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The displacement of the free liquid surface in geothermal and hydrologic reservoirs is an important capacitance factor. An iterative approach to determining the drawdown of the free liquid surface for a single sink region in a homogeneous, isotropic, Darcy-type porous mediums is discussed. The iterative approach involves a stepwise adjustment of the pressure on the reference surface which replaces the time-dependent free surface condition by a fixed plane Dirichlet type condition so that readily availiable, standard techniques can be applied. Grouping of producing wells into a single analogous well may be used to treat multiple well cases with the iterative approach.

An analytic solution for the infinite half space situation is used to compare solutions with the iterative technique. The analytic solution is derived for a point sink within an infinite, homogeneous, isotropic, Darcy-type porous half space. It is obtained by linearizing the free liquid boundary condition provided that the free surface deviates from its equilibrium reference position by only a small slowly undulating displacement h. The flow pressure at the equilibrium surface is then approximated by the hydrostatic

pressure for a column of height h.

A standard model is designed to be analogous to the analytic solution. Testing the iterative-procedure calculations for this model against the derived analytic solution produces very satisfactory results provided that the numerical grid spacing is adequately chosen for the problem. Calculations of the linear and quadratic terms of the free surface condition indicate that the neglected quadratic terms are in general small, and the approximation is reasonable.

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ITERATIVE TECHNIQUES IN LINEARIZED FREE SURFACE FLOW

I. INTRODUCTION

The production of fluid from a liquid dominated geothermal reservoir is generally associated with changes in the free liquid surface position. Changes in the liquid level depend largely on how the production has affected the reservoir pressure field. The lowering of the pressure field releases stored fluid for production. The three main storage capacitances within the reservoir are the compressibility of the reservoir rock and fluid, the displacement of the liquid surface, and the vaporization of liquid (Zais and Bodvarsson, 1980). The free surface capacitance is of particular importance, and the mechanism of the liquid surface response is very important in estimating the production capabilities of a reservoir.

The purpose of this study is to develop an iterative technique which can be used with existing numerical potential field programs to model the changes in the free liquid surface. A linearized form of the free-surface boundary condition is used in an iterative technique that calculates the pressure field solution at finite time increments incorporating an existing program obtained from the Lawrence Livermore Laboratory. We compare the iterative solution to an availiable analytic form for a particular case, and illustrate how the iterative technique can be used.

Relative Importance of the Free Liquid Surface Capacitance

We want to compare the relative importance of the free liquid surface with that of compressibility and liquid vaporization which can occur within a reservoir. Each of these physical processes releases fluid upon a decrease in pressure. Knowing the relative amount of fluid released for each will give us an idea as to which process is most important for the production capacity of the reservoir. Zais and Bodvarsson (1980) have carried out an investigation of this type.

Starting with a reservoir of thickness H in equilibrium, the production of fluid results in a change in pressure that propagates through the system. The amount of fluid that can be produced upon a given unit reduction in pressure is the fluid capacitance of the system.

First, compare the relative fluid storage or release by the compressibility of the reservoir formation with that of the level of the free liquid surface. The reservoir is assumed to be an isotropic and homogeneous slab with a porosity ϕ, storage capacitivity s, and fluid density ρ. The amount of fluid mass released due to compressibility for a small change in pressure Δp on a column of unit area and height H is

$$\Delta q_c = \rho s \Delta p H \quad . \tag{1.1}$$

The specific capacitance per unit area is defined as

$$dq_c/dp = \rho s H \quad . \tag{1.2}$$

Secondly, the displacement of the liquid surface by

$$\Delta z = \Delta p/\rho g \quad , \tag{1.3}$$

corresponding to a pressure reduction of Δp releases

$$\Delta q_f = (\Delta p/\rho g)(\rho \phi) = \frac{\Delta p \phi}{g} \tag{1.4}$$

and hence the specific capacitance of the free surface

$$dq_f/dp = \phi/g \quad . \tag{1.5}$$

The relative amounts of fluid released for these two mechanisms can be compared by the ratio

$$\frac{dq_f/dp}{dq_c/dp} = \phi/\rho s H g \quad . \tag{1.6}$$

Common reservoir field cases may involve a thickness $H = 1 \times 10^3 m$, capacitivity $s = 2 \times 10^{-11} Pa^{-1}$, and a porosity in the range $\phi = 0.01$ to 0.2. Substituting into the ratio (eq. 1.6) we find that the displacement of the free surface releases about 50 to 1000 times as much liquid mass as the compressibility effect.

Finally we will want to examine the significance of the fluid release due to liquid vaporization within the reservoir. The relationship for the change in vapor pressure p_s with temperature T degrees Kelvin along the saturation line denoted by the subscript v is given by the Clausius-Clapeyron equation that we can

approximate by

$$\left(\frac{dp_s}{dT}\right)_v = \rho_s L/T \qquad (1.7)$$

where ρ_s is the vapor density, and L is the latent heat of vaporization for the liquid. Rearranging and assuming saturation conditions the temperature change is

$$\Delta T = T\Delta p/\rho_s L \qquad (1.8)$$

for a change in the pressure Δp . If C_f is the heat capacitivity or specific heat and ρ_r the density of the wet formation then the heat released per unit volume of wet formation is

$$\Delta h = \rho_r C_f T\Delta p/\rho_s L \quad . \qquad (1.9)$$

The mass of vapor released by a unit volume upon a change in pressure Δp at saturation conditions is

$$\Delta q_v = \Delta h/L = (\Delta p \rho_r C_f T)/\rho_s L^2 \quad . \qquad (1.10)$$

The specific capacitance for a column of height H is then

$$dq_v/dp = (\rho_r C_f TH)/\rho_s L^2 \quad . \qquad (1.11)$$

We compare the vaporization with the compressibility effects on the basis of

$$\left(\frac{dq_v/dp}{dq_c/dp}\right) = (\rho_r C_f T)/(\rho s \rho_s L^2) \quad . \qquad (1.12)$$

Assuming some typical values of T=200°C (473°K), ρ_r=2500 kg/m^3 , C_f= 1 X10^3J/kg°K , s= 2X10^{-11} Pa^{-1}, ρ_s=7 kg/m^3 , L= 2X10^6J/kg a value of about 2000 is obtained for the ratio.

Comparing this ratio to the values obtained previously for the free surface and the compressibility this result indicates that unlike the compressibility effects, total vaporization within the reservoir material could theoretically release as much fluid mass as the free surface effect. In most cases, however, in liquid dominated systems vaporization is confined to the immediate vicinity of the producing boreholes. The relative effects of the free surface would then be about 10 times that of vaporization.

In the following discussion we will restrict ourselves to cases in which vaporization is not significant and compression will be negligible. This will involve not only geothermal situations but also many hydrologic reservoirs as well. The iterative technique we describe is equally suited for both situations provided that a free liquid surface effect remains dominant.

II. CONSIDERATIONS WITH AN ANALYTIC SOLUTION

In the most general sense, a reservoir is a collection or storage place for anything in quantity. The particular situation which we will consider involves the collection of a liquid within a large volume of porous and permeable rock that extends to the surface. Small interconnecting cavities or openings between grains of the rock provide the space for the fluid storage and movement. In some instances a reservoir may be bounded by fault planes, lithologic changes, or layers of low permeability which restrict fluid flow into or out of the reservoir area. The interconnection of pore space within the rock allows for fluid movement and the formation of a fluid level or liquid surface. Particuarly in our case, the fluid surface is a liquid interface free to move and respond to pressure changes exerted on it.

Basically the reservoir to be considered here is composed of a porous, isotropic, homogeneous rock volume within the earth, containing a liquid (water) with a freely moving liquid-air interface and possibly bounded from above by some impermeable layer. We assume that the fluid motion within the reservoir obeys Darcy's law, that is, the fluid mass flow density is proportional to the local pressure gradient induced. For the analytic model used, we will consider the reservoir as infinite in extent.

The Concept of Drawdown

The free liquid surface of the reservoir responds to changes in the pressure field. The gas phase above the liquid in our model can be at atmospheric pressure or be confined by impermeable overlying material. A homogeneous static pressure field acts on the liquid surface, and by the principle of superposition we can subtract the gas pressure from the pressure field within the fluid.

The ambient pressure field around a borehole withdrawing liquid from the reservoir will be depressed. The decrease in local pressure results in an observable lowering of the liquid surface in the vicinity of the borehole. The drawdown is defined as the lowering of the free liquid surface below the equilibrium position.

Analytic Solution for a Point Sink with a Linearized Free Surface Condition

The following development is in part an overview of a paper written by Bodvarsson (1977) with an emphasis on particular items that are essential to the understanding of the material in the next chapters. Although our notation will be adapted for the specific case of radial symetry, we will maintain similar notational symbols. In all cases the z-axis is defined as positive downward using cylindrical coordinates.

Consider an isotropic, incompressible, homogeneous half space of porous material with an area porosity ϕ . The area porosity is defined as the fractional area of fluid conductive pores in a given

cross section surface. The material is saturated with a liquid of density ρ . Choose an equilibrium reference surface Σ which corresponds to the initial liquid surface position at z=0 and let Ω represent the liquid surface at a later time. The points on Σ are $S=(r,0)$, and let $P=(r,z)$ be the general field point within the half space $z>0$.

Our basic assumption is that flow within the porous medium obeys Darcy's law

$$\vec{q}(P,t) = -C(\nabla\bar{p} - \rho\vec{g}) \tag{2.1}$$

where $C=\kappa/\gamma$= conductivity ; $\bar{p}(P,t)$ = total fluid pressure

κ= permeability ; $\vec{q}$ = mass flow vector

γ= kinematic viscosity ; g = acceleration of gravity

The second term on the right of equation 2.1 accounts for the gravitational pull on a fluid flowing in the z-direction. If there is given a mass flow source density as a function of position and time $f(P,t)$ then,

$$\nabla\cdot\vec{q} = f \tag{2.2}$$

We now assume that the total pressure is the superposition of the fluid hydrostatic pressure p_h and the flow or perturbation pressure p due to the source density, that is,

$$\bar{p} = p_h + p \tag{2.3}$$

where initially $p_h = \rho gz$. The external pressure on Ω will be zero. Since the second partial derivative with respect to z of

p_h is zero, equation 2.2 reduces in a homogeneous isotropic space to

$$-\nabla^2 p = f/C \tag{2.4}$$

The boundary condition on the free surface is

$$\left.\frac{D\bar{p}}{Dt}\right|_{\bar{p}=0} = 0 \tag{2.5}$$

where D/Dt is the material (or total) derivative. This is a non-linear condition which results in the loss of the principle of superposition. Consequently we are interested in a method of linearizing this condition without placing too rigid a restriction on the reservoir model.

If the position of the free surface Ω deviates from Σ by only a small, slowly undulating amplitude $h(S,t)$ that is positive up, we can modify the free surface condition by moving the fluid half space boundary to Σ and replacing the condition of zero pressure on Ω by the condition

$$p = \rho g h \tag{2.6}$$

on Σ. In other words, we replace the undulating surface Ω by the plane Σ and assume that to the first order the pressure difference between the two surfaces is only hydrostatic. Moreover, assuming that $h(S,t)$ is very small compared to the source depth, we can take that the flow pressure p at Σ is a small perturbation of the total pressure and that to the first order p on Ω is equal

to p on Σ.

In addition, the motion of the fluid at the free surface can be approximated as strictly vertical. Then the kinematic condition on the boundary represents the strictly vertical flow in a column just below the free surface. Hence, the mass flow to raise the free surface is

$$\partial_t(\phi\rho h) = \phi\rho\partial_t h \tag{2.7}$$

and since by equation 2.1

$$q_z = -C\partial_z p \tag{2.8}$$

we obtain the relation

$$\rho\phi\partial_t h = C\partial_z p \tag{2.9}$$

or
$$\partial_t p - a\partial_z p = 0 \quad \text{at} \quad z=0 \tag{2.10}$$

where $a = Cg/\phi$ is a characteristic fluid velocity for a porous medium.

Equation 2.10 represents the linearization of equation 2.5 where the only major restrictions imposed have been that the free surface Ω deviate only by a small amplitude from the horizontal reference surface Σ, and that the hydrostatic reservoir pressure be much larger than the flow pressure in the neighborhood of the free surface. A peculiarity about these conditions is that $h(P,t)$ takes on negative values when the Ω surface is below Σ, and the total pressure on the reference surface Σ is then negative.

However, negative pressures in this situation represent a mathematical abstraction and do not present any physical problems.

Now that we have gained a physical feeling for how the free surface condition (eq. 2.5) is approximated by equation 2.10 , it will be helpful to look at the form of the terms neglected by the linearization. The magnitude of the neglected terms will be useful later to determine the error involved in using this approximation.

Starting with the original boundary condition (eq. 2.5) for the free surface and the definition of the total pressure (eq. 2.3) we expand the material derivative into its component terms. In our case of axial symmetry

$$\frac{D\bar{p}}{Dt} = \partial_t \bar{p} + s\partial_r \bar{p} + w\partial_z \bar{p} \qquad (2.11)$$

where $$\frac{\partial \bar{p}}{\partial r} = \frac{\partial p}{\partial r} \quad ; \quad \frac{\partial \bar{p}}{\partial z} = (\rho g + \frac{\partial p}{\partial z}) \quad . \qquad (2.12)$$

$\vec{V}=(s,w)$ is the velocity vector of the pore fluid within the reservoir. The pore fluid velocity can be expressed in terms of the mass flow $\vec{q}$ by first dividing by the density to obtain a volume flow $\vec{q}/\rho$, then by dividing by the area porosity ϕ we get an expression which represents the velocity of the fluid elements.

$$\vec{V} = \vec{q}/\rho\phi \qquad (2.13)$$

Rewriting the mass flow due to the flow pressure as

$$\vec{q} = -C\nabla p = -C[\ \partial_r p\ \hat{r} + \partial_z p\ \hat{z}\ ] \qquad (2.14)$$

the pore velocity is then

$$\vec{V} = - \frac{C}{\rho\phi} [\partial_r p \, \hat{r} + \partial_z p \, \hat{z}] \tag{2.15}$$

where $$s = - \frac{C}{\phi\rho} \frac{\partial p}{\partial r} \quad ; \quad w = - \frac{C}{\rho\phi} \frac{\partial p}{\partial z} \quad . \tag{2.16}$$

The free surface condition then becomes

$$\frac{D\bar{p}}{Dt} = 0 = \partial_t \bar{p} + (- \frac{C}{\rho\phi} \frac{\partial p}{\partial r}) \frac{\partial \bar{p}}{\partial r} + (- \frac{C}{\rho\phi} \frac{\partial p}{\partial z}) \frac{\partial \bar{p}}{\partial z} \tag{2.17}$$

$$0 = \frac{\partial \bar{p}}{\partial t} - \frac{C}{\rho\phi} (\frac{\partial p}{\partial r})^2 - \frac{C}{\rho\phi} [\rho g \frac{\partial p}{\partial z} + (\frac{\partial p}{\partial z})^2] \tag{2.18}$$

$$0 = [\frac{\partial \bar{p}}{\partial t} - a(\frac{\partial p}{\partial z})] \quad - \quad \frac{a}{\rho g} [(\frac{\partial p}{\partial r})^2 + (\frac{\partial p}{\partial z})^2] \tag{2.19}$$

Linearized Free Surface Approximation — Neglected Quadratic Term

Thus in cases where the quadratic derivative term is small compared to the linear derivative term

$$- a \frac{\partial p}{\partial z} \tag{2.20}$$

we may neglect the quadratic and obtain the linearized free-surface condition just as before.

Using the basic equation 2.4 and the linearized free-surface condition (eq. 2.10) we can solve for the pressure field within the half space. Consider first the solution obtained for the source-free case where $f_o = 0$.

The basic equations are

$$\nabla^2 p = 0 \qquad z \geq 0 \tag{2.21}$$

and $$p = \rho g h_o \qquad \text{for} \quad t = 0 \,,\; z = 0 \;. \tag{2.22}$$

where $h_0(S)$ is the initial vertical amplitude for the free surface Ω relative to Σ, and S is some general point on the Σ surface. Bodvarsson (1977) observes that the solution for the pressure field will be of the form

$$p = p(r, z+at) \tag{2.23}$$

which satisfies the boundary condition (eq. 2.10) for all time. Since p as expressed by equation 2.23 is for $t>0$ a potential function of $(r,z+at)$ in $z>0$, we can continue the function into $z>0$ by standard potential theoretical formulas (Duff and Naylor, 1966) and hence

$$p(P,t) = [\rho g(z+at)/2\pi] \int_{\Sigma} (1/r_{PUt}^3)\, h_o(U)\, d\Sigma \tag{2.24}$$

for $t \geq 0$, $z \geq 0$, $U(r',0)$

with $$r_{PUt} = [\,(r-r')^2 + (z+at)^2\,]^{\frac{1}{2}} \tag{2.25}$$

and $$d\Sigma = 2\pi r' dr' \;. \tag{2.26}$$

The corresponding motion of the fluid surface Ω is found by setting $z=0$ in equation 2.24 and 2.25 giving

$$h(S,t) = (at/2\pi) \int_{\Sigma} (1/r_{SUt}^3) h_o(U)\, d\Sigma \quad ; \; t>0 \qquad (2.27)$$

and $$r_{SUt} = [\, (r-r')^2 + (at)^2 \,]^{\frac{1}{2}} \quad . \qquad (2.28)$$

Next we will examine the pressure field due to a concentrated point sink of strength f_o at a position Q=(0,d) within the half space. At t=0 the fluid is in a static equilibrium with the fluid surface corresponding to the reference surface Σ . The point sink begins withdrawing fluid at t=0+ with a constant rate f_o . The basic equation to be solved then takes the form

$$-\nabla^2 p = (-f_o/C)\, \delta(P-Q)\, I(t) \qquad (2.29)$$

where I(t) is the causal unit step function that takes the value I(0)=0 for t=0 and I(t)=1 for t>0 . Equation 2.10 represents the boundary condition placed on the free surface, and there is now an initial condition of p=0 in z>0 , at t=0 . Bodvarsson (1977) solves this problem by applying the method of images. To obtain the stationary pressure field as $t \to \infty$ an image of strength f_o is placed at Q'=(0,-d) giving the Neumann type solution of no flow through Σ ,

$$p_s(P) = (-\, f_o/4\pi C)[\, (1/r_{PQ}) + (1/r_{PQ'}) \,] \quad , \quad t \to \infty \qquad (2.30)$$

and on the basis of equation 2.6 , the surface amplitude is

$$h_s(S) = -\, f_o/2\pi C \rho g r_{SQ} \qquad (2.31)$$

where $$r_{PQ} = [\, r^2 + (z-d)^2 \,]^{\frac{1}{2}} \qquad (2.32)$$

$$r_{PQ'} = [\, r^2 + (z+d)^2 \,]^{\frac{1}{2}} \qquad (2.33)$$

$$r_{SQ} = [\, r^2 + d^2 \,]^{\frac{1}{2}} \quad . \qquad (2.34)$$

The general solution for t>0 is obtained by adding to the stationary pressure field (eq. 2.30) a time varying component which is initially equal and opposite to the stationary field thus satisfying the initial condition of p=0 at t=0 . The time varying component is given by equation 2.24 with $h_o(U) = -h_s(S)$ (eq. 2.31) which upon adding to that of 2.30 results in the solution

$$p\,(P,t) = (-f_o/4\pi C)[\, (1/r_{PQ}) + (1/r_{PQ'}) - (2/r_{PQ't}) \,] \qquad (2.35)$$

where $$r_{PQ't} = [\, r^2 + (z+at+d)^2 \,]^{\frac{1}{2}} \quad . \qquad (2.36)$$

By setting z=0 we obtain the flow pressure at the reference surface and using our approximation (eq. 2.6) the vertical amplitude of the free surface Ω relative to the reference surface Σ is

$$h(S,t) = (-f_o/4\pi C\rho g)[\, (1/r_{SQ}) - (1/r_{SQ't}) \,] \qquad (2.37)$$

$$r_{SQ't} = [\, r^2 + (at+d)^2 \,]^{\frac{1}{2}} \qquad (2.38)$$

for S=(r,0) on the surface. Equations 2.35 and 2.36 represent the analytic solution which we will refer to later.

Discussion of Assumptions

In the previous developement we made several approximations.

Some of these have already been pointed out. One is using the hydrostatic approximation to derive the pressure on the reference surface, and another is assuming a constant liquid velocity in the z-direction during each incremental time. In addition to these, we have neglected capillary pressure as a force on the free liquid surface. This is probably quite reasonable provided that we restrict the applications to porous materials such as sandstone etc. in which capillary forces are known to be small.

The flow pressure within the reservoir is being calculated as a pure potential field neglecting compressibility and the resulting pressure diffusion. In other words, we have assumed that the pressure diffusion through the reservoir requires much less time than the response of the free liquid surface. This is a reasonable assumption because the time required for the pressure signal to diffuse to one half of its full value over a distance d within the reservoir is on the order of (Carslaw and Jaeger, 1959)

$$t_{D\frac{1}{2}} \simeq \frac{d^2}{\kappa} \tag{2.39}$$

where κ is the diffusivity of the reservoir formation. Conversely equations 2.31 and 2.37 indicate that the time required for the free surface to reach one half of its stationary drawdown is on the order

$$t_{F\frac{1}{2}} \simeq \frac{d}{a} \tag{2.40}$$

where a is the characteristic velocity of the reservoir. The ratio of the two times is

$$t_{F\frac{1}{2}}/t_{D\frac{1}{2}} = (d\phi/Cg)(C/\rho sd^2) \simeq \phi/g\rho sd \qquad (2.41)$$

where s is the capacitivity or storage coefficient that has values of a few times 10^{-11} Pa^{-1} (Bodvarsson, 1970). Hence for such common values as $\phi=0.1$ and $d=10^3$ m, the ratio is a few times 10^2.

