

AN ABSTRACT OF THE THESIS OF

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Title: Uses of Microcomputers to Enhance Precalculus Mathematics

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Microcomputers are being used in teaching and learning mathematics. This paper examines ways that computers can be used to enhance teaching in precalculus mathematics. The advantages of computers discussed are: 1) instant access to graphs, 2) examples that are not oversimplified to make computations manageable, 3) encouragement of hypothesizing and testing, 4) reinforcement of problem-solving techniques, 5) increasing students' confidence in using mathematics, and 6) changing classroom dynamics to make teachers more like guides.

A literature review reveals two particularly noteworthy studies. The first one looks at the positive effects that computer use had on a beginning calculus class. The second one follows one student and shows how computer use did not enhance her progress in mathematics. Comparison of these two studies indicates some characteristics of both effective and ineffective uses of computers in mathematics.

Specific examples covering five topics are discussed to show how computers can be used advantageously in mathematics classes. The topics covered are 1) local extrema and roots of polynomials, 2) simple word problems, 3) Gaussian elimination, 4) linear programming, and 5) polar graphs. The paper concludes with a discussion of how computers should be used in teaching and learning mathematics, along with some suggestions for additional research directions.

Uses of Microcomputers
to Enhance Precalculus Mathematics

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Uses of Microcomputers to Enhance Precalculus Mathematics

Introduction

In recent years, there has been an continuing call for reform in mathematics education. One area that has received attention is the use of computers in teaching mathematics. The use of technology has been a concern for mathematics reformers at all levels. At the school level and the precalculus college level, reformers are looking at how to use computers both as remedial tutors and as discovery and demonstration tools. Even the National Science Foundation is funding several calculus reform projects that are studying the types of technology available and appropriate uses of them for teaching calculus (Ross, 1989). At higher levels, graphing packages and symbolic manipulation software are being used in courses like differential equations, linear algebra, and probability.

Computers can have beneficial effects on mathematics teaching and learning, if they are used properly, but

. . . like all tools, the computer itself does not supply the solution to our problems. . . . There is no automatic beneficial effect linked to a computer. . . . It is essential, therefore, that we should develop a serious programme of research, experimentation and reflective criticism into the use of informatics and the computer as an aid. (Howson & Kahane, 1986, p. 24)

This paper will examine the current position of mathematics education in such a program of research, experimentation, and criticism of computers in mathematics. In section 1, I will examine the potential benefits and some problems of computer use in mathematics teaching. I will show that computers can be beneficial in learning mathematics. In section two, I will examine the research literature and some results obtained from the use of computers in mathematics. Section three will present some specific examples of ways the computer can be used in precalculus mathematics classes. Specific topics include 1) exploring

polynomials, 2) simple word problems, 3) polar graphs, 4) linear programming, and 5) Gaussian elimination. The last section discusses conclusions and offers suggestions for additional research.

Using Computers in Mathematics

Computers are generally accepted as a potentially positive resource in teaching mathematics at all levels. This section will present some advantages and potential disadvantages of using computers in mathematics classes. I will focus on the effects that computers can have on (a) students' conceptual understanding, (b) their manipulation skills, (c) their attitudes, and (d) the content and methods of instruction.

Advantages

If computers are used properly, they can lead to creative intellectual activity. Alternate ways of approaching problems are facilitated by three features of current computers. First, graphs can be produced quickly for a wide variety of relations. The adage that "a picture is worth a thousand words" may not directly translate to "a graph is worth a thousand equations," but it is true that instant access to graphs can be beneficial. Consider the number of graphs found in mathematics texts, particularly in the lower levels, that illustrate various definitions and theorems. Graphs can also be used as a problem-solving device.

Second, examples need not be oversimplified to make computations manageable. When students study linear programming, most problems involve relatively few constraints, usually no more than two or three. Yet in a real life problem, many more constraints would be reasonable. (See section three for a specific example.) Using a computer to aid in graphing multiple constraints and in finding intersection points would reduce the time needed both in and out of class for these more relevant problems.

Third, computers can aid in discovery and exploration of mathematics. One of the computer's best features is its "what-if" capacity. Students can make hypotheses and test them, ultimately leading to conjectures that require proof. In this process, students must learn to organize information, generalize observations and analyze results. All of these activities are useful skills and are

applicable to any problem. Furthermore, in this exploration mode, students can learn that lots of true examples don't prove anything, while a single counterexample disproves a conjecture. So proof may become a more meaningful activity.

A fourth benefit is that the use of computers to propose and test various solutions to a problem can reinforce problem-solving techniques (Demana & Waits, 1987). Suppose a student, who is unfamiliar with the Binomial Theorem, is asked to find a general formula to calculate $(1 + x)^n$ for any integer value of n . The student could use the computer to generate expansions for several values of n . By examining the coefficients of the various expansions, the student could develop a hypothesis about the general form of the expansion of $(1 + x)^n$. Once the hypothesis is formed, several additional examples could be checked to test its validity. The number of tedious calculations necessary and the possibility of algebraic errors makes exploration of the binomial expansion by hand infeasible; yet, with aid by a computer, such an exploration is possible.

Another beneficial effect of the "what-if" nature of a computer is the aesthetic experience of "aha." Discovery can increase a student's self-confidence in using math to solve problems, and in turn, it can make math more attractive. The student who has discovered some mathematical property (even one that is well-known) has an incentive to study mathematics further (Howson & Kahane, 1986).

Used properly, computers can contribute to the advantages outlined above: 1) instant access to graphs, 2) examples that are not oversimplified to make computations manageable, 3) encouragement of hypothesizing and testing, 4) reinforcement of problem-solving techniques, and 5) increasing students' confidence in using mathematics. A sixth benefit of computers is the change computers can cause in classroom dynamics. Students become more actively engaged in mathematics and teachers become more like guides who help students discover mathematics, rather than gods who disseminate knowledge. Participants at two recent conferences on the use of computers in mathematics agreed that computers were

instrumental in encouraging students to think about and to do mathematics. At the St. Olaf conference on calculus reform in October 1989,

. . .[a] common observation was that, with the availability of computers and symbolic manipulating calculators, calculus classes become occasions not only for instructors to present mathematics, but also for students to "do" mathematics. A lecturer becomes more like a "coach" or guide; a healthy atmosphere of student engagement--and even experimentation--was reported by some. (Hundhausen, 1990, p. 5)

At the Second Annual Conference on Technology in Collegiate Mathematics in November 1989, a similar view was expressed.

Whether high tech or low tech--whether some combination of CAS [computer algebra system] and graphics packages or graphing calculators--students are being actively engaged in the learning process, coming to class early, discussing mathematics, asking questions. Teachers are seeing dramatic changes in themselves, too; they have quit lecturing so much and begun acting more like coaches, less like disseminators of dogma. (Technology, 1990, p. 5)

It is this effect of computers that not only combats passive learning, but also helps present mathematics as an area in which exciting things can be done. Taken together, these six advantages of computer use in mathematics are powerful arguments that computers should be used in mathematics instruction.

Potential Disadvantages

M. F. Atiyah (1986) outlines what he sees as the greatest dangers that computers present to mathematics. He states that, "the most central and insidious problem . . . is essentially the challenge the computer presents to the human intellect" (p.50). As

an example, he notes the solution of the famous four-color problem. (Basically, this theorem says that any planar map, like a map of the world, can be colored using four or fewer colors where adjacent sections must be colored differently.) The solution was obtained by "brute-force" since a computer was used to check hundreds of examples. Even though the specific problem was solved, no new insights were gained that might be applied to other problems. He sees an analogous difficulty in the widespread use of computers and calculators in teaching mathematics. The fear is that an "over-reliance on machines can lead to the atrophy of the human faculties involved" (Atiyah, 1986, p. 50). Why learn the multiplication tables if a calculator can give the answer correctly and easily? Some educators maintain that drill in basic skills aids in understanding mathematical concepts much like practice riding a bicycle aids in understanding balance. (See, for example, Engel, 1979.) The argument here is two-pronged. First, will students just learn to push the right buttons to get the right answers without understanding the concepts? And second, will they lose insights into the workings of mathematics by not performing the calculations themselves? Embodied in these questions are the first two possible problems arising from computer use in mathematics classes.

For the majority of students, a third danger of computer use is the passivity that inflexible, computer-controlled programs can produce. If computers are used solely as teachers, as in the Computer Assisted Instruction (CAI) approach, or as a management system for prescribing programs of study and assessing achievement, as in the Computer Managed Instruction (CMI) approach, they can induce passivity in students (Engel, 1979). This situation contributes to the view that math is something one does in a math class, for no discernable reason, and that mathematics is totally unconnected to anything in "real life."

When computerized education was introduced in the 1960's, there were two powerful influences on its conception and use. One influence was the type and availability of computers. Computers were still large, room-filling, relatively unsophisticated machines. Their graphics capabilities were primitive at best, and they were

not powerful enough to be truly interactive. So while educators saw great potential in computers, the first uses were largely inflexible with the computer in control. Much of the CAI courseware fell into the drill and practice category, further reinforcement that math was not a part of real life (Murakami & Hata, 1986, p.86). The second influence, behavioral psychology, stressed observable behaviors as evidence of understanding. In a defense of CMI, M.W. Allen stated,

Accountability encompasses the need to demonstrate the effectiveness of educational programs in stimulating learning. It includes the need to define the behaviors to be acquired by the learner and the procedures through which the ability to perform those behaviors is measured. (In Heimer, 1979, p. 225)

Since behaviors had to be observable, the intangible aspects of learning, such as attitudes toward mathematics, or transfer of problem-solving methods to new areas, were ignored (Heimer, 1979).

For students who might continue studies in math and eventually do creative work in math, there is a fourth danger: the lure of "entering a field [computer science] which is in an explosive stage of its development and where the opportunities to make your mark are much greater" (Atiyah, 1986). While not all potential mathematicians would be enticed away, the ones that are lost to computers weaken the field by their absence.

One way that computers are used in teaching math is to have students write programs to make a computer perform some task. The rationale is that if a student understands something well enough to teach it to someone else, in this case the computer, then he or she has learned the desired concept. However, too much emphasis can be placed on computer programming instead of mathematics. Students can get caught up in fixing the bugs in their code rather than analyzing the math necessary to complete the task.

It simply isn't true, as we used to think, that students *can't fake it* when writing programs. In fact, in most computer languages, students can fix programs one bug

at a time, without enhancing understanding. (Technology, 1990, p. 5)

So while writing programs can be a way to use computers in learning mathematics, it shouldn't be the only way or even the predominant way.

There are potential disadvantages to the indiscriminate use of computers in mathematics classes: 1) atrophying the human intellect, 2) losing insights into the workings of mathematics by not performing the calculations, 3) inducing student passivity, 4) luring away potential mathematicians into computer science, and 5) de-emphasizing mathematics. All of these are pitfalls to avoid when using computers in mathematics.

Conclusion

There are two questions that remain unanswered in these philosophical discussions. Are the advantages and disadvantages of computer use in mathematics teaching caused by the software, or are they simply the result of how the software is used within the classroom? If they are caused by the software, are packages available, or can packages be constructed, that avoid the disadvantages while accruing the advantages? In the next section, I will review research results and observations on current computer uses that help to answer these questions. In section four, I will return to these questions.

Literature Review

While several articles have been written about how computers are used or can be used in mathematics education, little actual research has been conducted on the effects of computer based instruction on student learning. Most research that has focussed on computer based instruction has been concerned with its effect on students' computational proficiency. In this section, I will detail two studies that have been conducted that look at the effects of computer use on learning mathematics. One study focused on the beneficial effects a computer-based calculus course had on students' conceptual understanding, and the other study shows the ineffectiveness of CAI in arithmetic for one particular student. I will also describe several ways that computers are currently being used in mathematics courses.

