However, the results obtained by Hino and Borgman numerically converged to the same value as that of Eq. (6.3.14) for the case investigated in this thesis. Furthermore, Hino included the steady current in the drag force and the effect of this current to the value of the autocovariance function of the wave forces. Further study is necessary to investigate the uniqueness of the solution to Eq. (6.3.1).

We may now approximate the autocovariance function for the wave forces by linearizing the drag term in the Morison equation. A reasonable approach would be to use the linear form of the horizontal water particle velocity replacing the nonlinear drag term; i.e.,
\[ u|u| = \sigma u \quad (6.3.15.a) \]

in which the constant, \( \sigma \), is estimated by using a least-squares estimate. This may be done by minimizing the variance

\[ E\{[u|u| - \sigma u]^2\} \quad (6.3.15.b) \]

and the result of this minimization is given by (Borgman, 1972)

\[ \sigma = \sigma_u (8/\pi)^{\frac{1}{3}} \quad (6.3.15.c) \]

Substituting Eq. (6.13.15.c) into Eq. (6.1.3) yields

\[ F_T(z,t) = (2/\pi)^{\frac{3}{4}} \rho C_D D \sigma u_x + \rho C_M \frac{\pi D^2}{4} a_x \quad (6.3.16) \]

The autocovariance function for the linearized wave forces may be expressed as a linear combination of the autocovariance function for velocity, \( C_{uu}(\tau) \), with the autocovariance function for acceleration, \( C_{aa}(\tau) \); i.e.,

\[ C_{FTT}(\tau) = (2/\pi)^{\frac{3}{4}} (\rho C_D D \sigma u)^2 C_{uu}(\tau) + (\rho C_M \pi D^2/4)^2 C_{aa}(\tau) \quad (6.3.17) \]

The expression given by Eq. (6.3.17) is exactly similar in form to the first series approximation of the autocovariance expression given by Borgman (1965).

The Fourier transform of Eq. (6.3.17) yields the spectral density function for the linearized forces

\[ S_{FTT}(\omega) = (2/\pi)^{\frac{3}{4}} (\rho C_D D \sigma u)^2 S_{uu}(\omega) + (\rho C_M \pi D^2/4)^2 C_{aa}(\tau) \quad (6.3.18) \]
6.4 Simulation in Time Domain

This method for digital simulation is very straightforward; the wave kinematics at specified depths may be calculated from Eq. (5.1.1.a) and (5.1.1.b) for linear waves, and from Eq. (5.1.5.a) and (5.1.5.b) or Eq. (5.2.4) and (5.2.10) for waves correct to second-order perturbation expansion. Using the Morison equation given by Eq. (6.1.3), we may predict the wave forces acting on a specified location as a function of time (drag and inertia coefficients, $C_M$, $C_D$, are assumed to be known).

As noted earlier, the statistical properties of the simulated wave forces are dependent upon the statistical properties of the wave kinematics; hence, the assumptions made for the probability distribution for wave kinematic realizations will directly affect the wave force statistics. As we have stated earlier, we did not make any attempt to derive the probability distribution for the wave forces due to the nonlinear waves.

6.5 Frequency-domain Simulation

Due to the nonlinearity of the drag component of wave forces, expressing a random wave force by a Fourier series becomes a very complicated problem. It is very useful to linearize this drag force component as was done in section 6.3. The linearized form of the Morison equation is given by Eq. (6.3.16).

$$F_T(z,t) = \left(\frac{2}{\pi}\right)^\frac{3}{2} (\rho C_D D a_x) u_{x} + \rho C_M \frac{\pi D^2}{4} a_{x} \quad \text{(6.3.16)}$$
The advantage of this type of simulation is that we do not need to calculate the wave kinematics in the time domain in order to calculate the wave forces. The wave forces may be obtained directly from the inverse Fourier transform of the product between Fourier coefficients for the design wave spectrum and the transfer function. Mathematically, this may be expressed as:

\[ F_T(z,t) = \sum_{n=-N}^{N} T_{FT}(z; \omega_n) F(n) \exp(i \omega_n t) \]  \hspace{1cm} (6.5.2)

In which \( F(n) \) is given by Eq. (4.1.1.b) for linear waves and Eq. (5.1.2) for nonlinear waves, and the complex-valued transfer function \( T_{FT}(z; \omega_n) \) for linearized wave force may be expressed as:

\[ T_{FT}(z; \omega_n) = \rho D \frac{g k_n \cosh[k_n (h+z)]}{\omega_n \cosh(k_n h)} \left[ (2/\pi)^{1/2} \sigma u C_D \right] + i \frac{\pi}{4} \omega_n C_M D ] \]  \hspace{1cm} (6.5.3)

The plot of this transfer function versus frequency is illustrated in Figure 6.5.1. Note that the plot consists of two parts which give magnitude and phase angle.
water depth $h = 99$ ft
vertical location $z = -43.6$ ft
pile diameter $D = 3.71$ ft

$C_D = 0.92$
$C_M = 1.18$

Figure 6.5.1. Transfer function for linearized wave forces.
7.0 APPLICATIONS AND RESULTS OF SIMULATION METHODS

Three different wave energy spectral representations were examined in order to compare the effect of spectral shape on the simulation of random sea surface realizations and random wave pressure forces at a particular vertical elevation beneath the free surface which were simulated by DSA and NSA methods. The first of these spectra was obtained from the water surface elevations recorded by Wave Force Project II during Hurricane Carla in the Gulf of Mexico during the period September 9-10, 1961. The other two spectra were two theoretical two-parameter spectra which were fitted to the smoothed measured spectrum from Data Record No. 6887 using the parameters of the area under spectrum, \( m_o \), and the peak frequency, \( \omega_p \) (Hudspeth, 1974). The two theoretical spectral representations selected for comparison were the Bretschneider and Scott spectra (cf. Table 4.4.1).

The water particle kinematics were predicted by LFD, HFD, and NFD transfer functions which operated on the spectra of the simulated random sea waves. These water particle kinematics were used to compute the random wave forces realizations by the Morison equation. These simulated wave forces were compared with the measured wave forces from WP II Data Record No. 6887.

7.1 Random Sea Surface Simulations by NSA and DSA Methods

The simulation procedures for NSA and DSA simulations are given in detail in section 4.2. The algorithms for generating uniform random
numbers were investigated by Hudspeth and Chen (1979), and the one which he found the most representative was used to produce the uniform random numbers. The normally distributed random numbers were generated by the sum of uniform deviates method (Zelen, M., Severo, N.C., 1972, p. 953). These normal random numbers were tested by chi-square, $\chi^2$, goodness-of-fit test (Bendat, J. S. and Piersol, A. G., 1971, pp. 119-125). From the 2000 random numbers tested, the reduced $\chi^2$, with a 5% level of significance based on equal class intervals, was 1.00073. This is acceptable to represent the normally distributed random numbers.

Characteristics of the wave simulation are identical with those of data records no. 6887 and are tabulated in Table 7.1.1.

Table 7.1.1. Characteristics of wave simulation realizations (from Hudspeth, 1974).

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>Length of Time Series</th>
<th>Cut off frequency</th>
<th>Peak frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time (min)</td>
<td>No. of points (NP)</td>
<td>$\Delta t$ (sec)</td>
</tr>
<tr>
<td>Measured</td>
<td>11.00</td>
<td>4096</td>
<td>0.16</td>
</tr>
<tr>
<td>Scott</td>
<td>11.00</td>
<td>4096</td>
<td>0.16</td>
</tr>
<tr>
<td>Bretschneider</td>
<td>11.00</td>
<td>4096</td>
<td>0.16</td>
</tr>
</tbody>
</table>

A set of 4096 independent uniform random numbers with initial seed no. 82356917 and a set of 9192 independent normally distributed random numbers with two initial seed nos. 82176329 and 22179102 were generated to simulate the DSA sea surface realizations and NSA sea surface realizations, respectively. The wave energy spectral densities were obtained via FFT computation from simulated realizations.
Table 7.1.2 summarizes the spectral moments and the spectral bandwidth parameter values for both the measured and computed spectra. The spectral moment of order $j$ is defined by

$$m_j = \int_0^\infty \omega^j S(\omega) d\omega ; \quad j \geq 0 \quad (7.1.1)$$

in which $\omega = \text{radial frequency}$

$S(\omega) = \text{one-sided energy density spectrum}$

The zeroeth moment is the energy per unit density due to the wave fluctuation.

The following parameter may be useful for the interpretation of the $j^{th}$ spectral moment (Vanmarcke, 1972):

$$\tilde{m}_j = \left( \frac{m_j}{m_0} \right)^{1/j} ; \quad j \geq 1 \quad (7.1.2)$$

$\tilde{m}_j$ has the dimension of radial frequency. $\tilde{m}_1$ may be interpreted as the distance of the centroid of the spectral mass, $S(\omega)$, from the origin or the mean frequency of the spectrum, $S(\omega)$ (Longuet-Higgins, 1975). The average frequency of the zero crossing waves may be given by $\tilde{m}_2$; whereas, $m_2$ may be interpreted as the total variance of the slope of the time series. The total variance of the second derivative of the time series is given by $m_4$. The two spectral bandwidth parameters which are proposed by Vanmarcke (1972) and by Cartwright and Longuet-Higgins (1956), respectively, are defined as

$$q = (1 - \frac{m_2}{m_0 \tilde{m}_2})^{1/2} \quad (7.1.3.a)$$
Table 7.1.2. Spectral moments and bandwidth parameters for simulated linear wave spectra for WPII data 6887 (cf. Fig. 4.1). Cut-off frequency $\omega_c = 2.4$ rad/sec and the peak frequency $\omega_p = 0.52$ rad/sec.