Other Reservoir Models and the Motivation of the Present Investigation

The above results are quite simple and have been obtained by an elementary method. Several alternative procedures including Hankel-Laplace transform techniques are also applicable for the same purpose. The transform techniques are of a more general scope and can also be applied to models of other geometries, in particular, to the case of a reservoir of a finite thickness with the same free surface boundary condition as above. This case is of considerable practical interest. Bodvarsson (personal communication, 1981) has investigated such cases and shown that a solution is readily availiable in the transform space, but the Hankel-inversion is not elementary and can not be expressed in a closed analytic form. It is a complex double series where convergence poses a non-trivial problem.

It is interesting that because of the parculiarities of the free surface condition a simple method of images technique breaks down in the case of the finite thickness reservoir. To cope with such problems, it may in many cases appear to be of interest to modify the free surface condition such that the Hankel inversion

becomes elementary. As will be elaborated on below, we will present one such technique that replaces the free surface condition by an iterative approach and has the following two advantages.

(1) The time-dependent free-surface condition is replaced by a stationary fixed plane Dirichlet type condition that moves the problem back to conventional potential theory where availiable standard solution techniques and numerical procedures can be applied.

(2) The iterative approach is equally applicable to models with more complex boundary conditions.

Our principle goal is to test the iterative approach on the infinite reservoir case where the computational results can be compared with a simple analytic solution.

The Reference Numerical Model (RM1)

In this section a model will be developed that will approximate the analytic situation for the infinite reservoir with the iterative technique. The magnitudes of the parameters selected for the numerical model are strictly relative and represent a reference case called Reference Model 1 (RM1) upon which other models can be based. Parameter values are chosen as reasonable values for a particular case of a Darcy flow type sandstone reservoir with very distant lateral boundaries from a borehole penetrating below a free liquid surface within the reservoir. The multi-hole case will be discussed later. The selected parameters are listed in Table 1 .

Table 1. Selected parameters for model RM1 .

Parameter			
Area Porosity	ϕ	0.2	
Conductivity	C	$1 X10^{-7}$	sec
Radius (to boundary)	r	14.4	km
Depth (to bottom)	z	14.4	km
Borehole Flow Strength	f_o	50.0	kg/sec
Depth of Borehole Flow	d	1.4	km
Flow Across Lower Boundaries	-	0.0	$kg/km^2 sec$
Pressure at Free Liquid Surface	-	0.0	$kg/km\ sec^2$

The radius and depth of the reservoir are selected as convienient approximations to the infinite half space of the analytic model. They are convienient because they give a grid spacing of 0.1 km for the numerical model when 144 numerical divisions are used to calculate the solution values. The borehole flow strength chosen is a reasonable figure for the rate of production by a single well within a geothermal field.

In the numerical case the allowed drawdown is limited by the requirement that it be small compared to the borehole depth and the radial scale of the free surface response. Preferably, the total drawdown of the free fluid surface not exceed 1/5 of the borehole depth. The 1/5 limitation is based on experience with the perturbation techniques (Dr. G. Bodvarsson, personal communication, 1981).

The depth of the borehole was chosen to be approximately 1/10 the distance to the bottom and side boundaries. This puts the bottom and sides sufficiently far away from the borehole that their

overall influence on the pressure solution values will be small.

The flow across the bottom and side boundaries is set to zero. Since the boundaries are relatively far from the borehole, the effects of this assumption are small within the borehole vicinity.

Borehole Grouping

A practical problem which may arrise is to determine at what distance R_o can a particular group of boreholes be replaced by a single borehole having a flow strength equal to the combined strength of the group of boreholes and yielding nearly the same pressure field. In particular, we want to know the distance beyond which the pressure field due to the single borehole will be approximately equal (within 10% for example) to the value from the borehole group. Take m_f as the flow strength of each of the grouped boreholes and M_f as the flow strength for the single borehole representing the group. Assume for example that we have a distribution of boreholes such that the flow points are placed at the vertices of a cube. At some distance R_o the pressure field due to this group of wells will be within a given fraction of that due to a single well with a flow point located at the center of the grouping. The distance at which the group of wells can be replaced by a single well will depend on the group spacing.

Solving for the whole-space, homogeneous, isotropic, Darcy-type porous medium pressure solution with the origin at the center of the cubic grouping

$$-\nabla^2 p = \frac{M_f \delta_+(r_s)}{4\pi C r_s^2} \tag{2.42}$$

we obtain the pressure field for the single grouped well

$$p_s = M_f/4\pi C r_s \tag{2.43}$$

where
$$r_s = (x^2 + y^2 + z^2)^{\frac{1}{2}} \quad . \tag{2.44}$$

Likewise the solution for each of the distributed boreholes is

$$p_i = m_f/4\pi C r_i \tag{2.45}$$

where
$$r_i = [\,(x-x')^2 + (y-y')^2 + (z-z')^2\,]^{\frac{1}{2}} \quad . \tag{2.46}$$

Summing up the pressure due to the distributed boreholes we have

$$p_g = (m_f/4\pi C) \sum_i^8 (1/r_i) \quad . \tag{2.47}$$

The spacing of the cubic group will be $2L$ on edges. Consider a point $P(x,y,z)$ directly out from the center of one of the faces of the cubic group and at a distance R_o from the center of the cube. p_g can be rewritten as

$$p_g = \left(\frac{m_f}{\pi C R_o}\right)\left[\,1+\left(\frac{3L^2-2R_oL}{R_o^2}\right)\,\right]^{-\frac{1}{2}} + \left[\,1+\left(\frac{3L^2+2R_oL}{R_o^2}\right)\,\right]^{-\frac{1}{2}}\Bigg]. \tag{2.48}$$

Using an expansion for $(1+X)^{-\frac{1}{2}}$, $|X|<1$ and neglecting third order and higher terms we obtain

$$p_g \simeq \frac{m_f}{\pi C R_o}\left[\, 2+ \frac{27}{4}\frac{L^4}{R_o^4}\,\right] \quad . \tag{2.49}$$

The pressure at P(x,y,z) due to a single well at the center of the cubic array with flow strength M_f is then

$$p_s = M_f/4\pi C R_o \quad . \tag{2.50}$$

The difference between the pressure field at P(x,y,z) from substituting a single well at the center of the cube compared with that of the field from the cubic group of wells will be within 10% provided

$$R_o \geq L(2.34763) \tag{2.51}$$

where P(x,y,z) is at a distance R_o from the center of the cubic group. If the spacing is L=100 m then R_o=235 m .

For a point on the free surface the drawdown from a cubic borehole group will appear nearly the same as that of a single well of equal strength if the distance to the surface is larger than that given by equation 2.51 . In cases in which a group of boreholes are producing from the same vicinity the group can be approximated by a single well which can be examined using the iteration technique to be discussed in the next chapter.

We can do a similar analysis for the fluid flow field in a whole space situation. Consider a homogeneous, isotropic, whole space Darcy type porous medium such that the fluid flow field is given by

$$\nabla\cdot\vec{q}_s = \frac{M_f \delta_+(r_s)}{4\pi r_s^2} \quad . \tag{2.52}$$

The flow field for a single well at the origin is then

$$\vec{q}_s = \frac{M_f \vec{r}_s}{4\pi r_s^3} \tag{2.53}$$

where r_s is defined by equation 2.44 . The coordinate axis x,y,z are oriented perpendicular to the faces of the cube with the origin at the center of the cubic grouping.

Consider a point on the z-axis P(0,0,z) and at a distance R_o from the origin. The fluid flow field along the z-axis from each of the boreholes in the cubic configuration of strength m_f is

$$q_{zi} = \frac{m_f}{4\pi} \frac{(z-z_i)}{r_i^3} \quad . \tag{2.54}$$

The total flow field in the z-direction at $P(0,0,R_o)$ from the cubic configuration will be

$$q_{zg} = \left(\frac{m_f}{4\pi} \right) \sum_{i}^{8} \frac{(z-z_i)}{r_i^3} \quad . \tag{2.55}$$

For a cubic spacing of 2L equation 2.55 can be rewritten as

$$q_{zg} = \frac{m_f}{\pi R_o^2} \left[\frac{(R_o-L)}{R_o\left[1+\left(\frac{3L^2-2R_oL}{R_o^2}\right)\right]^{\frac{3}{2}}} + \frac{(R_o+L)}{R_o\left[1+\left(\frac{3L^2+2R_oL}{R_o^2}\right)\right]^{\frac{3}{2}}} \right] . \tag{2.56}$$

Using a second order Taylor series expansion for $(1+X)^{-\frac{3}{2}}$, $|X|<1$ and neglecting third order and higher terms we obtain

$$q_{zg} \simeq \frac{m_f}{\pi R_o^2} \left[2+ 78.75\left(\frac{L^4}{R_o^4}\right) \right] \quad . \tag{2.57}$$

The fluid flow field at $P(0,0,R_o)$ for a single well of flow strength M_f at the origin will be

$$q_{zs} = \frac{M_f}{4\pi R_o^2} \quad . \tag{2.58}$$

Taking the difference between q_{zs} and q_{zg} we find that if the group of wells were replaced by a single well at the origin, then the distance at which the two flow fields differ by no more than 10% is related to the group spacing by

$$R_o \geq L(4.33876) \tag{2.59}$$

If the error in determining the flow field is at least 10% then for some region of interest at a distance R_o=434 m from the origin a cubic group of producing wells with a spacing L=100 m could be replaced by a single well without any significant effect on the flow field results at R_o .

III. THE TIME ITERATION TECHNIQUE

Time dependent effects of the free liquid surface can be approximated using an incremental or iterative proceedure. This involves replacing the free liquid surface by a stationary reference surface at $z=0$ and stepwise adjusting the pressure on the reference surface such that the free surface condition is approximated as closely as possible.

Technique

The proceedure begins by solving for the pressure field due to a point sink (source) placed on the symmetry axis of the reservoir and assuming the initial free surface condition $p=0$ at $z=0$ and $t=0$. We then obtain a fluid velocity at the surface and use this to determine an incremental displacement of the surface during a sufficiently small time interval. Next a hydrostatic approximation is used to adjust the pressure on the reference surface for the new position of the free surface. These pressure values form a new boundary condition for recalculating the pressure field within the reservoir during an additional small time increment. The process can then be repeated stepwise.

Considering the process in more detail, after the pressure field is found for the initial free surface condition we can calculate the fluid flow at the free surface and subsequently the free surface velocity. Since the free surface is sufficiently far from the sink and the displacement sufficiently small then the fluid flow in this region can be approximated as strictly vertical. Thus

only the z-component of the mass flow vector need to be considered in order to obtain an expression for the velocity of the free surface.

In chapter (2) we obtained a relation for the pore liquid velocity

$$w_o = -(C/\rho\phi)\ \partial p/\partial z \quad ; \quad z = 0 \quad . \tag{3.1}$$

If we assume that the calculated velocity does not change significantly during the selected time increment then multiplication of w_o by the incremental time gives the displacement Δh of the liquid surface in the z-direction. Due to the fact that we have defined h as positive up

$$\Delta h = (C/\rho\phi)(\partial p/\partial z)\Delta t \quad . \tag{3.2}$$

In order to complete the time step procedure, the boundary values for the pressure of the reference surface at z=0 must be calculated. As discussed in chapter (2) we can take that if Δh is small in comparison to the scale of the undulation of the liquid surface, the difference in the total pressure for the reference surface and the free liquid surface is essentially due to the hydrostatic pressure of the fluid column between them. Thus we can approximate the flow pressure on the reference surface by the hydrostatic pressure of a fluid column of height Δh

$$p = \rho g \Delta h = \frac{Cg}{\phi} \frac{\partial p}{\partial z} \Delta t \quad . \tag{3.3}$$

Using this relation a new set of boundary values is generated for the reference surface, and the pressure field within the

reservoir can then be calculated during the following time step. Repeating this process allows us to derive an approximation of the pressure field as a function of time. The schematic cartoon of figure 1 illustrates several such time steps showing the relative positions of the reference surface Σ and the free liquid surface Ω from both an analytical and a numerical point of view.

The Analytic Time Step Approximation

The analytic solution can be used to examine the criteria under which the time step technique approximation is a valid representation for the solution. We begin with a developement that is analogous to the developement of the computer technique given above with the analytic pressure field at t=0+ . Upon completing a derivation of the first time step flow pressure for a point on the reference surface we show how this result is related to a first order Taylor series expansion of equation 2.37 at (0,0). From the Taylor series expansion we gain some insight as to the restrictions which must be placed on the iteration technique.

Equation 2.35 for t=0+ is

$$p(P,0+) = \frac{-f_o}{4\pi C} \left[1/r_{PQ} - 1/r_{PQ'} \right] \quad . \tag{3.4}$$

Taking the gradient with respect to z ,

$$\frac{\partial p}{\partial z} = - \frac{f_o}{4\pi C} \left[\frac{(z+d)}{r_{PQ'}^3} - \frac{(z-d)}{r_{PQ}^3} \right] \quad , \tag{3.5}$$

the vertical flow of liquid can be obtained from equation 2.8 . A relation for the initial vertical pore liquid velocity (eq. 2.16) is

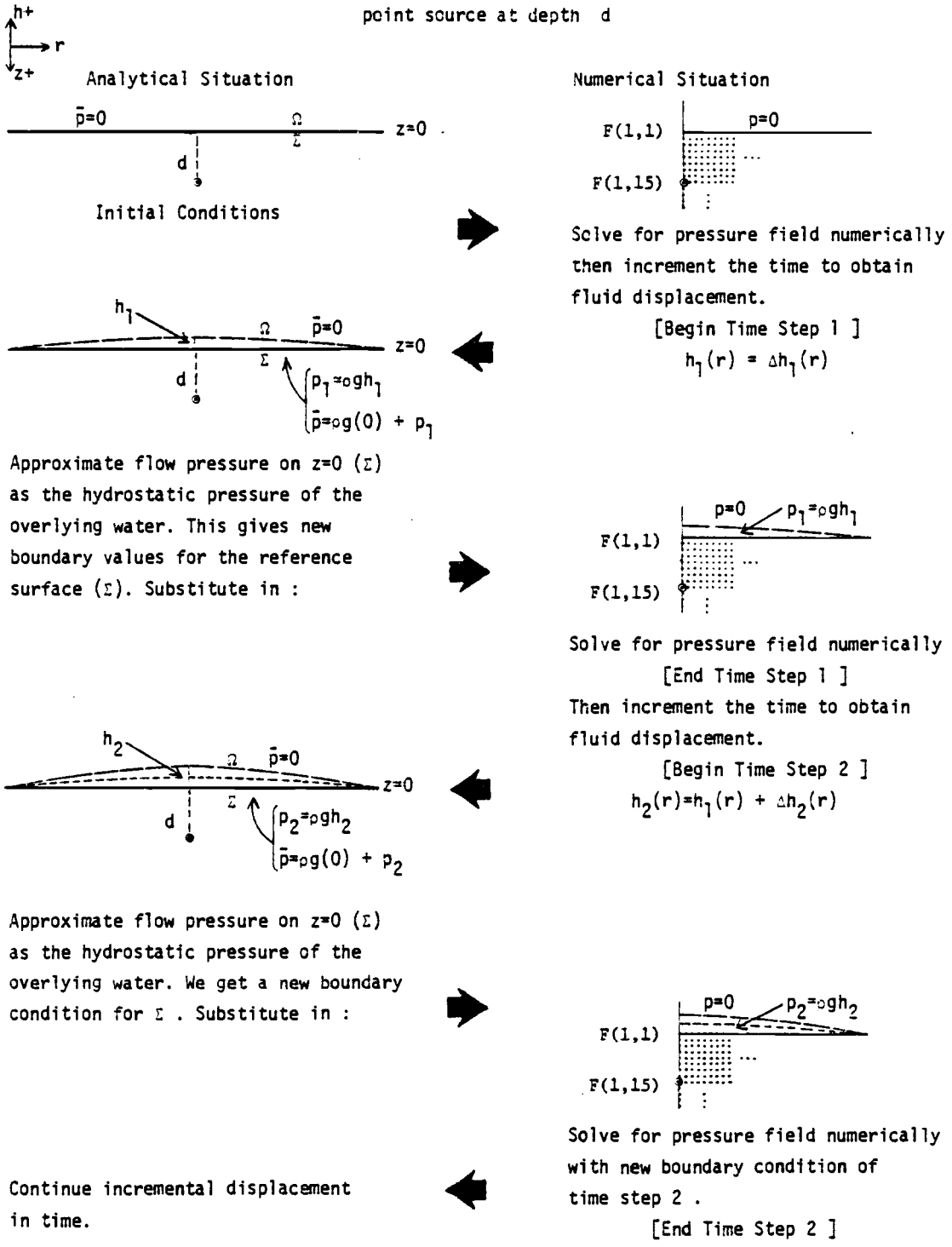

Figure 1. Schematic cartoon of time step process.

then

$$w = q_z/\rho\phi = - \frac{f_o}{4\pi\rho\phi} \left[\frac{(z+d)}{r^3_{PQ'}} - \frac{(z-d)}{r^3_{PQ}} \right] \quad , \quad t=0 \quad . \qquad (3.6)$$

Assuming that the velocity is approximately constant over some small time interval Δt the free surface will be displaced vertically by a distance Δh . Consider a point (0,0) on the reference surface directly above the point sink, the displacement will be

$$\Delta h_1 = w_o(0,0)\Delta t_1 \qquad (3.7)$$

which is analogous to equation 3.2 . Substituting in gives

$$h_1 = - \frac{f_o}{2\pi\rho\phi} \left(\frac{\Delta t}{d^2}\right) = - \frac{f_o}{2\pi\rho g C} \left(\frac{at_1}{d^2} \right) \qquad (3.8)$$

for $t_1 = \Delta t$ and $r = 0$. The iterative flow pressure at (0,0) is obtained using a hydrostatic approximation on the reference surface as

$$p_1 = \rho g h_1 = - \frac{f_o}{2\pi C} \left(\frac{at_1}{d^2} \right) \quad . \qquad (3.9)$$

From the complete analytic solution the vertical amplitude of the free surface Ω relative to the reference surface Σ is given by equations 2.37 and 2.38 as

$$h(S,t) = -(f_o/2\pi C\rho g) \left[1/r_{SQ} - 1/r_{SQ't} \right] \quad . \qquad (3.10)$$

For the point (0,0) on the reference surface we have

$$h(S,t) = -(f_o/2\pi C\rho g) \left[1/d - 1/(at+d) \right] \qquad (3.11)$$

which can be rewritten as

$$h(0,t) = -(f_o/2\pi C\rho gd)[\ 1 - 1/(1+ \frac{at}{d})\ ] \quad . \tag{3.12}$$

The first order Taylor series expansion for $1/(1 + \frac{at}{d})$ is

$$1/(1+ \frac{at}{d}) \simeq 1 - \frac{at}{d} \tag{3.13}$$

provided that $$|\frac{at}{d}| << 1 \quad . \tag{3.14}$$

Then substituting into equation 3.12 for the time $t = t_1$ we get

$$h(0,t_1) \simeq - \frac{f_o}{2\pi\rho gC} \left(\frac{at_1}{d^2} \right) \tag{3.15}$$

and using the hydrostatic approximation we obtain the flow pressure on the surface at (0,0) ,

$$p \simeq - \frac{f_o}{2\pi C} \left(\frac{at_1}{d^2} \right) \quad , t=t_1 \tag{3.16}$$

which is identical to equation 3.9 that we obtained using the iteration technique.

In making the Taylor series approximation we found that it is necessary to have the restriction stated in equation 3.14 to obtain a solution equivalent to equation 3.9 . Equation 3.14 gives a measure of the magnitude of the time step that can be allowed.

Computer Programs

The iteration technique described here is implimented by a main program supplemented by a group of subroutines. The basic iterative subroutines are VELOC, DIFTBL, and the subroutine package PWSCYL along with the function DERVZO. These subroutines with the assistance of the calling program POSGO perform the iteration technique under the direction of the user. Additional subroutines RADF, QUADR, and RZDIFT along with the function DERVR assist the user in obtaining information about the iterative solutions calculated.

Before getting into a more detailed discussion of the time step technique used, it will be helpful for later reference to briefly explain the notation and subroutines which are directly involved with the iteration technique. A more thourough discussion is supplied in Appendix B along with program documentation.

The main program and all subroutines except the subroutine group PWSCYL were written specifically for use in studying the time iterative technique. PWSCYL is a group of subroutines which was obtained through Dr. Jonathan Hanson from Lawrence Livermore Laboratory as part of their mathematical software library. Its original authors are Paul Swarztrauber and Roland Sweet (Technical note TN/IA-109 July 1975) of the National Center for Atmospheric Research , Boulder Colorado 80307 .

Initially, the subroutine PWSCYL calculates the pressure field solution for the case of zero pressure on the free surface z=0 .

PWSCYL solves a finite difference approximation to the Poisson equation in cylindrical coordinates. Solution values are then stored in a matrix array set up by the main program. Once this is accomplished the user then prompts the program to calculate a new set of boundary conditions using the subroutines VELOC and DIFTBL along with the function DERVZO .

VELOC sets up the conditions whereby DIFTBL and DERVZO can calculate a fourth order polynomial derivative at each point along the free surface. VELOC then calculates a new pressure boundary value by equation 3.3 , and stores the value in the corresponding matrix locations.

The main program then resets the input matrix array with the new free surface boundary values. A prompt from the user calls PWSCYL again to recalculate the pressure field solution values using the new boundary conditions. This process may be repeated until the job is completed.

Other features within the main program allow the user to output or view solution values at any stage in the procedure. Some additional features also allow the user to output a comparison of solution values at any stage of the process with the analytic solution given by equations 2.35 and 2.36 . The user is also able to obtain information on both the radial and z-derivative values for points along the free surface.

In the next chapter we will be discussing some equations which use and relate the notation of both the programs discussed here and the analytic solution derived in chapter (2) . Table 2 is set up to

relate these two notational schemes and briefly explain specific parameters utilized in the iteration process.

Table 2. Brief discussion of numerical parameters used in the time iterative technique.