Computer-based Calculus

A study that shows the beneficial effects of computers was conducted by Kathleen Heid at Pennsylvania State University in 1984 (Heid, 1988). In this study, Heid taught two classes of introductory applied calculus using graphical and symbol-manipulation computer programs to perform routine calculations. The students enrolled in these courses were majoring in business, architecture, or life science at a large public university. These college students spent the first twelve weeks of a fifteen week course learning the concepts of calculus. The last three weeks emphasized skill development. On the final exam, which emphasized routine skills, these students performed almost as well as those students who had taken a traditional skill-oriented course. In addition, the students in the experimental, computer-based course had performed better on some conceptually-oriented questions than the students in the traditional course.

Both the experimental and traditional courses covered the same topics: graphs, derivatives, modeling, the fundamental theorem of calculus, Riemann sums, and partial derivatives. An example of

the differences in development of these topics is found in the treatment and use of graphs. The traditional course concentrated on using zeros of first derivatives to sketch graphs of second and third degree functions, and on using linear asymptotes to sketch simple rational functions. Graphs were an end, not a means for making comparisons or drawing conclusions. In the experimental classes, graphs were an important tool for finding similarities and differences between functions and for determining properties of a given function. For example, comparisons were made between the graphs of a function and its first and second derivatives to determine the information given by the usual first derivative test for extrema and the second derivative test for concavity. Analogous differences were present in the other topics covered.

Computers served three major functions for the experimental classes:

1. Computers decreased the time and attention usually directed toward mastery of computational skills.
2. Computers provided concrete data for the discussion of calculus ideas. They were used to provide data that students could examine in their search for patterns, to generate initial representations on which students could base their reasoning, and to display examples and counterexamples with which students could corroborate or disprove their conjectures.
3. Computers lent flexibility to the analysis of problem situations. Their easy display of concepts in a large range of representations made feasible the consideration of more difficult problems, opened avenues for exploring several methods of solution for a single problem, and created an environment amenable to convenient exploration of the effects of changing parameters. (Heid, 1988, pp 10-11)

Two ways of assessing conceptual understanding were used on both the experimental and the traditional groups. First, seven

questions were agreed on by the two instructors and were used on quizzes and examinations throughout the term. These questions were unlike either the class exercises or the text exercises. These questions tested concepts and applications, but they did not require computational ability. For example, the following question tested the students' understanding of the concept of derivative.

Suppose transportation specialists have determined that $g(v)$, the number of miles per gallon that a vehicle gets, is a function of the vehicle's speed, in miles per hour.

- a. Interpret, in terms of mileage and speed, the fact that $G'(55) = .4$.
- b. How might that fact be used in a debate about setting an appropriate national speed limit? (Heid, 1988, p. 20)

Another of these concept-oriented questions tested the students' knowledge of the properties of graphs and applications of these properties.

$G(t)$ is the number of people unemployed in a country t weeks after the election of a fiscally conservative President. Translate each of the following facts about the graph $y = G(t)$ into statements about the unemployment situation:

- a. The y-intercept of $y = G(t)$ is 2,000,000.
- b. $G(20) = 3,000,000$.
- c. The slope of $y = G(t)$ at $t = 20$ is 10,000.
- d. $G''(36) = 800$ and $G'(36) = 0$. (Heid, 1988, p. 19)

The seven conceptual questions were divided into sixteen parts. On fourteen of the sixteen parts, the experimental group did better than

the traditional group. (On all parts of the two examples given, the experimental group performed better than the traditional group.)

Second, Heid conducted individual interviews with students from both groups. The questions in these interviews were intended to probe the students' understanding of calculus. In the interviews, several differences were noted between the two groups. Two of these differences particularly revealed deeper conceptual understanding by the experimental group. First, students from the experimental group tended to give more detailed answers in their own words. For example, when asked "What is a derivative?" students from the experimental group typically discussed both slope of a curve at a point and rate of change, in their own words. Students in the traditional group tended to give answers involving the kind of problem a derivative is useful in, or a short textbook description. Very little detail was given by this group, while the experimental group provided more details. Second, the experimental group showed more inclination to reason from basic principles to reconstruct forgotten facts. One student in the experimental group was talking about the second derivative. He paused over whether or not a positive second derivative meant that the graph was concave up at that point. He then drew a concave up curve and reasoned that since the slope of the curve was increasing from left to right that the second derivative would have to be positive. Conversely, a student from the traditional group was explaining how to optimize a function for one variable. When he started discussing the use of the second derivative, he too paused. In this case, however, he did not try to reason out the forgotten information; he simply tried to remember. He decided that a positive second derivative meant that the function had a maximum at that point. These two observations from the interviews convinced me that the experimental group had a more internalized notion of calculus. Heid concluded that the computer-based course had produced better conceptual understanding and only a slight deficit in skills development.

Current Uses

Graphs.

The most commonly mentioned use for computers in mathematics is graphing. The ability to produce accurate graphs quickly and easily has been hailed by many as the most important application of computers. Instead of spending valuable time plotting points (in precalculus math) for graphs that are usually only gross approximations, students and teachers can concentrate instead on analyzing the mathematics involved (Jinich, 1987). Duke University has installed a computer classroom that is equipped with a computer terminal and several ceiling-mounted monitors. One of the goals of this project is to make the computer easy to incorporate into traditional mathematics instruction. One of its suggested uses is to display graphs of functions that illustrate the point under discussion. Student questions that prompt the need to graph unanticipated functions can be dealt with immediately (Kraines & Smith, 1988).

Franklin Demana and Bert Waits describe another use of computer graphics in "Problem Solving Using Microcomputers" (1987). They have developed an interactive, graphing program called *Master Grapher*. This program will graph a variety of standard functions, user-defined functions, and conic, parametric and polar equations. The user can easily change the function or functions being graphed, the scale on the axes, and the portion of the graph shown. Further options allow for rotations, translations, other transformations, and inverse graphs. Some of their suggested uses include determining the number of real solutions to equations and systems of equations and numerical approximations to those solutions. It is possible to approximate maxima and minima of functions, and to simulate "real world" problems too complex for most precalculus classes. (Demana, et al., 1988). Their main interest is encouraging "precalculus students to explore enough examples of graphs of functions to build strong intuition and understanding of their properties." (Demana & Waits, 1987, p.241).

Through a series of what would normally be considered difficult problems, their students explore graphical means of solving problems. For example, given the system of equations

$$y = 3x^2 - 2x + 1$$

$$y = 3 - x + x^5$$

not many students could find solutions to it. But with the aid of a graphing package like *Master Grapher*, students can approximate the real solution. The graphs of these equations are shown in Figure 1.

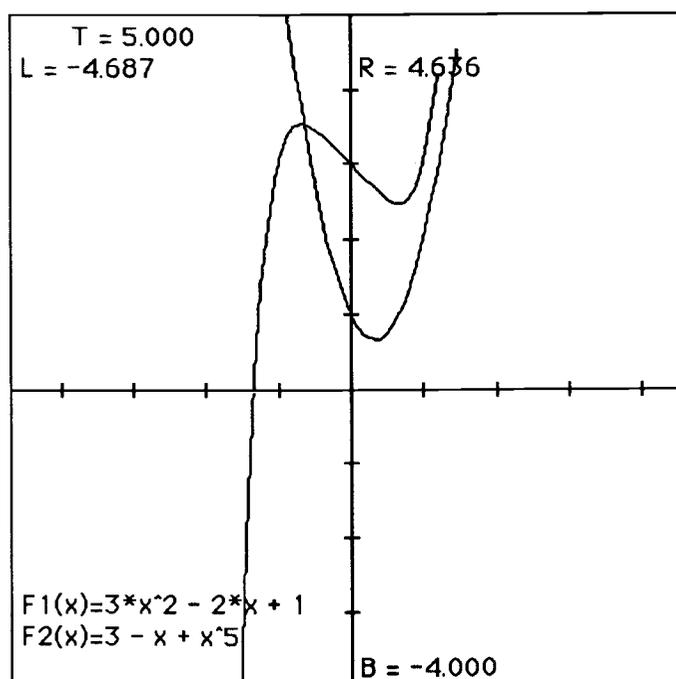


Figure 1. Graph of a system of equations.

A student can zoom in to get a more and more accurate reading of the point of intersection. This method reinforces the idea that a solution to a system of equations must be a solution of each individual equation in the system.

Whether the graphs are a means of approximating a solution or an impetus to finding an exact analytic solution, they serve a valuable purpose. By their nature, good graphing utilities encourage

active mathematical work from students. The students must hypothesize, test, generalize, and then demonstrate their solutions.

Manipulation.

Two types of manipulation which computers can aid are algebraic and arithmetic. One reason for using a computer to perform these manipulations is students' lack of manipulation skills. A common complaint among instructors is that their students do not know enough algebra or basic arithmetic (Young, 1989). Two choices are possible. Remedial instruction can be given, either within a more advanced course or in a specifically remedial course, or a computer can be used to perform the algebra or arithmetic. There are strong arguments both for and against the latter option. Basically their proponents differ on the need to perform manipulations. On one hand, these skills have long been deemed a necessary part of mathematics education. On the other hand, the ability to use mathematics to solve problems is an equally compelling argument even if the computer does the algebra or the arithmetic.

There is, however, a second reason for using a symbol manipulator or a number processor. A symbol manipulator or number processor can allow teachers and students to consider more complex examples. Often, examples are chosen because they illustrate the topic, but are simple enough that time consuming calculations are not required. For example, Heid reports that the traditional group learned to graph only a few types of equations--quadratic, cubic and simple rational functions. The limiting factor in choosing what types of graphs to study was the time necessary to produce the data used in the graphing method. The experimental group was not limited by that factor and so was able to study a variety of polynomial and rational functions. In this way, Heid's study challenges the notion that a knowledge of manipulation skills is necessary for conceptual understanding. Her students used *MuMath*, a high level symbol manipulation system that calculates exact answers to algebra and calculus problems, to perform the

algorithmic operations. They were then able to concentrate on what the results indicated, rather than on the symbol manipulation needed to get the results. In one of the interviews that Heid conducted as part of her study, a student commented on this point in contrasting her work in solving a problem by hand and in solving it with the aid of the computer.

When you first get the problem, you have to do a lot of thinking and analyzing and "what if" situations. And then all of a sudden you get your formula, and you plug in your variables, and then it's like you go into high gear or something. Blinders go on, and your head goes down, and your numbers get put in your mind's calculator, and that's it--you don't think. I only start thinking again when I get my answer, when I've done all my calculations and got my x equals or y equals.

On the machine there's never really that blinder time. Because, for example, if you have your formula and just plug in your variable, it's so quick that you're already thinking about what it could be and how it'll tie into your problem. . . .Because you're not actually doing the crunching and calculations of it, you don't become mechanical, I don't think. (Heid, 1988, p. 23)

When students do not have to "plug and chug," every problem forces them to concentrate on the concepts, through both simple and complex examples.

Similar uses can be made of the arithmetical operations of many computer programs. Programs, like *Matrix*, *Reduce*, *Maple*, and *Mathematica*, are available to perform matrix arithmetic, function evaluations and similar numerical and algebraic calculations. Often students lose the idea of what's being done in their frustration over sign errors and arithmetic mistakes. For example, students in many precalculus courses learn to use Gaussian elimination as a means of solving simultaneous linear equations. Systems with more than two or three equations are seldom attempted because of the time required and the high probability of arithmetic errors. Yet, most "real" examples of systems contain many more than three equations.

