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Title: Evaluation of a Pole Placement Controller for a Planar Manipulator

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The effectiveness of linear control of a planar manipulator is presented for robot operation markedly exceeding the limits of linearity assumed in the design of the linear controller. Wolovich's frequency domain pole placement algorithm is utilized to derive the linear controller. The control scheme must include state estimation since only link position is measured in the planar manipulator studied. Extensive simulations have been conducted not only to verify the linear control design but also to examine the behavior of the controlled system when inputs greatly exceed those assumed for linear design. The results from these studies indicate the linear model performs exactly as designed. The non-linear realistic simulation reveals that the linear model results are obtained when the inputs do not exceed linearity limits. However, when large inputs are applied, the nature of the system response

changes significantly. Regardless of the change in behavior, for the cases considered, there was no instability detected and steady-state values were realized with reasonable settling times which increased in length as the size of the inputs were increased. From the simulation results, it is concluded that the linear controller scheme studied is suitable for use in moving objects from one position to another but would not work well in the rapid drawing of lines and curves.

Evaluation of a Pole Placement Controller for a Planar Manipulator

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NOMENCLATURE

m_i = mass of link i.

$\bar{I}_i$ = moment of inertia of link i with respect to its center of mass.

I_1 = moment of inertia of link 1 about its center of rotation.

$\bar{L}_i$ = distance from center of rotation of link i to its center of mass.

L_i = length of link i.

$\bar{x}_i$ = x coordinate of the center of mass of link i.

$\bar{y}_i$ = y coordinate of the center of mass of link i.

k_1, k_2, k_3 = positive constants defined in Equation II.2.2.

k_f = coefficient of friction of each link.

k_τ = constant shown in Figure II.1 which is determined from stepper motor specification and the gear ratio.

k_d = k_f/k_τ

k_v = constant relating the input to the output of the voltage to frequency converter (see Figure II.1).

a, b, c = significant elements of the linearized matrix A defined in Equation II.3.9.

γ_1 = positive constant defined as $\gamma = ac - b^2$

x = state vector defined in Equation II.3.2.

x′ = state derivative vector defined in Equation II.3.3.

NOMENCLATURE *continued*

$\partial[.]$ = degree of the polynomial element of highest degree in matrix(.)

$\partial_{cj}[.]$ = degree of the polynomial element of highest degree in the j-th column of matrix (.).

$\Gamma_c[.]$ = the constant matrix consisting of the coefficient of the highest degree s terms in each column of (.).

EVALUATION OF A POLE PLACEMENT CONTROLLER FOR A PLANAR MANIPULATOR

I. INTRODUCTION

I.1 BACKGROUND

This thesis is concerned with the effectiveness of pole placement linear control of a two joint robot arm constrained to move in a horizontal plane (planar manipulator).[22] A robot is a mechanism, composed of links connected by joints into an open kinematic chain, which can be directed to do a variety of tasks without human supervision. The number of joints determines the manipulator's degrees-of-freedom (DOF). The Robot Institute of America (RIA) defines a robot as " a reprogrammable, multi-functional manipulator designed to move material, parts, tools, or specialized devices, through variable programmed motions for the performance of the variety of tasks."[14] The robot manipulator is a highly coupled nonlinear multivariable system. Controller design for the manipulator is concerned with correctly positioning the end effector in the manipulator's work space during the time allotted for a task.

In this thesis a ***pole placement linear control algorithm*** is applied to the planar manipulator dynamic model linearized around an equilibrium point.[Wolovich, 22] In general, linear control is valid for a neighborhood of an equilibrium point. Consequently, the main objective of the thesis is to determine the linear controller's capability to effectively control the planar manipulator outside the linear operating region.

I.2 STATEMENT OF THE PROBLEM

The system under consideration is a planar manipulator constrained to move in a horizontal plane as shown in Figure I.1.

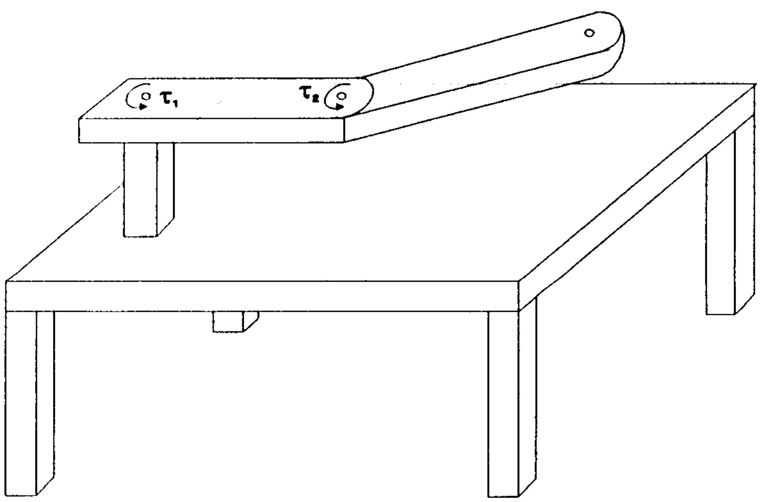


Figure I.1: Two Joint Planar Manipulator.

The inputs to the planar manipulator are voltages coming from the controller. The outputs are voltages which measure the joint angles. It is assumed that the only measurable signals (states) of the system are the inputs and the outputs. Since for the compensation of the system, knowledge of other states are also required, a ***frequency domain pole placement algorithm*** based on the ***frequency domain state estimation and feedback*** is employed.[22] This control system is shown Figure I.2.

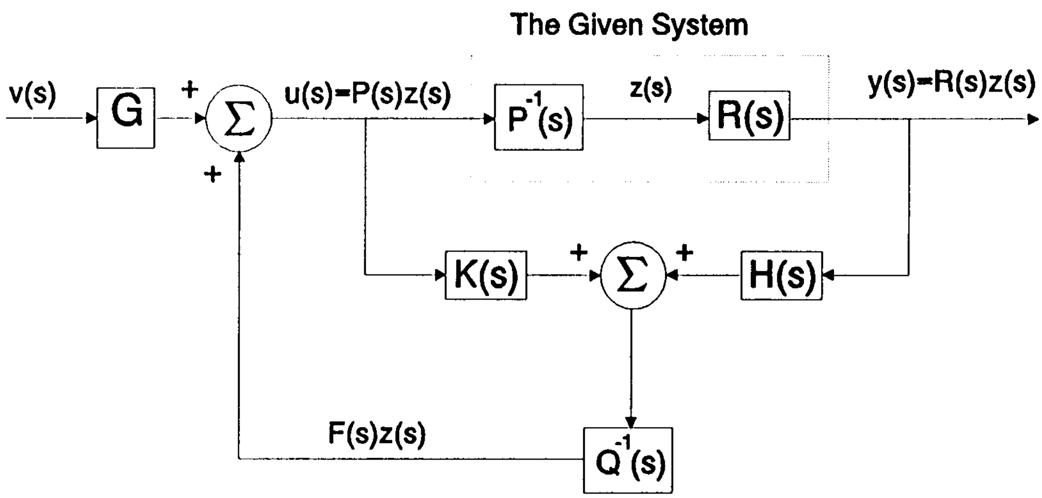


Figure I.2: Frequency Domain Compensation Scheme.

In Figure I.2, H(s), K(s), $Q^{-1}(s)$ are polynomial matrices of the complex frequency $s=\sigma+jw$ and G is a constant matrix. The goal is to choose these four matrices such that the overall closed loop system performs as desired. The open loop transfer matrix of the system must be of the form of $T(s)=R(s)P^{-1}(s)$ where R(s) and P(s) must be relatively right prime polynomial matrices.[22] Furthermore R(s), and P(s) must have certain properties as discussed later.

I.3 ORGANIZATION OF THE STUDY

The first attempt at linear control of the planar manipulator utilized a linear output feedback (l.o.f.) algorithm to arbitrarily assign the n closed loop poles of the system. The algorithm is defined as the control law

$$u(s) = Hy(s)+Gv(s) \tag{I.3.1}$$

where H and G are constant $(m \times p)$, and $(m \times m)$ gain matrices respectively, and v(s) is a m-vector external input. The advantage of l.o.f. method over the linear state estimation feedback (l.s.e.f) algorithm is that it does not increase the system order and therefore less complexity is associated with it. However, it can be shown that if for a given system $pm<n$, then the l.o.f. algorithm fails.[22] For the planar manipulator considered here $m=2$, $p=2$, and (as shown later) $n=6$. Hence $pm<6$. Consequently the l.o.f. algorithm was abandoned.

The state model of the planar manipulator under consideration is both controllable and observable. Observability of the system model implies the ability for estimating those states of the system model which are not directly measurable. Controllability together with observability enables one to employ the l.s.e.f. algorithm to arbitrarily place the n closed loop poles of the system at any desired position in the left half s plane. But why not use adaptive control, or self-tuning control, or etc? The major reason is the cost for a controller. After all if the cost was not of concern, all the states of the open loop system could be measured directly using some expensive transducers, and perhaps a simpler algorithm could have been employed.

The planar manipulator is discussed in Section II.1. Its dynamic equations are derived in Section II.2. The planar manipulator linearized model and state representation are derived in Section II.3. The pole placement frequency domain algorithm [22] is presented in Chapter III. Section III.1 includes four major steps involved in deriving the linear controller. Section III.2 includes some preliminaries and the main theorem for the compensation algorithm which is proved in Section III.4. A single-input single-output linear second order system is considered in Section III.5. Linear controller for the planar manipulator is derived in Section IV.1. Section IV.2 includes the planar manipulator control scheme used for simulation. Simulation results are presented in Chapter V. Conclusions and recommendations are presented in Chapter VI. Certain algebraic manipulations are included in Appendices A through E.

II. PLANAR MANIPULATOR

II.1 INTRODUCTION

The Planar manipulator is shown in Figure I.1. The arm is driven by two input torques provided by two identical stepper motors. Arm motion is constrained to a horizontal plane. Figure II.1 shows the two-input two-output planar manipulator open loop block diagram together with its stepper motors and drives. The stepper motor and controller constants k_τ, and k_v can be obtained from motor specifications or determined experimentally. Figure II.2 shows a reasonable approximation to the stepper motor torques versus torque angle character. For a permanent magnet stepper motor, the torque angle is the angle between the stator resultant magnetic field vector and the magnetic field vector of the rotor permanent magnet. For a reluctance stepper motor, the torque angle is the angle between the stator resultant magnetic field vector and its rotor position for minimum reluctance. If the torque angle $\delta\theta$ is too large, then not enough torque can be generated to meet load torques and slipping occurs. The stepping motor model is linearized around $\delta\theta = 0$ and does not include the capability for slipping which occurs if the acceleration is too high, i.e. if $|\delta\theta| > \delta\theta_{max}$. The linear range of operation is considered to be $45^\circ < \delta\theta < 45^\circ$ (electrical degrees) as indicated in Figure II.2.

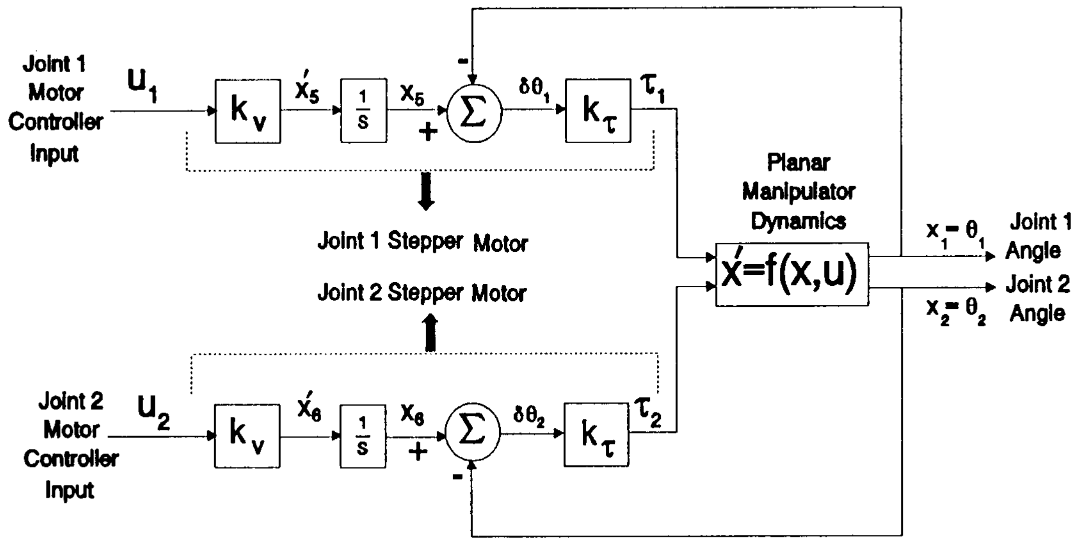


Figure II.1: Block Diagram of a Planar Manipulator Including the Stepper Motors and Drives.

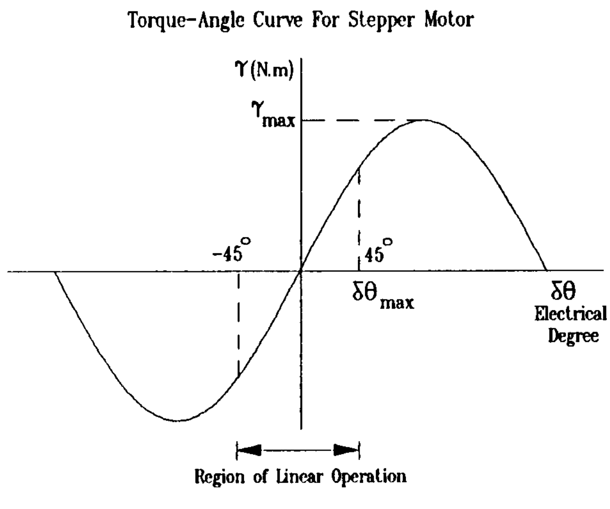


Figure II.2: Torque-Angle Characteristic of Stepper Motor.

For the particular stepper motors available in the laboratory, $\tau_{max}=.8$ (N.m)=115 (oz.in). In obtaining the constant k_τ, τ_{max} of the stepper motor at 45° together with the gear ratio and the number of poles inside the stepper motor have been taken into account. k_v is determined assuming the maximum speed of the stepper motor is achieved when the inputs u_1, and u_2 are 10 volts.

II.2 DYNAMIC EQUATIONS OF MOTION

There are two categories of manipulator modeling equations which apply to the control of a manipulator. ***Kinematic equations*** describe relationships, including position, orientation, and velocity, as well as acceleration of the links of the manipulator. These equations are used for the trajectory planning of robot motion and for deriving the dynamic equations of motion. ***Dynamic equations*** are the expressions of the necessary forces or the torques to be applied to the different joints of a manipulator as a function of position, velocity, and joint acceleration. The planar manipulator dynamic equations are discussed now.

Using Cartesian coordinates, the manipulator top view is shown in Figure II.3. The motion of the arm is constrained to a horizontal plane. The arm is driven by the two input torques produced by two identical stepper motors. The links are considered to be rigid bodies. Table II.1 gives the data which have been

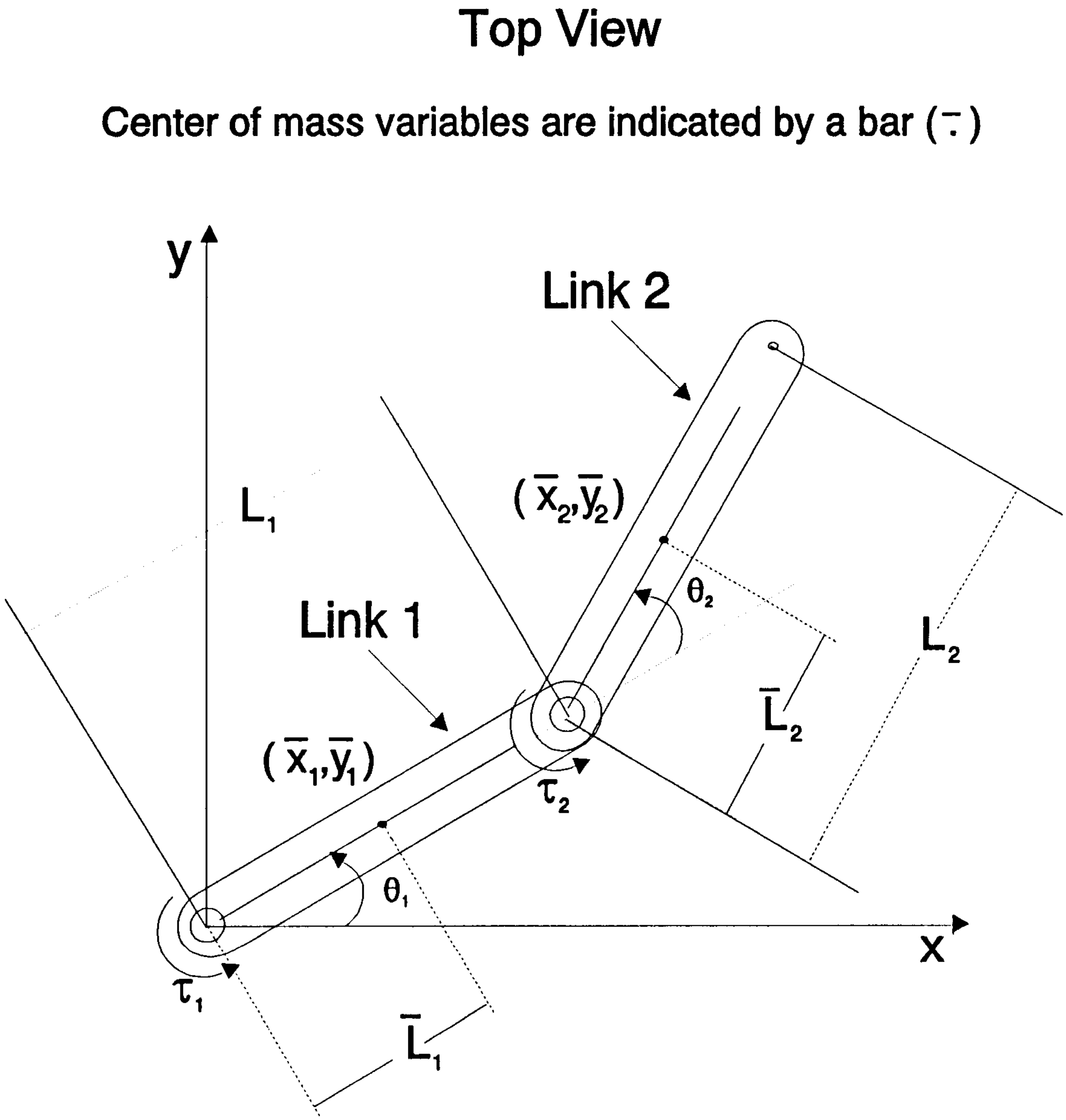


Figure II.3: Two-degree of freedom manipulator.

experimentally determined for a particular planar manipulator available in the Electrical and Computer Engineering Control System Laboratory. The links were taken apart and their masses were measured. The moment of inertias, were calculated assuming that the links are made of aluminum and then using the formula for the inertia of a rectangular bar with respects to its center of mass. The moment of inertia of link 1 was also calculated with respect to its center of rotation. The center of mass was located, using a very sharp edge to balance each link. To calculate the coefficient of friction, the table was positioned vertically with only one link attached to it (imagine a vertical pendulum). Then, the link was released at different initial angles. The trajectory of the link was observed. Data points were taken. The vertical link was simulated for different values of k_f until the same trajectory as the experimental one was obtained.

m_1	= 3.343	Kg-mass	m_2	= 4.813	Kg-mass
$\overline{I}_1$	= 0.073	$Kg.m^2$	$\overline{I}_2$	= 0.156	$Kg.m^2$
$\overline{L}_1$	= 0.084	m	$\overline{L}_2$	= 0.16	m
L_1	= 0.42	m	L_2	= 0.42	m
I_1	= 0.0966	$Kg.m^2$ (moment of inertia of link 1 about its center of rotation. Used to find k_f).			
k_f	= 0.2	$N.m.s^2/rad^2$ (coefficient of friction).			
k_v	= 34.2	rad/v.s	k_τ	= 949.2	N.m/rad

Table II.1: Data Obtained in the Laboratory.

The planar manipulator dynamic equations are derived in Appendix A using Lagrange's method. This method is based on the relation between the potential and the kinetic energy of the system. The planar manipulator dynamic equations of motion are as following:

$$\begin{bmatrix} \theta''_1(k_2+2k_1\cos\theta_2)-2\theta'_1\theta'_2k_1\sin\theta_2+\theta''_2(k_1\cos\theta_2+k_3)-(\theta'_2)^2k_1\sin\theta_2 \\ \theta''_1(k_1\cos\theta_2+k_3)+\theta''_2k_3+(\theta'_1)^2k_1\sin\theta_2 \end{bmatrix} = \begin{bmatrix} \tau_1-k_f\theta'_1 \\ \tau_2-k_f\theta'_2 \end{bmatrix} \qquad \textbf{II.2.1}$$

where k_1, k_2, and k_3 are constants defined as

$$\begin{aligned} k_1 &\triangleq m_2L_1\bar{L}_2 \\ k_2 &\triangleq \bar{I}_1+\bar{I}_2+m_1\bar{L}_1^2+m_2(L_1^2+\bar{L}_2^2) \\ k_3 &\triangleq \bar{I}_2+m_2\bar{L}_2^2 \end{aligned} \qquad \textbf{II.2.2}$$

Equation II.2.1 can be expressed as

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & k_3 \end{bmatrix}\begin{bmatrix} \theta''_1 \\ \theta''_2 \end{bmatrix} = \begin{bmatrix} a_4 \\ a_5 \end{bmatrix} \qquad \textbf{II.2.3}$$

where

$$\begin{aligned} a_1 &= k_2+2k_1\cos\theta_2 \\ a_2 &= k_3+k_1\cos\theta_2 \\ a_3 &= -k_1\sin\theta_2 \\ a_4 &= \tau_1-a_3(\theta'_2)^2-2a_3\theta'_1\theta'_2-k_f\theta'_1 \\ a_5 &= \tau_2+a_3(\theta'_1)^2-k_f\theta'_2 \end{aligned} \qquad \textbf{II.2.4}$$

Note that a_1 through a_5 are function of τ_1, τ_2, θ'_1, θ'_2, and θ_2.

θ_1'', and θ_2'' can be expressed as

$$\theta''_1 = (a_1k_3-a_2^2)^{-1}(k_3a_4-a_2a_5)$$
$$\theta''_2 = (a_1k_3-a_2^2)^{-1}(a_1a_5-a_2a_4) \qquad \text{II.2.5}$$

Notice that Equation II.2.5 is always valid since

$$a_1k_3-a_2^2 = \bar{I}_1\bar{I}_2+m_1\bar{I}_2(\bar{L}_1)^2+m_2\bar{I}_1(\bar{L}_2)^2+m_1m_2(\bar{L}_1)^2(\bar{L}_2)^2+$$
$$m_2\bar{I}_2L_1^2+m_2^2L_1^2(\bar{L}_2)^2\sin^2(\theta_2) \qquad \text{II.2.6}$$

is always greater than zero. Equation II.2.5, is used for nonlinear simulation. In the next section the dynamic equation linearization and system state representation are discussed.

II.3 LINEARIZED MODEL AND STATE REPRESENTATION OF THE SYSTEM

The general form of the state representation for a dynamical system can be expressed as

$$x' = f(x,u)$$
$$y = g(x,u) \qquad \text{II.3.1}$$

where x (n×1) is the state vector, y (p×1) is the output vector, and u (m×1) is the input vector. f and g are vector functions of the system state and input. The state vector for the open loop system given in Figure II.1, is defined as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta'_1 \\ \theta'_2 \\ x_5 \\ x_6 \end{bmatrix} \qquad \text{II.3.2}$$

The state vector derivative is

$$x' = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta''_1 \\ \theta''_2 \\ x'_5 \\ x'_6 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \theta''_1 \\ \theta''_2 \\ k_v u_1 \\ k_v u_2 \end{bmatrix} \qquad \text{II.3.3}$$

where

$$\begin{aligned}\theta''_1 = {} & [k_2k_3-k_3^2-k_1^2\cos^2(x_2)]^{-1} * \\ & [k_3k_\tau((x_5-x_1)-(x_6-x_2))+ \\ & k_1k_3\sin(x_2)(x_3^2+x_4^2+2x_3x_4)- \\ & k_1k_\tau\cos(x_2)(x_6-x_2)+ \\ & \tfrac{1}{2}k_1^2\sin(2x_2)x_3^2+ \\ & k_f(k_3(x_4-x_3)+k_1x_4\cos(x_2))]\end{aligned} \qquad \text{II.3.4}$$

$$\begin{aligned}\theta''_2 = {} & [k_2k_3-k_3^2-k_1^2\cos^2(x_2)]^{-1} * \\ & [k_\tau(k_2(x_6-x_2)-k_3(x_5-x_1))+ \\ & k_1k_\tau\cos(x_2)(2(x_6-x_2)-(x_5-x_1))- \\ & k_1^2\sin(2x_2)(x_3^2+\tfrac{1}{2}x_4^2+x_3x_4)- \\ & k_1\sin(x_2)(k_2x_3^2+k_3x_4^2+2k_3x_3x_4)+ \\ & k_f(k_1\cos(x_2)(x_3-2x_4)+k_3x_3-k_2x_4)]\end{aligned}$$

The output vector is

$$y = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{II.3.5}$$

To employ the pole placement linear controller algorithm depicted in Figure I.2, Equation II.3.3 must be linearized around an equilibrium point. The linearized state variable equations have the form

$$\begin{aligned} x' &= Ax+Bu \\ y &= Cx+Du \end{aligned} \qquad \text{II.3.6}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_o \;;\; B = \frac{\partial f}{\partial u}\Big|_o \qquad C = \frac{\partial g}{\partial x}\Big|_o \;;\; D = \frac{\partial g}{\partial u}\Big|_o \qquad \text{II.3.7}$$

In Equation II.3.7, subscript "o" indicates evaluations at the equilibrium point.

In Appendix B, it is shown that the planar manipulator linearized system matrices around an equilibrium point are

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_6} \\ \vdots & & & \vdots \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}_o = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a & b & k_d a & k_d b & -a & -b \\ b & c & k_d b & k_d c & -b & -c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{II.3.8}$$

where

$$\begin{aligned} a &\triangleq -k_3 k_\tau \; H_1 \\ b &\triangleq (k_1 \cos(x_{2n}) + k_3) k_\tau \; H_1 \\ c &\triangleq -(2k_1 \cos(x_{2n}) + k_2) k_\tau \; H_1 \\ k_d &\triangleq \frac{k_f}{k_\tau} \\ H_1 &\triangleq [k_2 k_3 - k_3^2 - k_1^2 \cos^2(x_{2o})]^{-1} \\ &= [a_1 k_3 - a_2^2]^{-1} \big|_o \end{aligned} \qquad \text{II.3.9}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial f_6}{\partial u_1} & \frac{\partial f_6}{\partial u_2} \end{bmatrix}_o = k_v \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{II.3.10}$$

$$C = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_6} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_6} \end{bmatrix}_o = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{II.3.11}$$

$$D = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix}_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{II.3.12}$$

Therefore the state representation of the linearized planar manipulator model is given by Equation II.3.6 where the system matrices, A, B, C, and D are given by Equation II.3.8 and Equations II.3.10 through II.3.12 respectively. The pole placement algorithm is discussed in the next chapter.

III. DERIVATION OF POLE PLACEMENT CONTROLLER

III.1 INTRODUCTION

As stated in Section I.2, the pole placement state estimation algorithm relies on the ability to determine a transfer matrix, T(s), as the product $R(s)P^{-1}(s)$. The polynomial matrices R(s) and P(s) *must be relatively right prime*, defined shortly. *This implies the (complete state) observability of any equivalent time domain realization and therefore the ability for estimating the entire state of the system.*[22] The following are four major steps involved in deriving the linear controller.

1. The system matrices must be transformed into a controllable companion form which implies that the system must be state controllable.
2. The structure theorem must be employed to the controllable companion form to find T(s) as the product of $R(s)P^{-1}(s)$.
3. It must be shown that the polynomial matrices R(s) and P(s) obtained from structure theorem are relatively right prime.
4. The frequency domain state estimation and feedback must be employed to derive the linear controller.

The procedure for steps 1 and 2 are discussed in Appendix C. To establish steps 3 and 4 the material presented in the next section are essential.

III.2 PRELIMINARIES

The following definitions are established. These definitions are repeated in Appendix C.

DEFINITION III.2.1: The degree of a polynomial matrix M(s) is defined as the degree of the polynomial element of highest degree in M(s). The degree of the j-th column of M(S) denoted by the scalar $\partial_{cj}[M(s)]$, is defined as the degree of the polynomial element of highest degree in the j-th column of M(s). The constant matrix consisting of the coefficients of the highest degree terms in each column of M(s) is denoted by $\Gamma_c[M(s)]$. Subscript "c" implies column. To illustrate, consider the following example:

EXAMPLE III.2.1: If

$$M(s) = \begin{bmatrix} s^2-3 & 1 & 2s \\ 4s+2 & 2 & 0 \\ -s^2 & s+3 & -3s+2 \end{bmatrix} \qquad \text{III.2.1}$$

then $\partial_{c1}=2$, $\partial_{c2}=\partial_{c3}=1$, and

$$\Gamma_c[M(s)] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ -1 & 1 & -3 \end{bmatrix} \qquad \text{III.2.2}$$

Note that $\Gamma_c[M(s)]$ is not of full rank since $|\Gamma_c[M(s)]|=0$. The column j zeros in $\Gamma_c[M(s)]$ indicate that the corresponding polynomials are of lesser degree than $\partial_{cj}[M(s)]$.

DEFINITION III.2.2: A $n \times m$ polynomial matrix, M(s), is called column proper if and only if $\Gamma_c[M(s)]$ has full rank; i.e. $\text{rank}\{\Gamma_c[M(s)]\}=\min(n,m)$. Hence a *square* polynomial matrix M(s), is column proper if and only if $|\Gamma_c[M(s)]| \neq 0$.

DEFINITION III.2.3: If three polynomial matrices satisfy the relation; $P(s)=H(s)G_r(s)$, then $G_r(s)$ is called a right divisor of P(s), and P(s) is called a left multiple of $G_r(s)$. A greatest common right divisor (g.c.r.d.) of two polynomial matrices P(s) and R(s) is a common right divisor which is a left multiple of every common right divisor of P(s) and R(s).

DEFINITION III.2.4: A *unimodular matrix* U(s) is defined as any *square polynomial matrix* whose determinant is a *nonzero constant.*

DEFINITION III.2.5: Two polynomial matrices R(s) and P(s) which have the same number of columns, are said to be relatively right prime if and only if their g.c.r.d.

are unimodular matrices.

EXAMPLE III.2.2: For the following two polynomial matrices R(s) and P(s)

$$R(s) = \begin{bmatrix} s & -s \\ 0 & 1 \end{bmatrix} \quad ; \quad P(s) = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \end{bmatrix} \qquad \text{III.2.3}$$

then

$$T(s) = R(s)P^{-1}(s) = \begin{bmatrix} s & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 1 \\ s & s^2 \end{bmatrix} \left(\frac{1}{s^4 - s} \right) = \begin{bmatrix} \dfrac{s}{s^2+s+1} & \dfrac{-(s+1)}{s^2+s+1} \\ \dfrac{1}{s^3-1} & \dfrac{s}{s^3-1} \end{bmatrix} \qquad \text{III.2.4}$$

Note that the system characteristic equation is determined only by |P(s)| since R(s) and P(s) are both polynomial matrices. Also notice that pole zero cancellations occur in all elements of T(s). It can be shown that the following square matrix is one of the greatest common right divisors of R(s) and P(s)

$$G_r(s) = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{III.2.5}$$

since

$$R(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} ;$$

$$P(s) = \begin{bmatrix} s & -1 \\ -1 & s^2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{III.2.6}$$

Notice that $G_r(s)$ is not a unimodular matrix since $|G_r(s)|=s$ is not a nonzero constant. The transfer matrix of a system described by $T(s)=R(s)P^{-1}(s)$ does not satisfy the pole placement algorithm used later since $G_r(s)$ is not unimodular.

