## AN ABSTRACT OF THE THESIS OF

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ISOTONIC REGRESSION, UNIVARIATE AND MULTIVARIATE. Redacted for Privacy
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Statistical inference sometimes involves order restrictions which are usually due to prior knowledge. Such order restrictions whenever they occur, will be a major factor in performing a good and reliable statistical analysis. How to make use of these order restrictions is one of the most interesting and the most facinating subjects in statistics nowadays.

Isotonic regression has been formulated and studied for the past 25 years by many statisticians. Their researches are very successful and very fruitful. The theory of such a statistical analysis is called the conditional expectation given a $\sigma$-lattice which is an extension of the conditional expectation. Conditional expectation given a $\sigma-l a t t i c e ~ h a s ~ b e e n ~ a n a l y z e d ~ i n ~ t h e ~ s a m e ~ d i r e c t i o n ~ a s ~ t h a t ~ o f ~ t h e ~ c o n d i-~$ tional expectation. The understanding of the concept of the latter will be very helpful for the study of the former. When several
measurements have been made at each given sample and each measurement has its own restriction, the point estimation of this type is called the multivariate isotonic regression.

The structures, the properties and the algorithms of isotonic regression and of multivariate isotonic regression are the major research in this thesis. Conditional expectation given a $\sigma$-lattice and isotonic regression are presented in separate chapters. They shall be considered as a single unit. The former emphasizes properties and the latter emphasizes algorithms. Multivariate isotonic regression is treated in the simplest case. Only bivariate isotonic regression with linear ordering in each variate will be considered.

The fundamental concept is the generalized projection. Some necessary and sufficient conditions have been presented. Isotonic regression and multivariate isotonic regression are discussed in the finite case. In such a situation, they are generalized projections to finitely generated cones. However, in such general structures, the monotonicity and the averaging property will not be preserved. Although the algorithms are different from each other, they are presented in the same pattern, i.e., as successive projections to linear spaces.

# Conditional Expectation Given a $\sigma$-Lattice. Isotonic Regression, Univariate and Multivariate. 

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# CONDITIONAL EXPECTATION GIVEN A $\sigma-L A T T I C E$. ISOTONIC REGRESSION, UNIVARIATE AND MULTIVARIATE. 

## I. INTRODUCTION

## I. 1 Forward

Over the past 25 years, statisticians have formulated and studied problems of statistical inference in the presence of order conditions arising in various contexts. For the most part, these problems may be interpreted in terms of isotonic regression over a quasi-ordered set. Some examples will be presented in the next section. Because of its theoretical interest and broad applications, the author devoted himself to research in this field.

The regression function of one random variable $\mathbf{X}$ on another, $Y$, is the conditional expectation $\mu(y)=E(X \mid Y=y)$, and furnishes the best fit to the distribution of $X$ by a function of $Y$ in the sense of least-squares. Isotonic regression is introduced in Chapter III by means of least-squares, as a generalization of the regression function. As a generalization of conditional expectation, this concept is called conditional expectation given a $\sigma$-lattice. Since it involves least-squares, isotonic regression is an instance of quadratic programming. Examining the objective function and the feasible region of an isotonic regression problem, we shall quickly
discover that the arithmetical manipulations involved are mainly the routine calculation of means.

Following the isotonic regression is the multivariate isotonic regression. The latter is quite complicated as we shall soon find out in Chapter V. But its nature is essentially the same as that of the former. Thus its research may take similar directions to that of the isotonic regression. Generalized projection, isotonic regression, conditional expectation given a $\sigma$-lattice, and multivariate isotonic regression are studied respectively from Chapter II to Chapter V. The author hopes that these results will provide a general outlook for the problems of least-squares under order restrictions.

## I. 2 Statistical Problems Under Order Restrictions

The (daily maximum) temperatures measured (in Fahrenheit) at Oregon State University during the two periods, March 16th, 17th, 18th and April 16th, 17th, 18th, 19th, 20th of 1974 , were $68^{\circ}, 58^{\circ}$, $56^{\circ}$ and $56^{\circ}, 59^{\circ}, 63^{\circ}, 58^{\circ}, 55^{\circ}$ respectively. The three days' average temperature in March is $x=60.7^{\circ}$ and the five days' average temperature in April is $y=58.2^{\circ}$. One wishes to estimate long run average (daily maximum) temperature in March $\mu$ and that in April $v$ with these eight observations. If we believe that $\mu \leq v$, then the estimates $\hat{\mu}$ and $\hat{v}$ for $\mu$ and $v$ respectively should therefore satisfy the constraing $\hat{\mu} \leq \hat{\nu}$.

The statistical hypothesis testing for $\mu=v$ against its alternative $\mu<\nu$ based on the eight observations will favor the former. Therefore, $\hat{\mu}=\hat{\nu}$ and $\hat{\mu}=\hat{v}=59.1^{\circ}$, the average of the above eight observations, is the best fit to $x$ and $y$ in the sense that $3(x-\hat{\mu})^{2}+5(y-\hat{\nu})^{2}$ has the smallest value among $3(x-\mu)^{2}+5(y-\nu)^{2}$ with $\mu \leq \nu$. (2.4) shows that $3(x-\mu)^{2}+5(y-\nu)^{2} \geq\left[3(x-\hat{\mu})^{2}+5(y-\hat{\nu})^{2}\right]+\left[3(\hat{\mu}-\mu)^{2}+5(\hat{\nu}-\nu)^{2}\right]$. It follows that whenever $x$ and $y$ are consistent estimators for $\mu$ and $v$ then so are $\hat{\mu}$ and $\hat{v}$.

The average (daily maximum) temperature measured at Oregon State University in March 1974 was $54^{\circ}$ and that in April 1974 was $57.5^{\circ}$ (cf. Appendix I). If we compare a three consecutive days' average temperature in March $x$ with a five consecutive days' average in April $y$ of that year, we shall find out that there are 284 pairs such that $\mathrm{x}>\mathrm{y}, 2$ pairs such that $\mathrm{x}=\mathrm{y}$ and 468 pairs such that $\mathrm{x}<\mathrm{y}$. This shows that $\mu=54^{\circ}$ is less than $v=57.5^{\circ}$ by as much as $3.5^{\circ}$, but there is about $3 / 8$ chance that one may observe $x>y$.

Experimental investigations sometimes deal with continuous variables which can not be measured in practice. For example, Dixon and Massey (1957) describe a procedure for testing the sensitivity of explosives to shock. A weight is dropped on specimens of the same explosive mixture from various heights. Suppose that a
given specimen, if it is dropped at some chosen height, will explode; so will it if dropped from any greater height. On the other hand, if it does not explode at that height, neither will any lesser height cause it to explode. Therefore, we may assume that there is a critical height associated with each specimen. The investigator's interest is in the rates of explosions from a population of such specimens dropped from various heights.

In such an experiment it is impossible to make more than one observation on a given specimen. Once a test has been made, and the specimen does not explode, one may suspect that its critical height will be altered. Thus a valid result can not be obtained from a second test. An experimental designer usually divides the sample of specimens into several groups and tests one group at one height, a second group at another height, etc. The data gathered are the numbers of those exploded and of those not exploded at each height.

Let X be the random variable, critical height, with distribution function $F(h)=P_{r}\{X \leq h\}$. Suppose 50 experiments have been made with ten tests at each height $\quad h_{i}, i=1,2, \ldots, 5$; for convenience suppose the heights $h_{1}, h_{2}, \ldots, h_{5}$ are arranged in increasing order. The 50 tests may be regarded as a set of 50 independent trials of events having probabilities $p_{i}=F\left(h_{i}\right), i=1,2, \ldots, 5$, of success if success means that the given specimen will explode at the height. If a large number of trials is made at each height $h_{i}, i=1,2, \ldots, 5$,
the ratios $\quad r_{i}$, number of success divided by number of trials, each determined for a particular height, will with high probability be in non-decreasing order. The best estimates of the probability are then these ratios. Suppose that a small number of trials will be allowed; these ratios may not be in monotone order. In such a case, the maximum likelihood estimates of the probabilities $p_{1}, p_{2}, \ldots, p_{5}$ are determined under order restrictions. This is a typical example of an isotonic regression problem over a linearly ordered finite set.

Ayer and coworkers (1955) showed that the maximum likelihood estimates of $\mathrm{p}_{1}, \mathrm{p}_{2}, \ldots, \mathrm{p}_{\mathrm{n}}$ subject to $\mathrm{p}_{1} \leq \mathrm{p}_{2} \leq \cdots \leq \mathrm{p}_{\mathrm{n}}$ are the same as the least-squares estimates of $p_{1}, p_{2}, \ldots, p_{n}$ subject to the same constraints. Suppose that $r_{1}=0.3, r_{2}=0.2, r_{3}=0.7$, $r_{4}=0.8$ and $r_{5}=0.5$. Then the optimal solution is that $\hat{\mathrm{p}}_{1}=\hat{\mathrm{p}}_{2}=0.25$ and $\hat{\mathrm{p}}_{3}=\hat{\mathrm{p}}_{4}=\hat{\mathrm{p}}_{5}=0.67$ (cf. Example 3.1). There are ten tests at the fifth height with five specimens exploded, 20 tests at the fifth height and the fourth height with 13 specimens exploded, 30 tests at the fifth, fourth and third heights with 20 specimens exploded, etc. The explosive ratios in such combinations of heights are respectively $.5, .65, .67, .55$ and .5 with .67 the largest value. If we drop the 30 specimens at the fifth height which were dropped at the third, fourth and fifth heights, then the explosive ratio will be at least.67. On the other hand, if we drop these 30 specimens at the third height, then the explosive ratio will be no more than 67 .

Similarly for the situation at the first and second heights (cf. Appendix II).

Let $\quad q_{i}=\left(p_{i+1}-p_{i}\right) /\left(h_{i+1}-h_{i}\right), i=1, \ldots, 4$. The ordering $\mathrm{p}_{1} \leq \mathrm{p}_{2} \leq \ldots \leq \mathrm{p}_{5}$ is equivalent to that $\mathrm{q}_{\mathrm{i}} \geq 0, \mathrm{i}=1,2,3,4$. In some situations, we require that $q_{i} \geq a_{i}, i=1,2,3,4$ for a set of non-negative real numbers $a_{1}, a_{2}, a_{3}$ and $a_{4}$. Such a consideration was introduced by Reid (1968). The maximum likelihood estimates of $p_{1}, p_{2}, \ldots, p_{5}$ subject to $p_{1}+0.05 \leq p_{2}, p_{2}+0.05 \leq p_{3}$, $\mathrm{p}_{3}+0.05 \leq \mathrm{p}_{4}, \mathrm{p}_{4}+0.05 \leq \mathrm{p}_{5}$ are $\hat{\mathrm{p}}_{1}=0.225, \hat{\mathrm{p}}_{2}=0.275$, $\hat{\mathrm{p}}_{3}=0.617, \hat{\mathrm{p}}_{4}=0.667, \hat{\mathrm{p}}_{5}=0.717$.

Estimation of variance components in random models is also an example with order restrictions. Consider the two -way random model

$$
y_{i j k}=\mu+a_{i}+b_{j}+c_{i j}+e_{i j k}
$$

$i=1, \ldots, r ; j=1, \ldots, s ; k=1, \ldots, t$ where $\left\{a_{i}\right\},\left\{b_{j}\right\},\left\{c_{i j}\right\}$ and $\left\{\mathrm{e}_{\mathrm{ijk}}\right\}$ are sets of mutually independent normal variates with means zero and variance $\sigma_{A}^{2}, \sigma_{B}^{2}, \sigma_{A B}^{2}$ and $\sigma^{2}$ respectively. Let

$$
\begin{aligned}
& S S_{A}=\operatorname{st} \Sigma_{i=1}^{r}\left(y_{i .} \cdot-y \ldots\right)^{2} \\
& S S_{B}=r t \Sigma_{j=1}^{s}\left(y_{. j},-y \ldots\right)^{2} \\
& S S_{A B}=t \Sigma_{i=1}^{r} \Sigma_{j=1}^{s}\left(y_{i j}-y_{i .}{ }^{-y} . j .+y \ldots\right)^{2}
\end{aligned}
$$

and

$$
\mathrm{SS}_{\mathrm{e}}=\Sigma_{\mathrm{i}=1}^{\mathrm{r}} \Sigma_{\mathrm{j}=1}^{\mathrm{s}} \sum_{\mathrm{k}=1}^{\mathrm{t}}\left(\mathrm{y}_{\mathrm{ijk}}-\mathrm{y}_{\mathrm{ij}}\right)^{2}
$$

and let $\mathrm{MS}_{\mathrm{A}}=\mathrm{SS}_{\mathrm{A}} /(\mathrm{r}-1), \mathrm{MS}_{\mathrm{B}}=\mathrm{SS}_{\mathrm{B}} /(\mathrm{s}-1), \mathrm{MS}_{\mathrm{AB}}=\mathrm{SS}_{\mathrm{AB}} /(\mathrm{r}-1)(\mathrm{s}-1)$ and $\mathrm{MS}_{\mathrm{e}}=\mathrm{SS}_{\mathrm{e}} / \mathrm{rs}(\mathrm{t}-1)$. Then the expectations of $\mathrm{MS}_{\mathrm{A}}, \mathrm{MS}_{\mathrm{B}}$,
$\mathrm{MS}_{\mathrm{AB}}$ and $\mathrm{M} S_{\mathrm{e}}$ are $\sigma^{2}+t \sigma_{\mathrm{AB}}^{2}+\operatorname{st\sigma }_{\mathrm{A}}^{2}, \sigma^{2}+t \sigma_{\mathrm{AB}}^{2}+r t \sigma_{\mathrm{B}}^{2}$, $\sigma^{2}+t \sigma_{A B}^{2}$ and $\sigma^{2}$ respectively. Since variances are non-negative, $E\left(M S_{A}\right) \geq E\left(M S_{A B}\right) \geq E\left(M S_{e}\right)$ and $E\left(M S_{B}\right) \geq E\left(M S_{A B}\right) \geq E\left(M S_{e}\right)$. However, the sample estimates $\mathrm{MS}_{A}, \mathrm{MS}_{B}, \mathrm{MS}_{\mathrm{AB}}$ and $\mathrm{MS}_{\mathrm{e}}$ may not be in such order. Estimating these variance components subject to order constraints is a problem of an isotonic regression over a tree (cf. Appendix III).

Let $(X(t), Y(t))$ be a random vector having a bivariate normal distribution with mean vector $(\mu(t), \nu(t))$ for each $t$ and with a known constant covariance matrix. It has been indicated that the means $\mu(t)$ and $\nu(t)$ are respectively a monotone increasing function of $t$ and a monotone decreasing function of $t$. Suppose a random sample of size $N$ has been given with $N_{i}$ observations at each parameter value $t_{i}, i=1, \ldots, n$. Multivariate isotonic regres sion will yield the maximum likelihood estimates to the sample by a monotone vector-valued function.

For instance, Bhattacharyya and Kotz (1966) use freezing dates and thawing dates for Lake Mendota for a period of 111 years to test
against warming trend. Suppose there is a warming trend in the Lake Mendota during that period and suppose the covariance matrix is constant for each year and it is given. Then the means $\mu(t)$ and $v(t)$ can be obtained by using the Simplified Projection algorithm.

## I. 3 Organization

As the title implies, the core of the study is the conditional expectation given a $\sigma$-lattice and its applications to the isotonic regression and to the multivariate isotonic regression. The application to the latter is indirect.

The algorithms that appear in this paper, Projection of Minimum Violators, Projection of Violators, Pool-Adjacent Violators, Minimum Violators, Minimum Upper Sets, Maximum Lower Sets, and Simplified Projection have the similar structures. Making use of the smoothing property, we obtain the projections of a given vector $X$ to a strictly decreasing sequence of linear spaces successively until we have the desired solution. Therefore, the justification for one algorithm can be applied to that of others. The only difference is the way we identify those pivotal elements and the way we obtain the projection to a linear space. Since the Projection of Minimum Violators algorithm comes first, its justification is given in more detail.

Conditional expectation of a square-integrable random variable given a $\sigma$-lattice, isotonic regression and multivariate isotonic regression are generalized projections to closed convex cones. The generalized projection presented in Chapter II furnishes the background for these topics. Let $H$ be a Hilbert space, let $C$ be a closed convex cone in $H$ and let $S$ be a line ar space in $H$ such that $S \supset C$. The smoothing property shows that $P(X \mid C)=P(P(X \mid S) \mid C)$ for every $X \in H$. This is the property that we shall use constantly for various algorithms in this thesis. The chapter is by itself an extension of results in Brunk (1965).

The aim of Chapter III, isotonic regression, is to develop some efficient algorithms for various types of isotonic regression problems through the use of indicators. We introduce a novel approach, according to which the isotonic regression can be seen as an orthogonal projection to a linear space. But which linear space is a proper one is uncertain. Therefore, we have to use the smoothing property and pivotal elements successively to obtain a linear space we want. Once a proper linear space is found, the isotonic regression for a given function $X$ is the orthogonal projection of $X$ to that linear space. Similarly for the multivariate isotonic regression. For the problems of isotonic regression over a linearly ordered set and isotonic regres sion over a tree, pivotal elements can be identified easily. For the isotonic regression over a partially ordered set, Alexander (1970)
introduced a method which rewrites a partial ordering into a consistent linear ordering.

The properties of isotonic regression are presented in Chapter IV, conditional expectation given a $\sigma$-lattice. Observing the similarities between the isotonic regression and the regression, one can generate a parallel concept, conditional expectation given a $\sigma$-lattice, by that of conditional expectation. Johansen (1967), and Brunk and Johansen (1970) introduced that conditional expectation given a $\sigma$-lattice as the Lebesgue-Radon-Nikodym derivative given a $\sigma$-lattice. Conditional expectation appears as a special case of conditional expectation given a $\sigma$-lattice. A special interest attaches to the case when the $\sigma$-lattice is linearly ordered. In such a situation, there are martingale and submartingale structures.

Although multivariate isotonic regression is the title of Chapter V, all the work has been done on the case of bivariate isotonic regression over a linearly ordered finite set. The author hopes that these results can serve as the "frontier" for the general case. The averaging property and the monotonicity in the bivariate problem have quite different significance as compared with the univariate case, which makes the problem a lot more complicated.

The monograph of Barlow and coworkers (1972), which contains a bibliography listing of 247 published works referred to this subject,
is a complete reference for this thesis, especially its Chapters 1, 2 and 7.

The sequences of theorems, lemmas, and examples will be numbered for each chapter ind ividually. Each corollary will be numbered according to that of the theorem it follows. Theorems and formulas which are quoted will be presented without proof.

Terminologies and notations are standard. Terms will be underlined when they are introduced for the first time in the thesis. Capital letters $X, Y$ and $Z$ will be used to denote functions, vectors, matrices or random variables. Small letters $a, \beta, \gamma, x$, $y, z$ will be used to denote real numbers. The inequality $\leq$ between a pair of functions stands for the same inequality between corresponding components. The same notation $\leq$ between a pair of Greek letters $\mu$ and $\nu$ stands for a quasi-ordering. The meet $a \wedge \beta$ is the smaller number of $a$ and $\beta$. The meet $X \wedge Y$ between a pair of functions is a function such that $(X \wedge Y)(\omega)=X(\omega) \wedge Y(\omega)$ at each argument $\omega$.

## II. GENERALIZED PROJECTION

## II. 1 On a Closed Convex Set

Let $H$ be a Hilbert space: a linear space with an inner product $(\cdot, \cdot)$ such that every Cauchy sequence converges in the space, where the norm of the element $X$ is defined by $\|X\|:=(X, X)^{1 / 2}$. Let $C$ be a subset of $H$. An element $Y$ is said to be a boundary point of $C$ if for every $\delta>0$, there exist $Z \in C$ and $X \notin C$ such that $\|Z-Y\|<\delta$ and $\|X-Y\|<\delta . \quad C$ is said to be closed if it contains all its boundary points. It therefore follows that the limit of a convergent sequence from a closed set is in the set. $C$ is said to be convex if $Y, Z \in C$ implies $\lambda Y+(1-\lambda) Z \in C$ for each real number $\lambda$ between zero and one. Throughout the rest of the section, the letter $C$ will denot a nonempty closed convex set in the Hilbert space $H$. The following result is well known.

Theorem 2.1. Let $C$ be a nonempty closed convex set in the Hilbert space $H$. For every $X \in H$, there exists a unique $X * C$ which minimizes $\|\mathbf{X}-\mathbf{Z}\|$ among all $\mathbf{Z} \in \mathrm{C}$.

The closedness of $C$ yields the existence of $X *$ and the convexity of $C$ yields the uniqueness of $X *$. Such an element $X *$, denoted by $P(X \mid C)$, is called a projection of $X$ on $C$ and the
operator $P(\cdot \mid C)$ is called a generalized projection. The generalized projection $P(\cdot \mid C)$ depends on $C$ and also depends on the inner product $(\cdot, \cdot)$ associated to the Hilbert space $H$. It is trivial that

$$
\begin{equation*}
P(Y+X \mid Y+C)=Y+P(X \mid C) \text { for each } \quad Y \in H \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
P(a X \mid a C)=a P(X \mid C) \text { for each real } a, \tag{2.2}
\end{equation*}
$$

where $Y+C:=\{Y+Z: Z \in C\}$ and $a C:=\{a Z: Z \in C\}$. The sets $Y+C$ and $a C$ are closed and convex.

If $X \in C$, then $P(X \mid C)=X$. A well known result which plays an important role in the theory of projection is the following theorem.

Theorem 2.2. Let $C$ be a nonempty closed convex set in the Hilbert space $H$. If $X \notin C$, then $P(X \mid C)$ is a boundary point of $C$.