"Eventually we must all (even self-proclaimed computer-haters) give in a little and admit that there are just some things that computers can do better than we can" (Frantz, 1986, p.172). What computers *should not* do is make all the decisions. The student must decide what should be done, and then the computer can be the tool that does it. In this way, students can learn the mathematics without simply becoming poor computers.

Conjectures.

This final use of computers can encompass both graphing packages and manipulation programs. In this category, students use programs that let them make guesses about some mathematical idea and then test those guesses. By graphing a function and using some manipulation package on a computer, students can develop hypotheses about properties of the function. Some programs are actually designed to provide such a guess and test situation. One example of this kind of program is the *Geometric Supposer*.

The *Geometric Supposer* allows students to make Euclidean constructions on primitive shapes, and it can repeat a series of constructions on a new figure with a single command. Furthermore, students can take measurements of various parts of the figure. The purpose of the *Geometric Supposer* is to provide enough examples so that a conjecture can be formed that the student may then attempt to prove. Consider an example of how this program might be used by students to explore mathematics. Suppose a student has directed the *Supposer* to draw an arbitrary triangle. Then the student draws in the medians of each side and notices that the medians are concurrent. The *Supposer* allows the student to repeat this construction on many triangles--acute, obtuse, and right. On each triangle, the student sees that the medians are concurrent. Thus the student has discovered a well-known geometric concept.

Will the student now want a proof of this conjecture? The authors of this program, Judah Schwartz and Michal Yerushalmy, comment on the effect its use had on a high-school geometry class. The students became highly involved in making and proving

conjectures. "Rather than disdain the need for proof, as some feared they might, they internalized the need for a way of knowing that was more powerful than demonstration in particular cases" (Schwartz & Yerushalmy, 1987, p. 59). This utility helped students to actually do mathematics--a kind of exploratory mathematics usually shunned or feared.

Another example of this type of computer use can be found in spreadsheet applications. Some common spreadsheets are *Lotus 1-2-3* and *Excel*. A typical spreadsheet example is budget preparation. The user can establish categories and define the mathematical relationship between them, and then change specific values to see what effect the change has on the rest of the values. Deane Arganbright provides numerous examples of how spreadsheets can be used to set up a guess and test problem-solving experience in *Mathematical Applications of Electronic Spreadsheets*, (1985), some of which are included in section three.

CAI and Sigal

An illustration of an ineffective program that did not enhance mathematical learning is found in a case study by Hativa (1988). In this study, Sigal, a second-grade girl in Israel, was observed using a CAI system for practicing arithmetic. This particular system had been evaluated extensively and results showed "that the average rate of progress in arithmetic of Israeli schoolchildren from disadvantaged areas more than doubled with the introduction of the CAI system" (Osin, 1984). For Sigal, however, the results were not good.

The mathematics curriculum was divided into fifteen strands and each strand was subdivided into ten levels for each academic year. Not all strands were active at all grade levels. Each student was given exercises from all strands currently active for that student at a level determined by previous and current performance. Students were expected to progress through an average of ten levels per year in each active strand. During one 5-minute period, Sigal

was given 13 problems representing 5 strands. Some examples of these problems, one from each of the 5 strands, follow.

$$\begin{array}{r} 1. \ 16 \\ +17 \\ \hline ? \end{array}$$

$$\begin{array}{r} 2. \ 81 \\ - 40 \\ \hline ? \end{array}$$

$$3. \ 7 \times 2 = ?$$

4. What number is missing? 450 ? 470 480 490

5. In the evening Avi had 2 marbles. How many marbles did he lose during the day if in the morning he had 11 marbles? (Hativa, 1988, p. 197)

Each student worked with the CAI system twice a week for two consecutive ten minute sessions in addition to three forty-minute sessions of regular, in-class arithmetic instruction per week.

Hativa observed Sigal using the CAI system for several months, and also tested Sigal with paper-and-pencil tests. Comparison of Sigal's performance on CAI and on paper-and-pencil tests showed that Sigal was consistently being given lower level problems on CAI than her paper-and-pencil work indicated that she needed. Several characteristics of the CAI program resulted in Sigal repeating many types of problems that she had already mastered.

1) Each answer had to be entered within a certain time, usually 20 to 30 seconds, so students had to calculate mentally. 2) Each digit was checked against the correct answer as it was entered, so no corrections were possible. 3) When a particular level was not mastered, the CAI system recycled the student through the levels leading up to the one that was not mastered. No remedial tutoring was provided. Sigal had difficulty performing mental arithmetic, and she frequently entered digits in left-right order instead of right-left order as required by the system. Sigal became "stuck" several times on a particular level problem, and she spent much time recycling through problems that she had mastered without improving her performance on the problematic level.

Hativa continued to study Sigal's work with CAI for two more years. At the end of third grade, Sigal was given a paper-and-pencil test that indicated a mastery of a particular strand at the maximum level. However, she was receiving CAI practice at ten levels below the maximum. At the end of the fourth grade, she was still practicing at three levels below the maximum in the CAI system in that particular strand. Four classes of students were given similar tailored paper-and-pencil tests at the same time Sigal was tested. Each student was given problems below, on, and above his or her current level of CAI problems in that strand. All students showed mastery beyond their current level in CAI.

In Sigal's case, CAI did not enhance her progress in mathematics. Hativa concludes that the primary problem with CAI is its inability to adjust to different abilities and difficulties. In short, CAI was too inflexible to be beneficial in this case.

Conclusion

In all of these uses of computers, the single most important characteristic is the way the application involves the student directly in doing and learning mathematics. As was shown in Sigal's case, a computer controlled system can lead to an ineffectual learning environment. On the other hand, when the computer is not in charge, but only used as a tool as in Heid's study, it can be a beneficial tool in learning mathematics. Good uses of computers encourage student activity and conceptual learning; they allow for current, real-life examples; and they improve students attitudes toward math. In the next section, I will offer specific activities that illustrate each of the types of effective computer use discussed in this section.

Sample uses of computers in precalculus mathematics

This section contains five examples of ways computers can be used in a precalculus course. Each example includes a description of its purpose and goals, the name and description of the commercial program used or, in one case, a description of my own program, and sample questions to be answered or problems to be solved using the computer. (A complete listing of the original program is given in the appendix.) These activities require active participation from the student, and they utilize the computer as a tool for learning concepts.

Exploring polynomials

Purpose and goals.

One technique often used in precalculus mathematics to teach graphing of polynomials is finding "key" points and graphing them. Then, assuming continuity, students connect the points, like dot-to-dot pictures. Two kinds of points commonly used in this method are the roots of the polynomial, or x-intercepts, and the local maxima and/or minima. With the advent of easy computer graphics, graphing polynomials "by hand" is not as vital a skill as it once was. However, understanding the graph is just as important as it's always been. These key points that have been used for graphing purposes also have other significance, such as maximizing or minimizing some function that has physical significance, and so we should continue to teach methods for finding these points. By combining a graphics program with a spreadsheet, students can determine approximations to the roots and local extrema of polynomials, and use this information to approximate answers to a variety of problems. All decisions of what to do, and how to interpret the data are left to the student. The computer only organizes information and calculates functional values.

Generally speaking, instructors choose polynomials so that they factor nicely to make the roots easy to find. But most

polynomials do not factor, so we also teach the idea that a root occurs between two x values whose functional values have opposite sign. If two x values can be found with this property, a bisection method can be used to continually refine an approximation to a root of the given polynomial. The bisection method is based on the Intermediate Value Property for continuous functions. That property insures that if the functional value of a continuous function changes sign on a closed interval then a root of the function occurs in the closed interval. The bisection method successively refines the closed interval. Suppose $[a, b]$ is a closed interval such that $f(a)$ and $f(b)$ are of opposite sign, and f is a continuous function. Let m be the midpoint of $[a, b]$. If $f(m) = 0$, then m is a root of the function. If $f(m) \neq 0$, then $f(x)$ must change sign either on $[a, m]$ or $[m, b]$. The interval that $f(x)$ changes sign on contains a root of the function. That interval can now be bisected itself. Using these ideas, students are able to approximate roots as accurately as desired (up to machine precision) for any polynomial.

The main drawback to this technique is the amount of time needed to calculate all the functional values for the endpoints and midpoints of each successive interval. A spreadsheet makes this bisection method a feasible means of approximating roots of polynomials that don't factor nicely. One method of approximating the roots of a polynomial is to read them directly from a graph produced by a graphing utility. This method requires that the scale of the displayed graph be refined so that the x -coordinate of the root is of the desired accuracy. The bisection method using a spreadsheet also refines the interval but does not require repeated changes of the scale of the graph. Using a graphing package to display the graph of the polynomial makes the choice of the initial interval containing a root easy.

The second kind of key points are the local maxima and minima. Without the use of derivatives, students can determine local extrema by looking at where the function changes from increasing to decreasing or vice-versa. Again a graphing package can be used to approximate these points by repeatedly changing the scale of the graph and reading the coordinates of the apparent maxima and

minima. Another method for finding approximations to these points is to use a spreadsheet to calculate functional values to determine where the function is increasing and where it is decreasing. A graph is used to choose an interval that contains a rough estimate that can be refined.

Program

This example explores the polynomial $f(x) = x^5 - x^4 - 2x^3 - 3x^2 + x + 3$. A graph of this function, from *Master Grapher*, appears in Figure 2. The spreadsheet that I use in this example is *Excel*, (see Figures 3 and 4). This example is similar to demonstration 3 in Arganbright (1985). The polynomial is entered in cell D3. The values of the left endpoint, a , and the right endpoint, b , of the initial interval are entered in cells B3 and B5, respectively. This interval is chosen from a graph of the given polynomial, and contains an x intercept. The computer calculates the midpoint, m , of the interval. The computer also displays the three x values: a , m , and b , in that order along with the functional value of each point. The student decides whether the root is between a and m , or between m and b . In the first case, the student replaces the value of b with the value of m , and so defines a new interval half as long that still contains a root of the function. In the second case, he or she replaces the value of a with the value of m , and again defines a new interval half as long that contains a root of the function. This process can be repeated until the values of a and b differ by less than the desired accuracy. The width of the columns in the spreadsheet must be adjusted to display enough digits to allow distinction of the two values, a and b , at the desired accuracy.

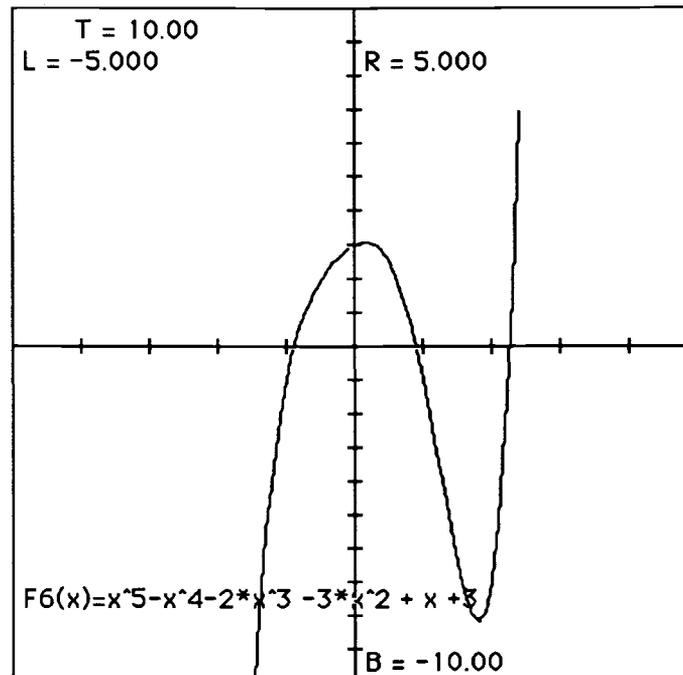


Figure 2. Graph of $f(x) = x^5 - x^4 - 2x^3 - 3x^2 + x + 3$.