DEFINITION III.2.6: A polynomial matrix, T(s), is called proper if the numerator degree of each entry of T(s), i.e. $T_{ij}(s)$, is less than or equal to the corresponding denominator degree. In the case of strictly proper transfer matrix, the degree of the numerator of each entry, $T_{ij}(s)$, of T(s) is equal to the corresponding denominator degree.

The dynamical behavior of an m-input, p-output, linear time-invariant physical system can always be represented by a *proper* $p \times m$ *transfer matrix*, T(s), where

$$y(s) = T(s)u(s) \qquad \textbf{III.2.7}$$

and

$$T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) & \cdots & T_{1m}(s) \\ T_{21}(s) & T_{22}(s) & \cdots & T_{2m}(s) \\ \vdots & & & \\ T_{p1}(s) & T_{p2}(s) & \cdots & T_{pm}(s) \end{bmatrix} \qquad \textbf{III.2.8}$$

where $T_{ij}(s)$ is a *proper* transfer function, i.e. the degree of the numerator of $T_{ij}(s)$ is less than or equal to the degree of its denominator.

Given a state controllable model of a system having a proper transfer matrix T(s), then the structure theorem of Appendix C guarantees that it is always possible to express T(s) as

$$T(s) = R(s)P^{-1}(s) \qquad \text{III.2.9}$$

where R(s) and P(s) are polynomial matrices, viz.

$$R(s) = \begin{bmatrix} R_{11}(s) & R_{12}(s) & \cdots & R_{1m}(s) \\ R_{21}(s) & R_{22}(s) & \cdots & R_{2m}(s) \\ \vdots & & & \\ R_{p1}(s) & R_{p2}(s) & \cdots & R_{pm}(s) \end{bmatrix} ;$$

$$\text{III.2.10}$$

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) & \cdots & P_{1m}(s) \\ P_{21}(s) & P_{22}(s) & \cdots & P_{2m}(s) \\ \vdots & & & \\ P_{m1}(s) & P_{m2}(s) & \cdots & P_{mm}(s) \end{bmatrix}$$

P(s) must be column proper and the degree of each column of R(s) must be less than or equal to the degree of the corresponding column in P(s). Define d_j as the degree of the j-th column of P(s). i.e.

$$\partial_{cj}[P(s)] = d_j \qquad \text{III.2.11}$$

then

$$\partial_{cj}[R(s)] \le d_j \qquad \text{III.2.12}$$

The following theorem due to Wolovich, is essential in determining the closed loop transfer matrix of the compensated system shown in Figure I.2.

THEOREM III.2.1 (page 239, [22]): Given the p×m open loop transfer matrix, $T(s)=R(s)P^{-1}(s)$, of Figure I.2 where $\partial_c[R(s)] \leq \partial_c[P(s)]$ and **P(s) is a m×m column proper polynomial matrix**, i.e. $|\Gamma_c[P(s)]| \neq 0$, with $\partial_{cj}[P(s)]=d_j \geq 1$ for all $j=1,2,\ldots,m$, **if R(s) and P(s) are relatively right prime polynomial matrices**, then for any **arbitrary m×m polynomial matrix F(s)** which fulfills

$$\partial_c[F(s)] < \partial_c[P(s)] \qquad \textbf{III.2.13}$$

polynomial matrices H(s), K(s), and Q(s) of Figure I.2 exist which satisfy the following:

1- The zeros of $|Q(s)|$ lie in the stable half-plane $Re(s)<0$ which implies that $Q^{-1}(s)$ is a stable transfer matrix.

2- $H(s)R(s)+K(s)P(s) = Q(s)F(s)$ **III.2.14**

3- Both $Q^{-1}(s)H(s)$ and $Q^{-1}(s)K(s)$ are (stable) proper transfer matrices.

Results obtained in the proof of this theorem play a significant role in designing the linear controller. Its significance from a point of view of the frequency domain compensation scheme of Figure I.2 is presented in the next section. Then the theorem is derived.

III.3 DERIVATION OF THE DESIRED CLOSED LOOP TRANSFER MATRIX

It is assumed that the matrices H(s), K(s), Q(s), and F(s) satisfy the requirement of Theorem III.2.1. Equating the signals at the first summing junction in Figure I.2 results in the following:

$$u(s) = P(s)z(s) = Gv(s) + Q^{-1}(s)[K(s)P(s)z(s) + H(s)R(s)z(s)] \qquad \textbf{III.3.1}$$

or

$$[Q(s)P(s) - K(s)P(s) - H(s)R(s)]z(s) = Q(s)Gv(s) \qquad \textbf{III.3.2}$$

By substituting Equation III.1.2 into Equation III.3.2 the following is realized:

$$z(s) = [P(s) - F(s)]^{-1}Q^{-1}(s)Q(s)Gv(s) \qquad \textbf{III.3.3}$$

Note that Q(s) has stable poles (Theorem III.2.1). Consequently any pole zero cancellation in $Q^{-1}(s)Q(s)$ do not lead to problems from a dynamical point of view. Since y(s) = R(s)z(s), it follows that

$$y(s) = R(s)P_F^{-1}(s)Gv(s) \qquad \textbf{III.3.4}$$

where

$$P_F(s) \triangleq P(s) - F(s) \qquad \textbf{III.3.5}$$

Note that

$$\partial_c[P_F(s)] = \partial_c[P(s)] \qquad \textbf{III.3.6}$$

since $\partial_c[F(s)] < \partial_c[P(s)]$ from Equation III.2.13.

The closed loop system satisfies $y(s) = T_c(s)v(s)$. Hence, from Equations III.3.4 and III.3.5 the closed loop transfer matrix of the compensated system is given by

$$T_c(s) = R(s)P_F^{-1}(s)G = R(s)[G^{-1}P_F(s)]^{-1} \qquad \textbf{III.3.7}$$

Since Theorem III.2.1 is satisfied then stable physically realizable compensation scheme of Figure I.2 can be employed to achieve ***any desired closed loop transfer matrix*** of the form given by Equation III.3.7. The proof of the theorem given in the next section also yields the procedure for selecting the appropriate polynomial matrices H(s), K(s), and Q(s).

III.4 PROOF OF THEOREM III.2.1

Wolovich's theorem is rederived here for completeness. The following example helps in understanding the proof of Theorem III.1.1.

Example III.4.1: Let R(s) and P(s) be the following polynomial matrices with $\partial_c[R(s)] \leq \partial_c[P(s)]$

$$R(s) = \begin{bmatrix} s+1 & s \\ s^2+2 & s^2 \end{bmatrix} \quad ; \quad P(s) = \begin{bmatrix} s+4 & s^2+4 \\ s^3+2 & s^2 \end{bmatrix} \qquad \text{III.4.1}$$

Form the polynomial matrix W(s) as

$$W(s) = \begin{bmatrix} R(s) \\ P(s) \end{bmatrix} = \begin{bmatrix} s+1 & s \\ s^2+2 & s^2 \\ --- & --- \\ s+4 & s^2+4 \\ s^3+2 & s^2 \end{bmatrix} \qquad \text{III.4.2}$$

Define $d_j = \partial_{cj}[P(s)]$; i.e. $d_1 = 3$, $d_2 = 2$. Then is it possible to find a constant matrix M such that $W(s) \stackrel{2}{=} MS$ where

$$S = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ s^3 & 0 \\ -- & -- \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix} \qquad \text{III.4.3}$$

Note that each column of S is of degree $\partial_{cj}[P(s)]$. Also note that the scalar product is zero for all different column vectors in S. Having defined S as such the following can be established

$$W(s) = \begin{bmatrix} s+1 & s \\ s^2+2 & s^2 \\ --- & --- \\ s+4 & s^2+4 \\ s^3+2 & s^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & | & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 & | & 4 & 0 & 1 \\ 2 & 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ s^3 & 0 \\ -- & -- \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix} \qquad \text{III.4.4}$$

In general, it is always possible to find a unique constant matrix M such that for any R(s) and P(s) which satisfy $\partial_c[R(s)] \leq \partial_c[P(s)]$, $W(s) = [R(s)^T \; P(s)^T]^T$ can be written as $W(s) = MS = M[S_1, S_2, \cdots, S_m]$ where

$$S = \begin{bmatrix} 1 & 0 & & 0 \\ s & \vdots & & \vdots \\ \vdots & & & \\ s^{d_1} & & & \\ -- & -- & & \\ 0 & 1 & & \\ \vdots & s & & \\ & \vdots & & \\ & s^{d_2} & & \\ & -- & \ddots & -- \\ & 0 & & 1 \\ & \vdots & & s \\ & & & \vdots \\ & & & s^{d_m} \end{bmatrix} ; \qquad \begin{aligned} d_j &= \partial_{cj}[P(s)], \; j=1,2,\cdots,m \\ S_i^T S_j &= 0 \quad \text{for } i \neq 0 \end{aligned} \qquad \text{III.4.5}$$

Now consider the polynomial matrices R(s) and P(s) of dimensions $p \times m$ and $m \times m$, respectively, with $\partial_c[R(s)] \leq \partial_c[P(s)]$, P(s) column proper, and $\partial_{cj}[P(s)] = d_j \geq 1$ for all $j = 1,2,\ldots,m$ where n is defined by

$$n \triangleq \sum_{j=1}^{m} d_j \qquad \text{III.4.6}$$

In view of these assumptions it follows that for $k = 1,2,\cdots$ the $k(m+p) \times m$ polynomial matrix

$$[R^T(s), sR^T(s), \ldots, s^{k-1}R^T(s), P^T(s), \ldots, s^{k-1}P^T(s)]^T \qquad \text{III.4.7}$$

can be expressed as the product of a constant $k(m+p) \times (n+mk)$ matrix, M_{ck}, and an $(n+mk) \times m$ matrix, $S_{ck}(s)$, consisting of monic single term polynomial elements; i.e.

$$\begin{bmatrix} R(s) \\ sR(s) \\ \vdots \\ s^{k-1}R(s) \\ P(s) \\ sP(s) \\ \vdots \\ s^{k-1}P(s) \end{bmatrix} = M_{ck}S_{ck}(s) = M_{ck} \begin{bmatrix} 1 & 0 & \dots & 0 \\ s & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ s^{d_1+k-1} & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ 0 & s & & \\ \vdots & \vdots & & \\ 0 & s^{d_2+k-1} & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ & & & 0 \\ & & & 1 \\ & & & s \\ & & & \vdots \\ 0 & 0 & \dots & s^{d_m+k-1} \end{bmatrix} \qquad \text{III.4.8}$$

for a unique constant matrix M_{ck} which depends on k. Note that M_{ck} is a square matrix if and only if $n=kp$.

DEFINITION III.4.1: The ***eliminant matrix***, M_c, of the two polynomial matrices R(s) and P(s) with P(s) column proper and $\partial_c[R(s)] \leq \partial_c[P(s)]$ is defined as $M_{c\nu}$, where ν is the least integer k in Equation III.4.8 for which $n+mk\text{-rank}[M_{ck}]$ is a minimum. $S_c(s)$ is then defined as $S_{c\nu}(s)$.

THEOREM III.4.1: The polynomial matrices R(s) and P(s) employed in Definition III.4.1 are relatively right prime if and only if their eliminant matrix has full rank

$(n+m\nu)$.[22]

COROLLARY III.4.1: Given the relatively right prime polynomial matrices R(s) and P(s) with $\partial_c[R(s)] \leq \partial_c[P(s)]$, P(s) column proper, and $\partial_{cj}[P(s)] = d_j \geq 1$ for all $j = 1,2,\cdots,m$, then a constant gain matrix [H,K] can be chosen such that $[H,K]M_eS_e(s) = H(s)R(s) + K(s)P(s)$ equals any $m \times m$ polynomial matrix $\beta(s) = \beta S_e(s)$ where β is an arbitrary constant matrix which satisfies:

$$\partial_{cj}[\beta(s)] \leq d_j + \nu - 1 \qquad \textit{for } j = 1,2,\ldots,m. \tag{III.4.9}$$

It is assumed that R(s) and P(s) satisfy the requirements of Corollary III.4.1. Since R(s) and P(s) are relatively right prime polynomial matrices, then by Theorem III.4.1 their eliminant matrix has a full rank; i.e. $\mathrm{rank}[M_e] = n + m\nu$. To establish the corollary, $\hat{M}_e$ is defined as the nonsingular matrix consisting of the first $n + m\nu$ linearly independent rows of M_e and $[\hat{H},\hat{K}]$ is defined as the $m \times (n + m\nu)$ matrix obtained from [H,K] by deleting those columns of [H,K] which correspond to the same numbered rows of M_e which were eliminated to form $\hat{M}_e$. In view of Equation III.4.8, any arbitrary $m \times m$ polynomial matrix, $\beta(s) = \beta S_e(s)$ which satisfies Equation III.4.9 can be obtained by solving:

$$[\hat{H},\hat{K}]\hat{M}_e S_e(s) = \beta S_e(s) \tag{III.4.10}$$

for $[\hat{H},\hat{K}]$; i.e.

$$[\hat{H},\hat{K}] = \beta \hat{M}_e^{-1} \qquad \text{III.4.11}$$

To find an appropriate [H,K], identically zero columns, corresponding to those columns of [H,K] which were eliminated to form $[\hat{H},\hat{K}]$ are now inserted into $[\hat{H},\hat{K}]$. Then it follows that $[\hat{H},\hat{K}]\hat{M}_e = [H,K]M_e = \beta$ and therefore in view of Equation III.4.8 and Equation III.4.10, the following can be established for some appropriate polynomial matrices H(s) and K(s).

$$[H,K]M_eS_e(s) = [H,K]\begin{bmatrix} R(s) \\ sR(s) \\ \vdots \\ s^{v-1}R(s) \\ P(s) \\ sP(s) \\ \vdots \\ s^{v-1}P(s) \end{bmatrix} \qquad \text{III.4.12}$$

$$= H(s)R(s)+K(s)P(s) = \beta S_e(s)$$

From the results obtained above, the fact that the polynomial matrices H(s),K(s), and Q(s) can be chosen to satisfy the three conditions of theorem III.1.1 can be verified now. In particular, by setting

$$Q(s) = \begin{bmatrix} s^{\nu-1} & 0 & \dots & 0 & q_{1m}(s) \\ -1 & s^{\nu-1} & 0 \; \dots & 0 & q_{2m}(s) \\ 0 & -1 & & . & . \\ 0 & 0 & . & . & . \\ . & . & . & & . \\ . & . & & . & . \\ 0 & 0 & 0 \; \dots & -1 & s^{\nu-1}+q_{mm}(s) \end{bmatrix} \qquad \textbf{III.4.13}$$

and evaluating $|Q(s)|$ by last column minors, the following expression can be derived

$$|Q(s)| = \sum_{i=1}^{m} q_{im}(s)s^{(i-1)(\nu-1)} + s^{m(\nu-1)} \qquad \textbf{III.4.14}$$

Define $q_{im}(s)$ for $i=1,2,\cdots,m$ as follows

$$q_{im}(s) = \sum_{j=0}^{\nu-2} \left(q_{(i-1)(\nu-1)+j}\right)s^j \quad ; \quad i=1,2,\cdots,m \qquad \textbf{III.4.15}$$

where $q_{(i-1)(\nu-1)+j}$ are constants. Hence $|Q(s)|$ is given by

$$|Q(s)| = q_0+q_1s+\dots+q_{m\nu-m-1}s^{m\nu-m-1}+s^{m\nu-m} \qquad \textbf{III.4.16}$$

where the real constants $q_0, q_1, \cdots, q_{m\nu-m-1}$ are chosen such that the roots of $|Q(s)|$ are in the left half s plane. Therefore, any arbitrary polynomial of degree $m\nu$-m can be chosen as $|Q(s)|$. If F(s) is *any arbitrary $m \times m$ polynomial matrix which satisfies Equation III.2.13*; i.e. $\partial_c[F(s)]<\partial_c[P(s)]$, then it can be verified that the product of Q(s) and F(s) is a polynomial matrix of column (j) degree < $d_j+\nu-1$. Therefore,

$$Q(s)F(s) = \beta(s) = \beta S_e(s) \qquad \text{III.4.17}$$

for some constant matrix β. If [H,K] is now chosen such that

$$[H,K]M_e S_e(s) = Q(s)F(s) = \beta S_e(s) \qquad \text{III.4.18}$$

then for this particular choice of $\beta(s)$, given by Equation III.4.17, it follows in view of Equation III.4.12, that

$$H(s)R(s)+K(s)P(s) = Q(s)F(s) \qquad \text{III.4.19}$$

where both $\partial_{ri}[H(s)] \leq \nu\text{-}1$ and $\partial_{ri}[K(s)] \leq \nu\text{-}1$ for all $i=1,2,...,m$ (see Equation III.4.12). Since $\partial_{ri}[Q(s)] = \nu\text{-}1$, it can be shown that if all of the zeros of $|Q(s)|$ are chosen to lie in the stable half-plane, $Re(s)<0$, then both $Q^{-1}(s)K(s)$ and $Q^{-1}(s)H(s)$ will be stable proper transfer matrices, and as a result Equation III.4.19 is satisfied. Theorem III.2.1 is therefore established.

The preceding can be summarized by noting that if $T(s)=R(s)P^{-1}(s)$ is a proper transfer matrix and if R(s) and P(s) are relatively right prime polynomial matrices where $\partial_c[R(s)] \leq \partial_c[P(s)]$ and P(s) is column proper with $\partial_{cj}[P(s)] = d_j \geq 1$ for all $j=1,2,...,m$, then one can achieve any desired stable closed loop transfer matrix $T_{F,G}(s) = R(s)P_F^{-1}(s) = R(s)[G^{-1}P_F(s)]^{-1}$ via the compensation scheme depicted in Figure I.2, where the only requirements on $G^{-1}P_F(s)$ are the following:

1. The determinant of $G^{-1}P_F(s)$ is the desired characteristic equation of the closed loop system.
2. $G^{-1}P_F(s)$ is a column proper polynomial matrix which shares the same ordered d_j as P(s) (see Equations III.2.13 and III.3.5).
3. G^{-1} exists which implies that G must be nonsingular.

To facilitate understanding the linear controller derivation for the planar manipulator, a linear single-input single-output second order system is considered in the next section.

III.5 POLE PLACEMENT LINEAR CONTROLLER DESIGN FOR A LINEAR SINGLE-INPUT SINGLE-OUTPUT SECOND ORDER SYSTEM

The intention of this section is to facilitate understanding the problem of the planar manipulator which is presented in the next chapter. Consider a controllable and observable linear single-input single-output second order system (p = m = 1) which is described by the following transfer function:

$$T(s) = \frac{s+2}{(s+1)(s-3)} = \frac{s+2}{s^2-2s-3} \qquad \text{III.5.1}$$

Note the presence of a pole in the right half s plane (s=3). The objective is to use the ***frequency domain pole placement algorithm*** to derive a linear controller for the system such that the closed loop poles of the system can be placed at any desired location in the left half s plane. The compensator scheme depicted in III.5.1 is used to achieve this objective.

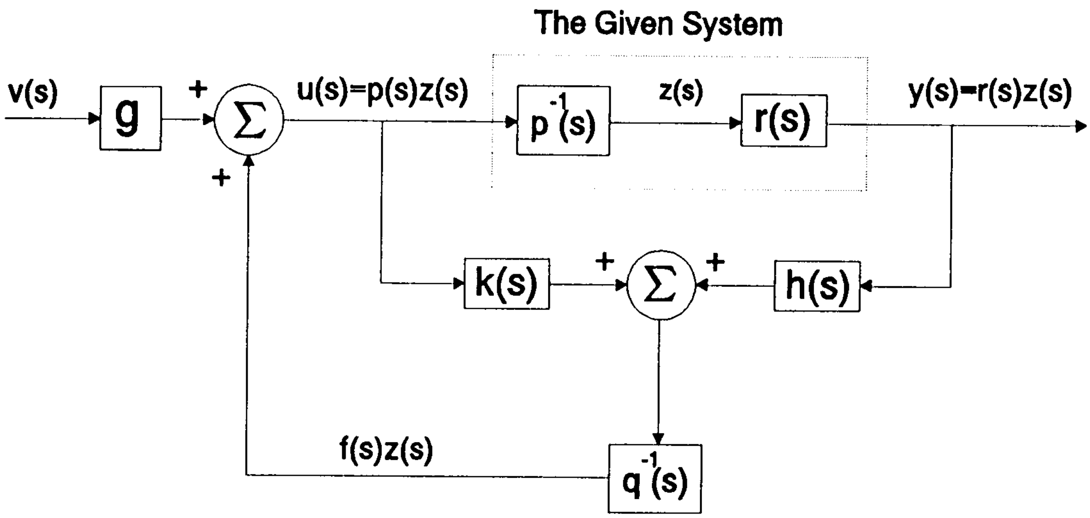


Figure III.1: The Scalar Compensation Scheme

To implement the compensation scheme given in Figure III.1, the open loop system transfer function must be transformed into the form $T(s)=r(s)p^{-1}(s)$ where r(s) and p(s) must be relatively prime polynomials. For the scalar case this implies that r(s) and p(s) must not have any common polynomial factors. By inspection of T(s) the following can be established:

$$\begin{aligned} r(s) &= s+2 \quad ; \quad p(s) = (s+1)(s-3) = s^2-2s-3 \\ d_1 &= 2 \quad ; \quad n = 2 \end{aligned} \qquad \text{III.5.2}$$

For the compensation scheme depicted in Figure III.1, notice that if k(s), h(s), and q(s) are chosen such that for any ***arbitrary polynomial, f(s), of degree no greater than (n-1)***, the following are satisfied

1- q(s) is a stable polynomial.

2- $k(s)p(s)+h(s)r(s)=q(s)f(s)$ **III.5.3**

3- Both $q^{-1}(s)k(s)$ and $q^{-1}(s)h(s)$ are (stable) proper transfer functions.

then it follows that this scalar compensation scheme yields ***any desired closed loop transfer function of the form***

$$t_c(s)=r(s)p_f^{-1}(s)g = r(s)[g^{-1}p_f(s)]^{-1} \qquad \textbf{III.5.4}$$

where $p_f(s)$ is defined as

$$p_f(s) \triangleq p(s)-f(S) \qquad \textbf{III.5.5}$$

From Equation III.5.5 notice that the zero of the system has not been affected by the pole placement algorithm. Also the following two equations are satisfied

$$\partial[p_f(s)] = \partial[p(s)] \qquad \textbf{III.5.6}$$

$$\Gamma[p_f(s)] = \Gamma[p(s)] \qquad \textbf{III.5.7}$$

(∂=degree of a polynomial and Γ=coefficient of the highest degree term in a polynomial). This is due to that fact that $\partial[f(s)] \leq n-1$ while $\partial[p(s)] = n$ (recall that p(s) is the characteristic equation of the open loop system). From this observation the nonsingularity of $p_f(s)$ is noticeable.

Now, suppose it is required to place the two poles of the closed loop system at s=-3 and s=-4. This implies that the characteristic equation of the closed loop system must have the following form

$$\Delta(s) = k(s+3)(s+4) = k(s^2+7s+12) \qquad \textbf{III.5.8}$$

where k is a constant to be chosen such that the design requirement is satisfied. From Equation III.5.4 notice that the characteristic equation of the closed loop system is given by

$$\Delta(s) = g^{-1}p_f(s) \qquad \textbf{III.5.9}$$

Equating Equations III.5.8 and III.5.9 the following is obtained

$$g^{-1}p_f(s) = k(s^2+7s+12) \qquad \textbf{III.5.10}$$

Consequently

$$p_f(s) = s^2+7s+12 \quad ; \quad g=\frac{1}{k} \qquad \textbf{III.5.11}$$

since from Equation III.5.7 the coefficient of the highest degree in $p_f(s)$ must equal to the coefficient of the highest degree in p(s) given by Equation III.5.2.

To derive k(s), h(s), and q(s) the eliminant matrix of r(S) and p(s) must be obtained first. From Definition III.4.1 and Equation III.4.8, since m=1 and d_1=degree of p(s)=2, it can be shown that for k=2, n+mk-rank[M_{ck}] is minimum and hence $\nu=2$. Consequently

$$M_e = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -3 & -2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{bmatrix} \quad ; \quad S_e(s) = \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix} \qquad \text{III.5.12}$$

Since r(s) and p(s) are relatively prime polynomials their eliminant matrix must have full rank. Indeed, this is the case and M_c^{-1} is given by

$$M_e^{-1} = \begin{bmatrix} 0.8 & -0.2 & 0.2 & 0 \\ -0.6 & 0.4 & -0.4 & 0 \\ 1.2 & 0.2 & 0.8 & 0 \\ 0.6 & 1.6 & 0.4 & 1 \end{bmatrix} \qquad \text{III.5.13}$$

Existence of M_c^{-1} indicates that $\hat{M}_c^{-1} = M_c^{-1}$.

Since $\nu=2$, from Equations III.4.13 and III.4.15 the following expression for q(s) is obtained:

$$q(s) = s^{(\nu-1)} + q_{11}(s) = s + q_0 \qquad \textbf{III.5.14}$$

where q_0 must be chosen such that the pole of q(s) is in the left half s plane. For the problem under consideration q_0 is chosen to be 5. Hence

$$q(s) = s+5 \qquad \textbf{III.5.15}$$

From Equations III.5.2, III.5.5, and III.5.11 the following expression for f(s) is derived

$$f(s) = p(s) - p_f(s) = -(9s+15) \qquad \textbf{III.5.16}$$

Notice that f(s) is a polynomial of first degree which satisfies the requirement $\partial[f(s)] \leq n-1$ since $n=2$ for the given system. The expression for q(s)f(s) follows:

$$\beta(s) = q(s)f(s) = -(9s^2+60s+75) \qquad \textbf{III.5.17}$$

At this point a constant 1×4 matrix β can be chosen such that Equation III.4.17 is satisfied. In particular

$$\beta(s) = q(s)f(s) = -(9s^2+60s+75) = [-75 \;\; -60 \;\; -9 \;\; 0]\begin{bmatrix}1\\ s\\ s^2\\ s^3\end{bmatrix} \Rightarrow \qquad \textbf{III.5.18}$$

$$\beta = [-75 \;\; -60 \;\; -9 \;\; 0]$$

Now, Equation III.4.11 (recall that $\hat{M}_e^{-1} = M_e^{-1}$ which implies that $[\hat{h},\hat{k}] = [h,k]$) together with Equation III.4.12 can be employed to derive h(s) and k(s). In particular

$$[h,k] = \beta M_e^{-1} = [-75 \quad -60 \quad -9 \quad 0]\begin{bmatrix} 0.8 & -0.2 & 0.2 & 0 \\ -0.6 & 0.4 & -0.4 & 0 \\ 1.2 & 0.2 & 0.8 & 0 \\ 0.6 & 1.6 & 0.4 & 1 \end{bmatrix} \qquad \text{III.5.19}$$

$$= [-34.8 \quad -10.8 \quad 1.8 \quad 0]$$

and since from Equation III.4.12 the following must be satisfied

$$[h,k]\begin{bmatrix} r(s) \\ sr(s) \\ p(s) \\ sp(s) \end{bmatrix} = h(s)r(s)+k(s)p(s)\text{ ; or}$$

$$[-34.8 \quad -10.8 \quad 1.8 \quad 0]\begin{bmatrix} r(s) \\ sr(s) \\ p(s) \\ sp(s) \end{bmatrix} = -(10.8s+34.8)r(s)+1.8p(s) \qquad \text{III.5.20}$$

$$= h(s)r(s)+k(s)p(s)$$

therefore h(s) and k(s) are given by

$$h(s) = -(10.8s+34.8) \quad ; \quad k(s) = 1.8 \qquad \text{III.5.21}$$

Hence, if g, q(s), and {h(s),k(s)} given by Equations III.5.11, III.5.15, and III.5.21 are employed in the compensation scheme of Figure III.1 the desired closed loop poles of the system will be placed at s=-3 and s=-4; i.e the closed loop transfer function will have the following form

$$t_c(s) = \frac{r(s)}{g^{-1}p_f(s)} = \frac{s+2}{k(s+3)(s+4)} \qquad \text{III.5.22}$$

There is a pole zero cancellation in $t_c(s)$, namely the zero (s+5) is canceled by the pole (s+5). Hence, the poles of the compensated system are at s=-3, -4, and -5. The transfer function $t_c(s)$ does not accurately reflect the value of the time initial condition response which must include terms arising from the pole at s=-5. The pole placement linear controller design for the planar manipulator is discussed in the next chapter.

IV. LINEAR CONTROLLER DESIGN FOR THE PLANAR MANIPULATOR

IV.1 LINEAR CONTROLLER DESIGN

The open loop transfer matrix of the planar manipulator is derived in Appendix C. It is shown that the open loop transfer matrix of the system can be expressed as $T(s)=R(s)P^{-1}(s)$ where R(s) and P(s) are polynomial matrices given by the following

$$R(s) = -k_v\begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad ; \quad P(s) = \begin{bmatrix} s^3-k_d as^2-as & -(k_d bs^2+bs) \\ -(k_d bs^2+bs) & s^3-k_d cs^2-cs \end{bmatrix} \qquad \text{IV.1.1}$$

Later, when the eliminant matrix of the two polynomial matrices R(s) and P(s) is derived, it will be shown that R(s) and P(s) are relatively right prime polynomial matrices and therefore the correct form of the planar manipulator open loop transfer matrix has been obtained. Consequently the compensation scheme depicted in Figure I.2, can be applied to derive the linear controller. Derivation of the linear controller is discussed now.

To derive the linear controller for the planar manipulator, one can select any desired closed loop transfer matrix which satisfies the three conditions stated in the Section III.4. It was shown that for the compensation scheme of Figure I.2, the closed loop transfer matrix is, $T_c(s)=R(s)[G^{-1}P_F(s)]^{-1}$. This implies that the closed loop poles of the compensated system are given by the determinant of $G^{-1}P_F(s)$. In particular $G^{-1}P_F(s)$ can be chosen as *any arbitrary polynomial of degree six ($n=6$)*, such that the six closed loop poles of the system are placed at $s=p_{d1}$, p_{d2}, p_{d3}, p_{d4}, p_{d5}, and p_{d6}.