Therefore, if $X \& C$, then the candidates for $P(X \mid C)$ are boundary points of $C$. Let $X \notin C, Z \in C$ and let $B(X, Z):=\{Y:\|Y-(X+Z) / 2\| \leq\|X-Z\| / 2\}$. From a geometric point of view, if $Y \in B(X, Z)$ and $Y \neq Z$ then $\|Y-X\|<\|Z-X\|$. It follows that if $X *$ is $P(X \mid C)$ then $B(X, X *) \cap C$ is a singleton $\{X *\}$. Since both $C$ and $B(X, X *)$ are convex, there is a separating hyperplane which separates $C$ from $B(X, X *)$. The hyperplane
supports $B(X, X *)$ at $X *$ so it can be characterized as $\{\mathrm{Y}:(\mathrm{X}-\mathrm{X} *, \mathrm{Y})=(\mathrm{X}-\mathrm{X} *, \mathrm{X} *)\}$. From the separation by the hyperplane, we have $(X-X *, X *-Z) \geq 0$ for every $Z \in C$. On the other hand, the inequality $\|\mathrm{Y}-(\mathrm{X}+\mathrm{Z}) / 2\| \leq\|\mathrm{X}-\mathrm{Z}\| / 2$ is equivalent to $(\mathrm{X}-\mathrm{Y}, \mathrm{Y}-\mathrm{Z}) \geq 0$. If we replace Y by $\mathrm{X}^{*}$, then $\|\mathrm{X}-\mathrm{X} *\| \leq\|\mathrm{X}-\mathrm{Z}\|$. We have thus proved the following theorem.

Theorem 2.3. Let $C$ be a closed convex set in the Hilbert space $H$ and let $X \in H$. An element $X * \in C$ is the projection of $X$ on $C$ if and only if

$$
\begin{equation*}
(X-X *, X *-Z) \geq 0 \quad \text { for every } \quad Z \in C . \tag{2.3}
\end{equation*}
$$

Analytic proofs of the theorem have been given by Brunk (1965) and by Barlow and coworkers (1972). By the identity $\|X-Z\|^{2}=\|X-X *\|^{2}+\|X *-Z\|^{2}+2(X-X *, X *-Z)$, we have the following result.

Corollary 2.3.1. An element $X * \in C$ is the projection of $X$ on $C$ if and only if

$$
\begin{equation*}
\|X-Z\|^{2} \geq\|X-X *\|^{2}+\|X *-Z\|^{2} \quad \text { for every } \quad Z \in C . \tag{2.4}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\|P(X \mid C)-Z\| \leq\|x-Z\| \quad \text { for every } \quad Z \in C \tag{2.5}
\end{equation*}
$$

and if the equality holds for an element $Z \in C$, then $P(X \mid C)=X$.

After a simple manipulation, an immediate result from (2.3) is that

$$
\begin{equation*}
\left\|P\left(X_{1} \mid C\right)-P\left(X_{2} \mid C\right)\right\| \leq\left\|X_{1}-X_{2}\right\| \tag{2.6}
\end{equation*}
$$

for each pair $X_{1}, X_{2} \in H$. This yields the following corollary.

Corollary 2.3.2. The generalized projection $P(\cdot \mid C)$
reduces distance. Therefore, it is a continuous mapping on $H$.

Let $C_{1}$ and $C_{2}$ be closed convex sets in $H$ with $C_{1} \subset C_{2}$. For every $X \in H, \quad$ (2.4) showed that

$$
\begin{align*}
\left\|P\left(X \mid C_{2}\right)-P\left(X \mid C_{1}\right)\right\|^{2} \leq & \left\|x-P\left(X \mid C_{1}\right)\right\|^{2}  \tag{2.7}\\
& -\left\|x-P\left(X \mid C_{2}\right)\right\|^{2}
\end{align*}
$$

Inequalities (2.3), (2.4), (2.6) and (2.7) are given by Brunk (1965). They are fundamental to the concept of the generalized projection $P(\cdot \mid C)$.

Let $S$ be a closed linear space. The closed convex set $Y+S$ is called an affine space. Let $A=Y+S$. Then $A=Y_{1}+S$ for any $Y_{l} \in A$. Write $A=P(X \mid A)+S . B y\left(\begin{array}{ll}2 & 3\end{array}\right)$, we have that $X *=P(X \mid A) \quad$ if and only if $X * \in A$ and $(X-X *, Z)=0 \quad$ for every $Z \in S$ or equivalently, for any $Y_{1} \in A,\left(X-X *, Y_{1}-Z\right)=0$ for every
$Z \in A$. If $A$ is also a closed linear space, i.e., $0 \in A$, then $X *=P(X \mid A) \quad$ if and only if $\quad(X-X *, Z)=0$ for every $Z \in A$. It is obvious that $P\left(P\left(X \mid S_{2}\right) \mid S_{1}\right)=P\left(X \mid S_{1}\right) \quad$ if $\quad S_{1} \quad$ and $\quad S_{2}$ are closed linear spaces and $S_{1} \subset S_{2}$. A similar result under a weaker hypothesis is given below.

Theorem 2.4. Let $C_{1}$ and $C_{2}$ be two closed convex sets in $H$. For every $X \in H$, if either $C_{2}$ is an affine space and $P\left(P\left(X \mid C_{2}\right) \mid C_{1}\right) \in C_{2}$, or $C_{1}$ is an affine space and $C_{1} C C_{2}$, then $P\left(P\left(X \mid C_{2}\right) \mid C_{1}\right)=P\left(X \mid C_{1} \cap C_{2}\right)$.

Proof. The intersection of closed convex sets is also closed and convex. Therefore, the operator $P\left(\cdot \mid C_{1} \cap C_{2}\right)$ is well defined. Let $Y_{2}=P\left(X \mid C_{2}\right)$ and let $Y_{1}=P\left(Y_{2} \mid C_{1}\right)$. We are going to show that $Y_{1}=P\left(X \mid C_{1} \cap C_{2}\right)$ if either one of the above two hypotheses is true.

Suppose $C_{2}$ is an affine space and $Y_{1} \in C_{2}$. Then $\left(X-Y_{2}, Y_{1}-Z\right)=0$ for every $Z \in C_{2}$ by the fact that $Y_{2}=P\left(X \mid C_{2}\right)$, and $\left(Y_{2}-Y_{1}, Y_{1}-Z\right) \geq 0$ for every $Z \in C_{1}$ by the fact that $Y_{1}=P\left(Y_{2} \mid C_{1}\right)$. Hence, $\quad\left(X-Y_{1}, Y_{1}-Z\right) \geq 0$ for every $Z \in C_{1} \cap C_{2}$. By Theorem 2.3, $\mathrm{Y}_{1}=\mathrm{P}\left(\mathrm{X} \mid \mathrm{C}_{1} \cap \mathrm{C}_{2}\right)$.

Suppose $C_{1}$ is an affine space and $C_{1} \subset C_{2}$. Then there exists a closed linear space $S$ in $H$ such that $C_{1}=Y_{1}+S$. If we can show that $\left(X-Y_{1}, Z\right)=0$ for every $Z \in S$, then $\mathrm{Y}_{1}=\mathrm{P}\left(\mathrm{X} \mid \mathrm{C}_{1}\right)=\mathrm{P}\left(\mathrm{X} \mid \mathrm{C}_{1} \cap \mathrm{C}_{2}\right)$. We claim that $\mathrm{Y}_{2}+\mathrm{S}$ is a subset of
$C_{2}$. By the convexity of $C_{2}, \lambda Y_{2}+(1-\lambda)\left(Y_{1}+a Z\right) \in C_{2}$ for each $\lambda \in(0,1)$, for each real $a$ and for each $Z \in S$. Set $a=(1-\lambda)^{-1}$. By letting $\lambda \rightarrow 1$ and using the fact that $C_{2}$ is closed, we find that $Y_{2}+Z \in C_{2}$.

Since $Y_{2}=P\left(X \mid C_{2}\right),\left(X-Y_{2}, Y_{2}-Y\right) \geq 0$ for each $Y \in C_{2}$. It follows that $\left(X-Y_{2}, Z\right)=0$ for each $Z \in S$. Since $Y_{1}=P\left(Y_{2} \mid C_{1}\right)$ and $C_{1}$ is affine, $\left(Y_{2}-Y_{1}, Z\right)=0$ for all $Z \in S$. The last two equalities yield that $\left(X-Y_{1}, Z\right)=0$ for each $Z \in S$. This completes the proof. $]$

Corollary 2.4.1. Let $C_{1}$ and $C_{2}$ be two closed convex sets with $C_{1} \subset C_{2}$. If either $C_{1}$ or $C_{2}$ is an affine space, then

$$
\begin{equation*}
P\left(P\left(X \mid C_{2}\right) \mid C_{1}\right)=P\left(X \mid C_{1}\right) \tag{2.8}
\end{equation*}
$$

The identity (2.8) is called the smoothing property. Most of the algorithms developed in Chapter III and Chapter V make use of this identity.

The important convergence theorems, Theorem 2.5, Corollary 2.5.1 and Theorem 2.6, are introduced by Brunk (1965) except that Theorem 2.5 is given here under a weaker hypothesis. Theorem 2.5 and Corollary 2.5.1 given below show that $P(X \mid C)$ can be obtained as the limit of $\left\{P\left(X \mid C_{n}\right)\right\}$ or $\left\{P\left(Y_{n} \mid C_{n}\right)\right\}$ as the monotone sequence $\left\{C_{n}\right\}$ converges to $C$.

Theorem 2.5. Let $\left\{\mathrm{C}_{\mathrm{n}}\right\}$ be a monotone sequence of closed convex sets in $H$, let $X \in H$ and let $X_{n}=P\left(X \mid C_{n}\right)$. If the sequence is monotone increasing, then $\lim _{n \rightarrow \infty} X_{n}$ exists and the limit is $P\left(X \mid C_{\infty}\right)$ where $C_{\infty}$ is the closure of $\cup_{n=1}^{\infty} C_{n}$. If the sequence is monotone decreasing and $\cap_{n=1}^{\infty} C_{n}$ is nonempty, then $\lim _{\mathrm{n} \rightarrow \infty} \mathrm{X}_{\mathrm{n}}$ exists and the limit is $\mathrm{P}\left(\mathrm{X} \mid \cap_{\mathrm{n}=1}^{\infty} C_{\mathrm{n}}\right)$.

A countable union of a monotone increasing sequence of convex sets is convex and the closure of convex set is convex. A countable intersection of closed convex sets is closed and convex. Therefore, $C_{\infty}$ and $\cap_{n=1}^{\infty} C_{n}$ are closed and convex under the ir corresponding assumptions. If $\left\{C_{n}\right\}$ is monotone increasing then by (2.7), the sequence $\left\{\left\|\mathrm{X}-\mathrm{X}_{\mathrm{n}}\right\|\right\}$ is monotone decreasing and $\left\{\mathrm{X}_{\mathrm{n}}\right\}$ is Cauchy. If $\left\{\mathrm{C}_{\mathrm{n}}\right\}$ is monotone decreasing and $\cap_{\mathrm{n}=1}^{\infty} \mathrm{C}_{\mathrm{n}} \neq \phi$, then by (2.7), the sequence $\left\{\left\|X-X_{n}\right\|\right\}$ is monotone increasing, it is bounded from above by $\|\mathrm{X}-\mathrm{Z}\|$ for any $\mathrm{Z} \in \cap_{\mathrm{n}=1}^{\infty} \mathrm{C}_{\mathrm{n}}$ and $\left\{\mathrm{X}_{\mathrm{n}}\right\} \quad$ is Cauchy.

Corollary 2.5.1. Additional to the assumptions in Theorem
2.5, let $\left\{\mathrm{Y}_{\mathrm{n}}\right\}$ be a convergent sequence with X as its limit. Then the sequence $\left\{P\left(Y_{n} \mid C_{n}\right)\right\}$ converges to $P\left(X \mid C_{\infty}\right)$ or $P\left(X \mid \cap_{n=1}^{\infty} C_{n}\right)$ according as $\left\{C_{n}\right\}$ is monotone increasing or monotone decreasing.

A sequence $\left\{X_{n}, C_{n}\right\}$ is called a martingale if for each $n$,
$C_{n}$ is a closed convex set in $H, X_{n} \in H, C_{n} \subset C_{n+1}$ and $X_{n}=P\left(X_{n+k} \mid C_{n}\right)$ for any non-negative integer $k$. If for each $n$, $C_{n}$ is a closed linear space in $H$ and $X_{n}=P\left(X \mid C_{n}\right)$ for an element $X \in H$, then $\left\{X_{n}, C_{n}\right\}$ is a martingale provided that $\left\{C_{n}\right\}$ is a monotone increasing sequence. An example of a martingale which is not composed of closed linear spaces will be given in Corollary 4.18.1.

Theorem 2.6. Let $\left\{X_{n}, C_{n}\right\}$ be a martingale in $H$ with the sequence $\left\{X_{n}\right\}$ being bounded. Then $\lim _{n \rightarrow \infty} X_{n}=X_{\infty}$ exists and $X_{n}=P\left(X_{\infty} \mid C_{n}\right)$ for each $n$. Consequently, every bounded martingale is of the form $\left\{P\left(X \mid C_{n}\right), C_{n}\right\}$ for some $X \in C_{\infty}$, where $C_{\infty}$ is the closure of $\cup_{n=1}^{\infty} C_{n}$.

A necessary and sufficient condition for the bounded sequence $\left\{X_{n}, C_{n}\right\}$ to be a martingale when the $C_{n}$ 's are cones is given by Corollary 2.11.1.

## II. 2 On a Closed Convex Cone

A subset $C$ of the Hilbert space $H$ is said to be cone if $X \in C$ implies $\delta X \in C$ for each $\delta \geq 0$. The family of isotonic functions described in Chapter III, the family of isotonic functions described in Chapter $V$, and the family of square-integrable $\Sigma$-measurable random variables described in Chapter IV are closed
convex cones with respect to their corresponding Hilbert spaces.
Throughout this section we shall let $C$ denote a non-empty closed convex cone in the Hilbert space H. From (2.2) we have

$$
\begin{equation*}
P(\delta X \mid C)=\delta P(X \mid C) \quad \text { for each } \quad \delta \geq 0 \tag{2.9}
\end{equation*}
$$

An immediate result from Theorem 2.3 is the following theorem.

Theorem 2.7. Let $C$ be a closed convex cone in $H$ and let $X \in H$. An element $X * \in C$ is the projection of $X$ on $C$ if and only if it satisfies

$$
\begin{equation*}
(X-X *, X *)=0 \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
(X-X *, Z) \leq 0 \quad \text { for each } \quad Z \in C . \tag{2.11}
\end{equation*}
$$

The generalized projection $P(\cdot \mid C)$ is positive homogeneous in the sense of (2.9) and orthogonal in the sense of (2.10) provided that $C$ is a closed convex cone.

Corollary 2.7.1. Let $X *$ be the projection of $X$ on $C$. Then

$$
\begin{equation*}
\|\mathbf{X} *\|=\left(X, X^{*}\right) \leq\|\mathbf{X}\|^{2} \tag{2.12}
\end{equation*}
$$

and if $\|X *\|=\|X\|$, then $X *=X$.

A geometric interpretation of (2.10) can be given as below.
Let $B_{X}:=\{Y \in H:(X-Y, Y) \geq 0\}$. Consider a simple case first. Let $H=R^{2}$ and let the inner product (.,.) be defined by $(X, Y)=X^{t} V Y$ for a given positive definite matrix $V$. Then $B_{X}$ is the region bounded by the ellipse $Y^{t} V(X-Y)=0 \quad$ which passes through $X$ and the origin. Let $C$ be a closed convex cone in $R^{2}$ and set $C_{X}:=C \cap B_{X}$. Then for each $X \in C_{X}$, we have $\|Z\| \leq\|P(X \mid C)\|, \quad 0 \leq(X, Z) \leq(X, P(X \mid C)) \quad$ and $\quad(P(X \mid C), Z) \geq 0$. The above results also hold in general; the proof follows.

Theorem 2.8. Let $C$ be a closed convex cone in $H$, let $X \in H$ and let $C_{X}:=\{Z \in C:(X-Z, Z) \geq 0\}$. Then $C_{X}$ is closed and convex, and for each $Z \in C_{X}$ we have

$$
\begin{equation*}
\|\mathrm{Z}\|^{2} \leq(\mathrm{X}, \mathrm{Z}) \leq(\mathrm{P}(\mathrm{X} \mid \mathrm{C}), \mathrm{Z}) \leq(\mathrm{P}(\mathrm{X} \mid \mathrm{C}), \mathrm{X})=\|\mathrm{P}(\mathrm{X} \mid \mathrm{C})\|^{2} \tag{2.13}
\end{equation*}
$$

and if the last inequality becomes an equality, then $Z=P(X \mid C)$.

Proof. The set $B_{X}=\{Y \in H:(X-Y, Y) \geq 0\}$ is a closed ball with center at $X / 2$ and with radius $\|x\| / 2$. Therefore, $C_{X}=C \cap B_{X}$ is closed and convex. For each $Z \in C_{X}$, the first inequality holds by the definition of $Z$, the second inequality is from (2.11) and the identity is from (2.12). Since $\|Z\|^{2} \leq(P(X \mid C), Z), \quad$ by the Holder's inequality $(P(X \mid C), Z) \leq\|P(X \mid C)\|\|Z\|$, we have $\|Z\| \leq \mid P(X \mid C) \|$ and hence
the last inequality follows.
If any one of the identities $\quad\|Z\|^{2}=(P(X \mid C), X)$, $(X, Z)=(P(X \mid C), X)$ and $(P(X \mid C), Z)=(P(X \mid C), X)$ holds, then $(P(X \mid C), Z)=\|P(X \mid C)\|\|Z\|=\|P(X \mid C)\|^{2}$ and hence $Z=P(X \mid C) . \quad \square$

The theorem shows that if $Z \in C$ satisfies (2 10), then $\|Z\| \leq\|P(X \mid C)\| ;$ and the equality holds if and only if $Z$ is the projection of $X$ on $C$. This conclusion can also be made by observing that $\|x\|^{2}=\|x-z\|^{2}+\|z\|^{2}$.

Let $C$ be a closed convex cone in $H$ and let
$C *:=\{Y:(Y, Z) \leq 0$ for each $Z \in C\}$. The set $C *$ is a closed convex cone and $C * \cap C=\{0\}$. Such a set $C^{*}$ is called the dual cone of $C$. A simple and obvious result about the projection of $X$ to C* is given below.

Theorem 2.9. Let $C$ be a closed convex cone in $H$ and let $C *$ be the dual cone of $C$. For each $X \in H$, we have

$$
\begin{equation*}
P(\mathbf{X} \mid \mathrm{C})+\mathrm{P}(\mathbf{X} \mid \mathrm{C} *)=\mathbf{X} \tag{2.14}
\end{equation*}
$$

Consequently, $P(X \mid C)=0$ if and only if $(X, Z) \leq 0$ for each $Z \in C$.

If $Y$ satisfies (2.l1), i.e., $(X-Y, Z) \leq 0$ for each $Z \in C$, then $X-Y \in C *$. Therefore, the collection $\{Y \in H:(X-Y, Z) \leq 0\}$
is simply $\mathbf{X}-\mathrm{C} \%$.

Theorem 2.10. Let $C$ be a closed convex cone in $H$ and let $C *$ be the dual cone of $C$. For every $X \in H$, we have

$$
\begin{equation*}
\|P(X \mid C)\|^{2}=(P(X \mid C), X) \leq(P(X \mid C), Y) \leq\|Y\|^{2} \tag{2.15}
\end{equation*}
$$

for every $\quad Y \in X-C *$ and if $(P(X \mid C), Y)=\|Y\|^{2}$, then $Y=P(X \mid C)$. Further, if $Y \in C \cap(X-C *)$, then $(X-Y, Y) \leq 0$.

Proof. If $Y \in X-C *$ then $(X-Y, P(X \mid C)) \leq 0$ and hence the first inequality in (2.15) follows. By the Holder's inequality $(P(X \mid C), Y) \leq\|P(X \mid C)\|\|Y\|$, we have $\|P(X \mid C)\| \leq\|Y\| \quad$ and the second inequality follows. If $(P(X \mid C), Y)=\|Y\|^{2}$, then $\|Y\|^{2}=(P(X \mid C), Y) \leq\|P(X \mid C)\|\|Y\|$. But we have just shown that $\|P(X \mid C)\| \leq\|Y\|, \quad$ so $(P(X \mid C), Y)=\|P(X \mid C)\|\|Y\|$ and hence $Y=P(X \mid C)$. If $Y \in C \cap(X-C *)$ then $(X-Y, Y) \leq 0$ since $X-Y \in C *$ and $Y \in C .[]$

By Theorem 2.8, the closed convex set $C_{X}$ is contained in the closed ball with center at origin and with radius $\|P(X \mid C)\|$, while from Theorem 2. 10 it follows that the closed convex set $X-C *$ is contained in the half-space $\left\{Y:(P(X \mid C), Y) \geq\|P(X \mid C)\|^{2}\right\}$. The intersection of $C_{X}$ and $X-C *$ is a singleton, $\quad\{P(X \mid C)\}$.

Theorem 2.11. Let $C_{1}$ and $C_{2}$ be closed convex cones in H with $\mathrm{C}_{1} \subset \mathrm{C}_{2}$ and let $\mathrm{X} \in \mathrm{H}$. Set $\mathrm{Y}_{\mathrm{i}}=\mathrm{P}\left(\mathrm{X} \mid \mathrm{C}_{\mathrm{i}}\right), \mathrm{i}=1,2$ and set $\quad Y_{0}=P\left(Y_{2} \mid C_{1}\right)$. Then

$$
\begin{equation*}
\left\|Y_{1}\right\| \leq\left\|Y_{0}\right\| \leq\left\|Y_{2}\right\| \leq\|x\| \tag{2.16}
\end{equation*}
$$

and if any equality holds, then the corresponding two elements in $H$ are identical.