	A	B	C	D
1	Roots of	polynomials		
2				$[f(a)] =$
3	a=	0	=B\$3	$=x^5 - x^4 - 2x^3 - 3x^2 + x + 3$
4	m=	=(B\$5+B\$3)/2		
5	b=	1		
6				
7	x=	f(x)=		
8	=B3	=D3		
9	=B4	"=TABLE(,C3)"		
10	=B5	"=TABLE(,C3)"		

Figure 3. Formula template for the bisection method.

	A	B	C	D
1	Roots of polynomials			
2				[f(a)] =
3	a=	-0.8843682	-0.8843682	-3E-08
4	m=	-0.8843682		
5	b=	-0.8843681		
6				
7	x=	f(x)=		
8	-0.8843682	-2.993E-07		
9	-0.8843682	7.2648E-08		
10	-0.8843681	4.4458E-07		

Figure 4. Spreadsheet view of the bisection method.

From the graph of the polynomial (Figure 2), a rough estimate can be made of an interval containing an apparent maximum or minimum. This example is an adaptation of a spreadsheet developed for *Lotus 1-2-3* by Ramamurthi (1988). The polynomial is entered in cell D3 of the spreadsheet as before (see Figure 5). The endpoints, a and b , of the initial interval are entered into cells B3 and B4. The computer then calculates ten equal subdivisions of the interval from a to b . The length of each subinterval is shown as h . At the bottom of the display, the points a , $a + h$, $a + 2h$, . . . , b are listed along with their functional values. At the right, the values -1 and 1 indicate that the functional value in the current row is smaller or larger, respectively, than the value in the row above. The student chooses a refined interval containing the local extrema by picking the x value above and the x value below the row where the right-hand column changes from 1 to -1 or from -1 to 1 . These two x values become the new endpoints, a and b , for a refined interval. The process is repeated on this new interval until a and b differ by the desired accuracy. As before, the column width must be adjusted according to the desired accuracy.

	A	B	C	D
1	Max/min of	polynomials		
2				[f(a)]
3	a=	- 1	=\$B\$3	=x ⁵ - x ⁴ - 2x ³ - 3x ² + x + 3
4	b=	0		
5	h=	=(B\$4-B\$3)/10		
6				
7	x=	f(x)=		higher/lower
8	=\$B\$3	=D3		
9	=A8+\$B\$5	=TABLE(,C3)		=IF(B9>B8,1,-1)
10	=A9+\$B\$5	=TABLE(,C3)		=IF(B10>B9,1,-1)
11	=A10+\$B\$5	=TABLE(,C3)		=IF(B11>B10,1,-1)
12	=A11+\$B\$5	=TABLE(,C3)		=IF(B12>B11,1,-1)
13	=A12+\$B\$5	=TABLE(,C3)		=IF(B13>B12,1,-1)
14	=A13+\$B\$5	=TABLE(,C3)		=IF(B14>B13,1,-1)
15	=A14+\$B\$5	=TABLE(,C3)		=IF(B15>B14,1,-1)
16	=A15+\$B\$5	=TABLE(,C3)		=IF(B16>B15,1,-1)
17	=A16+\$B\$5	=TABLE(,C3)		=IF(B17>B16,1,-1)
18	=A17+\$B\$5	=TABLE(,C3)		=IF(B18>B17,1,-1)

Figure 5. Formula template for approximating a local extremum without calculus.

	A	B	C	D
1	Max/min of	polynomials		
2				[f(a)]
3	a=	0.144225	0.144225	3.07545217
4	b=	0.14423		
5	h=	5E-07		
6				
7	x=	f(x)=		higher/lower
8	0.144225	3.07545217		
9	0.1442255	3.07545217		1
10	0.144226	3.07545217		1
11	0.1442265	3.07545217		-1
12	0.144227	3.07545217		-1
13	0.1442275	3.07545217		-1
14	0.144228	3.07545217		-1
15	0.1442285	3.07545217		-1
16	0.144229	3.07545217		-1
17	0.1442295	3.07545217		-1
18	0.14423	3.07545217		-1

Figure 6. Spreadsheet view of approximating a local extremum without calculus.

Problems.

Any of the usual type of max/min problems can be solved numerically using this spreadsheet. But problems do not need to have integral or even rational extrema to use this method. Likewise, this method allows students to find approximations of roots of polynomials even when those roots are not rational. The example shown in Figures 2 - 5 uses the polynomial $y = x^5 - x^4 - 2x^3 - 3x^2 + x + 3$. The spreadsheet view (Figure 4) shows a root at -0.884368 accurate to 6 decimal places. Figure 6 shows a minimum at $x = 0.14423$.

Because students are not limited in the types of graphs that they explore, they do not see only "easy" graphs like quadratics. The concepts of local extrema and roots of a function can be examined for many polynomials. The combination of a visual representation (in the graph) and a numerical representation (in the spreadsheet) encourages students to consider problems from more than one viewpoint. In this instance, the computer is an aid to discovering some important properties of polynomials as well as an aid to reducing the number of calculations needed. Some additional problems to explore follow.

Find all roots and any local extrema.

1. $f(x) = x^4 + 2x - 1$

2. $f(x) = x^3/3 - 3x^2 + x$

(Can you find a root of problem 2 without a graph?)

3. $f(x) = 3e^x + 4x^3$

Simple Word Problems

Purpose and goals

Many students are frustrated by word problems in mathematics. They have the most difficulty with these problems setting up the equation. Students will write equations that mix

quantities of different types without understanding why they don't get the "right" answers. This exercise provides students with experience in manipulating values within the context of coin, mixture, interest, and distance problems. With more experience within a structured setting, students should develop more of a "feel" for combining quantities properly. It also reinforces the guess-and-test problem solving strategy. (For another approach to using spreadsheets to model algebra word problems, see Arganbright (1985), demonstration 26.)

Program

This example also uses the spreadsheet *Excel*. I have constructed a template to model a certain type of problem (see Figures 7 - 14). A student can experiment with different values in the cells to guess a solution. After some practice with this strategy, a series of questions leads the student to discover the relationships between the different parts of the problem and to write an algebraic equation to solve this problem. For the mixture problem, I have included a series of spreadsheet views of how a student might proceed to a solution to the problem using this method.

Problems.

Below are examples of one problem for each of the four spreadsheets in this application. Additional problems of each type can be found in most college algebra texts. In actual use, several problems of a certain type should be solved using the spreadsheet before questions (b) - (e) are introduced.

1. Use the spreadsheet called *Coins* to find a solution to problem (a). Guess a solution, and try to refine your guess until you get an answer that meets the conditions of the problem. Then answer questions (b) - (e).

- a. John has \$3.25 in change. He has 22 quarters and dimes. How many dimes and how many quarters does John have?
- b. What is the relationship between the number of dimes, the number of quarters, and the total number of coins?
- c. How could you write the number of dimes in terms of the number of quarters?
- d. How does the value of the coins relate to the amount of money John has?
- e. Based on your experiments in part (a), can you write an algebraic statement that represents the problem in part (a)?

	A	B	C	D
1			<i>Coins</i>	
2		number x value	= total	
3	type of coin	number of coins	value of each coin	total money
4				
5	pennies		0.01	=B5*C5
6	nickels		0.05	=B6*C6
7	dimes		0.10	=B7*C7
8	quarters		0.25	=B8*C8
9	half-dollars		0.50	=B9*C9
10				
11	total	=sum(B5:B9)		=sum(D5:D9)

Figure 7. Formula template for a coin problem.

	A	B	C	D
1			<i>Coins</i>	
2		number x value	= total	
3	type of coin	number of coins	value of each coin	total money
4				
5	pennies		0.01	
6	nickels		0.05	
7	dimes	15	0.10	1.50
8	quarters	7	0.25	1.75
9	half-dollars		0.50	
10				
11	total	22		3.25

Figure 8. Spreadsheet view of the solved sample coin problem.

2. Use the spreadsheet called *Mixtures* to find a solution to problem (a). Guess a solution, and try to refine your guess until you get an answer that meets the conditions of the problem. Then answer questions (b) - (d).

a. Suzy has some 70% antifreeze solution. How much pure water and how much 70% solution should she mix together to get 14 liters of 50% antifreeze solution?

b. What is the relationship between the number of liters of water, the number of liters of 70% solution, and the number of liters of 50% solution?

c. How could you write the number of liters of water in terms of the number of liters of 70% solution?

d. Based on your experiments in part (a), can you write an algebraic statement that represents the problem in part (a)?

	A	B	C	D
1		<i>Mixtures</i>		
2	Liters of	solution x %	solute =	Liters of solute
3				
4	Solutions	Liters of solution	% solute	Liters of solute
5				
6	1			=B6 * C6
7	2			=B7 * C7
8	final			=B8 * C8
9				
10	total 1&2	= B6 + B5		=D6 + D7

Figure 9. Formula template for a solution mixture problem.

	A	B	C	D
1		<i>Mixtures</i>		
2	Liters of	solution x %	solute =	Liters of solute
3				
4	Solutions	Liters of solution	% solute	Liters of solute
5				
6	1	7	.70	4.90
7	2	7	0	0.00
8	final	14	.50	7.00
9				
10	total 1&2	14		4.90

Figure 10.a. Initial guess to a solution.

	A	B	C	D
1		<i>Mixtures</i>		
2	Liters of	solution x %	solute =	Liters of solute
3				
4	Solutions	Liters of solution	% solute	Liters of solute
5				
6	1	12	.70	8.40
7	2	2	0	0.00
8	final	14	.50	7.00
9				
10	total 1&2	14		8.40

Figure 10.b. Second guess at a solution.

	A	B	C	D
1		<i>Mixtures</i>		
2	Liters of	solution x %	solute =	Liters of solute
3				
4	Solutions	Liters of solution	% solute	Liters of solute
5				
6	1	9	.70	6.30
7	2	5	0	0.00
8	final	14	.50	7.00
9				
10	total 1&2	14		6.30

Figure 10.c. Third guess at a solution.

	A	B	C	D
1		<i>Mixtures</i>		
2	Liters of	solution x %	solute =	Liters of solute
3				
4	Solutions	Liters of solution	% solute	Liters of solute
5				
6	1	10	.70	7.00
7	2	4	0	0.00
8	final	14	.50	7.00
9				
10	total 1&2	14		7.00

Figure 10.d. Spreadsheet view of the solution to the sample mixture problem.

3. Use the spreadsheet called *Investments* to find a solution to problem (a). Guess a solution, and try to refine your guess until you get an answer that meets the conditions of the problem. Then answer questions (b) - (e).

a. Dave and Clara invested \$20,000, some in bonds that pay 9% interest and the rest in Treasury notes that pay 7.5% interest. How much did they invest in bonds, if their total interest for the first year was \$1710?

b. What is the relationship between the amount invested in bonds, the amount invested in Treasury notes, and the total amount invested?

c. How could you write the amount invested in Treasury notes in terms of the amount invested in bonds?

d. How does the interest earned on each type of investment relate to the total interest earned?

e. Based on your experiments in part (a), can you write an algebraic statement that represents the problem in part (a)?