Parameters			
Numerical	Analytic		
B	r	-	The radial range of the reservoir $0 \leq r \leq B$ [A=0 always]
M	-	-	Number of radial subdivisions for the interval [A,B]
D	z	-	The range of z (depth) for the reservoir $0 \leq z \leq D$
N	-	-	Number of vertical subdivisions for the interval [C,D]
BDA,BDB BDC,BDD	-	-	Specified values for the normal derivative on the boundaries A,B,C,D respectively
F	*	-	Input - matrix which specifies the source/sink information along with specified boundary data Output - matrix of pressure solution value
MBDCND NBDCND	-	-	Specifies the boundary condition (see Appendix B)
ELMBDA	-	-	Specifies the calculation of the Poisson equation by PWSCYL (ELMBDA = 0 always)
TINC	Δt	-	Time step increment
TIM	t	-	Time (sum of time step increments plus initial time)
COND	C	-	Fluid conductivity of medium
PHI	ϕ	-	Porosity of medium
RHO	ρ	-	Density of liquid

* Relationship between F and f_o given later in chapter (4).

IV. COMPARISON BETWEEN ANALYTIC AND NUMERICAL MODELS

Basically the analytic model and the time step reference model (RM1) are very similar. Both use a linearized form of the free surface condition, and both approximate the flow pressure on the reference surface Σ by the hydrostatic approximation in the space between Σ and Ω . The major differences between them are due to the inherent problems of the finite difference computer techniques. Some of these problems, like the infinite reservoir in the analytic case can only be approximated by putting the sides and bottom boundaries very far from the region of interest. Also the analytic point source has to be approximated by a small source region in the numerical situation. Yet the similarity of the calculated solutions for each case indicates that the approximation techniques are quite adequate.

Relationship Between the Analytic and Numerical Models

The source (sink) term of the analytic model (f_o) can be related to the source term of the numerical model (F) by integrating over the space surrounding the point source in the analytic case and summing over all of the grid points for the numerical case.

$$\text{Analytic} \quad : \quad -\nabla^2 p = (-f_o/2\pi rC)\ \delta_+(r)\ \delta(d-z) \tag{4.1}$$

$$\int \nabla^2 p\ dv = \int [(f_o/2\pi rC)\ \delta_+(r)\ \delta(d-z)\ ]\ dv \tag{4.2}$$

$$dv = rd\theta drdz \tag{4.3}$$

$$\int \nabla^2 p\ dv = f_o/C \tag{4.4}$$

Numerical : $$\nabla^2 p = F(r,z) \tag{4.5}$$

$$F(r,z) = \begin{cases} \text{constant} = F_o \; ; \begin{cases} 0 \leq r \leq (\frac{B}{2M}) \\ (d - \frac{D}{2N}) \leq z \leq (d + \frac{D}{2N}) \end{cases} \\ 0 \qquad ; \text{Otherwise} \end{cases} \tag{4.6}$$

$$\int \nabla^2 p \; dv = \int F(r,z) \; dv \tag{4.7}$$

$$\int \nabla^2 p \; dv = -F_o \pi r^2 \Big|_0^{\frac{B}{2M}} z \Big|_{d-\frac{D}{2N}}^{d+\frac{D}{2N}} \tag{4.8}$$

Equating equations 4.4 and 4.8 we obtain a relationship between f_o and F_o

$$\frac{f_o}{C} = - F_o \pi \left(\frac{B}{2M} \right)^2 \left(\frac{D}{2N} \right) \tag{4.9}$$

which allows us to determine an input value F_o for the numerical model that corresponds to some selected mass flow rate f_o . The minus sign in equation 4.9 results from an interchange of sources and sinks.

One other peculiarity of the computer technique is that we have expressed the input reservoir dimensions in kilometers (km) rather than meters (m) . This is done so as to reduce the magnitude of the parameters B (radius) and D (depth) . It has been suggested that the finite difference technique used to calculate the solution for Poisson's equation will give better results if the magnitude of these parameters is small (Dr. F.T. Lindstrom, personal communication,1980). However, this implies an unusual set of units where

all length dimensions are in km (mass in kg and time in sec remain the same). F_o used as an input then has dimensions of kg/km^3sec^2, and the output values of pressure have dimensions $kg/km \cdot sec^2$. Translating the pressure values into the standard MKS system is fortunately a simple task. Simply by dividing p by 10^3 we obtain the equivalent MKS value.

Comparison of the Models

Comparison between the numerical and analytic solutions can be made using several techniques. The most obvious and quite satisfactory procedure is to plot the position of the free surface (or some surface of interest) for both models in the same scale directly over one another.

Another way to compare the two situations would be to consider the ratio of the pressure values. On this procedure we operate with the Ratio (R)

$$R = \frac{\text{Numerical Solution}}{\text{Analytic Solution}} \qquad (4.10)$$

at selected locations within the reservoir.

In most of our comparisons we consider specific profiles which are common to both situations. Particuarly three specific profiles are chosen to illustrate essential changes within the reservoir. One of these is obviously along the reference surface (or slightly below the surface for the t=0 case) starting at the borehole and extending radially out. This profile will be of special interest because the boundary conditions along this surface define

the pressure values in the reservoir during the succeeding iteration. Another profile of interest is along the central axis extending from the reference surface down. A third profile can be located below the reference surface and extend radially out. In general we have chosen to look at a profile beginning somewhere near the point sink. These three areas give a fairly good coverage of the changes taking place in the reservoir. In the particular case below we have extended these profiles out to a radius or depth of about 3 km for a reservoir with an outer radius of over 14 km . At this distance from the numerical boundaries any boundary effects will clearly be negligible, and yet a good understanding of the significant characteristics of the pressure field can be obtained.

The Homogeneous Boundary Value Solution

Initially we begin our iteration procedure by setting the pressure on the free surface to zero. The solution to this situation corresponds to the half space homogeneous boundary value case p=0 at t=0 and z=0 as we discussed in chapter (2) for the analytic solution. However with the side and bottom boundaries having a no-flow condition, derivative normal to boundary equal zero, the numerical solution represents a deviation from the half space case.

The homogeneous boundary value solution is a simple case and will provide a good situation to obtain a measure of how well the numerical solution matches the analytic solution. One thing to consider is the effect of the lower reservoir boundary on the solution values. By keeping the grid spacing constant and varying

the number of grid points between the sink point and the boundaries we can examine the effect of the no-flow condition on solution values. This changes the ratio d/L (ratio of point sink depth d to the boundary distance L) for which we concluded in chapter (2) a reasonable value would be 1/10 or less.

Looking at the plots generated in figures 2, 3, and 4 the most apparent effect of fewer grid points is a poorer fit between the two models. In particular, the R value increases at a greater rate for larger radii. For M=N=144 or 100 the similarity of the R curve shapes and their closeness to unity indicate that the boundary effects are minor for these values. Considering that using M=N=100 rather than 144 requires about 30% less computational time it is reasonable to consider the lower value as an adequate choice for many cases. However, the choice of the M,N values should depend on the specific requirements of the situation.

One other feature which we should point out is the way in which the numerical and analytic solutions close to the borehole sink match. Since the analytic point sink is of infinite density we really can't compare the two situations at the sink position, but the solution values are finite elsewhere. Within about two grid spacings of the borehole sink, the two models become strikingly different. However, beyond this distance the solution values compare much better. This effect is observed in each of the graphs in figures 2 and 3 regardless of the M and N values.

In the graphs shown we do not compare the two models out past 3 km (30 grid points) although they compare relatively well at

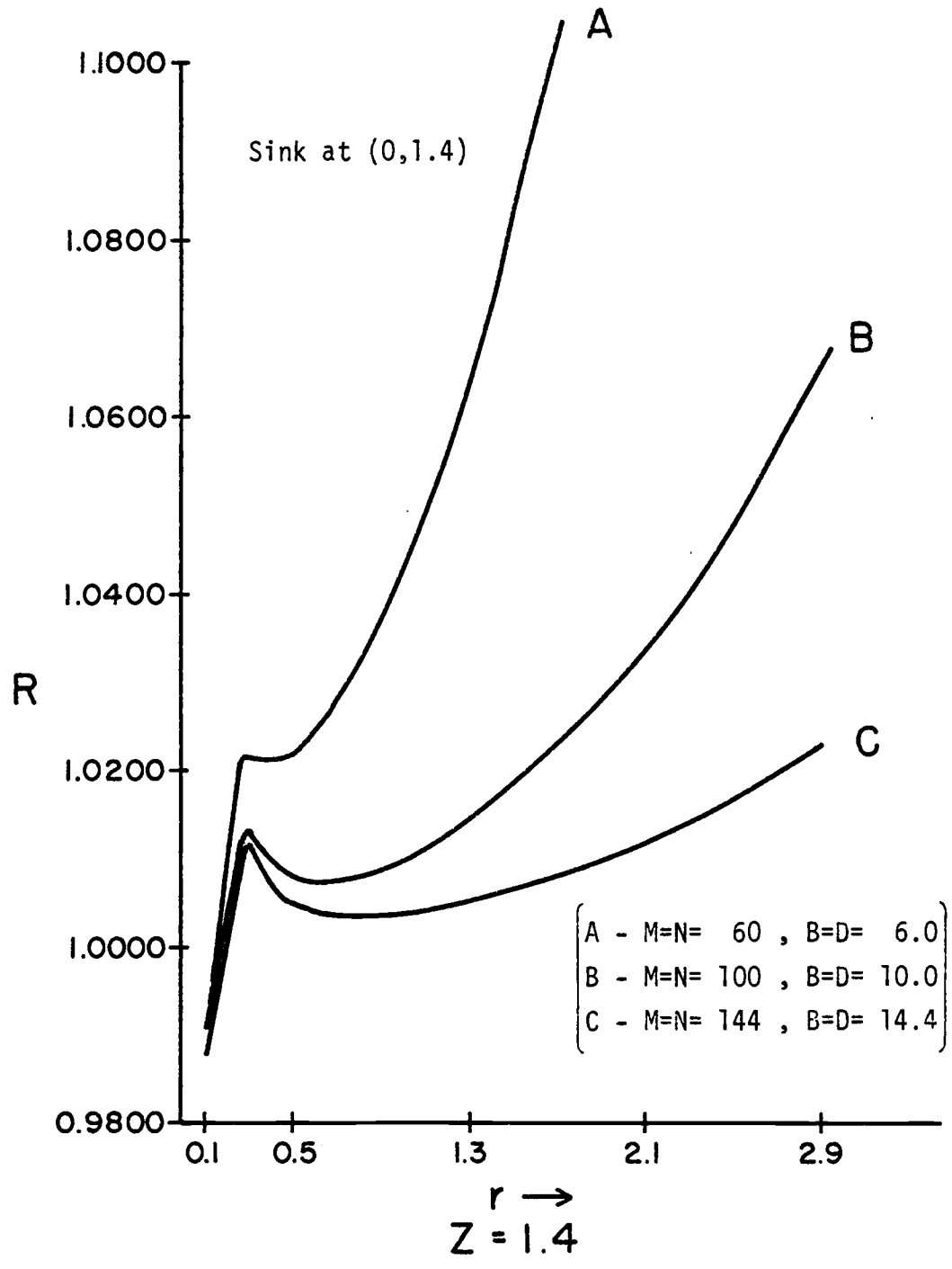

Figure 2. Profile of R values for the homogeneous boundary value solution for three different boundary distances relative to a constant point sink location.

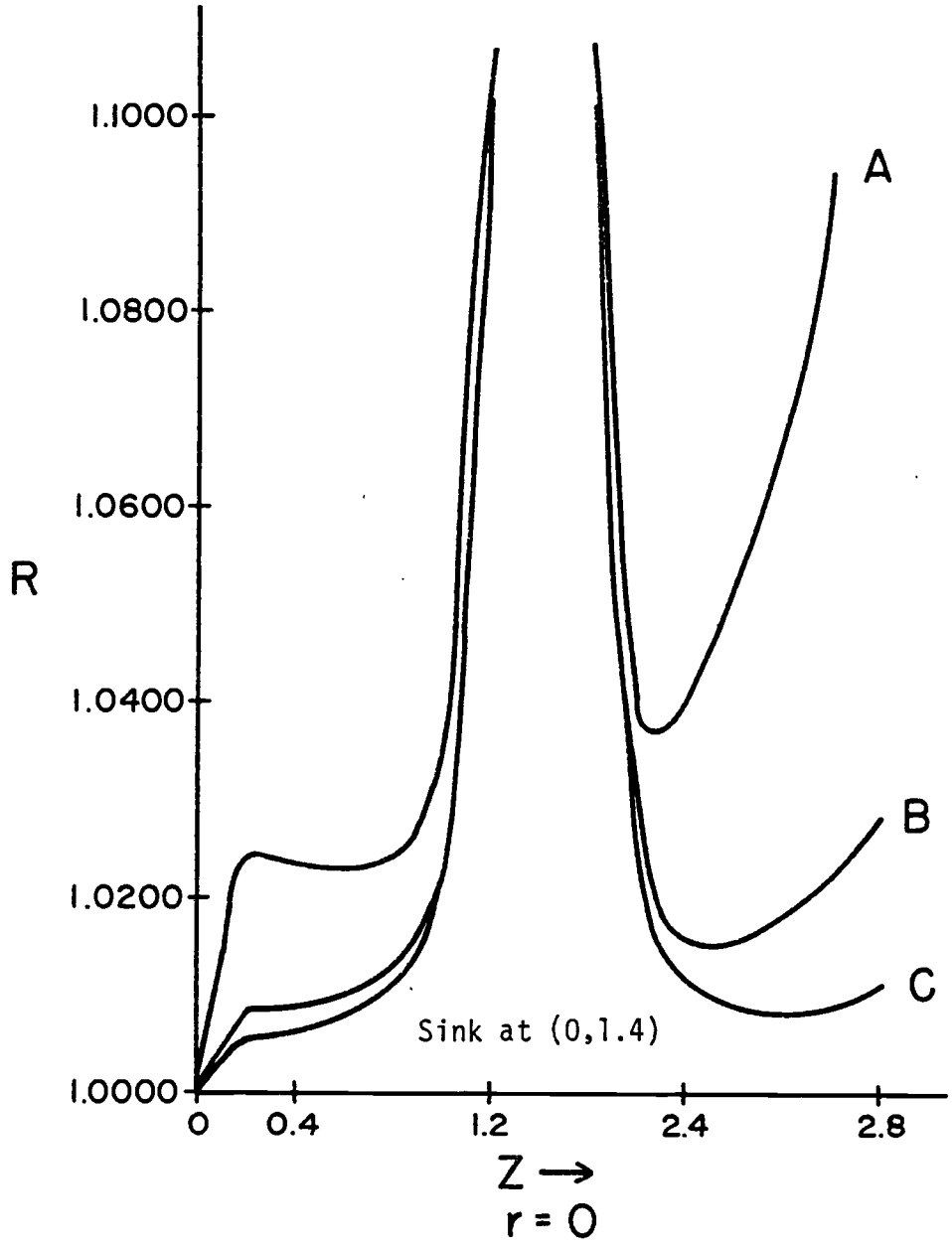

Figure 3. Profile of R values for the homogeneous boundary value solution along the radial axis for three different boundary distances relative to a constant point sink location.

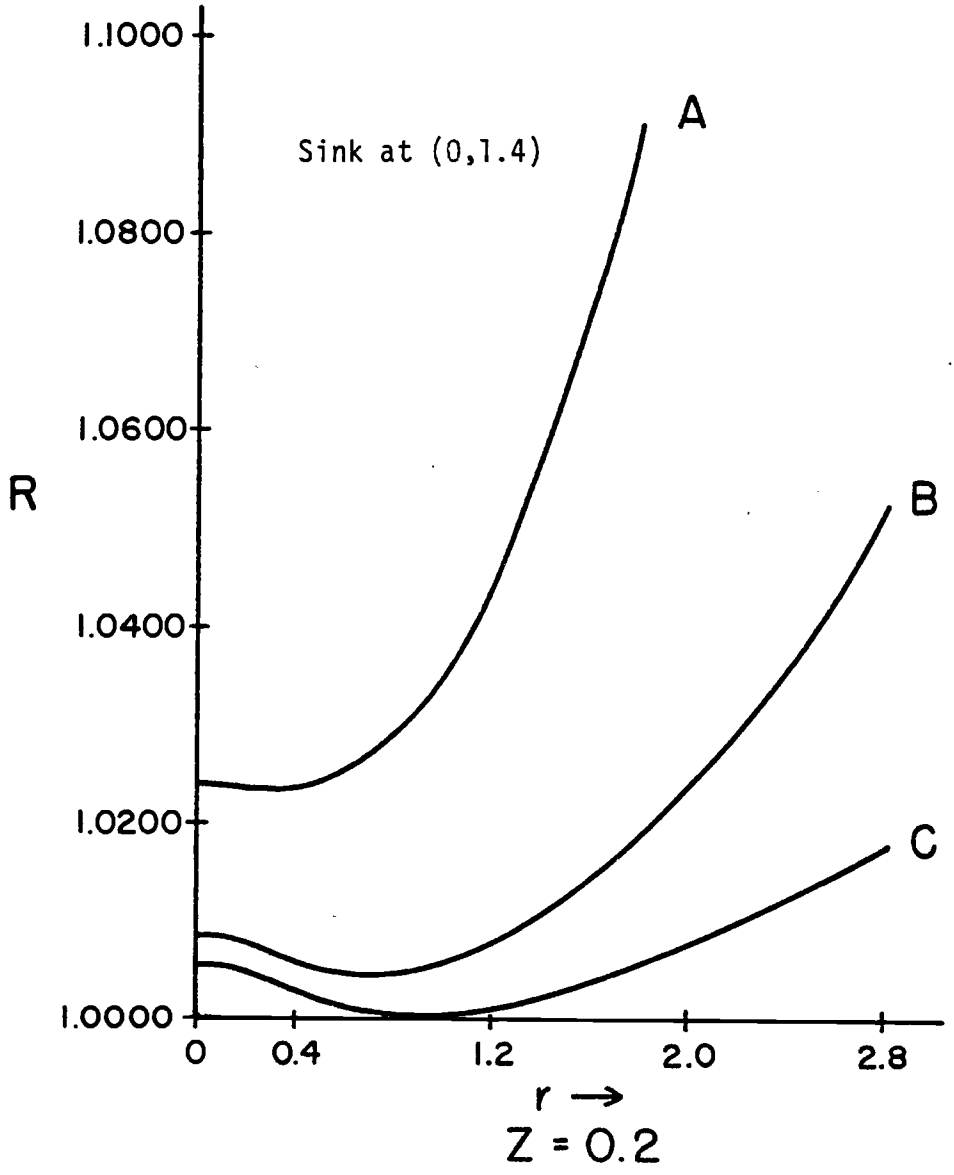

Figure 4. Profile of R values for the homogeneous boundary value solution radially just below the free surface for three different boundary distances relative to a constant point sink location.

larger distances up to the vicinity of the boundaries where the fit becomes poorer again. From an examination of R values at constant depth radially out from the point sink we find that the pressure values of the two models are within 10% of each other as far out as 5.1 km .

Because of linearity these results are independent of the sink strength. When the strength is multiplied by some constant, the solution values are uniformly also multiplied by some constant. The analytic solution most clearly exemplifies this situation because the strength of the sink is a simple multiple in the pressure solution (see eq. 2.35), and any multiple of this value then merely multiplies the solution values likewise. However, the validity of the linearization depends on the drawdown amplitude.

Approximation of dp/dz by a Polynomial

In order to find the liquid velocity at the free surface in the iteration technique we have to calculate the vertical pressure gradient at the reference surface. An analytic expression for the pressure as a function of depth at z=0 is obtained by fitting a polynomial curve to points there. The positions used consist of a point on the reference surface and adjacent points along the z-direction. By taking the derivative of this polynomial at the z=0 position we obtain an expression which approximates dp/dz at z=0. Appendix A describes the developement of the fourth order polynomial derivative using Newton's Divided-Difference Method. A brief discussion is also included on the subroutines used to calculate

the polynomial derivative.

The selection of a fourth order polynomial was based mainly on a trial and error procedure in which a higher order polynomial was found to offer no significant improvements. For example, in a particular situation we calculated the incremental reference surface pressure to be $p= -9.9822X10^{6}$ at $z=0$ using a fourth order polynomial and $p= -9.9949X10^{6}$ at $z=0$ using a fifth order polynomial. The change in the two surface values for an increase in polynomial order is less than one percent.

The Validity of the Linearized Free Surface Approximation

There are at least two major factors to consider in deciding on the validity of the time step technique. First we need to know how well the linearization approximates the true situation, and second, how well the time step technique approximates the analytic form of the linearization. The second factor can be evaluated relatively easily by simply selecting the model parameters such that they fit the analytic model. Then by calculating some values and comparing them we obtain a measure of how closely the two models match.

However, no analytic solution is known which satisfies the complete non-linear free surface condition in equation 2.5 . Thus we really have no good analytical procedure at our disposal to test how well the linearization would fit an exact solution. Nevertheless by explicitly calculating the quadratic and linear boundary terms in given cases it is possible to obtain a measure of their relative magnitudes.

A convienient way to compare these two terms is by taking the ratio which we call Quadratio QR ,

$$QR = \frac{\text{Quadratic Term}}{\text{Linear Term}} \qquad (4.11)$$

where the quadratic and linear terms are give by equations 2.19 and 2.20 respectively. QR is convieniently expressed as a percentage value. If this value does not exceed 10% then we consider that a valid justification for excluding the quadratic term since its contribution to the overall free surface condition is small. The 10% limitation is based on common experience with perturbation techniques.

Testing out this technique with the reference model RM1 discussed in chapter (2), we find that, in fact, the quadratic terms are small in comparison to the linear term provided that the sink strength and the total time of drawdown are within reasonable bounds. Table 3 gives some selected examples that have been calculated using this model with a time step of $4X10^7$ sec (463 days). We find that QR tends to limit the admissible sink strength (f_o) in this particular model. In addition, if we know QR for some specific f_o value and some specified model parameters, then within the present approximation the QR' for some other f_o' value with the same model parameters is simply

$$QR' \simeq \left(\frac{f_o'}{f_o} \right) QR \quad . \qquad (4.12)$$

It is also of interest that the surface point having the highest QR value tends to change with time and approaches in the case of RM1 a stable position near (r,z)=(1.0,0) . Calculation of QR for the

analytic solution also indicates a similar change of position for the maximum QR with time. The position it migrates to depends on the depth of the point sink.