Proof. Since $Y_{2}=P\left(X \mid C_{2}\right)$ and $Y_{0}=P\left(Y_{2} \mid C_{1}\right)$, the results for the sequence $\left\|Y_{0}\right\| \leq\left\|Y_{2}\right\| \leq\|X\|$, follow from Corollary 2.7.1. Since $C_{1}\left(C_{2}\right.$, by (2.11), we have $\left(X-Y_{2}, Z\right) \leq 0$ and $\left(Y_{2}-Y_{0}, Z\right) \leq 0$ for each $Z \in C_{1}$. Therefore, $\left(X-Y_{0}, Z\right) \leq 0$ for each $Z \in C_{1}$ and the results follow immediately from Theorem 2.10. []

Corollary 2.11.1. Under the same assumptions as in Theorem
2.11, a necessary and sufficient condition for $P\left(P\left(X \mid C_{2}\right) \mid C_{1}\right)=P\left(X \mid C_{1}\right)$ is that $\left(P\left(X \mid C_{2}\right)-P\left(X \mid C_{1}\right), Z\right) \leq 0$ for each $Z \in C_{1}$.

Proof. Let $Y_{i}=P\left(X \mid C_{i}\right), i=1,2$ and let $Y_{0}=P\left(Y_{2} \mid C_{1}\right)$. If $Y_{0}=Y_{1}$, then by (2.11) we have $\left(Y_{2}-Y_{1}, Z\right) \leq 0$ for each $Z \in C_{1}$. On the other hand, if $\left(Y_{2}-Y_{1}, Z\right) \leq 0$ for each $Z \in C_{1}$, then $Y_{1} \in Y_{2}-C_{1}^{*}$ and by (2.15), $\left\|Y_{0}\right\| \leq\left\|Y_{1}\right\|$. We may
conclude our result by applying the above theorem. []

By the identity (2.12), the identity
$\|X-P(X \mid C)\|^{2}=\|X\|^{2}-\|P(X \mid C)\|^{2}$ and the inequality (2.7), we may obtain that

$$
\begin{align*}
\left\|P\left(X \mid C_{2}\right)-P\left(X \mid C_{1}\right)\right\|^{2} & \leq\left\|P\left(X \mid C_{2}\right)\right\|^{2}-\left\|P\left(X \mid C_{1}\right)\right\|^{2}  \tag{2.17}\\
& =\left(P\left(X \mid C_{2}\right)-P\left(X \mid C_{1}\right), X\right)
\end{align*}
$$

for each pair of closed convex cones $\quad C_{1}$ and $C_{2}$ with $C_{1} C C_{2}$. Theorem 2.7, Corollary 2.7.1, the identity (2.9) and the inequality (2.17) were introduced by Brunk (1965).

## II. 3 On a Finitely Generated Cone

Let $\quad Z_{1}, Z_{2}, \ldots, Z_{m}$ be a finite sequence of elements in the Hilbert space $H$ and let $C$ be the set of all non-negative linear combinations of the sequence. It has been shown that $C$ is a closed convex cone (cf. Rockafellar (1970)). Such a closed convex cone is said to be finitely generated. And we denote it by $C\left[Z_{1}, \ldots, Z_{m}\right]$. Let $Z \in C\left[Z_{1}, \ldots, Z_{m}\right]$. Then there exists a set of non-negative real numbers $a_{1}, \ldots, a_{m}$ such that $Z=\sum_{i=1}^{m} a_{i} Z_{i}$. The representation need not be unique and we may even have a representation which is composed of some non-negative coefficients and some negative coefficients. An element $Y$ of a closed convex cone $C$
is said to be an extreme vector of $C$ if there do not exist linearly independent elements $Y_{1}$ and $Y_{2}$ in $C$ such that $Y=Y_{1}+Y_{2}$. If $Y$ is an extreme vector of the cone $C\left[Z_{1}, \ldots, Z_{m}\right]$, then $Y$ must be a scalar multiple of $Z_{i}$ for an $i$ between 1 and $m$. If $Z_{1}, \ldots, Z_{m}$ are linearly independent, then they form a complete set of extreme vectors for $C\left[Z_{1}, \ldots, Z_{m}\right]$. Let $S$ be the smallest linear space in $H$ which contains the finitely generated cone $C$.

Then $S$ is closed in $H$. By the smoothing property (2.8), $P(P(X \mid S) \mid C)=P(X \mid C)$ for each $X \in H$. Since our interest here is to obtain $P(X \mid C)$ and the projection $P(X \mid S)$ can be obtained very easily, without loss of generality we may assume that $H$ is the smallest linear space containing $C$. The following theorem is an immediate result of Theorem 2.7.

Theorem 2.12. Let $C$ be the cone $C\left[Z_{1}, \ldots, Z_{m}\right]$ and let $X \in H$. An element $X * \in C$ is the projection of $X$ on $C$ if and only if $X *$ satisfies $(X-X *, X *)=0$ and

$$
\begin{equation*}
\left(X-X *, Z_{i}\right) \leq 0 \quad \text { for } \quad i=1,2, \ldots, m \tag{2.18}
\end{equation*}
$$

If $x \notin C\left[Z_{1}, \ldots, Z_{m}\right]$, then by Theorem 2.2,
$P\left(X \mid C\left[Z_{1}, \ldots, Z_{m}\right]\right)$ is a boundary point of $C\left[Z_{1}, \ldots, Z_{m}\right]$.
Boundary points of a finitely generated cone may not be eas ily identified. An element in a cone is said to be an interior point if it is not a
boundary point. Let $Y=\Sigma_{i=1}^{m} a_{i} Z_{i}$ with $a_{1}, a_{2}, \ldots, a_{m}$ positive. We claim that $Y$ is an interior point. Without loss of generality, we may assume that $Z_{1}, Z_{2}, \ldots, Z_{n}$ are linearly independent where $n \leq m$ and the dimension of $H$ is $n$. Let $Z_{0}=\sum_{i=n+1}^{m} a_{i} Z_{i}$. Then $Z_{0} \in C\left[Z_{1}, \ldots, Z_{m}\right]$. It can be shown that there is an open ball $B$ containing $Y$ such that if $Z \in B$, then $Z$ can be represented by $Z=Z_{0}+\Sigma_{i=1}^{n} \beta_{i} Z_{i}$ where $\beta_{i}, \ldots, \beta_{n}$ are positive. Therefore, the claim is established. Let $S$ be a linear subspace of $H$ and let C be a closed convex cone contained in S. Relative boundary point and relative interior point of $C$ with respect to the linear space $S$ are defined in the same way except that only elements of $S$ are considered.

$$
\text { Let } C=C\left[Z_{1}, \ldots, Z_{m}\right] \text { be a cone in } H \text { and let } X \in H \text {. }
$$

Then there is a sequence of non-negative real numbers $a_{1}, a_{2}, \ldots, a_{m}$ such that $P(X \mid C)=\sum_{i=1}^{m} a_{i} Z_{i}$. Let us define $\Lambda:=\left\{i: a_{i}>0\right\}$, $C_{\Lambda}:=\left\{\Sigma_{i \in \Lambda} \beta_{i} Z_{i}: \beta_{i} \geq 0\right.$ for each $\left.i \in \Lambda\right\}$ and $S_{\Lambda}:=\left\{\Sigma_{i \in \Lambda} \beta_{i} Z_{i}: \beta_{i}\right.$ is real for each $\left.i \in \Lambda\right\}$. Since $P(X \mid C) \in C_{\Lambda^{\prime}}$ we have $P(X \mid C)=P\left(X \mid C_{\Lambda}\right)$. By the smoothing property, $P\left(X \mid C_{\Lambda}\right)=P\left(P\left(X \mid S_{\Lambda}\right) \mid C_{\Lambda}\right)$. We claim that $P\left(X \mid S_{\Lambda}\right) \in C_{\Lambda}$ and hence $P\left(X \mid S_{\Lambda}\right)=P(X \mid C)$. If it were not true, i. e., $P\left(X \mid S_{\Lambda}\right) \notin C_{\Lambda}$, then by Theorem 2.2, $P\left(X \mid C_{\Lambda}\right)$ would be a relative boundary point of $C_{\Lambda}$ with respect to $S_{\Lambda}$. But on the other hand,
$P\left(X \mid C_{\Lambda}\right)=P(X \mid C)=\Sigma_{i \in \Lambda} a_{i} Z_{i}$ with $a_{i}>0$ for each $i \in \Lambda$, so $P\left(X \mid C_{\Lambda}\right)$ must be a relative interior point of $C_{\Lambda}$ with respect to $S_{\Lambda}$. Therefore, we have a contradiction, and thus the following theorem has been proved.

Theorem 2.13. Let $C$ be the cone $C\left[Z_{1}, \ldots, Z_{m}\right]$ and let $X \in H$. Then there exists a linear space $S$ generated by a subset of $\left\{Z_{1}, \ldots, Z_{m}\right\}$, such that $P(X \mid C)=P(X \mid S)$.

The linear space $S$ satisfying $P(X \mid C)=P(X \mid S)$, is not unique. Since the representation of $X$ by $Z_{1}, \ldots, Z_{m}$ is not unique, the linear space $S_{\Lambda}$ described above need not be unique. If we can identify a subset $\Gamma$ of $\{1,2, \ldots, m\}$ such that $P(X \mid C) \in C_{\Gamma}$, then $P\left(X \mid S_{\Gamma}\right)$ is the solution we want, provided that $P\left(X \mid S_{\Gamma}\right) \in C_{\Gamma} . \quad$ Such an identification can be achieved for the problems of isotonic regression over a linearly ordered set, isotonic regression over a tree, and bivariate isotonic regression with $\rho \geq 0$. However, in these three cases, the closed convex cones are each generated by a set of linearly independent vectors.

Corollary 2.13.1. Let $C$ be the cone $C\left[Z_{1}, \ldots, Z_{m}\right]$ and let $X \in H$. Let $\Lambda$ and $S_{\Lambda}$ be defined as above and let $\Gamma$ be a subindex set of $\{1,2, \ldots, m\}$. If $S_{\Gamma} \supset S_{\Lambda}$, then $P\left(X \mid S_{\Gamma}\right)=P(X \mid C)$ provided that $P\left(X \mid S_{\Gamma}\right) \in C_{\Gamma}$.

Proof. If $S_{\Gamma} \supset S_{\Lambda}$, then by (2.17) we have $\left\|P\left(X \mid S_{\Gamma}\right)\right\| \geq\|P(X \mid C)\| . \quad$ Recall that $\left(X-P\left(X \mid S_{\Gamma}\right), \quad P\left(X \mid S_{\Gamma}\right)\right)=0$, so if $P\left(X \mid S_{\Gamma}\right) \in C_{\Gamma}$ then by Theorem 2.8 we have $P\left(X \mid S_{\Gamma}\right)=P(X \mid C) . \quad \square$

It is obvious that if $P\left(X \mid S_{\Gamma}\right)=P(X \mid C)$, then $P(X \mid S)=P(X \mid C)$ for any linear space such that $C_{\Lambda} \subset S \subset S_{\Gamma}$. If $S_{\Gamma} \supset S_{\Lambda}$ and $P\left(X \mid S_{\Gamma}\right) \notin C_{\Gamma}$, then by the smoothing property, we have $P\left(X \mid C_{\Lambda}\right) \subset P\left(P\left(X \mid S_{\Gamma}\right) \mid C_{\Lambda}\right)$. Therefore, we may suppose that our Hilbert space at this stage is $S_{\Gamma}$. As the dimension of $S_{\Gamma}$ goes down, we shall eventually obtain $P(X \mid C)$ by successive projections on linear spaces and that is the process employed in Chapter III and Chapter V.

Suppose that $P(X \mid C)$ is unknown and there is no way to identify a subindex set $\Gamma$ such that either $P(X \mid C) \in C_{\Gamma}$ or $P(X \mid C) \in S_{\Gamma}$. Theorem 2.8 shows that $P(X \mid C)$ has the largest norm among those $P\left(X \mid S_{\Gamma}\right)$ 's such that $P\left(X \mid S_{\Gamma}\right) \in C_{\Gamma}$. We state this formally as the following theorem.

Theorem 2.14. Let $C$ be the cone $C\left[Z_{1}, \ldots, Z_{m}\right]$, let $X \in H$ and let $S_{X}:=\left\{P\left(X \mid S_{\Gamma}\right): \Gamma\right.$ is a subset of $\{1, \ldots, m\}$ and $\left.P\left(X \mid S_{\Gamma}\right) \in C\right\}$. If $P\left(X \mid S_{\Gamma}\right)$ has the largest norm in $S_{X}$, then $P\left(X \mid S_{\Gamma}\right)=P(X \mid C)$.

There are $2^{m}$ possible linear spaces of the form $S_{\Gamma}$.
$P(X \mid C)=P(X \mid H) \quad$ if and only if $X \in C, \quad$ and $P(X \mid C)=P(X \mid\{0\}) \quad$ if and only if $\left(X, Z_{i}\right) \leq 0$ for $i=1,2, \ldots, m$. A further reduction of candidates for $P(X \mid C)$ of the form $P\left(X \mid S_{\Gamma}\right)$ can be described as follows. $P\left(X \mid S_{\Gamma}\right)$ satisfies (2.10). Let $(P 1)$ be the property that $P\left(X \mid S_{\Gamma}\right) \in C$. If $(P 1)$ is true, then by Theorem 2.8, $\left\|P\left(X \mid S_{\Gamma}\right)\right\| \leq\|P(X \mid C)\|$. Let (P2) be the property that $\left(X-P\left(X \mid S_{\Gamma}\right), Z_{i}\right) \leq 0 \quad$ for $\quad i=1, \ldots, m$. If $(P 2)$ is true, then by Theorem 2.10, $\left\|P\left(X \mid S_{\Gamma}\right)\right\| \geq\|P(X \mid C)\|$. A program to obtain $P(X \mid C)$ can be made through calculation of $P\left(X \mid S_{\Gamma}\right)$ by the following theorem.

Theorem 2.15. Let $C$ be the cone $C\left[Z_{1}, \ldots, Z_{m}\right]$, let $X \in H$ and let $\Gamma$ be a subindex set. Then we have the following statements.
(1) $P\left(X \mid S_{\Gamma}\right)=P(X \mid C)$ if and only if $P\left(X \mid S_{\Gamma}\right)$ satisfies (P1) and (P2).
(2) If (P1) holds but (P2) fails, then for any $\Delta$ such that $\left\|P\left(X \mid S_{\Delta}\right)\right\| \leq\left\|P\left(X \mid S_{\Gamma}\right)\right\|$, we have $P\left(X \mid S_{\Delta}\right) \neq P(X \mid C)$. In particular, $P\left(X \mid S_{\Delta}\right) \neq P(X \mid C)$ for each $\Delta \subset \Gamma$.
(3) If (P1) fails but (P2) holds, then for any $\Delta$ such that $\left\|P\left(X \mid S_{\Delta}\right)\right\| \geq\left\|P\left(X \mid S_{\Gamma}\right)\right\|, \quad$ we have $P\left(X \mid S_{\Delta}\right) \neq P(X \mid C)$. In particular, $P\left(\left.X\right|_{\Delta}\right) \neq P(S \mid C)$ for each $\Delta \supset \Gamma$.

Proof. The projection $P\left(X \mid S_{\Gamma}\right)$ satisfies (2.10), i.e., $\left(X-P\left(X \mid S_{\Gamma}\right), P\left(X \mid S_{\Gamma}\right)\right)=0$. Statement (1) follows from Theorem 2.12. If (P1) holds but (P2) fails, then $P\left(X \mid S_{\Gamma}\right) \in C_{X}$ and $P\left(X \mid S_{\Gamma}\right) \neq P(X \mid C)$. Statement (2) follows from Theorem 2.8. If (P2) holds but (P1) fails, then $P\left(X \mid S_{\Gamma}\right) \in X-C^{*}$ and $P\left(X \mid S_{\Gamma}\right) \neq P(X \mid C)$. Statement (3) follows from Theorem 2. 10. []

Let $C$ be the cone generated by a linearly independent sequence $Z_{1}, \ldots, Z_{n}$ of elements in $H$ and let the dimension of $H$ be $n$. For each $i$, there exists an element $Y_{i} \in H$ such that $\left(Y_{i}, Z_{i}\right)<0$ and $\left(Y_{i}, Z_{j}\right)=0$ for each $j \neq i$. The sequence $Y_{1}, \ldots, Y_{n}$ is linearly independent and it is unique up to scalar multiplication. The cone $C\left[Y_{1}, \ldots, Y_{n}\right]$ is the dual cone of $C$.

Theorem 2.16. Let $C$ be the cone $C\left[Z_{1}, \ldots, Z_{n}\right]$ and let $C *=C\left[Y_{1}, \ldots, Y_{n}\right]$ be the dual cone of $C$. For every $X \in H$, there exists a unique representation $X=\Sigma_{i \in \Lambda} a_{i} Z_{i}+\Sigma_{j \in \Lambda^{c}} \beta_{j} Y_{j}$ for some index set $\Lambda$ such that $a_{i} \geq 0$ for each $i \in \Lambda$ and $\beta_{j} \geq 0$ for each $j \in \Lambda^{C}$. Furthermore $P(X \mid C)=\Sigma_{i \in \Lambda} a_{i} Z_{i}$ and $P(X \mid C *)=\Sigma_{j \in \Lambda^{c}} \beta_{j} Y_{j}$.

Proof. Let $P(X \mid C)$ be represented by $P(X \mid C)=\Sigma_{i=1}^{n} a_{i} Z_{i}$, where $a_{i} \geq 0$ foreach $i$. Let $\Lambda:=\left\{i: a_{i}>0\right\}$. Then $P(X \mid C)=P\left(X \mid S_{\Lambda}\right)$ and hence $\left(X-P(X \mid C), Z_{i}\right)=0 \quad$ for each $\quad i \in \Lambda$.

Let $P(X \mid C *)$ be represented by $P(X \mid C *)=\sum_{j=1}^{n} \beta_{j} Y_{j}$, where $\beta_{j} \geq 0$ for each $j$. By Theorem 2.9, $P(X \mid C *)=X-P(X \mid C)$ and hence $\left(P(X \mid C *), Z_{i}\right)=\sum_{j=1}^{n} \beta_{j}\left(Y_{j}, Z_{i}\right)=\beta_{i}\left(Y_{i}, Z_{i}\right)=0$ for each $i \in \Lambda$. It follows that $\beta_{i}=0$ for each $i \in \Lambda$ and therefore, $X=\Sigma_{i \in \Lambda} a_{i} Z_{i}+\Sigma_{j \in \Lambda^{c}} \beta_{j} Y_{j}$.

If $X$ has a representation $\Sigma_{i \in \Lambda} a_{i} Z_{i}+\Sigma_{j \in \Lambda c} \beta_{j} Y_{j}$ with $a_{i}$ 's and $\beta_{j}^{\prime} s$ non-negative, then it is trivial that $P(X \mid C)=\Sigma_{i \in \Lambda} a_{i} Z_{i}$ and $P(X \mid C *)=\Sigma_{j \in \Lambda c} \beta_{j} Y_{j}$. Since $\left\{Z_{1}, \ldots, Z_{n}\right\}$ and $\left\{Y_{1}, \ldots, Y_{n}\right\}$ are each linearly independent, the representations of $P(X \mid C)$ by $Z_{1}, \ldots, Z_{n}$ and of $P(X \mid C *)$ by $Y_{1}, \ldots, Y_{n}$ are unique. For each $i$, the product of the corresponding coefficients $a_{i}$ and $\beta_{i}$ must be zero. The uniqueness of the above representation for $X$ must therefore be satisfied. []

Let $\Lambda$ be an index set. Theorem 2.9 shows that
$P(X \mid C)=P\left(X \mid S_{\Lambda}\right)$ if and only if $P(X \mid C *)=P\left(X \mid S_{\Lambda}^{\perp}\right)$, where $S_{\Lambda}^{\perp}=\left\{Y:(Y, Z)=0\right.$ for each $\left.Z \in S_{\Lambda}\right\} \quad$ is the orthogonal complement of $S_{\Lambda}$. Under the hypothesis of Theorem 2.16,
$S_{\Lambda}^{\perp}=\left\{\Sigma_{j \in \Lambda C} \beta_{j} Y_{j}: \beta_{j}\right.$ real $\}$. The property that $a_{i} \beta_{i}=0$ for each $i$ reminds us that $\beta_{1}, \ldots, \beta_{n}$ are Lagrangian multipliers. The inequalities $\left(Y_{j}, P(X \mid C)\right) \leq 0$ for $j=1, \ldots, n$ are the constraints. Therefore, the theorem is equivalent to the Kuhn-Tucker condition.

## III. ISOTONIC REGRESSION

## III. 1 Preliminaries

Isotonic regression problems discussed in this chapter are those defined on some quasi-ordered finite sets. We shall leave the general case to the next chapter. The problems are standard mathematical programming problems. Their objective functions are the weighted sums of squares described in (3.1). Their feasible regions are the intersections of some closed half-spaces which are each determined by a pair of arguments.

The binary relation $\leq$ defined on a set $\Omega$ is said to be a partial ordering if
(1) it is reflexive; $\omega \leq \omega$ for each $\omega$ in $\Omega$
(2) it is transitive; $\omega, \mu, \nu \in \Omega, \omega \leq \mu$ and $\mu \leq \nu$ imply $\omega \leq \nu$
and (3) it is antisymmetric; $\omega, \mu \in \Omega, \omega \leq \mu$ and $\mu \leq \omega$ imply $\omega=\mu$.

A quasi-ordering is reflexive and transitive but not necessarily antisymmetric. A pair of elements $\mu$ and $\nu$ in $\Omega$ is said to be comparable if either $\mu \leq \nu$ or $v \leq \mu$. A linear ordering is a partial ordering such that each pair of elements is comparable.