	A	B	C	D	E
1			Investments		
2		P x	R x T = I		
3	Type of investment	amount invested	interest rate	time	amount of interest
4					
5	Bonds			1	=B5*C5*D5
6	Treasury notes			1	=B6*C6*D6
7					
8	Total	=B5 + B6			=E5 + E6

Figure 11. Formula template for an investment problem.

	A	B	C	D	E
1			Investments		
2		P x	R x T = I		
3	Type of investment	amount invested	interest rate	time	amount of interest
4					
5	Bonds	14000.00	.09	1	1260
6	Treasury notes	6000.00	.075	1	450
7					
8	Total	20000.00			1710

Figure 12. Spreadsheet view of the solved sample investment problem.

4. Use the spreadsheet called *Distance* to find a solution to problem (a). Guess a solution, and try to refine your guess until you get an answer that meets the conditions of the problem. Then answer questions (b) - (e).

a. The Jones family drove to Portland and back in 5 hours. If they averaged 54 mph going and 46 mph returning, how long did it take for them to get to Portland?

b. What is the relationship between the time it took to get to Portland, the time it took to return, and the total time for the roundtrip?

c. How could you write the time it took to return in terms of the time it took to go to Portland?

d. How are the distance traveled to and from Portland related?

e. Based on your experiments in part (a), can you write an algebraic statement that represents the problem in part (a)?

	A	B	C	D
1		Distance		
2		$R \times T = D$		
3	Traveler	Rate	Time	Distance
4				
5	To			$=B5 * C5$
6	From			$=B6 * C6$
7				
8	Total		$=C5 + C6$	

Figure 13. Formula template for a distance problem.

	A	B	C	D
1		Distance		
2		$R \times T = D$		
3	Traveler	Rate	Time	Distance
4				
5	To	54	2.3	124.2
6	From	46	2.7	124.2
7				
8	Total		5	

Figure 14. Spreadsheet view of the solved sample distance problem.

In this example, the computer is used to assist students in developing a sense of what quantities go together in four common types of word problems. The chance to guess at solutions and then to see how well that guess meets the conditions of the problem should give the student a better intuitive notion about solving word problems. As shown in Figure 10, the student tries several combinations of solution and water until the conditions of the problem are met. By comparing the bottom two rows of the spreadsheet, the student decides how to refine the guess at each step. The questions focus the student's attention on the means of reaching an algebraic solution based on his or her experience with a numerical solution. The following ideas about word problems are reinforced with this method. A relationship must be established among the quantities in the problem. The same relationship must hold for like quantities. Totals can only be formed from one quantity at a time; mixed totals are not allowed. These same basic facts are used in all word problems, and the computer provides an easy means of concretely demonstrating these concepts.

Gaussian Elimination

Purpose and goals

Several techniques for solving systems of linear equations are introduced in precalculus mathematics courses. In one method, Gaussian elimination, computers are particularly useful. The major difficulty students have with Gaussian elimination is arithmetic errors. The process of elementary row operations is not troubling, but arithmetic errors frequently make this method very frustrating. Use of a computer can help relieve this frustration, as well as allow for more in-depth study of this common numerical technique.

Program

Several computer programs include a Gaussian elimination routine. Some programs even have partial pivoting within the algorithm. But, for teaching and learning purposes, a program is needed that performs the necessary arithmetic based on instructions from the student. There are spreadsheet templates that implement such a program (see, for example, Arganbright 1985). But the ones that I've seen were confusing, or did not look like the work a student would do by hand. Consequently, I wrote a program for the Apple II computer in Basic. This program asks the student to input the number of equations to be solved, the coefficient matrix, and the right-hand side. At each step, the student must choose to multiply a row by a constant, add one row to another, or switch two rows. The result of each elementary row operation is printed. A student can either continue the process to a row echelon matrix and perform back substitution by hand, or to a reduced row echelon form and read the answers directly from the matrix. In this program, the computer only performs the arithmetic operations; the student must make the decisions on what to do next. Because there are no arithmetic mistakes, students could examine the process more deeply.

Problems

1. Solve the following system of equations:

$$\begin{array}{rclcl} x & + & 2y & + & 3z & = & 4 \\ & & -1y & - & 2z & = & -3 \\ -4x & - & 5y & - & 2z & = & 3 \end{array}$$

In this example, the computer is used primarily as a tool to keep information organized and to keep computations accurate. Even the most proficient student will become frustrated in attempting this technique if forced to do the computations by hand. They are not difficult computations. They are the type of computations that are most likely to result in errors. In this method of solving systems of equations, one computational error at any step can result in a solution that is highly inaccurate. Once a student wades through one 3 by 3 system by hand (which can involve 45 separate additions or multiplications) and finds all the computational errors, he or she is probably ready to consign this solution technique to the trash. Yet this method is a good means of solving relatively small (to a computer) systems. By using the computer to make the computations, the student can learn the concepts of solving systems without the frustration of computational errors.

The printout shown in Figure 15 is what is printed as a record of the student's work. The original matrix is shown, and each elementary row operation that the student chooses to perform is recorded along with the results of that operation. On the computer screen, the student only sees the most recent matrix along with the choices of row operations.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ -4 & -5 & -2 & 3 \end{array}$$

MULTIPLY ROW 1 BY 4 AND ADD THE RESULT TO ROW 3.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 3 & 10 & 19 \end{array}$$

MULTIPLY ROW 2 BY -1.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 10 & 19 \end{array}$$

MULTIPLY ROW 2 BY -2 AND ADD THE RESULT TO ROW 1.

$$\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 10 & 19 \end{array}$$

MULTIPLY ROW 2 BY -3 AND ADD THE RESULT TO ROW 3.

$$\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 10 \end{array}$$

MULTIPLY ROW 3 BY .25.

$$\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2.5 \end{array}$$

MULTIPLY ROW 3 BY 1 AND ADD THE RESULT TO ROW 1.

$$\begin{array}{cccc} 1 & 0 & 0 & .5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2.5 \end{array}$$

MULTIPLY ROW 3 BY -2 AND ADD THE RESULT TO ROW 2.

$$\begin{array}{cccc} 1 & 0 & 0 & .5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2.5 \end{array}$$

Figure 15. Sample printout from Gaussian Elimination with Student Control.

Some additional problems and applications of this method follow.

2. Solve the following system of equations:

$$\begin{array}{rccccrcr} x & + & y & + & z & + & w & = & 6 \\ 2x & + & 3y & & & - & w & = & 0 \\ -3x & + & 4y & + & z & + & 2w & = & 4 \\ x & + & 2y & - & z & + & w & = & 0 \end{array}$$

3. You have studied consistent, consistent-dependent, and inconsistent systems. How can you tell from the print-out whether a system is consistent, consistent-dependent, or inconsistent?

4. One use for solving systems of equations is to fit an n^{th} degree polynomial function to $n + 1$ data points where every data point has a distinct x -coordinate. The n^{th} degree polynomial has the form $f(x) = ax^n + bx^{n-1} + \dots + dx + e$. Since each of the $n + 1$ data points should solve this polynomial, a system of n equations and n unknowns can be created by substituting the values of x and $f(x)$ into the general formula for each data point. The solution of this system gives the coefficients for the n^{th} degree polynomial that fits the $n + 1$ data points. Find a polynomial equation that fits the following data points. $(-2, 4)$, $(-1, 5)$, $(0, 2)$, $(1, -3)$, $(2, -4)$.

Polar Graphs

Purpose and goals.

In precalculus students learn to recognize and to sketch a variety of graphs. While a good graphing package will produce accurate graphs quickly, students who become familiar with common types of graphs have one more way of thinking about common types of problems. Most students have experience with basic linear, quadratic, and cubic graphs, but little or no experience with graphing in polar coordinates, and they find the usual method of plotting points more difficult. Because they lack experience with polar

coordinates, students find polar graphs confusing. This example is designed to provide students with sufficient experience with some common types of polar graphs to allow the students to recognize and sketch these basic types. I have included exercises on cardioids and roses that lead the user to discover the basic shape, orientation, and size of these two types of polar graphs.

Program.

This example utilizes the graphing program *Master Grapher* by Demana and Waits, described earlier, to create graphs in polar coordinates. This program has a default viewing screen of -10 to 10 on both the horizontal and vertical axes. The zoom-in feature allows the student to quickly change the viewing rectangle to include only the graph. *Master Grapher* will graph in either a dot-to-dot mode or a connect-the-dots mode. In most cases, the latter mode produces better graphs.

Problems.

Cardioids

1. Graph on the computer, and copy a sketch onto your paper.
(Type in the equations exactly as I have them written.)
 - a. $r(t) = 1 - \cos(t)$
 - b. $r(t) = 1 - \cos(t - \pi/2)$
 - c. $r(t) = 1 - \cos(t - \pi/6)$

What is the difference between the graph in a and the one in b?

What is the difference between the graph in a and the one in c?

What would happen if you graphed the equation $r(t) = 1 - \cos(t - \theta)$ for any value of θ ? What would the graph look like if the equation were $r(t) = 1 + \cos(t)$?