<table>
<thead>
<tr>
<th>Spectral measure</th>
<th>Smoothed measured</th>
<th>Linear Simulation Spectrum</th>
<th>Bretschneider</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DSA (2)</td>
<td>NSA (3)</td>
<td>DSA (4)</td>
</tr>
<tr>
<td>$m_2 [ft^2-rad^2/sec^2]$</td>
<td>17.500</td>
<td>17.866</td>
<td>17.355</td>
</tr>
<tr>
<td>$m_4 [ft^2-rad^4/sec^4]$</td>
<td>24.683</td>
<td>22.613</td>
<td>23.268</td>
</tr>
<tr>
<td>$q$</td>
<td>0.407</td>
<td>0.377</td>
<td>0.411</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.751</td>
<td>0.727</td>
<td>0.739</td>
</tr>
</tbody>
</table>
and
\[ \varepsilon = \left(1 - \frac{m_2}{m_0^2}\right)^{\frac{1}{2}} \]  

(7.1.3.b)

The Vanmarcke spectral bandwidth parameter, \( q \), may be interpreted as the degree of the dispersion of \( S(\omega) \) about its mean frequency. Its values are bounded by \( 0 \leq q \leq 1 \). The Cartwright and Longuet-Higgins spectral bandwidth parameter, \( \varepsilon \), may be obtained by the following integral:

\[ \varepsilon^2 = \frac{1}{m_4} \int_0^\infty (\omega^2 - m_2^2)^2 S(\omega) \, d\omega \]

This parameter represents the peakedness of \( S(\omega) \) about the mean frequency of zero upcrossing waves. The spectral bandwidth, \( \varepsilon \), is more sensitive to spectral bandwidth measure than its counterpart, \( q \). This statement may be proven by the data collected from simulated random sea surface realizations in which \( \varepsilon \) is consistently larger than \( q \).

The NSA-simulated time series contains more energy than the DSA-simulated time series does. Higher energy content for NSA method perhaps is due to the randomness of the sample spectrum. This is demonstrated in Figs.(7.1.1, 2, 3).

The statistical measures for simulated random sea surfaces are also computed. Two of these measures, the mean sea level, \( \mu \), and the total variance, \( \sigma^2 \), are imposed on the simulation. The skewness for a zero mean process is defined by

\[ \lambda_3 = \frac{1}{(\sigma^2_\eta)^{3/2}} \int_{-\infty}^{\infty} \eta^3 \beta(\eta) \, d\eta \]  

(7.1.4)
Table 7.1.3. Statistical measures and reduced $\chi^2$ goodness-of-fit with 95% confidence interval based on equal class intervals for random waves for WPII data 6887. Class intervals NC = 105.

<table>
<thead>
<tr>
<th>Spectrum Type</th>
<th>Method of Simulation</th>
<th>$\mu x 10^5$ (ft)</th>
<th>$\sigma^2_n$ (ft$^2$)</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>Reduced $\chi^2$ (Gaussian)</th>
<th>Reduced $\chi^2$ (Gram-Charlier)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured T.S.</td>
<td></td>
<td>0.220</td>
<td>29.019</td>
<td>0.326</td>
<td>0.755</td>
<td>6.560</td>
<td>2.350</td>
</tr>
<tr>
<td>Smoothed Measured</td>
<td>Linear</td>
<td>DSA</td>
<td>0.000</td>
<td>28.370</td>
<td>0.057</td>
<td>-0.112</td>
<td>2.028</td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>-0.099</td>
<td>29.911</td>
<td>0.112</td>
<td>0.365</td>
<td>1.622</td>
<td>2.501</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>DSA</td>
<td>0.000</td>
<td>30.215</td>
<td>0.184</td>
<td>0.026</td>
<td>2.542</td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>0.194</td>
<td>31.360</td>
<td>0.215</td>
<td>0.545</td>
<td>3.931</td>
<td>1.863</td>
</tr>
<tr>
<td>Scott</td>
<td>Linear</td>
<td>DSA</td>
<td>0.000</td>
<td>28.405</td>
<td>0.047</td>
<td>-0.193</td>
<td>1.929</td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>0.114</td>
<td>31.584</td>
<td>0.073</td>
<td>0.170</td>
<td>1.977</td>
<td>2.425</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>DSA</td>
<td>0.000</td>
<td>29.368</td>
<td>0.222</td>
<td>-0.133</td>
<td>2.368</td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>-0.178</td>
<td>33.489</td>
<td>0.195</td>
<td>0.380</td>
<td>4.325</td>
<td>1.917</td>
</tr>
<tr>
<td>Bretschneider</td>
<td>Linear</td>
<td>DSA</td>
<td>0.000</td>
<td>28.533</td>
<td>0.037</td>
<td>-0.181</td>
<td>1.913</td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>0.135</td>
<td>31.565</td>
<td>0.072</td>
<td>0.246</td>
<td>1.743</td>
<td>2.267</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>DSA</td>
<td>0.000</td>
<td>29.900</td>
<td>0.205</td>
<td>-0.130</td>
<td>2.065</td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>0.075</td>
<td>32.992</td>
<td>0.198</td>
<td>0.388</td>
<td>3.968</td>
<td>1.597</td>
</tr>
</tbody>
</table>
Figure 7.1.1. Smoothed measured linear wave energy spectra for WPII Data No. 6887. $m_0 = 28.450\text{ ft}^2$, $\omega_c = 2.4\text{ rad/sec}$, $\omega_p = 0.52\text{ rad/sec}$. 
Figure 7.1.2. Scott linear wave energy spectra for WPII Data No. 6887. \( m_0 = 28.525 \text{ ft}^2 \), \( \omega_c = 2.4 \text{ rad/sec} \), \( \omega_D = 0.52 \text{ rad/sec} \).
Figure 7.1.3. Bretschneider linear wave energy spectra for WPII Data No. 6887. \( m_0 = 28.397 \, \text{ft}^2, \) \( \omega_c = 2.4 \, \text{rad/sec}, \) \( \omega_p = 0.52 \, \text{rad/sec}. \)
This is the most important characteristic of the nonlinearity of the sea surfaces (Longuet-Higgins, 1963). For random sea surface realizations the skewness is positive which means the trough is flattened and the peak is sharpened. This fact is demonstrated in Table 7.1.2. The skewness for nonlinear waves simulated by both NSA and DSA methods is always lower than the skewness of the measured waves. The skewness is heavily affected by the directionality of the wave. Sharma (1979) has shown the least skewness occurs when the wave is unidirectional and may account for these lower values.

The excess of kurtosis for a zero mean process is defined by

\[ \lambda_4 = \frac{1}{(\sigma_\eta^4)^2} \int_{-\infty}^{\infty} \eta^4 p(\eta) d\eta - 3 \]  (7.1.5)

This is the measure of the peakedness of a density distribution compared with the normal distribution. The excess of kurtosis computed from sea surface realizations simulated by NSA gives better approximations to the measured sea surfaces, as is shown in Table 7.1.3.

The reduced chi square, \( \chi^2 \), goodness-of-fit test measures for both Gaussian and Gram-Charlier distributions were computed for each simulation. The number of class interval, \( NC \), was determined according to the relationship given by Williams (1950) using 5% level of significance according to

\[ NC = 4 \text{ antilog} \left[ 0.2 \log \left( \frac{2(NP - 1)^2}{1.64^2} \right) \right] \]  (7.1.5)

in which \( NP = \) number of points.
Values of the reduced $\chi^2$ are summarized in Table 7.1.3. The number of degrees-of-freedom was determined by the method of Bevington (1969, pp. 66-89) according to

$$\text{DOF} = NC - 2$$  \hspace{1cm} (7.1.6)

Based on the reduced $\chi^2$ goodness-of-fit test, the NSA method of simulation gives a better fit to the theoretical distribution than does the DSA method; and, the Gram Charlier distribution gives a better representation for nonlinear and measured sea surfaces than does the Gaussian distribution (cf., Sec. 4.2).

### 7.2 Wave Pressure Forces

The water particle kinematics were computed 43.6 feet below the mean sea surface in water with a total depth of 99 feet. Three types of transfer functions presented in Chapter 5 were employed to predict the water particle velocities and accelerations from the simulated random sea surfaces. The horizontal pressure forces were computed from the Morison equation using drag and modified inertia coefficients suggested by Dean and Aagaard (1970; Hudspeth, 1974). These coefficients are given in Table 7.2.1. The density of sea water was assumed to be 2.0 slugs/ft$^3$, and the pile diameter, 3.71 feet.

Table 7.2.2 demonstrates the comparisons of the spectral measures determined by employing the wave kinematics predicted by three different types of transfer functions. Surprisingly, the total variance of the simulated wave forces are in reasonable agreement with the total variance of the measured wave forces, even though no
Table 7.2.1. Drag and modified inertia force coefficients for resultant pressure forces (from Hudspeth, 1974).

<table>
<thead>
<tr>
<th>Reynold No. (x10^-5)</th>
<th>( R_e &lt; 3 )</th>
<th>( 3 \leq R_e \leq 10 )</th>
<th>( 10 \leq R_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_D )</td>
<td>1.34</td>
<td>0.98</td>
<td>0.92</td>
</tr>
<tr>
<td>( C_M )</td>
<td>1.18</td>
<td>1.18</td>
<td>1.18</td>
</tr>
</tbody>
</table>

attempt was made to equate them. However, the first, second, and the fourth spectral moments differ significantly from the measured force spectrum which has larger values than does the simulated wave forces spectrum. Consequently, both bandwidth parameters, \( q \) and \( \epsilon \), of the measured spectrum are larger than those of simulated wave forces spectrum. This demonstrates that the measured spectrum disperses from its mean frequency with a higher degree than do the simulated wave force spectra.