Let $\leq$ be a quasi-ordering on $\Omega$ and let $\Gamma$ be a subset of $\Omega$. An element $\omega \in \Gamma$ is maximal in $\Gamma$ if $\mu \in \Gamma$ and $\omega \leq \mu$
imply $\mu \leq \omega ; \quad \omega \in \Gamma$ is minimal in $\Gamma$ if $\mu \in \Gamma$ and $\mu \leq \omega$ imply $\omega \leq \mu . \quad \Gamma \quad$ is said to be bounded from above if there exists an element $\mu \in \Omega$ such that $\omega \leq \mu$ for each $\omega$ in $\Gamma$; such an element $\mu$ is called an upper bound of $\Gamma$. If the set $\Lambda$ of all upper bounds of $\Gamma$ is non-empty and it has a unique minimal element, then such an element is said to be the least upper bound of $\Gamma$, and is denoted by VI. Similarly for the definitions of "bounded from below", "lower bound" and the "greatest lower bound". The greatest lower bound of $\Gamma$ is denoted by $\quad \wedge \Gamma$.

A partial ordering is said to have a tree structure if every non-comparable pair has a greatest lower bound but does not have an upper bound. A set $\Omega$ with an ordering $\leq$ is said to be an ordered set; such a pair is denoted by $(\Omega, \leq)$. The pair $\left(\Omega, \leq_{t}\right)$ is called a tree if $\leq_{t}$ is a partial ordering having a tree structure. Some problems involving trees have been given by Thompson (1962) and Hartigan (1967). A finite tree, by its definition, has a unique minimal element. A simple tree is a tree such that every element is either maximal or minimal. A tree with a unique maximal element is a linearly ordered set. From the definitions, a linearly ordered set is a tree, a tree is a partially ordered set and a partially ordered set is a quasi-ordered set.

Let $\left(\Omega, \leq_{q}\right)$ be a quasi-ordered finite set. The finiteness of $\Omega$ is our assumption throughout this chapter. Let $\Gamma_{1}, \ldots, \Gamma_{k}$ be
subsets of $\Gamma$. They are unrelated if every pair of elements from different subsets is not comparable. Let $\Lambda$ be a subset of $\Omega$ and let $\mu \in \Lambda$. An element $\nu \in \Lambda$ can be reached from $\mu$ in $\Lambda$ if there exist $\omega_{1}, \omega_{2}, \ldots, \omega_{k}$ in $\Lambda$ such that $\omega_{i} \leq \omega_{i+1}$ or $\omega_{i} \geq \omega_{i+1}$ for $i=0,1, \ldots, k$ where $\omega_{0}=\mu$ and $\omega_{k+1}=\nu$. Let $\Lambda(\mu):=\{\nu \in \Lambda: \nu$ can be reached from $\mu$ in $\Lambda\}$. It is obvious that if $v \in \Lambda(\mu)$, then $\Lambda(\nu)=\Lambda(\mu) ;$ and if $\nu \in \Lambda$ and $\nu \notin \Lambda(\mu)$, then $\Lambda(v)$ and $\Lambda(\mu)$ are unrelated. $\Lambda$ is said to be connected if $\quad \Lambda(\mu)=\Lambda$ for some $\mu \in \Lambda . \Lambda$ is said to be separable if it is not connected. If $\Lambda$ is separable, then there exist $\omega_{1}, \ldots, \omega_{k}$ in $\Lambda$ such that $\Lambda=\cup_{i=1}^{k} \Lambda\left(\omega_{i}\right)$ and $\Lambda\left(\omega_{1}\right), \ldots, \Lambda\left(\omega_{k}\right)$ are unrelated. Each subset $\Lambda\left(\omega_{i}\right)$ is called a component of $\Lambda$. Linearly ordered sets and trees are connected.

Let $\leq_{1}$ and $\leq_{2}$ be two quasi-orderings on $\Omega . \leq_{1}$ is said to be the reversal of $\leq_{2}$ provided that for $\mu, v$ in $\Omega$ $\mu \leq L_{1} v$ if and only if $v \leq_{2} \mu$. The reversal of a quasi-ordering $\leq$ is denoted by $\leq_{r}$. The quasi-ordered set $\left(\Omega, \leq_{r}\right)$ is called the reversal of $(\Omega, \leq)$. When there is no ambiguity, we shall use $\Omega$ to denote a quasi-ordered set and $\Omega_{r}$ to denote its reversal. The quasi-ordered set $\Omega$ is connected or separable if and only if its reversal is connected or separable. The reversal of a tree is called a reversed tree.

A pair of elements $\mu$ and $\nu$ in $\Omega$ is said to be an immediately comparable pair if $\mu \leq \nu$ and there does not exist an element $\omega$ other than $\mu$ and $v$ such that $\mu \leq \omega$ and $\omega \leq \nu$. We denote such a pair by $[\mu \leq \nu]$. If $[\mu \leq \nu]$ is an immediately comparable pair, then $\mu$ is called an immediate predecessor of $v$, and $v$ is called an immediate successor of $\mu$. For a partially ordered set $\Omega$, the ordering on each component can be described by listing all immediately comparable pairs. Let $\Gamma$ be a component of $\Omega$ with $m$ elements. If $\Gamma$ is a linearly ordered set or a tree, then there are m-1 immediately comparable pairs in $\Gamma$; and $\mathrm{m}-1$ is the smallest number of immediately comparable pairs that a connected partially ordered set $\Gamma$ may have.

A real-valued function $\quad Z$ defined on a quasi-ordered set is isotonic if $\mu, \nu \in \Omega$ and $\mu \leq \nu$ imply $Z(\mu) \leq Z(\nu)$. Isotonic functions are constant over each equivalence class $[\omega]:=\{\mu: \mu \leq \omega, \omega \leq \mu\}$. Let $M(\Omega)$ be the collection of all isotonic functions defined on $\Omega$. It is obvious that $M\left(\Omega_{r}\right)=-M(\Omega)$. When there is no ambiguity, we shall use $M$ instead of $M(\Omega)$. A weight function $W$ is a non-negative function defined on $\Omega$. The pair $(\Omega, W)$, where $\Omega$ is a quasi-ordered set and $W$ is a weight function, will furnish the structure for the isotonic regression problems described below.

Let $X$ be a given real-valued function defined on $\Omega$. An isotonic regression of $X$ over $(\Omega, W)$ is an element in $M$ which minimizes

$$
\begin{equation*}
f(Z):=\Sigma_{\omega \in \Omega}[X(\omega)-Z(\omega)]^{2} W(\omega) \tag{3.1}
\end{equation*}
$$

among all functions $Z$ in $M$. If $\Omega$ is separable, then the minimization problem can be studied in each component of $\Omega$ independently. If in addition, each component is an equivalence class $[\omega]$, then it is a regression problem. Let $\Gamma$ be a subset of $\Omega$ and let $X \mid \Gamma$ and $W \mid \Gamma$ be the restrictions of $X$ and of $W$ to $\Gamma$. Then $\Gamma$ is by itself a quasi-ordered set. An isotonic regression of $X \mid \Gamma$ over $(\Gamma, W \mid \Gamma)$ is called a restricted isotonic regression of $X$ to $\Gamma$.

Let $\Omega_{0}=\{\mu \in \Omega: W(\mu)>0\}$. If $\Omega_{0}$ is an empty set, the problem is trivial. Suppose $\Omega_{0}$ is non-empty and $X_{0}^{*}$ is a restricted isotonic regression of $X$ to $\Omega_{0}$. The function $X^{*}$ defined by $\quad X^{*}(\omega)=\min \left\{X_{0}^{*}(\mu): \mu \in \Omega_{0}\right\} \quad$ if $\quad\left\{\mu \in \Omega_{0}: \omega \geq \mu\right\}=\varnothing \quad$ and $\mathrm{X}^{*}(\omega)=\max \left\{\mathrm{X}_{0}^{*}(\mu): \mu \leq \omega, \mu \in \Omega_{0}\right\} \quad$ otherwise, is an isotonic regres sion of X over $(\Omega, W)$. Since isotonic functions are constant on each equivalence class, the objective function can be written as

$$
\begin{aligned}
\mathrm{f}(\mathrm{Z}) & =\Sigma_{\omega \in \Omega_{0}}[\mathrm{X}(\omega)-\mathrm{Z}(\omega)]^{2} \mathrm{~W}(\omega) \\
& =\Sigma_{\omega \in \Omega_{0}}[\overline{\mathrm{X}}(\omega)-\mathrm{Z}(\omega)]^{2} \overline{\mathrm{~W}}(\omega)+\Sigma_{\omega \in \Omega_{0}}[\mathrm{X}(\omega)-\overline{\mathrm{X}}(\omega)]^{2} \mathrm{~W}(\omega)
\end{aligned}
$$

where $\bar{X}(\omega)$ is the weighted average of $X$ over the equivalence class $[\omega]$ in $\Omega_{0}$ and $\bar{W}(\omega)$ is the average of $W$ over the same equivalence class $[\omega]$. The second term on the right-hand side of the above equation is independent of $Z$ and $[\overline{\mathrm{X}}(\omega)-\mathrm{Z}(\omega)]^{2} \overline{\mathrm{~W}}(\omega)$ is constant on each equivalence class. Therefore, without loss of generality, we may assume that $\Omega$ is a connected partially ordered set and $W$ is a positive weight function.

Let $H$ be the linear space of all real-valued functions defined on the connected partially ordered set $\Omega$, let $W$ be a positive weight function on $\Omega$ and let $(\cdot, \cdot)$ be a bilinear functional defined on $H \times H$ by

$$
(X, Y)=\Sigma_{\omega \in \Omega} X(\omega) Y(\omega) W(\omega) \quad \text { for each pair } \quad X, Y \in H
$$

The bilinear functional (.,.) is an inner product on $H$, and the linear space $H$ with the inner product described above is a Hilbert space. The objective function $f(Z)$ can be represented as $\|X-Z\|^{2}$ where $\|Y\|=(Y, Y)^{1 / 2}$.

For each immediately comparable pair $[\mu \leq \nu]$, let us define a linear functional $g_{\mu, v}$ on $H$ by

$$
g_{\mu, v}(\mathbf{Z})=\mathbf{Z}(\mu)-Z(v)
$$

The set $\left\{\mathrm{Y}: \mathrm{g}_{\mu, \nu}(\mathrm{Y}) \leq 0\right\}$ is a closed half-space containing 0 and hence is a closed convex cone. Since an element $Z \in H$ is isotonic
if and only if $g_{\mu, \nu}(Z) \leq 0$ for each immediately comparable pair, the family $M$ of all isotonic functions can be characterized as the intersection of all such closed half-spaces, i.e.,
$M=\left\{Z \in H: g_{\mu, v}(Z) \leq 0\right.$ for each $\left.[\mu \leq \nu]\right\}$. Therefore, the family $M$ is a closed convex cone and the isotonic regression is the generalized projection to $M$. The uniqueness and existence of the isotonic regression of a given $X$ follows from Theorem 2.1.

Let $Z$ be a non-constant isotonic function and let $c$ be a constant function. Then $c$ and $Z-c$ are linearly independent and $Z=c+Z-c$. Since $M$ contains constant functions, $Z$ is not an extreme vector of $M$ and hence the only extreme vectors of $M$ are constant functions, positive or negative. Let $M *$ be the dual cone of $M$, i.e., $Y \in M^{*}$ if $(Y, Z) \leq 0$ for each $Z \in M$. For each $[\mu \leq \nu]$, let $Y_{\mu, \nu}$ be the function in $H$ defined by $\mathrm{Y}_{\mu, \nu}(\mu)=\mathrm{W}(\mu)^{-1}, \mathrm{Y}_{\mu, \nu}(\nu)=-\mathrm{W}(\nu)^{-1}$ and $\mathrm{Y}_{\mu, \nu}(\omega)=0$ elsewhere. Since $\left(Y_{\mu, \nu}, Z\right)=g_{\mu, \nu}(Z), \quad Y_{\mu, \nu} \in M *$ for each $[\mu \leq \nu]$. The dual cone of the closed half-space $\left\{\mathrm{Y}: \mathrm{g}_{\mu, \nu}(\mathrm{Y}) \leq 0\right\}$ is the ray $\left\{\delta Y_{\mu, \nu}: \delta \geq 0\right\}$. The convex hull of a given set A is the intersection of all convex sets containing $A$, and is denoted by conh(A). Let $C_{1}, C_{2}, \ldots, C_{k}$ be a finite sequence of closed convex cones in $H$. It is known that $\left(C_{1} \cap C_{2} \cap \cdots \cap C_{k}\right)^{*}=\operatorname{conh}\left(C_{1}^{*} \cup C_{2}^{*} \cup \cdots \cup C_{k}^{*}\right)$ (cf. Rockafeller (1970)). Therefore, $M^{*}=\operatorname{conh} \cup\left\{\delta Y_{\mu, \nu}:[\mu \leq \nu]\right.$ an immediately comparable pair $\}$. In
other words, $M *$ is the cone generated by the $Y_{\mu, \nu}{ }^{\prime} s,[\mu \leq \nu]$ an immediately comparable pair.

If $Y$ is an extreme vector of $M *$, then $Y$ is a scalar multiple of $Y_{\mu, \nu}$ for some $[\mu \leq \nu]$. We shall show that $Y_{\mu, \nu}$ is an extreme vector of $M^{*}$ for each $[\mu \leq \nu]$. If $Z \in M *$, then $\left(\mathrm{Z}, 1_{\omega}\right) \leq 0$ and $\left(\mathrm{Z}, 1_{\omega}^{0}\right) \leq 0$ for each $\omega \in \Omega$ because $1_{\omega}$ and ${ }_{1}^{1}{ }_{\omega}^{0}$ are in $M$ where $l_{\omega}(\nu)=1$ if $\omega \leq \nu$ and $l_{\omega}(\nu)=0$ otherwise; $\quad{ }_{\omega}^{1}(\nu)=1$ if $\omega \leq \nu, \omega \neq \nu$ and ${ }_{\omega}^{1}(\nu)=0$ otherwise. It follows that $M *$ does not contain any line $\{a Z:$ a real\}. Suppose that there is an immediately comparable pair $[\xi \leq \eta]$ such that $Y_{\xi, \eta}$ is not an extreme vector of $M *$. Then the cone generated by other $Y_{\mu, \nu}$ 's is $M *$ because $M *$ does not contain a line and it is the cone generated by the $\mathrm{Y}_{\mu, \nu}$ 's (cf. Rockafeller (1970)). By the identities $\left(C_{1} \cap C_{2} \cap \cdots \cap C_{k-1}\right) *=\operatorname{conh}\left(C_{1}^{*} \cup C_{2}^{*} \cup \ldots \cup C_{k-1}^{*}\right)$ and $\quad M=(M *) *$, it will then follow that $M$ can be represented as an intersection with one less closed half-space $\left\{Y: g_{\xi, \eta}(Y) \leq 0\right\}$. Therefore, $g_{\xi, \eta}$ is a redundant constraint. On the other hand, let $Z$ be defined by $Z(\omega)=1$ if $\omega \geq \xi, \omega \neq \eta \quad$ with zeros elsewhere. Then $Z$ satisfied all other constraints but $Z$ is not isotonic. This contradicts that $g_{\xi, \eta}$ is redundant. It follows that $Y_{\mu, \nu}$ is an extreme vector of $M *$ for each $[\mu \leq \nu]$.

Let $S_{0}$ be the linear space $\{Y:(Y, l)=0\}$. Since $\left(Y_{\mu, \nu}, l\right)=0$ for each $[\mu \leq \nu], M *$ is a subset of $S_{0}$.

## III. 2 Upper Sets and Ind icators

A subset $U$ of a quasi-orderes set $\Omega$ is said to be an upper set if $\mu \in U$ and $v \geq \mu$ imply $v \in U$. A subset $L$ of $\Omega$ is said to be a lower set if $v \in \mathrm{~L}$ and $\mu \leq \nu$ imply $\mu \in \mathrm{L}$. It is obvious that if $U$ is an upper set then $U^{C}$ is a lower set and if $L$ is a lower set than $L^{C}$ is an upper set. A non-empty connected upper set is called a basic set. If an upper set is separable, then each of its components is an upper set and hence a basic set. For each $\omega \in \Omega$, the set $U(\omega):=\{\mu: \mu \geq \omega\}$ is a basic set. Such a basic set $U(\omega)$ is said to be determined by $\omega$. $\Omega$ and $\phi$ are upper sets and lower sets. $\Omega$ is separable if and only if there exists a proper subset of $\Omega$ which is both an upper set and a lower set. Further discussion of upper sets and lower sets will be given in Section IV.1. In the remainder of the chapter, we shall let $\Omega$ denote a connected partially ordered finite set. The intersection of an uper set and a lower set is called a level set. A subset $\Gamma$ of $\Omega$ is a level set if and only if $\mu, v \in \Gamma$ and $\mu \leq \omega \leq v$ imply that $\omega \in \Gamma$. A component of a level set is a level set. The intersection of level sets is a level set.

Let $\Gamma$ be a subset of $\Omega$. The indicator of $\Gamma$, denoted by ${ }^{1}{ }_{\Gamma}$, is a real-valued function which assumes the value one at each element in $\Gamma$ and zero elsewhere. Indicators of $\Omega$ and $\phi$ are denoted by 1 and 0 respectively. The indicator $l_{\Gamma}$ is isotonic
if and only if $\Gamma$ is an upper set. The indicator of a basic set is called a basic function. For a basic set $U(\omega)$ determined by $\omega$, its basic function is denoted by $l_{\omega}$. If $Z$ is an isotonic function, then for each real $a,[Z \geq a]:=\{\omega: Z(\omega) \geq a\}$ is an upper set. Suppose $Z$ assumes the values $a_{1} \leq a_{2} \leq \cdots \leq a_{k}$. Then $\left[Z=a_{i}\right]:=\left\{\omega: Z(\omega)=a_{i}\right\}$ is a level set for each $i$. Let $\beta_{i}=a_{i}-a_{i-1}, i=1, \ldots, k$ where $a_{0}=0$. Then $Z=\Sigma_{i=1}^{k} \beta_{i} l^{1}\left[Z \geq a_{i}\right]$. Since an indicator of apper set is the sum of the indicators of its components, every non-negative isotonic function is a non-negative linear combination of basic functions and every isotonic function is a linear combination of basic functions such that the coefficient associated with each indicator other than 1 is non-negative. But such a representation need not be unique.

Let $M_{+}$be the family of all non-negative isotonic functions. Then $M_{+}$is the cone generated by the basic functions. Therefore $M_{+}$has finitely many extreme vectors. Let $U$ be a basic set and let $l_{U}$ be it indicator. We clain that $l_{U}$ is an extreme vector of $M_{+}$. Suppose there are $Z_{1}$ and $Z_{2}$ in $M_{+}$such that $l_{U}=Z_{1}+Z_{2}$. Then $Z_{1}(\omega)=Z_{2}(\omega)=0$ for all $\omega \notin U$. Let $\mu$ and $\nu$ be two elements in $U$ with $\mu \leq \nu$. Then $Z_{i}(\mu) \leq Z_{i}(\nu)$, $i=1,2$ and $Z_{1}(\mu)+Z_{2}(\mu)=Z_{1}(\nu)+Z_{2}(\nu)=1$. Therefore $Z_{i}(\mu)=Z_{i}(\nu), i=1,2$. If $\omega \in U$ can be reached from $\mu$ in $U$,
then it follows that $Z_{i}(\omega)=Z_{i}(\mu), i=1,2$. Since a basic set is a non-empty connected upper set, every element $\omega \in \mathbb{U}$ can be reached from $\mu$ and hence $Z_{1}=\lambda l_{U}$ and $Z_{2}=(1-\lambda) 1_{U}$ for some $\lambda$ between zero and one. Our claim therefore has been established and basic functions form a complete set of extreme vectors for $M_{+}$. If $\Omega$ is a tree, then every upper set with more than one minimal element is separable. It follows that basic functions must be of the form $l_{\omega}$, and $M_{+}$has exactly the same number of extreme vectors as the number of elements in $\Omega$. Thus we have proved the following theorem.

Theorem 3.1. Let $\Omega$ be a connected partially ordered finite set, let $M$ be the family of isotonic functions and let $M_{+}$be the family of non-negative isotonic functions. Then $M_{+}$is the cone generated by the basic functions, and basic functions form a complete set of extreme vectors for $M_{+}$. The family $M$ is the cone generated by basic functions and -1 .

Following the same procedure, one can show that the above theorem also holds for general quasi-ordered finite sets connected or separable.

The isotonic regression of $\mathbf{X}$ over $\Omega$ is the projection $P(X \mid M)$ of $X$ on the closed convex cone $M$. In the finite case, $L_{2}(\Sigma)$, introduced in Chapter IV is the same as $M$ and the
conditional expectation $\mathrm{E}(\mathrm{X} \mid \Sigma)$, also introduced in Chapter IV, is the same as $P(X \mid M)$. Therefore, properties of isotonic regression can be found in Chapter II in the form of $P(X \mid M)$ and also can be found in Chapter IV in the form of either $P\left(X \mid L_{2}(\Sigma)\right)$ or $E(X \mid \Sigma)$. The main interest in the remaining part of this chapter is in algorithms for various isotonic regression problems and some results which lead to these algorithms.

Theorem 3.2. Let $X *=\Sigma_{U \text { basic }} a_{U} l_{U}$ with $a_{U} \geq 0$ for each basic set $U$ different from $\Omega$, and let $\Lambda=\left\{\mathrm{U}: \mathrm{U}\right.$ is basic, $\left.\mathrm{a}_{\mathrm{U}}>0\right\} \cup\{\Omega\}$. Then $\mathrm{X}^{*}$ is the isotonic regression of $X$ if and only if $\left(X-X *, 1_{U}\right)=0$ for each $U \in \Lambda$ and $\left(X-X *, 1_{U}\right) \leq 0 \quad$ for each basic set $U$.