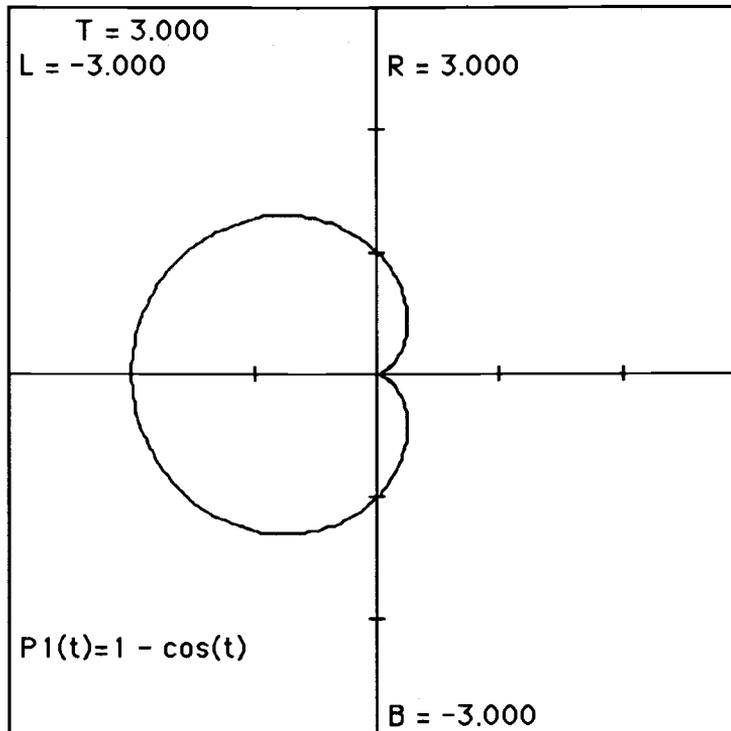


Figure 16.a. Graph of problem 1.a. : $P(t) = 1 - \cos(t)$

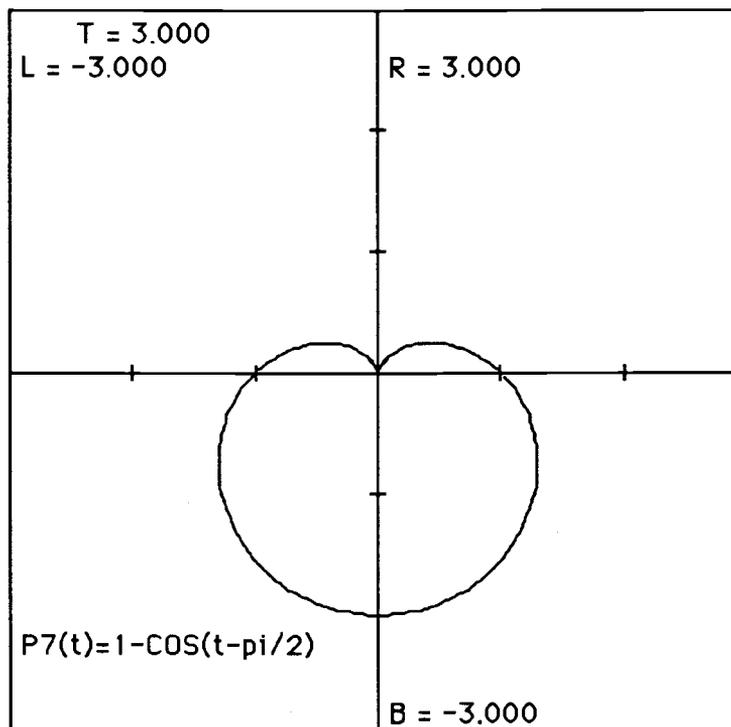


Figure 16.b. Graph of problem 1.b. : $P(t) = 1 - \cos(t - \pi/2)$

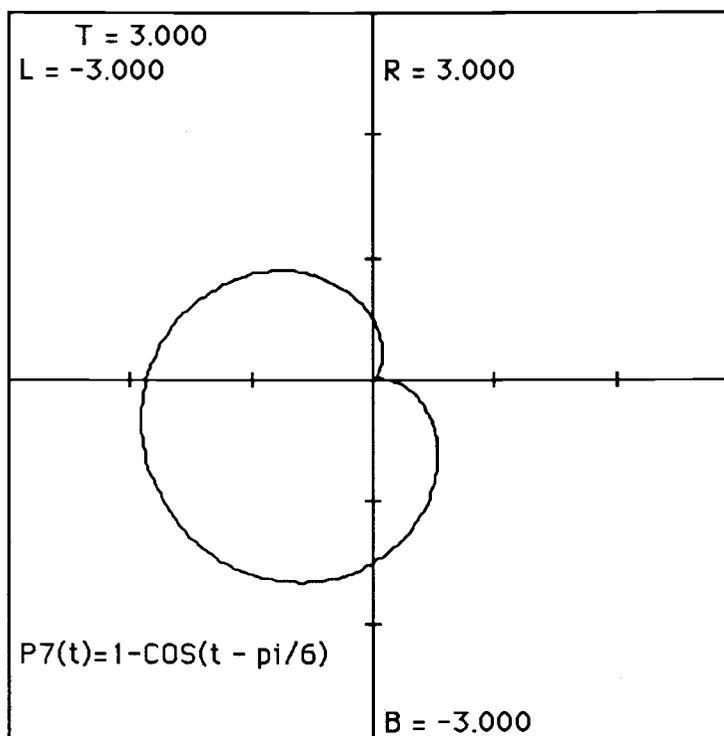


Figure 16.c. Graph of problem 1.c. : $P(t) = 1 - \cos(t - \pi/6)$

2. Graph on the computer, and copy a sketch onto your paper.

- a. $r(t) = 2 - 2\cos(t)$
- b. $r(t) = 3 - 3\cos(t)$
- c. $r(t) = 5 - 5\cos(t)$
- d. $r(t) = 4(1 - \cos(t - \pi/2))$

What effect does a, b, or c have on the basic graph of 1.a.? Which graph in 1 is d. most like, and how does d differ from it?

3. Graph on the computer, and copy a sketch onto your paper.

- a. $r(t) = 1 - \sin(t)$

Which of the previous graphs is this one like?

- b. $r(t) = 3 - 3\sin(t)$

How could you write an equation for this graph in terms of cos?

4. Describe a general means for sketching a graph without the aid of the computer if the equation is $r(t) = a(1 - \cos(t))$ or $r(t) = a(1 + \cos(t))$.

Roses

1. Graph on the computer, and copy a sketch onto your paper.

- $r(t) = \cos(t)$
- $r(t) = \cos(2*t)$
- $r(t) = \cos(3*t)$
- $r(t) = \cos(n*t)$ for several integer values of n .

Find a pattern for the type of graph shown. (Hint: You need to try several values of n in part d. Look at even n 's and odd n 's.)

(Figure 17 shows the graphs of $n = 2, 3, 4, 5$. After graphing several more, the pattern becomes obvious.)

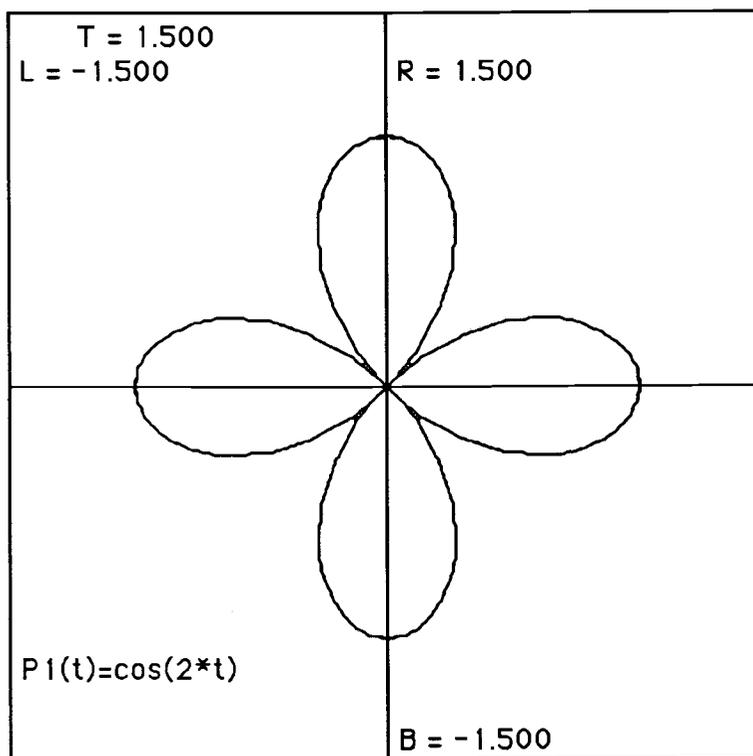


Figure 17.a. Graph of a 4-petaled rose, $n = 2$.

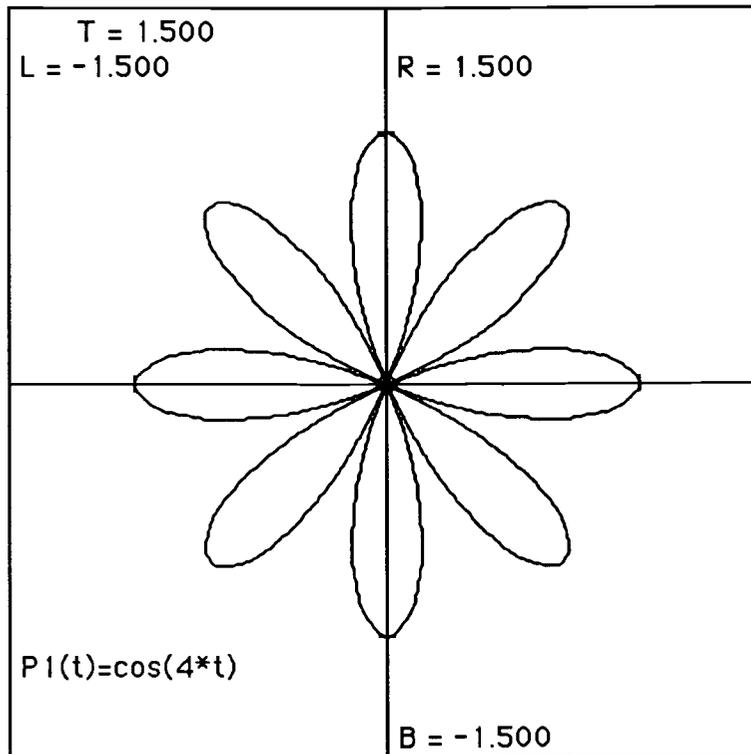


Figure 17.b. Graph of an 8-petaled rose, $n = 4$.

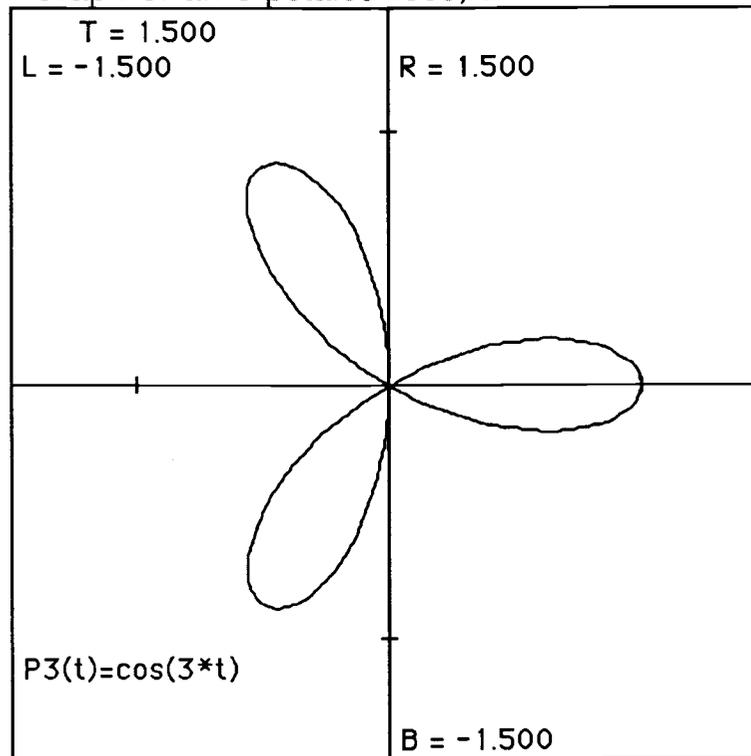


Figure 17.c. Graph of a 3-petaled rose, $n = 3$.

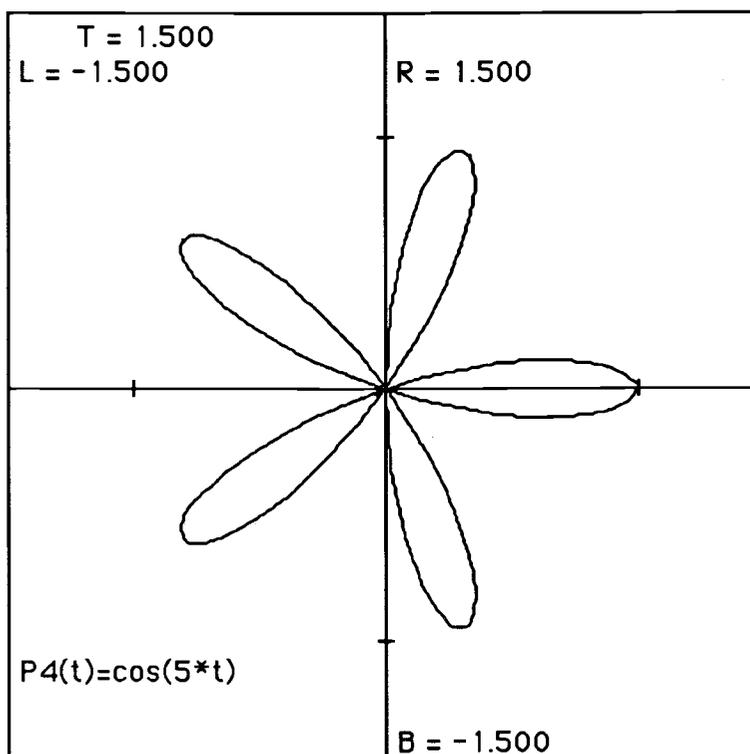


Figure 17.d. Graph of a 5-petaled rose, $n = 5$.