Figures 7.2.1-6 demonstrate how the measured spectrum compares with the wave force spectra simulated by two different methods. The comparisons would perhaps be improved if the low and the high frequency forces present in the measured wave spectrum were removed. As expected, a high degree of randomness is apparent in the wave forces spectra simulated by the NSA method. Figures 7.2.7-10 are the measured wave forces time series and linear and nonlinear wave forces time series simulated from theoretical wave energy spectra.

Statistical measures computed both from measured and simulated wave forces realizations and from theory are tabulated in Table 7.2.3. The standard deviation of the simulated wave forces and the measured wave forces agree favorably with the theoretical standard.
Table 7.2.2. Spectral moments and Bandwidth parameters for simulated wave force spectra for WP II data 6887.

<table>
<thead>
<tr>
<th>Spectrum Type</th>
<th>Method of Simulation</th>
<th>$m_0$ (ft$^2$) (4)</th>
<th>$m_1$ (ft$^2$-rad) (5)</th>
<th>$m_2$ (ft$^2$-rad$^2$) (6)</th>
<th>$m_4$ (ft$^2$-rad$^4$) (7)</th>
<th>q (8)</th>
<th>e (9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured T.S.</td>
<td>-</td>
<td>49.326</td>
<td>51.393</td>
<td>183.390</td>
<td>0.584</td>
<td>0.833</td>
<td></td>
</tr>
<tr>
<td>Linear</td>
<td>DSA</td>
<td>46.399</td>
<td>30.880</td>
<td>22.531</td>
<td>0.346</td>
<td>0.692</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>50.071</td>
<td>33.489</td>
<td>24.853</td>
<td>0.345</td>
<td>0.693</td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>DSA</td>
<td>60.263</td>
<td>38.528</td>
<td>29.451</td>
<td>0.404</td>
<td>0.723</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>51.922</td>
<td>34.082</td>
<td>27.090</td>
<td>0.353</td>
<td>0.697</td>
<td></td>
</tr>
<tr>
<td>Nonlinear</td>
<td>DSA</td>
<td>59.761</td>
<td>37.798</td>
<td>28.796</td>
<td>0.407</td>
<td>0.723</td>
<td></td>
</tr>
<tr>
<td>T.F.</td>
<td>NSA</td>
<td>53.412</td>
<td>35.512</td>
<td>27.105</td>
<td>0.364</td>
<td>0.698</td>
<td></td>
</tr>
<tr>
<td>Scott</td>
<td>Linear</td>
<td>49.044</td>
<td>31.350</td>
<td>21.465</td>
<td>0.353</td>
<td>0.709</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>54.345</td>
<td>34.834</td>
<td>24.348</td>
<td>0.354</td>
<td>0.712</td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>DSA</td>
<td>51.688</td>
<td>32.240</td>
<td>21.503</td>
<td>0.366</td>
<td>0.718</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>56.764</td>
<td>36.811</td>
<td>27.535</td>
<td>0.366</td>
<td>0.717</td>
<td></td>
</tr>
<tr>
<td>Nonlinear</td>
<td>DSA</td>
<td>54.219</td>
<td>33.407</td>
<td>21.873</td>
<td>0.375</td>
<td>0.719</td>
<td></td>
</tr>
<tr>
<td>T.F.</td>
<td>NSA</td>
<td>58.249</td>
<td>37.115</td>
<td>26.860</td>
<td>0.376</td>
<td>0.717</td>
<td></td>
</tr>
<tr>
<td>Bretschneider</td>
<td>Linear</td>
<td>50.978</td>
<td>32.561</td>
<td>21.186</td>
<td>0.338</td>
<td>0.700</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>57.034</td>
<td>36.735</td>
<td>25.087</td>
<td>0.342</td>
<td>0.706</td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>DSA</td>
<td>53.617</td>
<td>33.501</td>
<td>21.277</td>
<td>0.349</td>
<td>0.708</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSA</td>
<td>59.324</td>
<td>38.403</td>
<td>27.527</td>
<td>0.356</td>
<td>0.713</td>
<td></td>
</tr>
<tr>
<td>Nonlinear</td>
<td>DSA</td>
<td>56.057</td>
<td>34.620</td>
<td>21.599</td>
<td>0.359</td>
<td>0.709</td>
<td></td>
</tr>
<tr>
<td>T.F.</td>
<td>NSA</td>
<td>61.325</td>
<td>39.001</td>
<td>27.295</td>
<td>0.367</td>
<td>0.713</td>
<td></td>
</tr>
</tbody>
</table>
Table 7.2.3 Statistical measures and \( \chi^2 \) goodness-of-fit with 95% confidence interval and equal class interval for dimensionless simulated wave forces for WP II data 6887. Time* = time series; Dist* = theoretical distribution.

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>Type</th>
<th>Method of Simulation</th>
<th>( \mu )</th>
<th>( \chi^2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_{4Y} )</th>
<th>Reduced ( \chi^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Msd. Force T.S.</td>
<td>Linear</td>
<td>DSA</td>
<td>-0.004</td>
<td>1.306</td>
<td>1.300</td>
<td>0.129</td>
<td>1.170</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>NSA</td>
<td>0.014</td>
<td>1.358</td>
<td>1.312</td>
<td>0.170</td>
<td>3.377</td>
</tr>
<tr>
<td></td>
<td>Hybrid</td>
<td>DSA</td>
<td>-0.025</td>
<td>1.410</td>
<td>1.305</td>
<td>-0.402</td>
<td>2.409</td>
</tr>
<tr>
<td></td>
<td>Hybrid</td>
<td>NSA</td>
<td>0.007</td>
<td>1.364</td>
<td>1.301</td>
<td>-0.254</td>
<td>3.901</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>DSA</td>
<td>-0.038</td>
<td>1.413</td>
<td>1.390</td>
<td>-0.642</td>
<td>2.493</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>NSA</td>
<td>0.024</td>
<td>1.390</td>
<td>1.314</td>
<td>-0.875</td>
<td>6.252</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>DSA</td>
<td>-0.003</td>
<td>1.363</td>
<td>1.331</td>
<td>-0.351</td>
<td>2.836</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>NSA</td>
<td>0.007</td>
<td>1.403</td>
<td>1.368</td>
<td>-0.176</td>
<td>3.590</td>
</tr>
<tr>
<td></td>
<td>Hybrid</td>
<td>DSA</td>
<td>-0.021</td>
<td>1.396</td>
<td>1.343</td>
<td>-0.674</td>
<td>3.907</td>
</tr>
<tr>
<td></td>
<td>Hybrid</td>
<td>NSA</td>
<td>0.012</td>
<td>1.412</td>
<td>1.368</td>
<td>-0.215</td>
<td>4.065</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>DSA</td>
<td>-0.036</td>
<td>1.428</td>
<td>1.352</td>
<td>-1.111</td>
<td>5.803</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>NSA</td>
<td>0.028</td>
<td>1.438</td>
<td>1.375</td>
<td>-0.852</td>
<td>6.216</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>DSA</td>
<td>-0.005</td>
<td>1.354</td>
<td>1.340</td>
<td>-0.332</td>
<td>2.159</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>NSA</td>
<td>0.007</td>
<td>1.405</td>
<td>1.372</td>
<td>0.084</td>
<td>3.400</td>
</tr>
<tr>
<td>Breitschneider</td>
<td>Hybrid</td>
<td>DSA</td>
<td>-0.022</td>
<td>1.388</td>
<td>1.353</td>
<td>-0.600</td>
<td>3.100</td>
</tr>
<tr>
<td></td>
<td>Hybrid</td>
<td>NSA</td>
<td>0.012</td>
<td>1.416</td>
<td>1.372</td>
<td>-0.330</td>
<td>4.217</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>DSA</td>
<td>-0.036</td>
<td>1.415</td>
<td>1.361</td>
<td>-0.985</td>
<td>4.698</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>NSA</td>
<td>0.029</td>
<td>1.445</td>
<td>1.380</td>
<td>-0.973</td>
<td>6.772</td>
</tr>
</tbody>
</table>
Figure 7.2.1. Spectral comparison of wave force simulated by linear DSA method for WPII Data No. 6887.
Figure 7.2.2. Spectral comparison of wave force simulated by DSA-HFD method for WPII Data No. 6887.
Figure 7.2.3. Spectral comparison of wave force simulated by DSA-NFD method for WPII Data No. 6887.
Figure 7.2.4. Spectral comparison of wave force simulated by linear NSA method for WPII Data No. 6887.
Figure 7.2.5. Spectral comparison of wave force simulated by NSA-HFD method for WPII Data No. 6887.
Figure 7.2.6. Spectral comparison of wave force simulated by NSA-NFD method for WPII Data No. 6887.
Figure 7.2.7. Wave force ensemble comparison between measured realization and linear and nonlinear realizations simulation by DSA-HFD method for WPII Data No. 6887.
Figure 7.2.8. Wave force ensemble comparison between measured realization and linear and nonlinear realizations simulated by DSA-NPD method for WPII Data No. 6887.
Figure 7.2.9. Wave force ensemble comparison between measured realization and linear and nonlinear realizations simulated by NSA-HFD method for WPII Data No. 6887.
Figure 7.2.10. Wave force ensemble comparison between measured realization and linear and nonlinear realizations simulated by NSA-NFD method for WPII Data No. 6887.
deviation. Except for linear waves simulated from the smoothed measured wave energy spectrum and simulated by the NSA method, all of the simulated wave spectra demonstrate the negative skewness which is not consistent with those of the simulated sea surfaces. This discrepancy remains an enigma.