Proof. The function $X *$ given above is obviously an isotonic function. Let $S_{\Lambda}=\left\{Z=\Sigma_{U \in \Lambda} \beta_{U}{ }^{1}{ }_{U}: \beta_{U}\right.$ real $\}$. The condition $\left(X-X *, l_{U}\right)=0$ for each $U \in \Lambda$ is equivalent to $X *=P\left(X \mid S_{\Lambda}\right)$. Therefore, the result follows from Theorem 2.12 and Theorem 2.13. The only difference is that we have $\Omega \in \Lambda$ no matter what value $a_{\Omega}$ is. This gap can be filled by considering (4.2), which can be interpreted as $(X-X *, 1)=0$ in our present situation. []

Corollary 3.2.1. Let $X * \in M$. Then $X *$ is the isotonic regression of $X$ if and only if $X *$ satisfies

$$
\begin{equation*}
\Sigma_{\left.\omega \in[X *=\gamma]^{X(\omega) W}(\omega)=\Sigma_{\omega \in[X *=\gamma]}\right]^{*} *(\omega) W(\omega) \text { for each } \gamma} \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\Sigma_{\omega \in U} X(\omega) W(\omega) \leq \Sigma_{\omega \in U} X *(\omega) W(\omega) \tag{3.3}
\end{equation*}
$$

for each upper set $U$.

Proof. Let $X *$ be an isotonic function which assumes values $\gamma_{1}<\gamma_{2}<\ldots<\gamma_{k}$ and let $a_{i}=\gamma_{i}-\gamma_{i-1}, i=1, \ldots, k$ where $\gamma_{0}=0$. Then $X *=\Sigma_{i=1}^{k} a_{i} l^{1}\left[X * \gamma_{i}\right]$ with $a_{i}$ positive for $i>1$. Let $X *$ satisfy (3.2) and (3.3). Then

$$
(X-X *, X *)=\Sigma_{i=1}^{k} \gamma_{i} \Sigma_{\omega \in\left[X *=\gamma_{i}\right]}(X(\omega)-X *(\omega)) W(\omega)=0
$$

and $(X-X *,-1)=0$. Since $M$ is the cone generated by upper sets and -1 , by Theorem 2.12, $X^{*}$ is the isotonic regression of $X$.

On the other hand, let $X *$ be the isotonic regression of $X$. Since each upper set is the disjoint union of its components, by Theorem 3.2 we have $\left(X-X *, l_{U}\right) \leq 0$ for each upper set $U$. Similarly, $\left.\quad\left(X-X^{*}, l_{\left[X_{*} \geq \gamma_{i}\right.}\right]\right)=0 \quad$ for $i=1, \ldots, k$. For each $\gamma$ different from $\gamma_{1}, \ldots, \gamma_{k}, X *$ satisfies (3.2) automatically. For each i,

$$
\left(X-X *, 1\left[X *=\gamma_{i}\right]\right)=\left(X-X *, 1\left[X * \geq \gamma_{i}\right]\right)-\left(X-X *, 1\left[X *>\gamma_{i}\right]\right)=0
$$

This completes the proof. []

Corollary 3.2.2. Let $X *$ be the isotonic regression of $X$. If $\nu$ is minimal in $[a \leq X * \leqslant]$, then $X *(\nu) \leq X(\nu)$. If $\mu$ is maximal in $[a \leq X * \leq \beta]$, then $X *(\mu) \geq X(\mu)$.

Proof. If $v$ is minimal in $[a \leq X * \leq \beta]$, then $v$ is minimal in $[a \leq X *]$. By the minimality of $v$ in $[a \leq X *]$, $U=\{\omega: X *(\omega) \geq a, \omega \neq \nu\}$ is an upper set. From Corollary 3.2.1,

$$
\left.\left(X-X *, l_{[a \leq X *}\right]\right)=0
$$

and

$$
\left(X-X *, 1_{U}\right) \leq 0,
$$

which implies $X(\nu)-X *(v) \geq 0$. Similarly we have the second statement. []

Corollary 3.2.3. Let $X *$ be the isotonic regression of $X$, let $[\mu \leq \nu]$ be an immediately comparable pair and let $X(\mu) \geq X(\nu)$. Then either there exists an immediate successor $\omega$ of $\mu$ such that $\mathrm{X} *(\omega)=\mathrm{X} *(\mu)$ or there exists an immediate predecessor $\tau$ of $v$ such that $\quad X *(\tau)=X *(\nu)$.

Proof. Suppose on the contrary that neither $X *(\omega)=X *(\mu)$ for any immediate successor $\omega$ of $\mu$ nor $X *(\tau)=X *(\nu)$ for any immediate predecessor $\tau$ of $v$. Then $X *(\nu)>X^{*}(\mu), \quad v$ is minimal in $[\mathrm{X} * \geq \mathrm{X} *(\nu)]$ and $\mu$ is maximal in $[\mathrm{X} * \leq \mathrm{X} *(\mu)]$. By

Corollary 3.2.2, we have $X(\nu) \geq X *(\nu)>X *(\mu) \geq X(\mu)$. This contradicts that $\quad \mathbf{X}(\mu) \geq \mathbf{X}(\nu) . \quad[]$

Theorem 3.3. Let $\Omega$ be a partially ordered set, let $X *$ be the isotonic regression of $X$ and let $\mu$ be an immediate predecessor of $v$. If any one of the following three statements is true:
(1) $\mu$ is the only immediate predecessor of $\nu, \quad \nu$ is the only immediate successor of $\mu$, and $X(\mu) \geq X(\nu)$,
(2) $\mu$ is the only immediate predecessor of $\nu$ and $X(\nu) \leq X(\omega)$ for every $\quad \omega \geq \mu$,
(3) $\nu$ is the only immediate successor of $\mu$ and $X(\mu) \geq X(\omega)$ for every $\omega \leq \nu$,
then $X *(\mu)=X *(\nu)$.

The theorem is an extension of Theorem 2.6 given by Barlow and coworkers (1972). That $X *(\mu)=X *(\nu)$ follows from the first statement is an immediate result of Corollary 3.2.3. If $\Omega$ is a linearly ordered set, then each element has at most one immediate predecessor and at most one immediate successor. Thereofre, statment (1) is enough to provide an algorithm in such a situation. If $\Omega$ is a tree, then each element has at most one immediate predeces sor, and if $\Omega$ is a reversed tree, then each element has at most one immediate successor. Statement (2) and statement (3) will be sufficient for obtaining isotonic regressions in such situations. Statement
(2) and statement (3) are symmetric with respect to reversal, because $\mathrm{M}(\Omega)=-\mathrm{M}\left(\Omega_{\mathbf{r}}\right)$ and $\mathrm{P}(\mathrm{X} \mid \mathrm{M}(\Omega))=-\mathrm{P}\left(-\mathrm{X} \mid \mathrm{M}\left(\Omega_{\mathrm{r}}\right)\right)$.

Proof. Only the result following from statement (2) will be considered. Let $\mu$ be the only immediate predecessor of $v$ and $X(v) \leq X(\omega)$ for every $\omega \geq \mu$. Suppose $X *(\mu)<X *(v)$. Then $v$ is minimal in $[X *(v) \leq X *]$. By Corollary 3.2.2, $X *(v) \leq X(v)$. On the other hand, $U(\mu)=\{\omega: \omega \geq \mu\}$ is an upper set and $X(\omega) \geq X(\nu)$ for each $\omega \in U(\mu)$. By Corollary 4.15.2 which does not depend on Theorem 3. 3, $X *(\mu) \geq X(v)$ and hence $X *(\mu) \geq X(\nu) \geq X *(\nu)$. This contradicts the assumption $\mathrm{X} *(\mu)<\mathrm{X} *(\nu)$. []

## III. 3 On a Linearly Ordered Set

The minimization problem discussed here is to minimize $f(Z)$ subject to $\quad z_{i} \leq z_{i+1}$ for $i=1,2, \ldots, n-1$ where $Z=\left(z_{1}, \ldots, z_{n}\right)$ and

$$
f(Z)=\Sigma_{i=1}^{n}\left(x_{i}-z_{i}\right)^{2} w_{i}
$$

with given $X=\left(x_{1}, \ldots, x_{n}\right)$ and given $w_{i}>0, i=1, \ldots, n$. The problem is known as an isotonic regression over the linearly ordered set $\{1,2, \ldots, n\}$. It can be solved by the Pool-AdjacentViolators algorithm which was introduced by Ayer and coworkers (1955). The algorithm, which is an immediate result of Theorem
3.3, will be presented later in this section.

Let $\quad M=\left\{Z: z_{i} \leq z_{i+1}, i=1,2, \ldots, n-1\right\} \quad$ be the feasible region, let $M *$ be the dual cone of $M$ with respect to the inner product $(X, Y)=\Sigma_{i=1}^{n} X_{i} y_{i} W_{i}$ and let $Y_{1}, Y_{2}, \ldots, Y_{n-1}$ be $n-c o m p o n e n t$ vectors such that for each $k, \quad Y_{k}=\left(y_{k l}, \ldots, y_{k n}\right)$ is defined by $y_{k k}=w_{k}^{-1}, y_{k, k+1}=-w_{k+1}^{-1}$ and $y_{k i}=0$ if $i \neq k, k+1$. The dual cone $M *$ is the cone generated by the linearly ind ependent vectors $Y_{1}, \ldots, Y_{n-1}$, and it is a subset of the linear space $S_{0}=\left\{Z: \Sigma_{i=1}^{n} z_{i} w_{i}=0\right\}$. The inner products of $Y_{i}^{\prime} s$ among themselves are $\left(Y_{j-1}, Y_{j}\right)=-w_{j}^{-l}, \quad\left(Y_{j}, Y_{j}\right)=w_{j}^{-1}+w_{j+1}^{-1}$, $\left(Y_{j+1}, Y_{j}\right)=-w_{j+1}^{-1} \quad$ and $\quad\left(Y_{i}, Y_{j}\right)=0 \quad$ if $\quad|i-j|>1$.

The isotonic regression of $X, P(X \mid M)$, is $X-P(X \mid M *)$ as given by Theorem 2.9. By the smoothing propery, $P(X \mid M *)=P\left(P\left(X \mid S_{0}\right) \mid M *\right)$. The projection of $X$ on $S_{0}$ is

$$
P\left(X \mid S_{0}\right)=X-\bar{x} \quad \text { where } \quad \bar{x}=\sum_{i=1}^{n} x_{i} w_{i} / \Sigma_{i=1}^{n} w_{i}
$$

Let $X_{0}=X-\bar{x}$. Since $S_{0}$ is the linear space generated by $Y_{1}, \ldots, Y_{n-1}$, there exists a unique set of real numbers $a_{1}, \ldots, a_{n-1}$ such that

$$
X_{0}=\Sigma_{i=1}^{n-1} a_{i} Y_{i} .
$$

The projection $P\left(X_{0} \mid M *\right)$ is in $M *$, so there exists a unique set
of non-negative real numbers $\quad a_{1}^{*}, \ldots, a_{n-1}^{*}$ such that

$$
P\left(X_{0} \mid M *\right)=\Sigma_{i=1}^{n-1} a_{i}^{*} Y_{i} .
$$

Let $\Lambda=\left\{\mathrm{i}: \mathrm{a}_{\mathrm{i}}^{*}>0\right\}$ and let $\mathrm{N}_{\Lambda}:=\left\{\mathrm{Y}=\Sigma_{\mathrm{i} \in \Lambda} \beta_{\mathrm{i}} \mathrm{Y}_{\mathrm{i}}: \beta_{\mathrm{i}} \geq 0\right\}$. Then $P\left(X_{0} \mid M^{*}\right)=P\left(X_{0} \mid N_{\Lambda}\right)$. We shall soon see that $P\left(X_{0} \mid M^{*}\right)$ can be obtained by successive projections on linear spaces, each of which is generated by a subset of $\left\{Y_{1}, \ldots, Y_{n-1}\right\}$.

Let $\Gamma$ be a subindex set of $\{1,2, \ldots, n-1\}$. The linear space generated by $\left\{Y_{i}: i \in \Gamma\right\}$, is denoted by $T_{\Gamma}$. If $j \in \Gamma$, the projection of $Y_{j}$ on $T_{\Gamma}, P\left(Y_{j} \mid T_{\Gamma}\right)$, is $Y_{j}$ itself. For fixed $\mathrm{i}, \mathrm{j}, \mathrm{k}$ with $\mathrm{i}<\mathrm{j}<\mathrm{k}$, let $\Gamma_{1}=\{\mathrm{i}, \mathrm{i}+1, \ldots, \mathrm{j}-1, \mathrm{j}+1, \ldots, \mathrm{k}\}$, let $\Gamma_{2}=\{1,2, \ldots, \mathrm{i}-2, \mathrm{k}+2, \ldots, \mathrm{n}-1\}$ and let $\Gamma_{3}=\Gamma_{1} \cup \Gamma_{2}$. The linear spaces ${ }^{T} \Gamma_{1}$ and ${ }^{T} \Gamma_{2}$ are orthogonal and $Y_{j}$ is orthogonal to $T_{\Gamma_{2}}$. It follows that $P\left(Y_{j} \mid T_{\Gamma_{3}}\right)=P\left(Y_{j} \mid T_{\Gamma_{1}}\right)$.

If $\Gamma$ is such that $\Gamma_{1} \subset \Gamma \subset \Gamma_{3}$, then by the smoothing property we have $P\left(Y_{j} \mid T_{\Gamma}\right)=P\left(P\left(Y_{j} \mid T_{\Gamma_{3}}\right) \mid T_{\Gamma}\right)=P\left(P\left(Y_{j} \mid T_{\Gamma_{1}}\right) \mid T_{\Gamma}\right)$ $=P\left(Y_{j} \mid T_{\Gamma_{1}}\right)$. Let

$$
\begin{equation*}
a_{q}=\Sigma_{m=i}^{q} w_{m} / \Sigma_{m=i}^{j} w_{m}, \quad b_{q}=\Sigma_{m=q+1}^{k+1} w_{m} / \Sigma_{m=j+1}^{k+1} w_{m} \tag{3.4}
\end{equation*}
$$

and let $Y=-\Sigma_{q=i}^{j-1} a_{q} Y_{q}-\Sigma_{q=j+1}^{k} b_{q} Y_{q}$. Then for each $h=i, i+1, \ldots, j-2$, we have

$$
\begin{aligned}
\left(Y_{j}-Y, Y_{h}\right) & =-\left(Y, Y_{h}\right) \\
& =a_{h-1}\left(Y_{h-1}, Y_{h}\right)+a_{h}\left(Y_{h}, Y_{h}\right)+a_{h+1}\left(Y_{h+1}, Y_{h}\right) \\
& =-a_{h-1} w_{h}^{-1}+a_{h} w_{h}^{-1}+a_{h} w_{h+1}^{-1}-a_{h+1} w_{h+1}^{-1} \\
& =w_{h}^{-1}\left(a_{h}-a_{h-1}\right)-w_{h+1}^{-1}\left(a_{h+1}-a_{h}\right) \\
& =0
\end{aligned}
$$

and similarly for each $h=j+2, \ldots, k$, we have $\left(Y_{j}-Y, Y_{h}\right)=0$ by changing the $a$ 's to $b$ 's. For $h=j-1$, we have

$$
\begin{aligned}
\left(Y_{j}-Y, Y_{j-1}\right) & =\left(Y_{j}, Y_{j-1}\right)+a_{j-2}\left(Y_{j-2}, Y_{j-1}\right)+a_{j-1}\left(Y_{j-1}, Y_{j-1}\right) \\
& =-w_{j}^{-1}-a_{j-2} w_{j-1}^{-1}+a_{j-1}\left(w_{j-1}^{-1}+w_{j}^{-1}\right) \\
& =w_{j-1}^{-1}\left(a_{j-1}^{-a}{ }_{j-2}\right)-w_{j}^{-1}\left(1-a_{j-1}\right) \\
& =0
\end{aligned}
$$

and similarly, we have $\left(Y_{j}-Y, Y_{j+1}\right)=0$. Therefore, $Y$ is the projection $P\left(Y_{j} \mid T_{\Gamma_{1}}\right)$ and

$$
\begin{gather*}
P\left(Y_{j} \mid T_{\Gamma}\right)=-\Sigma_{q=i}^{j-1} a_{q} Y_{q}-\Sigma_{q=j+1}^{k} b_{q} Y_{q} .  \tag{3.5}\\
\text { If } \Gamma_{1}=\{j+1, \ldots, k\}, \quad \Gamma_{3}=\Gamma_{1} \cup\{1, \ldots, j-2, k+2, \ldots, n-1\} \text { and }
\end{gather*}
$$ $\Gamma$ is such that $\Gamma_{1} \subset \Gamma \subset \Gamma_{3}$, then following the same procedure we have

$$
\begin{equation*}
P\left(Y_{j} \mid T_{\Gamma}\right)=-\Sigma_{q=j+1}^{k} b_{q} Y_{q} \tag{3.6}
\end{equation*}
$$

Similarly, if $\Gamma_{1}=\{\mathrm{i}, \mathrm{i}+1, \ldots, \mathrm{j}-\mathrm{l}\}, \quad \Gamma_{3}=\Gamma_{1} \cup\{1, \ldots, \mathrm{i}-2, \mathrm{j}+2, \ldots, \mathrm{n}-1\}$ and $\Gamma$ is such that $\Gamma_{1} \subset \Gamma \subset \Gamma_{3}$, then

$$
\begin{equation*}
P\left(Y_{j} \mid T_{\Gamma}\right)=-\Sigma_{q=i}^{j-1} a_{q} Y_{q} \tag{3.7}
\end{equation*}
$$

If $\mathrm{j}-1, \mathrm{j}$ and $\mathrm{j}+1$ are not in $\Gamma$, then $P\left(\mathrm{Y}_{\mathrm{j}} \mid \mathrm{T}_{\Gamma}\right)=0$.
Let $Z=\Sigma_{j=1}^{n-1} \beta_{j} Y_{j}$. The projection of $Z$ on $T_{\Gamma}$, for some subindex set $\Gamma$, is $P\left(Z \mid T_{\Gamma}\right)=\Sigma_{j=1}^{n-1} \beta_{j} P\left(Y_{j} \mid T_{\Gamma}\right)$. If $j \in \Gamma$, $P\left(Y_{j} \mid T_{\Gamma}\right)=Y_{j}$. Let $j \notin \Gamma$. If $j-1 \in \Gamma$, let $i$ be the smallest index such that $i, i+1, \ldots, j-1$ arein $\Gamma$ and if $j+l \in \Gamma$, let $k$ be the largest index such that $j+1, j+2, ., k$ are in $\Gamma$. If $j-1$ and $j+1$ are not in $\Gamma$, we have $P\left(Y_{j} \mid T_{\Gamma}\right)=0$. We identify $\Gamma_{1}$ as $\{i, i+1, \ldots, j-1\}, \quad\{j+1, \ldots, k\}$ or their union according as only $j-1 \in \Gamma$, only $j+1 \in \Gamma$ or both $j-1$ and $j+1$ in $\Gamma$. Therefore, the projection of $Z$ on $T_{\Gamma}$ can be calculated by (3.5), (3.6) and (3.7).

Let $P\left(Z \mid T_{\Gamma}\right):=\Sigma_{h \in \Gamma} \beta_{h}^{*} Y_{h}$. For each $h \in \Gamma$, let $p=\max \{m: m<h, m \notin \Gamma\} \quad$ and let $q=\min \{m: m>h, m \notin \Gamma\}$. Suppose $p$ and $q$ exist. Then for each $j \neq h$ which is less than $p$, greater than $q$ or belonging to $\Gamma$, we have $P\left(Y_{j} \mid T_{\Gamma}\right)=\Sigma_{m \in \Gamma} \gamma_{m} Y_{m}$ with $\gamma_{h}=0$. It follows that

$$
\begin{equation*}
\beta_{h}^{*}=\beta_{h}-\left(\beta_{p} b_{n}+\beta_{q} a_{h}\right) \tag{3.8}
\end{equation*}
$$

where $a_{h}=\Sigma_{m=p+1}^{h} w_{m} / \Sigma_{m=p+1}^{q} w_{m}$ and
$b_{h}=\Sigma_{m=h+1}^{q} w_{m} / \Sigma_{m=p+1}^{q} w_{m}$. The $a_{h}$ and the $b_{h}$ are the same as in (3.4); for the $a_{h}$, we identify $i$ and $j$ as $p+1$ and $q$ respectively; for the $b_{h}$, we identify $j$ and $k$ as $p$ and $q-1$ respectively. If $p$ does not exist, set $p=0$ and $\beta_{p}=0$, and if $q$ does not exist, set $q=n$ and $\beta_{q}=0$. Then $\beta_{h}^{*}$ is still given by (3.8).

Theorem 3.4. Let $X_{0}=\Sigma_{i=1}^{n-1} a_{i} Y_{i}$ and let $P\left(X_{0} \mid M *\right)=\Sigma_{i=1}^{n-1} a_{i}^{*} Y_{i}$. If $a_{h}=\min \left\{a_{1}, \ldots, a_{n-1}\right\} \quad$ and $\quad a_{h}<0$, then $a_{h}^{*}=0$.