2. Graph on the computer, and copy a sketch onto your paper.
 - a. $r(t) = 3 \cdot \cos(t)$
 - b. $r(t) = 3 \cdot \cos(2 \cdot t)$
 - c. $r(t) = a \cdot \cos(n \cdot t)$ for several values of a and n .

What effect do the values of a and of n have on the graph?

3. Graph on the computer, and copy a sketch of two specific examples onto your paper.
 - a. $r(t) = a \cdot \sin(n \cdot t)$ for several values of a and n .

What other graph does each of these graphs look like? How do they differ? What effect do a and n have on the graph?

4. How are $r(t) = a \cdot \sin(n \cdot t)$ and $r(t) = a \cdot \cos(n \cdot t)$ related?
5. How could you write $r(t) = 5 \cdot \sin(3 \cdot t)$ in terms of \cos , and still get the same graph?
6. Based on your observations from the work above, describe a general means for sketching a graph without the aid of the computer if the equation is $r(t) = a \cdot \cos(n \cdot t)$ or $r(t) = a \cdot \sin(n \cdot t)$.

These examples use the computer to help students discover information about polar graphs that is usually just memorized. By examining many examples of polar graphs, students can become more familiar with the form of the equations and the shapes that those equations produce. For the cardioids, the computer can help students see the effects that various constant parameters have on the basic graph. In the first problem, students are asked to graph several equations. Each of the subsequent graphs is a rotation of the first one. The only difference in the equations in question one is that a constant has been added to the argument of the cosine. By graphing enough examples of this type, students should be able to determine the amount of rotation from the standard position just by examining the equation. Question two on cardioids leads students through a series of graphs that illustrate the effect of multiplying the basic equation by a constant. The size change in these examples is enlargement, but fractions could be used to show reduction. The third question on cardioids combines the ideas of rotation and size change to show the relationship between the graphs defined by cosine and the ones defined by sine. The last question ties together all of the ideas in the first three questions. Students should be able to determine the orientation, amount of rotation, and size based on their explorations with these graphs. In actual use, students would probably need several additional graphs at each step.

The questions concerning the roses also encourage students to determine the major components of each graph. In the first question, students discover that the number of petals on each rose is determined by n in the equations $P(t) = a \cos(nt)$ or $P(t) = a \sin(nt)$. Additional questions explore the length of each petal and the placement of those petals. After working with numerous examples of this type of polar graph, students should be able to tell the basic features of the graph by inspecting the equation. The computer enables the student to have the concrete, visual experience of seeing lots of examples of these graphs.

Another way that *Master Grapher* can be useful in studying polar graphs is in determining the number of points of intersection

and their approximate locations for systems of polar equations. Figure 18 shows two examples of such systems. In Figure 18.a. this system of equations is considered:

$$P1(t) = 2\cos(t)$$

$$P2(t) = 2 - 2\cos(t).$$

This system has three solutions. In 18.b. the system shown is:

$$P1(t) = 1$$

$$P2(t) = 2\sin(2t).$$

This system has eight solutions. Since points do not have unique representations in polar coordinates, the algebraic method for finding the number of solutions is complex. With a picture to start from, at least the student knows how many solutions to look for and their approximate coordinates.

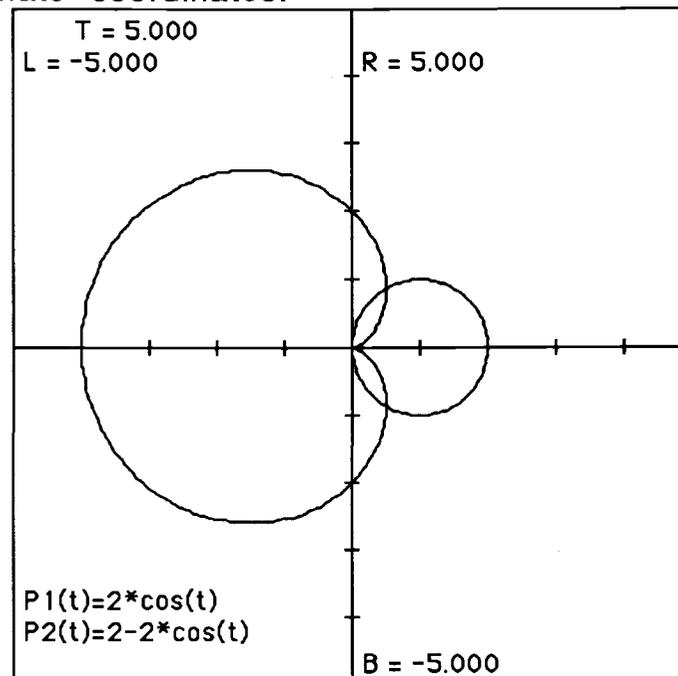


Figure 18.a. Graph of a system of polar equations.

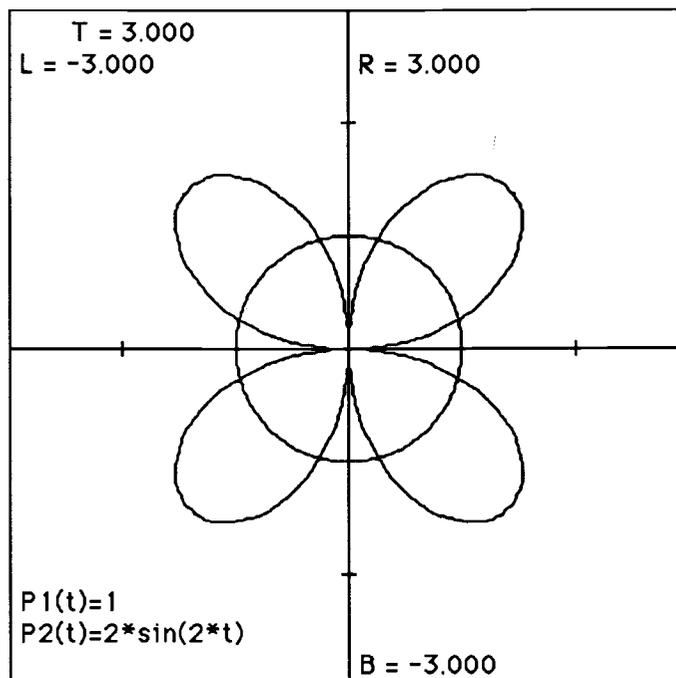


Figure 18.b. Graph of a system of polar equations.

Linear programming

Purpose and goals.

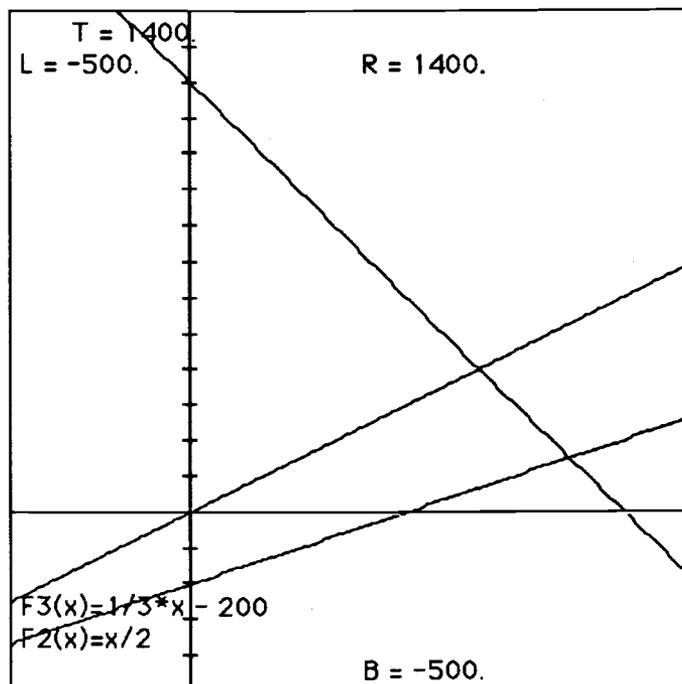
Linear programming is taught in many precalculus and business calculus classes. Students learn to maximize or minimize a function of two variables subject to several constraints. In the process, students must solve several pairs of equations to find the intersection points of the region of possibilities. Since this process takes considerable time, the number of constraints is usually small. By using a graphing package, students can have a quick picture of the region of possibilities. The computer will graph the line, and the student must choose which half-plane fits the constraints. Intersection points can be closely approximated, and these points can be checked in the objective function to choose the maximum or minimum value.

Program.

I have used *Master Grapher* and *Excel* in this example. *Master Grapher* allows up to eight functions to be graphed simultaneously. It also has a "read position" choice that makes it easy to determine the coordinates of a specific point. Once the possible extreme points are determined, a spreadsheet can be used to quickly find the functional value at each point.

Problems.

1. A manufacturer wants to maximize the profit for two products. The first product yields a profit of \$1.50 per unit, and the second product yields a profit of \$2.00 per unit. Market tests and available resources have indicated the following constraints: (1) The combined production level should not exceed 1200 units per month, (2) the demand for product II is at most half of that for product I, and (3) the production level of product I can exceed three times the production level of product II by at most 600 units. Find the number of units of product I and the number of units of product II that the manufacturer should produce in order to maximize the profit and meet the constraints (see Figures 19 and 20).



$$\begin{aligned}
 x + y &< 1200 \\
 y &< x/2 \\
 x &< 3y + 600 \\
 x &> 0 \\
 y &> 0
 \end{aligned}$$

Figure 19. Sample graph of a linear programming problem.

The sample problem is a common type of problem for students to be given. The objective function is $C = 1.5x + 2y$. The constraints are all inequalities. *Master Grapher* will graph only lines so the student must determine which half-plane is indicated by each of the inequalities by checking a point in one half-plane. After all of the constraints are drawn and the polygonal region of possibility is determined, the student must check the coordinates of each vertex of the polygon. The "read position" choice in the menu of *Master Grapher* allows the student to find those coordinates quickly. The spreadsheet in Figure 20 can then be used to check each of these points in the objective function.

	A	B	C
1	<i>Linear</i>	<i>Programming</i>	
2			
3	input	the objective	function in C7
4	and	copy down the	column
5			
6	x=	y=	C=
7	0	0	=1.5*A7+2*B7
8	600	0	=1.5*A8+2*B8
9	1050	150	=1.5*A9+2*B9
10	800	400	=1.5*A10+2*B10

Figure 20.a. Formula template for linear programming function evaluation.

	A	B	C
1	<i>Linear</i>	<i>Programming</i>	
2			
3	input	the objective	function in C7
4	and	copy down the	column
5			
6	x=	y=	C=
7	0	0	0
8	600	0	900
9	1050	150	1875
10	800	400	2000

Figure 20.b. Spreadsheet view of linear programming function evaluation.

The computer aids the student in the mechanical tasks only. The student is left with all of the decisions that involve conceptual

ideas. Another problem like the ones commonly given to students, but with more constraints, is shown below.