Table 3. Selected examples of the maximum QR for different times using a time step of $4X10^7$sec with model RM1.

Time $X10^7$ sec	Position (r,z) km	QR (f_o=50)	QR (f_o=1000)
0	(0.0,0)	0.00416	0.0832
4	(0.0,0)	0.00299	0.0598
12	(0.7,0)	0.00221	0.0442
20	(0.9,0)	0.00244	0.0488
40	(1.0,0)	0.00405	0.0810
60	(1.0,0)	0.00634	0.1268
80	(1.0,0)	0.00902	0.1804

The Time-Iterative Solution

The time-iterative solution is the next step beyond the homogeneous boundary value case since it begins with a zero pressure on the free surface boundary then perturbs the situation and recalculates a new solution. We still must consider the fact that the time step procedure depends on a pressure field solution which contains boundaries at distance. However, comparing this to the analytic solution through many time steps will be a good test of the iteration method.

Figure 5 shows a profile of the free liquid surface for RM1 as set up in chapter (2) over a time period of $80XI0^7$sec . Here we have selected a time step of $4X10^7$sec as an example. The surface drawdown is in a conical shape around the borehole with the maximum at the

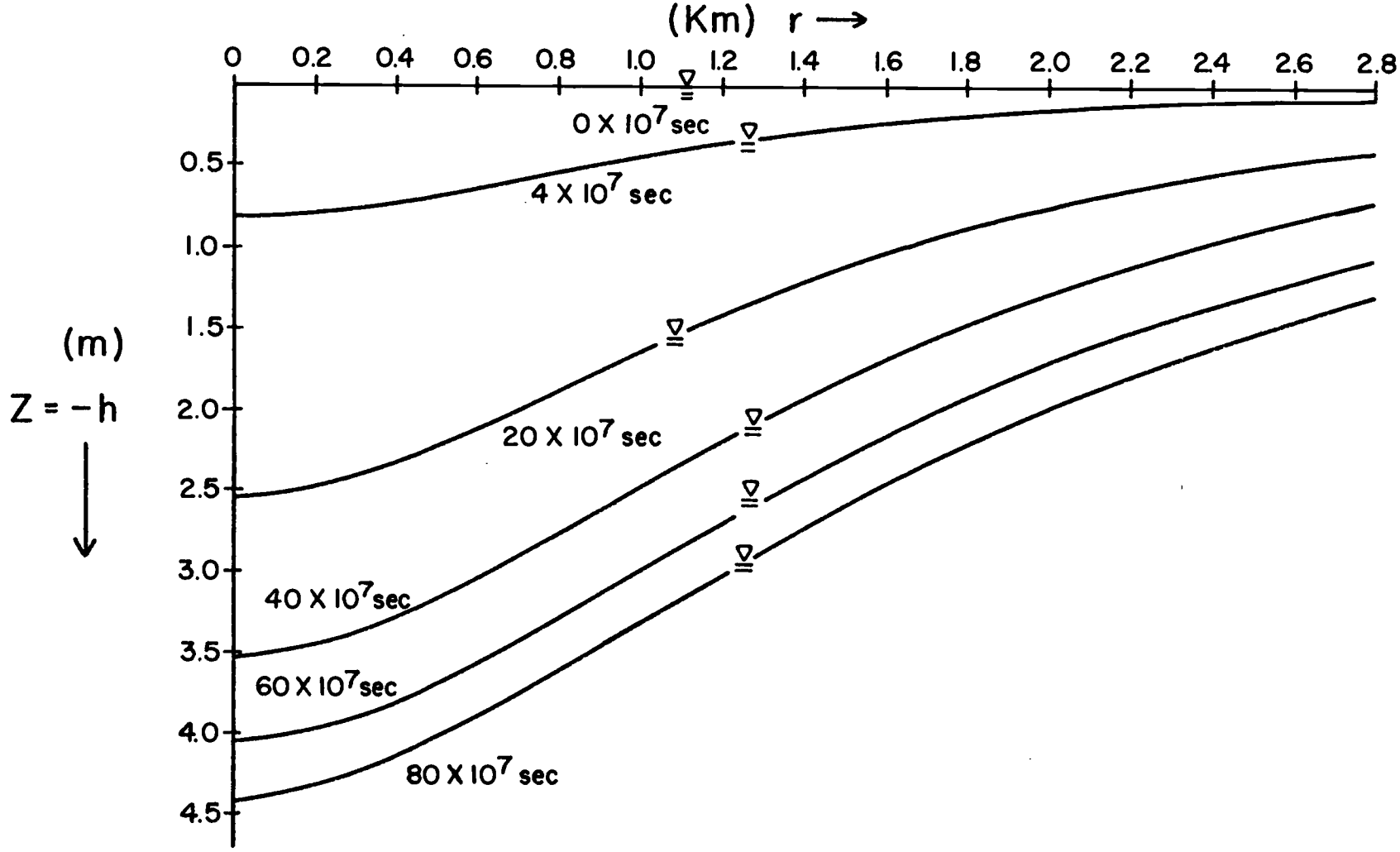

Figure 5. Profile of the free liquid surface radially outward from the central axis for several different times of the model RM1 with $\Delta t = 4X10^7$sec .

central axis and smoothly becomes smaller radially outward. The initial drawdown rate is large, but rapidly slows down with increasing time as indicated by the time lines being closer together for large t .

The profiles shown here are scaled with a rather large vertical exaggeration of 1000 to emphasize the drawdown features. As noted earlier for the homogeneous boundary value case, the magnitude of the pressure field within the reservoir is directly proportional to the value of the sink strength (f_o) . This effect holds also through the time step procedure. Consequently if a larger withdrawal (e.g. 500 kg/sec) were considered the curves would still be the same only the scale on the z-axis would be multiplied by a constant factor (equivalent to a vertical exaggeration of 100).

The next problem is to find out how well figure 5 represents the analytic situation which RM1 is designed to approximate. Figures 6 and 7 show the ratio R for two of the selected profiles (radially out from the sink and along the reference surface, respectively) over the time interval of the drawdown. After the first time step (time = $4X10^7$sec) the free surface has significantly overshot the drawdown of the analytic solution near the borehole. However, it quickly settles down and achieves a fairly close approximation to the analytic solution after several time steps. Farther out from the borehole the R values indicate that the difference between the analytic and numerical solution becomes notably larger, and in addition the trend over time is for this situation to become worse. Yet figures 6 and 7 indicate that the divergence of the analytic

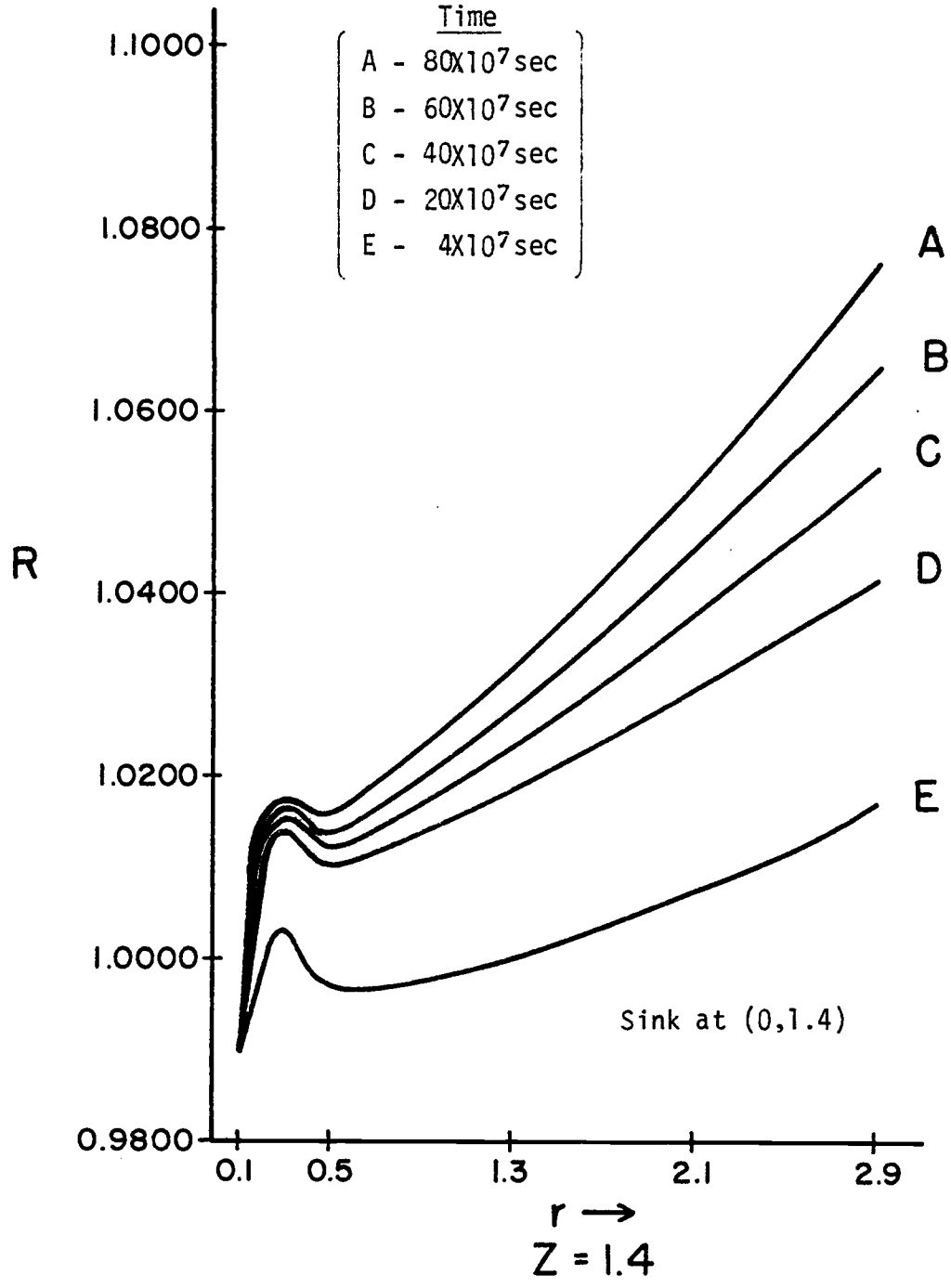

Figure 6. R values plotted radially out from the point sink at several different times for RM1 with $\Delta t = 4X10^7$sec.

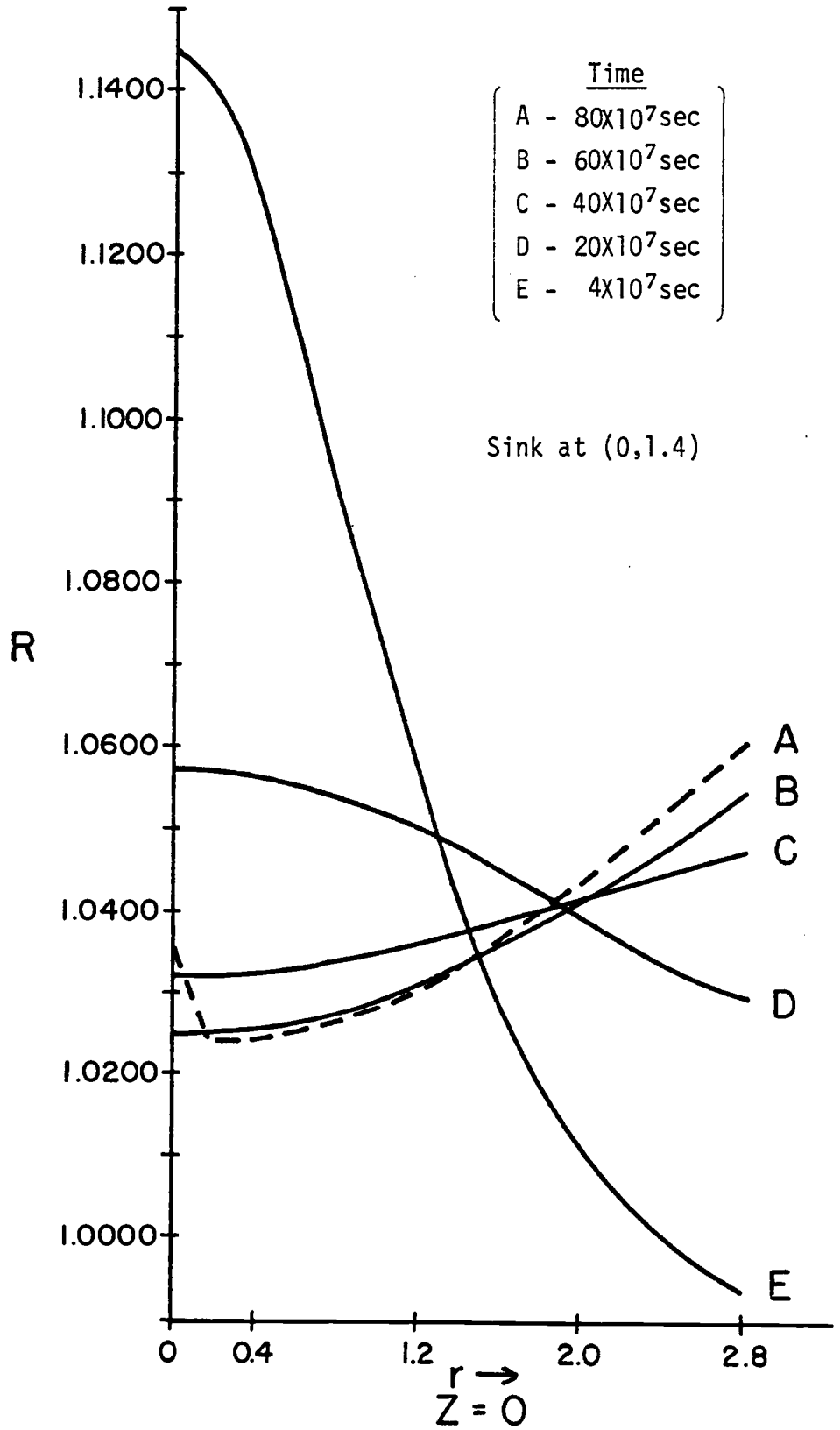

Figure 7. R values plotted radially out along the reference surface at several different times for RM1 with $\Delta t=4X10^7$.

and numerical models with time is not very rapid.

We also are interested in how well the polynomial derivatives of the numerical solutions compare with the analytically derived derivatives. To illustrate the similarity over time for several surface points Table 4 list some calculated values for a particular case ($\Delta t = 4X10^7$sec). Comparison indicates that the numerical polynomial derivative values are fairly good approximations to the analytic solutions.

Table 4. Comparison between analytic and RM1 polynomial derivative values for $\Delta t = 4X10^7$sec at two selected points on the reference surface.

	dp/dr			(r,z)=(0.4,0) dp/dz		
Time	Analytic	Numerical	%Diff	Analytic	Numerical	%Diff
0	0	0	0	$-36.09X10^6$	$-36.19X10^6$	<1
$20X10^7$	$8.05X10^6$	$8.60X10^6$	7	$-13.47X10^6$	$-12.27X10^6$	9
$40X10^7$	$9.49X10^6$	$9.76X10^6$	3	$-6.90X10^6$	$-6.38X10^6$	8
$60X10^7$	$9.93X10^6$	$10.11X10^6$	2	$-4.17X10^6$	$-3.99X10^6$	4
				(r,z)=(2.4,0)		
0	0	0	0	$-5.19X10^6$	$-5.26X10^6$	1
$20X10^7$	$3.96X10^6$	$4.15X10^6$	5	$-4.90X10^6$	$-5.08X10^6$	4
$40X10^7$	$6.19X10^6$	$6.41X10^6$	4	$-3.80X10^6$	$-3.89X10^6$	2
$60X10^7$	$7.34X10^6$	$7.49X10^6$	2	$-2.83X10^6$	$-2.92X10^6$	3

In Chapter (2) we established that a drawdown of 1/5 the borehole depth would be an upper bound for the applicability of the linearized free surface approximation. Calculating out the maximum drawdown at the borehole for RM1 after a time of $80X10^7$sec

($\Delta t = 4X10^7$sec) we get 4.42 m for a flow strength f_o=50 kg/sec . This compares well with the analytically determined value of 4.27 m for the same time, and it is well within the established limit. Another way to view this situation is to use the maximum drawdown limit for this model and calculate the maximum flow strength required to produce that drawdown over the time interval. Since the pressure values calculated for a particular flow strength f_o can be used to determine the pressure values for any other flow strength f_o' (as a simple multiple f_o'/f_o), this task becomes an easy problem. We find that a flow rate of 3168 kg/sec would produce the drawdown limit for RM1 ($\Delta t = 4X10^7$sec) after $80X10^7$sec .

However, the maximum drawdown is not the only limiting factor for the flow rate. The Quadratio (QR) also changes as approximately a simple multiple (f_o'/f_o) of the flow rate. Taking the maximum QR value calculated along the reference surface of the numerical model, we use this to set up a proportionality and find the limiting f_o value. Table 5 list the maximum f_o values allowed based on the QR=0.1 limit defined earlier. The limitations placed on the quadratic term specify an upper limit to the flow rate of 554 kg/sec for this model. Thus the QR value becomes the limiting factor at least in this case.

The iterative technique we have discussed works adequately for most physical models. It is however limited by an inherent complication which may arrise in certain cases. Generally these computational artifacts are easily distinquished. In the cases we have dealt with, a sudden reversal in the pressure gradient caused anomalous values to occur at the point (0,0) (or nearby points on

Table 5. The maximum flow rate (f_{om} kg/sec) possible for RM1 to maintain the established limits on QR through successive time iterations. Based on the largest QR value for each iteration (Time in sec. $X10^7$).

Δt =	$4X10^7$		$5X10^7$		$8X10^7$		$9X10^7$		$10X10^7$	
Itera.#	Time	f_{om}	Time	f_{om}	Time	f_{om}	Time	f_{om}	Time	f_{om}
1	4	1672	5	1849	8	2108	9	2060	10	1960
2	8	2116	10	2234	16	2048	18	1905	20	1735
3	12	2267	15	2204	24	1735	27	1600	30	1501
4	16	2202	20	2009	32	1345	36	1132		
5	20	2048	25	1778	40	1112	45	949		
6	24	1863	30	1563	48	946				
7	28	1684	35	1368	56	712				
8	32	1516	40	1204	64	62				
9	36	1365	45	1067						
10	40	1233	50	951						
11	44	1118	55	855						
12	48	1018	60	771						
13	52	931	65	702						
14	56	855								
15	60	788								
16	64	729								
17	68	677								
18	72	632								
19	76	591								
20	80	554								

the reference surface) after several time steps. Continuing the time step procedure resulted in similar discrepancies at other points along the surface. A possible cause for this condition may be that the numerical polynomial derivative values become small after some given number of time steps and the calculated value then becomes subjected to irregularities of the polynomial approximation. Consequently when the polynomial derivative becomes small it may begin taking on incorrect negative values which result in the observed pressure gradient reversals. The a priori predicting of such gradient reversals would require a rather involved analysis.

The parameter b

$$b = \frac{Cg}{\phi} \Delta t \tag{4.13}$$

has been found to be related to the maximum number of time steps taken before an anomalous pressure gradient reversal occurs. It is a factor which appears in the calculation of the free surface pressure values for a time step (see eq. 3.3), and we have called it the characteristic interval. Its name is chosen because Cg/ϕ is refered to as the characteristic fluid velocity (Bodvarsson,1977). Figure 8 shows how this is related to the number of time steps taken for a point sink. The values of porosity, conductivity, and the time increment can be varied for convenience without affecting the program operation provided that b remains the same. We can also change the magnitude of the point sink (source) strength f_o without affecting the number of time steps taken before the pressure gradient reversal occurs.