It is desirable to have a diagonal (decoupled) closed loop transfer matrix. In Appendix D, it is shown that the following choice of $G^{-1}P_F(s)$

$$G^{-1}P_F(s) = -k_v \begin{bmatrix} \dfrac{aw_1(s)}{c_1} & \dfrac{bw_1(s)}{c_1} \\ \dfrac{bw_2(s)}{c_2} & \dfrac{cw_2(s)}{c_2} \end{bmatrix} \qquad \text{IV.1.2}$$

where

$$\begin{aligned} w_1(s) &= (s-p_{d1})(s-p_{d2})(s-p_{d3}) \quad ; \quad c_1 = -p_{d1}p_{d2}p_{d3} \\ w_2(s) &= (s-p_{d4})(s-p_{d5})(s-p_{d6}) \quad ; \quad c_2 = -p_{d4}p_{d5}p_{d6} \end{aligned} \qquad \text{IV.1.3}$$

not only satisfies the three conditions, but also results in a diagonal (decoupled) closed loop transfer matrix of the form

$$T_c(s) = \begin{bmatrix} \frac{c_1}{w_1(s)} & 0 \\ 0 & \frac{c_2}{w_2(s)} \end{bmatrix} \qquad \textbf{IV.1.4}$$

To find G, and F(s) such that the closed loop transfer matrix of Equation IV.1.4 is achieved, notice that from Equation III.2.13 and Equation III.3.5 the following can be established

$$\Gamma_c[P_F(s)] = \Gamma_c[P(S)] \qquad \textbf{IV.1.5}$$

Consequently

$$\Gamma_c[G^{-1}P_F(s)] = G^{-1}(\Gamma_c[P_F(s)]) = G^{-1}\Gamma_c[P(s)] \qquad \textbf{IV.1.6}$$

and therefore G is given by

$$G = \Gamma_c[P(s)](\Gamma_c[G^{-1}P_F(s)])^{-1} \qquad \textbf{IV.1.7}$$

From Equation IV.1.1 it follows that

$$\Gamma_c[P(s)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \textbf{IV.1.8}$$

and from Equation IV.1.2 and Equation IV.1.3:

$$(\Gamma_c[G^{-1}P_F(s)])^{-1} = \left(-k_v \begin{bmatrix} \frac{a}{c_1} & \frac{b}{c_1} \\ \frac{b}{c_2} & \frac{c}{c_2} \end{bmatrix} \right)^{-1} \qquad \textbf{IV.1.9}$$

therefore

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{1}{k_v \gamma_1} \begin{bmatrix} -cc_1 & bc_2 \\ bc_1 & -ac_2 \end{bmatrix} \qquad \textbf{IV.1.10}$$

where $\gamma_1 = ac-b^2$ (it can be shown that $\gamma_1 = ac-b^2 = (k_T H_1)^2(a_1 k_3 - a_2^2) > 0$; see Equation II.2.6). At this point, only the general procedure for determining H(s), K(s), and Q(s) is discussed. More details are given in Appendix E.

To derive $P_F(s)$, matrix G given by Equation IV.1.10 is premultiplied by $G^{-1}P_F(s)$ as given by Equation IV.1.2. Once $P_F(s)$ has been derived, Equation III.3.5 together with Equation IV.1.1, are employed to obtain an expression for F(s). For the system under the consideration it is shown that $\nu = 3$, and therefore employing Equation III.4.13 together with Equation III.4.15 yield the following expression for Q(S)

$$Q(s) = \begin{bmatrix} s^2 & q_1 s + q_0 \\ -1 & s^2 + q_3 s + q_2 \end{bmatrix} \qquad \textbf{IV.1.11}$$

where q_0, q_1, q_2, and q_3 are arbitrary real constants to be chosen such that the roots of the $|Q(s)| = s^4 + q_3 s^3 + q_2 s^2 + q_1 s + q_0$ remain in the left half s plane. At this point, since Q(s)F(s) is known, corollary III.4.1 can be employed to find the constant matrix β. In particular, since for the system under study, $S_e(s)$ is given by the following

$$S_e(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ s^3 & 0 \\ s^4 & 0 \\ s^5 & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \\ 0 & s^3 \\ 0 & s^4 \\ 0 & s^5 \end{bmatrix} \qquad \textbf{IV.1.12}$$

then by setting the arbitrary polynomial matrix $\beta(s)=\beta S_e(s)$ equal to Q(s)F(s), an expression for the constant matrix β is found. For the planar manipulator under the consideration, the eliminant matrix of R(s) and P(s) is a nonsingular 12×12 square matrix. Therefore $\hat{M}_e = M_e$ and hence $[H,K]=[\hat{K},\hat{H}]$ (see Appendix E). Equation III.4.11 and Equation III.4.12 can now be employed to derive the expressions for H(s) and K(s). The final expressions for H(s), and K(s) are as following

$$H(s) = \begin{bmatrix} \mu_{15}s^2+\mu_{13}s+\mu_{11} & \mu_{16}s^2+\mu_{14}s+\mu_{12} \\ \mu_{25}s^2+\mu_{23}s+\mu_{21} & \mu_{26}s^2+\mu_{24}s+\mu_{22} \end{bmatrix} \triangleq \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \qquad \textbf{IV.1.13}$$

and

$$K(s) = \begin{bmatrix} \mu_{19}s+\mu_{17} & \mu_{1,10}s+\mu_{18} \\ \mu_{29}s+\mu_{27} & \mu_{2,10}s+\mu_{28} \end{bmatrix} \triangleq \begin{bmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{bmatrix} \qquad \text{IV.1.14}$$

where the constant coefficients μ's are given in Appendix E.

If the polynomial matrices H(s), K(s), and Q(s) given by Equations IV.1.13, IV.1.14, and IV.1.11 together with G given by Equation IV.1.10 are now employed in the feedback scheme depicted in Figure I.2, then the desired (decoupled) closed loop transfer matrix given by Equation IV.1.4 is achieved. For the simulation purposes Figure I.2 must be rearranged to its more suitable form. This is discussed in the next section.

IV.2 BLOCK DIAGRAM FOR SIMULATION PURPOSES

For the planar manipulator there are two inputs, u_1 and u_2, and two outputs, y_1 and y_2. For the controller presented here, two command inputs v_1 and v_2 are used (see Figure I.2). Thus:

$$V(s) = \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix}; \quad U(s) = \begin{bmatrix} u_1(S) \\ u_2(s) \end{bmatrix} \qquad \text{IV.2.1}$$

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \qquad \text{IV.2.2}$$

$$K(s) = \begin{bmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{bmatrix}$$

$$H(s) = \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \qquad \text{IV.2.3}$$

$$Q^{-1}(s) = \begin{bmatrix} \bar{q}_{11}(s) & \bar{q}_{12}(s) \\ \bar{q}_{21}(s) & \bar{q}_{22}(s) \end{bmatrix}$$

where for all $i,j \epsilon (1,2)$ $k_{ij}(s)$ and $h_{ij}(s)$ are polynomials given in Equations IV.1.13 and IV.1.14, and $\bar{q}_{ij}(s)$ are rational polynomials in s given by the following

$$\bar{q}_{11}(s) = \frac{s^2+q_3s+q_2}{\Delta Q(s)} \qquad \bar{q}_{12}(s) = \frac{-(q_1s+q_0)}{\Delta Q(s)}$$

$$\bar{q}_{21}(s) = \frac{1}{\Delta Q(s)} \qquad \bar{q}_{22}(s) = \frac{s^2}{\Delta Q(s)} \qquad \text{IV.2.4}$$

$$\Delta Q(s) = s^4+q_3s^3+q_2s^2+q_1s+q_0$$

Consequently

$$U(s) = GV(s)+Q^{-1}(s)K(s)U(s)+Q^{-1}(s)H(s)Y(s) \qquad \text{IV.2.5}$$

or

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} + \begin{bmatrix} \bar{q}_{11}(s) & \bar{q}_{12}(s) \\ \bar{q}_{21}(s) & \bar{q}_{22}(s) \end{bmatrix} \begin{bmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} + \\ \begin{bmatrix} \bar{q}_{11}(s) & \bar{q}_{12}(s) \\ \bar{q}_{21}(s) & \bar{q}_{22}(s) \end{bmatrix} \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} \qquad \textbf{IV.2.6}$$

Expanding Equation IV.2.6 gives

$$\begin{aligned} u_1(s) = {} & g_{11}v_1(s) + g_{12}v_2(s) + \\ & \left(\bar{q}_{11}(s)k_{11}(s) + \bar{q}_{12}(s)k_{21}(s)\right)u_1(s) + \\ & \left(\bar{q}_{11}(s)k_{12}(s) + \bar{q}_{12}(s)k_{22}(s)\right)u_2(s) + \\ & \left(\bar{q}_{11}(s)h_{11}(s) + \bar{q}_{12}(s)h_{21}(s)\right)y_1(s) + \\ & \left(\bar{q}_{11}(s)h_{12}(s) + \bar{q}_{12}(s)h_{22}(s)\right)y_2(s) \end{aligned} \qquad \textbf{IV.2.7}$$

and

$$\begin{aligned} u_2(s) = {} & g_{21}v_1(s) + g_{22}v_2(s) + \\ & \left(\bar{q}_{21}(s)k_{11}(s) + \bar{q}_{22}(s)k_{21}(s)\right)u_1(s) + \\ & \left(\bar{q}_{21}(s)k_{12}(s) + \bar{q}_{22}(s)k_{22}(s)\right)u_2(s) + \\ & \left(\bar{q}_{21}(s)h_{11}(s) + \bar{q}_{22}(s)h_{21}(s)\right)y_1(s) + \\ & \left(\bar{q}_{21}(s)h_{12}(s) + \bar{q}_{22}(s)h_{22}(s)\right)y_2(s) \end{aligned} \qquad \textbf{IV.2.8}$$

Figure IV.1 shows the block diagram for the complete control system when Equations IV.2.7 and IV.2.8 are employed to determine inputs to the planar manipulator using the rational polynomials defined in Equation IV.2.9.

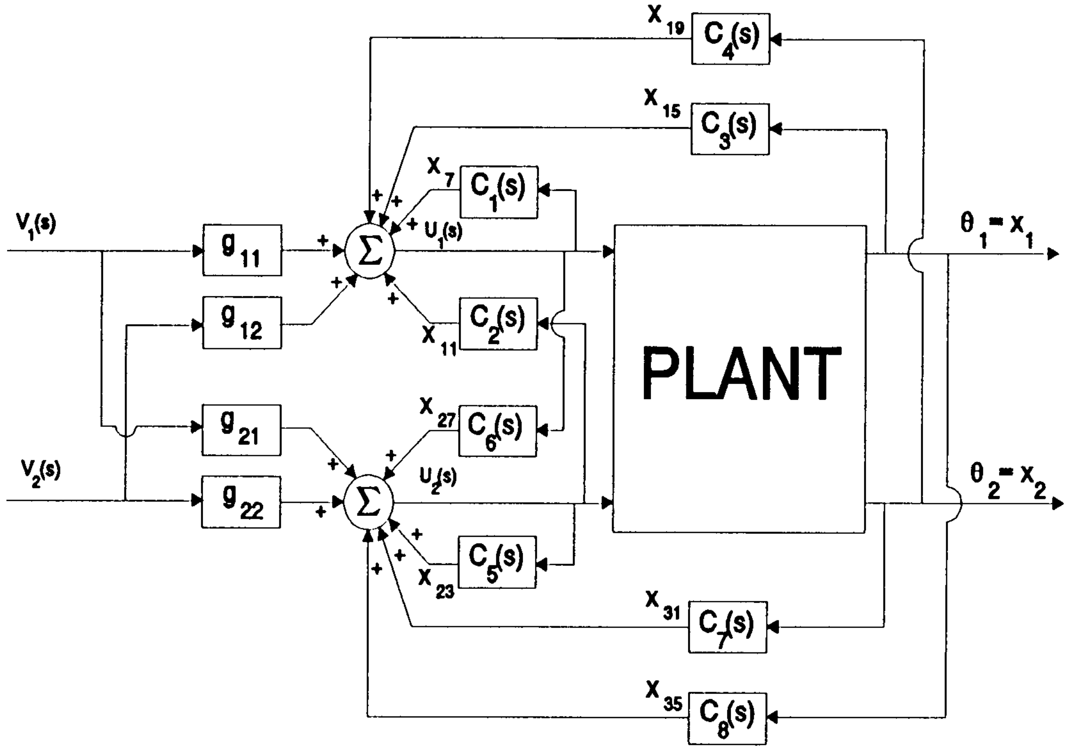


Figure IV.1: Planar Manipulator Closed Loop Compensation Scheme.

$$
\begin{aligned}
C_1(s) &\triangleq \bar{q}_{11}(s)k_{11}(s)+\bar{q}_{12}(s)k_{21}(s)\\
C_2(s) &\triangleq \bar{q}_{11}(s)k_{12}(s)+\bar{q}_{12}(s)k_{22}(s)\\
C_3(s) &\triangleq \bar{q}_{11}(s)h_{11}(s)+\bar{q}_{12}(s)h_{21}(s)\\
C_4(s) &\triangleq \bar{q}_{11}(s)h_{12}(s)+\bar{q}_{12}(s)h_{22}(s)\\
C_5(s) &\triangleq \bar{q}_{21}(s)k_{12}(s)+\bar{q}_{22}(s)k_{22}(s)\\
C_6(s) &\triangleq \bar{q}_{21}(s)k_{11}(s)+\bar{q}_{22}(s)k_{21}(s)\\
C_7(s) &\triangleq \bar{q}_{21}(s)h_{12}(s)+\bar{q}_{22}(s)h_{22}(s)\\
C_8(s) &\triangleq \bar{q}_{21}(s)h_{11}(s)+\bar{q}_{22}(s)h_{21}(s)
\end{aligned}
\qquad \text{IV.2.9}
$$

From Equations IV.1.13, IV.1.14, IV.2.3, IV.2.4, and IV.2.9 the following expressions for $C_j(s)$ rational polynomials are obtained

$$
\begin{aligned}
C_1(s) &= \frac{1}{\Delta Q(s)}(\sigma_3 s^3+\sigma_2 s^2+\sigma_1 s+\sigma_0)\\
C_2(s) &= \frac{1}{\Delta Q(s)}(\sigma_7 s^3+\sigma_6 s^2+\sigma_5 s+\sigma_4)\\
C_3(s) &= \frac{1}{\Delta Q(s)}(\sigma_{12} s^4+\sigma_{11} s^3+\sigma_{10} s^2+\sigma_9 s+\sigma_8)\\
C_4(s) &= \frac{1}{\Delta Q(s)}(\sigma_{17} s^4+\sigma_{16} s^3+\sigma_{15} s^2+\sigma_{14} s+\sigma_{13})\\
C_5(s) &= \frac{1}{\Delta Q(s)}(\sigma_{21} s^3+\sigma_{20} s^2+\sigma_{19} s+\sigma_{18})\\
C_6(s) &= \frac{1}{\Delta Q(s)}(\sigma_{25} s^3+\sigma_{24} s^2+\sigma_{23} s+\sigma_{22})\\
C_7(s) &= \frac{1}{\Delta Q(s)}(\sigma_{30} s^4+\sigma_{29} s^3+\sigma_{28} s^2+\sigma_{27} s+\sigma_{26})\\
C_8(s) &= \frac{1}{\Delta Q(s)}(\sigma_{35} s^4+\sigma_{34} s^3+\sigma_{33} s^2+\sigma_{32} s+\sigma_{31})
\end{aligned}
\qquad \textbf{IV.2.10}
$$

where σ's are constants given by the following

$$
\begin{aligned}
\sigma_0 &= q_2\mu_{17}-q_0\mu_{27} & \sigma_{18} &= \mu_{18} \\
\sigma_1 &= q_3\mu_{17}+q_2\mu_{19}-q_1\mu_{27}-q_0\mu_{29} & \sigma_{19} &= \mu_{1,10} \\
\sigma_2 &= q_3\mu_{19}-q_1\mu_{29}+\mu_{17} & \sigma_{20} &= \mu_{28} \\
\sigma_3 &= \mu_{19} & \sigma_{21} &= \mu_{2,10} \\
\sigma_4 &= q_2\mu_{18}-q_0\mu_{28} & \sigma_{22} &= \mu_{17} \\
\sigma_5 &= q_3\mu_{18}+q_2\mu_{1,10}-q_1\mu_{28}-q_0\mu_{2,10} & \sigma_{23} &= \mu_{19} \\
\sigma_6 &= q_3\mu_{1,10}-q_1\mu_{2,10}+\mu_{18} & \sigma_{24} &= \mu_{27} \\
\sigma_7 &= \mu_{1,10} & \sigma_{25} &= \mu_{29} \\
\sigma_8 &= q_2\mu_{11}-q_0\mu_{21} & \sigma_{26} &= \mu_{12} \\
\sigma_9 &= q_3\mu_{11}+q_2\mu_{13}-q_1\mu_{21}-q_0\mu_{23} & \sigma_{27} &= \mu_{14} \\
\sigma_{10} &= q_3\mu_{13}+q_2\mu_{15}-q_1\mu_{23}-q_0\mu_{25}+\mu_{11} & \sigma_{28} &= \mu_{16}+\mu_{22} \\
\sigma_{11} &= q_3\mu_{15}-q_1\mu_{25}+\mu_{13} & \sigma_{29} &= \mu_{24} \\
\sigma_{12} &= \mu_{15} & \sigma_{30} &= \mu_{26} \\
\sigma_{13} &= q_2\mu_{12}-q_0\mu_{22} & \sigma_{31} &= \mu_{11} \\
\sigma_{14} &= q_3\mu_{12}+q_2\mu_{14}-q_1\mu_{22}-q_0\mu_{24} & \sigma_{32} &= \mu_{13} \\
\sigma_{15} &= q_3\mu_{14}+q_2\mu_{16}-q_1\mu_{24}-q_0\mu_{26}+\mu_{12} & \sigma_{33} &= \mu_{21}+\mu_{15} \\
\sigma_{16} &= q_3\mu_{16}-q_1\mu_{26}+\mu_{14} & \sigma_{34} &= \mu_{23} \\
\sigma_{17} &= \mu_{16} & \sigma_{35} &= \mu_{25}
\end{aligned}
\qquad \text{IV.2.11}
$$

In Equation IV.2.11 μ's are constants given in Appendix E. At this point all the parameters of the compensators given in Figure IV.1 are known and the closed loop compensated system can be simulated. The simulation results are given in the next chapter.

V. SIMULATION RESULTS

A program was written to simulate the planar manipulator dynamics. The variable step size Runge-Kutta Fehlberg integration method was used. Inputs to the simulation program are:

1- Initial position of the links in degrees.

2- Magnitude of the command signals in degrees for different type of inputs.

3- q_i's for $i=0,1,2,3$ (see Section IV.1).

4- Location of desired closed loop poles ($p_{d1},p_{d2},\ldots,p_{d6}$).

Simulations for different values of closed loop poles and q_i's were performed. After considerable experimenting with the program, the following parameters were selected since the controlled system responded favorably:

$q_0 = 0.8250 \quad q_1 = 3.7750 \quad q_2 = 6.050 \quad q_3 = 4.10$

$p_{d1} = (-3.20, -.20) \qquad p_{d4} = (-3.00, -1.00)$
$p_{d2} = (-3.20, .20) \qquad p_{d5} = (-3.00, 1.00)$
$p_{d3} = (-900.00, .00) \qquad p_{d6} = (-800.00, .00)$

The particular values for q_i's were selected such that the poles of $|Q(s)|$ are placed at $s=-0.5,-1,-1.1,-1.5$ ($Q(s)$ is given by Equation III.4.13).

For the particular stepper motors in the laboratory the maximum torque for the linear region of operation is 0.63 N.m. The maximum external torque that can be applied at each joint is directly proportional (through the gear ratio constant=6.9) to the maximum torque of the stepper in the linear region. Hence, the magnitude of the maximum torque available at each joint is 4.3 N.m.

For the compensators parameters given in Table V.1 the simulation results around $\theta_{1o}=0^\circ$ and $\theta_{2o}=0^\circ$ are shown in Figure V.1 through Figure V.3. Figure V.1-(a) shows the response of the linearized model of the planar manipulator for command step changes of 10° to both joints; i.e. $\delta v_1=\delta v_2=10^\circ$. Figure V.1-(b) shows the corresponding response of the non-linear model. The error between the non-linear simulation and the linearized model simulation is less than 0.12°. Notice that the torques do not exceed 4.3 N.m. and hence the stepper motors do not slip. Figure V.2 shows the same simulations for joint command step changes of $\delta v_1=\delta v_2=30^\circ$. Note the difference between the linear and non-linear model responses. The linear model has no overshoot while the non-linear model demonstrates a peak overshoot of less than 0.5° for link 1 and a peak overshoot of about 2.5° for link 2. The settling time for the nonlinear case is about 7 seconds longer than the linear model. From Figure V.2-(c) notice that the error in this case is larger than the error in the previous case by about 3°. Also From Figure V.2-(d) notice that the maximum torque available (4.3 N.m) is exceeded. This implies that the stepper motor is operating out of region of linearity and therefore slippage will occur. To overcome the slippage problem one can apply ramp inputs instead of step inputs. This is discussed later. Figure V.3 shows the same simulations for joint command step changes of $\delta v_1 = \delta v_2 = 50^{\circ}$ for which the maximum error between the linear and non-linear model is about 17°. Note the very different response for the non-linear model. The linear region has been "exceeded". However, notice that the commanded angles are realized after about 9 seconds.

When the arms are initially at:

θ_{1o} = .00 (deg.) θ_{2o} = .00 (deg.)

If the following parameters are selected to compensate the system;

g_{11}	=	.5335	q_3	=	4.1000
g_{12}	=	.1486	q_2	=	6.0500
g_{21}	=	.1718	q_1	=	3.7750
g_{22}	=	.0689	q_0	=	.8250
σ_3	=	-1134.644	σ_7	=	709.052
σ_2	=	-10014.253	σ_6	=	3590.971
σ_1	=	-9904.339	σ_5	=	-29336.570
σ_0	=	-30396.398	σ_4	=	-15116.413
σ_{12}	=	32.260	σ_{17}	=	-20.924
σ_{11}	=	290.688	σ_{16}	=	-105.330
σ_{10}	=	285.371	σ_{15}	=	856.639
σ_9	=	886.519	σ_{14}	=	441.366
σ_8	=	-.440	σ_{13}	=	-.123
σ_{21}	=	-575.060	σ_{25}	=	-106.708
σ_{20}	=	7418.300	σ_{24}	=	-5432.814
σ_{19}	=	709.052	σ_{23}	=	-1134.644
σ_{18}	=	-1486.994	σ_{22}	=	-5765.036
σ_{30}	=	16.623	σ_{35}	=	2.493
σ_{29}	=	-217.176	σ_{34}	=	157.805
σ_{28}	=	-21.192	σ_{33}	=	31.754
σ_{27}	=	43.209	σ_{32}	=	167.831
σ_{26}	=	-.057	σ_{31}	=	-.142

then the closed loop poles of the linearized system are:

p_{d1}	=	(-3.20, -.20)	p_{d4}	=	(-3.00, -1.00)
p_{d2}	=	(-3.20, .20)	p_{d5}	=	(-3.00, 1.00)
p_{d3}	=	(-900.00, .00)	p_{d6}	=	(-800.00, .00)

Table V.1: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=0^\circ$ and $\theta_{2o}=0^\circ$.

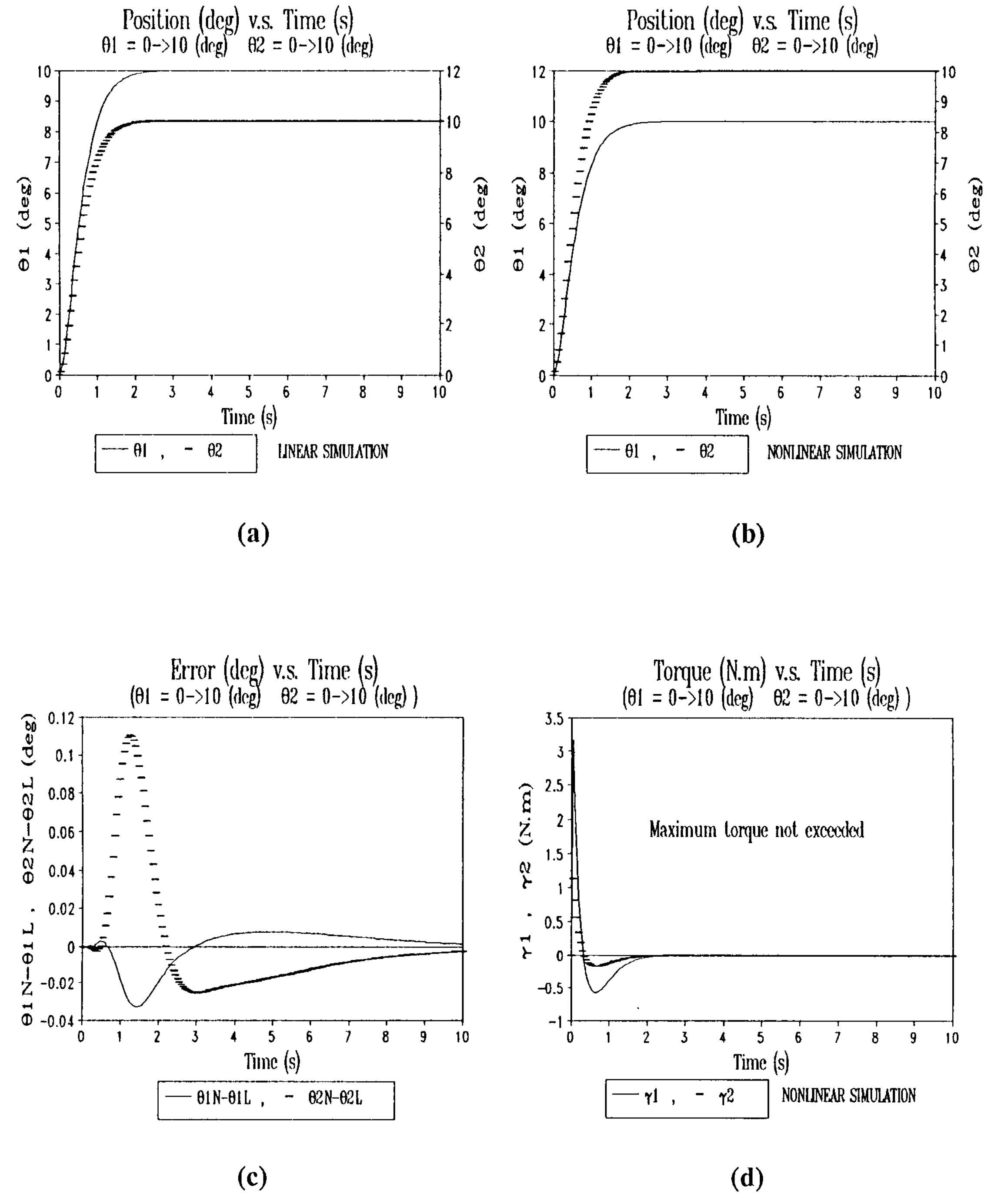


Figure V.1: Planar Manipulator Response: 10° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=0°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

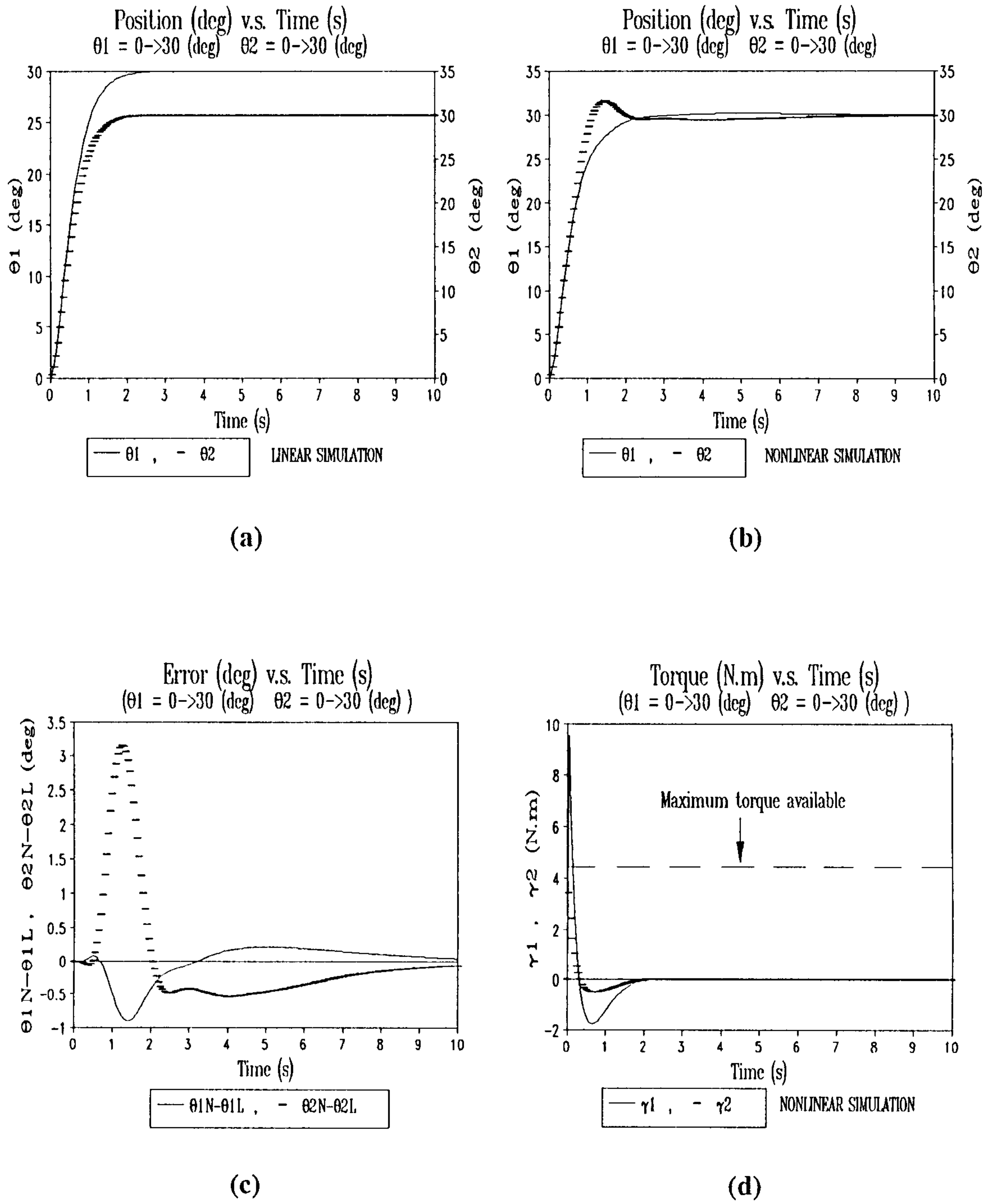


Figure V.2: Planar Manipulator Response: 30° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=0°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

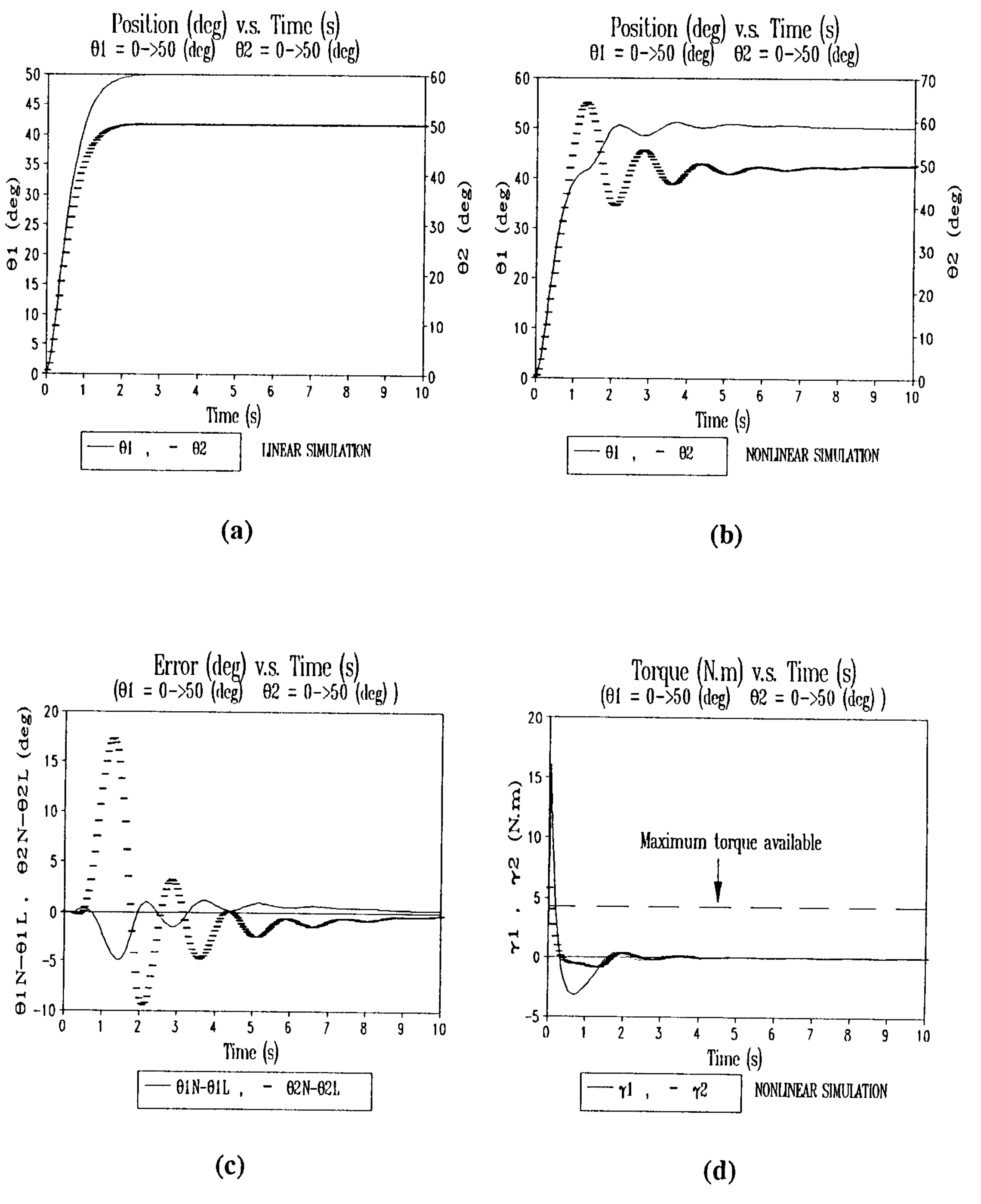


Figure V.3: Planar Manipulator Response: 50° Step Command Applied to Both Joints Around $\theta_{1o}=0^\circ$ and $\theta_{2o}=0^\circ$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

For the compensators parameters given in Table V.2 the simulation results around $\theta_{1o}=0^\circ$ and $\theta_{2o}=30^\circ$ are shown in Figure V.4 through Figure V.6. The overall closed loop poles of the system are the same as the previous case. Figure V.4-(a) shows the linear response for $\delta v_1=\delta v_2=10^\circ$. Figure V.4-(b) shows the corresponding non-linear response. The error between the non-linear simulation and the linearized model simulation is shown in Figure V.4-(c). The torque applied to each link is shown in Figure V.4-(d). Notice that the torques do not exceed 4.3 N.m. and hence the stepper motor will not slip. Comparison of Figure V.4-(d) with Figure V.1-(d) shows that when the initial position of link 2 is $\theta_{2o}=30^\circ$, less torque is required for the same amount of angular movement. Figure V.5 shows the same simulations for $\delta v_1=\delta v_2=30^\circ$. Notice that the non-linear response shown in (b) is not as nice as the previous case. From Figure V.5-(c) notice that the error in this case is larger than the error in the previous case. Also from Figure V.5-(d) notice that the maximum torque available is exceeded. Figure V.6 shows the same simulations for $\delta v_1=\delta v_2=50^\circ$.