The positive value of the excess of kurtosis shows that the normalized histogram of the frequency distribution of the wave force realizations is concentrated near the mean value. The application of the $\chi^2$ goodness-of-fit test provides no information to indicate that the wave force probability distribution will obey the theoretical derivation of the wave force distribution. Figures 7.2.11-16 show that a large discrepancy between the sample distribution and the theoretical distribution occurs at the tails.
Figure 7.2.11. Comparison of cumulative probability distributions for wave forces simulated from measured wave spectrum by DSA method for WPII Data No. 6887.
Figure 7.2.12. Comparison of cumulative probability distributions for wave forces simulated from Scott spectrum by DSA method for WPII Data No. 6887.
Figure 7.2.13. Comparison of cumulative probability distributions for wave forces simulated from Bretschneider spectrum by DSA method for WPII Data No. 6887.
Figure 7.2.14. Comparison of cumulative probability distributions for wave forces simulated from measured wave spectrum by NSA method for WPII Data No. 6887.
Figure 7.2.15. Comparison of cumulative probability distributions for wave forces simulated from Scott spectrum by NSA method for WPII Data No. 6887.
Figure 7.2.16. Comparison of cumulative probability distributions for wave forces simulated from Bretschneider spectrum by NSA method for WPII Data No. 6887.
8.0 SUMMARY AND CONCLUSIONS

8.1 Random Sea Surfaces Simulations

In the present study two types of simulations, NSA (Nondeterministic Spectral Amplitude) and DSA (Deterministic Spectral Amplitude) are utilized to simulate both the Gaussian sea surfaces and slightly non-Gaussian sea surfaces. The superposition procedure, in which the ocean wave processes are considered as the sum of infinite number of sinusoidal waves with different amplitude, frequency, and phase, is applied for both the NSA and DSA methods. Although both methods produce realizations with similar distributions, the NSA method provides a wave energy spectrum with randomness which characterizes the real sea wave energy spectrum. The randomness in the wave energy spectrum is due to the fact that the amplitudes and the phase angles in the NSA method are treated as nondeterministic quantities. In contrast to the NSA method, the DSA method considers only the phase angles as the random quantities while the amplitudes and the frequencies are deterministic.

The expected spectrum obtained by averaging the sample spectrum will, theoretically, be equal to the target spectrum. This statement has not, however, been demonstrated by the data presented by this thesis. Further investigations are needed to support this theoretical prediction. Furthermore, the directional effects due to the randomness of the ocean waves may also be taken into account for the NSA method of simulation.
8.2 Wave Forces Simulations

Three distinct transfer functions were developed in Chapter 5; viz. LFD (Linear Frequency Domain), HFD (Hybrid Frequency Domain), and NFD (Nonlinear Frequency Domain). These transfer functions operate on the complex-valued wave spectral amplitudes which are simulated by both NSA and DSA methods to compute the complex-valued wave kinematics spectral amplitudes. The inverse Fourier transform of the complex-valued wave kinematics spectral amplitude is computed by an FFT algorithm. The efficiency of the FFT algorithm makes the simulation of wave kinematics in the frequency domain much faster than that in the time domain.

The HFD transfer function is a linear transfer function which operates on the nonlinear sea. It is the Fourier transform of the linear filter given by Reid (1958) and Wheeler (1970). This transfer function is more intuitive than exact. However, the wave kinematics predicted by HFD transfer function closely approximate the nonlinear waves and the real sea wave kinematics as shown in Chapter 7 (also refer to Reid, 1958; Wheeler, 1970; Hudspeth, 1974). On the contrary, the NFD transfer function is the exact derivation of the nonlinear second-order wave kinematics in which the convective terms are also included. Although the velocity potential nonlinear interaction matrix required to compute the NFD transfer function is symmetric and antisymmetric (cf. Section 3.3), the prediction of the wave kinematics by the NFD transfer function requires more computer time than the HFD transfer function. Table 8.2.1 compares the times required to compute
the wave kinematics by both the HFD and NFD transfer functions.

Table 8.2.1. The computer execution time required to compute the wave kinematics (CDC CYBER 80 NOS. 1-3).

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>HFD (sec)</th>
<th>NFD (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured</td>
<td>140</td>
<td>200</td>
</tr>
<tr>
<td>Scott</td>
<td>110</td>
<td>160</td>
</tr>
<tr>
<td>Bretschneider</td>
<td>130</td>
<td>190</td>
</tr>
</tbody>
</table>

The wave forces for all three simulation methods were calculated by the Morison equation in the time domain. The nonlinearity due to the velocity squared drag force component causes the computation in the frequency domain to be less efficient than in the time domain. However, wave pressure force simulations in the time domain are only possible if the coefficients $C_D$ and $C_M$ are independent of the wave frequency. In contrast, wave force simulations in the frequency domain may be carried out with coefficients $C_D$ and $C_M$ which are functions of wave frequency. The linearized wave drag force method discussed in Chapter 6 may be a compromise method when the coefficients $C_D$ and $C_M$ are functions of wave frequency.

The wave force realization simulated from the measured spectrum by the DSA method yields the spectral bandwidth parameters, $q$ (Vanmarcke) and $\varepsilon$ (Cartwright-Longuet-Higgins), which are slightly higher than for the Scott and Bretschneider spectra. This is because for the record from WPII Data No. 6887 the $q$ and $\varepsilon$ of the measured spectrum are also higher than those of the Scott and Bretschneider.
The skewnesses of the simulated pressure forces are generally negative except for the linear simulation by the NSA method. These negative values are not expected considering the skewnesses of the simulated wave realizations are always positive. However, a positive skewness resulted from the Scott spectrum by the Linear Transfer Function (LFD) for the NSA method closely approximates that of the measured wave force realizations.
REFERENCES


APPENDICES
APPENDIX A

List of Notations

A = cross sectional area of a circular cylinder

$A_m$ = amplitude of a sinusoidal wave with frequency $\omega_m$

$a = \text{water particle acceleration}$

$a_n$ = coefficient of the nth term of a polynomial

$a_x$ = x component of water particle acceleration

$a_1 = \text{defined as } a_x(t)$

$a_2 = \text{defined as complex conjugate of } a_x(t+\tau)$

$B(.,.)$ = complex-valued nonlinear spectral amplitude of velocity potential

$B_1, B_2, B_4$ = dimensionless parameter in Bretschneider spectrum

$b_n$ = coefficient of the nth term of a polynomial

$c(p)$ = constant defined by Eq. (4.4.4.b)

$C(t_1, t_2)$ = autocovariance of random sequences $z(t_1)$ and $z(t_2)$

$C_{aa}$ = autocovariance of water particle acceleration

$C_D$ = empirical drag coefficient

$C_{FT,T}$ = autocovariance of wave forces on a circular pile

$C_{jk}$ = cross covariance of random variables $z_j$ and $z_k$

$C_M$ = empirical inertial coefficient

$C_{nn}$ = autocovariance of water surface elevations

$C_{ua}$ = cross covariance of water particle velocity and water particle acceleration

$C_{uu}$ = autocovariance of water particle velocity

$c = \text{real-valued constant defined by Eq. (6.3.15.a)}$

$c_a = \text{constant for inertial wave force component}$
Appendix A (continued)

\( a_n \) = coefficient of \( n \)th term of a polynomial
\( c_u \) = constant for drag wave force component
\( D \) = diameter of a circular pile

\( D(\omega_n, \omega_m, \hat{k}_n, \hat{k}_m) \) = nonlinear interaction matrix for velocity potential

\( D_{\alpha\beta}(\cdot) \) = parabolic cylinder function

DOF = degree of freedom

\( d_n \) = coefficient of \( n \)th term of a polynomial

\( E(\cdot) \) = expected value of random variable (\( \cdot \))

\( \hat{\imath}(\cdot) \) = unit vector in (\( \cdot \)) direction

\( F(m) \) = complex-valued random wave amplitude correct to second-order perturbation expansion

\( F(n) \) = discrete random Fourier coefficient of a random process; complex-valued linear random wave amplitude

\( \bar{F}(\hat{k}_n, \omega_n) \) = complex-valued linear random amplitude of velocity potential

\( F(z_1, \ldots, z_n) \) = probability distribution function

\( \vec{F}_T \) = total wave force on a circular pile

\( \vec{F}_D \) = drag component of wave force

\( \vec{F}_I \) = inertial component of wave force

\( \vec{F}_M \) = recorded wave force on a circular pile

\( _2F_1(\cdot) \) = hypergeometric function

\( f \) = temporal frequency

\( f(\omega/\omega_p) \) = shape function for theoretical wave energy spectrum

\( G_a(\omega) \) = linear filter for water particle acceleration

\( G_u(\omega) \) = linear filter for water particle velocity

\( g \) = gravitational acceleration

\( g(\cdot) \) = function of a random variable
Appendix A (continued)

\( H = \) twice the amplitude of the motion of cylinder

\( H_{1/3} = \) significant wave height

\( H(\omega_n, \omega_m; k_n, k_m) = \) nonlinear interaction matrix for water surface elevation

\( h = \) water depth

\( \hat{h}(t) = \) data window

\( \hat{h}(\cdot) = \) function of a random variable

\( I_v = \) modified Bessel function of the first kind and of order \( v \)

\( i = \) the imaginary number which is defined as \( \sqrt{-1} \)

\( |J| \) or \( J(\cdot) = \) Jacobian of transformation

\( j = \) integer index

\( K_v = \) modified Bessel function of the second kind and of order \( v \)

\( k = \) integer index; summation indices

\( \kappa_n = \) wave number vector for linear random sea surfaces

\( k_n = \) magnitude of linear wave number

\( k_{nx}, k_{ny} = \) wave number component in \( x \) and \( y \) direction

\( L = \) wave length; \( L_c = \) length of cylinder

\( L(\cdot) = \) Laplace transform

\( l = \) integer index; summation indices

\( M = \) total mass

\( M(\cdot) = \) directional spreading function

\( m = \) integer index; summation indices

\( \bar{m} = \) Vanmarcke frequency

\( m_{12\cdots n} = \) statistical moment

\( m_0 = \) area under energy spectral density
Appendix A (continued)