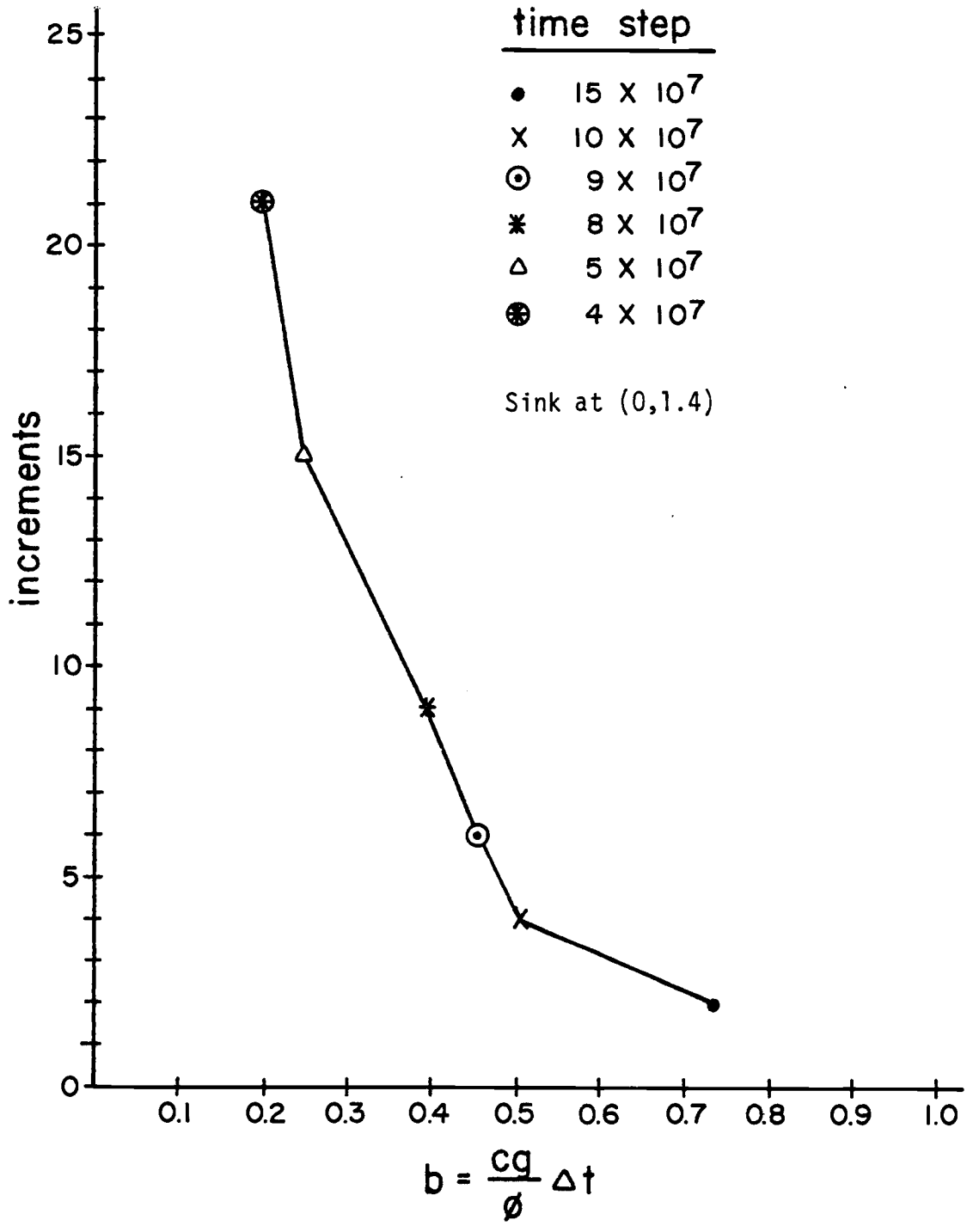

Figure 8. Maximum time steps before reversal of pressure gradient as a function of the characteristic interval for RM1 .

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APPENDICES

APPENDIX A

Development of a Fourth Order Polynomial Derivative

Development of an interpolating polynomial using Newton's Divided-Difference Method is given by Carnahan, et al. (1969). Adapting a similar notation we can write out a fourth order interpolating polynomial without the remainder term as:

$$y(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{A.1}$$

where,

$$a_4 = y[x_4,x_3,x_2,x_1,x_0] \tag{A.2}$$

$$a_3 = y[x_3,x_2,x_1,x_0] - (x_0+x_1+x_2+x_3)y[x_4,x_3,x_2,x_1,x_0] \tag{A.3}$$

$$a_2 = \Big[(x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3)y[x_4,x_3,x_2,x_1,x_0] - (x_0+x_1+x_2)y[x_3,x_2,x_1,x_0] + y[x_2,x_1,x_0]\Big] \tag{A.4}$$

$$a_1 = \Big[(x_0x_1+x_0x_2+x_1x_2)y[x_3,x_2,x_1,x_0] - (x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3)y[x_4,x_3,x_2,x_1,x_0] - (x_0+x_1)y[x_2,x_1,x_0] + y[x_1,x_0]\Big] \tag{A.5}$$

$$a_0 = \Big[(x_0x_1x_2x_3)y[x_4,x_3,x_2,x_1,x_0] - (x_0x_1x_2)y[x_3,x_2,x_1,x_0] + (x_0x_1)y[x_2,x_1,x_0] - (x_0)y[x_1,x_0] + y[x_0]\Big] \ . \tag{A.6}$$

$$y[x_n,x_{n-1}, \ldots,x_0] = \frac{y[x_n,x_{n-1}, \ldots,x_1] - y[x_{n-1},x_{n-2}, \ldots,x_0]}{x_n - x_0} \tag{A.7}$$

Taking the derivative we have,

$$y'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1 \tag{A.8}$$

for the sample values y_4, y_3, y_2, y_1, y_0 corresponding to the positions x_4, x_3, x_2, x_1, x_0 . In this situation the spacing of the y values determines the denominator of the difference thus only relative magnitudes of the x positions are significant.

We need to know the derivative at the reference surface. Consequently, for y_0 at the surface z=0 we choose x_0=0 for convenience such that y_1,y_2,y_3,y_4 and x_1,x_2,x_3,x_4 are successive points below the surface. The derivative is then taken at x=0 which helps simplify the form of equation A.8 to

$$y'(0) = a_1 \qquad (A.9)$$

To calculate the derivative values on the reference surface with the time step technique a subroutine called DIFTBL is used to set up the Newton's divided-difference table for the selected values. Input parameters are listed in Table 6 .

Table 6. Input for subroutines DIFTBL and DERVZO to calculate the polynomial derivative of the free surface.

Parameter		
X	-	Array containing position values x_0,x_1,x_2,x_3,x_4 with x_0=x(1)
F	-	Matrix array of solution values for the reservoir
IC	-	Radial position F(IC,1) of first polynomial point x_0 (x_1,x_2,x_3,x_4 are along the z-axis from x_0)
TABLE	-	Matrix array of the divided difference table values (output)
MP	-	Polynomial order
NP	-	Number of points used
K	-	Dimension of arrays X and TABLE
OK	-	Check for polynomial order errors

Arrays F and X must be set up prior to use of the subroutine. Output values in matrix array TABLE are used with Function DERVZO to calculate equation A.9 for the point F(IC,1) using a polynomial fit to points F(IC,J) J=1,5 .

APPENDIX B

Program Listings

The interactive calling program for the iteration technique is POSGO . Using a set of command prompts from a terminal the user directs the computational actions and checks progress or results. The program documentation contains a complete list of these commands with explanation. In addition, all parameters required are explained in the program documentation for each section or subroutine used. Some commands require additional information from the user before the job can be completed. In this case, the program prompts the user with appropriate questions or format required for input.

The program operation of POSGO begins by a prompt for a command. Entering a command from the terminal results in the program operation being shifted to the specific command section. If additional information is required (e.g. input of a parameter value, profile position, etc.) a prompt will appear on the screen indicating the type of information required and/or the format specification expected on input. When sufficient information is obtained the appropriate subroutines are called and jobs performed. After completion of the calculations or task for a command section the program returns to the command selection and prompts for another request.