When the arms are initially at:

θ_{1o} = .00 (deg.) θ_{2o} = 30.00 (deg.)

If the following parameters are selected to compensate the system;

g_{11}	=	25.4405	q_3	=	4.1000
g_{12}	=	6.8944	q_2	=	6.0500
g_{21}	=	7.9734	q_1	=	3.7750
g_{22}	=	3.4430	q_0	=	.8250
σ_3	=	-1075.292	σ_7	=	539.240
σ_2	=	-9721.232	σ_6	=	2831.178
σ_1	=	-13026.752	σ_5	=	-20468.911
σ_0	=	-30476.822	σ_4	=	-14456.266
σ_{12}	=	1528.943	σ_{17}	=	-797.291
σ_{11}	=	14111.740	σ_{16}	=	-4156.325
σ_{10}	=	18843.999	σ_{15}	=	29871.829
σ_9	=	44448.836	σ_{14}	=	21105.446
σ_8	=	-20.988	σ_{13}	=	-5.688
σ_{21}	=	-634.885	σ_{25}	=	-85.096
σ_{20}	=	4495.861	σ_{24}	=	-4372.722
σ_{19}	=	539.240	σ_{23}	=	-1075.292
σ_{18}	=	-1776.393	σ_{22}	=	-5633.772
σ_{30}	=	919.470	σ_{35}	=	95.161
σ_{29}	=	-6586.547	σ_{34}	=	6344.497
σ_{28}	=	-811.226	σ_{33}	=	1506.145
σ_{27}	=	2583.567	σ_{32}	=	8202.306
σ_{26}	=	-2.840	σ_{31}	=	-6.578

then the closed loop poles of the linearized system are:

p_{d1}	=	(-3.20, -.20)	p_{d4}	=	(-3.00, -1.00)
p_{d2}	=	(-3.20, .20)	p_{d5}	=	(-3.00, 1.00)
p_{d3}	=	(-900.00, .00)	p_{d6}	=	(-800.00, .00)

Table V.2: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=0^\circ$ and $\theta_{2o}=30^\circ$.

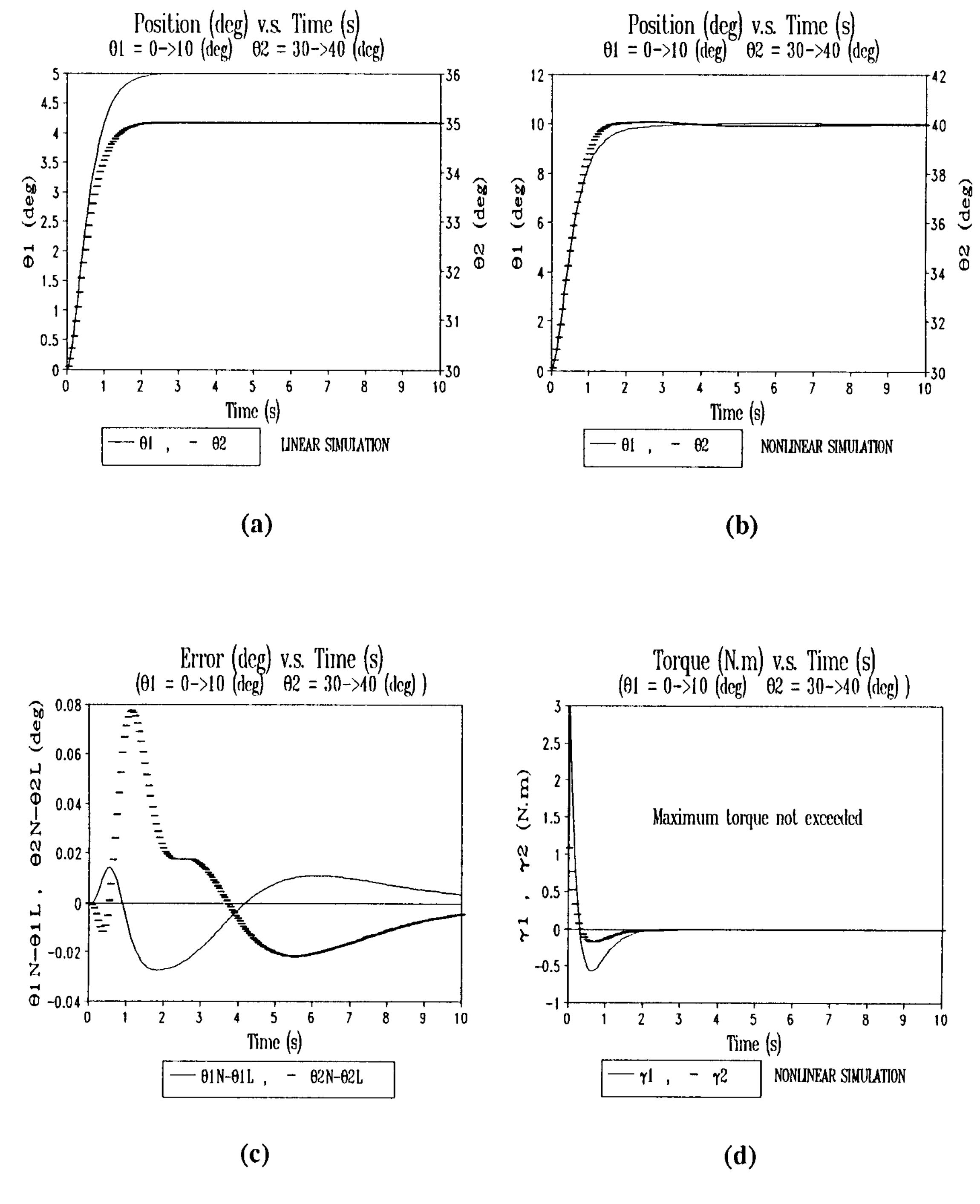


Figure V.4: Planar Manipulator Response: 10° Step Command Applied to Both Joints Around $\theta_{1o}=0^\circ$ and $\theta_{2o}=30^\circ$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

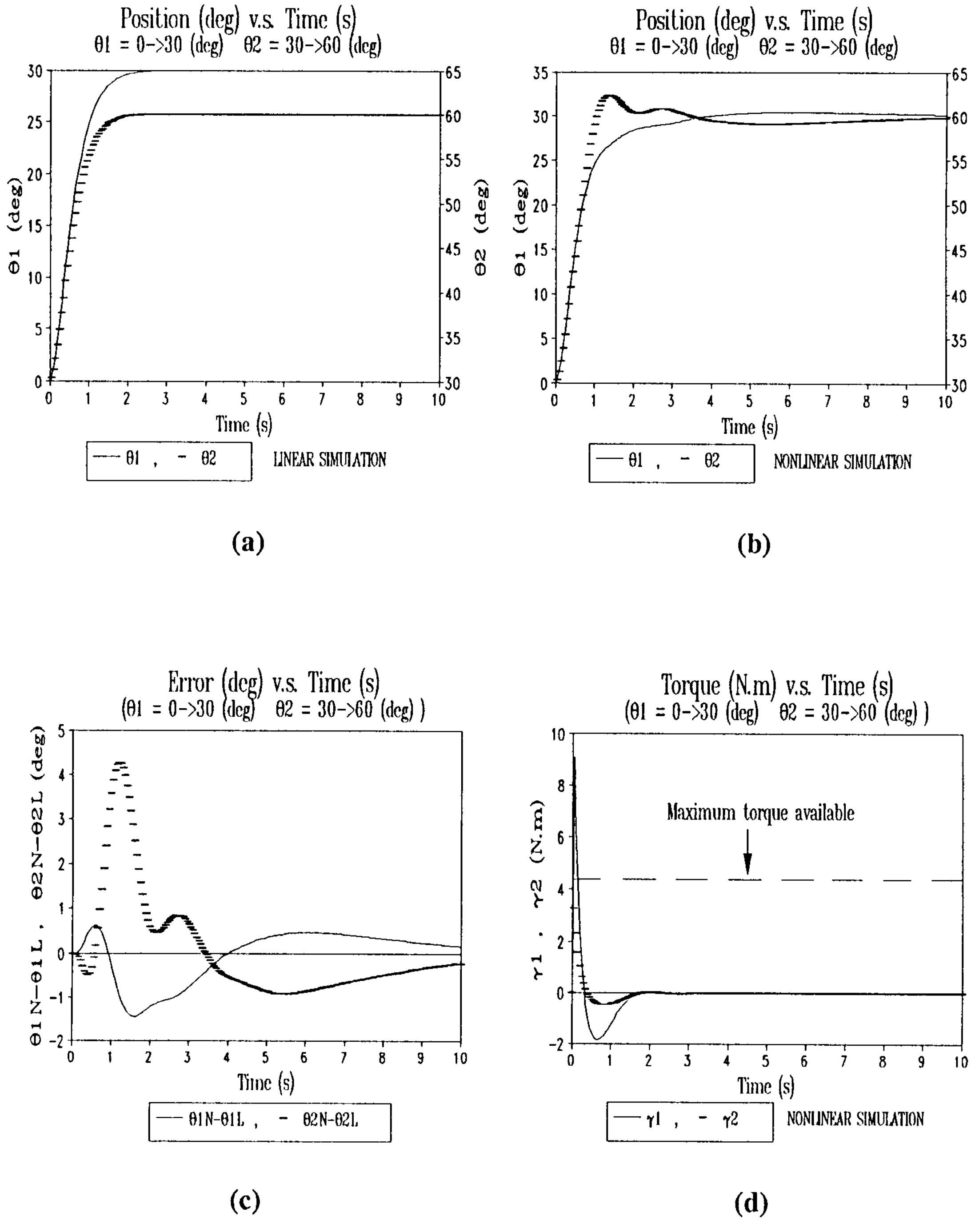


Figure V.5: Planar Manipulator Response: 30° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=30°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

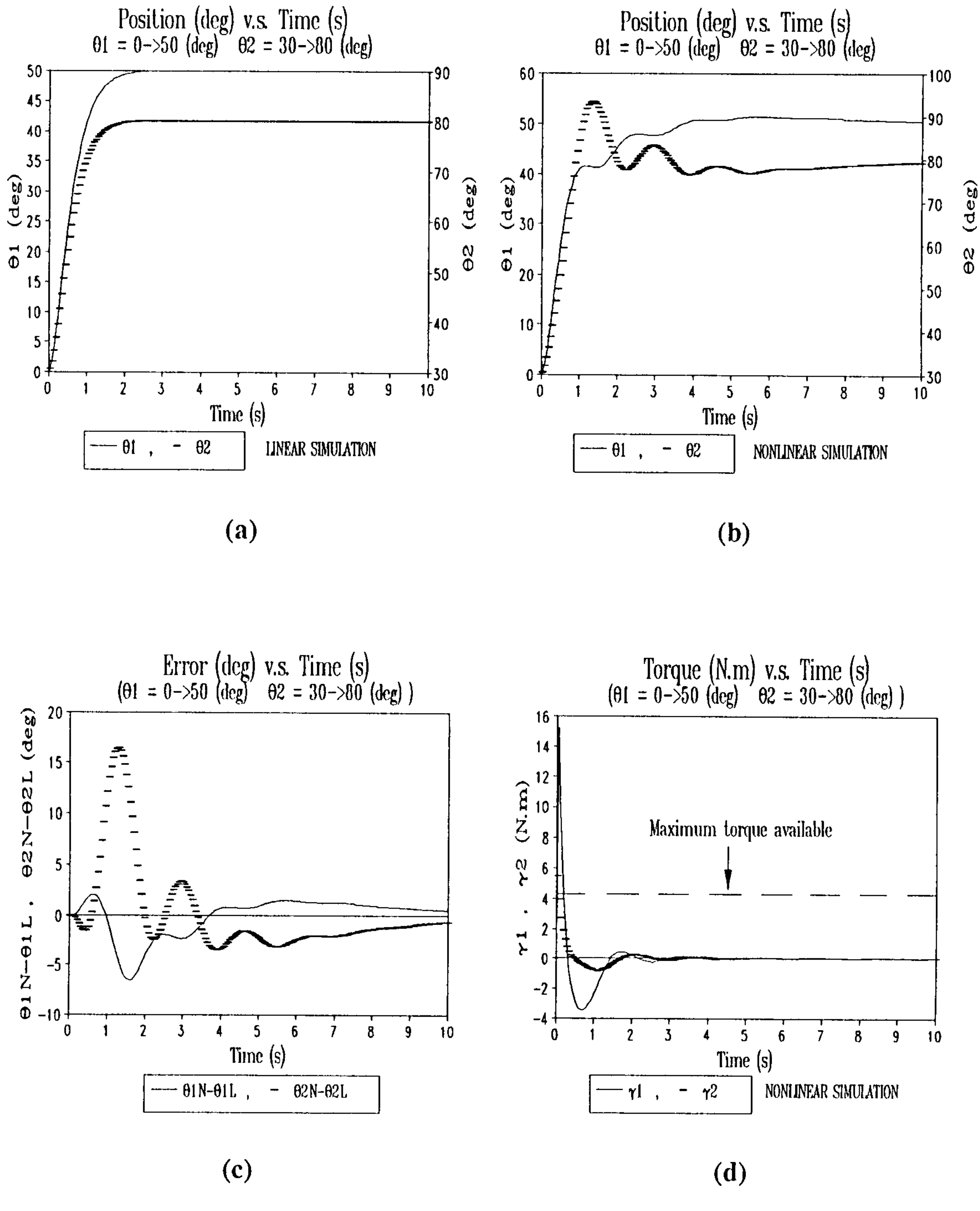


Figure V.6: Planar Manipulator Response: 50° Step Command Applied to Both Joints Around $\theta_{1o}=0^\circ$ and $\theta_{2o}=30^\circ$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

Figure V.7 through Figure V.14 are simulation results for different joint command inputs, δv's, and different initial θ_2's. The explanations are similar to the previous cases. In all of these simulations, the closed loop poles of the system are kept the same. One important observation from the graphs is that as the initial θ_2 gets closer to 180° or -180°, less torque is required for the angular movements. This is due to the fact that the effective inertia "seen" at joint 1 gets smaller as the arm "bends" towards that joint.

When the arms are initially at:

θ_{1o} = .00 (deg.) θ_{2o} = 90.00 (deg.)

If the following parameters are selected to compensate the system;

g_{11}	=	17.4574	q_3	=	4.1000
g_{12}	=	3.4430	q_2	=	6.0500
g_{21}	=	3.9818	q_1	=	3.7750
g_{22}	=	3.4430	q_0	=	.8250
σ_3	=	-935.855	σ_7	=	129.855
σ_2	=	-8661.861	σ_6	=	454.310
σ_1	=	-19963.543	σ_5	=	-1111.696
σ_0	=	-28579.399	σ_4	=	-16043.449
σ_{12}	=	1342.577	σ_{17}	=	-194.946
σ_{11}	=	12595.464	σ_{16}	=	-672.599
σ_{10}	=	29052.660	σ_{15}	=	1601.298
σ_9	=	41709.069	σ_{14}	=	23440.629
σ_8	=	-14.402	σ_{13}	=	-2.840
σ_{21}	=	-775.406	σ_{25}	=	-29.878
σ_{20}	=	-2591.923	σ_{24}	=	-1567.703
σ_{19}	=	129.855	σ_{23}	=	-935.855
σ_{18}	=	-3005.254	σ_{22}	=	-4937.645
σ_{30}	=	1126.747	σ_{35}	=	29.351
σ_{29}	=	3773.636	σ_{34}	=	2270.039
σ_{28}	=	-212.333	σ_{33}	=	1335.944
σ_{27}	=	4380.147	σ_{32}	=	7201.699
σ_{26}	=	-2.840	σ_{31}	=	-3.285

then the closed loop poles of the linearized system are:

p_{d1}	=	(-3.20, -.20)	p_{d4}	=	(-3.00, -1.00)
p_{d2}	=	(-3.20, .20)	p_{d5}	=	(-3.00, 1.00)
p_{d3}	=	(-900.00, .00)	p_{d6}	=	(-800.00, .00)

Table V.3: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=0^\circ$ and $\theta_{2o}=90^\circ$.

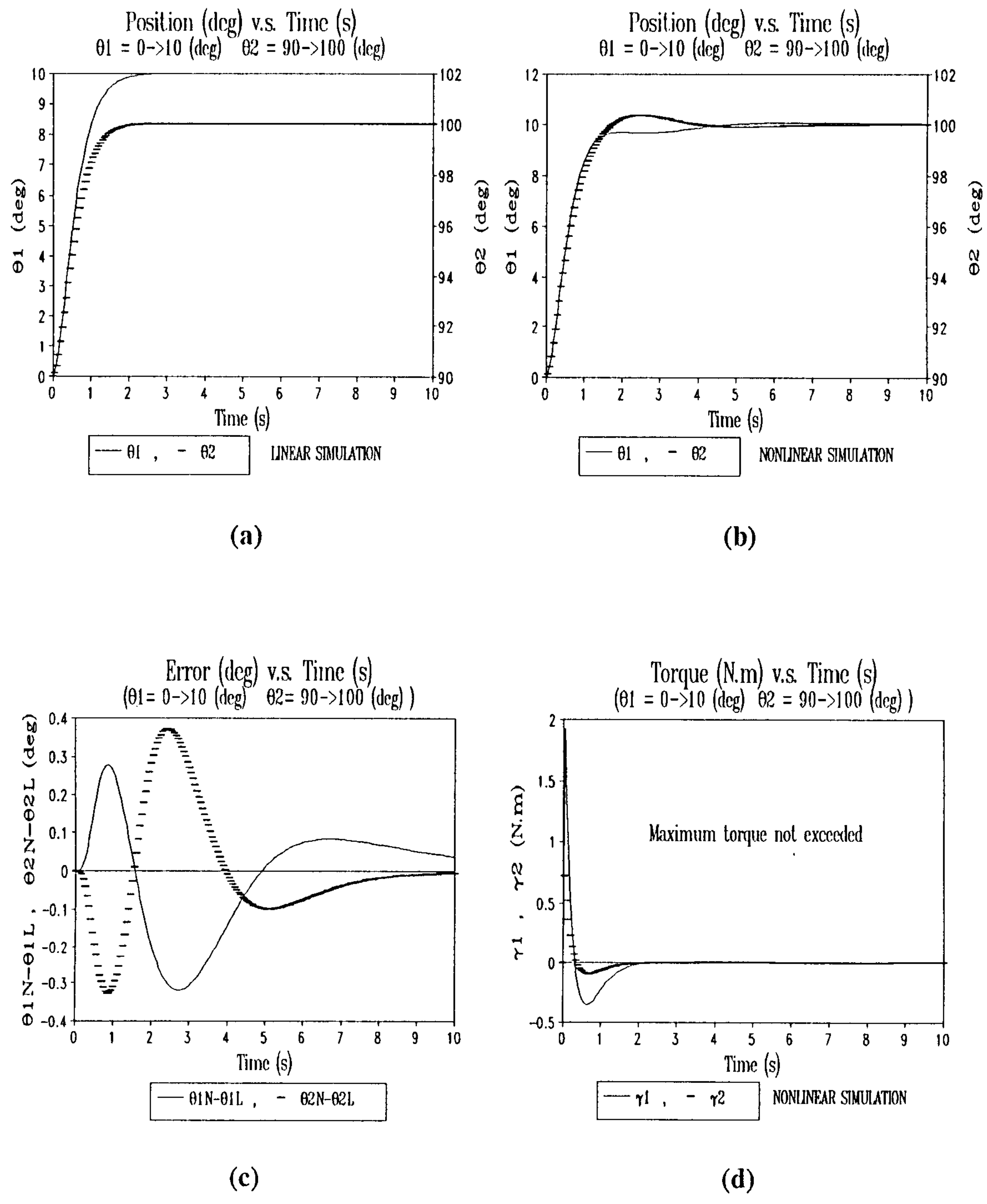


Figure V.7: Planar Manipulator Response: 10° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=90°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

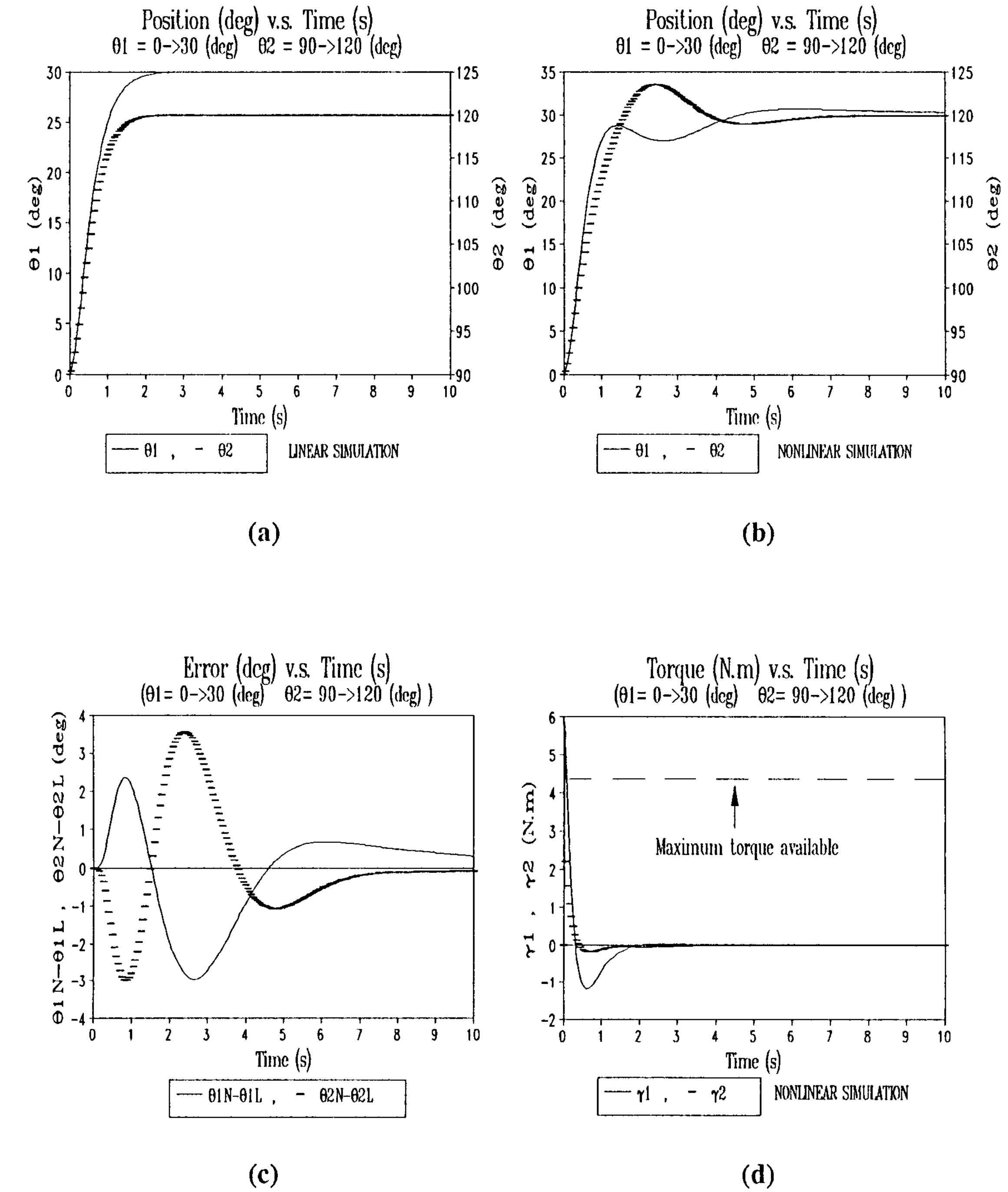


Figure V.8: Planar Manipulator Response: 30° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=90°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

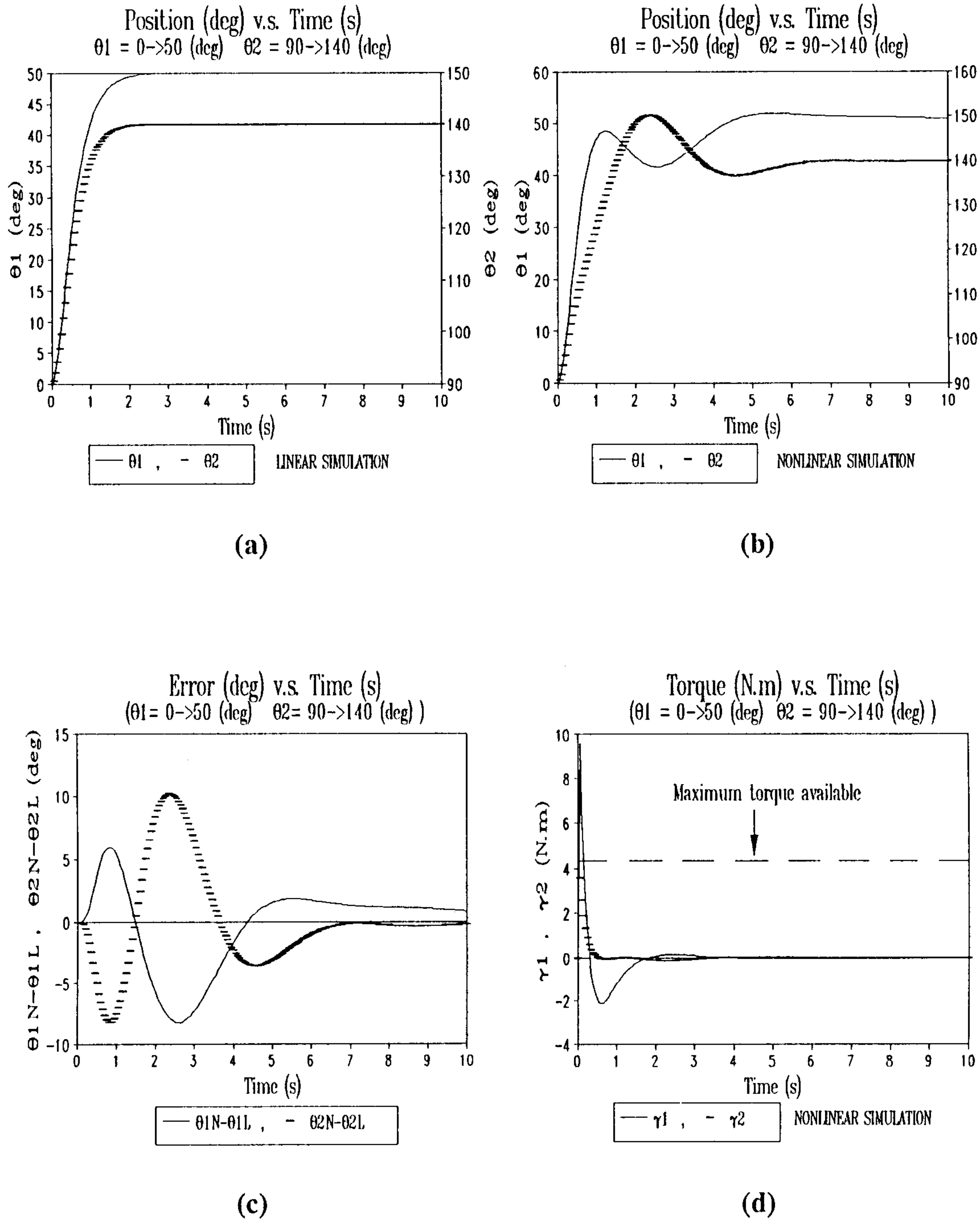


Figure V.9: Planar Manipulator Response: 50° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=90°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

When the arms are initially at:

θ_{1o} = .00 (deg.) θ_{2o} = 135.00 (deg.)