2. Find the minimum value of the objective function $C = 4x + 5y$ subject to the constraints:

$$x + y \geq 8$$

$$4x + 3y \geq 27$$

$$3x + 5y \geq 30$$

$$x \geq 0$$

$$y \geq 0$$

(Larson & Hostetler, 1985, p. 373, 376).

Conclusion

All of the examples given in this section can be used within current precalculus mathematics classes. Each of these examples offers a means of giving students concrete examples without requiring excessive numbers of routine calculations. New courses need not be designed in order to include these kinds of computer activities. The computer is beneficial in these examples because it requires active participation from the student. These examples could be used to supplement the usual lecture format or could be used as a primary focus for learning new material. In either case, the students are confronted with the concepts under discussion and do not have to spend most of their time performing calculations.

Discussion and Conclusions

Computers can be used to enhance learning in mathematics. They provide another means of exploring alternate methods of solving problems. The examples given in section three are intended to typify the ways computers are being used successfully in mathematics classrooms. Graphs are used to help students to understand the form of polar equations and the relationships among the different types of equations. Graphs are also used to help students focus on concepts such as local extrema and zeros of polynomials. Students are encouraged to explore word problems with a guess-and-test strategy. The computer is used as a number processor and data organizer. It also aids students in approximating solutions to different types of problems. The examples in section three offer direct, specific experience with mathematical concepts. In the polynomial example, students can see the zeros or local extrema that are being sought. The mathematical computations that refine the estimates of the location of these points are directly related to the graph that the student can see. Both the example involving polar graphs and the ones based on word problems are primarily intended to provide direct experience with mathematical concepts through problem solving. The linear programming and Gaussian elimination examples show how the computer can make routine computations manageable without eliminating student control over solving the problem. All of these examples reinforce problem solving by focussing the student's attention on the mathematical processes involved rather than the calculations. Overall, these uses have one main, common characteristic: The student is forced to think about and to do mathematics. This outcome is the overwhelming advantage that computers offer to us.

Active involvement in thinking about and doing mathematics is not inherent in the machine. The type of software used is important. The computer should be more than just an automatic dispenser of problems. The inflexible nature of the software that Sigal used did not enhance her mathematical progress. Most of her classmates agreed that in-class arithmetic allowed them to explore different

ways of solving problems, while arithmetic on the computer required a specific technique (Hativa, 1988). In this way, the computer reinforced skill-building, not conceptual learning. Conversely, Heid's students, who used computers to study calculus, showed more conceptual understanding than the comparison group who did not use computers. If the software is inflexible, and concentrates only on skill building, there is not much hope that its use will produce anything more than improved computational skills, if that. If, however, the computer can be incorporated into the classroom as another tool with which to investigate and explore mathematics, it can produce better conceptual understanding, and aid in computation.

A difficult question to answer about the use of computers in mathematics is how much impact the way the computer is used has on its effectiveness as a learning tool in mathematics. Research has little to say on this topic so far. Most of what is believed about the impact of the way computers are used in the classroom comes from self-observation by teachers who are using them. The experiences of the instructors quoted in section one show that we must be willing to change the role that we have played for many years. We must help students find different ways of examining a problem, and we must help them to gain experience in thinking about mathematics. Computers can provide a means to achieve these aims if we consider their use carefully.

To accomplish these goals, information is needed in three major areas. First, further studies like Heid's must show that if computers are used properly, they actually improve student learning. These studies must control for teacher effectiveness. If the advantages demonstrated are not the results of using computers but rather are the results of the specific teacher's instruction, then there is no compelling reason to include computers in mathematics. So while anecdotal evidence suggests the benefits that computers can offer, studies need to demonstrate this point. Assuming studies validate the efficacy of computer use in teaching and learning mathematics, there is a second area that must be investigated. We must have more information about specific ways to use the

computer in the classroom. Further studies like Heid's and Hativa's need to be conducted to demonstrate effective and ineffective uses of computers in teaching mathematics so that we will know what constitutes proper use. More information on how computers can be used effectively in the classroom needs to be available to guide instructors in choosing or designing good software and using it to the students' advantage. Specific examples using computers, like the ones I've offered in section three, should be constructed and tested for effectiveness. Along with these specifics in how to use computers effectively, there is a third area to examine. Studies should be conducted that examine the role that teachers must take to make the best use of this new tool. It is clear that using computers will require changes in the standard lecture format of definition - theorem - proof - example. The answer to "What changes?" is not as clear. Several questions need to be explored in this area. What teaching methods are best for incorporating computers into the mathematics classroom effectively? Should computers be used as a primary means of instruction, or should they be used in a laboratory setting to demonstrate specific examples of abstract concepts? What changes are necessary in the training of new instructors to accommodate this new tool? Answers to these questions would help to determine what changes need to be made in the teacher's role.

Computers appear to offer several definite advantages to teaching and learning mathematics. How many of these advantages are actually realized and to what extent remains to be seen. Additional work is needed in this area to ascertain the best way to utilize computers in teaching and learning mathematics.

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APPENDIX

PROGRAM WRITTEN IN BASIC FOR APPLE II TO PERFORM
GAUSSIAN ELIMINATION WITH STUDENT CONTROL

```

5  HOME
10  REM *****
20  REM WRITTEN BY KATHY IVEY
30  REM APRIL 1990
40  REM GAUSSIAN ELIMINATION WITH STUDENT CONTROL
50  REM *****
90  REM THIS SECTION SETS THE NUMBER OF EQUATIONS AND
    REM INITIALIZES THE MATIX.
93  REM THE COEFF. MATRIX IS A, AND THE PERMUT. VECTOR IS P.
95  REM *****
100 INPUT "HOW MANY EQUATIONS ARE IN THE SYSTEM?";N
110 DIM A(N,N+1)
120 FOR I = 1 TO N
125 P(I) = I
130 FOR J = 1 TO N+1
140 A(I,J) = 0
150 NEXT J
160 NEXT I
190 REM *****
192 REM THIS SECTION ENTERS THE AUGMENTED COEFFICIENT
    REM MATRIX.
195 REM *****
199 PRINT : PRINT
200 PRINT "ENTER THE COEFFICIENTS IN ORDER FROM LEFT TO RIGHT
    AND PRESS RETURN AFTER EACH ONE. ENTER 0 FOR ANY MISSING
    POWERS. FINALLY, ENTER THE RIGHT HAND SIDE."
210 FOR I = 1 TO N
220 FOR J = 1 TO N+1
230 INPUT A(I,J)
240 NEXT J
250 NEXT I

```

```
255 PRINT: PRINT "ALL NUMBERS ARE PRINTED WITH 2 DECIMAL
    PLACES OF          ACCURACY SHOWN.": PRINT : PRINT "PRESS
    ANY KEY TO BEGIN."
257 GET K$
260 REM *****
270 REM THIS SECTION PRINTS THE MATRIX A.
280 REM *****
290 GOSUB 500
300 REM *****
310 REM THIS SECTION PRESENTS CHOICES OF ELEMENTARY ROW
    OPERATIONS.
320 REM *****
330 PRINT : PRINT
340 PRINT "PRESS THE NUMBER OF YOUR CHOICE AND THEN PRESS
    RETURN."
350 PRINT
360 PRINT "1. MULTIPLY A ROW BY A CONSTANT"
365 PRINT "2. DIVIDE A ROW BY A CONSTANT"
370 PRINT : PRINT "3. ADD A CONSTANT MULTIPLE OF ONE ROW TO
    ANOTHER"
380 PRINT : PRINT "4. SWITCH TWO ROWS"
390 PRINT : PRINT "5. QUIT"
400 INPUT C
410 IF C <1 OR C>5 THEN PRINT "YOU MUST CHOOSE 1, 2, 3, 4, OR 5.
    PLEASE RE-ENTER.": INPUT C: GOTO 410
420 ON C GOSUB 650, 650, 800, 950, 1100
430 IF A$ = "Y" THEN GOTO 450
440 GOTO 300
450 PRINT CHR$(4); "RUN HELLO"
499 END
500 REM *****
510 REM PRINTING SUBROUTINE
515 REM FIRST ON THE SCREEN, THEN ECHOS ON THE PRINTER
520 REM *****
525 HOME : GOSUB 610
530 FOR I = 1 TO N
```

```

540 FOR J = 1 TO N+1
545 W = INT (A(P(I),J) * 100 + .5) / 100
550 D = 1: IF W < 0 THEN D = 0
555 IF ABS (W) < 1 THEN D = D + 1: IF W = 0 THEN D = D - 1
560 IF ABS (W) < 10 THEN D = D + 1
565 HTAB (7 * (J - 1) + D): PRINT " ";W'
570 NEXT J
580 PRINT
590 NEXT I
595 PRINT : PRINT
600 IF B = 1 THEN PR# 0: GOTO 630
605 B = 1: PR#1
610 IF C = 1 THEN PRINT "MULTIPLY ROW ";R;" BY ";S;". "
615 IF C = 2 THEN PRINT "DIVIDE ROW ";R;" BY ";S;". "
620 IF C = 3 THEN PRINT "MULTIPLY ROW ";T;" BY ";S;" AND ADD THE
    RESULT TO ROW ";U;". "
625 IF C = 4 THEN PRINT "SWITCH ROW ";T;" WITH ROW ";U;". "
630 PRINT
635 RETURN
650 REM *****
660 REM MULTIPLYING OR DIVIDING A ROW BY A CONSTANT
670 REM *****
675 IF C = 1 THEN O$ = "MULTIPLY"
678 IF C = 2 THEN O$ = "DIVIDE"
680 PRINT "FILL IN THE BLANKS."
685 INPUT "MULTIPLY ROW ";R: INPUT "BY THE DECIMAL CONSTANT
    ";S
690 IF R < 1 OR R > N THEN PRINT "RE-ENTER A NUMBER BETWEEN 1
    AND ";N;" FOR THE ROW NUMBER.": INPUT R: GOTO 690
695 IF C = 2 THEN T = S : S = 1/S
700 FOR J = 1 TO N + 1
710 A(P(R),J) = S * A(P(R),J)
720 NEXT J
730 GOSUB 500
740 RETURN
750 REM *****

```

```

800 REM *****
810 REM ADDING TWO ROWS TOGETHER
820 REM *****
830 PRINT : PRINT
835 PRINT "FILL IN THE BLANKS IN THE FOLLOWING SENTENCE."
840 INPUT "MULTIPLY ROW ";T
845 INPUT "BY ";S
850 INPUT "AND ADD THE RESULT TO ROW ";U
860 FOR J = 1 TO N + 1
870 A(P(U),J) = S * A(P(T),J) + A(P(U),J)
880 NEXT J
890 GOSUB 500
900 RETURN
910 REM *****
950 REM *****
960 REM SWITCHING TWO ROWS
970 REM *****
980 PRINT : PRINT "FILL IN THE BLANKS IN THE FOLLOWING
SENTENCE."
990 INPUT "SWITCH ROW ";T
1000 INPUT "WITH ROW ";U
1010 IF T < 1 OR T > N OR U < 1 OR U > N THEN PRINT "PLEASE
RE-ENTER YOUR CHOICES.": GOTO 990
1020 W = P(T)
1030 P(T) = P(U)
1040 P(U) = W
1050 GOSUB 500
1060 RETURN
1070 REM *****
1100 REM *****
1120 REM QUITTING
1130 REM *****
1140 PRINT "DO YOU WANT TO WORK ANOTHER PROBLEM?
PRESS Y IF YOU DO; PRESS N IF YOU DON'T WANT ANOTHER
PROBLEM."
1150 INPUT A$

```

```
1160   IF A$ = "N" THEN END
1170   RETURN
1180   REM   *****
```