\( m_n = \) nth moment of energy spectral density

\( N = \) large integer number

\( N_C = \) number of class intervals

\( N_P = \) number of points

\( n = \) integer index; summation indices

\( O(\cdot) = \) order of magnitude

\( P = \) constant value defined in Appendix B

\( p = \) real-valued constant defined by Eq. (4.4.4.b); summation indices

\( P(\cdot) = \) probability distribution

\( p(\cdot) = \) probability density function

\( Q = \) constant value defined in Appendix B

\( Q(t) = \) Bernoulli's constant

\( q = \) dummy variable; Vanmarcke spectral bandwidth

\( R(t_1, t_2) = \) autocorrelation of random sequences \( z(t_1) \) and \( z^*(t_2) \)

\( \hat{R}(\tau) = \) estimate of autocorrelation of a stationary random process

\( R = \) criteria for estimating the suitability of the data for determining \( C_D \) and \( C_M \)

\( \Re = \) Reynolds number

\( R_{jk} = \) cross correlation of random variables \( z_j \) and \( z^*_k \)

\( R_m = \) nondeterministic amplitude associated with frequency \( \omega_m \)

\( r = \) variable defined as \( \frac{C_{uu}}{\sigma_u^2} \)

\( r_n = \) coefficient of the nth term of a polynomial

\( r_1, r_2 = \) constants
Appendix A (continued)

\[ S = \text{event space} \]

\[ S(n) = \text{discrete energy spectral density} \]

\[ S_{\text{aa}}(n) = \text{acceleration energy spectral density} \]

\[ S(\omega_n) = \text{continuous energy spectral density} \]

\[ \hat{S}(\omega) = \text{estimate of the energy spectrum} \]

\[ S_{FTFT}(\omega) = \text{theoretical energy spectral density for total wave forces} \]

\[ S_{\text{m}}(\omega) = \text{two-sided discrete energy spectral density for water surface elevation} \]

\[ S_{\text{mm}}(\omega) = \text{two-sided continuous energy spectral density for water surface elevation} \]

\[ S_{\text{m}}(\omega) = \text{theoretical one-sided wave energy spectral density} \]

\[ \mathbf{s} = \text{position vector} \]

\[ s = |\mathbf{s}|; \text{variable of moment generating function} \]

\[ T = \text{length of temporal data record; the periodicity of a periodic function} \]

\[ \bar{T} = \text{average wave period} \]

\[ T_n(\omega) = \text{equivalent energy spectral density filter} \]

\[ T_{xn} = \text{transfer function in which } x \text{ denotes the variable on which } T_{xn} \text{ operates and } n \text{ denotes the order} \]

\[ T_{1/3} = \text{significant wave period} \]

\[ t = \text{time variable; dummy variable} \]

\[ t_n = \text{time sequence} \]

\[ U = \text{wind velocity measured at 10 m above the ocean surface} \]

\[ U_g = \text{wave group velocity} \]

\[ U_m = \text{amplitude of the horizontal component of water particle velocity} \]
Appendix A (continued)

\[ \dot{\mathbf{u}} = \text{water particle velocity} \]

\( u_x, u_y, u_z \) = water particle velocities in \( x, y, \) and \( z \) direction

\( u_1 = \text{defined as } u_x(t) \)

\( u_2 = \text{defined as complex conjugate of } u_x(t+\tau) \)

\( V = \text{constant defined in Appendix B} \)

\( W = \text{constant defined in Appendix B} \)

\( x = \text{horizontal coordinate axis} \)

\( x_1, x_2 = \text{dummy variables} \)

\( Y = \text{dimensionless random wave forces} \)

\( y = \text{lateral coordinate axis} \)

\( y_n(\cdot) = \text{random sequence} \)

\( Z(m) = \text{complex-valued Fourier coefficients of time series } z(t) \)

\( z = \text{vertical coordinate axis measured positive upwards from still water level} \)

\( z' = \text{vertical stretched coordinate axis} \)

\( z(n) = \text{discrete time sequence} \)

\( z(t) = \text{continuous time series} \)

\( z_n(\xi) = \text{random sequence} \)

\( z_n(\xi) \) or \( z(t) = \text{random realization} \)

\( * = \text{complex conjugate} \)

\( \alpha = \text{real-valued constant} \)

\( \alpha_m, \beta_m = \text{two independent random variables normally distributed with zero mean and unit variance} \)

\( \beta = \text{real-valued constant} \)

\( \Gamma(\cdot) = \text{gamma function} \)
Appendix A (continued)

\[ \gamma = \text{real-valued constant} \]

\[ \vec{\nabla} = \text{gradient operator} \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \]

\[ \nabla^2 = \text{Laplacian operator} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \]

\[ \delta_{mn}, \delta_{m,n} = \text{Kronecker delta} \]

\[ \zeta = \text{real-valued constant} \]

\[ \Delta_f = \text{fundamental temporal frequency} \]

\[ \Delta \omega = \text{fundamental angular frequency} \]

\[ \Delta t = \text{incremental time spacing} \]

\[ \varepsilon = \text{Longuet-Higgins bandwidth parameter; the square root of the sum of the square of the difference between measured and theoretical wave forces} \]

\[ \eta(\cdot) = \text{water surface elevations} \]

\[ \theta = \text{angle between } \hat{k}_n \text{ and } x \text{ axis} \]

\[ \theta_m = \text{random phase angle associated with frequency } \omega_m \]

\[ \vec{k}_k = \text{wave number vector for random sea surface correct to second-order perturbation expansion} \]

\[ \lambda_3 = \text{skewness} \]

\[ \lambda_4 = \text{excess of kurtosis} \]

\[ \lambda_{4Y} = \text{excess of kurtosis of dimensionless wave forces} \]

\[ \hat{\mu} = \text{estimate value for mean of a stationary random process} \]

\[ \mu_m = \text{mean of random variable } z_m(\xi) \]

\[ \mu_t \text{ or } \mu(t) = \text{mean of a random process} \]

\[ \xi = \text{possible outcome; constant value defined as } \frac{1}{\sigma_u^2} \cdot c_{uu} \]

\[ \pi = 3.14159265 \ldots \]

\[ \rho = \text{mass density of fluid} \]
Appendix A (continued)

$\sigma(\cdot) = \text{standard deviation of random variable } (\cdot)$

$\sigma^2(\cdot) = \text{variance of random variable } (\cdot)$

$\sigma_m^2 = \text{variance associated with frequency } \omega_m$

$\tau = \text{time lag defined by } (t_2-t_1)$

$\phi = \text{velocity potential}$

$\phi(t) = \text{random process defined by } z(t)z^*(t+)$

$\chi(n) = \text{Fourier transform of } \psi\left(-\frac{\mathcal{C}_{uu}(\tau)}{\sigma^2}\right)$

$\chi^2 = \text{chi-square error}$

$\psi(r) = \text{autocovariance function of drag wave forces}$

$\psi(\cdot) = \text{joint characteristic function}$

$\omega = 2\pi f, \text{angular frequency,}$

$\omega_c = \text{cutoff frequency}$

$\omega_p = \text{peak frequency}$

$\omega_n = n\Delta\omega$

$\Omega_k = \text{angular frequency for second-order perturbation expansion}$

$\infty = \text{infinity}$
APPENDIX B

Central Limit Theorem

We wish to determine the probability density function for a random variable which is given by the sum of many independent random variables. We assume that the individual probability density functions for each of these random variables is known.

The central limit theorem states that the distribution of this sum tends to become Gaussian as the number of independent random variables is increased without limit, regardless of their probability distribution function as long as they have finite mean, finite variance and satisfy the condition (Davenport and Root, 1958, pp. 81-84).

\[
\lim_{N \to \infty} \frac{\sum_{m=1}^{N} E(|z_m - \mu|^2 + \xi)}{N} \to 0
\]

in which

- \( z_m \) = the independent random variable
- \( \mu_m \) = the mean of \( z_m \)
- \( \sigma^2_m \) = the variance of \( z_m \)
- \( \xi \) = any positive number

For linear random waves at fixed points, the sea surface random variable is given by

\[
\eta(t) = \sum_{m=1}^{N} A_m \cos(\omega_m t - \theta_m)
\]
Appendix B (continued)

in which \( A_m = \) a deterministic amplitude which is equal \( [2S(\omega_m)\Delta\omega]^{\frac{1}{2}} \);
\( S(\omega_m) = \) two-sided wave energy spectral density
\( \omega_m = \) the angular frequency
\( \theta_m = \) random phase angle which is uniformly distributed between \((-\pi, \pi)\)

Let \( z_m = A_m \cos(\omega_m t - \theta_m) \), then since \( \mu_m = 0 \), it follows that

\[
V = \sum_{m=1}^{N} E(\left| z_m \right|^2 + \xi)
\]

\[
= \sum_{m=1}^{N} A_m^{2+\xi} E(\left| \cos(\omega_m t - \theta_m) \right|^2 + \xi)
\]

\[
< \sum_{m=1}^{N} [2S(\omega_m)\Delta\omega]^{1+\xi/2} E(\cos^2(\omega_m t - \theta_m))
\]

\[
\approx \sum_{m=1}^{N} \frac{1}{2}(\Delta\omega)^{\xi/2} (2S(\omega_m))^{1+\xi/2}\Delta\omega
\]

and

\[
W = \left( \sum_{m=1}^{N} \sigma_m^2 \right)^{1+\xi/2}
\]

\[
= \left( \sum_{m=1}^{N} 2S(\omega_m)\Delta\omega \right)^{1+\xi/2}
\]

Since \( S(\omega_m)\Delta\omega \approx 0 \), for small \( \Delta\omega \), \( V < (\Delta\omega)^{\xi/2} W \).