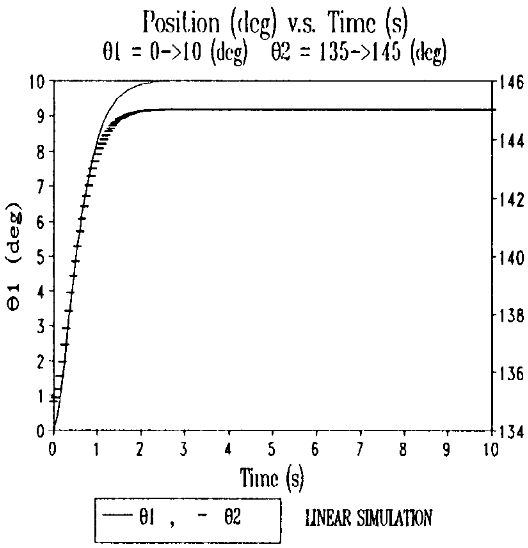
If the following parameters are selected to compensate the system;

g_{11} =	10.9393	q_3 =	4.1000
g_{12} =	.6250	q_2 =	6.0500
g_{21} =	.7228	q_1 =	3.7750
g_{22} =	3.4430	q_0 =	.8250
σ_3 =	-907.355	σ_7 =	18.398
σ_2 =	-8010.432	σ_6 =	-32.078
σ_1 =	-21835.497	σ_5 =	3270.428
σ_0 =	-25817.058	σ_4 =	-15652.593
σ_{12} =	1315.173	σ_{17} =	-29.136
σ_{11} =	11668.306	σ_{16} =	49.570
σ_{10} =	31843.892	σ_{15} =	-4781.757
σ_9 =	37698.519	σ_{14} =	22881.239
σ_8 =	-9.025	σ_{13} =	-.516
σ_{21} =	-804.057	σ_{25} =	-6.762
σ_{20} =	-4074.530	σ_{24} =	-355.818
σ_{19} =	18.398	σ_{23} =	-907.355
σ_{18} =	-3142.823	σ_{22} =	-4315.803
σ_{30} =	1168.807	σ_{35} =	8.068
σ_{29} =	5939.495	σ_{34} =	519.872
σ_{28} =	-49.341	σ_{33} =	1321.739
σ_{27} =	4581.272	σ_{32} =	6306.553
σ_{26} =	-2.840	σ_{31} =	-.596

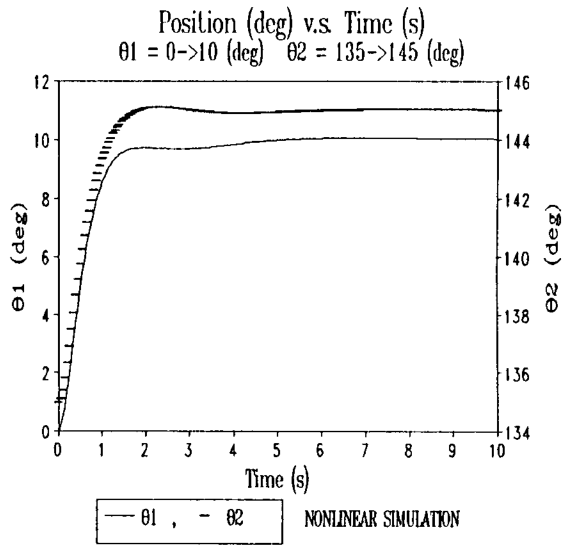
then the closed loop poles of the linearized system are:

p_{d1} =	(-3.20, -.20)	p_{d4} =	(-3.00, -1.00)
p_{d2} =	(-3.20, .20)	p_{d5} =	(-3.00, 1.00)
p_{d3} =	(-900.00, .00)	p_{d6} =	(-800.00, .00)

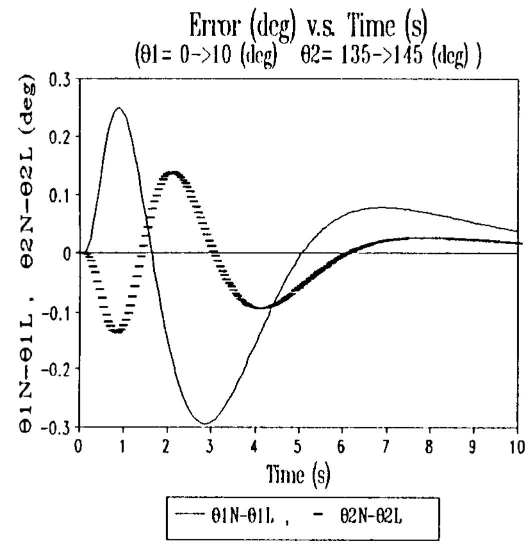
Table V.4: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=0°$ and $\theta_{2o}=135°$.



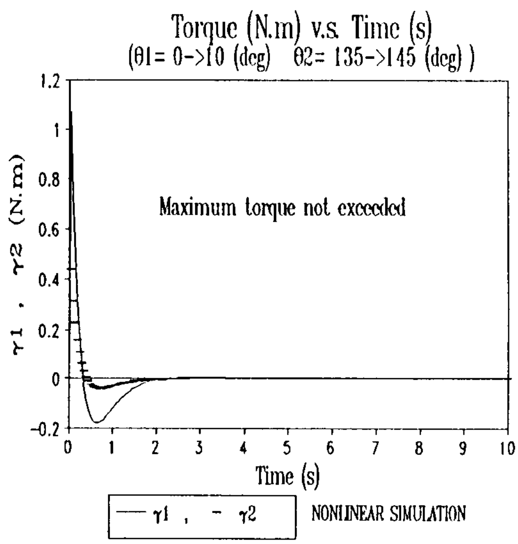
(a)



(b)



(c)



(d)

Figure V.10: Planar Manipulator Response: 10° Step Command Applied to Both Joints Around $\theta_{1o}=0^\circ$ and $\theta_{2o}=135^\circ$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

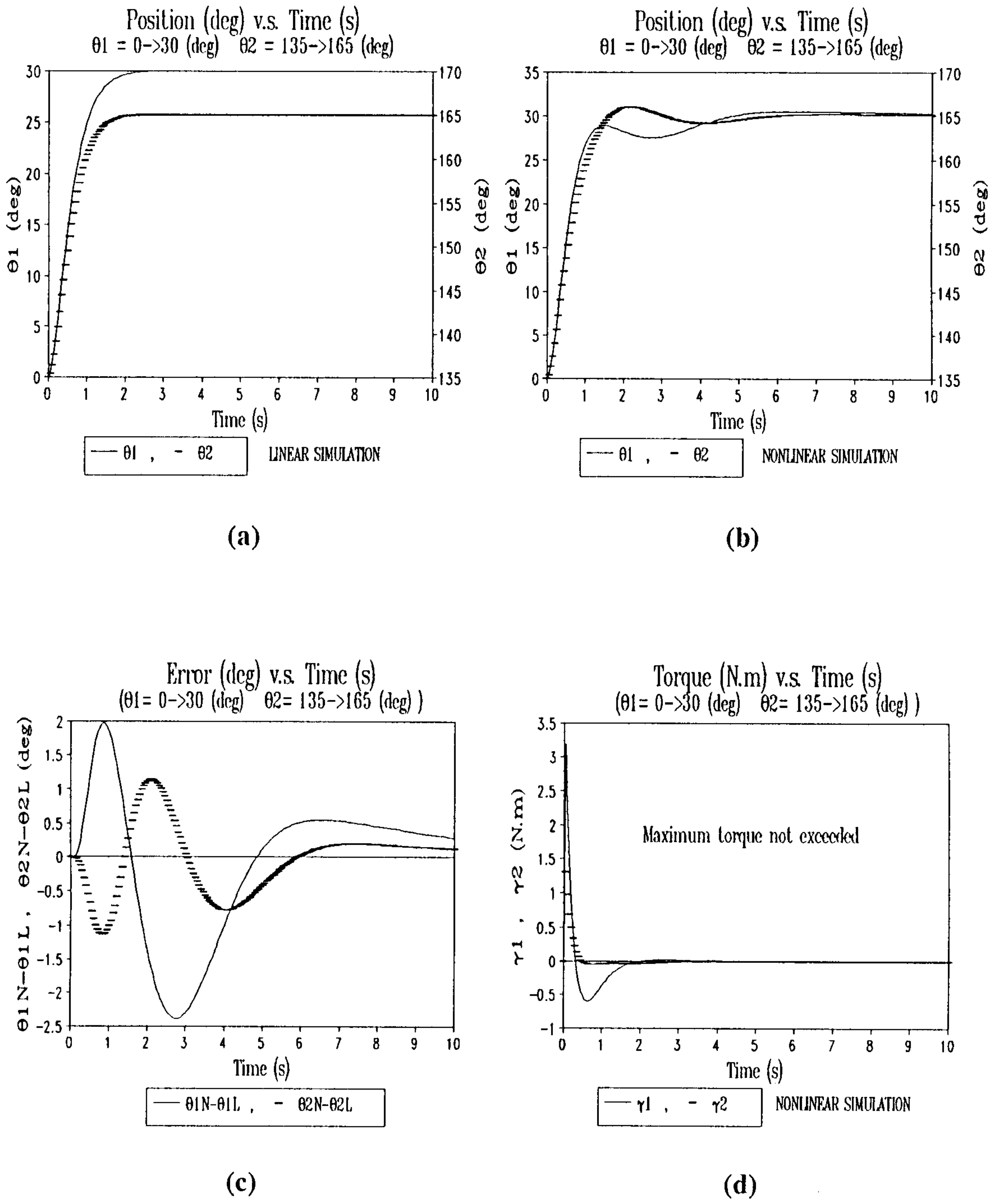


Figure V.11: Planar Manipulator Response: 30° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=135°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

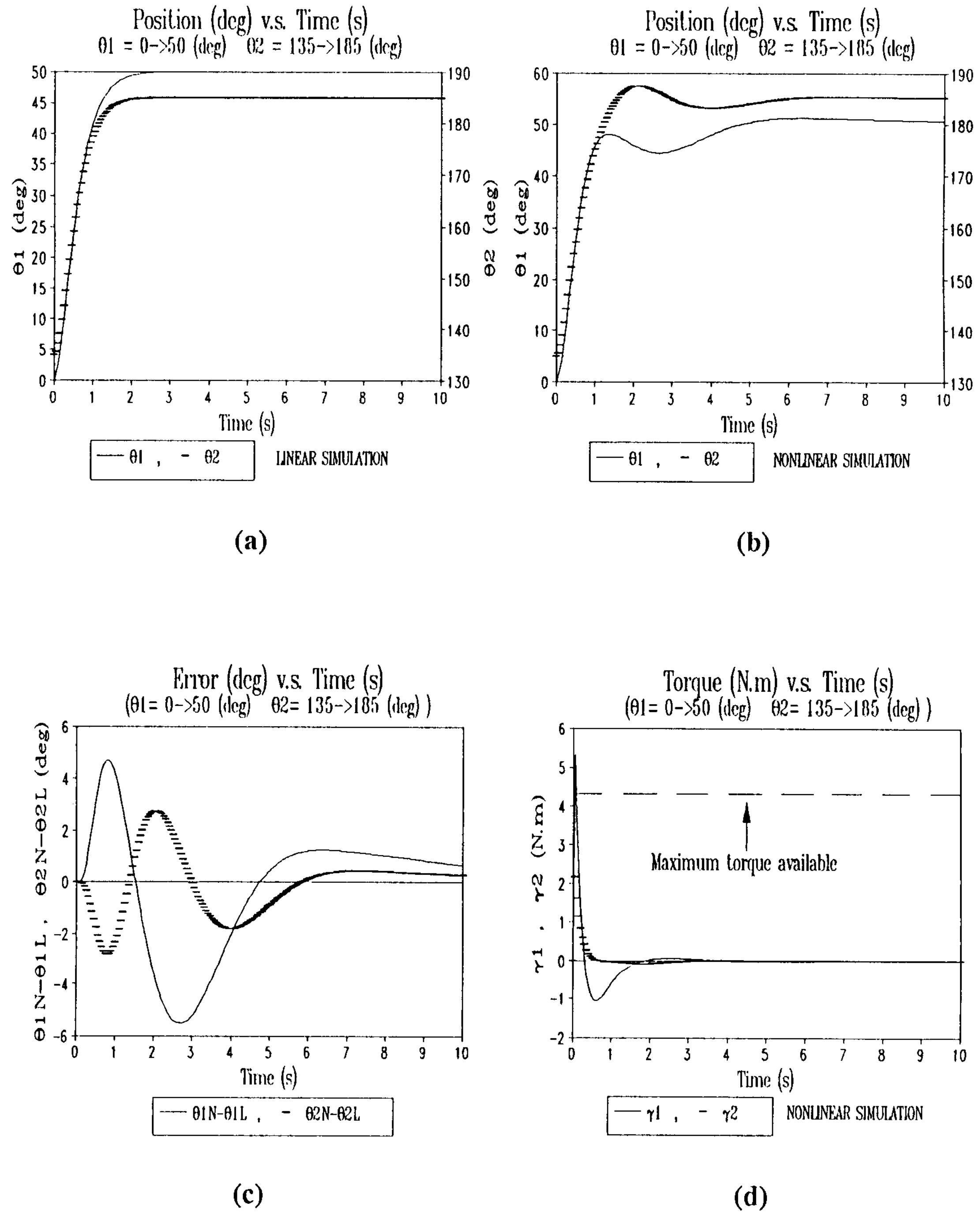


Figure V.12: Planar Manipulator Response: 50° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=135°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

When the arms are initially at:

θ_{1o} = .00 (deg.) θ_{2o} = -45.00 (deg.)

If the following parameters are selected to compensate the system;

g_{11}	=	23.9755	q_3	=	4.1000
g_{12}	=	6.2611	q_2	=	6.0500
g_{21}	=	7.2409	q_1	=	3.7750
g_{22}	=	3.4430	q_0	=	.8250
σ_3	=	-1028.453	σ_7	=	404.527
σ_2	=	-9440.135	σ_6	=	2120.897
σ_1	=	-15395.273	σ_5	=	-13765.620
σ_0	=	-30256.619	σ_4	=	-14554.364
σ_{12}	=	1463.666	σ_{17}	=	-599.605
σ_{11}	=	13707.120	σ_{16}	=	-3117.369
σ_{10}	=	22319.111	σ_{15}	=	20077.049
σ_9	=	44133.185	σ_{14}	=	21251.569
σ_8	=	-19.780	σ_{13}	=	-5.165
σ_{21}	=	-682.095	σ_{25}	=	-67.743
σ_{20}	=	2149.454	σ_{24}	=	-3506.152
σ_{19}	=	404.527	σ_{23}	=	-1028.453
σ_{18}	=	-2112.573	σ_{22}	=	-5479.206
σ_{30}	=	989.151	σ_{35}	=	72.420
σ_{29}	=	-3156.507	σ_{34}	=	5082.435
σ_{28}	=	-614.175	σ_{33}	=	1443.834
σ_{27}	=	3075.058	σ_{32}	=	7979.475
σ_{26}	=	-2.840	σ_{31}	=	-5.974

then the closed loop poles of the linearized system are:

p_{d1}	=	(-3.20, -.20)	p_{d4}	=	(-3.00, -1.00)
p_{d2}	=	(-3.20, .20)	p_{d5}	=	(-3.00, 1.00)
p_{d3}	=	(-900.00, .00)	p_{d6}	=	(-800.00, .00)

Table V.5: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=0^\circ$ and $\theta_{2o}=-45^\circ$.

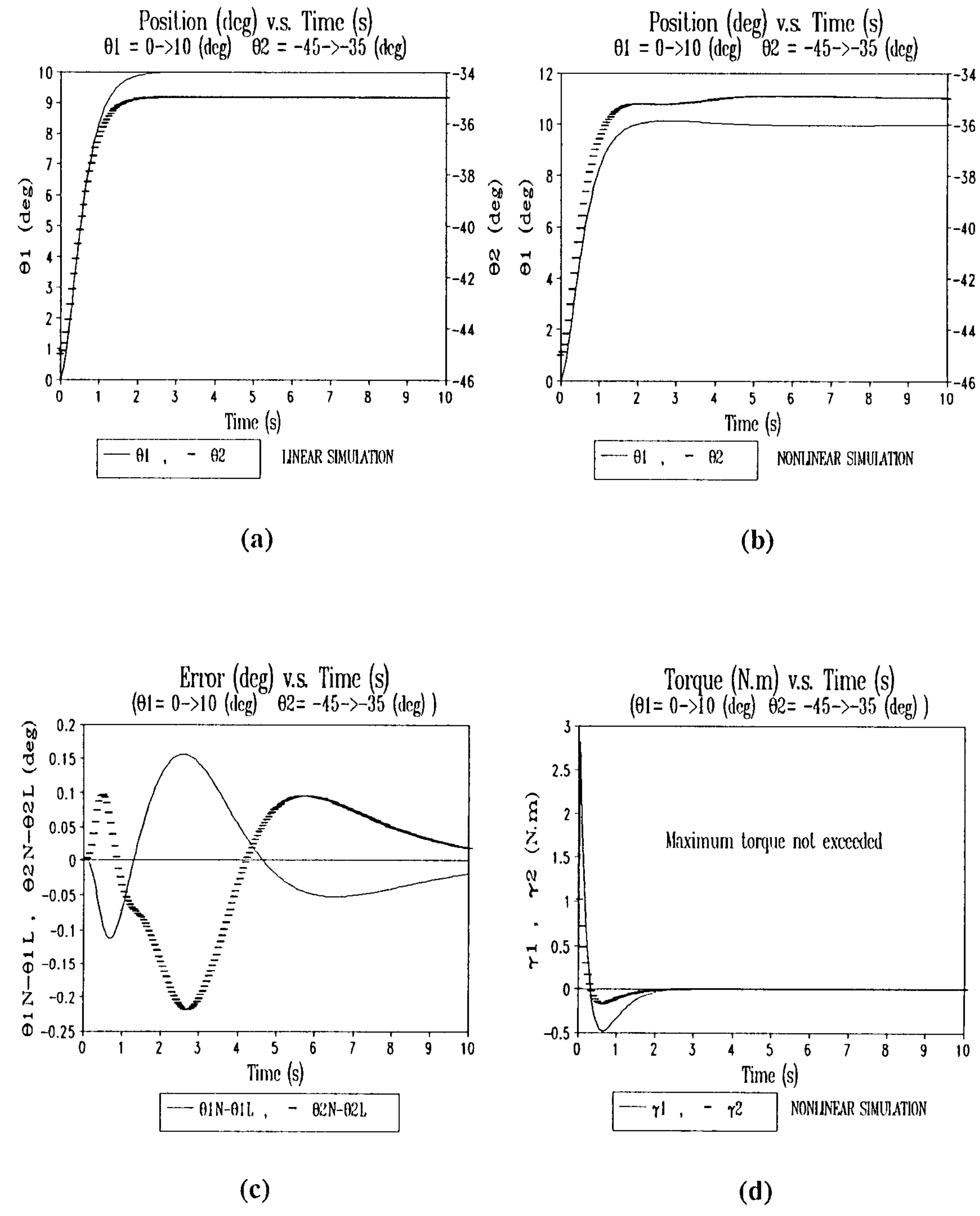
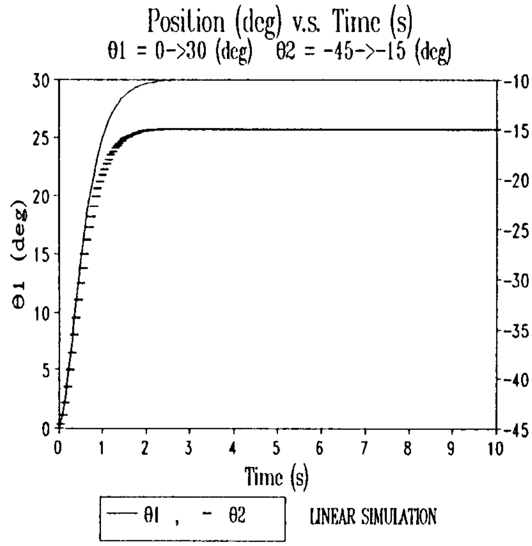
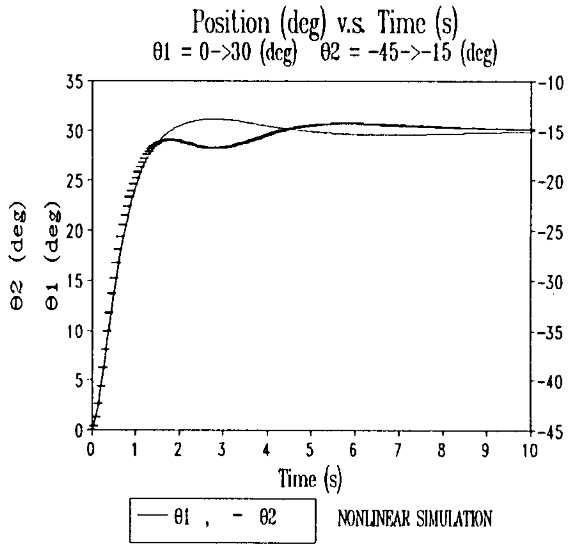


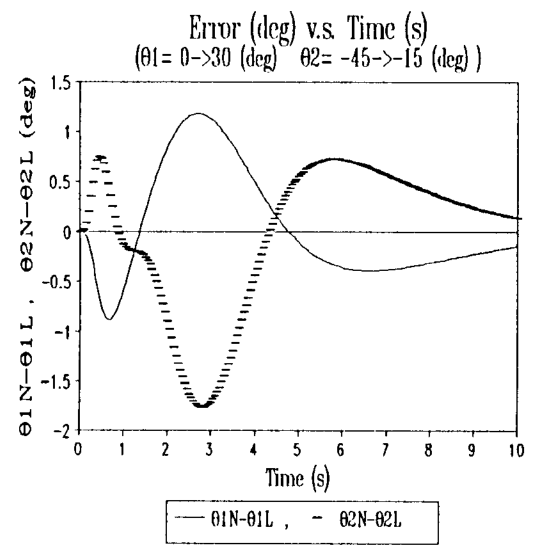
Figure V.13: Planar Manipulator Response: 10° Step Command Applied to Both Joints Around $\theta_{1o}=0^\circ$ and $\theta_{2o}=-45^\circ$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.



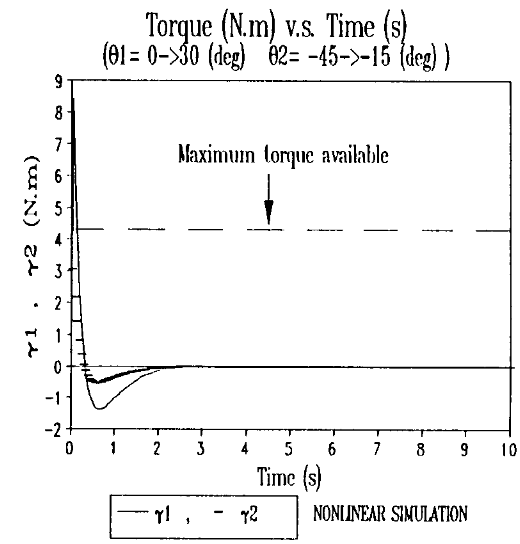
(a)



(b)



(c)



(d)

Figure V.14: Planar Manipulator Response: 30° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=-45°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

To overcome the slippage problem of the stepper motor, the input given in Figure V.15-(a) is applied to the system. The interval for the ramp part of the input is 0.625 second. The compensators parameters are identical to those of Table V.1. Comparing Figure V.15-(d) with Figure V.3-(d), it is observed that by applying the ramp input, the maximum external torque applied to each link has been decreased significantly. The maximum $\delta\theta$ of the stepper motor in this case was 43.2°. Also from Figure V.15-(b) and Figure V.3-(b) notice that the non-linear response has not been changed significantly; i.e. for the ramp input the response has been slowed down only a little bit.

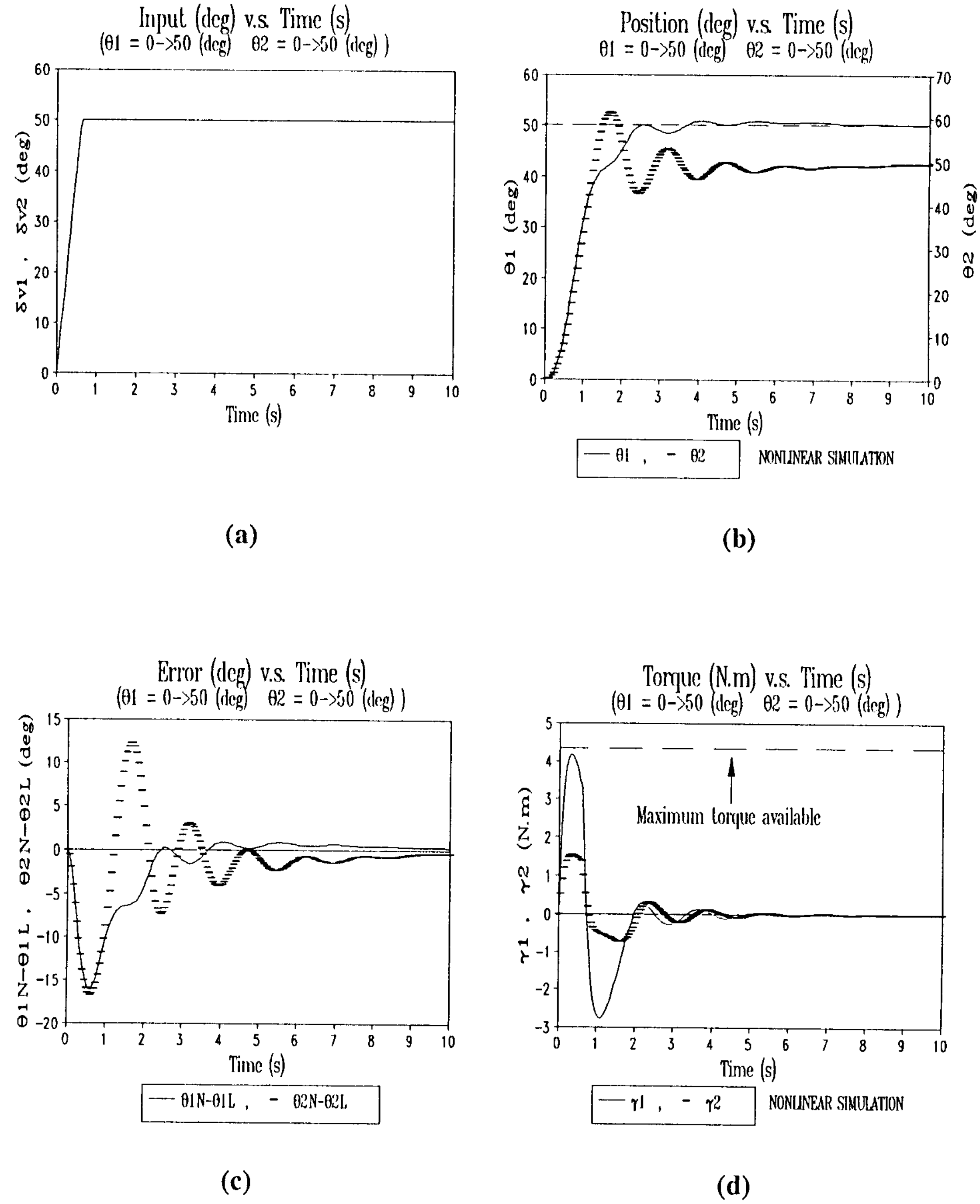


Figure V.15: Planar Manipulator Response: Ramp Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=0°$. (a) Command Applied to Both Joints. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

The compensators parameters given in Table V.6 are for the case when the planar manipulator has been commanded to draw a straight line. The simulation results for two different time intervals are shown in Figures V.16 and V.17. Figure V.16-(a) shows the command inputs to the system such that the end effector moves in x direction (straight line) from -70 cm to 70 cm in 20 seconds. Notice that both v_1 and v_2 have changed more than 130°. The end effector trajectory and the error is shown in Figure V.16-(b). Figure V.16-(c) and Figure V.16-(d) show the external torques applied to link 1 and link 2 respectively. Notice that the torques are small and therefore no stepper motor slippage is present. Figure V.17 is similar to Figure V.16 except that the robot arm has been commanded to draw the same straight line in 60 seconds. Comparison of Figure V.17-(b) with Figure V.16-(b) indicates that when the time interval has been longer the trajectory is much better.

When the arms are initially at:

θ_{1o} = 133.95 (deg.) θ_{2o} =32.61 (deg.)

If the following parameters are selected to compensate the system;

g_{11}	=	.5045	q_3	=	4.1000
g_{12}	=	.1360	q_2	=	6.0500
g_{21}	=	.1573	q_1	=	3.7750
g_{22}	=	.0689	q_0	=	.8250
σ_3	=	-1067.052	σ_7	=	515.601
σ_2	=	-9675.195	σ_6	=	2713.240
σ_1	=	-13449.214	σ_5	=	-19271.199
σ_0	=	-30457.453	σ_4	=	-14434.692
σ_{12}	=	30.347	σ_{17}	=	-15.252
σ_{11}	=	280.908	σ_{16}	=	-79.678
σ_{10}	=	389.270	σ_{15}	=	562.431
σ_9	=	888.429	σ_{14}	=	421.486
σ_8	=	-.416	σ_{13}	=	-.112
σ_{21}	=	-643.190	σ_{25}	=	-82.069
σ_{20}	=	4085.635	σ_{24}	=	-4222.549
σ_{19}	=	515.601	σ_{23}	=	-1067.052
σ_{18}	=	-1828.767	σ_{22}	=	-5610.092
σ_{30}	=	18.635	σ_{35}	=	1.822
σ_{29}	=	-119.737	σ_{34}	=	122.513
σ_{28}	=	-15.533	σ_{33}	=	29.900
σ_{27}	=	53.203	σ_{32}	=	163.363
σ_{26}	=	-.057	σ_{31}	=	-.130

then the closed loop poles of the linearized system are:

p_{d1}	=	(-3.20, -.20)	p_{d4}	=	(-3.00, -1.00)
p_{d2}	=	(-3.20, .20)	p_{d5}	=	(-3.00, 1.00)
p_{d3}	=	(-900.00, .00)	p_{d6}	=	(-800.00, .00)

Table V.6: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=133.95°$ and $\theta_{2o}=32.61°$.

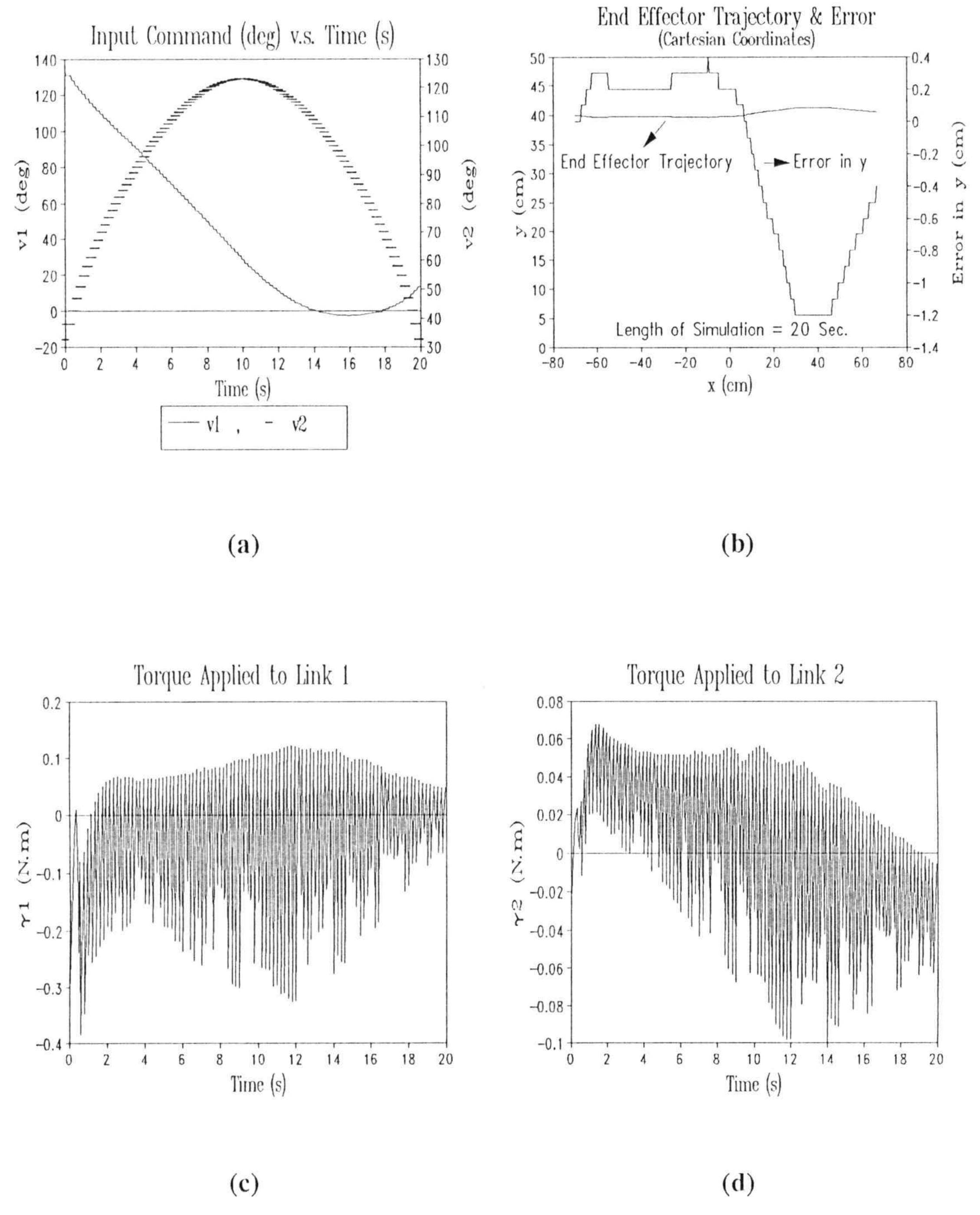


Figure V.16: Planar Manipulator Response: Straight Line Trajectory Is Desired in 20 Second. (a) Command Applied to Each Joint. (b) End Effector Trajectory and Error. (c) Torque Applied to Link 1. (d) Torque Applied to Link 2.