POSGO and associated subroutines are written in single precision FORTRAN IV and designed to be used on a CYBER computer. To use on some other computer the array allocations and input/output sections must be checked for compatability.

```
      PROGRAM POSGO(INPUT,OUTPUT,TAPE9,TAPE12,TAPE11=500,TAPE14=500)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C     POSGO IS AN INTERACTIVE CALLING PROGRAM FOR THE SUBROUTINES
C     PWSCYL,...  IT GIVES THE USER A WAY OF CHANGING PARAMETERS
C     AND RECALULATING THE SOLUTION FOR THE 2-DIM POISSON EQ. (OR THE
C     MORE GENERAL HELMHOLTZ EQ.)
C
C
C            WRITTEN BY :   WILLIAM E. AVERA
C
C            TIME ITERATIVE PROCEEDURE
C            DEVISED BY :   DR. GUNNAR BODVARSSON
C
C            VERSION    :   3   :   MARCH 1981
C
C
C     INPUT / OUTPUT SPECIFICATIONS
C
C     TAPE9=INTIAL PARAMETERS INPUT TO POSGO
C     TAPE11=OUTPUT FILE USED TO MAKE A SECOND RUN OR IS REALLY
C            JUST A COPY OF F(I,J) AS CALLED ON INPUT TO PWSCYL.
C     TAPE12=OUTPUT FILE CONTAINING SOME PARAMETERS USED ALONG
C            WITH THE OUTPUT VALUES   (AS FOLLOWS)
C
C                      PARAMETERS
C            LINE 1    A,B,M,MBDCND,COND,IFLG1,IFLG2 (2X,2F15.4,2I10,E15.8,1X,2I1)
C            LINE 2    C,D,N,NBDCND,PHI,IFLG3,IFLG4 (2X,2F15.4,2I10,E15.8,1X,2I1)
C            LINE 3    ELMBDA,RHO,TINC,TIM        (2X,4F15.4)
C            LINE 4 THROUGH LINE (N+1)+3 OR LINE (M+1)+3 WHICHEVER LARGEST
C                        BDA(I), BDB(I), BDC(I), BDD(I)                 (2X,4F15.4)
C            FROM THIS POINT TILL END OF FILE
C                         OUTPUT FROM PWSCYL [F(I,J)]
C                           (2X,E15.8,11X,E15.8,11X,E15.8,11X,E15.8,11X,E15.8,11X)
C
C     TAPE14=COPY OF THE PARAMETERS WHICH CAN BE USED LATER AS
C            A PARAMETER INPUT FILE
C
C     OUTPUT FILES  TAPE14  AND  TAPE12  INCLUDE LINE PRINTER CARRAGE CONTROL
C
C     PRESENTLY  IDIMF=155 ALWAYS  (THE  I  DIMENSION OF THE  F  ARRAY)
C
C     M  IS IN THE RADIAL DIRECTION. N  IS IN THE  Z  DIRECTION
C     (USING CYLINDRICAL CORD.)
C                                 P    Q    R
C     N  MUST BE OF THE FORM (2) (3) (5) , P,Q,R ARE NON-NEGATIVE INTEGERS
C     LIMITS ON  N  AND  M  ARE SET AT 150
C
C     OUTPUT TO TERMINAL MAY BE OBTAINED FOR ANY SIZE BLOCK OF THE
C     MATRIX BY SPECIFYING THE BEGINNING AND ENDING  (I)S  AND  (J)S.
C                                                (I3,4X,I3,1X,I3,4X,I3)
C
C     F(I,J) OUTPUT TO TAPE12 AND TAPE14 IS IN  5  COLUMNS WHERE THE
C     (I) DIMENSION OF THE BLOCK REQUESTED HAS BEEN BROKEN UP INTO
C     A MULTIPLE OF  5 .
C
C     INPUT INFORMATION FROM THE TERMINAL IS GENERALLY PROMPTED BY A GROUP
C     OF LETTERS ABOVE THE INPUT LINE THAT INDICATE THE FORMAT,POSITION,
C     AND TYPE OF INFORMATION REQUESTED.
C
C     [ NOTE ] CAUTION IS REQUIRED WHEN INPUTTING VALUES TO PROGRAM
C              INTEGERS MUST BE RIGHT SHIFTED AND NON-INTEGERS MUST
C              INCLUDE A PERIOD.
C
C     FLG3 - FLAG TO ALLOW THE USER TO DECIDE IF THE DERIVATIVE VALUES
C       BDA,BDB,BDC,BDD  WILL BE OUTPUT WITH THE SOLUTION VALUES(POSLST)
C       OUTPUT OF THESE VALUES IS SURPRESSED UNLESS SPECIFIED EACH TIME
C       COMMAND   P   IS USED.
C       FLG3 = 0  DERIVATIVE OUTPUT SURPRESSED FROM POSLST
C       FLG3 = 1  DERIVATIVE VALUES OUTPUT WITH POSLST
C
C     [ NOTE ] EXPLANATION OF THE MEANING AND USAGE OF PARAMETERS IS
C              GIVEN IN THE  PWSCYL  DOCUMENTATION.
C
C     A POLYNOMIAL FIT TO THE FIRST FIVE POINTS ALONG THE  Z  DIRECTION
C     OF EACH COLUMN FOR  F(I,J)  IS USED TO CALCULATE THE DERIVATIVE
C     OF THE PRESSURE FIELD AT  Z=0. A TIME INCREMENT IS THEN USED TO
C     DETERMINE THE MOVEMENT OF THE FLUID SURFACE.  USING A HYDROSTATIC
C     PRESSURE WE THEN CALCULATE A NEW PRESSURE VALUE FOR  Z=0 .
C     AN ITERATIVE PROCEDURE ALLOWS THIS TO BE DONE FOR AS LONG AS
C     NECCESARY.
C
C     PARAMETERS WITH SPECIFIC USAGES ( NOT FOUND IN PWSCYL DOCUMENTATION )
```

```
C
C       TINC - TIME INCREMENT VALUE USED TO EVALUATE THE INCREMENTAL
C              PRESSURE CHANGE FOR THE FREE SURFACE ( F(I,1) I=1,M+1)
C              DURING ONE TIME STEP.          UNITS SECONDS
C
C       RHO  - DENSITY OF FLUID IN CYLINDRICAL RESERVOIR
C                                            UNITS MASS/VOLUME
C
C       PHI  - EFFECTIVE POROSITY OF RESERVOIR MATERIAL
C
C       TIM  - TIME OF WHICH THE CURRENT SOLUTIONS HAVE BEEN
C              INCREMENTED TO.
C               TIM = SUM OF TIME INCREMENTS USED
C                                            UNITS SECONDS
C
C       COND - FLUID CONDUCTIVITY OF RESERVOIR
C                 COND = PERMEABILITY / ( KINEMATIC VISCOSITY )
C                                            UNITS SECONDS
C
C
C            EXPLANATION OF COMMANDS
C
C     R - (RUN) CAUSES EXECUTION OF SUBROUTINE GROUP  PWSCYL  TO
C         CALCULATE THE NUMERICAL SOLUTION BASED ON THE INPUT VALUES
C         IN ARRAY  F  , ALONG WITH THE BOUNDARY CONDITIONS (A,B,C,D,MBDCND,
C          NBDCND,BDA,BDB,BDC,BDD), AND PARAMETERS (M,N,ELMBDA)
C
C     I - (INCREMENT) DIRECTS THE SYSTEM TO INCREMENT PRESSURE VALUES
C         ALONG THE UPPER SURFACE (REFERENCE SURFACE) OF THE CYLINDRICAL
C         REGION (I.E.  F(I,1) , I=1,M+1 ). PRESSURE INCREMENTS ARE OBTAINED
C         BY TAKING THE SOLUTION VALUES OF A COLUMN IN THE F-MATRIX (J=1,5)
C         AND CALCULATING A FOURTH ORDER POLYNOMIAL APPROXIMATION TO THE
C         GRADIENT (DERIVATIVE) AT THE REFERENCE SURFACE (J=1). USING THIS
C         A VERTICAL VELOCITY IS CALCULATED FOR THE FREE SURFACE AT EACH
C         OF THE COLUMNS BY
C
C             VEL = COND*(PRESSURE GRADIENT) / (RHO * PHI) .
C                                            UNITS  LENGTH / TIME
C
C           [ NOTE ] IT HAS BEEN SUGGESTED THAT REDUCING THE MAGNITUDE OF
C                    THE  A,B,C,D  PARAMETERS WILL PRODUCE A MORE ACCURATE
C                    SOLUTION. FOR THIS REASON WE HAVE CHOOSEN TO REPRESENT
C                     A,B,C,D  IN KILOMETERS RATHER THAN METERS. THIS RESULTS
C                    IN SOMEWHAT STRANGE UNITS FOR PRESSURE OF
C                     KG./((KM.**3)*(SEC**2))  SINCE ALL LENGTH DIMENSIONS
C                    ARE THEN INPUT AS  KM.  INSTEAD OF METERS. LIKEWISE THE
C                    VELOCITY IS THEN  KM./SEC . WE HAVE ALSO CHOOSEN A
C                    CYLINDRICAL COORDINATE SYSTEM WITH OROGIN (0,0) AT F(1,1)
C
C         THE VELOCITY IS MULTIPLIED BY THE TIME INCREMENT TO OBTAIN AN
C         INCREMENTAL LOWERING  DELTA(H)  OF THE FREE SURFACE.
C
C               DELTA(H) = VEL * TINC
C
C         A NEW PRESSURE VALUE IS THEN CALCULATED FOR EACH POINT ON THE
C         SURFACE AT  F(I,1) I=1,M+1  USING THE HYDROSTATIC RELATION
C
C            NEW PRESSURE = (OLD PRESSURE) + DELTA(H)*GRAV*RHO
C
C                    GRAV = 9.80665E-3  KM/SEC
C
C         THE REMAINING  F(I,J) I=1,M+1 , J=2,N+1  ARE RESET TO ZERO FOR
C         THE NEXT ITERATION.
C
C         [ NOTE ] FOR THE SPECIFIC PURPOSE OF DEALING WITH POINT SOURCE(SINK)
C         SOLUTIONS THE LAST  F(I,J)  CHANGED WITH THE  CHANGE  COMMAND IS
C         INSERTED INTO THE SAME F(I,J) LOCATION WITH THIS COMMAND TO
C         PRESERVE THE SOURCE VALUE DURING AN ITERATION.
C
C     L - (LIST) ALLOWS THE USER TO VIEW ANY PORTION OF THE  F(I,J)  MATRIX
C         DESIRED AT THE TERMINAL BY SIMPLY RESPONDING TO A PROMPT WITH
C         THE BEGINNING  I  VALUE FOLLOWED BY THE ENDING  I  VALUE FOLLOWED
C         BY THE BEGINNING AND ENDING  J  VALUES. USE FORMAT
C                                        (I3,4X,I3,1X,I3,4X,I3).
C
C     P - (PRINT SOLUTION) CREATES A FILE THROUGH  TAPE12  OF SOME PARAMETERS
C         VALUES USED AND THE SOLUTION VALUES  F(I,J)  CREATED BY PWSCYL.
C         ANY PORTION OF THE  F(I,J)  MATRIX MAY BE OUTPUT AS SPECIFIED
C         BY A RESPONCE TO A PROMPT WITH THE BEGINNING AND ENDING  I  AND
C          J  VALUES.  ALSO AS SPECIFIED BY THE SETTING OF  FLG3  .
C
C     T - (THEORETICAL VALUES) COMPUTES THE ANALYTICAL SOLUTION TO POISSON'S
C         EQUATION FOR A POINT SOURCE(SINK) IN AN INFINITE HALF SPACE
C         BOUNDED BY A FREE SURFACE ALONG A SELECTED PROFILE. THE PROFILE
```

```
C           IS SELECTED BY SUPPLYING THE BEGINNING  I  AND  J  VALUES [F(I,J)]
C           FOLLOWED BY A VALUE TO INCREMENT THE  I  SUBSCRIPT FOLLOWED BY
C           A VALUE TO INCREMENT THE  J  SUBSCRIPT FOLLOWED BY THE TOTAL
C           NUMBER OF POINTS (INCREMENTS) TO BE USED IN THE PROFILE
C            (THIS CAN NOT EXCEED  155 ).  (I3,1X,I3,1X,I3,1X,I3,1X,I3)
C           THE PROFILE VALUES ARE THEN OUTPUT TO  TAPE12 .
C           A RATIO OF THE NUMERICAL RESULTS WITH THE THEORETICAL VALUES
C           IS OUTPUT FOR COMPARISON PURPOSES.
C
C                RATIO = NUMERICAL / THEORETICAL
C
C        Q - (QUADRATIC RATIO) COMPUTES THE FOURTH ORDER POLYNOMIAL DERIVATIVES
C            [ DP/DR , DP/DZ ], AND CALCULATES BOTH THE LINEAR AND QUADRATIC
C           TERMS. THEN FORMS THE RATIO
C
C                 QR = QUADRATIC / LINEAR
C
C           COMPUTATIONAL POINTS  F(I,1)  ALONG THE REFERENCE SURFACE ARE
C           SELECTED BY INPUT OF BEGINNING  I  FOLLOWED BY ENDING  I  FOLLOWED
C           BY INTERVAL SPACING OF CALCULATED  F(I,1)  POINTS.
C                                                  (I3,1X,I3,1X,I3)
C
C        D - (DATA FILE) OUTPUTS A PARAMETER FILE TO  TAPE14  WHICH CAN BE
C           USED AS AN INPUT FILE. THE PWSCYL INPUT  F-MATRIX MUST HAVE
C           BEEN SAVED WITH COMMAND  SF  (TO CREATE FILE  TAPE11) IN ORDER
C           FOR THE PARAMETER FILE TO BE OUTPUT. USE OF THIS COMMAND DOES
C           NOT ALTER THE CONTENTS OF THE  F-MATRIX.
C
C        E - (END) ALLOWS THE USER TO EXIT THE PROGRAM OPERATIONS WHEN WORK
C           IS COMPLETED.
C
C        C - (CHANGE) A [ C ] PLACED BEFORE ANY OF THE PARAMETER NAMES
C            [ F,M,N,A,B,C,D,TINC,COND,TIM,PHI,RHO,BDA,BDB,BDC,BDD,
C             MBDCND,NBDCND,ELMBDA,FLG3 ] ALLOWS THE USER TO CHANGE THE
C           VALUE OF THAT PARAMETER.  (EXAMPLE : CTINC )
C           THE USER IS PROMPTED FOR THE INPUT WITH AN INTEGER OR NON-INTEGER
C           DESIGNATING THE LOCATION OF INPUT FIELD.
C
C       SF - (SAVE  F ) CREATES A FILE  (TAPE11)  OF THE CURRENT  F-MATRIX
C           (USUALLY USED TO SAVE THE  F-MATRIX INPUT TO PWSCYL BEFORE A
C            PASS THROUGH PWSCYL IN THE EVENT THAT THOSE PARAMETER VALUES
C            WOULD NEED TO BE USED LATER)
C
C       RF - (RESET  F ) UTILIZES THE FILE OF INPUT  F  VALUES CREATED
C           WITH THE  SF  COMMAND TO RELOAD THE  F-MATRIX WITH THESE VALUES
C           (USEFUL IF ONLY A FEW VALUES ARE CHANGED IN THE INPUT  F-MATRIX
C            FOR SUCCESSIVE RUNS WITH PWSCYL)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DIMENSION BDA(155),BDB(155),BDC(155),BDD(155)
      DIMENSION F(155,155),W(2170),RATIO(155)
      DIMENSION DRVR(155),DRVZ(155)
      INTL=0
C   ZERO ALL ARRAYS EXCEPT  W
      DO 21 I=1,155
      BDA(I)=0.
      BDB(I)=0.
      BDC(I)=0.
      BDD(I)=0.
      RATIO(I)=0.1E-19
      DO 22 J=1,31
      JC1=J+31
      JC2=J+62
      JC3=J+93
      JC4=J+124
      F(I,J)=0.
      F(I,JC1)=0.
      F(I,JC2)=0.
      F(I,JC3)=0.
      F(I,JC4)=0.
   22 CONTINUE
   21 CONTINUE
      FSAVE=0.
      IFLG1=0
      IFLG2=0
      IFLG3=0
      IFLG4=0
      IFLG=0
C
C
C   SET IDIMF=155
C
      IDIMF=155
```

```
C
C     READ IN THE INITIAL PARAMETER LIST
C
        READ(9,103) A,B,M,MBOCNO,CONO,IFLG1,IFLG2
        READ(9,103) C,D,N,NBOCNO,PHI,IFLG3,IFLG4
        READ(9,109) ELMBOA,RHO,TINC,TIM
        M1=M+1
        N1=N+1
C
C     DETERMINE THE OPTIMUM F(I,J) LIST LENGTH
C
        RRM=M1
        RRM=(RRM/5.)+0.9
C     0.9 ADDED TO ROUND-OFF TO NEXT INTEGER
C
        MRR=RRM
        MRR2=2*MRR
        MRR3=3*MRR
        MRR4=4*MRR
C
C     CONTINUE READING PARAMETERS
C
        IF(M1.GE.N1) IT=M1
        IF(N1.GE.M1) IT=N1
        DO 23 I=1,IT
        READ(9,109) BOA(I),BOB(I),BOC(I),BOD(I)
   23   CONTINUE
        DO 24 I=1,MRR
        IC1=I+MRR
        IC2=I+MRR2
        IC3=I+MRR3
        IC4=I+MRR4
        DO 25 J=1,N1
        READ(9,126) F(I,J),F(IC1,J),F(IC2,J),F(IC3,J),F(IC4,J)
   25   CONTINUE
   24   CONTINUE
C
CEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFC
C                                                                        C
C     COMMAND AND EDITOR FUNCTIONS                                       C
C       (THE ORDER OF COMMAND CHECKS IS BASED ON EXPECTED MOST USED      C
C       COMMANDS FIRST)                                                  C
C                                                                        C
   26   WRITE 101                                                        C
        READ 107,RESP1                                                   C
        ICHNG=1                                                          C
        IF(RESP1.EQ.7HR        ) GO TO 53                                C
        IF(RESP1.EQ.7HI        ) GO TO 83                                C
        IF(RESP1.EQ.7HL        ) GO TO 66                                C
        IF(RESP1.EQ.7HP        ) GO TO 62                                C
        IF(RESP1.EQ.7HT        ) GO TO 94                                C
        IF(RESP1.EQ.7HQ        ) GO TO 98                                C
        IF(RESP1.EQ.7HD        ) GO TO 60                                C
        IF(RESP1.EQ.7HE        ) GO TO 80                                C
        IF(RESP1.EQ.7HCTINC    ) GO TO 44                                C
        IF(RESP1.EQ.7HCF       ) GO TO 32                                C
        IF(RESP1.EQ.7HCM       ) GO TO 30                                C
        IF(RESP1.EQ.7HCN       ) GO TO 31                                C
        IF(RESP1.EQ.7HCCONO    ) GO TO 46                                C
        IF(RESP1.EQ.7HCA       ) GO TO 38                                C
        IF(RESP1.EQ.7HCB       ) GO TO 39                                C
        IF(RESP1.EQ.7HCC       ) GO TO 40                                C
        IF(RESP1.EQ.7HCD       ) GO TO 41                                C
        IF(RESP1.EQ.7HCPHI     ) GO TO 47                                C
        IF(RESP1.EQ.7HCTIM     ) GO TO 45                                C
        IF(RESP1.EQ.7HCRHO     ) GO TO 48                                C
        IF(RESP1.EQ.7HCBOA     ) GO TO 33                                C
        IF(RESP1.EQ.7HCBOB     ) GO TO 34                                C
        IF(RESP1.EQ.7HCBOC     ) GO TO 35                                C
        IF(RESP1.EQ.7HCBOD     ) GO TO 36                                C
        IF(RESP1.EQ.7HCMBOCNO) GO TO 42                                  C
        IF(RESP1.EQ.7HCNBOCNO) GO TO 43                                  C
        IF(RESP1.EQ.7HCELMBOA) GO TO 37                                  C
        IF(RESP1.EQ.7HSF       ) GO TO 50                                C
        IF(RESP1.EQ.7HRF       ) GO TO 75                                C
        IF(RESP1.EQ.7HCFLG3    ) GO TO 49                                C
        WRITE 115,RESP1                                                  C
        GO TO 26                                                         C
C                                                                        C
CEFCEFCFECEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFCEFC
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                        C
C             COMMAND  :  CHANGE = C  :                                  C
```

```
C
CCC    PARAMETER EDITOR SECTION
CCC    (EACH GROUP OF STATEMENTS REPRESENTS THE REFERENCING  CHANGE  COMMAND)
C
   30 WRITE 108
      READ 105,XM
      M=XM
      M1=M+1
      RRM=M1
      RRM=(RRM/5.)+0.9
      MRR=RRM
      MRR2=2*MRR
      MRR3=3*MRR
      MRR4=4*MRR
      GO TO 26
   31 WRITE 108
      READ 105,XN
      N=XN
      N1=N+1
      INTL=0
      GO TO 26
   32 WRITE 110
      READ 111,ICHNG
      DO 27 I=1,ICHNG
      WRITE 112
      READ 113,IVALF,JVALF
      WRITE 108
      READ 105,F(IVALF,JVALF)
   27 CONTINUE
      ICHNG=1
C
CCC    FOR SPECIAL ITERATIVE USES THE LAST F(I,J) VALUE CHANGED IS SAVED
C
      FSAVE=F(IVALF,JVALF)
      IFLG4=1
C
      GO TO 26
   33 WRITE 110
      READ 111,ICHNG
      DO 28 I=1,ICHNG
      WRITE 114
      READ 111,IVAL
      WRITE 108
      READ 105,BOA(IVAL)
   28 CONTINUE
      ICHNG=1
      GO TO 26
   34 WRITE 110
      READ 111,ICHNG
      DO 29 I=1,ICHNG
      WRITE 114
      READ 111,IVAL
      WRITE 108
      READ 105,BOB(IVAL)
   29 CONTINUE
      ICHNG=1
      GO TO 26
   35 WRITE 110
      READ 111,ICHNG
      DO 92 I=1,ICHNG
      WRITE 114
      READ 111,IVAL
      WRITE 108
      READ 105,BOC(IVAL)
   92 CONTINUE
      ICHNG=1
      GO TO 26
   36 WRITE 110
      READ 111,ICHNG
      DO 93 I=1,ICHNG
      WRITE 114
      READ 111,IVAL
      WRITE 108
      READ 105,BOD(IVAL)
   93 CONTINUE
      ICHNG=1
      GO TO 26
   37 WRITE 108
      READ 105,ELMBOA
      GO TO 26
   38 WRITE 108
      READ 105,A
      GO TO 26
   39 WRITE 108
```

```
      READ 105,B
      GO TO 26
   40 WRITE 108
      READ 105,C
      GO TO 26
   41 WRITE 108
      READ 105,D
      GO TO 26
   42 WRITE 108
      READ 105,XMBCO
      MBDCND=XMBCO
      GO TO 26
   43 WRITE 108
      READ 105,XNBCO
      NBDCND=XNBCO
      INTL=0
      GO TO 26
   44 WRITE 108
      READ 105,TINC
      GO TO 26
   45 WRITE 108
      READ 105,TIM
      GO TO 26
   46 WRITE 108
      READ 105,COND
      GO TO 26
   47 WRITE 108
      READ 105,PHI
      GO TO 26
   48 WRITE 108
      READ 105,RHO
      GO TO 26
   49 WRITE 108
      READ 105,FLG3
      IFLG3=FLG3
      GO TO 26
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
CSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFS
C
C          COMMAND  :  SAVE  F  = SF  :
C
C
C   CREATE FILE OF CURRENT  F(I,J)  FOR LATER USE
C
C
C
   50 REWIND 11
      IFLG2=1
      DO 51 I=1,MRR
      IC1=I+MRR
      IC2=I+MRR2
      IC3=I+MRR3
      IC4=I+MRR4
      DO 52 J=1,N1
      WRITE(11,126) F(I,J),F(IC1,J),F(IC2,J),F(IC3,J),F(IC4,J)
   52 CONTINUE
   51 CONTINUE
      GO TO 26
C
CSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFSFS
C
CRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
C
C          COMMAND  :  RUN = R  :
C
C
C   USING THE CURENT PARAMETERS TO OBTAIN A SOLUTION
   53 CONTINUE
C
      CALL PWSCYL(INTL,A,B,M,MBDCND,BDA,BDB,C,D,N,NBDCND,BDC,BDD,
     1 ELMBDA,F,IDIMF,PERTRB,IERROR,W)
C
C   RESETTING  INTL  FOR MORE RUNS WITH NEW PARAMETERS
C
      INTL=1
C
C   NOW WRITING OUT ERROR INFORMATION  IERROR  AND  PERTRB
C   ALONG WITH INFORMATION ON THE # OF MATRIX (GRID) POINTS
C   AND THE TIME INCREMENT
C
      WRITE 116,IERROR,PERTRB,M1,N1,TINC,TIM
      IFLG1=0
      GO TO 26
```

```
C                                                                                  R
CRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
C
CLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLL
C             COMMAND  :  LIST = L  :                                             L
C                                                                                 L
C   OUTPUTTING SOLUTION VALUES TO TERMINAL                                        L
C                                                                                 L
C   OUTPUT MAY BE OBTAINED IN BLOCKS OF SIZE DEPENDING ON THE INPUT               L
C   GIVEN AS FOLLOWS: BEGINNING  (I)  VALUE FOLLOWED BY THE ENDING                L
C   (I)  VALUE FOLLOWED BY THE BEGINNING  (J)  VALUE AND ENDING                   L
C   (J)  VALUE.                                                                   L
C                                                                                 L
C   FURTHER BLOCKS MAY THEN BE INSPECTED                                          L
C    (THE PROCESS MAY THEN BE REPEATED)                                           L
C                                                                                 L
   66 WRITE 127                                                                   L
      READ 128,IB,IE,JB,JE                                                        L
      DO 63  I=IB,IE                                                              L
      DO 64  J=JB,JE                                                              L
      WRITE 122,F(I,J),I,J                                                        L
   64 CONTINUE                                                                    L
   63 CONTINUE                                                                    L
      WRITE 129                                                                   L
      READ 102,RESP6                                                              L
      IF(RESP6.EQ.1HY) GO TO 66                                                   L
      GO TO 26                                                                    L
C                                                                                 L
CLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLLL
C
CPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
C                                                                                 P
C             COMMAND  :  PRINT SOLUTIONS = P  :                                  P
C                                                                                 P
C   PRODUCING A LISTING OF SOLUTION VALUES AND SOME PARAMETERS USED               P
C                                                                                 P
C   OUTPUT OF F(I,J) IS IN BLOCKS WHICH ARE ARRANGED IN  5  COLUMNS               P
C   WHERE THE  (I)  DIMENSION HAS BEEN BROKEN UP INTO THE  5  DIVISIONS           P
C                                                                                 P
   62 WRITE(12,123)  A,B,M,MBDCND,COND,IFLG1,IFLG2                                P
      WRITE(12,103)  C,D,N,NBDCND,PHI,IFLG3,IFLG4                                 P
      WRITE(12,109)  ELMBDA,RHO,TINC,TIM                                          P
C                                                                                 P
CJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJ                          P
C                                                                                 P
C   STATEMENT INSERTED TO JUMP CURRENTLY UNNECESSARY   OUTPUT                     P
C                                                                                 P
      IF(IFLG3.NE.1) GO TO 73                                                     P
      IF(M1.GE.N1)  IT=M1                                                         P
      IF(N1.GE.M1)  IT=N1                                                         P
      DO 70  I=1,IT                                                               P
      WRITE(12,109)  BDA(I),BDB(I),BDC(I),BDD(I)                                  P
   70 CONTINUE                                                                    P
C                                                                                 P
CJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJ                          P
C                                                                                 P
   73 WRITE 127                                                                   P
      READ 128,IB,IE,JB,JE                                                        P
      RIE=IE-IB+1                                                                 P
      RIE=(RIE/5.)+0.9                                                            P
      IIE1=RIE                                                                    P
      IIE=IIE1+IB-1                                                               P
      IIE2=2*IIE1                                                                 P
      IIE3=3*IIE1                                                                 P
      IIE4=4*IIE1                                                                 P
      DO 71  I=IB,IIE                                                             P
      IC1=I+IIE1                                                                  P
      IC2=I+IIE2                                                                  P
      IC3=I+IIE3                                                                  P
      IC4=I+IIE4                                                                  P
      DO 72  J=JB,JE                                                              P
      WRITE(12,122)  F(I,J),I,J,F(IC1,J),IC1,J,F(IC2,J),IC2,J,                    P
     1 F(IC3,J),IC3,J,F(IC4,J),IC4,J                                              P
   72 CONTINUE                                                                    P
   71 CONTINUE                                                                    P
      WRITE 129                                                                   P
      READ 102,RESP7                                                              P
      IF(RESP7.EQ.1HY) GO TO 73                                                   P
      IFLG3=0                                                                     P
      GO TO 26                                                                    P
C                                                                                 P
CPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
C
```

```
C
C          COMMAND  :  INCREMENT (TIME) = I  :
C
C    THIS SECTION FITS DATA TO A POLYNOMIAL,CALCULATES THE DERIVATIVE
C    AND THE FLOW VELOCITY. THEN IT ASSIGNS NEW PRESSURE VALUES
C    AT  Z=0  [F(I,1)]  BASED ON THE TIME INCREMENT INPUT. F(I,J) J>1
C    IS THEN RESET TO ZERO FOR THE NEXT ITERATION.
C
   83 CALL VELOC(F,M1,D,N,COND,TINC,TIM,PHI,RHO)
C
C    RESET  F(I,J) J>1
C
      DO 81 J=2,155
      DO 82 I=1,31
      IC1=I+31
      IC2=I+62
      IC3=I+93
      IC4=I+124
      F(I,J)=0.0
      F(IC1,J)=0.
      F(IC2,J)=0.
      F(IC3,J)=0.
      F(IC4,J)=0.
   82 CONTINUE
   81 CONTINUE
      F(IVALF,JVALF)=FSAVE
      IFLG1=1
      GO TO 26
C
C
C          COMMAND :  THEORETICAL SOLUTION = T  :
C
C    THIS SECTION PRODUCES A PROFILE OF RATIO VALUES BY COMPARING THE
C    NUMERICAL SOLUTION FOR A POINT SOURCE (SINK) TO THAT OF AN ANALYTICAL
C    SOLUTION GIVEN IN
C
C    (REF. UNCONFINED AQUIFER FLOW WITH A LINEARIZED FREE SURFACE
C      CONDITION : BY DR. G. BODVARSSON : 1977 )
C
C    [[NOTE]]  THE LAST F(I,J) VALUE  CHANGED  IS TAKEN AS THE VALUE AND
C              LOCATION OF THE POINT SOURCE.
C              SEE  FSAVE  COMMAND STATEMENT #32
C
C    SELECT PROFILE
   94 IF(IFLG4.NE.1) GO TO 95
   90 WRITE 106
      READ 117,IB,JB,IINC,JINC,NNP
      XM=M
      XN=N
      XIB=IB-1
      XJB=JB-1
      RSP=(B-A)/XM
      ZSP=(D-C)/XN
      R=XIB*RSP
      Z=XJB*ZSP
      RINC=RSP*IINC
      ZINC=ZSP*JINC
      DQVAL=JVALF-1
      DQ=DQVAL*ZSP
C
      CALL RADF(COND,PHI,FSAVE,RINC,ZINC,R,Z,NNP,RATIO,DQ,TIM,
     1 B,D,XM,XN,IFLG)
      IF(IFLG.NE.0) GO TO 96
C
C    WRITE OUT THEORETICAL SOLUTION VALUES FOR PROFILES
C
      DO 91 I=1,NNP,5
      IIB=IB+((I-1)*IINC)
      IIB1=IIB+IINC
      IIB2=IIB1+IINC
      IIB3=IIB2+IINC
      IIB4=IIB3+IINC
      JJB=JB+((I-1)*JINC)
      JJB1=JJB+JINC
      JJB2=JJB1+JINC
      JJB3=JJB2+JINC
      JJB4=JJB3+JINC
      WRITE(12,115) RATIO(I),IIB,JJB,RATIO(I+1),IIB1,JJB1,RATIO(I+2)
     1 ,IIB2,JJB2,RATIO(I+3),IIB3,JJB3,RATIO(I+4),IIB4,JJB4
C
```

```
C     COMPUTE THE    RATIO = (NUMERICAL/THEORETICAL)
C
        RATIO(I)=F(IIB,JJB)/RATIO(I)
        RATIO(I+1)=F(IIB1,JJB1)/RATIO(I+1)
        RATIO(I+2)=F(IIB2,JJB2)/RATIO(I+2)
        RATIO(I+3)=F(IIB3,JJB3)/RATIO(I+3)
        RATIO(I+4)=F(IIB4,JJB4)/RATIO(I+4)
        WRITE(12,119) RATIO(I),IIB,JJB,RATIO(I+1),IIB1,JJB1,
      1 RATIO(I+2),IIB2,JJB2,RATIO(I+3),IIB3,JJB3,RATIO(I+4),
      2 IIB4,JJB4
        RATIO(I)=.1E-19
        RATIO(I+1)=.1E-19
        RATIO(I+2)=.1E-19
        RATIO(I+3)=.1E-19
        RATIO(I+4)=.1E-19
   91   CONTINUE
        WRITE 129
        READ 102,RESP2
        IF(RESP2.EQ.1HY) GO TO 90
        GO TO 26
   95   WRITE 120
        GO TO 26
   96   WRITE 121
        IFLG=0
        GO TO 26
C
CTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTT
C
C
CQQQQQQQQQQQQCQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQ
C                                                                        Q
C                  COMMAND  :   QUADRATIO = Q    :                       Q
C                                                                        Q
C     QUADRATIO IS THE RATIO OF THE QUADRIATIC TERM NEGLECTED IN THE     Q
C     LINEARIZATION ( SEE THEISIS REFERENCE ) TO THE LINEAR TERM         Q
C     OF THE LINEARIZED FREE SURFACE APPROXIMATION.                      Q
C                                                                        Q
C                                                                        Q
C       LINEAR TERM      : -(CONO*GRAV/PHI) * (DP/DZ)   ,   Z=0  .       Q
C                                                                        Q
C       QUADRIATIC TERM : -(CONO/(RHO*PHI))*[(DP/DZ)**2. +(DP/DR)**2.] ,Z=0
C                                                                        Q
C       IST = IST = STARTING  F(IST,1)  POINT                            Q
C       ISTP= ISP = STOPPING  F(ISP,1)  POINT                            Q
C       IINT= INT = INTERVAL SPACING OF CALCULATED POINTS                Q
C                                                                        Q
C                                                                        Q
C     SELECT WINDOW LENGTH                                               Q
   98 WRITE 125                                                          Q
      READ 117,IST,ISTP,IINT                                             Q
      CALL QUADR(F,M1,N1,B,D,M,N,IST,ISTP,IINT,CONO,PHI,RHO,             Q
     1 DRVR,DRVZ,RATIO)                                                  Q
C                                                                        Q
C     WRITE OUT DERIVATIVE AND QUADRATIO VALUES                          Q
C       ( QUADRATIO VALUES ARE STORED IN ARRAY "RATIO" )                 Q
C                                                                        Q
        INP=((ISTP-IST)/IINT)+1
        DO 99 I=1,INP,5
        ISTT=IST+((I-1)*IINT)
        IST1=ISTT+IINT
        IST2=IST1+IINT
        IST3=IST2+IINT
        IST4=IST3+IINT
        I1=I+1
        I2=I+2
        I3=I+3
        I4=I+4
        WRITE(12,130) DRVR(I),ISTT,DRVR(I1),IST1,DRVR(I2),IST2,
      1 DRVR(I3),IST3,DRVR(I4),IST4
        WRITE(12,131) DRVZ(I),ISTT,DRVZ(I1),IST1,DRVZ(I2),IST2,
      1 DRVZ(I3),IST3,DRVZ(I4),IST4
        WRITE(12,132) RATIO(I),ISTT,RATIO(I1),IST1,RATIO(I2),IST2,
      1 RATIO(I3),IST3,RATIO(I4),IST4
        DRVR(I)=0.0
        DRVR(I1)=0.0
        DRVR(I2)=0.0
        DRVR(I3)=0.0
        DRVR(I4)=0.0
        DRVZ(I)=0.0
        DRVZ(I1)=0.0
        DRVZ(I2)=0.0
        DRVZ(I3)=0.0
        DRVZ(I4)=0.0
        RATIO(I)=0.1E-19
```

```
      RATIO(I1)=0.1E-19                                                    Q
      RATIO(I2)=0.1E-19                                                    Q
      RATIO(I3)=0.1E-19                                                    Q
      RATIO(I4)=0.1E-19                                                    Q
   99 CONTINUE                                                             Q
      WRITE 129                                                            Q
      READ 102,RESP2                                                       Q
      IF(RESP2.EQ.1HY) GO TO 98                                            Q
      GO TO 26                                                             Q
C                                                                          Q
CQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQQ
C
CDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDD
C                                                                          D
C          COMMAND  :  DATA FILE = D  :                                    D
C                                                                          D
C                                                                          D
C   WRITE OUT PARAMETER LIST IF NECESSARY                                  D
C     (IFLG1 IS USED TO DETECT IF F(I,J) HAS BEEN RENEWED OR NOT)          D
C     (IFLG2 IS USED TO DETECT IF FILE  11  HAS BEEN CREATED)              D
C                                                                          D
   60 REWIND 11                                                            D
      IF(IFLG2.NE.1) GO TO 85                                              D
      WRITE(14,124) A,B,M,MBDCND,COND,IFLG1,IFLG2                          D
      WRITE(14,103) C,D,N,NBDCND,PHI,IFLG3,IFLG4                           D
      WRITE(14,109) ELMBDA,RHO,TINC,TIM                                    D
      IF(M1.GE.N1) IT=M1                                                   D
      IF(N1.GE.M1) IT=N1                                                   D
      DO 55 I=1,IT                                                         D
      WRITE(14,109) BDA(I),BDB(I),BDC(I),BDD(I)                            D
   55 CONTINUE                                                             D
      DO 58 I=1,MRR                                                        D
      IC1=I+MRR                                                            D
      IC2=I+MRR2                                                           D
      IC3=I+MRR3                                                           D
      IC4=I+MRR4                                                           D
      DO 59 J=1,N1                                                         D
      READ(11,126) FF1,FF2,FF3,FF4,FF5                                     D
      WRITE(14,122) FF1,I,J,FF2,IC1,J,FF3,IC2,J,FF4,IC3,J,                 D
     1 FF5,IC4,J                                                           D
   59 CONTINUE                                                             D
   58 CONTINUE                                                             D
      GO TO 26                                                             D
C                                                                          D
CDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDDD
C
CRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFR
C                                                                          R
C          COMMAND  :  RESET  F  = RF  :                                   R
C                                                                          R
C   RESET  F  VALUES FROM  SAVE F  FILE FOR MORE RUNS                      R
C                                                                          R
   75 IF(IFLG1.EQ.1) GO TO 26                                              R
      IF(IFLG2.NE.1) GO TO 85                                              R
      REWIND 11                                                            R
      DO 76 I=1,MRR                                                        R
      IC1=I+MRR                                                            R
      IC2=I+MRR2                                                           R
      IC3=I+MRR3                                                           R
      IC4=I+MRR4                                                           R
      DO 77 J=1,N1                                                         R
      READ(11,126) F(I,J),F(IC1,J),F(IC2,J),F(IC3,J),F(IC4,J)              R
   77 CONTINUE                                                             R
   76 CONTINUE                                                             R
      IFLG1=1                                                              R
      GO TO 26                                                             R
   85 WRITE 104                                                            R
      GO TO 26                                                             R
C                                                                          R
CRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFRFR
C
  101 FORMAT(" QUEST ?")
  102 FORMAT(A1)
  103 FORMAT(2X,2F15.4,2I10,E15.8,1X,2I1)
  104 FORMAT(" FILE OF  F  VALUES NOT CREATED")
  105 FORMAT(E15.8)
  106 FORMAT("PROFILE",/2X,"IIB,JJB,INC,JNC,NNP")
  107 FORMAT(A7)
  108 FORMAT("  S.XXXXXXXXXESYY")