Proof. Let $\Lambda=\left\{i: a_{i}^{*}>0\right\}$. Theorem 2.13 shows that $P\left(X_{0} \mid M^{*}\right)=P\left(X_{0} \mid T_{\Lambda}\right)$. Suppose $h \in \Lambda$. By (3.8), $a_{h}^{*}=a_{h}-\left(a_{p} b_{h}+a_{q} a_{h}\right)$. Since $a_{p} \geq a_{h}, \quad b_{h}>0, a_{q} \geq a_{h}, a_{h}>0$ and $b_{h}+a_{h}=1$, we have $a_{h}^{*} \leq 0$. This contradicts that $h \in \Lambda$. $]$

The Projection of Minimum Violators is an algorithm to obtain the isotonic regression of $X$ over the linearly ordered set $\{1,2, \ldots, n\}$ through the dual cone $M *$. By a violator we mean an index $j$ such that $a_{j} \leq 0$.
Step 1. Set $\bar{x}=\Sigma_{i=1}^{n} x_{i} w_{i} / \Sigma_{i=1}^{n} w_{i}$.
Set $\quad a_{1}=w_{1}\left(x_{1}-\bar{x}\right), \quad a_{i}=a_{i-1}+w_{i}\left(x_{i}-\bar{x}\right)$ for $i=2, \ldots, n-1$.
Set $\Gamma=\{1, \ldots, n-1\}$.

Step 2. If $a_{1}, \ldots, a_{n-1}$ are non-negative, go to Step 4;
otherwise, choose $j$ such that $a_{j}=\min \left\{a_{1}, \ldots, a_{n-1}\right\}$.
Set $\Gamma=\Gamma-\{j\}$.
If $j-1 \in \Gamma$, then find the smallest $i$ such that
i, i+1, ..,j-1 $\in \Gamma$ and set $\gamma_{m}=0$ for $m=1, \ldots, i-1$
and set $\quad \gamma_{m}=\Sigma_{h=i}^{m} w_{h} / \sum_{h=i}^{j} w_{h}$ for $m=i, i+1, \ldots, j-1$;
otherwise, set $\gamma_{m}=0$ for $m=1, \ldots, j-1$. If $j+1 \in \Gamma$,
then find the largest $k$ such that $j+1, \ldots, k \in \Gamma$ and
set $\quad \gamma_{m}=\Sigma_{h=m+1}^{k+1} w_{h} / \Sigma_{h=j+1}^{k+1} w_{h}$ for $m=j+1, \ldots, k$ set
$\gamma_{m}=0$ for $m=k+1, \ldots, n-1$; otherwise, set $\gamma_{m}=0$
for $m=j+1, \ldots, n-1$.
Set $\quad \gamma_{j}=1$.
Step 3. Set $a_{m}=a_{m}-a_{j} \gamma_{m}$ for $m=1, \ldots, n-1$.
Go to Step 2.
Step 4. Set $y_{1}=w_{1}^{-1} a_{1}, \quad y_{i}=w_{i}^{-1}\left(a_{i}-a_{i-1}\right)$ for $i=2, \ldots, n-1$, $y_{n}=-w_{n}^{-1} a_{n-1}$.
Set $x_{i}^{*}=x_{i}-y_{i}$ for $i=1,2, \ldots, n$.
End.
When we have more than one $j$ such that
$a_{j}=\min \left\{a_{1}, \ldots, a_{n-1}\right\}$, we may compute $\gamma_{m}$ 's simultaneously for such a set of $j$ 's. The algorithm is not efficient as evidenced by Example 3.1; yet, it is the method developed through the dual cone. The justification of the algorithm can be described as follows. Let
$X_{0}=\Sigma_{i=1}^{n-1} a_{i} Y_{i}$, where $X_{0}=P\left(X \mid S_{0}\right)$. If $a_{1}, \ldots, a_{n-1}$ are non-negative, then $X_{0}=P\left(X_{0} \mid M *\right)=P(X \mid M *)$. In any case, let $\Lambda_{0}:=\left\{i: a_{i} \geq 0\right.$ or $a_{i}>a_{j}$ for some $\left.j \neq i\right\}$. Theorem 3.4 shows that $\Lambda_{0} \supset \Lambda$, where $\Lambda=\left\{i: a_{i}^{*}>0\right\}$ and $P\left(X_{0} \mid M *\right)=\Sigma_{i=1}^{n-1} a_{i}^{*} Y_{i}$. Since $P\left(X_{0} \mid M *\right)=P\left(X_{0} \mid N_{\Lambda}\right)$, by the smoothing property $P\left(X_{0} \mid M *\right)=P\left(P\left(X_{0} \mid T_{\Lambda_{0}}\right) \mid N_{\Lambda}\right)$. Let $\quad X_{1}:=P\left(X_{0} \mid T_{\Lambda_{0}}\right)$ be represented by $\Sigma_{i \in \Lambda_{0}} a_{1 i} Y_{i}$ and let $\Lambda_{1}:=\left\{i \in \Lambda_{0}: a_{1 i} \geq 0\right.$ or $a_{1 i}>a_{1 j}$ for some $j \neq i\}$. If $a_{1 i} \geq 0$ for each $i \in \Lambda_{0}$, then $\Lambda_{1}=\Lambda_{0}$ and $X_{1} \in N_{\Lambda}$. Since $P\left(X_{0} \mid M *\right)=P\left(X_{0} \mid N_{\Lambda}\right)$, the smoothing property shows that $P\left(X_{0} \mid N_{\Lambda_{1}}\right)=P\left(P\left(X_{0} \mid T_{\Lambda_{0}}\right) \mid N_{\Lambda_{1}}\right)=P\left(X_{1} \mid N_{\Lambda_{1}}\right)$. It follows that $X_{1}=P\left(X_{0} \mid M *\right)$. If $a_{1 i}<0$ for some $i \in \Lambda_{0}$, then $\Lambda_{0} \neq \Lambda_{1} \supset \Lambda$. Since $P\left(X_{0} \mid M *\right)=P\left(X_{1} \mid N_{\Lambda_{1}}\right)$, the set $\Lambda$ stays the same, and $P\left(X_{0} \mid M *\right)=P\left(P\left(X_{1} \mid T_{\Lambda_{1}}\right) \mid N_{\Lambda}\right)$. Therefore, applying the above procedure inductively, we have a strictly decreasing sequence $\Lambda_{0} \supset \Lambda_{1} \supset \Lambda_{2} \supset \cdots \supset \Lambda$. The sequence will terminate at some index $k$ such that $\Lambda_{k}=\Lambda_{k+1}$. At that stage, $a_{k i} \geq 0$ for each $i \in \Lambda_{k}, X_{k} \in N_{\Lambda_{k}}$ and $X_{k}=P\left(X_{0} \mid M *\right)$. The index set $\Lambda_{k}$ need not be the same as $\Lambda$, and we may reach the situation $\Lambda_{k}=\Lambda=\phi$ which indicates $X_{k}=0$ and $N_{\Lambda_{k}}=T_{\Lambda_{k}}=\{0\}$. The situation occurs only when $X \in M$. Once $P\left(X_{0} \mid M *\right)$ is obtained,
$P(X \mid M)=X-P(X \mid M *)=X-P\left(X_{0} \mid M *\right)$.
A similar technique can be applied to $M$ also. For each $i$, let $l_{i}$ be the vector such that the ith,..., nth entries have values
one with zeros elsewhere. The vectors $1_{1}, 1_{2}, \ldots, l_{n}$ are linearly ind ependent, and every vector in $M$ is a linear combination of the se indicators such that the coefficients are non-negative except the one corresponding to $1_{1}$.

Theorem 3.5. Let $X=\Sigma_{i=1}^{n} a_{i} 1_{i}$, let the isotonic regression of $X, X *$, be represented by $X *=\Sigma_{i=1}^{n} a_{i}^{*} l_{i}$ where $a_{i}^{*} \geq 0$ for $i>1$ and let $\Lambda=\left\{i: a_{i}^{*}>0\right.$ or $\left.i=1\right\}$. If $a_{j} \leq 0$ for some $j>1$, then $j \xi \Lambda$.

The theorem is essentially the same as Theorem 2.1 of Ayer and coworkers (1955), and it is a consequence of Theorem 3.2,

Corollary 3.2.2 or Theorem 3.3. Suppose there is an index $j>1$ such that $a_{j} \leq 0$ and $a_{j}^{*}>0$. Let $x=\left(x_{1}, \ldots, x_{n}\right)$ and let $X^{*}=\left(x_{1}^{*}, \ldots, x_{n}^{*}\right)$. Then $a_{j}=x_{j}-x_{j-1}$ and $a_{j}^{*}=x_{j}^{*}-x_{j-1}^{*}$ for $j>1$. Since $a_{j}^{*}>0, j$ is minimal in $\left[X^{*} \geq x_{j}^{*}\right]$ and $j-1$ is maximal in $\left[X * \leq x_{j-1}^{*}\right]$, by Corollary 3.2.2 we have $x_{j} \geq x_{j}^{*}$ and $x_{j-1}^{*} \geq x_{j-1}$. This contradicts that $x_{j}^{*}>x_{j-1}^{*}$ and $x_{j} \leq x_{j-1}$.

For each subindex set $\Gamma$ of $\{1,2, \ldots, n\}$, let $S_{\Gamma}$ denote the linear space generated by $\quad\left\{1_{i}: i \in \Gamma\right\} \quad$ and let $\quad M_{\Gamma}=S_{\Gamma} \cap M$. Let $\Lambda_{0}=\left\{i: a_{i}>0\right.$ or $\left.i=1\right\}$. Then $\Lambda_{0} \supset \Lambda$, $P\left(P\left(X \mid S_{\Lambda_{0}}\right) \mid M_{\Lambda^{\prime}}\right)=P(X \mid M)$ and $P\left(X \mid S_{\Lambda_{0}}\right)=\Sigma_{i=1}^{n} a_{i} P\left(1_{i} \mid S_{\Lambda}\right)$. If one computes $P\left(1_{i} \mid S_{\Lambda_{0}}\right)$ for each $i$ and then does a succession of
projections on certain linear spaces $S_{\Gamma}$ with $\Gamma \supset \Lambda$ in a manner similar to the Projection of Minimum Violators algorithm, one obtains the Projection of Violators algorithm. However, there is a better method to compute projections on the linear spaces $S_{\Gamma}$.

Let $\Gamma$ be a subindex set of $\{1,2, \ldots, n\}$. Suppose $p$ and $q$ are two consecutive indices in $\Gamma$. If $Y=\left(y_{1}, \ldots, y_{n}\right)$ is the projection of $Z=\left(z_{1}, \ldots, z_{n}\right)$ on $S_{\Gamma}$, then $\left(Z-Y, 1_{j}\right)=0$ for each $j \in \Gamma$ and $y_{p}=y_{p+1}=\ldots=y_{q-1}$. Therefore, $\left(Z-Y, l_{q}-1_{p}\right)=0$ or equivalently

$$
\Sigma_{i=p}^{q-1}\left(z_{i}-y_{i}\right) w_{i}=0
$$

and hence

$$
y_{p}=\Sigma_{i=p}^{q-1} z_{i} w_{i} / \Sigma_{i=p}^{q-1} w_{i} .
$$

This modified version of the Projection of Violators algorithm is known as the Pool-Adjacent-Violators algorithm, which was introduced by Ayer and coworkers (1955). Let $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a vector of $n$ entries.

Step 1. If $x_{1} \leq x_{2} \leq \cdots \leq x_{n}$, stop; otherwise, divide the sequence into several monotone decreasing sequences from left to right such that each subsequence is as long as possible, i.e.,

$$
\begin{aligned}
& x_{1} \geq x_{2} \geq \cdots \geq x_{i_{1}-1}, x_{i_{1}} \geq x_{i_{1}+1} \geq \cdots \geq x_{i_{2}-1}, \ldots, x_{i_{k}} \geq \\
& x_{i_{k}+1} \geq \cdots \geq x_{n} \quad \text { with } \quad x_{i_{j}-1}<x_{i_{j}} \quad \text { for each } j=1, \ldots, k
\end{aligned}
$$

$$
\text { Set } \Gamma=\left\{1, \mathrm{i}_{1}, \mathrm{i}_{2}, \ldots, \mathrm{i}_{\mathrm{k}}, \mathrm{n}+1\right\} .
$$

Step 2. Set $x_{h}=\sum_{i=p}^{q-1} x_{i} w_{i} / \Sigma_{i=p}^{q-1} w_{i}$ for $h=1, \ldots, n$ where $p$
and $q$ are two consecutive indices in $\Gamma$ such that $\mathrm{p} \leq \mathrm{h}<\mathrm{q}$.

Go to Step 1.
The justification of the algorithm is similar to that of the Projection of Minimum Violators algorithm. Let $X, X *$ and $\Lambda$ be defined as in Theorem 3.5. Let $\Lambda_{0}=\left\{i: a_{i}>0\right.$ or $\left.i=1\right\}$. The condition $a_{i}>0$ is the same as $x_{i-1}<x_{i}$. Theorem 3.5 shows that $\Lambda_{0} \supset \Lambda$. If $x_{1} \leq x_{2} \leq \cdots \leq x_{n}$, then $X \in M$ and $X *=X$. Otherwise, $\quad X *=P(X \mid M)=P\left(X \mid M_{\Lambda}\right)$. By the smoothing property, we have $P\left(X \mid M_{\Lambda}\right)=P\left(P\left(X \mid S_{\Lambda_{0}}\right) \mid M_{\Lambda}\right)$. Let $\quad X_{1}:=P\left(X \mid S_{\Lambda_{0}}\right)$ be represented by $X_{1}=\Sigma_{i \in \Lambda_{0}} a_{1 i} l_{i}$ and let $\Lambda_{1}=\left\{i \in \Lambda_{0}: a_{1 i}>0\right.$ or $\mathrm{i}=1\}$. If $\mathrm{x}_{11} \leq \mathrm{x}_{12} \leq \cdots \leq \mathrm{x}_{1 n}$, then $\mathrm{X}_{1} \in \mathrm{M}$ and $\mathrm{X} *=\mathrm{X}_{1}$ as indicated by Corollary 2.13.1. Otherwise, $\Lambda_{0} \neq \Lambda_{1} \supset \Lambda$. Applying the above procedure inductively, we have a strictly decreasing sequence $\Lambda_{0} \supset \Lambda_{1} \supset \Lambda_{2} \supset \ldots \supset \Lambda_{k} \supset \Lambda$. The sequence will terminate at some index $k$ such that $x_{k 1} \leq x_{k 2} \leq \cdots \leq x_{k n}$. The situation that $\Lambda_{k}=\{1\}$ is the case when $X-\bar{x} \in M *$ where $\overline{\mathrm{x}}=\Sigma_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \mathrm{w}_{\mathrm{i}} / \Sigma_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w}_{\mathrm{i}}$.

The similarity of the Projection of Minimum Violators algorithm and the Pool-Adjacent-Violators algorithm can be observed from Theorem 2. 16 by considering $M *$ and $M_{0}:=M \cap S_{0}$. The closed convex cone $M_{0}$ is the dual cone of $M^{*}$ with respect to the linear space $S_{0}$. There are $n-1$ extreme vectors in $M_{0}$ each of which is determined by $P\left(l_{i} \mid S_{0}\right)$ for some $i=2, \ldots, n$. It is obvious that for each $i,\left(Y_{i}, l_{j}\right)=\left(Y_{i}, P\left(l_{j} \mid S_{0}\right)\right)=0$ if $j \neq i+1$ and $\left(Y_{i}, l_{i+1}\right)=\left(Y_{i}, P\left(l_{i+1} \mid S_{0}\right)\right)=-1$.

Example 3.1. Let $X=(3,2,7,8,5)$ and let $w_{1}=w_{2}=w_{3}=w_{4}=w_{5}=10$. The isotonic regression of $X$ will be the same if we let $w_{1}=w_{2}=w_{3}=w_{4}=w_{5}=1$. By the Projection of Minimum Violators algorithm, we have $\overline{\mathrm{x}}=5$ and $X-\bar{x}=(-2,-3,2,3,0)$.

Step 1. $a_{1}=-2, a_{2}=-5, a_{3}=-3, a_{4}=0$

$$
\Gamma=\{1,2,3,4\}
$$

Step 2. $\quad j=2, \Gamma=\{1,3,4\}, i=1, k=4$

$$
\gamma_{1}=1 / 2, \gamma_{2}=1, \gamma_{3}=2 / 3, \gamma_{4}=1 / 3
$$

Step 3. $\quad a_{1}=1 / 2, a_{2}=0, a_{3}=1 / 3, a_{4}=5 / 3$
Now, $a_{1}, a_{2}, a_{3}$ and $a_{4}$ are non-negative
Step 4. $\quad y_{1}=1 / 2, y_{2}=-1 / 2, y_{3}=1 / 3, y_{4}=4 / 3, y_{5}=-5 / 3$

$$
\begin{aligned}
& x_{1}^{*}=2 \frac{1}{2}, x_{2}^{*}=2 \frac{1}{2}, x_{3}^{*}=6 \frac{2}{3}, x_{4}^{*}=6 \frac{2}{3}, x_{5}^{*}=6 \frac{2}{3} \\
& (Y=P(X \mid M *), X *=P(X \mid M))
\end{aligned}
$$

By the Pool-Adjacent-Violators algorithm, we have
Step 1. $\{3,2\},\{7\},\{8,5\}$

$$
\Gamma=\{1,3,4,6\}
$$

Step 2. $\quad X_{1}=\left(2 \frac{1}{2}, 2 \frac{1}{2}, 7,6 \frac{1}{2}, 6 \frac{1}{2}\right)$
Step 1. $\quad\left\{2 \frac{1}{2}, 2 \frac{1}{2}\right\},\left\{7,6 \frac{1}{2}, 6 \frac{1}{2}\right\}$
$\Gamma=\{1,3,6\}$
Step 2. $\quad X_{2}=\left(2 \frac{1}{2}, 2 \frac{1}{2}, 6 \frac{2}{3}, 6 \frac{2}{3}, 6 \frac{2}{3}\right)$

$$
X_{2}=P(X \mid M)
$$

III. 4 On a Tree

A partially ordered set is a tree if each non-comparable pair has a greatest lower bound but does not have an upper bound. A finite tree has a unique minimal element $r$, called the root, and each element other than $\mathbf{r}$ has exactly one immediate predeces sor. In a partially ordered set, if $\mu \leq \nu$, then there exists a chain $\ddot{\omega}_{0} \leq \omega_{1} \leq \cdots \leq \omega_{j+1}$ such that $\left[\omega_{i} \leq \omega_{i+1}\right]$ is an immediately comparable pair, $i=0, l, \ldots, j$, where $\mu=\omega_{0}$ and $v=\omega_{j+1}$. The chain connecting $\mu$ and $v$ need not be unique. The intersection of chains need not be a chain. Let $\Omega$ be a finite tree, let $m_{1}, m_{2}, \ldots, m_{k}$ be its maximal elements and let $\mathrm{Ch}\left(\mathrm{m}_{1}\right), \mathrm{Ch}\left(\mathrm{m}_{2}\right), \ldots, \mathrm{Ch}\left(\mathrm{m}_{\mathrm{k}}\right)$ be the chains which connect r and $m_{i}, i=1, \ldots, k$. The chain $C h\left(m_{i}\right)$ connecting $r$ and $m_{i}$ is
unique. The intersection $\mathrm{Ch}\left(\mathrm{m}_{\mathrm{i}}\right) \cap \mathrm{Ch}\left(\mathrm{m}_{\mathrm{j}}\right)$ is also a chain which connects $r$ and $m_{i} \wedge m_{j}$. The union $\cup_{i=1}^{k} \operatorname{Ch}\left(m_{i}\right)$ is simply $\Omega$ itself. The basic sets are upper sets with a unique minimal element; they are denoted by $U(\omega)$ with $\omega$ the minimal element in the set. The indicator of $U(\omega)$ is denoted by $l_{\omega}$. An example of a tree will be given below.

Let $Y$ be a function defined on $\Omega$, let $a(r)=Y(r)$ and for each $v$ other than $r$, let $a(\nu)=Y(v)-Y(\mu)$ where $\mu$ is the immediate predecessor of $\nu$. Let $\omega$ be an element other than $r$ and let $\operatorname{Ch}(\omega)=\left\{r, \omega_{1}, \omega_{2}, \ldots, \omega_{j}, \omega\right\}$ be the chain connecting $r$ and $\omega$. The $\operatorname{sum} \Sigma_{\nu \in C h(\omega)} a(\nu)$ is $Y(\omega)$. Let $Z=\sum_{\omega \in \Omega} a(\omega) l_{\omega}$. Then $Z(\omega)=\Sigma_{\nu \leq \omega}{ }^{a(\nu)}$. The set $\{\nu: \nu \leq \omega\}$ is the chain $\mathrm{Ch}(\omega)$. Therefore, we have $Y=\Sigma_{\omega \in \Omega} a(\omega){ }_{\omega}$. Since every function $Y$ on $\Omega$ has a unique linear representation by $\left\{1_{\omega}: \omega \in \Omega\right\}$, it follows that $\left\{l_{\omega}: \omega \in \Omega\right\}$ is linearly independent. If $Z$ is isotonic, then $Z$ is a linear combination of $\left\{1_{\omega}: \omega \in \Omega\right\}$ such that the coefficients are non-negative except possibly the one corresponding to $l_{r}$. The coefficient of $l_{\nu}$ is $Z(\nu)-Z(\mu)$ where $\mu$ is the immediate predecessor of $v$ provided that $v$ differs from $r$. If $\Omega$ has $n$ elements, then there are $n-1$ constraints (immediately comparable pairs) and $M_{+}$has $n$ extreme vectors $\left\{l_{\omega}: \omega \in \Omega\right\}$.