From Eq. (B.3.a) and (B.3.b), the condition (B.1) is

\[
P/Q < \frac{1}{3}(\Delta\omega)^{\xi/2} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]

Consequently, the distribution of \( \eta(t) \) given in Eq. (B.2) approaches a normal law.
APPENDIX C

The Probability Density Function for Nondeterministic Amplitude and Phase Angle From the NSA Method of Simulation

The representation of the linear random waves at fixed point may also be given by (Rice, 1944, pp. 328-330)

\[ \eta(t) = \sum_{m=1}^{N} \left( \alpha_m \cos \omega_m t + \beta_m \sin \omega_m t \right) \]  

(C.1)

in which \( \alpha_m \) and \( \beta_m \) are independent random variables which are distributed normally about zero with equal standard deviation \( \sqrt{S(\omega_m^2 \Delta \omega)} \).

Another representation for Eq. (C.1) is given by the following expression

\[ \eta(t) = \sum_{m=1}^{N} R_m \cos(\omega_m t - \theta_m) \]  

(C.2)

in which \( R_m = (\alpha_m^2 + \beta_m^2)^{\frac{1}{2}} \)  

(C.3.a)

\[ \theta_m = \arctan(\beta_m/\alpha_m) \]  

(C.3.b)

In Eq. (C.2), \( \eta(t) \) is determined from a finite sum of linear sinusoidal components with random amplitudes and random phase angles.

The joint probability density function of amplitude, \( R_m \), and phase angle, \( \theta_m \), may be calculated according to

\[ p(R_m, \theta_m) = \frac{\partial^2}{\partial R_m \partial \theta_m} \left| \begin{array}{cc} \partial \alpha_m & \partial \beta_m \\ \partial R_m & \partial \theta_m \end{array} \right| p(\alpha_m, \beta_m) \]  

(C.4)

in which the joint probability density function of \( \alpha_m \) and \( \beta_m \) is given by
Appendix C (continued)

\[
p(\alpha_m, \beta_m) = p(\alpha_m)p(\beta_m)
\]

\[
= \frac{1}{2\pi \sigma_m^2} \exp \left[ -\frac{\alpha_m^2 + \beta_m^2}{2\sigma_m^2} \right]
\]

\[
= \frac{1}{2\pi \sigma_m^2} \exp \left(-\frac{R_m^2}{2\sigma_m^2}\right)
\]

(C.5)

in which \( \sigma_m = \sqrt{\frac{1}{\Delta \omega}} \).

The Jacobian of transformation in Eq. (C.4) may be obtained from Eq. (C.3.a) and (C.3.b) and substituted along with Eq. (C.5) into Eq. (C.4) to obtain

\[
p(R_m, \theta_m) = \left(2\pi \sigma_m^2\right)^{-1} R_m \exp\left(-\frac{R_m^2}{2\sigma_m^2}\right)
\]

(C.6)

Noting that the domain of the amplitude \( R_m \) is given by any real number greater than zero, we may calculate the marginal probability density function by

\[
p(\theta_m) = \int_0^\infty p(R_m, \theta_m) \, dR_m
\]

\[
= \int_0^\infty \left(\frac{R_m}{2\pi \sigma_m^2}\right) \exp\left(-\frac{R_m^2}{2\sigma_m^2}\right) \, dR_m
\]

\[
= \frac{1}{2\pi} \exp\left(-\frac{R_m^2}{2\sigma_m^2}\right) \bigg|_0^\infty
\]

\[
= \frac{1}{2\pi}
\]

(C.7)

which demonstrates that the random variables \( \theta_m \) are uniformly distributed over \((-\pi, \pi)\). In the same manner, we may calculate the probability density function for the random amplitudes \( R_m \),
Appendix C (continued)

\[ p(R_m) = \int_{-\pi}^{\pi} \left( \frac{R_m}{2\pi \sigma_m^2} \exp\left(-\frac{R_m^2}{2\sigma_m^2}\right) \right) \, d\theta_m \]

\[ = \frac{R_m}{\sigma_m^2} \exp\left(-\frac{R_m^2}{2\sigma_m^2}\right) \]

which is known as the Rayleigh distribution.
APPENDIX D

Methods for Determining $C_D$ and $C_M$

There have been two basic types of empirical evaluations in determining the force coefficients $C_D$ and $C_M$:

I. Laboratory test
   A. Standing and progressive waves
   B. "U" tube test
   C. Oscillating cylinder

II. Oscillatory ocean waves
   A. Deterministic fitting methods
      1. Crest/zero crossing methods
      2. Least-squares fitting
         a. Numerical wave theory
         b. Linear filter
         c. Measured kinematics
   B. Nondeterministic fitting methods
      1. Drag/inertia dominate frequency fitting
      2. Method of moments
      3. Spectral least-squares fitting

I. Laboratory test

I.A Standing or progressive waves

Keulegan and Carpenter (1958) investigated the $C_D$ and $C_M$ for both circular and vertical flat plates. They placed a cylinder or a
Appendix D (continued)

plate at the midsection of a rectangular basin at the node of a single
frequency standing wave. Progressive waves may be used in a wave tank
with an energy absorbing beach instead of standing waves. The force
per unit length at a fixed point on a cylinder is given by

\[ F(t) = F_D(t) + F_I(t) \]

\[ = C_D \rho/2 D U_m \cos \omega t | U_m \cos \omega t | \]

\[ - C_M \rho \frac{\pi D^2}{4} \omega U_m \sin \omega t \]  

(D.1)

in which \( \rho \) = the mass density of the water

D = the diameter of the cylinder

\( U_m \) = the amplitude of the horizontal velocity

\( \omega = \frac{2\pi}{T} \)

\( T \) = the wave period

By Fourier analysis, the force coefficients \( C_D \) and \( C_M \) may be cal-
culated from measured forces according to

\[ C_D = \frac{3\pi}{2T} \int_0^T \frac{F_M(t)}{\rho L_c U_m^2 D} \cos \frac{2\pi t}{T} \, dt \]  

(D.2.a)

\[ C_M = -\frac{4U_m}{\pi^2 D} \int_0^T \frac{F_M(t)}{\rho L_c U_m^2 D} \sin \frac{2\pi t}{T} \, dt \]  

(D.2.b)

in which \( F_M(t) \) = recorded force on the cylinder as a function of time

\( L_c \) = the length of the cylinder
This standing or progressive wave method is difficult to use for high Reynolds number due to the difficulties in creating the waves with high Reynolds number in the laboratory. "U" tube test or oscillating cylinders may be utilized to determine $C_D$ and $C_M$ at high Reynolds numbers.

### I.B "U" tube test

Sarpkaya (1975) first introduced this method in determining the $C_D$, $C_M$. The equipment which he used was a U-shaped duct filled with water in which a cylinder was placed in the valley inside the U tube. The water was oscillated with fixed amplitude $\frac{H}{2}$. The force coefficients $C_D$ and $C_M$ may be determined according to the Eq. (D.2.a) and (D.2.b).

### I.C Oscillating cylinder

When the fluid is incompressible, the flow past a stationary object is hydrodynamically equivalent to the flow caused by the same object in otherwise still fluid (Batchelor, 1974) with the exception that the inertia force is equal to the sum of the inertia force due to the accelerated object, instead of fluid, and the inertia of the fluid surrounding it (Le Méhauté, 1976, pp. 174-179).

Yamamoto and Nath (1976) gave the following formula to calculate the force coefficients $C_D$ and $C_M$:

$$C_D = \frac{F_D}{\frac{1}{2} \rho DL U_m^2} \quad (D.3.a)$$
Appendix D (continued)

\[ C_M = \frac{F_M}{\left(\frac{1}{2} \rho \pi D^2 L \frac{\partial U_m}{\partial t}\right)} - \frac{M}{\left(\frac{1}{4} \rho \pi D^2 L_c\right)} + 1 \]  
(D.3.b)

in which the definition for \( C_D, C_M, D, \rho, L_c \) are given in Eq. (D.1), and

\[ F_D = \text{measured horizontal force at the instant of maximum horizontal velocity} \]
\[ F_M = \text{measured horizontal force at the instant of maximum horizontal acceleration} \]
\[ U_m = \text{the maximum horizontal velocity which is equivalent to} \frac{\pi H}{T} \]
\[ H = \text{twice the amplitude of the motion of the cylinder} \]
\[ T = \text{the wave period} \]
\[ \frac{\partial U_m}{\partial t} = \text{measured maximum horizontal acceleration equivalent to} \frac{2\pi^2 H}{T} \]
\[ M = \text{the mass of the test cylinder} \]

II. Oscillatory ocean waves

II.A Deterministic fitting method

II.A.1 Crest/zero crossing linear theory fitting

This method was used by Wiegel (1957) to analyze the data measured near Davenport, California. The procedure may be listed as follows

(a) Select the regular-looking waves from wave records and determine the wave height, \( H \), and period, \( T \).

(b) Pick out the crest and still water level forces which are read from corresponding portions of force records.
Appendix D (continued)

The force coefficients \( C_D \) and \( C_M \) may be determined, using linear wave theory, according to

\[
C_D = \frac{2}{\rho \pi H \cosh k(h+z)} \left[ T \sinh kh \right]^2 F_D(z) \tag{D.4.a}
\]

\[
C_M = \frac{2}{\rho} \left[ \frac{T^2}{3} \sinh kh \right] \frac{F_I(z)}{D \cosh k(h+z)} \tag{D.4.b}
\]

in which \( C_D, C_M, T, D, \rho \) are defined in Eq. (D.1) and Eq. (D.3), and

\( F_D(z) = \) measured force corresponding to the wave crest at elevation \( z \)

\( F_I(z) = \) measured force corresponding to the still water level at elevation \( z \)

\( H = \) the wave height

\( h = \) water depth

\( z = \) the location of the measured force acting on cylinder

This method is not recommended for determining \( C_D \) and \( C_M \) for irregular waves from the ocean.