  109 FORMAT(2X,4E15.8)
  110 FORMAT(" HOW MANY (I) VALUES DO YOU NEED TO CHANGE",/
     1"  III")
  111 FORMAT(I3)
  112 FORMAT(" WHAT F(I,J) VALUE WOULD YOU LIKE TO CHANGE",/
```

```
     1"   III,JJJ")
 113 FORMAT(I3,1X,I3)
 114 FORMAT(" WHAT (I) VALUE WOULD YOU LIKE TO CHANGE",/
     1"   III")
 115 FORMAT(" PARAMETER  ",A7,"  NOT FOUND")
 116 FORMAT(" IERROR=",I3,"    PERTRB=",E12.6,/
     1 "SOLUTIONS IN F(1,1) THROUGH F(",I3,I4,")",/
     2 "   TINC=",F15.4,"      TIME=",F15.4)
 117 FORMAT(I3,1X,I3,1X,I3,1X,I3,1X,I3)
 118 FORMAT(2X,E15.8,"T(",I3,I4,") ",E15.8,"T(",I3,I4,") ",
     1 E15.8,"T(",I3,I4,") ",E15.8,"T(",I3,I4,") ",E15.8,"T(",
     2 I3,I4,") ")
 119 FORMAT(2X,E15.8,"R(",I3,I4,") ",E15.8,"R(",I3,I4,") ",
     1 E15.8,"R(",I3,I4,") ",E15.8,"R(",I3,I4,") ",
     2 E15.8,"R(",I3,I4,")")
 120 FORMAT(" F(IVALF,JVALF)  NOT ASSIGNED")
 121 FORMAT(" [ ERROR ]  PHI .EQ. ZERO ")
 122 FORMAT(2X,E15.8,"F(",I3,I4,") ",E15.8,"F(",I3,I4,") ",
     1 E15.8,"F(",I3,I4,") ",E15.8,"F(",I3,I4,") ",E15.8,
     2 "F(",I3,I4,") ")
 123 FORMAT("1",1X,2F15.4,2I10,E15.8,1X,2I1,15X,"POSLST")
 124 FORMAT("1",1X,2F15.4,2I10,E15.8,1X,2I1,15X,"PARLST")
 125 FORMAT("QUADRATIO",/2X,"IST,ISP,INT")
 126 FORMAT(2X,E15.8,11X,E15.8,11X,E15.8,11X,E15.8,11X,E15.8)
 127 FORMAT("WHAT F(I,J) VALUES",/
     1 "   III TO III,JJJ TO JJJ")
 128 FORMAT(I3,4X,I3,1X,I3,4X,I3)
 129 FORMAT("MORE ? Y OR N")
 130 FORMAT(2X,E15.8,"DR(",I3,",1) ",E15.8,"DR(",I3,",1) ",
     1 E15.8,"DR(",I3,",1) ",E15.8,"DR(",I3,",1) ",E15.8,"DR(",I3,",1)")
 131 FORMAT(2X,E15.8,"DZ(",I3,",1) ",E15.8,"DZ(",I3,",1) ",
     1 E15.8,"DZ(",I3,",1) ",E15.8,"DZ(",I3,",1) ",E15.8,"DZ(",I3,",1)")
 132 FORMAT(2X,E15.8,"QR(",I3,",1) ",E15.8,"QR(",I3,",1) ",
     1 E15.8,"QR(",I3,",1) ",E15.8,"QR(",I3,",1) ",E15.8,"QR(",I3,",1)")
C
CEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEE
C                                                                      E
C          COMMAND  :  END = E  :                                      E
C                                                                      E
   80 STOP                                                             E
C                                                                      E
CEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEEE
C
      END
```

```
      SUBROUTINE VELOC(F,M1,D,N,COND,TINC,TIM,PHI,RHO)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                         C
C   VELOC UTILIZES  SUBROUTINE DIFTBL  AND  FUNCTION DERVZ0               C
C   TO DETERMINE THE VELOCITY AND DISPLACEMENT OF THE FREE SURFACE AT     C
C    Z=0  FOR A RESERVOIR.                                                C
C                                                                         C
C   VELOC OPERATES ON THE MATRIX  F(I,J)  IN PROGRAM  POSGO  TO           C
C   RESET VALUES AND ITERATE ON THE SOLUTION.                             C
C                                                                         C
C                                                                         C
C   THE  Z=0  SURFACE PRESSURE  F(I,1)  IS  CALCULATED WITH               C
C                                                                         C
C                 P = (RHO)(GRAV) DELTA(H)                                C
C                                                                         C
C   WHERE DELTA(H) IS DETERMINED BY THE TIME STEP CHOOSEN AND THE         C
C   CALCULATED VELOCITY.                                                  C
C                                                                         C
C        CONDUCTIVITY = COND                                              C
C        DENSITY      = RHO                                               C
C        GRAVITY      = GRAV = 9.80665E-03   ( KM. /(SEC*SEC) )           C
C        POROSITY     = PHI                                               C
C        TIME         = TIM                                               C
C        TIME  INCRMT = TINC                                              C
C                                                                         C
C                                                                         C
C   SURFACE PRESSURE VALUES CALCULATED ARE OUTPUT WITH SOLUTIONS OF       C
C   NEXT ITERATION AS THE  F(I,1) I=1,M1  VALUES.                         C
C                                                                         C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DIMENSION X(6),VELFL(155),TABLE(6,6),F(155,155)
      GRAV=9.80665E-03
      TIM=TINC+TIM
C
C    SET UP   X   SPACING OF F(I,J) J=1,5  POINTS
C
      XN=N
      SPAC=D/XN
      X(1)=0.0
      DO 21  I=2,5
      X(I)=X(I-1)+SPAC
   21 CONTINUE
C
C    START POLYNOMIAL FITTING
C
      DO 31  IC=1,M1
C
C    DETERMINE DIFFERENCE TABLE FOR POLYNOMIAL IN  Z  DIRECTION
C     - OK -  IS A PARAMETER TO CHECK FOR  MP.GE.NP : NOT USED HERE
C
      CALL DIFTBL(X,F,IC,TABLE,4,5,6,OK)
C
CJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJ
C
C    WRITE OUT TABLE FOR FIRST TIME THROUGH TO CHECK
C
      IF(IC.NE.1) GO TO 70
      DO 41  JT=1,5
      WRITE(12,101)  (TABLE(IT,JT),IT=1,5)
   41 CONTINUE
C
CJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJJ
C
C    CALCULATE THE DERIVATIVE
C
C    RHO  IS OMITTED FROM THE EQUATION OF VELFL(IC)
C     SINCE IT CANCELS IN THE EQUATION FOR PRESSURE  F(IC,1)
C
   70 DELTP=DERVZ0(X,TABLE,6)
      VELFL(IC)=(COND*DELTP)/PHI
      DELTH=TINC*VELFL(IC)
C
C    FOR A POINT SINK AT TIME>0 THE FREE SURFACE IS BELOW THE
C     Z=0  LEVEL AND  P=0  AT SOME POINT ABOVE  Z=0. THUS THE
C    PRESSURE AT  Z=0  MUST BE NEGATIVE.
C
C    RHO IS OMITTED FROM THE NEXT EXPRESSION
C
      F(IC,1)=(GRAV*DELTH)+F(IC,1)
   31 CONTINUE
C
  101 FORMAT(2X,6F15.4)
```

```
103 FORMAT(2X,F15.4,'' F('',I3,'',1)'',F15.4,'' F('',I3,'',1) '',
    1 F15.4,'' F('',I3,'',1)'',F15.4,'' F('',I3,'',1)'',F15.4,
    2 '' F('',I3,'',1)'')
     RETURN
     END
```

```
      SUBROUTINE DIFTBL(X,F,IC,TABLE,MP,NP,K,OK)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                        C
C   DIFTBL CALCULATES THE DIFFERENCE TABLE FOR NEWTONS                   C
C   DIVIDED DIFFERENCE METHOD. (ADAPTED FROM "APPLIED NUMERICAL METHODS" C
C    BOOK BY CARNAHAN ET.AL. 1969)                                       C
C      MP = POLYNOMIAL ORDER                                             C
C      NP = NUMBER OF POINTS USED                                        C
C                                                                        C
C   OK  DETECTS ERRORS IN POLYNOMIAL ORDER                               C
C                                                                        C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DIMENSION X(K),F(155,155),TABLE(K,K)
      OK=1
      IF(MP.GE.NP) GO TO 10
C
C   CALCULATE FIRST ORDER DIFFERENCES
C
      N1=NP-1
      DO 21  J=1,N1
      TABLE(1,J)=(F(IC,J+1)-F(IC,J))/(X(J+1)-X(J))
   21 CONTINUE
      IF(MP.LE.1) GO TO 11
C
C   NOW CALCULATE HIGHER ORDER DIFFERENCES
C
      DO 31  I=2,MP
      DO 41  J=I,N1
      IS=J+1-I
      TABLE(I,J)=(TABLE(I-1,J)-TABLE(I-1,J-1))/(X(J+1)-X(IS))
   41 CONTINUE
   31 CONTINUE
   11 OK=0
   10 RETURN
      END
```

```
      FUNCTION DERVZO(X,TABLE,K)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
C    CALCULATIONS ARE COMPLETED USING A SHORT FORM OF THE DERIVATIVE         C
C    FOR THE FOURTH ORDER POLYNOMIAL. THE DERIVATIVE IS CONSEQUENTLY         C
C    ONLY FOR THE SPECIAL CASE                                               C
C                                                                           C
C                        D( P(Z))/DZ    Z=0                                  C
C                                                                           C
C    WITH THE FIRST POLYNOMIAL FIT POINT  X(1)  AT  Z=0 .                    C
C                                                                           C
C    ALSO THE DIVIDED DIFFERENCE TABLE MUST BE SET UP FOR  TABLE(1,1)        C
C    AS THE FIRST DIFFERENCE USED AND ALL NEEDED DIFFERENCES ON THE          C
C    MATRIX DIAGONAL.                                                        C
C                                          2                                C
C     P(Z) = A(1) + A(2)Z + A(3)Z  + ...                                     C
C                                                                           C
C     DP/DZ = A(2)    Z=0                                                    C
C                                                                           C
C     WITH  X(1)=0                                                           C
C     A(2) REDUCES TO                                                        C
C                                                                           C
C     A(2)=(X(2)X(3))P[X(4)]-(X(2)X(3)X(4))P[X(5)]+P[X(2)]-(X(2))P[X(3)]     C
C                                                                           C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DIMENSION X(K),TABLE(K,K)
      X23=X(2)*X(3)
      X234=X23*X(4)
      DERVZO=X23*TABLE(3,3)-X234*TABLE(4,4)+TABLE(1,1)-
     1 X(2)*TABLE(2,2)
      RETURN
      END
```

```
      SUBROUTINE RAOF(COND,PHI,FSAVE,RINC,ZINC,R,Z,NNP,RATIO,DQ,TIM,
     1 B,D,XM,XN,IFLG)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
C   RAOF COMPUTES THE THEORETICAL HALF SPACE POINT SOURCE SOLUTION    C
C   FOR THE PRESSURE FIELD ASSOCIATED WITH A LINEARIZED FREE SURFACE. C
C                                                                     C
C      RATIO = FNOT * [(1/RPQ) + (1/RPQP) - (2/RPQTP))                C
C                                                                     C
C      RPQ = ( R*R + (Z-DQ)**2 )**0.5                                 C
C                                                                     C
C      RPQP = ( R*R + (Z-DQ)**2 )**0.5                                C
C                                                                     C
C      RPQTP = ( R*R + (Z+(A*TIM)+DQ)**0.5                            C
C                                                                     C
C   THIS IS THE SOLUTION TO POISSON*S EQUATION WITH A SOURCE TERM     C
C                                                                     C
C              - (FZ / COND)*DELT(P-Q)I(T)                            C
C                                                                     C
C      I(T)        = CAUSAL UNIT STEP FUNCTION                        C
C                                                                     C
C      DELT(P-Q) = DIRAC DELTA FUNCTION AT POINT  P  WITH             C
C                  SOURCE POINT  Q .                                  C
C                                                                     C
C      FZ          = CONCENTRATED SINK                                C
C                                                                     C
C      (REF.: DR. G. BODVARSSON , UNCONFINED AQUIFER FLOW WITH A LINEARIZEDC
C             FREE SURFACE CONDITION : 1977)                          C
C                                                                     C
C   IT TURNS OUT THAT THIS CAN BE RELATED TO  F  IN                   C
C                                                                     C
C                    LAPL(P) = F    (=FSAVE)                          C
C                                                                     C
C   WHERE  LAPL( )  STANDS FOR THE LAPLACIAN OPERATOR.                C
C                                                                     C
C   SO,                                                               C
C                                                                     C
C                FNOT = -FSAVE*(((RR2/2.)**2)*ZZ]/4.                  C
C                                                                     C
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - C
C                                                                     C
C         G = 9.80665E-03    ( KM. / SEC*SEC )                        C
C                                                                     C
C         A = CHARATERISTIC FLUID VELOCITY                            C
C                                                                     C
C      IFLG = CHECK TO PREVENT DIVISION BY ZERO                       C
C                                                                     C
C        DQ = DEPTH TO POINT SOURCE                                   C
C                                                                     C
C                                                                     C
C        ( NOTE )  RATIO MUST BE DIMENSIONED THE SAME AS IN CALLING PROGRAM C
C                                                                     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DIMENSION RATIO(155)
      IF(PHI.EQ.0.0) GO TO 40
      RR=B/(XM*2.)
      ZZ=D/XN
      RRZZ=(RR*RR*ZZ)/4.
      G=9.80665E-03
      A=(COND*G)/PHI
      FNOT=-(FSAVE*RRZZ)
      IFLG=0
      DO 21 I=1,NNP
      XI=I
      RSQ=R*R
      ZAD=Z+(A*TIM)+DQ
      ZMD=Z-DQ
      ZPLD=Z+DQ
      RPQ=(RSQ+(ZMD*ZMD))**0.5
      RPQP=(RSQ+(ZPLD*ZPLD))**0.5
      RPQTP=(RSQ+(ZAD*ZAD))**0.5
      IF(RPQ.EQ.0.0 .OR. RPQP.EQ.0.0 .OR. RPQTP.EQ.0.0) GO TO 30
      RATIO(I)=FNOT*((1./RPQ)+(1./RPQP)-(2./RPQTP))
      IF(RATIO(I).EQ.0.0) RATIO(I)=.1E-19
   31 CONTINUE
      R=R+RINC
      Z=Z+ZINC
   21 CONTINUE
      RETURN
   30 WRITE 105
      RATIO(I)= .1E21
      GO TO 31
```

```
 40 IFLG=1
    RETURN
105 FORMAT(" A SINGULARITY DETECTED, DENOMINATOR OF RATIO SET",/
   1 " EQUAL TO     .1E21   .")
    END
```

```
      SUBROUTINE QUADR(F,M1,N1,B,D,M,N,IST,ISTP,IINT,COND,PHI,RHO,
     1 DRVR,DRVZ,RATIO)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                       C
C    QUADR  UTILIZES SUBROUTINE  RZDIFT  AND FUNCTIONS  DERVR,DERVZO    C
C    TO DETERMINE THE RATIO OF THE NEGLECTED QUADRATIC TERM TO THE      C
C    LINEAR TERM IN THE LINEARIZED FREE SURFACE APPROXIMATION.          C
C                                                                       C
C                                  QUADRATIC   TERM                     C
C              QUADRATIO = -------------------------                    C
C                                  LINEAR TERM                          C
C                                                                       C
C      QUADRIATIC TERM = -(COND/(RHO*PHI))*[(DP/DZ)**2. + (DP/DR)**2.] , Z=0C
C                                                                       C
C      LINEAR TERM     = -(COND*GRAV/PHI) * (DP/DZ) ,     Z=0           C
C                                                                       C
C         CONDUCTIVITY = COND                                           C
C         DENSITY      = RHO                                            C
C         GRAVITY      = GRAV = 9.80665E-3                              C
C         POROSITY     = PHI                                            C
C                                                                       C
C                  IST  = STARTING  F(IST,1)   POINT                    C
C                  ISTP = STOPPING  F(ISTP,1)  POINT                    C
C                  IINT = INTERVAL SPACING OF CALCULATED POINTS         C
C                                                                       C
C       X  CORRESPONDS TO  Z COORDINATE                                 C
C       Y  CORRESPONDS TO  R COORDINATE                                 C
C                                                                       C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      DIMENSION X(6),Y(6),DRVR(155),DRVZ(155),TABLEX(6,6),TABLEY(6,6),
     1 F(155,155),RATIO(155),FRAY(155)
      GRAV=9.80665E-03
      CRP=COND/(RHO*PHI)
      CGP=(COND*GRAV)/PHI
C
C    SET UP ARRAY FOR RADIAL DERIVATIVE
C
      FRAY(1)=F(3,1)
      FRAY(2)=F(2,1)
      DO 31 I=3,155
      II=I-2
      FRAY(I)=F(II,1)
   31 CONTINUE
C
C    SET UP  X , Y  SPACING FOR  F(I,J)  POINTS
C
      XN=N
      YM=M
      SPACX=D/XN
      SPACY=B/YM
      X(1)=0.0
      Y(1)=-2.*SPACY
      DO 21 I=2,5
      X(I)=X(I-1)+SPACX
      Y(I)=Y(I-1)+SPACY
   21 CONTINUE
C
C    START POLYNOMIAL FITTING
C
      IP=0
      DO 41 IC=IST,ISTP,IINT
C
C    DETERMINE DIFFERENCE TABLES
C
      CALL RZDIFT(X,Y,F,FRAY,IC,TABLEX,TABLEY,4,5,6,OK)
C
C    CALCULATE DERIVATIVE VALUES AND QUADRATIO TERMS
C
      IP=IP+1
      DRVZ(IP)=DERVZO(X,TABLEX,5)
      DRVR(IP)=DERVR(Y,TABLEY,5)
      RQUAD=-CRP*(DRVZ(IP)*DRVZ(IP) + DRVR(IP)*DRVR(IP))
      RLIN=-CGP*DRVZ(IP)
      IF(RLIN.EQ.0.0) GO TO 45
      RATIO(IP)=RQUAD/RLIN
   46 CONTINUE
   41 CONTINUE
  105 FORMAT(" SINGULARITY DETECTED  QUADRATIO SET EQUAL TO  1.0E20")
      RETURN
   45 WRITE 105
      RATIO(IP)=0.1E21
      GO TO 46
      END
```

```
      SUBROUTINE RZDIFT(X,Y,F,FRAY,IC,TABLEX,TABLEY,MP,NP,K,OK)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                         C
C    RZDIFT CALCULATES THE DIFFERENCE TABLES FOR NEWTONS DIVIDED          C
C    DIFFERENCE METHOD.                                                   C
C     (ADAPTED FROM "APPLIED NUMERICAL METHODS" BOOK                      C
C       BY CARNAHAN ET.AL. 1969)                                          C
C                                                                         C
C        MP = POLYNOMIAL ORDER                                            C
C        NP = NUMBER OF POINTS USED                                       C
C         K = DIMENSION OF POINT ARRAYS USED                              C
C        IC =  R  POSITION , F(IC,1) , OF SPECIFIED POINT                 C
C                                                                         C
C    OK   DETECTS ERRORS IN POLYNOMIAL ORDER                              C
C                                                                         C
C        X  CORRESPONDS TO THE  Z COORDINATE                              C
C        Y  CORRESPONDS TO THE  R COORDINATE                              C
C                                                                         C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DIMENSION X(K),Y(K),F(155,155),TABLEX(K,K),TABLEY(K,K),FRAY(155)
      OK=1
      IF(MP.GE.NP) GO TO 10
C
C    CALCULATE FIRST ORDER DIFFERENCES
C
      N1=NP-1
      DO 21 J=1,N1
      TABLEX(1,J)=(F(IC,J+1)-F(IC,J))/(X(J+1)-X(J))
      TABLEY(1,J)=(FRAY(IC+J)-FRAY(IC+J-1))/(Y(J+1)-Y(J))
   21 CONTINUE
      IF(MP.LE.1) GO TO 11
C
C    NOW CALCULATE HIGHER ORDER DIFFERENCES
C
      DO 31 I=2,MP
      DO 41 J=I,N1
      IS=J+1-I
      TABLEX(I,J)=(TABLEX(I-1,J)-TABLEX(I-1,J-1))/(X(J+1)-X(IS))
      TABLEY(I,J)=(TABLEY(I-1,J)-TABLEY(I-1,J-1))/(Y(J+1)-Y(IS))
   41 CONTINUE
   31 CONTINUE
   11 OK=0
   10 RETURN
      END
```

```
      FUNCTION DERVR(Y,TABLEY,K)
C
C     CALCULATIONS ARE COMPLETED USING A SHORT FORM OF THE DERIVATIVE FOR A
C     FOURTH ORDER POLYNOMIAL AT  R=0 . THE DERIVATIVE IS CONSEQUENTLY ONLY
C     FOR THE SPECIAL CASE :
C
C                          D[P(R)]/DR        AT     R = 0   ,
C
C     WHERE THE CENTRAL POINT OF THE FIVE POINTS FITTED IS ASSIGNED   R=0   .
C
C     THE DIVIDED DIFFERENCE TABLE MUST BE SET UP FOR   TABLEY(1,1)  AS THE
C     FIRST DIFFERENCE USED AND ALL NEEDED DIFFERENCES ON THE MATRIX DIAGONAL
C
C       P(R) = A(1) + A(2)R + A(3)R**2 +  ...
C
C       DP/DR = A(2)   ,  R = 0   .
C
C     WITH  Y(3)=0 ,
C     A(2)  REDUCES TO
C
C  A(2)=P[Y(2)]-(Y(1)+Y(2))P[Y(3)]+(Y(1)*Y(2))P[Y(4)]-(Y(1)*Y(2)*Y(4))P[Y(5)]
C
      DIMENSION Y(K),TABLEY(K,K)
      Y12=Y(1)+Y(2)
      Y2=Y(1)*Y(2)
      Y23=Y2*Y(4)
      DERVR=TABLEY(1,1)-Y12*TABLEY(2,2)+Y2*TABLEY(3,3)-Y23*TABLEY(4,4)
      RETURN
      END
```

```
      SUBROUTINE PWSCYL (INTL,A,B,M,MBDCND,BDA,BDB,C,D,N,NBDCND,BDC,     PYL00005
     1                   BDD,ELMBDA,F,IDIMF,PERTRB,IERROR,W)             PYL00010
C                                                                        PYL00015
C*****************************************************************     PYL00020
C                                                                *     PYL00025
C                                                                *     PYL00030
C          VERSION  2  OCTOBER 1976  INCLUDING ERRATA  OCTOBER 1976*     PYL00035
C                                                                *     PYL00040
C         DOCUMENTATION FOR THIS PROGRAM IS GIVEN IN             *     PYL00045
C                                                                *     PYL00050
C        EFFICIENT FORTRAN SUBPROGRAMS FOR THE SOLUTION OF       *     PYL00055
C            ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS             *     PYL00060
C                                                                *     PYL00065
C                              BY                                *     PYL00070
C                                                                *     PYL00075
C           PAUL SWARZTRAUBER   AND  ROLAND SWEET                *     PYL00080
C                                                                *     PYL00085
C              TECHNICAL NOTE TN/IA-109   JULY 1975              *     PYL00090
C                                                                *     PYL00095
C        NATIONAL CENTER FOR ATMOSPHERIC RESEARCH  BOULDER,COLORADO 80307PYL00100
C                                                                *     PYL00105
C         WHICH IS SPONSORED BY THE NATIONAL SCIENCE FOUNDATION  *     PYL00110
C                                                                *     PYL00115
C*****************************************************************     PYL00120
C                                                                        
CC+--------------------------------------------------------------------+CC
CC+                                                                    +CC
CC+                      LAWRENCE LIVERMORE LABORATORY                 +CC
CC+       NUMERICAL MATHEMATICS GROUP -- MATHEMATICAL SOFTWARE LIBRARY +CC
CC+                                                                    +CC
CC+--------------------------------------------------------------------+CC
CC+                                                                    +CC
CC+                  CLASS TWO ROUTINE)  PWSCYL                        +CC
CC+                           REVISION)  0                             +CC
CC+                  DATE LAST CHANGED)  77-03-15                      +CC
CC+                                                                    +CC
CC+   CLASS TWO  ROUTINES ARE MADE AVAILABLE BY NMG AS A SERVICE TO THE LLL  +CC
CC+   COMPUTING COMMUNITY.  SUCH ROUTINES FAIL TO MEET ONE OR MORE OF THE    +CC
CC+   CRITERIA FOR INCLUSION IN THE  CLASS ONE (CERTIFIED)  LIBRARY.  IT IS  +CC
CC+   RECOMMENDED THAT A  CLASS TWO  ROUTINE BE USED ONLY WHEN THERE IS NO   +CC
CC+   APPROPRIATE  CLASS ONE  ROUTINE.                                      +CC
CC+                                                                    +CC
CC+   NMG WILL ATTEMPT TO PROVIDE CONSULTATION SERVICES FOR  CLASS TWO       +CC
CC+   ROUTINES, BUT SUPPORT IS NOT GUARANTEED.                             +CC
CC+                                                                    +CC
CC+   +----------------------------------------------------------+     +CC
CC+   +                          N O T I C E                     +     +CC
CC+   +                                                          +     +CC
CC+   +  THIS REPORT WAS PREPARED AS AN ACCOUNT OF WORK SPONSORED BY THE +     +CC
CC+   +  UNITED STATES GOVERNMENT.  NEITHER THE UNITED STATES NOR THE   +     +CC
CC+   +  UNITED STATES ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION,  +     +CC
CC+   +  NOR ANY OF THEIR EMPLOYEES, NOR ANY OF THEIR CONTRACTORS, SUB-  +     +CC
CC+   +  CONTRACTORS, OR THEIR EMPLOYEES, MAKES ANY WARRENTY, EXPRESS OR +     +CC
CC+   +  IMPLIED, OR ASSUMES ANY LEGAL LIABILITY OR RESPONSIBILITY FOR   +     +CC
CC+   +  THE ACCURACY, COMPLETENESS OR USEFULNESS OF ANY INFORMATION,    +     +CC
CC+   +  APPARATUS, PRODUCT OR PROCESS DISCLOSED, OR REPRESENTS THAT ITS +     +CC
CC+   +  USE WOULD NOT INFRINGE PRIVATELY-OWNED RIGHTS.                 +     +CC
CC+   +                                                          +     +CC
CC+   +----------------------------------------------------------+     +CC
CC+                                                                    +CC
CC+   PLEASE REPORT ANY SUSPECTED ERRORS IN THIS ROUTINE IMMEDIATELY TO NMG.  +CC
CC+                                                                    +CC
CC+--------------------------------------------------------------------+CC
C                                                                        
C                                                                        PYL00125
C                                                                        PYL00130
C                                                                        PYL00135
C     SUBROUTINE PWSCYL SOLVES A FINITE DIFFERENCE APPROXIMATION TO THE  PYL00140
C       MODIFIED HELMHOLTZ EQUATION IN CYLINDRICAL COORDINATES           PYL00145
C                                                                        PYL00150
C          (1/R)(D/DR)(R(D/DR)U) + (D/DZ)(D/DZ)U                         PYL00155
C                                                                        PYL00160
C                                + (LAMBDA/R**2)U = F(R,Z)               PYL00165
C                                                                        PYL00170
C     THIS TWO DIMENSIONAL MODIFIED HELMHOLTZ EQUATION RESULTS FROM      PYL00175
C     THE FOURIER TRANSFORM OF THE THREE DIMENSIONAL POISSON EQUATION    PYL00180
C                                                                        PYL00185
C                                                                        PYL00190
C     THE ARGUMENTS ARE DEFINED AS0                                      PYL00195
C                                                                        PYL00200
C     * * * * * * * * *     ON INPUT     * * * * * * * * *               PYL00205
C                                                                        PYL00210
C                                                                        PYL00215
C     INTL                                                               PYL00220
```

```
C       = 0  ON INITIAL ENTRY TO PWSCYL OR IF N AND NBDCND ARE CHANGED  PYL00225
C            FROM PREVIOUS CALL.                                        PYL00230
C       = 1  IF N AND NBDCND ARE UNCHANGED FROM PREVIOUS CALL TO PWSCYL.PYL00235
C                                                                       PYL00240
C       NOTE0  A CALL WITH INTL = 1 IS ABOUT 1 PERCENT FASTER THAN A    PYL00245
C              CALL WITH INTL = 0  .                                    PYL00250
C                                                                       PYL00255
C     A,B                                                               PYL00260
C       THE RANGE OF R, I.E., A .LE. R .LE. B.  A MUST BE LESS THAN B   PYL00265
C       AND A MUST BE NON-NEGATIVE.                                     PYL00270
C                                                                       PYL00275
C     M                                                                 PYL00280
C       THE NUMBER OF PANELS INTO WHICH THE INTERVAL (A,B) IS           PYL00285
C       SUBDIVIDED.  HENCE, THERE WILL BE M+1 GRID POINTS IN THE        PYL00290
C       R-DIRECTION GIVEN BY R(I) = A+(I-1)DR, FOR I = 1,2,...,M+1,     PYL00295
C       WHERE DR = (B-A)/M IS THE PANEL WIDTH.                          PYL00300
C                                                                       PYL00305
C     MBDCND                                                            PYL00310
C       INDICATES THE TYPE OF BOUNDARY CONDITIONS AT R = A AND R = B.   PYL00315
C                                                                       PYL00320
C       = 1  IF THE SOLUTION IS SPECIFIED AT R = A AND R = B.           PYL00325
C       = 2  IF THE SOLUTION IS SPECIFIED AT R = A AND THE DERIVATIVE OFPYL00330
C            THE SOLUTION WITH RESPECT TO R IS SPECIFIED AT R = B.      PYL00335
C       = 3  IF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO R IS     PYL00340
C            SPECIFIED AT R = A (SEE NOTE BELOW) AND R = B.             PYL00345
C       = 4  IF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO R IS     PYL00350
C            SPECIFIED AT R = A (SEE NOTE BELOW) AND THE SOLUTION IS    PYL00355
C            SPECIFIED AT R = B.                                        PYL00360
C       = 5  IF THE SOLUTION IS UNSPECIFIED AT R = A = 0 AND THE        PYL00365
C            SOLUTION IS SPECIFIED AT R = B.                            PYL00370
C       = 6  IF THE SOLUTION IS UNSPECIFIED AT R = A = 0 AND THE        PYL00375
C            DERIVATIVE OF THE SOLUTION WITH RESPECT TO R IS SPECIFIED  PYL00380
C            AT R = B.                                                  PYL00385
C                                                                       PYL00390
C       NOTE0  IF A = 0, DO NOT USE MBDCND = 3 OR 4, BUT INSTEAD USE    PYL00395
C              MBDCND = 1,2,5, OR 6  .                                  PYL00400
C                                                                       PYL00405
C     BDA                                                               PYL00410
C       A ONE-DIMENSIONAL ARRAY OF LENGTH N+1 THAT SPECIFIES THE VALUES PYL00415
C       OF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO R AT R = A.   PYL00420
C       WHEN MBDCND = 3 OR 4,                                           PYL00425
C                                                                       PYL00430
C            BDA(J) = (D/DR)U(A,Z(J)), J = 1,2,...,N+1  .               PYL00435
C                                                                       PYL00440
C       WHEN MBDCND HAS ANY OTHER VALUE, BDA IS A DUMMY VARIABLE.       PYL00445
C                                                                       PYL00450
C     BDB                                                               PYL00455
C       A ONE-DIMENSIONAL ARRAY OF LENGTH N+1 THAT SPECIFIES THE VALUES PYL00460
C       OF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO R AT R = B.   PYL00465
C       WHEN MBDCND = 2,3, OR 6,                                        PYL00470
C                                                                       PYL00475
C            BDB(J) = (D/DR)U(B,Z(J)), J = 1,2,...,N+1  .               PYL00480
C                                                                       PYL00485
C       WHEN MBDCND HAS ANY OTHER VALUE, BDB IS A DUMMY VARIABLE.       PYL00490
C                                                                       PYL00495
C     C,D                                                               PYL00500
C       THE RANGE OF Z, I.E., C .LE. Z .LE. D.  C MUST BE LESS THAN D.  PYL00505
C                                                                       PYL00510
C     N                                                                 PYL00515
C       THE NUMBER OF PANELS INTO WHICH THE INTERVAL (C,D) IS           PYL00520
C       SUBDIVIDED.  HENCE, THERE WILL BE N+1 GRID POINTS IN THE        PYL00525
C       Z-DIRECTION GIVEN BY Z(J) = C+(J-1)DZ, FOR J = 1,2,...,N+1,     PYL00530
C       WHERE DZ = (D-C)/N IS THE PANEL WIDTH.  N MUST BE OF THE FORM   PYL00535
C       (2**P)(3**Q)(5**R) WHERE P, Q, AND R ARE ANY NON-NEGATIVE       PYL00540
C       INTEGERS.  N MUST BE GREATER THAN 2  .                          PYL00545
C                                                                       PYL00550
C     NBDCND                                                            PYL00555
C       INDICATES THE TYPE OF BOUNDARY CONDITIONS AT Z = C AND Z = D.   PYL00560
C                                                                       PYL00565
C       = 0  IF THE SOLUTION IS PERIODIC IN Z, I.E., U(I,1) = U(I,N+1). PYL00570
C       = 1  IF THE SOLUTION IS SPECIFIED AT Z = C AND Z = D.           PYL00575
C       = 2  IF THE SOLUTION IS SPECIFIED AT Z = C AND THE DERIVATIVE OFPYL00580
C            THE SOLUTION WITH RESPECT TO Z IS SPECIFIED AT Z = D.      PYL00585
C       = 3  IF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO Z IS     PYL00590
C            SPECIFIED AT Z = C AND Z = D.                              PYL00595
C       = 4  IF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO Z IS     PYL00600
C            SPECIFIED AT Z = C AND THE SOLUTION IS SPECIFIED AT Z = D. PYL00605
C                                                                       PYL00610
C     BDC                                                               PYL00615
C       A ONE-DIMENSIONAL ARRAY OF LENGTH M+1 THAT SPECIFIES THE VALUES PYL00620
C       OF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO Z AT Z = C.   PYL00625
C       WHEN NBDCND = 3 OR 4,                                           PYL00630
C                                                                       PYL00635
C            BDC(I) = (D/DZ)U(R(I),C), I = 1,2,...,M+1  .               PYL00640
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```
C
C       WHEN NBDCNO HAS ANY OTHER VALUE, BDC IS A DUMMY VARIABLE.
C
C  BDD
C    A ONE-DIMENSIONAL ARRAY OF LENGTH M+1 THAT SPECIFIES THE VALUES
C    OF THE DERIVATIVE OF THE SOLUTION WITH RESPECT TO Z AT Z = D.
C    WHEN NBDCND = 2 OR 3,
C
C         BDD(I) = (D/DZ)U(R(I),D) , I = 1,2,...,M+1  .
C
C    WHEN NBDCND HAS ANY OTHER VALUE, BDD IS A DUMMY VARIABLE.
C
C  ELMBDA
C    THE CONSTANT LAMBDA IN THE HELMHOLTZ EQUATION.  IF
C    LAMBDA .GT. D, A SOLUTION MAY NOT EXIST.  HOWEVER, PWSCYL WILL
C    ATTEMPT TO FIND A SOLUTION.  LAMBDA MUST BE ZERO WHEN
C    MBDCND = 5 OR 6  .
C
C  F
C    A TWO-DIMENSIONAL ARRAY THAT SPECIFIES THE VALUES OF THE RIGHT
C    SIDE OF THE HELMHOLTZ EQUATION AND BOUNDARY DATA (IF ANY).  FOR
C    I = 2,3,...,M AND J = 2,3,...,N
C
C         F(I,J) = F(R(I),Z(J)).
C
C    ON THE BOUNDARIES F IS DEFINED BY
C
C         MBDCND   F(1,J)            F(M+1,J)
C         ------   ---------         ---------
C
C           1      U(A,Z(J))         U(B,Z(J))
C           2      U(A,Z(J))         F(B,Z(J))
C           3      F(A,Z(J))         F(B,Z(J))    J = 1,2,...,N+1
C           4      F(A,Z(J))         U(B,Z(J))
C           5      F(D,Z(J))         U(B,Z(J))
C           6      F(D,Z(J))         F(B,Z(J))
C
C         NBDCND   F(I,1)            F(I,N+1)
C         ------   ---------         ---------
C
C           D      F(R(I),C)         F(R(I),C)
C           1      U(R(I),C)         U(R(I),D)
C           2      U(R(I),C)         F(R(I),D)    I = 1,2,...,M+1
C           3      F(R(I),C)         F(R(I),D)
C           4      F(R(I),C)         U(R(I),D)
C
C    F MUST BE DIMENSIONED AT LEAST (M+1)*(N+1).
C
C    NOTE
C
C    IF THE TABLE CALLS FOR BOTH THE SOLUTION U AND THE RIGHT SIDE F
C    AT  A CORNER THEN THE SOLUTION MUST BE SPECIFIED.
C
C  IDIMF
C    THE ROW (OR FIRST) DIMENSION OF THE ARRAY F AS IT APPEARS IN THE
C    PROGRAM CALLING PWSCYL.  THIS PARAMETER IS USED TO SPECIFY THE
C    VARIABLE DIMENSION OF F.  IDIMF MUST BE AT LEAST M+1  .
C
C  W
C    A ONE-DIMENSIONAL ARRAY THAT MUST BE PROVIDED BY THE USER FOR
C    WORK SPACE.  THE LENGTH OF W MUST BE AT LEAST 6(N+1)+8(M+1).
C
C  * * * * * * * * * *     ON OUTPUT     * * * * * * * * * *
C
C  F
C    CONTAINS THE SOLUTION U(I,J) OF THE FINITE DIFFERENCE
C    APPROXIMATION FOR THE GRID POINT (R(I),Z(J)), I = 1,2,...,M+1,
C    J = 1,2,...,N+1  .
C
C  PERTRB
C    IF ONE SPECIFIES A COMBINATION OF PERIODIC, DERIVATIVE, AND
C    UNSPECIFIED BOUNDARY CONDITIONS FOR A POISSON EQUATION
C    (LAMBDA = D), A SOLUTION MAY NOT EXIST.  PERTRB IS A CONSTANT,
C    CALCULATED AND SUBTRACTED FROM F, WHICH ENSURES THAT A SOLUTION
C    EXISTS.  PWSCYL THEN COMPUTES THIS SOLUTION, WHICH IS A LEAST
C    SQUARES SOLUTION TO THE ORIGINAL APPROXIMATION.  THIS SOLUTION
C    IS NOT UNIQUE AND IS UNNORMALIZED. THE VALUE OF PERTRB SHOULD
C    BE SMALL COMPARED TO THE RIGHT SIDE F. OTHERWISE , A SOLUTION
C    IS OBTAINED TO AN ESSENTIALLY DIFFERENT PROBLEM. THIS COMPARISON
C    SHOULD ALWAYS BE MADE TO INSURE THAT A MEANINGFUL SOLUTION HAS
C    BEEN OBTAINED
C
C  IERROR
```