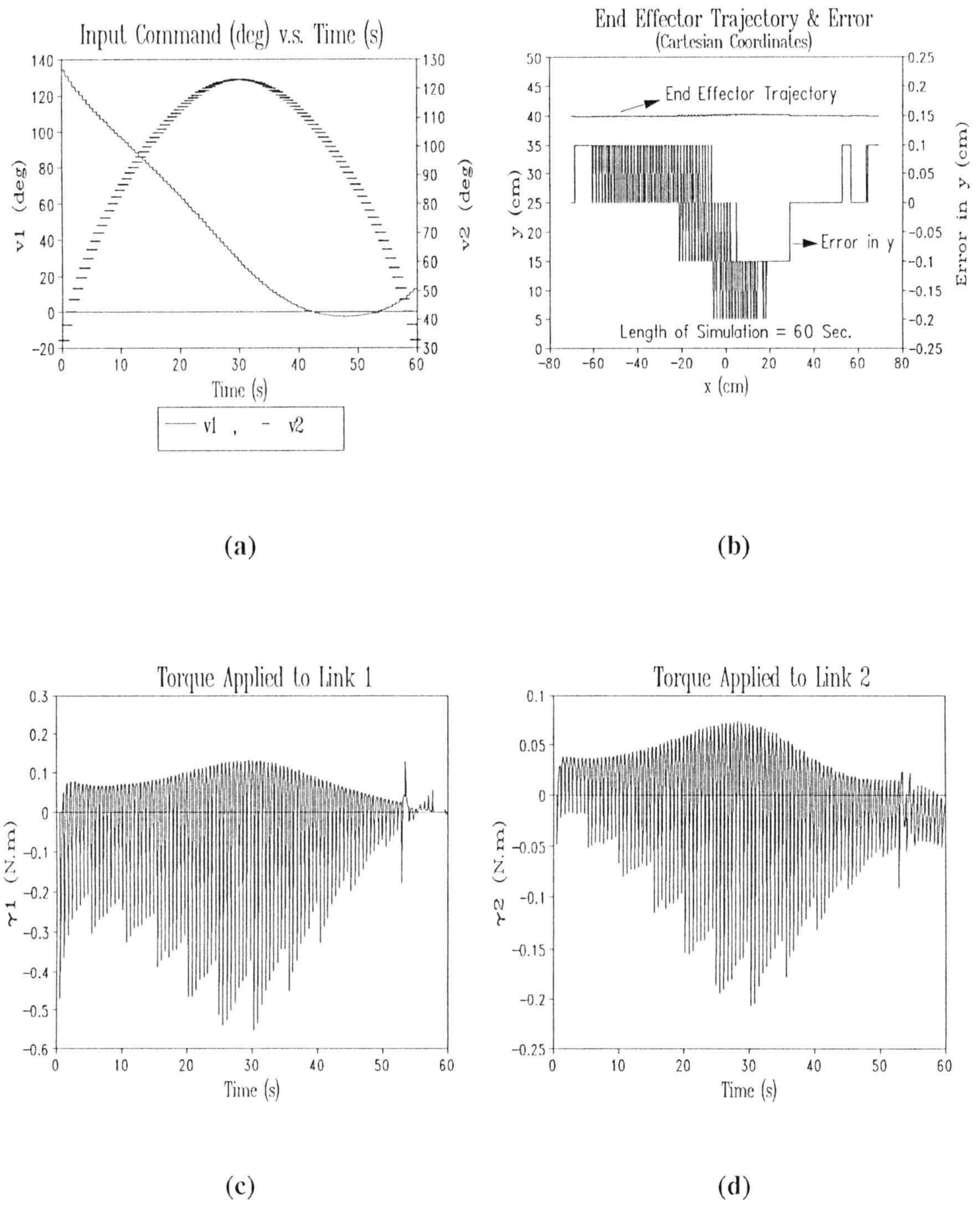


Figure V.17: Planar Manipulator Response: Straight Line Trajectory Is Desired in 60 Second. (a) Command Applied to Each Joint. (b) End Effector Trajectory and Error. (c) Torque Applied to Link 1. (d) Torque Applied to Link 2.

For the cases considered so far, both links have been commanded to move simultaneously. The intention has been to consider the worst cases. However, for completeness Figures V.18 and V.19 are also included. Figure V.18 shows the simulation results when link 1 has been commanded to move 50° from its initial position and link 2 has been commanded to stay at its initial position. Figure V.19 shows the simulation results when link 1 has been commanded to stay at its initial position and link 2 has been commanded to move 50° from its initial position. Comparing Figure V.18-(b) with Figure V.19-(b) it can be seen that the nonlinear model behaves more like the linear model for the case when only link 1 is commanded to move. Note that the maximum error is 0.003 degree in this case while the maximum error is -3.3 degree when only link 2 is commanded to move. Hence, it follows that the error between the linear model and the nonlinear model is introduced mostly due to the movement of link 2. Also from Figure V.18-(d) and Figure V.19-(d) note that when only link 2 is commanded to move, less torque is required as expected. This is because when only link 1 is commanded to move it has to carry link 2 also i.e. more mass is involved for the movement and therefore more torque is required.

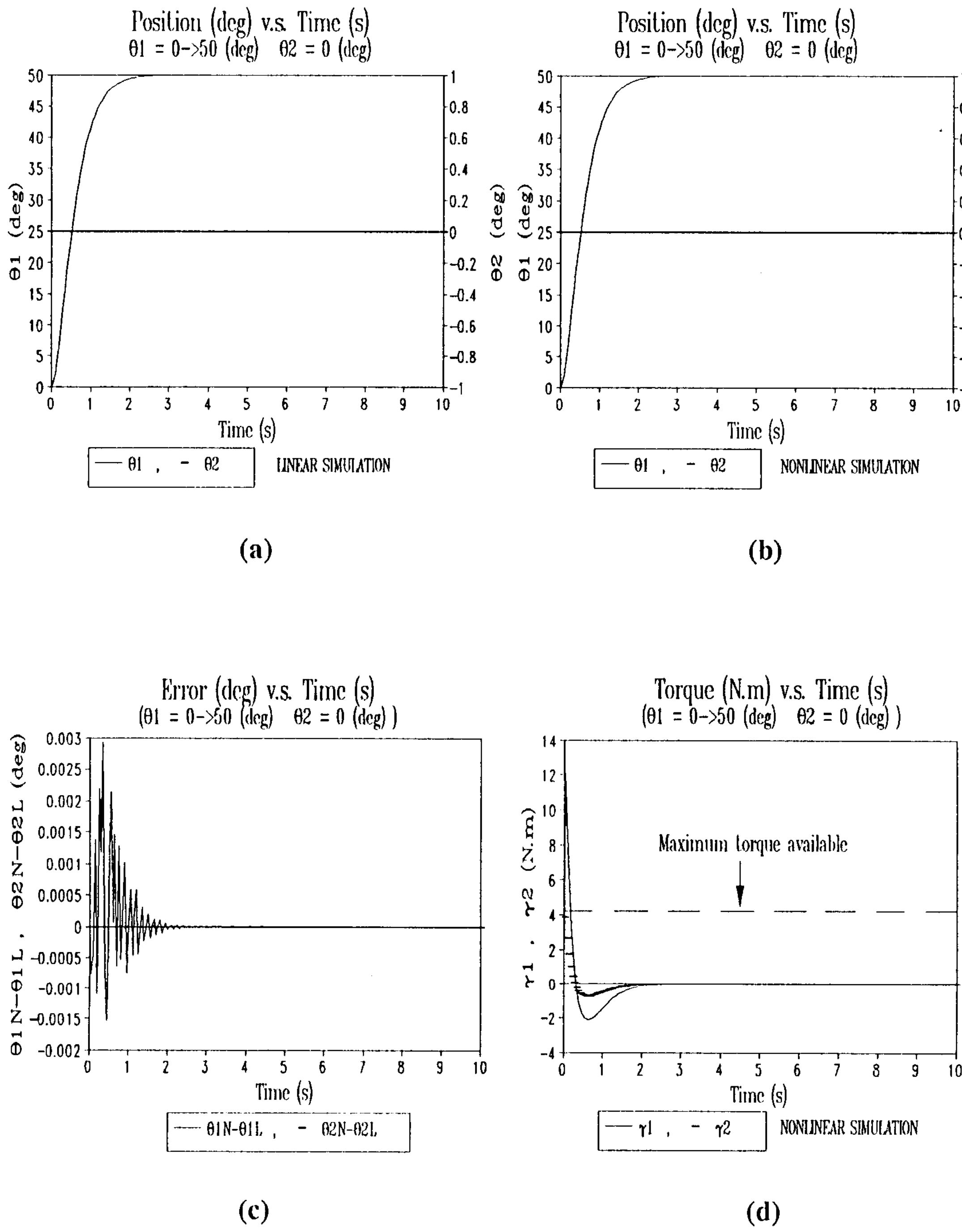


Figure V.18: Planar Manipulator Response: 50° Step Command Applied to Joint 1 Around $\theta_{1o}=0°$ and $\theta_{2o}=0°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

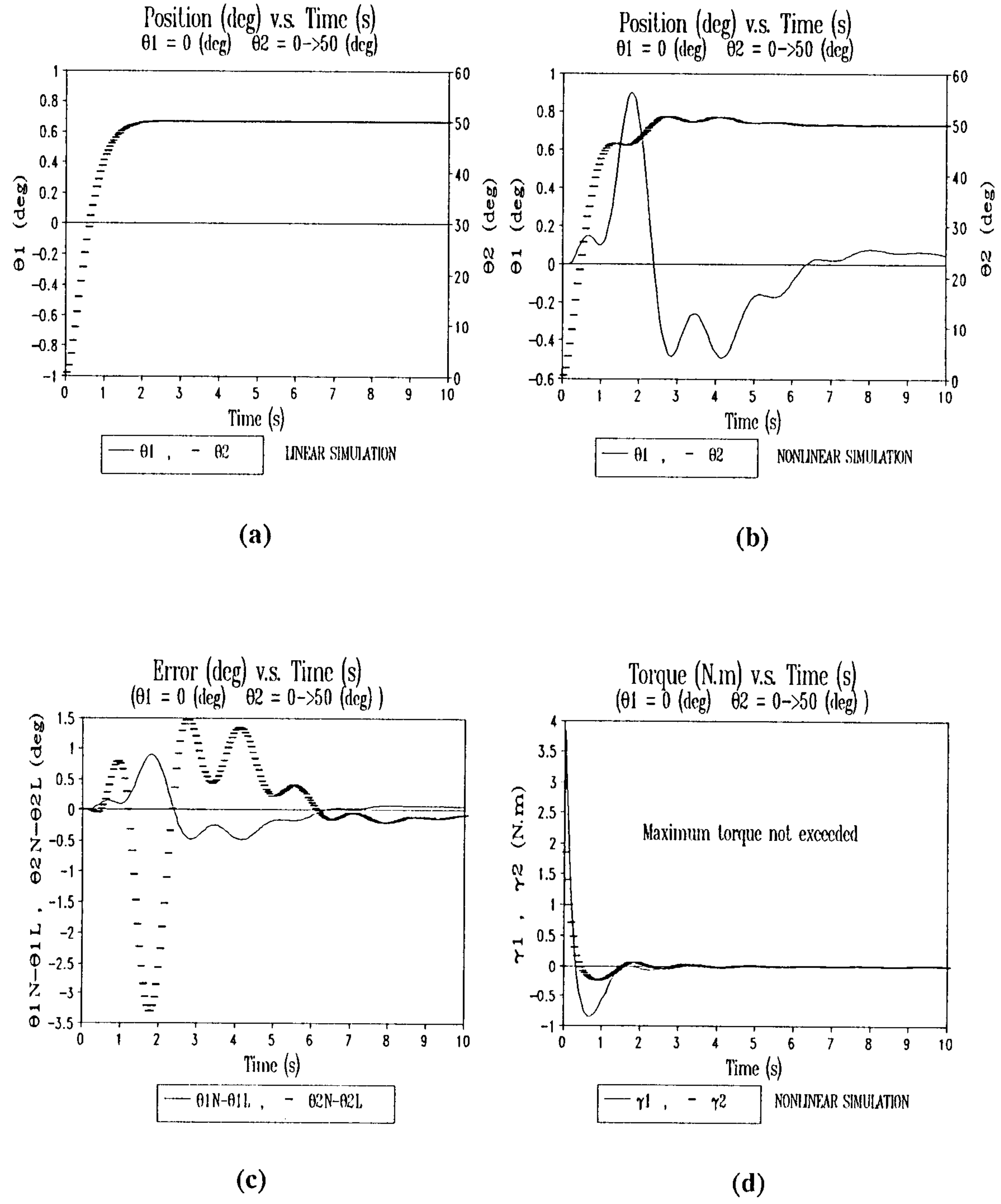


Figure V.19: Planar Manipulator Response: 50° Step Command Applied to Joint 2 Around $\theta_{1o}=0^\circ$ and $\theta_{2o}=0^\circ$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

For all the previous simulations the desired closed loop poles of the planar manipulator system have been selected as following

p_{d1}	= (-3.20, -.20)	p_{d4}	= (-3.00, -1.00)
p_{d2}	= (-3.20, .20)	p_{d5}	= (-3.00, 1.00)
p_{d3}	= (-900.00, .00)	p_{d6}	= (-800.00, .00)

Figure V.20 shows the simulation results around $\theta_1=0^\circ$ and $\theta_2=0^\circ$ for $\delta v_1=\delta v_2=10^\circ$ where the desired closed loop poles of the system have been changed as follows

p_{d1}	= (-3.20, -.20)	p_{d4}	= (-3.00, -1.00)
p_{d2}	= (-3.20, .20)	p_{d5}	= (-3.00, 1.00)
p_{d3}	= (-90.00, .00)	p_{d6}	= (-80.00, .00)

The compensators parameters are given in Table V.7. By comparing Figure V.20 with Figure V.1 the following two observations can be made; 1- as the real poles of the closed loop system are moved toward the origin of the s plane, the external torques applied to the links do not change significantly, 2- the nonlinear model response is not as well behaved as it was when the real poles were further from the origin. Perhaps this implies that the reduced ordered model can be used to describe the planar manipulator arm. By reduced order model it is meant that the two real poles of the system can be placed at infinity; i.e. they can be ignored.

When the arms are initially at:

θ_{1o} = .00 (deg.) $\qquad$ θ_{2o} = .00 (deg.)

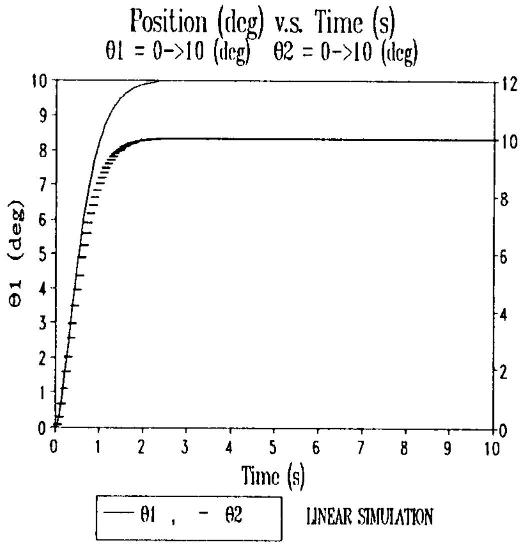
If the following parameters are selected to compensate the system;

g_{11}	=	.0534	q_3	=	4.1000
g_{12}	=	.0149	q_2	=	6.0500
g_{21}	=	.0172	q_1	=	3.7750
g_{22}	=	.0069	q_0	=	.8250
σ_3	=	-119.729	σ_7	=	72.770
σ_2	=	462.093	σ_6	=	-2866.974
σ_1	=	17264.549	σ_5	=	-53984.564
σ_0	=	8612.644	σ_4	=	-29385.917
σ_{12}	=	3.399	σ_{17}	=	-2.028
σ_{11}	=	-13.836	σ_{16}	=	84.133
σ_{10}	=	-505.209	σ_{15}	=	1578.552
σ_9	=	-252.037	σ_{14}	=	859.174
σ_8	=	-.044	σ_{13}	=	-.012
σ_{21}	=	-59.975	σ_{25}	=	-11.731
σ_{20}	=	10746.536	σ_{24}	=	-3775.779
σ_{19}	=	72.770	σ_{23}	=	-119.729
σ_{18}	=	-3391.740	σ_{22}	=	908.699
σ_{30}	=	1.780	σ_{35}	=	.299
σ_{29}	=	-314.094	σ_{34}	=	110.271
σ_{28}	=	-2.055	σ_{33}	=	3.349
σ_{27}	=	99.168	σ_{32}	=	-26.644
σ_{26}	=	-.006	σ_{31}	=	-.014

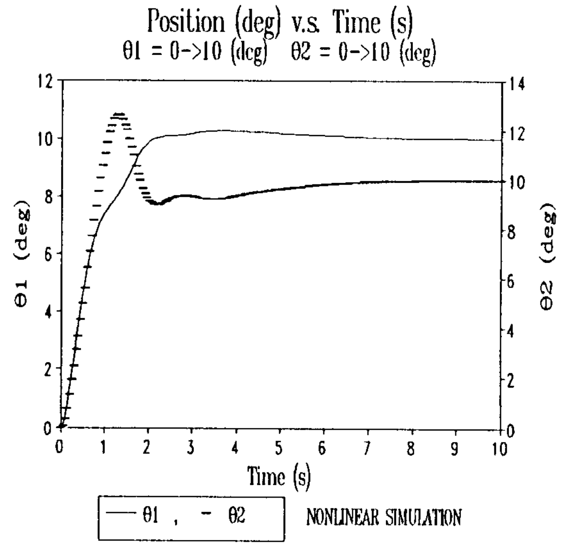
then the closed loop poles of the linearized system are:

p_{d1}	=	(-3.20, -.20)	p_{d4}	=	(-3.00, -1.00)
p_{d2}	=	(-3.20, .20)	p_{d5}	=	(-3.00, 1.00)
p_{d3}	=	(-900.00, .00)	p_{d6}	=	(-800.00, .00)

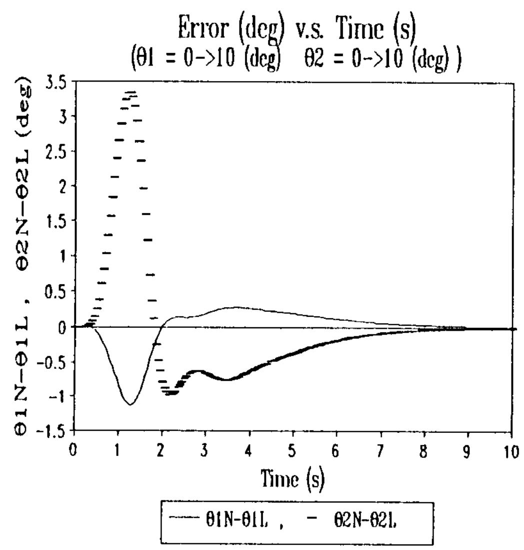
Table V.7: Parameters Used for Compensation Around Equilibrium Point $\theta_{1o}=0°$ and $\theta_{2o}=0°$ (Real Poles Shifted).



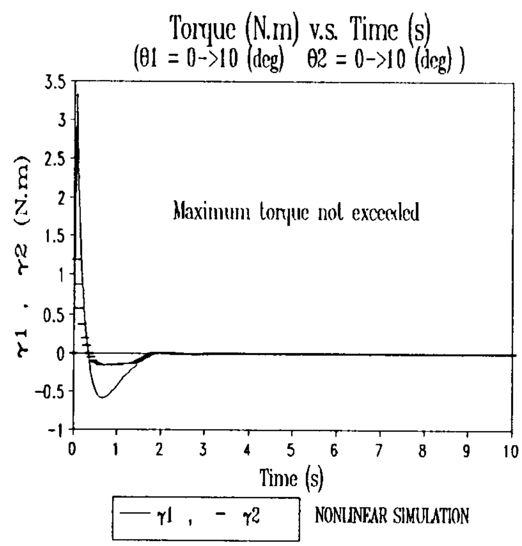
(a)



(b)



(c)



(d)

Figure V.20: Planar Manipulator Response: 10° Step Command Applied to Both Joints Around $\theta_{1o}=0°$ and $\theta_{2o}=0°$. (a) Linear Model Response. (b) Nonlinear Model Response. (c) Error Between Linear and Nonlinear Model Response. (d) Nonlinear Model Torque Applied to Each Link.

The results obtained from the simulations can be tabulated as following:

θ_{1o}	θ_{2o}	$\delta v_1 = \delta v_2$	L.M.R.		N.L.M.R.		Max. Err.	S.M. slip?
			P.O.	S.T.	P.O.	S.T.		
0°	0°	10°	0°	2.5	0°	4	.11°	N
0°	0°	30°	0°	2.5	2.2°	9	3.3°	Y
0°	0°	50°	0°	2.5	15°	10	18°	Y
0°	30°	10°	0°	2.5	.05°	8	.08°	N
0°	30°	30°	0°	2.5	2.5°	10	4.3°	Y
0°	30°	50°	0°	2.5	15°	12.5	17°	Y
0°	90°	10°	0°	2.5	.5°	8.2	.4°	N
0°	90°	30°	0°	2.5	3.°	11.5	3.7°	Y
0°	90°	50°	0°	2.5	10°	13	10°	Y
0°	135°	10°	0°	2.5	.5°	8	.3°	N
0°	135°	30°	0°	2.5	1.5°	8	2.5°	N
0°	135°	50°	0°	2.5	5°	9.8	6°	Y
0°	-45°	10°	0°	2.5	.1°	8	.2°	N
0°	-45°	30°	0°	2.5	1°	10	1.8°	Y

θ_{io} = Initial Position of Link i.
δv_i = Command Input to Link i.
L.M.R. = Linear Model Response.
N.L.M.R. = Non-Linear Model Response.
P.O. = Peak Overshoot.
S.T. = Settling Time in Second.
Max. Err. = Maximum Error Between L.M.R. & N.L.M.R.
S.M. Slip? = Stepper Motor Slippage Present? (Y = either one or both motors slip; N = neither motor slips)

Table V.8: Tabulated Results.

VI. CONCLUSIONS & RECOMMENDATIONS

In has been demonstrated that the multivariable pole placement algorithm for linear systems can be applied to the nonlinear planar manipulator system. The compensation procedure used is based on the ability to express the open loop transfer matrix of a system as the product $R(s)P^{-1}(s)$, where $P(s)$ and $R(s)$ are relatively right prime polynomial matrices with $P(s)$ column proper and degree of each column of $P(s)$ greater than or equal to the degree of the corresponding column in $R(s)$. It is shown by Wolovich [22] that for a controllable and observable system, the transformation of the open loop transfer matrix to $R(s)P^{-1}(s)$ where $R(s)$ and $P(s)$ satisfy the necessary requirements is guaranteed by employment of the structure theorem. The pole placement algorithm in general requires a lot of calculation. It might be noted that the compensation scheme has been done entirely in the frequency domain with no reference whatsoever to the time domain notion of state.

The main question asked in this thesis is "How effective is the linear pole placement controller for a nonlinear planar manipulator?" To answer this question, simulations at different equilibrium points were performed. Simulation results are summarized in Table V.8. Consider the case where the initial position of link 2 is 0° and command input of 30° is applied to each joint. Note that slippage is present i.e. at least one of the motors is slipping. Now consider the case where link 2 is initially at 135° and the same command input as the previous case is applied to each joint. Notice that the stepper motors are operating in their linear region i.e. slippage does not occur. As link 2 moves towards link 1 the effective inertia around joint 1 gets smaller and hence less torque occurs for the same angular movement. To compensate for the slippage of the stepper motor the following are recommended:

1- Use a stepper motor which is capable of producing higher torque.

2- Apply ramp inputs instead of step inputs.

Stability of the system even when the input command is as large as 50° is noticeable. This implies the robustness of the system relative to the perturbation around an equilibrium point. For further research one can study the robustness of the system relative to pay load.

The effectiveness of the linear pole placement controller for the planar manipulator was also demonstrated by commanding the planar manipulator to draw a straight line in the x direction for a distance of 1.4 meter first in 20 seconds and then in 60 seconds. It was shown that better result is obtained by allowing more time for the planar manipulator to perform its task.

It was shown that as the two real poles of the closed loop system are moved away from the origin of the s plane, better responses are obtained. This suggests that the sixth order system can be approximated by a fourth order system. However this is still questionable and it is left for further research. Simulations when only one link at a time is commanded to move were also performed. It was shown that the error introduced between the linear and the nonlinear model of the planar manipulator is mostly due to the movement of link 2.

Considering the rise times, peak overshoots, and settling times of all nonlinear model responses (see Table V.8), it can be concluded that the pole placement algorithm is effective for the nonlinear planar manipulator. As soon as the planar manipulator is completed in the Oregon State University control laboratory the results presented here should be experimentally verified.

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APPENDICES

APPENDIX A

A.1 LAGRANGE METHOD

The motion equations in the Lagrange method are derived in terms of generalized coordinates. Generalized coordinates are used to "locate" elements of the system with respect to a reference system (positions, angles, independent node potentials, independent loop currents, charges, etc.):

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad \text{A.1.1}$$

The generalized forces acting on the system are

$$F(q) = \begin{bmatrix} F_1(q) \\ \vdots \\ F_n(q) \end{bmatrix} \qquad \text{A.1.2}$$

If the generalized forces can be obtained from the gradient of a scalar function V=V(q), i.e:

$$F = -\frac{\partial V}{\partial q} = \begin{bmatrix} -\frac{\partial V}{\partial q_1} \\ \vdots \\ -\frac{\partial V}{\partial q_n} \end{bmatrix} \qquad \text{A.1.3}$$

then the system is called energy conservative and the function V(q) is called the potential energy. The kinetic energy, T, is defined in terms of the generalized coordinates and their derivatives, i.e:

$$T = T(q,q') \qquad \text{A.1.4}$$

The Lagrange function L, is now defined as

$$L = L(q,q') = T(q,q') - V(q) \qquad \text{A.1.5}$$

and the Lagrange equations of the motion have a form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q} = R \qquad \text{A.1.6}$$

where R is nonconservative force vector.

A.2 EQUATIONS OF MOTION FOR THE PLANAR MANIPULATOR

Consider the planar manipulator shown in Figure II.3. To derive the equations of motion the total kinetic energy of the manipulator must be computed. Notice that the change in potential energy of the manipulator is zero since the movement of the links are constrained in a horizontal plane. To derive the kinetic energy of the system, the following equations are established. For link 1

$$\begin{aligned} \bar{x}_1 &= \bar{L}_1\cos\theta_1 \\ \bar{y}_1 &= \bar{L}_1\sin\theta_1 \end{aligned} \qquad \text{A.2.1}$$

and for link 2

$$\begin{aligned} \bar{x}_2 &= L_1\cos\theta_1+\bar{L}_2\cos(\theta_1+\theta_2) \\ \bar{y}_2 &= L_1\sin\theta_1+\bar{L}_2\sin(\theta_1+\theta_2) \end{aligned} \qquad \text{A.2.2}$$

The total kinetic energy is given by

$$T = \frac{1}{2}(m_1\bar{v}_1^2+\bar{I}_1\omega_1^2+m_2\bar{v}_2^2+\bar{I}_2\omega_2^2) \qquad \text{A.2.3}$$

where $\bar{v}_i$ is velocity magnitude of the center of mass of link i, $\bar{I}_i$ is the moment of inertia of link i about its center of mass, and ω_i is the angular rotation rate of link i. ω_1 and ω_2 are given as

$$\begin{aligned} \omega_1 &= \theta'_1 \\ \omega_2 &= \theta'_1+\theta'_2 \end{aligned} \qquad \text{A.2.4}$$

Only horizontal planar motion is considered. Hence for link 1:

$$\bar{x}'_1 = -\theta'_1\bar{L}_1\sin\theta_1$$
$$\bar{y}'_1 = \theta'_1\bar{L}_1\cos\theta_1 \qquad \textbf{A.2.5}$$

$$\bar{v}_1^2 = \bar{x}'^2_1+\bar{y}'^2_1 = \theta'^2_1\bar{L}_1^2 \qquad \textbf{A.2.6}$$

For link 2:

$$\bar{x}'_2 = -\theta'_1L_1\sin\theta_1-(\theta'_1+\theta'_2)\bar{L}_2\sin(\theta_1+\theta_2)$$
$$\bar{y}'_2 = \theta'_1L_1\cos\theta_1+(\theta'_1+\theta'_2)\bar{L}_2\cos(\theta_1+\theta_2) \qquad \textbf{A.2.7}$$

$$\bar{v}_2^2 = \bar{x}'^2_2+\bar{y}'^2_2 = \theta'^2_1(L_1^2+\bar{L}_2^2+2L_1\bar{L}_2\cos\theta_2)+$$
$$\theta'^2_2\bar{L}_2^2+2\theta'_1\theta'_2(\bar{L}_2^2+L_1\bar{L}_2\cos\theta_2) \qquad \textbf{A.2.8}$$

The following positive constants are defined:

$$k_1 = m_2L_1\bar{L}_2$$
$$k_2 = \bar{I}_1+\bar{I}_2+m_1\bar{L}_1^2+m_2(L_1^2+\bar{L}_2^2) \qquad \textbf{A.2.9}$$
$$k_3 = \bar{I}_2+m_2\bar{L}_2^2$$

Since the arm is moving in a horizontal plane, the change in its potential energy is zero and therefore the Lagrange equations of motion given by Equation A.1.6, simplifies to the following form (notice that $q=\theta$)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta'}\right)-\frac{\partial T}{\partial \theta} = R \qquad \textbf{A.2.10}$$

where

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}; \quad \boldsymbol{\theta'} = \begin{bmatrix} \theta'_1 \\ \theta'_2 \end{bmatrix} \qquad \text{A.2.11}$$

and R, the total nonconservative forces, is given by

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \tau_1 - k_{f1}\theta'_1 \\ \tau_2 - k_{f2}\theta'_2 \end{bmatrix} \qquad \text{A.2.12}$$

For the manipulator in the control laboratory, the masses of the links are very close together. Hence it is assumed that the coefficient of friction, which is obtained experimentally, is the same for both links i.e. $k_{f1} = k_{f2} = k_f$. In fact several simulations were done using different values of k_f and nearly identical results were obtained. With this assumption Equation A.2.12 becomes

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \tau_1 - k_f\theta'_1 \\ \tau_2 - k_f\theta'_2 \end{bmatrix} \qquad \text{A.2.13}$$

In Equation A.2.13, τ_1 and τ_2 are the external torques applied to each link and $k_f\theta'_i$ is the assumed friction for link i.