The isotonic regression over a tree can be obtained very easily as we shall soon see in Example 3.2. Thompson (1962) introduced the

Minimum Violator algorithm. Let $X$ be a given function defined on $\Omega$ and let $\mathrm{X} *$ be the isotonic regression of X . If $v$ is an immediate successor of $\mu$ and $X(v) \leq X(\mu)$, then $v$ is said to be a violator. Among all the violators, if $X(v)$ attains the minimum value, then $v$ is called a minimum violator. Let $v$ be a minimum violator and let $\mu$ be its immediate predecessor. Then by Theorem 3.3 $X *(\mu)=X *(\nu)$. By the smoothing property, we may group $\mu$ and $v$ as a single element $\mu$ whose $X$ value is the weighted average of those at $\mu$ and $\nu$ and whose weight is the sum of the weights at $\mu$ and $\nu$. Now, we have $n-1$ elements in $\Omega-\{\nu\}$. The ordering on $\Omega-\{\nu\}$ is determined by all immediately comparable pairs in $\Omega$ except that we take out the pair $[\mu \leq \nu]$ and change all the pairs $[\nu \leq \omega]$ into $[\mu \leq \omega]$. Applying this procedure inductively, there will eventually be no violator. Let $\Omega_{0}$ be the set of remaining elements and let $\mathrm{X}_{0}^{*}$ be the function on $\Omega_{0}$ at this stage. Define the function $\mathrm{X}^{*}$ on $\Omega$ by $X *(\omega)=\max \left\{X_{0}^{*}(\mu): \mu \leq \omega, \mu \in \Omega_{0}\right\}$. Then $X^{*}$ is the isotonic regres sion of $X$.

A modified version of the Minimum Violator algorithm is the Minimum Upper Set algorithm which involves an improvement in the method of grouping. Let $v$ be a violator and let $\mu$ be its immediate predecessor. A violator $v$ is said to be pivotal if either $X(\nu)=\min \{X(\omega): \omega \geq \mu\} \quad$ or $\quad \nu \quad$ is the only immediate successor of $\mu$.

By Theorem 3.3 a pivotal element and its immediate predecessor will have the same $X *$ value. Instead of using minimum violators, we use pivotal elements in the Minimum Upper Set algorithm.

Example 3.2. Let $\Omega=\{(1,4),(2,3),(2,4),(3,1),(3,2),(3,4)$, $(4,3),(4,4),(5,4)\}$ be a tree with its ordering being specified by the five chains $\{(3,1),(3,2),(2,3),(1,4)\},\{(3,1),(3,2),(2,3),(2,4)\}$, $\{(3,1),(3,2),(2,3),(3,4)\},\{(3,1),(3,2),(4,3),(4,4)\}$ and $\{(3,1),(3,2),(4,3),(5,4)\}$, let the weight function $W$ have the value one at each element in $\Omega$ and let $X$ be the function $g$ iven as follows.


Let us say the pivotal element $v$ is type $I$, if $v$ is the only immediate successor of its immediate predecessor and is type II otherwise. The grouping of a pivotal element $v$ and its immediate predecessor $\mu$ into a single element $\mu$ is not necessary as long as we keep the pair $[\mu \leq \nu]$ having the same function value. The
pair $[\mu \leq \nu]$ will be connected by double line segments to indicate their special relation.

The violators are $(1,4),(3,2),(4,3),(4,4)$ and $(5,4)$. The element $(3,2)$ is a type I pivotal element and elements (1,4) and $(4,4)$ are type II pivotal elements. The first iter ation yields the function $X_{1}$ given below:


The element $(4,3)$ is the only violator. Therefore, the average is taken over $\{(3,1),(3,2),(4,3),(4,4)\}$ and the second iteration yields the function $X_{2}$.


A new violator occurs at $(5,4)$. We take the average over the set $\{(3,1),(3,2),(4,3),(4,4),(5,4)\}$. We see that the function $X_{3}$ is isotonic and hence it is the isotonic regression of $X$. $]$


If $\Omega$ is a reversed tree, then its reversal $\Omega_{r}$ is a tree.
One may use the reversal technique based on the relation, $\mathrm{P}(\mathrm{X} \mid \mathrm{M}(\Omega))=-\mathrm{P}\left(-\mathrm{X} \mid \mathrm{M}\left(\Omega_{\mathrm{r}}\right)\right)$, to obtain the isotonic regression. The procedure is that we change the ordering from less than to greater than and replace $X(\omega)$ by $-X(\omega)$ for each $\omega \in \Omega$. After we have the isotonic regression $Y$ of the latter problem, we define $X *(\omega)=-Y(\omega)$ for each $\omega \in \Omega$ and $X *$ is the isotonic regression of $X$ over $\Omega$. For convenience, we may use the Maximum Lower Set algorithm. The method is the same as the Minimum Upper Set algorithm except that pivotal elements are $\mu^{\prime} s$ such that either $X(\mu)=\max \{X(\omega): \omega \leq \nu\} \quad$ or $\quad \mu$ is the only immediate predecessor of $v$, where $v$ is the immediate successor of $\mu$. In this situation, we group $[\mu \leq \nu]$ into a single element $v$. Applications of the

Maximum Lower Set algorithm will be found in Example 3.3 and Example 3.5.

## III. 5 On a Partially Ordered Set

The structure of a general connected partially ordered finite set is very complicated and hence algorithms for isotonic regression of this type will not be easy to develop. Alexander (1970) introduced a method which rewrites a partial ordering into a consistent linear ordering. The method is much more complicated than the Pool-Adjacent-Violators algorithm and the Minimum Upper Set algorithm. In the remainder of this section we will try to reduce the minimization problem into several small problems, whenever it is possible, such that the combination of the solutions to the small problems is the desired solution. In a smaller problem, Alexander's algorithm will be easier to apply.

Let $\Omega$ be a connected partially ordered finite set, let $W$ be a given positive weight function, let $X$ be a given function and let $\Gamma$ be a subset of $\Omega$. If $Z$ is a function defined on $\Omega$, we denote

$$
f(Z ; \Gamma):=\Sigma_{\omega \in \Gamma}[X(\omega)-Z(\omega)]^{2} W(\omega)
$$

Let $\quad \Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{k}$ be a partition of $\Omega$, i.e., $\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{k}$ are mutually disjoint and $\cup_{i=1}^{k} \Gamma_{i}=\Omega$, let $Y_{1}, Y_{2}, \ldots, Y_{k}$ be the
restricted isotonic regression of $X$ to $\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{k}$ respectively and let $Y$ be defined by $Y(\omega)=Y_{i}(\omega)$ if $\omega \in \Gamma_{i}$. For any isotonic function $\quad Z, \quad f\left(Z ; \Gamma_{i}\right) \geq f\left(Y ; \Gamma_{i}\right), \quad i=1,2, \ldots, k \quad$ and hence $f(Z) \geq f(Y)$. Therefore, if $Y$ is isotonic, it is the isotonic regression of $X$.

Let $X *$ be the isotonic regression of $X$, let $[a \leq X * \beta]$ be non-empty, let $C$ be a component of $[a \leq X * \leq \beta]$ and let $Y$ be the restriction of $X *$ to $C, X * \mid C$. Let $U_{c}$ be an upper set in C, i.e., $\mu \in U_{c}, v \in C$ and $\mu \leq \nu$ imply $v \in U_{c}$. For any $U_{c}$, define $U=U_{c} \cup\left[X_{*}>\beta\right]$. Then $U$ is an upper set. By (3.2) and (3.3) we have

$$
\Sigma_{\omega \in U_{c}} X(\omega) W(\omega) \leq \Sigma_{\omega \in U_{c}} X *(\omega) W(\omega)=\Sigma_{\omega \in U_{c}} Y(\omega) W(\omega) .
$$

For any $\gamma$ between $a$ and $\beta$, let $V_{c}=\{\omega \notin C: \gamma \leq X *(\omega) \leq \beta\}$ and let $V=V_{c} \cup\left[X^{*}>\beta\right]$. Then $V$ is an upper set and similarly

$$
\Sigma_{\omega \in V_{c}} X(\omega) W(\omega) \leq \Sigma_{\omega \in V_{c}} X *(\omega) W(\omega) .
$$

The set $\{\omega \in C: X *(\omega) \geq \gamma\}$ is an upper set in $C$ and $V_{c} \cup\{\omega \in C: X *(\omega) \geq \gamma\}$ is the level set $[\gamma \leq X * \leq \beta]$. Therefore, by (3.2) we have

$$
\begin{aligned}
\Sigma_{\omega \in C}, X *(\omega) \geq \gamma^{X}(\omega) W(\omega) & =\Sigma_{\omega \in C, X *(\omega) \geq \gamma^{X}} *(\omega) W(\omega) \\
& =\Sigma_{\omega \in C ; X *(\omega) \geq \gamma^{Y}}(\omega) W(\omega)
\end{aligned}
$$

and it follows that $Y$ is the restricted isotonic regression of $X$ to $C$. The restriction of $X *$ to any non-empty level set $[\alpha \leq X * \leq \beta]$, to one of its component or to the union of some of its components is the restricted isotonic regression of $X$ to that set. Therefore, the following theorem has been established.

Theorem 3.6. Let $X *$ be the isotonic regression of $X$, let the non-empty level set $[\alpha \leq X * \leq \beta]$ have components $C_{1}, \ldots, C_{k}$ and let $\Gamma=\cup_{i=1}^{j} C_{i}$ where $l \leq j \leq k \quad$ Then the restriction of $X^{*}$ to $\Gamma$ is the restricted isotonic regression of $X$ to $\Gamma$.

Theorem 3.7. Let $\Gamma_{2}$ be a non-empty level set, let $\Gamma_{1} \subset \Gamma_{2}^{c}$ be a lower set such that $\omega \in \Gamma_{2}^{c}$ and $\omega \leq \mu$ for some $\mu \in \Gamma_{2}$ imply that $\omega \in \Gamma_{1}$, let $\Gamma_{3} \subset \Gamma_{1}^{c} \cap \Gamma_{2}^{c}$ be an upper set such that $\omega \in \Gamma_{2}^{c}$ and $\omega \geq \mu$ for some $\mu \in \Gamma_{2}$ imply that $\omega \in \Gamma_{3}$ and let $Y_{1}, Y_{2}$ and $Y_{3}$ be the restricted isotonic regressions of $X$ to $\Gamma_{1}, \Gamma_{2}$ and $\Gamma_{3}$ respectively. If there exist $a$ and $\beta$ such that $Y_{1} \leq a, a \leq Y_{2} \leq \beta$ and $\beta \leq Y_{3}$, then $X *(\omega) \leq Y_{1}(\omega)$ for each $\omega \in \Gamma_{1}, \quad X *(\omega)=Y_{2}(\omega)$ for each $\omega \in \Gamma_{2}$
and $X *(\omega) \geq Y_{3}(\omega)$ for each $\omega \in \Gamma_{3}$ where $X *$ is the isotonic regression of $X$.

Proof. Let $\Gamma_{4}=\Omega-\left(\Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3}\right)$, let $Z \quad$ be an isotonic function and let $Y$ be defined by $Y(\omega)=Z(\omega) \wedge Y_{1}(\omega)$ on $\Gamma_{1}$, $Y(\omega)=Y_{2}(\omega)$ on $\Gamma_{2}, Y(\omega)=Z(\omega) \vee Y_{3}(\omega)$ on $\Gamma_{3}$ and $\mathrm{Y}(\omega)=\mathrm{Z}(\omega)$ on $\Gamma_{4}$. We shall show that Y is isotonic and $f(Y) \leq f(Z)$. It will then follow that $X *(\omega) \leq Y_{1}(\omega)$ for each $\omega \in \Gamma_{1}$, $X *(\omega)=Y_{2}(\omega)$ for each $\omega \in \Gamma_{2}$ and $X *(\omega) \geq Y_{3}(\omega)$ for each $\omega \in \Gamma_{3}$. The order-preserving property of $Y$ on each $\Gamma_{i}$ is trivial. The sets $\Gamma_{2}$ and $\Gamma_{4}$ are unrelated. Since $\Gamma_{1}$ is a lower set, elements in $\Gamma_{1}$ will be no greater than ay element in $\Gamma_{2} \cup \Gamma_{3} \cup \Gamma_{4}$. Since $\Gamma_{3}$ is an upper set, elements in $\Gamma_{3}$ will be no less than any element in $\Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{4}$. Any element greater than an element in $\Gamma_{2}$ is in $\Gamma_{2} \cup \Gamma_{3}$ and any element less than an element in $\Gamma_{2}$ is in $\Gamma_{1} \cup \Gamma_{2}$. Since $Y \leq a$ on $\Gamma_{1}$, $a \leq Y \leq \beta$ on $\Gamma_{2}$ and $Y \geq \beta$ on $\Gamma_{3}$, it follows that $Y$ is isotonic. By (4.7), we have $f\left(Y ; \Gamma_{1}\right) \leq f\left(Z ; \Gamma_{1}\right)$ and $\mathrm{f}\left(\mathrm{Y} ; \Gamma_{3}\right) \leq \mathrm{f}\left(\mathrm{Z} ; \Gamma_{3}\right)$ when we regard $\Gamma_{1}$ or $\Gamma_{3}$ as our given partially ordered set. Since $f\left(Y ; \Gamma_{2}\right) \leq f\left(Z ; \Gamma_{2}\right), f\left(Y ; \Gamma_{4}\right)=f\left(Z ; \Gamma_{4}\right)$ and $f(Z)=\Sigma_{i=1}^{4} f\left(Z ; \Gamma_{i}\right)$, we have $f(Y) \leq f(Z)$. This completes the proof. []

Corollary 3.7.1. Let $U$ be a non-empty upper set, let $\Gamma_{1} \subset U^{\mathrm{C}}$ be a lower set such that $\omega \in \mathrm{U}^{\mathrm{C}}$ and $\omega \leq \mu$ for some $\mu \in U$ imply that $\omega \in \Gamma_{1}$ and let $Y_{U}$ and $Y_{1}$ be the restricted isotonic regressions of $X$ to $U$ and $\Gamma_{1}$ respectively. If there exists an $a$ such that $Y_{1} \leq a$ and $a \leq Y_{U}$, then $X *(\omega)=Y_{U}(\omega)$ for all $\omega \in U$ and $X *(\omega) \leq Y_{1}(\omega)$ for all $\omega \in \Gamma_{1}$. Similarly, let $L$ be a non-empty lower set, let $\Gamma_{3} \subset L^{c}$ be an upper set such that $\omega \in L^{c}$ and $\omega \geq \mu$ for some $\mu \in L$ imply that $\omega \in \Gamma_{3}$ and let $Y_{L}$ and $Y_{3}$ be the restricted isotonic regressions of $X$ to $L$ and $\Gamma_{3}$ respectively. If there exists a $\beta$ such that $Y_{L} \leq \beta$ and $\beta \leq Y_{3}$, then $X *(\omega)=Y_{L}(\omega)$ for each $\omega \in L$ and $X *(\omega) \geq Y_{3}(\omega)$ for each $\omega \in \Gamma_{3}$.

Proof. For the first statement, let $\Gamma_{2}=\mathrm{U}$ and $\Gamma_{3}=\phi$. For the second statement, let $\Gamma_{1}=\phi$ and $\Gamma_{2}=L$. The results follow from Theorem 3.7. []

Corollary 3.7.2. Let $\Gamma_{2}$ be a non-empty level set, let $\Gamma_{1}=\left\{\omega \in \Gamma_{2}^{c}: \omega \leq \mu\right.$ for some $\left.\mu \in \Gamma_{2}\right\}$ and let $\Gamma_{3}=\left\{\omega \in \Gamma_{2}^{c}: \omega \geq \mu\right.$ for some $\left.\mu \in \Gamma_{2}\right\}$. If there exist $a$ and $\beta$ such that $X \leq a$ on $\Gamma_{1}, a \leq X \leq \beta$ on $\Gamma_{2}$ and $X \geq \beta$ on $\Gamma_{3}$, then the isotonic regression of $X$ can be obtained by considering $\Gamma_{2}$ and $\Gamma_{2}^{c}$ independently, i.e., the restriction of the isotonic
regression of $X$ to $\Gamma_{2}$ is the restricted isotonic regression of $X$ to $\Gamma_{2}$.

Proof. $\quad \Gamma_{1}$ is a lower set, $\quad \Gamma_{3}$ is an upper set and $\Gamma_{1} \cap \Gamma_{3}=\phi$. Let $Y_{1}, Y_{2}$, and $Y_{3}$ be the restricted isotonic regressions of $X$ to $\Gamma_{1}, \Gamma_{2}$ and $\Gamma_{3}$ respectively. Then by the monotonicity in Theorem 4.5, we have $Y_{1} \leq a, a \leq Y_{2} \leq \beta$ and $Y_{3} \geq \beta$. It follows from Theorem 3.7 that $X *(\omega)=Y_{2}(\omega)$ for each $\omega \in \Gamma_{2} \quad[$

Corollary 3.7.3. Let $U$ be a non-empty upper set and let $\Gamma_{1}=\left\{\omega \in U^{C}: \omega \leq \mu\right.$ for some $\left.\mu \in U\right\}$. If there exists an a such that $X \geq a$ on $U$ and $X \leq a$ on $\Gamma_{1}$, then the isotonic regression of $X$ can be obtained by considering $U$ and $U^{c}$ independently. Similarly, let $L$ be a non-empty lower set and let $\Gamma_{3}=\left\{\omega \in L^{c}: \omega \geq \mu\right.$ for some $\left.\mu \in L\right\}$. If there exists a $\beta$ such that $X \leq \beta$ on $L$ and $X \geq \beta$ on $\Gamma_{3}$, then the isotonic regression of X can be obtained by considering $L$ and $L^{c}$ independently.

In particular, if a maximal element has a maximum $X$ value or a minimal element has a minimum $X$ value, then the value of the isotonic regression of $X$ at that element is the value of $X$ at that element.

Proof. For the first statement, let $\Gamma_{2}=\mathrm{U}$ and $\Gamma_{3}=\phi$. For
the second statement, let $\Gamma_{1}=\phi$ and let $\Gamma_{2}=L$. The results follow from Corollary 3.7.2. []

The level set $\Gamma_{2}$ in Theorem 3.7, if it is connected, will be a component of the level set $[\alpha \leq X * \leq \beta]$ where $X *$ is the isotonic regression of $X$. Such a component may be obtained by inspection of $X$ values as indicated by Corollary 3.7.2 or Corollary 3. 7. 3. Corollary 3.2.3 and Theorem 3. 3 may be able to take care of violations locally; Theorem 3.7 and its corollaries may be able to take care of the problem globally. Theoretically, these results are enough to obtain the isotonic regression of $X$. The difficulties are how to determine such a component $\Gamma_{2}$ and how to find the restricted isotonic regression of $X$ to $\Gamma_{2}$. Presumably, if the longest chain in $\Omega$ has a small number of elements or upper sets tend to have larger $X$ values than lower sets, then the difficulties mentioned above will not be serious.

When difficulties arise, the reader may want to refer to Alexander's algorithm. But Corollary 3.2.3, Theorem 3.3, Theorem 3. 7 and its corollaries may be greatly helpful in reducing the problem, as demonstrated by the following example which appears in Alexander (1970).

Example 3. 3. Let $\Omega=\{(i, j): i, j=1,2,3,4\}$ be a partially ordered set with the ordering $(i, j) \leq(h, k)$ if $i \leq h$ and $j \leq k$. For each
element $(i, j)$, the function value $x_{i j}$ and the weight $w_{i j}$ are respectively the fraction and the denominator as given below.

| 4 | $1 / 5$ | $2 / 11$ | $1 / 2$ | $1 / 3$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{x}$ | 3 | $1 / 6$ | $1 / 7$ | $1 / 8$ | $1 / 3$ |
| 2 | $1 / 8$ | $1 / 10$ | $1 / 7$ | $1 / 2$ |  |
|  |  | 1 | $1 / 16$ | $1 / 7$ | $4 / 39$ |

By inspection, one may find out that $\mathrm{U}=\{(3,4),(4,2),(4,3),(4,4)\}$ is an upper set and $X \geq 1 / 3$ on $U, X \leq 1 / 5$ on $U^{c} ;(1,1)$ is the minimal element with the minimum $X$ value; $\quad \Gamma_{2}=\{(1,4),(2,4)\}$ is a level set with $\Gamma_{3}=\{(3,4),(4,4)\}$ and $\Gamma_{1}=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$ such that $\mathrm{X}<2 / 11$ on $\Gamma_{1}, 2 / 11 \leq X \leq 1 / 5$ on $\Gamma_{2}$ and $X>1 / 5$ on $\Gamma_{3} ;$ and also $\Lambda_{2}=\{(4,1)\}$ is a level set with $\Lambda_{3}=\{(4,2),(4,3),(4,4)\}$ and $\Lambda_{1}=\{(1,1),(1,2),(1,3)\}$ such that $X<1 / 6$ on $\Lambda_{1}, X=1 / 6$ on $\Lambda_{2}$ and $X>1 / 6$ on $\Lambda_{3}$. From Corollary 3.7.2 and Corollary 3.7.3, $x_{11}^{*}=1 / 16$ where $X *$ is the isotonic regression of $X$ and we may consider $U, \Gamma_{2}$ and $\Lambda_{2}$ independently. The set $U$ is a reversed tree and by the Maximum Upper Set algorithm we have $x_{34}^{*}=x_{42}^{*}=x_{43}^{*}=x_{44}^{*}=2 / 5$. The set $\Gamma_{2}$ is a linearly ordered set and by the Pool-Adjacent-Violators algorithm we have $x_{14}^{*}=x_{24}^{*}=3 / 16$. The set $\Lambda_{2}$ is a singleton and hence $x_{41}^{*}=1 / 6$.