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C     AN ERROR FLAG WHICH INDICATES INVALID INPUT PARAMETERS.  EXCEPT PYL01065
C     FOR NUMBERS 0 AND 11, A SOLUTION IS NOT ATTEMPTED.       PYL01070
C                                                              PYL01075
C     =  0  NO ERROR.                                          PYL01080
C     =  1  A .LT. 0  .                                        PYL01085
C     =  2  A .GE. 8.                                          PYL01090
C     =  3  MBDCND .LT. 1 OR MBDCND .GT. 6  .                  PYL01095
C     =  4  C .GE. 0.                                          PYL01100
C     =  5  N .NE. (2**P)(3**Q)(5**R) OR N .LE. 2  .           PYL01105
C     =  6  NBDCND .LT. 0 OR NBDCND .GT. 4  .                  PYL01110
C     =  7  A = 0, MBDCND = 3 OR 4  .                          PYL01115
C     =  8  A .GT. 0, MBDCND .GE. 5  .                         PYL01120
C     =  9  A = 0, LAMBDA .NE. 0, MBDCND .GE. 5  .             PYL01125
C     = 10  IDIMF .LT. M+1  .                                  PYL01130
C     = 11  LAMBDA .GT. 0  .                                   PYL01135
C                                                              PYL01140
C     SINCE THIS IS THE ONLY MEANS OF INDICATING A POSSIBLY INCORRECT PYL01145
C     CALL TO PWSCYL, THE USER SHOULD TEST IERROR AFTER THE CALL.  PYL01150
C                                                              PYL01155
C     W                                                        PYL01160
C     CONTAINS INTERMEDIATE VALUES THAT MUST NOT BE DESTROYED IF PYL01165
C     PWSCYL WILL BE CALLED AGAIN WITH INTL = 1  .             PYL01170
C                                                              PYL01175
C                                                              PYL01180
C                                                              PYL01185
```