From Equation A.2.10 through Equation A.2.13, the following can be written:

$$\frac{\partial T}{\partial \theta'_1} = \theta'_1(k_2+2k_1\cos\theta_2)+\theta'_2(k_3+k_1\cos\theta_2)$$

$$\frac{\partial T}{\partial \theta'_2} = \theta'_2k_3+\theta'_1(k_3+k_1\cos\theta_2) \qquad \text{A.2.14}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta'_1}\right)= \theta''_1(k_2+2k_1\cos\theta_2)-2\theta'_1\theta'_2k_1\sin\theta_2+\theta''_2(k_3+k_1\cos\theta_2)-\theta'^2_2k_1\sin\theta_2 \qquad \text{A.2.15}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta'_2}\right) = \theta''_2k_3+\theta''_1(k_3+k_1\cos\theta_2)-\theta'_1\theta'_2k_1\sin\theta_2$$

$$\frac{\partial T}{\partial \theta_1} = 0.0$$

$$\frac{\partial T}{\partial \theta_2} = -\theta'^2_1k_1\sin\theta_2-\theta'_1\theta'_2k_1\sin\theta_2 \qquad \text{A.2.16}$$

Substitution of Equation A.2.13, and Equation A.2.15 through Equation A.2.16 into Equation A.2.10, will result in the following dynamical equations of the system

$$\begin{bmatrix}\theta''_1(k_2+2k_1\cos\theta_2)-2\theta'_1\theta'_2k_1\sin\theta_2+\theta''_2(k_1\cos\theta_2+k_3)-\theta'^2_2k_1\sin\theta_2\\ \theta''_1(k_1\cos\theta_2+k_3)+\theta''_2k_3+\theta'^2_1k_1\sin\theta_2\end{bmatrix} = \begin{bmatrix}\tau_1-k_f\theta'_1\\ \tau_2-k_f\theta'_2\end{bmatrix} \qquad \text{A.2.17}$$

Equation A.2.17, is used for the nonlinear simulation.

APPENDIX B

B. DERIVATION OF THE LINEARIZED MODEL

Consider the open loop block diagram of the robot arm, including the stepper motors and drives, shown in Figure II.1. In Section II.3, it was shown that, if the state vector is chosen as given by Equation II.3.2, then the first order differential state vector is described as

$$x' = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta''_1 \\ \theta''_2 \\ x'_5 \\ x'_6 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \theta''_1 \\ \theta''_2 \\ k_v u_1 \\ k_v u_2 \end{bmatrix} \qquad \textbf{B.1}$$

where

$$
\begin{aligned}
f_3 = \theta''_1 = {} & [k_2k_3-k_3^2-k_1^2\cos^2(x_2)]^{-1}* \\
& [k_3k_\tau((x_5-x_1)-(x_6-x_2))+ \\
& k_1k_3\sin(x_2)(x_3^2+x_4^2+2x_3x_4)- \\
& k_1k_\tau\cos(x_2)(x_6-x_2)+ \\
& \tfrac{1}{2}k_1^2\sin(2x_2)x_3^2+ \\
& k_f(k_3(x_4-x_3)+k_1x_4\cos(x_2))] \\
f_4 = \theta''_2 = {} & [k_2k_3-k_3^2-k_1^2\cos^2(x_2)]^{-1}* \\
& [k_\tau(k_2(x_6-x_2)-k_3(x_5-x_1))+ \\
& k_1k_\tau\cos(x_2)(2(x_6-x_2)-(x_5-x_1))- \\
& k_1^2\sin(2x_2)(x_3^2+\tfrac{1}{2}x_4^2+x_3x_4)- \\
& k_1\sin(x_2)(k_2x_3^2+k_3x_4^2+2k_3x_3x_4)+ \\
& k_f(k_1\cos(x_2)(x_3-2x_4)+k_3x_3-k_2x_4)]
\end{aligned}
\tag{B.2}
$$

Notice that f_3 and f_4 given by Equation B.2 are nonlinear. To be able to write Equation B.1 in a state representation form of $x' = Ax + Bu$, it must be linearized and evaluated around an equilibrium point.

Having defined the state vector by Equation II.3.2, it can be shown that a_1 through a_5 given by Equation II.2.4, can be rewritten as

$$
\begin{aligned}
a_1 &= 2k_1\cos(x_2)+k_2 \\
a_2 &= k_1\cos(x_2)+k_3 \\
a_3 &= -k_1\sin(x_2) \\
a_4 &= k_\tau(x_5-x_1)+k_1\sin(x_2)x_4^2+2k_1\sin(x_2)x_3x_4-k_fx_3 \\
a_5 &= k_\tau(x_6-x_2)-k_1\sin(x_2)x_3^2-k_fx_4
\end{aligned}
\tag{B.3}
$$

Substitution of a_1 through a_5 from Equation B.3 into the set of nonlinear equations given by Equation II.2.5, results in the following expressions for f_3 and f_4:

$$\begin{aligned} f_3 &= \theta''_1 = H_1 * H_2 \\ f_4 &= \theta''_2 = H_1 * H_3 \end{aligned} \qquad \text{B.4}$$

where

$$H_1 = [k_2 k_3 - k_3^2 - k_1^2 \cos^2(x_2)]^{-1}$$

$$\begin{aligned} H_2 = {} & k_3 k_\tau [(x_5 - x_1) - (x_6 - x_2)] + k_1 k_3 \sin(x_2)[x_3^2 + x_4^2 + 2x_3 x_4] - k_1 k_\tau \cos(x_2) * \\ & (x_6 - x_2) + \tfrac{1}{2} k_1^2 \sin(2x_2) x_3^2 + k_f [k_3(x_4 - x_3) + k_1 \cos(x_2) x_4] \end{aligned} \qquad \text{B.5}$$

$$\begin{aligned} H_3 = {} & k_\tau [k_2(x_6 - x_2) - k_3(x_5 - x_1)] + k_1 k_\tau \cos(x_2)[2(x_6 - x_2) - (x_5 - x_1)] - \\ & k_1^2 \sin(2x_2)(x_3^2 + \tfrac{1}{2} x_4^2 + x_3 x_4) - k_1 \sin(x_2)(k_2 x_3^2 + k_3 x_4^2 + 2k_3 x_3 x_4) + \\ & k_f [k_1 \cos(x_2)(x_3 - 2x_4) + k_3 x_3 - k_2 x_4] \end{aligned}$$

The general form of the state representation of a linearized model can be described by the following [12]

$$\begin{aligned} \delta x' &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u \end{aligned} \qquad \text{B.6}$$

where the A, B, C, and D matrices (system matrices) must be evaluated at the desired equilibrium point. It can be shown that the linearized system matrices for the system under consideration are given by

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_6} \\ \vdots & & & \vdots \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{B.7}$$

(the third and forth row of matrix A are derived shortly)

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial f_6}{\partial u_1} & \frac{\partial f_6}{\partial u_2} \end{bmatrix} = \frac{1}{k_v} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{B.8}$$

$$C = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_6} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{B.9}$$

$$D = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{B.10}$$

By inspection of Equation B.7 through Equation B.10, one can see that the only matrix which varies at different equilibrium points is the matrix A. The elements in the third and the forth row of matrix A are as following:

$$a_{31} = \frac{\partial f_3}{\partial x_1} = H_1(-k_3k_\tau) \qquad \textbf{B.11}$$

$$\begin{aligned} a_{32} = \frac{\partial f_3}{\partial x_2} = & -H_1^2(k_1^2\sin(2x_2))H_2 + \\ & H_1\,[k_3k_\tau + k_1k_3\cos(x_2)(x_3^2 + x_4^2 + 2x_3x_4) + \\ & k_1k_\tau\sin(x_2)(x_6 - x_2) + k_1k_\tau\cos(x_2) + \\ & k_1^2\cos(2x_2)x_3^2 - k_1k_f\sin(x_2)x_4] \end{aligned} \qquad \textbf{B.12}$$

$$a_{33} = \frac{\partial f_3}{\partial x_3} = H_1[k_1k_3\sin(x_2)(2x_3 + 2x_4) + k_1^2\sin(2x_2)x_3 - k_fk_3] \qquad \textbf{B.13}$$

$$a_{34} = \frac{\partial f_3}{\partial x_4} = H_1[k_1k_3\sin(x_2)(2x_3 + 2x_4) + k_f(k_3 + k_1\cos(x_2))] \qquad \textbf{B.14}$$

$$a_{35} = \frac{\partial f_3}{\partial x_5} = H_1k_3k_\tau \qquad \textbf{B.15}$$

$$a_{36} = \frac{\partial f_3}{\partial x_6} = H_1(-k_3k_\tau - k_1k_\tau\cos(x_2)) \qquad \textbf{B.16}$$

$$a_{41} = \frac{\partial f_4}{\partial x_1} = H_1(k_3k_\tau + k_1k_\tau\cos(x_2)) \qquad \textbf{B.17}$$

$$a_{42} = \frac{\partial f_4}{\partial x_2} = -H_1^2(k_1^2\sin(2x_2))H_3 +$$
$$H_1[-k_2k_\tau - k_1k_\tau\sin(x_2)(2(x_6-x_2)-(x_5-x_1)) -$$
$$2k_1k_\tau\cos(x_2) - 2k_1^2\cos(2x_2)(x_3^2+\tfrac{1}{2}x_4^2+x_3x_4) -$$
$$k_1\cos(x_2)(k_2x_3^2+k_3x_4^2+2k_3x_3x_4) + k_f(-k_1\sin(x_2)(x_3-2x_4))] \qquad \text{B.18}$$

$$a_{43} = \frac{\partial f_4}{\partial x_3} = H_1[-k_1^2\sin(2x_2)(2x_3+x_4) - k_1\sin(x_2) *$$
$$(2k_2x_3+2k_3x_4) + k_f(k_1\cos(x_2)+k_3)] \qquad \text{B.19}$$

$$a_{44} = \frac{\partial f_4}{\partial x_4} = H_1[-k_1^2\sin(2x_2)(x_4+x_3) - k_1\sin(x_2) *$$
$$(2k_3x_4+2k_3x_3) - k_f(2k_1\cos(x_2)+k_2)] \qquad \text{B.20}$$

$$a_{45} = \frac{\partial f_4}{\partial x_5} = H_1(-k_3k_\tau - k_1k_\tau\cos(x_2)) \qquad \text{B.21}$$

$$a_{46} = \frac{\partial f_4}{\partial x_6} = H_1(k_2k_\tau + 2k_1k_\tau\cos(x_2)) \qquad \text{B.22}$$

At any equilibrium point all the rates variable must be zero. This implies that the following must be satisfied at any equilibrium point (see Figure II.1):

$$u_1 = u_2 = 0 \qquad x_5 = x_1$$
$$x_3 = x_4 = 0 \qquad x_6 = x_2 \qquad \text{B.23}$$

Substitution of Equation B.23 into Equation B.11 through Equation B.22 results in the following matrix A around any particular equilibrium point.

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_6} \\ \vdots & & & \vdots \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}_o = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a & b & k_d a & k_d b & -a & -b \\ b & c & k_d b & k_d c & -b & -c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{B.24}$$

where

$$\begin{aligned} a &\triangleq -k_3 k_\tau \; H_1 \\ b &\triangleq (k_1 \cos(x_{2o}) + k_3) k_\tau \; H_1 \\ c &\triangleq -(2k_1 \cos(x_{2o}) + k_2) k_\tau \; H_1 \\ k_d &\triangleq \frac{k_f}{k_\tau} \end{aligned} \qquad \text{B.25}$$

Notice that the evaluation of matrix A around any equilibrium point is independent of the initial position of the first link.

The A, B, C, and D matrices given by Equation B.24, and Equation B.8 through Equation B.10, respectively, are the one which are used for the linearized model. Consequently these matrices are used to derive the compensator of the closed loop system.

APPENDIX C

The structure theorem establishes a fundamental structure of dynamical systems. In particular, if a given system is state controllable, the structure theorem can be employed to write the transfer function of the system as $T(s)=R(s)P^{-1}(s)$ where R(s) and P(s) have certain properties as discussed shortly. In order to establish this theorem, the following definitions, some of which has been stated in Section II.4, are presented first.

C.1 PRELIMINARY DEFINITIONS

DEFINITION C.1.1: The degree of a polynomial matrix P(s), denoted by the scalar $\partial[P(s)]$ is defined as the degree of the polynomial element of highest degree in P(s). The degree of the j-th column of P(S) denoted by the scalar $\partial_{cj}[P(s)]$, is defined as the degree of the polynomial element of highest degree in the j-th column of P(s). The constant matrix consisting of the coefficients of the highest degree terms in each column of P(s) is denoted by $\Gamma_c[P(s)]$. Subscript "c" implies column.

DEFINITION C.1.2: The degree of the i-th row of P(S), denoted by the scalar $\partial_{ri}[P(s)]$, is defined as the degree of the polynomial element of highest degree in the i-th row of P(s). The constant matrix consisting of the coefficients of the highest degree terms in each row of P(s) is denoted by $\Gamma_r[P(s)]$. Subscript "r" implies row. To illustrate, consider the following example:

EXAMPLE C.1.1: If

$$P(s) = \begin{bmatrix} p_{11}(s) & \cdots & p_{1m}(s) \\ \vdots & & \vdots \\ p_{n1}(s) & \cdots & p_{nm}(s) \end{bmatrix} \; ; \text{ with} \tag{C.1.1}$$

$$p_{ij}(s) \triangleq \alpha_{\partial_{ij}[P(s)]} s^{\partial_{ij}[P(s)]} + \sum_{k=0}^{\partial_{ij}[P(s)]-1} \alpha_k s^k$$

where

$$\alpha_{\partial_{ij}[P(s)]} \neq 0 \tag{C.1.2}$$

which implies $\partial[P_{ij}(s)] = \partial_{ij}[P(s)]$, then

$$\begin{aligned} \partial_{cj}[P(s)] &= Max\ \{(\partial_{ij}[P(s)])_{i=1\to n}\} \\ \partial_{ri}[P(s)] &= Max\ \{(\partial_{ij}[P(s)])_{j=1\to m}\} \end{aligned} \tag{C.1.3}$$

Note that

$$\begin{aligned} \partial_{ij}P(s) &\leq \partial_{cj}[P(s)] \quad \text{for all j} \\ \partial_{ij}P(s) &\leq \partial_{rj}[P(s)] \quad \text{for all i} \end{aligned} \tag{C.1.4}$$

In particular if

$$P(s) = \begin{bmatrix} s^2-3 & 1 & 2s \\ 4s+2 & 2 & 0 \\ -s^2 & s+3 & -3s+2 \end{bmatrix} \qquad \text{C.1.5}$$

then $\partial_{c1}=2$, $\partial_{c2}=\partial_{c3}=1$, $\partial_{r1}=\partial_{r3}=2$, $\partial_{r2}=1$ and

$$\Gamma_c[P(s)] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ -1 & 1 & -3 \end{bmatrix} \; ; \; \Gamma_r[P(s)] = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{C.1.6}$$

The column j zeros in $\Gamma_c[P(s)]$ indicate that the corresponding polynomials are of lesser degree than $\partial_{cj}[P(s)]$. The row i zeros in $\Gamma_r[P(s)]$ indicate that the corresponding polynomials are of lesser degree than $\partial_{ri}[P(s)]$.

DEFINITION C.1.3: A $n \times m$ polynomial matrix, P(s), is called column proper if and only if $\Gamma_c[P(s)]$ has full rank i.e. $\text{rank}\{\Gamma_c[P(s)]\}=\min(n,m)$. A $n \times m$ polynomial matrix, P(s), is called row proper if and only if $\Gamma_r[P(s)]$ has full rank i.e. $\text{rank}\{\Gamma_r[P(s)]\}=\min(n,m)$.

DEFINITION C.1.4: If three polynomial matrices satisfy the relation; $P(s)=H(s)G_r(s)$, then $G_r(s)$ is called a right divisor of P(s), and P(s) is called a left multiple of $G_r(s)$. A greatest common right divisor (g.c.r.d.) of two polynomial matrices P(s) and R(s) is a common right divisor which is a left multiple of every common right divisor of P(s) and R(s).

DEFINITION C.1.5: A *unimodular matrix* U(s) is defined as any *square polynomial matrix* whose determinant is a *nonzero constant.*

DEFINITION C.1.6: Two polynomial matrices R(s) and P(s) which have the same number of columns, are said to be relatively right prime if and only if their greatest common right divisors are unimodular matrices.

EXAMPLE C.1.2: For the following two polynomial matrices R(s), and P(s)

$$R(s) = \begin{bmatrix} s & -s \\ 0 & 1 \end{bmatrix} \quad ; \quad P(s) = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \end{bmatrix} \tag{C.1.7}$$

it can be shown that the following square matrix is one of the greatest common right divisors of the two polynomial matrices R(s) and P(s).

$$G_r(s) = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \tag{C.1.8}$$

In particular

$$\begin{aligned} R(s) &= \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} ; \\ P(s) &= \begin{bmatrix} s & -1 \\ -1 & s^2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{C.1.9}$$

Notice that $G_r(s)$ is not a unimodular matrix since $|G_r(s)|=s$ is not a nonzero constant.

DEFINITION C.1.7: A polynomial matrix, T(s), is called proper if the numerator degree of each entry of T(s), i.e. $T_{ij}(s)$, is less than or equal to the corresponding denominator degree. In the case of strictly proper transfer matrix, the degree of the numerator of each entry, $T_{ij}(s)$, of T(s) is equal to the corresponding denominator degree.

C.2 EQUIVALENT SYSTEMS

Consider a dynamical system represented by:

$$\begin{aligned} x'(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)+Du(t) \end{aligned} \qquad \text{C.2.1}$$

If the state of the system, x(t), is altered via the relationship

$$\hat{x}(t) = Qx(t) \qquad \text{C.2.2}$$

where Q is a $n \times n$ nonsingular real matrix, then

$$x(t) = Q^{-1}\hat{x}(t) \qquad \text{C.2.3}$$

Substitution of Equation C.2.3 into Equation C.2.1 yields the following

$$Q^{-1}\hat{x}'(t) = AQ^{-1}\hat{x}(t)+Bu(t)$$
$$y(t) = CQ^{-1}\hat{x}(t)+Du(t) \qquad \text{C.2.4}$$

Equation C.2.4 can be rewritten as:

$$\hat{x}'(t) = QAQ^{-1}\hat{x}(t)+QBu(t)$$
$$y(t) = CQ^{-1}\hat{x}(t)+Du(t) \qquad \text{C.2.5}$$

or

$$\hat{x}'(t) = \hat{A}\hat{x}(t)+\hat{B}u(t)$$
$$y(t) = \hat{C}\hat{x}(t)+\hat{D}u(t) \qquad \text{C.2.6}$$

where

$$\begin{array}{lcl} \hat{A} = QAQ^{-1} & & \hat{C} = CQ^{-1} \\ & ; & \\ \hat{B} = QB & & \hat{D} = D \end{array} \qquad \text{C.2.7}$$

Therefore the following can be established:

DEFINITION C.2.1: The state representations of Equation C.2.1 and Equation C.2.6 with states related by Equation C.2.2 are said to be equivalent and Q is called an equivalence transformation. In other words, the system {A,B,C,D} and {Â,B̂,Ĉ,D̂} are equivalent if and only if the following relationships hold for some nonsingular real matrix Q:

$$\hat{A} = QAQ^{-1}$$
$$\hat{B} = QB$$
$$\hat{C} = CQ^{-1} \qquad \textbf{C.2.8}$$
$$\hat{D} = D$$

the justification for the use of the term "equivalent" in Definition C.2.1 can be readily demonstrated by noting that the solution of either system, x(t) or $\hat{x}(t)$, immediately implies the solution of the other via Equation C.2.2.

C.3 CONTROLLABLE COMPANION FORM

Before starting the procedure for deriving the controllable companion form the following theorem is stated.

THEOREM C.3.1: The following statements regarding the linear, time invariant dynamical system $\dot{x}(t)=Ax(t)+Bu(t)$ are equivalent:

a. The system is completely state controllable.

b. The rank of the $n \times nm$ controllability matrix:

$$\mathbb{C} = [B, AB, \ldots, A^{n-1}B] \qquad \textbf{C.3.1}$$

is n.

If a system {A,B,C,D} in the state form given by Equation C.2.1 is completely state controllable ***with B of full rank $m \leq n$***, then it can be reduced via a nonsingular transformation Q to an equivalent controllable system in a certain structured form which is called a "controllable companion form". The procedure for deriving a controllable companion form is now discussed.[22]

Consider any completely state controllable system of the form given by Equation C.2.1. Since the system is assumed to be controllable, it follows from Theorem C.3.1 that $\mathbf{C}$ has full rank (n). $\bar{\mathbf{C}}$ is now defined as the $n \times n$ matrix obtained by selecting from left to right the first n linearly independent columns of the controllability matrix given by Equation C.3.1. Therefore, $\bar{\mathbf{C}}$ has full rank n and $|\bar{\mathbf{C}}| \neq 0$. Since it is assumed that matrix B has full rank, therefore ***the first m columns of $\bar{\mathbf{C}}$*** is the matrix B. The nonsingular $n \times n$ matrix L is now constructed by simply reordering the n columns of $\bar{\mathbf{C}}$, beginning with a "power ordering" of those first (d_1) columns of $\bar{\mathbf{C}}$ which involve b_1, the first column of B, and then employing those (d_2) columns of $\bar{\mathbf{C}}$ which involve b_2 next and so forth. In particular,

$$L = [b_1, Ab_1, \ldots, A^{d_1-1}b_1, b_2, Ab_2, \ldots, A^{d_2-1}b_2, \ldots, A^{d_m-1}b_m] \qquad \text{C.3.2}$$

Notice that $d_1, d_2, \ldots, d_m$ defined as such, satisfy the following condition

$$\sum_{i=1}^{m} d_i = n \qquad \text{C.3.3}$$

The m positive integers d_i ,for i=1,2,...,m, are now defined as the controllability indices of the system, and the following is established

$$\sigma_k = \sum_{i=1}^{k} d_i \quad for \quad k = 1,2,...,m \tag{C.3.4}$$

which implies that:

$$\begin{aligned} \sigma_1 &= d_1 \\ \sigma_2 &= d_1+d_2 \\ &\vdots \\ \sigma_m &= d_1+d_2+\cdots+d_m = n \end{aligned} \tag{C.3.5}$$

The controllability indices not only specify the dimensions of various diagonal companion-form submatrices of $\hat{A}$, but also determine the m ordered integers σ_k, for k=1,2,...,m, which denote the "nontrivial" rows of $\hat{A}$ and $\hat{B}$.

At this point q_k^T is set equal to the σ_k-th row of L^{-1} for k=1,2,...,m, and the following n×n matrix Q is defined:

$$Q = \begin{bmatrix} q_1^T \\ q_1^T A \\ \vdots \\ q_1^T A^{d_1-1} \\ q_2^T \\ \vdots \\ q_2^T A^{d_2-1} \\ \vdots \\ q_m^T A^{d_m-1} \end{bmatrix} \qquad \text{C.3.6}$$

If Q, defined as so, is postmultiplied by L, it can be shown that $|QL| = 1$ in "absolute" value which implies the nonsingularity of Q since $|QL| = |Q|\,|L|$. This particular choice of Q will reduces the given system to an equivalent state representation form given by Equation C.2.6, where the pair $\{\hat{A},\hat{B}\}$ assumes a particularly useful structured form, namely a multivariable controllable companion form; i.e.

$$\hat{A} = \left[\begin{array}{ccccc|ccccc|c|ccccc}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & 0 & \cdots & 0 \\
 & & & \ddots & & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots \\
 & & & & 1 & & & & & & & & & & & \\
x & x & x & \cdots & x & x & x & x & \cdots & x & & x & x & x & \cdots & x \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0\cdots & 0 & & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & & & & & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots \\
 & & & & & & & & & 1 & & & & & & \\
x & x & x & \cdots & x & x & x & x & \cdots & x & & x & x & x & \cdots & x \\
\hline
 & & & \vdots & & & & & \vdots & & \ddots & & & & \vdots & \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & 1 & 0\cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \cdots & & & & \ddots & \\
 & & & & & & & & & & & & & & & 1 \\
x & x & x & \cdots & x & x & x & x & \cdots & x & & x & x & x & \cdots & x
\end{array}\right] \qquad \text{C.3.7}$$

and

$$\hat{B} = \left[\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & & & \vdots \\
1 & x & \cdots & x \\
\hline
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & & & \vdots \\
0 & 1 & x & \cdots \\
\hline
\vdots & & & \vdots \\
0 & 0 & \cdots & 1
\end{array}\right] \qquad \text{C.3.8}$$

It is important to notice that all the information regarding the equivalent state matrix $\hat{A}$ can be derived from knowledge of the m ordered controllability indices d_i and the m ordered σ_k-th rows of $\hat{A}$. The same thing can be said of $\hat{B}$, since only these same ordered σ_k rows of $\hat{B}$ are nonzero.

C.4 CONTROLLABLE COMPANION FORM OF PLANAR MANIPULATOR

For the planar manipulator under consideration n=6 and therefore from Equation C.3.1 the controllability matrix for the system is given by

$$\mathbb{C} = [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B] \tag{C.4.1}$$

To find $\bar{\mathbb{C}}$, only the *first n (6) independent* columns of the controllability matrix are needed. It turns out that the first 6 columns of C are independent and therefore

$$\bar{\mathbb{C}} = [B \quad AB \quad A^2B] = k_v \begin{bmatrix} 0 & 0 & 0 & 0 & -a & -b \\ 0 & 0 & 0 & 0 & -b & -c \\ 0 & 0 & -a & -b & d & e \\ 0 & 0 & -b & -c & e & f \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{C.4.2}$$

where a,b, and c are given by Equation II.3.9, and d,e, and f are:

$$
\begin{aligned}
d &= -k_d(a^2+b^2) \\
e &= -k_d(a+c)b \\
f &= -k_d(b^2+c^2)
\end{aligned}
\tag{C.4.3}
$$

The nonsingular matrix L is now constructed as the following

$$
L = [b_1 \;\; Ab_1 \;\; A^2b_1 \;\; b_2 \;\; Ab_2 \;\; A^2b_2]
$$

$$
= k_v \begin{bmatrix} 0 & 0 & -a & 0 & 0 & -b \\ 0 & 0 & -b & 0 & 0 & -c \\ 0 & -a & d & 0 & -b & e \\ 0 & -b & e & 0 & -c & f \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\tag{C.4.4}
$$

Comparison of L with Equation C.3.2 suggests that the controllability indices are $d_1=d_2=3$. By employing Equation C.3.4 to the planar manipulator problem, the following can be established

$$
\begin{aligned}
\sigma_1 &= \sum_{i=1}^{1} d_i = d_1 = 3 \\
\sigma_2 &= \sum_{i=1}^{2} d_i = d_1+d_2 = 6
\end{aligned}
\tag{C.4.5}
$$

To find Q, inverse of L must be found first. It can be shown that L^{-1} is

$$L^{-1} = \frac{1}{k_v}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \mho_9 & \mho_{10} & \mho_{11} & \mho_{13} & 0 & 0 \\ \mho_{12} & \mho_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\mho_7}{\mho_3} & \frac{\mho_8}{\mho_3} & \mho_{13} & \frac{1}{\mho_3} & 0 & 0 \\ \mho_4 & \frac{1}{\mho_3} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{C.4.6}$$

where

$$\begin{aligned} \mho_1 &= e-\frac{bd}{a} \\ \mho_2 &= f-\frac{be}{a} \\ \mho_3 &= \frac{b^2-ac}{a} \\ \mho_4 &= \frac{b}{ac-b^2} \\ \mho_5 &= \frac{e}{a}-\mho_2\mho_4 \end{aligned} \quad ; \quad \begin{aligned} \mho_6 &= \frac{d}{a}-\mho_1\mho_4 \\ \mho_7 &= \mho_5-\frac{b}{a}\mho_6 \\ \mho_8 &= \frac{b}{a}\frac{\mho_1}{\mho_3}-\frac{\mho_2}{\mho_3} \\ \mho_9 &= -\frac{\mho_6}{a}-\frac{b}{a}\frac{\mho_7}{\mho_3} \end{aligned} \quad ; \quad \begin{aligned} \mho_{10} &= \frac{\mho_1-b\mho_8}{a\mho_3} \\ \mho_{11} &= -\frac{1}{a}+\frac{b^2}{a^2\mho_3} \\ \mho_{12} &= -\frac{c}{ac-b^2} \\ \mho_{13} &= \frac{b}{ac-b^2} \end{aligned} \qquad \text{C.4.7}$$

Now by defining the following

$$\begin{aligned} \gamma_0 &= a+c \\ \gamma_1 &= ac-b^2 \end{aligned} \qquad \text{C.4.8}$$

it can be shown that q_1^T and q_2^T which are the σ_1-th (third), and σ_2-th (sixth) rows of the L^{-1} matrix are

$$q_1^T = \frac{1}{k_v}\begin{bmatrix} -\frac{c}{\gamma_1} & \frac{b}{\gamma_1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$q_2^T = \frac{1}{k_v}\begin{bmatrix} \frac{b}{\gamma_1} & -\frac{a}{\gamma_1} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{C.4.9}$$

and therefore Q is

$$Q = \begin{bmatrix} q_1^T \\ q_1^T A \\ q_1^T A^2 \\ q_2^T \\ q_2^T A \\ q_2^T A^2 \end{bmatrix} = \frac{1}{k_v}\begin{bmatrix} \frac{-c}{\gamma_1} & \frac{b}{\gamma_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-c}{\gamma_1} & \frac{b}{\gamma_1} & 0 & 0 \\ -1 & 0 & -k_d & 0 & 1 & 0 \\ \frac{b}{\gamma_1} & \frac{-a}{\gamma_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{\gamma_1} & \frac{-a}{\gamma_1} & 0 & 0 \\ 0 & -1 & 0 & -k_d & 0 & 1 \end{bmatrix} \qquad \text{C.4.10}$$

It can also be shown that Q^{-1} is given by

$$Q^{-1} = -k_v\begin{bmatrix} a & 0 & 0 & b & 0 & 0 \\ b & 0 & 0 & c & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 \\ 0 & b & 0 & 0 & c & 0 \\ a & k_d a & -1 & b & k_d b & 0 \\ b & k_d b & 0 & c & k_d c & -1 \end{bmatrix} \qquad \text{C.4.11}$$

Consequently the matrices Â, B̂, and Ĉ are

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & k_d a & 0 & b & k_d b \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b & k_d b & 0 & c & k_d c \end{bmatrix} \qquad \text{C.4.12}$$

$$\hat{B} = QB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{C.4.13}$$

$$\hat{C} = CQ^{-1} = -k_v \begin{bmatrix} a & 0 & 0 & b & 0 & 0 \\ b & 0 & 0 & c & 0 & 0 \end{bmatrix} \qquad \text{C.4.14}$$

Comparison of Â and B̂ with the controllable companion form given by Equation C.3.7 and Equation C.3.8, indicates that the controllable companion form has been achieved through transformation matrix Q.

C.5 STRUCTURE THEOREM

Structure theorem establishes a fundamental structure of dynamical system and provides a most useful relationship between time and frequency domain representations for linear multivariable systems.

To establish the theorem $\hat{A}_{m,}$ and $\hat{B}_m$ must be defined first. Let $\hat{A}_{m,}$ and $\hat{B}_m$ be defined as the $m \times n$ matrix consisting of the m ordered σ_k-th rows of $\hat{A}$, and the $m \times m$ matrix consisting of the m ordered σ_k-th rows of $\hat{B}$ respectively. By inspection of Equation C.3.8, it is noticeable that $\hat{B}_m$, thus defined, is an upper right triangular matrix with ones along the diagonal; i.e.

$$\hat{B}_m = \begin{bmatrix} 1 & x & x & \cdots & x \\ 0 & 1 & x & \cdots & x \\ \vdots & & \ddots & & \vdots \\ & 0 & & & x \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \text{C.5.1}$$

and is nonsingular since, by inspection, $|\hat{B}_m| = 1$. $\hat{A}_m$ assumes no particular form. If now S(s) is defined as the following $n \times m$ polynomial matrix with n nonzero, monic, single-term entries

$$S(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ s & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ s^{d_1-1} & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & s & & \\ \vdots & \vdots & & \vdots \\ 0 & s^{d_2-1} & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ & & & 0 \\ & & & 1 \\ & & & s \\ & & & \vdots \\ 0 & 0 & \cdots & s^{d_m-1} \end{bmatrix} \qquad \text{C.5.2}$$

then the following theorem can be stated.