II.A.2 Least-squares method

This method is frequently used to graphically fit a curve to a set of measured data points. As a matter of fact, it can also be used for laboratory tests. The reason for grouping this method in oscillatory ocean waves is that it is commonly used to determine the force coefficients \( C_D \) and \( C_M \) in field tests.

The analysis is the determination of \( C_D \) and \( C_M \) by fitting the measured data with the Morison equation, i.e., Eq. (6.1.3); the variance of the error is given by
Appendix D (continued)

\[ \varepsilon^2 = \frac{1}{N} \sum_{n=1}^{N} \left[ F_M(n) - C_M \frac{D^2}{4} a_x(n) \right] \]
\[ - C_D \rho \frac{D}{2} u_x(n) |u_x(n)| \]

(D.5)

in which \( \varepsilon \) = the difference between measured force and theoretical force
\( N = \) number of points
\( a_x(n) = \) horizontal acceleration of water particles at time \( n \Delta t \)
\( u_x(n) = \) horizontal velocity of water particles at time \( n \Delta t \)
\( \bar{\cdot} = \) average values

It can be shown that \( \varepsilon \) is a minimum when

\[ \frac{\partial \varepsilon^2}{\partial C_D} = 0 \]

(D.6)

\[ \frac{\partial \varepsilon^2}{\partial C_M} = 0 \]

Substituting Eq. (D.5) into Eq. (D.6) leads to two simultaneous linear equations

\[ [\rho D/2 \sum_{n=1}^{N} u_x(n) |u_x(n)|] C_D + [\rho \pi D^2/4 \sum_{n=1}^{N} a_x(n)] C_M \]
\[ = \sum_{n=1}^{N} F_M(n) u_x(n) |u_x(n)| \]

\[ [\rho D/2 \sum_{n=1}^{N} u_x(n) |u_x(n)| |a_x(n)|] C_D + [\rho \pi D^2/4 \sum_{n=1}^{N} a_x^2(n)] C_M \]
\[ = \sum_{n=1}^{N} F_M(n) a_x(n) \]

(D.7)
Two unknown quantities $C_D$ and $C_M$ may be calculated by Cramer's rule.

Dean (1976) defined a parameter

$$R = \frac{2/\pi D}{\frac{\sum_{n=1}^{N} u_x(n)}{\sum_{n=1}^{N} a_x^2(n)}}$$

in which $R$ is the criteria for estimating the suitability of the data for determining $C_D$ and $C_M$.

The following table derived from linear wave theory kinematics provides a range of values for the parameter $R$, in which the data are either drag or inertia dominated.

<table>
<thead>
<tr>
<th>$R$</th>
<th>Well-conditional for determining</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;0.25</td>
<td>$C_M$</td>
</tr>
<tr>
<td>0.25-4.0</td>
<td>$C_M$ and $C_D$</td>
</tr>
<tr>
<td>&gt;4.0</td>
<td>$C_D$</td>
</tr>
</tbody>
</table>

a. Numerical wave theory

Dean and Aagaard (1970) used the water particle velocity and acceleration that was calculated from the kinematic flow field of waves using Dean's stream function (1974). The data analysis procedure as described by Dean and Aagaard (1970) involves:

1. Digitizing the regular-looking wave records and the corresponding measured wave forces
Appendix D (continued)

(2) Calculating the water particle velocities and accelerations according to Dean's stream function theory from digitized wave records,

(3) Grouping these velocities, accelerations and the corresponding wave forces within the specified Reynolds number range,

(4) Applying Eq. (D.7), (D.8) and Table (D.1) to determine $C_D$ and $C_M$.

This method gives good results for large waves, but loses accuracy for small velocities and accelerations. It is necessary to note that the waves are assumed to be undirectional, long crested, with no current superimposed in the wave field.

b. Linear filter

This method was introduced by Reid (1958) and applied by Wheeler (1969) by introducing a stretched vertical coordinate.

In this procedure, the irregular waves are assumed to be the result of a linear superposition of linear waves. The water particle horizontal velocities and accelerations may be calculated according to

\[ u_x(n) = \sum_{m=-N}^{N} G_u(m) \eta(n-m) \]

\[ a_x(n) = \sum_{m=-N}^{N} G_a(m) \eta(n-m) \]  

(D.9)

in which $G_u(m)$ and $G_a(m)$ are the inverse Fourier transform of the linear transfer function for horizontal velocity and horizontal
Appendix D (continued)

acceleration, respectively, which are given by Eq. (5.1.3) and (5.1.4). The procedures (3) and (4) given in method a are then followed.

This filtering method is equivalent to the method given by Eq. (5.1.5.a) and (5.1.5.b) using linear transfer function. In that method, the filtering operation is applied in the frequency domain instead of in the time domain.

Both the linear filter and the linear transfer function methods are improvements over the preceding methods for predicting the kinematics in the real ocean waves.

c. Measured kinematics

This method may be the most accurate method, so far, in applying the least squares method. The water particle velocities and accelerations are measured at almost the same location as that of the wave forces. The procedures (3) and (4) in method a may be employed to determine \( C_D \) and \( C_M \).

II.B Nondeterministic fitting methods

The statistical approach is used for estimating \( C_D \) and \( C_M \). These methods begin with functions characterizing the statistical behaviour of forces such as probability density function, covariance function, moments, or spectral density. These methods have been suggested by Brown and Borgman (1967).
Appendix D (continued)

II.B.1 Drag/inertia dominate frequency fitting

For drag dominated force, a simple and crude method for determining $C_D$ and $C_M$ for a circular cylinder is that first we assume that the value of $C_M$ is equal to $C_M$ for inviscid fluid, i.e. $C_M = 2.0$; the value of $C_D$ is determined by graphically fitting the theoretical spectrum with trial and error values of $C_D$ with the measured spectrum. The coefficient $C_D$ is the trial value when the distribution of the peak frequency of the theoretical spectrum is the same as that of the measured spectrum.

II.B.2 Method of moments

From Eq. (6.2.14) the standard deviation and the excess of kurtosis for random variable $Y$ are given by

$$
\sigma_Y = (3 \ c_u^2 \sigma_u^2 + 1)^{1/2}
$$

$$
\lambda_{4Y} = 105 \ c_u^4 \sigma_u^8 + 18 \ c_a^2 \sigma_a^2
$$

in which

$Y = F_T/\sigma_a \ C_M \rho \pi D^2/4$

$F_T$ = horizontal force on the cylinder

$\sigma_a^2$ = the variance of the horizontal acceleration

$\sigma_u^2$ = the variance of the horizontal velocity

$C_u = 2 \pi (C_D/\sigma_a \ C_M)$

Substituting the expression for $Y$ and $C_u$ into Eq. (D.12) results in two simultaneous equations in $C_D$ and $C_M$. 
Appendix D (continued)

\[ \sigma^2_{FT} = \left( \frac{\sigma_D^2}{4} \right)^2 \left[ \frac{12}{\pi^2} \sigma_u^4 C^2 D^2 + \frac{\sigma_a^2}{a M} \right] \]

\[ \lambda_{4FT} = \left( \frac{\sigma_D^2}{4} \right)^4 \left[ \frac{1680}{\pi^4} \sigma_u^8 C^4 D^4 + \frac{18\sigma_a^4 C^4}{a M} \right] \]  

(D.11)

\[ \sigma^2_{FT} \] and \[ \lambda_{4FT} \] are the variance and the excess of kurtosis of measured forces, and \[ \sigma_u^2 \] and \[ \sigma_a^2 \] are given by

\[ \sigma_u^2 = \sum_{n=-N}^{N} T_{ui}^2(n) S_{nn}(n) \]  

(D.12)

\[ \sigma_a^2 = \sum_{n=-N}^{N} T_{a1}^2(n) S_{nn}(n) \]

in which \[ T_{ui} \] and \[ T_{a1} \] are given by Eq. (5.1.3) and (5.1.4), respectively.

II.B.3 Spectral least-square fitting

The theoretical energy density for the wave force is given by Eq. (6.3.18) which is equal to

\[ S_{FTF_T}(n) = \left( \frac{\sigma_D^2}{2} \right)^2 C_D^2 \chi(n) + \]

\[ \left( \frac{\sigma_D^2}{4} \right)^2 C_{aa}^2 M \]  

(D.13)

in which \[ \chi(n) = \text{the Fourier transform of } \psi(\frac{C_{uu}(\tau)}{\sigma_u^2}) \text{ is given by eq. (6.3.8.a)} \]

\[ C_{uu}(\tau) = \text{the autocovariance function of horizontal velocity} \]

\[ S_{aa}(n) = \text{two-sided energy spectral density of horizontal acceleration} \]
Appendix D (continued)

The least-square estimates of $C_D$ and $C_M$ are the values which minimize the expression

$$
\varepsilon^2 = \sum_{n=-N}^{N} \left[ \tilde{S}_{F F} (n) - \left( \frac{u}{2} \right)^2 C_D^2 \chi(n) \right]^2 - \left( \frac{\pi D}{2} \right)^2 C_M^2 S_{aa} (n)^2
$$

in which $\tilde{S}_{F F} (n)$ = discrete two-sided measured energy spectral density of wave forces.

Differentiating $\varepsilon^2$ with respect to $C_D^2$ and $C_M^2$, respectively, and setting the two resulting equations equal to zero, leads to

$$
\frac{\rho D u^2}{2} \sum_{n=-N}^{N} \chi(n) C_D^2 + \left( \frac{\pi D}{2} \right)^2 S_{aa} (n) \chi(n) C_M^2
$$

$$
= \sum_{n=-N}^{N} \tilde{S}_{F F} (n) \chi(n)
$$

$$
\frac{\rho D u^2}{2} \sum_{n=-N}^{N} \chi(n) S_{aa} (n) C_D^2 + \left( \frac{\pi D}{2} \right)^2 \sum_{n=-N}^{N} S_{aa} (n) C_M^2
$$

$$
= \sum_{n=-N}^{N} \tilde{S}_{F F} (n) S_{aa} (n)
$$

(D.14)

$C_D^2$ and $C_M^2$ then may be solved from these two equations.
APPENDIX E

The Mathematical Expectation Involved in Calculating the Covariance Function of Random Ocean Waves

The expectation of the multiple product of functions of several random variables, \( g(\theta_1), g(\theta_2), \ldots, g(\theta_N) \), with the joint probability density function \( p(\theta_1, \theta_2, \ldots, \theta_N) \) is defined by

\[
E\{g(\theta_1)g(\theta_2)g(\theta_3) \ldots g(\theta_N)\} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(\theta_1) \ldots g(\theta_N) p(\theta_1, \theta_2, \ldots, \theta_N) d\theta_1 d\theta_2 d\theta_3 \ldots d\theta_N
\]  

(E.1)

Consider the random variables \( \theta_1, \theta_2, \ldots, \theta_N \) which are independent with probability density function

\[
p(\theta_n) = \frac{1}{2\pi} \text{ for } -\pi < \theta_n < \pi
\]

\[= 0 \text{ otherwise}
\]  

(E.2)

and their joint probability density function

\[
p(\theta_1, \theta_2, \ldots, \theta_n) = p(\theta_1)p(\theta_2) \ldots p(\theta_n)
\]

\[= \left(\frac{1}{2\pi}\right)^n \text{ for } -\pi < \theta_m < \pi \quad m = 1, 2, \ldots, n
\]

\[= 0 \text{ otherwise}
\]  

(E.3)

and the function

\[
g(\theta_n) = F(n) = |F(n)| e^{-i\beta n}
\]
The following expectation may now be calculated using Eqs. (E.1, 2, 3)

1. $E[F(n)F^*(m)] = |F(n)| |F(m)| \left(\frac{1}{2\pi}\right)^2 \frac{1}{2\pi} \int e^{-i(\theta n - \theta m)} d\theta_n d\theta_m$

$$= |F(n)| |F(m)| \delta_{nm}$$

in which Kronecker delta $\delta_{nm}$ is defined by

$$\delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n\neq m \end{cases}$$

2. $E[(F(n)F^*(\ell)F^*(k)] = |F(n)| |F(m)| |F(\ell)| \left(\frac{1}{2\pi}\right)^3$.

$$\cdot \frac{1}{2\pi} \int e^{-i(\theta n - \theta m - \theta \ell)} d\theta_n d\theta_m d\theta_\ell$$

$$= 0 \quad \text{for } (\theta_n - \theta_m - \theta_\ell) \neq 0$$

The value $(\theta_n - \theta_m - \theta_\ell)$ is never zero. If it is zero, then $\theta_n, \theta_m, \theta_\ell$ become dependent to each other which is contrary to the previous statement

3. $E[F(n)F(m)F^*(\ell)F^*(k)] = |F(n)| |F(m)| |F(\ell)| |F(k)| \left(\frac{1}{2\pi}\right)^4$.

$$\cdot \frac{1}{2\pi} \int e^{-i(\theta_n + \theta m - \theta \ell - \theta k)} d\theta_n d\theta_m d\theta_\ell d\theta_k$$

$$= |F(n)| |F(m)| |F(\ell)| |F(k)| \left(\delta_{nk}\delta_{mk} + \delta_{nk}\delta_{mk} - \delta_{\ell k}\delta_{nm}\right)$$

Let us now consider the two sequences of random variables $R_1, R_2, \ldots, R_n$ and $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ which are distributed according to Eq. (D.6); i.e.;
Appendix E (continued)

\[ p(R_m, \theta_m) = p(R_m)p(\theta_m) = (2\pi \sigma_m^2)^{-1} R_m \exp(-R_m^2/2\sigma_m^2) \]  
\( -\pi \leq \theta_m \leq \pi \)

\[ 0 \leq R_m < \infty \]

in which \( \sigma_m^2 = \frac{1}{4} \mathbb{E}[R_m^2] \), Papoulis (1965, p. 148). If the complex-valued amplitude spectrum, \( F(m) \), is expressed by

\[ F(m) = R_m \exp(-i\theta_m) \]  

then the function of random variables may now be identified by the following:

\[ g(R_m) = R_m \]

\[ g(\theta_m) = \exp(-i\theta_m) \]

The following expectation may now be calculated by using Eq. (E.1,2,3)

1. \( E[F(n)F^*(m)] = (2\pi \sigma_n \sigma_m)^{-2} \int_0^\infty R_n^2 R_m^2 \exp\left[-\frac{R_n^2}{2\sigma_n^2} + \frac{R_m^2}{2\sigma_m^2}\right] \int_0^\pi \int_0^\pi \exp(-i(\theta_n - \theta_m)) \ d\theta_n \ d\theta_m \)

\[ = 0 \quad \text{for } n \neq m \]

\[ = \frac{1}{\sigma_m^2} \int_0^\infty R_m^3 \exp\left(-\frac{R_m^2}{\sigma_m^2}\right) \ dR_m = 2\sigma_m^2 \quad \text{for } m = n \]

2. \( E[F(n)F^*(m)F^*(k)] = (2\pi)^{-3} (\sigma_n \sigma_m \sigma_k)^{-2} \)

\[ \int_0^\infty \int_0^\infty \frac{R_n^2 R_m^2 R_k^2}{2\sigma_n^2 2\sigma_m^2 2\sigma_k^2} \exp\left[-\frac{R_n^2}{2\sigma_n^2} + \frac{R_m^2}{2\sigma_m^2} + \frac{R_k^2}{2\sigma_k^2}\right] \ dR_n \ dR_m \ dR_k \]

\[ \int_\pi \int_\pi \int_\pi \exp(-i(\theta_n - \theta_m - \theta_k)) \ d\theta_n \ d\theta_m \ d\theta_k = 0 \quad \text{for } (\theta_n - \theta_m - \theta_k) \neq 0 \]
The value $\left( \theta_n^m - \theta_n^m - \theta_k^l \right)$ is never zero; therefore the expected value of the triple product as given by the above equation is always zero.

3. $E\left[ F(n)F(m)F^*(k)F^*(l) \right] = [(2\pi)^2 \sigma_n \sigma_m \sigma_k \sigma_l]^{-2} \cdot 

\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \exp \left[ - \frac{R_n^2}{2\sigma_n^2} + \frac{R_m^2}{2\sigma_m^2} + \frac{R_k^2}{2\sigma_k^2} + \frac{R_l^2}{2\sigma_l^2} \right] dR_n dR_m dR_k dR_l \cdot 

\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi e^{-i(\theta_n^m + \theta_m^n - \theta_k^l - \theta_l^k)} d\theta_n d\theta_m d\theta_k d\theta_l \cdot 

= 0 \quad \text{for } n \neq m \neq k \neq l 

= (\sigma_n \sigma_m \sigma_k \sigma_l)^{-2} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \exp \left[ - \frac{R_k^2}{2\sigma_k^2} + \frac{R_l^2}{2\sigma_l^2} \right] dR_k dR_l \cdot 

+ (\sigma_n \sigma_m \sigma_k \sigma_l)^{-2} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \exp \left[ - \frac{R_k^2}{2\sigma_k^2} + \frac{R_l^2}{2\sigma_l^2} \right] dR_k dR_l 

= 8\sigma_n^2 \sigma_m^2 \sigma_k^2 \sigma_l^2 \quad \text{for } n, k, m=l \text{ or } n=l, m=k \text{ but } n \neq m 

= \sigma_n^2 \int_0^\pi \int_0^\pi \exp \left( - \frac{R_n^2}{2\sigma_n^2} \right) dR_n 

= 8\sigma_n^4 \quad \text{for } m=n, l=n, k=n. 

in which $\sigma_l^2 = S(\omega_l) \Delta \omega 

S(\omega_l) = \text{two-sided spectral density evaluated at frequency } \omega_l$. 

Appendix F

Coefficients in the Polynomial Approximation for $I_{\frac{1}{4}}$ and $I_{-\frac{1}{4}}$

The values of coefficients in the polynomial approximation for the modified Bessel function of the first kind of order $\frac{1}{4}$ and $-\frac{1}{4}$ are tabulated below (section 6.2).

Table F.1. Coefficients for approximating $I_{\frac{1}{4}}$ and $I_{-\frac{1}{4}}$ for argument range $-8 < t < 8$ (from Luke, Y. L., 1975).

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<th>( b_n )</th>
<th>( c_n )</th>
</tr>
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<td>30087 15195</td>
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<td>1</td>
<td>113.42825</td>
<td>38420 08688</td>
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<tr>
<td>2</td>
<td>47.64374</td>
<td>69803 57541</td>
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<tr>
<td>3</td>
<td>12.42423</td>
<td>30269 63289</td>
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<tr>
<td>4</td>
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<td>94052 25023</td>
</tr>
<tr>
<td>5</td>
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<td>6</td>
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<td>83312 33806</td>
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<td>7</td>
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<td>16199 50850</td>
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<td>0.00009</td>
<td>23567 02672</td>
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<tr>
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<td>01495 62011</td>
</tr>
<tr>
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<td>00045 71268</td>
</tr>
<tr>
<td>12</td>
<td>0.00000</td>
<td>00001 11588</td>
</tr>
</tbody>
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Table 5.2. Coefficients for approximating $I_{\frac{1}{4}}$ and $I_{-\frac{1}{4}}$ for argument range $t > 8$ (from Luke, Y. L., 1975).

<table>
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<th>( d_n )</th>
<th>( r_n )</th>
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<tr>
<td>1</td>
<td>0.00631</td>
<td>99620 31140</td>
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<td>2</td>
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<td>0.00000</td>
<td>52052 40761</td>
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<td>03591 84411</td>
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<td>6</td>
<td>0.00000</td>
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<td>00001 18337</td>
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