THEOREM C.5.1 (The Structure Theorem): If a state representation {A,B,C,D} is controllable with B of full rank $m \leq n$, its transfer matrix given by $C(SI-A)^{-1}B+D$, can be expressed as: (proof of the structure theorem can be found in [23])

$$T(s) = \hat{C}S(s)\delta^{-1}(s)\hat{B}_m + D = [\hat{C}S(s) + D\hat{B}_m^{-1}\delta(s)]\,[\hat{B}_m^{-1}\delta(s)]^{-1} \qquad \text{C.5.3}$$

where

$$\delta(s) = \begin{bmatrix} s^{d_1} & 0 & . & . & 0 \\ 0 & s^{d_2} & 0 & . & . \\ . & & & . & . \\ . & & & . & . \\ 0 & . & . & . & s^{d_m} \end{bmatrix} - \hat{A}_m S(s) \qquad \text{C.5.4}$$

The most important aspect of the structure theorem, however, is that it enables one to express the transfer matrix T(s) of a time domain dynamical system as the product of a $p \times m$ polynomial matrix

$$R(s) = \hat{C}S(s) + D\hat{B}_m^{-1}\delta(s) \qquad \text{C.5.5}$$

and the inverse of another $m \times m$ polynomial matrix

$$P(s) = \hat{B}_m^{-1}\delta(s) \qquad \text{C.5.6}$$

i.e.

$$T(s) = R(s)P^{-1}(s) \qquad \text{C.5.7}$$

The two polynomial matrices have certain important properties. In particular (see Equation C.5.5 and Equation C.5.6):

1- P(s) is column proper since $\Gamma_c[P(s)] = \hat{B}_m^{-1}$

2- $\partial_{cj}[R(s)] \leq \partial_{cj}[P(s)]$ $\qquad j=1,2,...m$

C.6 DERIVATION OF THE PLANAR MANIPULATOR OPEN LOOP TRANSFER MATRIX

Structure theorem is employed to derive the open loop transfer matrix of the planar manipulator. For the planar manipulator under the consideration, from the results obtained in Section C.4 and from the definition of $\hat{A}_m$ and $\hat{B}_m$ given in Section C.5 the following can be established:

$$\hat{A}_m = \begin{bmatrix} 0 & a & k_d a & 0 & b & k_d b \\ 0 & b & k_d b & 0 & c & k_d c \end{bmatrix} \quad ; \quad \hat{B}_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{C.6.1}$$

Notice that m=2, and $d_1 = d_2 = 3$ therefore the following expressions for S(s) is obtained:

$$S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ o & s \\ 0 & s^2 \end{bmatrix} \qquad \text{C.6.2}$$

From Equation C.5.4, $\delta(s)$ can be expressed as:

$$\delta(s) = \begin{bmatrix} s^3 & 0 \\ 0 & s^3 \end{bmatrix} - \begin{bmatrix} 0 & a & k_d a & 0 & b & k_d b \\ 0 & b & k_d b & 0 & c & k_d c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ o & s \\ 0 & s^2 \end{bmatrix} \qquad \text{C.6.3}$$

$$= \begin{bmatrix} s^3 - k_d a s^2 - as & -k_d b s^2 - bs \\ -k_d b s^2 - bs & s^3 - k_d c s^2 - cs \end{bmatrix}$$

and since for the planar manipulator, D is a null matrix and $\hat{B}_m$ is an identity matrix (see Equation C.6.1) the following expressions for R(s) and P(s) are obtained:

$$R(s) = \hat{C}S(s) = -k_v \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad \text{C.6.4}$$

$$P(s) = \hat{B}_m^{-1}\delta(s) = \delta(s) = \begin{bmatrix} s^3 - k_d a s^2 - as & -(k_d b s^2 + bs) \\ -(k_d b s^2 + bs) & s^3 - k_d c s^2 - cs \end{bmatrix} \qquad \text{C.6.5}$$

APPENDIX D

D.1 DIAGONALIZED CLOSED LOOP TRANSFER MATRIX FOR THE LINEARIZED PLANAR MANIPULATOR

The objective is, to select $G^{-1}P_F(s)$ such that the overall closed loop transfer matrix of the robot arm system, $R(s)[G^{-1}P_F(s)]^{-1}$, is diagonal (decoupled). It can be shown that if one selects the following

$$G^{-1}P_F(s) = \begin{bmatrix} g_1 w_1(s) & g_3 w_3(s) \\ g_4 w_4(s) & g_2 w_2(s) \end{bmatrix} \qquad \text{D.1.1}$$

where g_1, g_2, g_3, g_4, $w_1(s)$, $w_2(s)$, $w_3(s)$, and $w_4(s)$ are to be determined such that all of the three requirements in section III.3 is satisfied, then

$$[G^{-1}P_F(s)]^{-1} = \frac{1}{g_1 g_2 w_1(S) w_2(s) - g_3 g_4 w_3(s) w_4(s)} \begin{bmatrix} g_2 w_2(s) & -g_3 w_3(s) \\ -g_4 w_4(s) & g_1 w_1(s) \end{bmatrix} \qquad \text{D.1.2}$$

Also from Equation IV.1.1:

$$R(s) = -k_v \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad \textbf{D.1.3}$$

therefore form Equation D.1.2 and Equation D.1.3 the following can be written

$$\begin{aligned} T_c(s) &= R(s)[G^{-1}P_F(s)]^{-1} \\ &= \frac{k_v}{\Delta_c}\begin{bmatrix} ag_2w_2(s)-bg_4w_4(s) & -ag_3w_3(s)+bg_1w_1(s) \\ bg_2w_2(s)-cg_4w_4(s) & -bg_3w_3(s)+cg_1w_1(s) \end{bmatrix} \end{aligned} \qquad \textbf{D.1.4}$$

where

$$\Delta_c \triangleq g_1g_2w_1(s)w_2(s)-g_3g_4w_3(s)w_4(s) \qquad \textbf{D.1.5}$$

Since a diagonal form of the closed loop transfer matrix is desired, the off diagonal entries are set equal to zero from which the following is obtained

$$w_3(s) = \frac{bg_1}{ag_3}w_1(s) \quad ; \quad w_4(s) = \frac{bg_2}{cg_4}w_2(s) \qquad \textbf{D.1.6}$$

Substitution of $w_3(s)$, and $w_4(s)$ from Equation D.1.6 into Equation D.1.4 will result in the following diagonal (decoupled) closed loop transfer matrix:

$$T_c(s) = \begin{bmatrix} \dfrac{ak_v}{g_1w_1(s)} & 0 \\ 0 & \dfrac{ck_v}{g_2w_2(s)} \end{bmatrix} = \begin{bmatrix} \dfrac{c_1}{w_1(s)} & 0 \\ 0 & \dfrac{c_2}{w_2(s)} \end{bmatrix} \qquad \textbf{D.1.7}$$

where

$$c_1 \triangleq \frac{ak_v}{g_1} \quad ; \quad c_2 \triangleq \frac{ck_v}{g_2} \qquad \text{D.1.8}$$

The nonzero constants c_1 and c_2 are derived later.

From Equation D.1.6 , Equation D.1.8, and Equation D.1.1 the following expression for $G^{-1}P_F(s)$ can be written

$$G^{-1}P_F(s) = -k_v \begin{bmatrix} \frac{aw_1(s)}{c_1} & \frac{bw_1(s)}{c_1} \\ \frac{bw_2(s)}{c_2} & \frac{cw_2(s)}{c_2} \end{bmatrix} \qquad \text{D.1.9}$$

Now by letting

$$\begin{aligned} w_1(s) &= (s-p_{d1})(s-p_{d2})(s-p_{d3}) \\ w_2(s) &= (s-p_{d4})(s-p_{d5})(s-p_{d6}) \end{aligned} \qquad \text{D.1.10}$$

where $p_{d1},\ldots,p_{d6}$ are the desired closed loop poles of the system, it can be shown that $G^{-1}P_F(s)$ satisfies all three conditions stated in section II.4. In particular $G^{-1}P_F(s)$ is column proper since

$$|\Gamma_c[G^{-1}P_F(s)]| = \begin{vmatrix} -\frac{ak_v}{c_1} & -\frac{bk_v}{c_1} \\ -\frac{bk_v}{c_2} & -\frac{ck_v}{c_2} \end{vmatrix} = k_v^2 \frac{\gamma_1}{c_1 c_2} \qquad \text{D.1.11}$$

is nonzero, and it also shares the same ordered d_j as P(s). Also the determinant of $G^{-1}P_F(s)$ is the desired characteristic equation of the closed loop system and consequently G^{-1} exists which implies that G is nonsingular.

D.2 DERIVATION OF CONSTANTS c_1 AND c_2 USING FINAL-VALUE THEOREM

The constants c_1 and c_2 are derived by applying the final- value theorem to the closed loop transfer matrix of the system given by Equation D.1.7. The theorem is stated now.

THEOREM D.2.1 (final-value theorem): If $y(t) \leftrightarrow Y(s)$ and if the limit of y(t) as $t \to \infty$ exists, then

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} [sY(s)] \qquad \text{D.2.1}$$

From Equation D.1.7 one can write

$$Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = T(s)\,U(s) = \begin{bmatrix} \frac{c_1}{w_1(s)} & 0 \\ 0 & \frac{c_2}{w_2(s)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \qquad \text{D.2.2}$$

or

$$Y_1(s) = \frac{c_1}{W_1(s)}U_1(s) \quad ; \quad Y_2(s) = \frac{c_2}{W_2(s)}U_2(s) \qquad \text{D.2.3}$$

For a step input, u(t), the following steady state step response is desired:

$$\lim_{t \to \infty} y(t) = 1 \qquad \text{D.2.4}$$

But notice that

$$u(t) \leftrightarrow \frac{1}{s} \qquad \text{D.2.5}$$

Considering Equation D.2.1 through Equation D.2.5 it can be shown that

$$\lim_{t \to \infty} y_1(t) = 1 = \lim_{s \to 0} sY_1(s) = \lim_{s \to 0} \frac{sc_1}{W_1(s)}U_1(s) = \lim_{s \to 0} \frac{c_1}{W_1(s)} = \frac{c_1}{-p_{d1}p_{d2}p_{d3}} \qquad \text{D.2.6}$$

and therefore c_1 must satisfy the following

$$c_1 = -p_{d1}p_{d2}p_{d3} \qquad \text{D.2.7}$$

Using a similar approach it can also be shown that c_2 must fulfil the following

$$c_2 = -p_{d4}p_{d5}p_{d6} \qquad \text{D.2.8}$$

Notice that determination of c_1, and c_2 automatically implies determination of g_1, and g_2 through Equation D.1.8.

APPENDIX E

E. DERIVATION OF COMPENSATORS IN DETAIL

It is shown in Section III.4 that the matrix G is given as:

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{1}{\gamma_1 k_v} \begin{bmatrix} -cc_1 & bc_2 \\ bc_1 & -ac_2 \end{bmatrix} \qquad \text{E.1}$$

If now, $w_1(s)$, and $w_2(s)$ given by Equation IV.1.3, are expanded, the following is obtained:

$$\begin{aligned} w_1(s) &= (s-p_{d1})(s-p_{d2})(s-p_{d3}) \triangleq s^3+\epsilon_2 s^2+\epsilon_1 s+\epsilon_0 \\ w_2(s) &= (s-p_{d4})(s-p_{d5})(s-p_{d6}) \triangleq s^3+\xi_2 s^2+\xi_1 s+\xi_0 \end{aligned} \qquad \text{E.2}$$

where

$$\begin{aligned} \epsilon_0 &\triangleq -p_{d1}p_{d2}p_{d3} & \zeta_0 &\triangleq -p_{d4}p_{d5}p_{d6} \\ \epsilon_1 &\triangleq p_{d1}p_{d2}+p_{d2}p_{d3}+p_{d3}p_{d1} & \zeta_1 &\triangleq p_{d4}p_{d5}+p_{d5}p_{d6}+p_{d6}p_{d4} \\ \epsilon_2 &\triangleq -(p_{d1}+p_{d2}+p_{d3}) & \zeta_2 &\triangleq -(p_{d4}+p_{d5}+p_{d6}) \end{aligned} \qquad \text{E.3}$$

and therefore from Equations IV.1.2, E.1, and E.2 it can be shown that

$$P_F(s) = G(G^{-1}P_F(s))$$

$$= \frac{1}{\gamma_1}\begin{bmatrix} acw_1(s)-b^2w_2(s) & bc(w_1(s)-w_2(s)) \\ ab(w_2(s)-w_1(s)) & acw_2(s)-b^2w_1(s) \end{bmatrix} \tag{E.4}$$

or

$$P_F(s) = \begin{bmatrix} P_{F_{11}}(s) & P_{F_{12}}(s) \\ P_{F_{21}}(s) & P_{F_{22}}(s) \end{bmatrix} \tag{E.5}$$

where

$$\begin{aligned} P_{F_{11}}(s) &= s^3+(\frac{ac\epsilon_2-b^2\zeta_2}{\gamma_1})s^2+(\frac{ac\epsilon_1-b^2\zeta_1}{\gamma_1})s+(\frac{ac\epsilon_0-b^2\zeta_0}{\gamma_1}) \\ P_{F_{12}}(s) &= \frac{bc}{\gamma_1}((\epsilon_2-\zeta_2)s^2+(\epsilon_1-\zeta_1)s+(\epsilon_0-\zeta_0)) \\ P_{F_{21}}(s) &= \frac{ab}{\gamma_1}((\zeta_2-\epsilon_2)s^2+(\zeta_1-\epsilon_1)s+(\zeta_0-\epsilon_0)) \\ P_{F_{22}}(s) &= s^3+(\frac{ac\zeta_2-b^2\epsilon_2}{\gamma_1})s^2+(\frac{ac\zeta_1-b^2\epsilon_1}{\gamma_1})s+(\frac{ac\zeta_0-b^2\epsilon_0}{\gamma_1}) \end{aligned} \tag{E.6}$$

From Equation III.3.5 the following can be written:

$$F(s) = P(s)-P_F(s) = \begin{bmatrix} \eta_3s^2+\eta_2s+\eta_1 & \eta_6s^2+\eta_5s+\eta_4 \\ \eta_9s^2+\eta_8s+\eta_7 & \eta_{12}s^2+\eta_{11}s+\eta_{10} \end{bmatrix} \tag{E.7}$$

where

$$
\begin{aligned}
\eta_1 &= \frac{b^2\zeta_0 - ace_0}{\gamma_1} & \eta_7 &= \frac{ab(e_0-\zeta_0)}{\gamma_1} \\
\eta_2 &= \frac{b^2\zeta_1 - ace_1}{\gamma_1} - a & \eta_8 &= \frac{ab(e_1-\zeta_1)}{\gamma_1} - b \\
\eta_3 &= \frac{b^2\zeta_2 - ace_2}{\gamma_1} - k_d a & \eta_9 &= \frac{ab(e_2-\zeta_2)}{\gamma_1} - k_d b \\
\eta_4 &= \frac{bc(\zeta_0 - e_0)}{\gamma_1} & \eta_{10} &= \frac{b^2 e_0 - ac\zeta_0}{\gamma_1} \\
\eta_5 &= \frac{bc(\zeta_1 - e_1)}{\gamma_1} - b & \eta_{11} &= \frac{b^2 e_1 - ac\zeta_1}{\gamma_1} - c \\
\eta_6 &= \frac{bc(\zeta_2 - e_2)}{\gamma_1} - k_d b & \eta_{12} &= \frac{b^2 e_2 - ac\zeta_2}{\gamma_1} - k_d c
\end{aligned}
\qquad \text{E.8}
$$

which is consistent with the requirement that $\partial_c[F(s)] < \partial_c[P(s)]$.

Since for the system under consideration $\nu = 3$, Q(s) is given by

$$
Q(s) = \begin{bmatrix} s^2 & q_{12}(s) \\ -1 & s^2 + q_{22}(s) \end{bmatrix} \qquad \text{E.9}
$$

with

$$
\begin{aligned}
q_{12}(s) &= \sum_{k=0}^{1} q_{(1-1)(3-1)+k}\, s^k = \sum_{k=0}^{1} q_k\, s^k = q_0 + q_1 s \\
q_{22}(s) &= \sum_{k=0}^{1} q_{(2-1)(3-1)+k}\, s^k = \sum_{k=0}^{1} q_{2+k}\, s^k = q_2 + q_3 s
\end{aligned}
\qquad \text{E.10}
$$

where q_0, q_1, q_2, and q_3 are arbitrary coefficients to be chosen such that the roots of the $|Q(s)| = s^4 + q_3 s^3 + q_2 s^2 + q_1 s + q_0$ remain in the left half s plane.

This particular choice of Q(s) will result in the following expression for Q(s)F(s):

$$Q(s)F(s) \triangleq \begin{bmatrix} (QF)_{11}(s) & (QF)_{12}(s) \\ (QF)_{21}(s) & (QF)_{22}(s) \end{bmatrix} \tag{E.11}$$

where

$$\begin{aligned}
(QF)_{11}(s) &= \eta_3 s^4+(\eta_2+q_1\eta_9)s^3+(\eta_1+q_0\eta_9+q_1\eta_8)s^2+ \\
&\quad (q_0\eta_8+q_1\eta_7)s+q_0\eta_7 \\
(QF)_{12}(s) &= \eta_6 s^4+(\eta_5+q_1\eta_{12})s^3+(\eta_4+q_0\eta_{12}+q_1\eta_{11})s^2+ \\
&\quad (q_0\eta_{11}+q_1\eta_{10})s+q_0\eta_{10} \\
(QF)_{21}(s) &= \eta_9 s^4+(\eta_8+q_3\eta_9)s^3+(\eta_7+q_3\eta_8+q_2\eta_9-\eta_3)s^2+ \\
&\quad (q_3\eta_7+q_2\eta_8-\eta_2)s+(q_2\eta_7-\eta_1) \\
(QF)_{22}(s) &= \eta_{12} s^4+(\eta_{11}+q_3\eta_{12})s^3+(\eta_{10}+q_3\eta_{11}+q_2\eta_{12}-\eta_6)s^2+ \\
&\quad (q_3\eta_{10}+q_2\eta_{11}-\eta_5)s+(q_2\eta_{10}-\eta_4)
\end{aligned} \tag{E.12}$$

but since:

$$S_e(s) = \begin{bmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s & s^2 & s^3 & s^4 & s^5 \end{bmatrix}^T \tag{E.13}$$

and $\beta(s)=\beta S_e(s)=Q(s)F(s)$, it can be shown that β which is a constant matrix, is given by

$$\beta \triangleq \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \cdots & \beta_{1,11} & \beta_{1,12} \\ \beta_{21} & \beta_{22} & \beta_{23} & \cdots & \beta_{2,11} & \beta_{2,12} \end{bmatrix} \tag{E.14}$$

where

$$
\begin{aligned}
\beta_{11} &= q_0\eta_7 & \beta_{21} &= q_2\eta_7-\eta_1 \\
\beta_{12} &= q_0\eta_8+q_1\eta_7 & \beta_{22} &= q_3\eta_7+q_2\eta_8-\eta_2 \\
\beta_{13} &= \eta_1+q_0\eta_9+q_1\eta_8 & \beta_{23} &= \eta_7+q_3\eta_8+q_2\eta_9-\eta_3 \\
\beta_{14} &= \eta_2+q_1\eta_9 & \beta_{24} &= \eta_8+q_3\eta_9 \\
\beta_{15} &= \eta_3 & \beta_{25} &= \eta_9 \\
\beta_{16} &= 0 & \beta_{26} &= 0 \\
\beta_{17} &= q_0\eta_{10} & \beta_{27} &= q_2\eta_{10}-\eta_4 \\
\beta_{18} &= q_0\eta_{11}+q_1\eta_{10} & \beta_{28} &= q_3\eta_{10}+q_2\eta_{11}-\eta_5 \\
\beta_{19} &= \eta_4+q_0\eta_{12}+q_1\eta_{11} & \beta_{29} &= \eta_{10}+q_3\eta_{11}+q_2\eta_{12}-\eta_6 \\
\beta_{1,10} &= \eta_5+a_1\eta_{12} & \beta_{2,10} &= \eta_{11}+q_3\eta_{12} \\
\beta_{1,11} &= \eta_6 & \beta_{2,11} &= \eta_{12} \\
\beta_{1,12} &= 0 & \beta_{2,12} &= 0
\end{aligned}
\qquad \text{E.15}
$$

The following can be derived as the eliminant matrix of the two polynomial matrices R(s) and P(s) for the planar manipulator:

$$
M_e = \begin{bmatrix}
-k_v a & 0 & 0 & 0 & 0 & 0 & -k_v b & 0 & 0 & 0 & 0 & 0 \\
-k_v b & 0 & 0 & 0 & 0 & 0 & -k_v c & 0 & 0 & 0 & 0 & 0 \\
0 & -k_v a & 0 & 0 & 0 & 0 & 0 & -k_v b & 0 & 0 & 0 & 0 \\
0 & -k_v b & 0 & 0 & 0 & 0 & 0 & -k_v c & 0 & 0 & 0 & 0 \\
0 & 0 & -k_v a & 0 & 0 & 0 & 0 & 0 & -k_v b & 0 & 0 & 0 \\
0 & 0 & -k_v b & 0 & 0 & 0 & 0 & 0 & -k_v c & 0 & 0 & 0 \\
0 & -a & -k_d a & 1 & 0 & 0 & 0 & -b & -k_d b & 0 & 0 & 0 \\
0 & -b & -k_d b & 0 & 0 & 0 & 0 & -c & -k_d c & 1 & 0 & 0 \\
0 & 0 & -a & -k_d a & 1 & 0 & 0 & 0 & -b & -k_d b & 0 & 0 \\
0 & 0 & -b & -k_d b & 0 & 0 & 0 & 0 & -c & -k_d c & 1 & 0 \\
0 & 0 & 0 & -a & -k_d a & 1 & 0 & 0 & 0 & -b & -k_d b & 0 \\
0 & 0 & 0 & -b & -k_d b & 0 & 0 & 0 & 0 & -c & -k_d c & 1
\end{bmatrix}
\qquad \text{E.16}
$$

It can be shown that

$$|M_e| = -(k_v^2\gamma_1)^3 \qquad \textbf{E.17}$$

which does not equal to zero. M_e^{-1} is given by:

$$M_e^{-1} = \begin{bmatrix} \xi_{11} & \xi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{23} & \xi_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_{35} & \xi_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{43} & 0 & \xi_{45} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{53} & \xi_{54} & \xi_{55} & \xi_{56} & \xi_{57} & \xi_{58} & 1 & 0 & 0 & 0 \\ 0 & 0 & \xi_{63} & \xi_{64} & \xi_{65} & \xi_{66} & \xi_{67} & \xi_{68} & \xi_{69} & \xi_{6,10} & 1 & 0 \\ \xi_{71} & \xi_{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{83} & \xi_{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_{95} & \xi_{96} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_{10,4} & 0 & \xi_{10,6} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{11,3} & \xi_{11,4} & \xi_{11,5} & \xi_{11,6} & \xi_{11,7} & \xi_{11,8} & 0 & 1 & 0 & 0 \\ 0 & 0 & \xi_{12,3} & \xi_{12,4} & \xi_{12,5} & \xi_{12,6} & \xi_{12,7} & \xi_{12,8} & \xi_{12,9} & \xi_{12,10} & 0 & 1 \end{bmatrix} \qquad \textbf{E.18}$$

where

$$\xi_{11} = -\frac{c}{k_v\gamma_1}$$

$$\xi_{12} = \frac{b}{k_v\gamma_1}$$

$$\xi_{23} = \xi_{11}$$

$$\xi_{24} = \xi_{12}$$

$$\xi_{35} = \xi_{11}$$

$$\xi_{36} = \xi_{12}$$

$$\xi_{43} = -\frac{1}{k_v}$$

$$\xi_{45} = -\frac{k_d}{k_v}$$

$$\xi_{53} = -\frac{k_d a}{k_v}$$

$$\xi_{54} = -\frac{k_d R}{k_v}$$

$$\xi_{55} = -\frac{1+k_d^2 a}{k_v}$$

$$\xi_{56} = -\frac{k_d^2 b}{k_v}$$

$$\xi_{57} = k_d a$$

$$\xi_{58} = k_d b$$

$$\xi_{63} = -\frac{a+k_d^2(a^2+b^2)}{k_v}$$

$$\xi_{64} = -\frac{b+k_d^2(ab+bc)}{k_v}$$

$$\xi_{65} = -\frac{k_d(2a+k_d^2(a^2+b^2))}{k_v}$$

$$\xi_{66} = -\frac{k_d(2b+k_d^2(ab+bc))}{k_v}$$

$$\xi_{67} = a+k_d^2(a^2+b^2)$$

$$\xi_{68} = b+k_d^2(ab+bc)$$

$$\xi_{69} = \xi_{57}$$

$$\xi_{6,10} = \xi_{58}$$

$$\xi_{71} = \frac{b}{k_v\gamma_1}$$

$$\xi_{72} = -\frac{a}{k_v\gamma_1}$$

$$\xi_{83} = \xi_{71}$$

$$\xi_{84} = \xi_{72}$$

$$\xi_{95} = \xi_{71}$$

$$\xi_{96} = \xi_{72}$$

$$\xi_{10,4} = -\frac{1}{k_v}$$

$$\xi_{10,6} = -\frac{k_d}{k_v}$$

$$\xi_{11,3} = -\frac{k_d b}{k_v}$$

$$\xi_{11,4} = -\frac{k_d c}{k_v}$$

$$\xi_{11,5} = -\frac{k_d^2 b}{k_v}$$

$$\xi_{11,6} = -\frac{1+k_d^2 c}{k_v} \qquad \text{E.19}$$

$$\xi_{11,7} = k_d b$$

$$\xi_{11,8} = k_d c$$

$$\xi_{12,3} = \xi_{64}$$

$$\xi_{12,4} = -\frac{c+k_d^2(b^2+c^2)}{k_v}$$

$$\xi_{12,5} = \xi_{66}$$

$$\xi_{12,6} = -\frac{kd(2c+k_d^2(b^2+c^2))}{k_v}$$

$$\xi_{12,7} = \xi_{68}$$

$$\xi_{12,8} = c+k_d^2(b^2+c^2)$$

$$\xi_{12,9} = \xi_{11,7}$$

$$\xi_{12,10} = \xi_{11,8}$$

Since M_e^{-1} exists therefore, $M_e = \hat{M}_e$ or $M_e^{-1} = \hat{M}_e^{-1}$. Also $[H,K] = [\hat{H},\hat{K}]$. In view of the preceding and using Equation III.4.11 the following can be employed to derive H(s) and K(s):

$$[H,K] = \beta M_e^{-1} \tag{E.20}$$

From Equations E.14, E.15, E.18, and E.20, it follows that

$$[H,K] = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1,11} & \mu_{1,12} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2,11} & \mu_{2,12} \end{bmatrix} = \beta M_e^{-1} \tag{E.21}$$

where μ's are given by

$$\begin{aligned}
\mu_{11} &= \beta_{11}\xi_{11}+\beta_{17}\xi_{71} \\
\mu_{12} &= \beta_{11}\xi_{12}+\beta_{17}\xi_{72} \\
\mu_{13} &= \beta_{12}\xi_{23}+\beta_{14}\xi_{43}+\beta_{15}\xi_{53}+\beta_{18}\xi_{83}+\beta_{1,11}\xi_{11,3} \\
\mu_{14} &= \beta_{12}\xi_{24}+\beta_{15}\xi_{54}+\beta_{18}\xi_{84}+\beta_{1,10}\xi_{10,4}+\beta_{1,11}\xi_{11,4} \\
\mu_{15} &= \beta_{13}\xi_{35}+\beta_{14}\xi_{45}+\beta_{15}\xi_{55}+\beta_{19}\xi_{95}+\beta_{1,11}\xi_{11,5} \\
\mu_{16} &= \beta_{13}\xi_{36}+\beta_{15}\xi_{56}+\beta_{19}\xi_{96}+\beta_{1,10}\xi_{10,6}+\beta_{1,11}\xi_{11,6} \\
\mu_{17} &= \beta_{14}+\beta_{15}\xi_{57}+\beta_{1,11}\xi_{11,7} \\
\mu_{18} &= \beta_{15}\xi_{58}+\beta_{1,10}+\beta_{1,11}\xi_{11,8} \\
\mu_{19} &= \beta_{15} \\
\mu_{1,10} &= \beta_{1,11} \\
\mu_{1,11} &= 0 \\
\mu_{1,12} &= 0
\end{aligned} \tag{E.22}$$

$$\mu_{21} = \beta_{21}\xi_{11}+\beta_{27}\xi_{71}$$
$$\mu_{22} = \beta_{21}\xi_{12}+\beta_{27}\xi_{72}$$
$$\mu_{23} = \beta_{22}\xi_{23}+\beta_{24}\xi_{43}+\beta_{25}\xi_{53}+\beta_{28}\xi_{83}+\beta_{2,11}\xi_{11,3}$$
$$\mu_{24} = \beta_{22}\xi_{24}+\beta_{25}\xi_{54}+\beta_{28}\xi_{84}+\beta_{2,10}\xi_{10,4}+\beta_{2,11}\xi_{11,4}$$
$$\mu_{25} = \beta_{23}\xi_{35}+\beta_{24}\xi_{45}+\beta_{25}\xi_{55}+\beta_{29}\xi_{95}+\beta_{2,11}\xi_{11,5}$$
$$\mu_{26} = \beta_{23}\xi_{36}+\beta_{25}\xi_{56}+\beta_{29}\xi_{96}+\beta_{2,10}\xi_{10,6}+\beta_{2,11}\xi_{11,6} \qquad \text{E.23}$$
$$\mu_{27} = \beta_{24}+\beta_{15}\xi_{57}+\beta_{2,11}\xi_{11,7}$$
$$\mu_{28} = \beta_{25}\xi_{58}+\beta_{2,10}+\beta_{2,11}\xi_{11,8}$$
$$\mu_{29} = \beta_{25}$$
$$\mu_{2,10} = \beta_{2,11}$$
$$\mu_{2,11} = 0$$
$$\mu_{2,12} = 0$$

To find H(s) and K(s) Equation III.4.12 is now employed i.e.

$$[H,K]\,M_e S_e(s) = [H,K]\begin{bmatrix} R(s) \\ sR(s) \\ s^2R(s) \\ --- \\ P(s) \\ sP(s) \\ s^2P(s) \end{bmatrix} = H(s)R(s)+K(s)P(s) \qquad \text{E.24}$$

$$= \beta S_e(s) = Q(s)F(s)$$

comparing Equation E.21 with Equation E.24 , it can be shown that:

$$H(s) = \begin{bmatrix} \mu_{15}s^2+\mu_{13}s+\mu_{11} & \mu_{16}s^2+\mu_{14}s+\mu_{12} \\ \mu_{25}s^2+\mu_{23}s+\mu_{21} & \mu_{26}s^2+\mu_{24}s+\mu_{22} \end{bmatrix} \tag{E.25}$$

and

$$K(s) = \begin{bmatrix} \mu_{19}s+\mu_{17} & \mu_{1,10}s+\mu_{18} \\ \mu_{29}s+\mu_{27} & \mu_{2,10}s+\mu_{28} \end{bmatrix} \tag{E.26}$$

If the polynomial matrices H(s), K(s), and Q(s) are now employed in the feedback scheme depicted in Figure I.2, the desired (decoupled) closed loop transfer matrix is obtained.