Therefore, the remaining problem is the following:

| 3 | $1 / 6$ | $1 / 7$ | $1 / 8$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{X}_{1} \quad 2$ | $1 / 8$ | $1 / 10$ | $1 / 6$ |
| 1 |  | $1 / 7$ | 4/39 |
| j/i | 1 | 2 | 3 |

Since $(1,3)$ and $(3,2)$ have the largest $X$ value $1 / 6$, Theorem 3.3. shows that $x_{13}^{*}=x_{23}^{*}$ and $x_{32}^{*}=x_{33}^{*}$. By the smoothing property as described in the Maximum Lower Set algorithm, we may replace $1 / 7$ and $1 / 8$ at $(2,3)$ and $(3,3)$ by $2 / 13$ and $2 / 14$ respectively and cross out elements $(1,3)$ and $(3,2)$. The weights at $(2,3)$ and $(3,3)$ now are 13 and 14 .

|  | 3 |  | $2 / 13$ | $2 / 14$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{X}_{2}$ | 2 | $1 / 8$ | $1 / 10$ |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

At this stage we have six elements. Since $X_{2} \geq 1 / 7$ in the upper set $U=\{(2,3),(3,3)\}$ and $X_{2} \leq 1 / 7$ on $U^{C}$, by Corollary 3.7.3 we may consider $U$ and $U^{c}$ independently. The set $U$ is linearly ordered and by the Pool-Adjacent-Violators algorithm we have $x_{23}^{*}=x_{33}^{*}=4 / 27$. Recall that $x_{13}^{*}=x_{23}^{*}$ and $x_{32}^{*}=x_{33}^{*}$, so
$x_{13}^{*}=x_{23}^{*}=x_{32}^{*}=x_{33}^{*}=4 / 27$.


For a set with four elements, the problem can always be solved very easily. If $[\mu \leq \nu]$ is an immediately comparable pair and $X(\mu) \geq X(\nu)$, then by Corollary 3.2 .3 we shall group $\mu$ with at least one of its immediate successors or group $v$ with at least one of its immediate predecessors. It follows that $x_{12}^{*}=x_{22}^{*} x_{21}^{*}=x_{22}^{*}$ or $x_{21}^{*}=x_{33}^{*}$. Let us consider $U=\{(1,2),(2,2)\}$. This is an upper set and the restricted isotonic regressions $Y$ and $Z \quad$ of $X_{3}$ to $U$ and $U^{c}$ are $y_{12}=y_{22}=2 / 18$ and $z_{21}=z_{31}=5 / 46$. The value $2 / 18$ is larger than $5 / 46$ and from Corollary 3.7.1 we have $x_{12}^{*}=x_{22}^{*}=2 / 18$ and $x_{21}^{*}=x_{22}^{*}=5 / 46$. Therefore, the isotonic regression $X *$ of $X$ has been obtained. Suppose we group (2, 1) and (2,2) first. Then we have a linearly ordered set with function values from the smallest element to the largest element being $1 / 8$, 2/17 and 4/39. The Pool-Adjacent-Violators algorithm shows that $x_{12}^{*}=x_{21}^{*}=x_{22}^{*}=x_{31}^{*}=7 / 64$. But $x_{12} w_{12}+x_{22} w_{22}=2$ and $\mathrm{x}_{12}^{*} \mathrm{w}_{12}+\mathrm{x}_{22}^{*} \mathrm{w}_{22}=63 / 32$ which violates (3.3). Therefore, the proper grouping should be $\{(1,2),(2,2)\}$ or $\{(2,1),(3,1)\}$ and either one will lead to the final solution. The isotonic regression $X *$ of

X is given below. []


## III. 6 Some Related Problems

Let $\Omega$ be a partially ordered set and let $W$ be a given weight function. For any $X$ there is an isotonic regression $X *$ of $X$. Let $\left\{a_{1}, \ldots, a_{k}\right\}$ be the range of $X *$ and let $\Gamma_{i}=\left[X *=a_{i}\right], i=1, \ldots, k$. The sequence $\Gamma_{1}, \ldots, \Gamma_{k}$ is a partition of $\Omega$. Such a partition may possibly be determined by Theorem 3.3, Corollary 3.7.2 and Corollay 3.7 .3 without regarding what the weight function is. Therefore, we will be able to select another weight function $W_{0}$ of interest such that the function $Y$ defined by $Y(\omega)=\Sigma_{\omega \in \Gamma_{i}} X(\omega) W_{0}(\omega) / \Sigma_{\omega \in \Gamma_{i}} W_{0}(\omega)$ for $\omega \in \Gamma_{i}$, is isotonic. Our interest could be the weight function, the isotonic regression or both. The problem we are going to study is to minimize

$$
f(Z, W)=\Sigma_{\omega \in \Omega}[Z(\omega)-X(\omega)]^{2} W(\omega)
$$

subject to $Z \in M$ and subject to some conditions on the weight
function $W$. For each weight function $W$, there is an isotonic regression $X *(W)$. Since $f(Z, W) \geq f(X *(W), W)$, the minimization problem can be studied as a function of $W$, i.e.,

$$
\mathrm{h}(\mathrm{~W})=\mathrm{f}(\mathrm{X} *(\mathrm{~W}), \mathrm{W})
$$

The isotonic regression $X *(W)$ may not heavily depend on $W$. If that is the case, the problem can be solved very easily.

Example 3.4. The problem is to minimize

$$
\begin{aligned}
f(Z, W)= & \left(z_{11}-1 / 7\right)^{2} \mathrm{w}_{11}+\left(\mathrm{z}_{21}-1 / 8\right)^{2} \mathrm{w}_{21}+\left(\mathrm{z}_{31}-1 / 3\right)^{2} \mathrm{w}_{31} \\
& +\left(\mathrm{z}_{12}-2 / 11\right)^{2} \mathrm{w}_{12}+\left(\mathrm{z}_{22}-1 / 2\right)^{2} \mathrm{w}_{22}+\left(\mathrm{z}_{32}-1 / 3\right)^{2} \mathrm{w}_{32}
\end{aligned}
$$

subject to

$$
\begin{aligned}
& z_{i j} \leq z_{h k} \quad \text { if } \quad i \leq h \quad \text { and } \quad j \leq k \\
& w_{i j} \geq 1 / 5 \quad i=1,2,3 ; \quad j=1,2
\end{aligned}
$$

and

$$
\mathrm{w}_{\mathrm{i} 1}+\mathrm{w}_{\mathrm{i} 2}=1, \quad \mathrm{i}=1,2,3
$$

Let $X$ be given by

| 2 | $2 / 11$ | $1 / 2$ | $1 / 3$ |
| :---: | :---: | :---: | :---: |
| 1 | $1 / 7$ | $1 / 8$ | $1 / 3$ |
| $\mathrm{j} / \mathrm{i}$ | 1 | 2 | 3 |

For each $W$, let $X *(W)=\left(x_{i j}^{*}\right)$ be the isotonic regression of $X$ with respect to $W$. Then $x_{11}^{*}=x_{21}^{*}=\left(w_{11} / 7+w_{12} / 8\right) /\left(w_{11}+w_{12}\right)$, $x_{31}^{*}=1 / 3, x_{12}^{*}=2 / 11$. and $x_{22}^{*}=x_{32}^{*}=\left(w_{22} / 2+w_{32} / 3\right) /\left(w_{22}+w_{23}\right)$.

Therefore,

$$
\begin{aligned}
\mathrm{h}(\mathrm{~W})= & \mathrm{f}(\mathrm{X} *(\mathrm{~W}), \mathrm{W}) \\
= & \left(1 / 7-\mathrm{x}_{11}^{*}\right)^{2} \mathrm{w}_{11}+\left(1 / 8-\mathrm{x}_{21}^{*}\right)^{2} \mathrm{w}_{21}+\left(1 / 2-\mathrm{x}_{22}^{*}\right)^{2} \mathrm{w}_{22} \\
& +\left(1 / 3-\mathrm{x}_{32}^{*}\right)^{2} \mathrm{w}_{32} \\
= & \mathrm{w}_{11} \mathrm{w}_{21}\left(\mathrm{w}_{11}+\mathrm{w}_{21}\right)^{-1} / 3136+\mathrm{w}_{22} \mathrm{w}_{32}\left(\mathrm{w}_{22}+\mathrm{w}_{32}\right)^{-1} / 36
\end{aligned}
$$

For fixed $w_{21}, w_{22}$ and $w_{32}, h$ is an increasing function of $w_{11}$ over the range $[1 / 5,4 / 5]$ and hence ${ }_{w}^{*}=1 / 5$ where $W *$ is the optimal solution. Similarly, $w_{32}^{*}=1 / 5$. Let $\lambda=w_{21}$. We have

$$
h(\lambda)=\lambda / 3136(1+5 \lambda)+(1-\lambda) / 36(6-5 \lambda) \quad \text { where } \quad \lambda \in[1 / 5,4 / 5] .
$$

Since $h$ is monotone decreasing, $\lambda=4 / 5$. The optimal solution $(\mathrm{X} *, \mathrm{~W} *)$ has been obtained and it is given below. $\square$
$\mathrm{X} * \mathrm{~W}$ *

| 2 | $2 / 11$ | $5 / 12$ | $5 / 12$ |
| :---: | :---: | :---: | :---: |
| 1 | $9 / 70$ | $9 / 70$ | $1 / 3$ |
| j | 1 | 2 | 3 |


| 2 | $4 / 5$ | $1 / 5$ | $1 / 5$ |
| :---: | :---: | :---: | :---: |
| 1 | $1 / 5$ | $4 / 5$ | $4 / 5$ |
| $\mathrm{j} / \mathrm{i}$ | 1 | 2 | 3 |

Another problem we are interested in is the situation when the ordering imposed on the set $\Omega$ could be one of $k$ given types. We want to minimize

$$
f(Z)=\Sigma_{\omega \in \Omega}[Z(\omega)-X(\omega)]^{2} W(\omega)
$$

subject to $Z \in \cup_{i=1}^{k} M\left(\Omega, \leq_{i}\right)$ where $M\left(\Omega, \leq_{i}\right) \quad$ is the family of isotonic functions with respect to the ordering $\leq_{i}, i=1, \ldots, k$. Let $X_{i}^{*}$ be the isotonic regression of $X$ with respect to the ordering $\leq_{i}$. Since $f(Z) \geq f\left(X_{i}^{*}\right)$ whenever $Z \in M\left(\Omega, \leq_{i}\right)$, the optimal solution is $X_{j}^{*}$ such that $f\left(X_{j}^{*}\right)=\min \left\{f\left(X_{i}^{*}\right): i=1, \ldots, k\right\}$.

Example 3.5. The problem is to minimize

$$
f(Z)=\left(z_{1}-3\right)^{2}+\left(z_{2}-2\right)^{2}+\left(z_{3}-7\right)^{2}+\left(z_{4}-8\right)^{2}+\left(z_{5}-5\right)^{2}
$$

subject to $Z$ being unimodal, i.e., $z_{1} \leq z_{2} \leq \cdots \leq z_{j}$ and $z_{j} \geq z_{j+1} \geq \cdots \geq z_{5}$ for $j$ from one to five. For each ordering $\leq_{i}$, the pair $\left(\Omega, \leq_{i}\right)$ is a reversed tree where $\Omega=\{1,2,3,4,5\}$. The isotonic regressions of $X=(3,2,7,8,5)$ with respect to these orderings can be found very easily and the optimal solution is $(2.5,2.5,7,8,5) . \quad[]$

Bounded isotonic regression is isotonic regression with an addition constraint:

$$
\mathrm{Z}_{1} \leq \mathrm{Z} \leq \mathrm{Z}_{2}
$$

where $Z_{1}$ and $Z_{2}$ are two given functions. The problem was introduced by van Eeden (cf. Barlow and coworkers (1972)). Let $X$ * and $\hat{X}$ be the isotonic regression and the bounded isotonic regres sion of a given $X$ over a partially ordered finite set and let $\widehat{M}:=\left\{Z \in M: Z_{1} \leq Z \leq Z_{2}\right\}$. Since $\widehat{M}$ is a cosed convex set, if it is non-empty then the bounded isotonic regression exists and is unique.

By the monotonicity (cf. Theorem 4.5) of the isotonic regression, if $Z_{1} \leq Z \leq Z_{2}$ and $Z \in M$, then $Z_{1}^{*} \leq Z \leq Z_{2}^{*}$ where $Z_{1}^{*}$ and $Z_{2}^{*}$ are respectively the isotonic regressions of $Z_{1}$ and $Z_{2}$. Without loss of generality, we may assume that $Z_{1}$ and $Z_{2}$ are isotonic. If there is an $\omega$ such that $Z_{2}(\omega)<Z_{1}(\omega)$, then there is no feasible solution. If $Z_{1} \leq Z_{2}$ and $Z_{1}(\omega)=Z_{2}(\omega)$ for some $\omega$, then $X(\omega)=Z_{1}(\omega)$. In the remainder of the section, we shall assume that $Z_{1}$ and $Z_{2}$ are isotonic and $Z_{1} \leq Z_{2}$.

Theorem 3.8. If $Z_{1}$ and $Z_{2}$ are constant functions with values $\alpha$ and $\beta$ respectively, then $X=(X * \vee a) \wedge \beta$.

Proof. Let $U=\left[X^{*}>\beta\right], L=[X *<a]$ and $\Gamma=[a \leq X * \leq \beta]$. From Theorem 3.6, $X *|U, X *| L$ and $X * \mid \Gamma$ are the restricted isotonic regressions of $X$ to $U, L$ and $\Gamma$ respectively. Let $Z \in \widehat{M}$, then $Z \leq \beta$ on $U$ and $X *>\beta$ on $U$. By (4.8), we
have $f(Z \vee \beta ; U) \leq f(Z ; U)$ where $f(Z ; U):=\Sigma_{\omega \in U}[X(\omega)-Z(\omega)]^{2} W(\omega)$. Similarly, $f(Z \wedge a ; L) \leq f(Z ; L)$. Since $f(Z ; \Gamma) \geq f(X * \Gamma)$,
$Z \vee \beta=\beta, \quad Z \wedge a=a$ and $f(Z)=f(Z ; U)+f(Z ; L)+f(Z ; \Gamma), \quad$ it follows that $\hat{X}=X^{*}$ on $\Gamma, \hat{X}=a$ on $L$ and $\hat{X}=\beta$ on $U . \quad \square$

Let $a=\min \{X *(\omega): \omega \in \Omega\}, \quad b=\max \{X *(\omega): \omega \in \Omega\}$,
$a=\min \left\{Z_{2}(\omega): \omega \in \Omega\right\} \quad$ and $\quad \beta=\max \left\{Z_{1}(\omega): \omega \in \Omega\right\} \quad$ Let
$U=\left\{\omega: Z_{1}(\omega)=\beta\right\}, \quad U_{b}=\left\{\omega: Z_{1}(\omega) \geq b\right\}, \quad L=\left\{\omega: Z_{2}(\omega)=a\right\} \quad$ and
$L_{a}=\left\{\omega: Z_{2}(\omega) \leq a\right\}$.
Lemma 3.1. If $U_{b}$ is non-empty, then $\widehat{X}(\omega)=Z_{1}(\omega)$ for each $\omega \in U$. If $L_{a}$ is non-empty, then $\hat{X}(\omega)=Z_{2}(\omega)$ for each $\omega \in L$.

Proof. If $U_{b}$ is non-empty, then $\beta \geq b$. For each $\omega \in U$, $\hat{X}(\omega) \geq \beta$. We are going to show $\hat{X}(\omega)=\beta$. Note that $X * \leq b$ and hence $X^{*} \leq \beta$. If $Z \in \widehat{M}$, then $Z \wedge \beta \in \widehat{M}$. By (4.8), $f(Z) \geq f(Z \wedge \beta)$. Therefore $\hat{X}=\hat{X} \wedge \beta$. Similarly if $L_{a}$ is nonempty then $\hat{X}(\omega)=Z_{2}(\omega)$ for each $\omega \in L$. []

Lemma 3.2. Let $X *$ be a constant function. If $L_{a}$ is empty, then $\widehat{X} \geq X *$. If $U_{b}$ is empty, then $X * \geq \widehat{X}$.

Proof. If $L_{a}=\phi$, then $Z_{2}>a=X *$ and it follows that $Z \in \widehat{M}$ implies $Z \vee X * \in \widehat{M}$. By (4.7), $f(Z) \geq f(Z \vee X *)$. Therefore
$\hat{X}=\hat{X} \vee X *$. Similarly, if $U_{b}$ is empty, then $X * \geq \hat{X} . \square$

Theorem 3.9. Let $\Omega=\left\{\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right\}$ be a linearly ordered set with the ordering $\omega_{i} \leq \omega_{i+1}, i=1,2, \ldots, n-1$. The bounded isotonic regression of $X$ can be obtained by considering each level set $[X *=c]$ independently, where $X *$ is the isotonic regression of X .
 $[\mathrm{X} *=\mathrm{b}]=\left\{\omega_{j+1}, \ldots, \omega_{k}\right\}$ be two adjacent level sets with $\mathrm{a}<\mathrm{b}$ and let $\hat{X}_{a}$ and $\hat{X}_{b}$ be the bounded restricted isotonic regressions of $X$ to $[X *=a]$ and $[X *=b]$. If we can show $\hat{X}_{a}\left(\omega_{j}\right) \leq \hat{X}_{b}\left(\omega_{j+1}\right)$, then $\hat{X}(\omega)=\hat{X}_{a}(\omega)$ for each $\omega \in[X *=a]$ and $\hat{X}(\omega)=\hat{X}_{b}(\omega)$ for each $\omega \in[X *=b]$. By Theorem 3.6, the restricted isotonic regres sion of $X$ to each level set $\left[X^{*}=c\right]$ is the constant c. Therefore, if $Z_{1}\left(\omega_{j}\right) \geq a$ then by Lemma 3.1 $\hat{X}_{a}\left(\omega_{j}\right)=Z_{1}\left(\omega_{j}\right) ;$ other wise, by Lemma $3.2 \hat{X}_{a}\left(\omega_{j}\right) \leq X *\left(\omega_{j}\right)=$ a. Similarly, if $Z_{2}\left(\omega_{j+1}\right) \leq b$ we have $\hat{X}_{b}\left(\omega_{j+1}\right)=Z_{2}\left(\omega_{j+1}\right)$ and otherwise $\hat{X}_{b}\left(\omega_{j+1}\right) \geq b \quad$ If $\quad Z_{1}\left(\omega_{j}\right) \geq a$, then $\quad \hat{X}_{a}\left(\omega_{j}\right)=Z_{1}\left(\omega_{j}\right) \leq Z_{1}\left(\omega_{j+1}\right)$ $\leq \hat{X}_{b}\left(\omega_{j+1}\right)$. If $Z_{2}\left(\omega_{j+1}\right) \leq b$, then
$\hat{X}_{a}\left(\omega_{j}\right) \leq Z_{2}\left(\omega_{j}\right) \leq Z_{2}\left(\omega_{j+1}\right)=\hat{X}_{b}\left(\omega_{j+1}\right)$. Otherwise $\hat{X}_{a}\left(\omega_{j}\right) \leq a<b \leq \hat{X}_{b}\left(\omega_{j+1}\right)$. This completes the proof. $]$

Example 3.6. Let $\quad X=(25,13,2,15,14,21,9,33,25,15)$,
$Z_{1}=(10,11.5,13,14.5,16,17.5,19,20.5,22,23.5)$ and
$Z_{2}=(13,14.5,16,17.5,19,20.5,22,23.5,25,26.5)$. The problem is to minimize

$$
f(Z)=\sum_{i=1}^{10}\left(x_{i}-z_{i}\right)^{2}
$$

subject to $z_{1} \leq z_{2} \leq \cdots \leq z_{10}$ and $Z_{1} \leq Z \leq Z_{2}$.
The isotonic regression $X *$ of $X$ is
$\mathbf{X}^{*}=(13 . \dot{3}, 13 . \dot{3}, 13 . \dot{3}, 14.5,14.5,15,15,24 . \dot{3}, 24 . \dot{3}, 24 . \dot{3})$. Theorem 3. 9 shows that the bounded isotonic regression $\widehat{X}$ of $X$ can be obtained by considering the set $\{25,13,2\},\{15,14\},\{21,9\}$ and $\{33,25,15\}$ of $X$ values independently. For the first partition $\{25,13,2\}$, the average is 13.3 and $L_{a}=\{1\}$. By Lemma 3.1, $\hat{x}_{1}=z_{21}=13$. Therefore, $\hat{x}_{2}$ and $\hat{x}_{3}$ are the optimal solution to the following problem. Minimize $\left(13-z_{2}\right)^{2}+\left(2-z_{3}\right)^{2}$ subject to $13 \leq z_{2} \leq 14.5, \quad 13 \leq z_{3} \leq 16$ and $z_{2} \leq z_{3}$. Hence $\hat{x}_{2}=13$ and $\hat{x}_{3}=13$. For the second partition $\{15,14\}$, the average is 14.5 and $U_{b}=\{4,5\}$. By Lemma 3. $1, \hat{x}_{5}=z_{15}=16$. Therefore, $\hat{x}_{4}$ is the optimal solution to the problem: minimize $\left(15-z_{4}\right)^{2}$ subject to $14.5 \leq z_{4} \leq 16$. Hence $\hat{x}_{4}=15$. Similarly, for the third partition $\{21,9\}$ we have $\hat{x}_{7}=19$ and it follows that $\hat{\mathbf{x}}_{6}=19$. For the last partition $\{33,25,15\}$, we have $\hat{\mathbf{x}}_{8}=23.5$ and it follows that $\hat{x}_{9}=\hat{x}_{10}=23.5$. The bounded isotonic regression is therefore
$\hat{X}=(13,13,13,15,16,19,19,23.5,23.5,23.5)$. In order to check that
$\widehat{X}$ has the correct values, one may refer to the Kuhn-Tucker condition (cf. Appendix IV). []
IV. CONDITIONAL EXPECTATION GIVEN A $\sigma-L A T T I C E$
IV. $1 \quad \sigma$-Lattices and $\Sigma$-Measurable Random Variables

Let $\Sigma$ be a family of subsets of a given set $\Omega$. It is said to be a lattice if it contains $\Omega$ and $\varnothing$ and it is closed under union and closed under intersection. A field is a lattice which contains complements of sets in the family. A $\underline{\sigma-l a t t i c e}$ is a lattice which is closed under countable union and closed under countable intersection. A $\underline{\sigma-f i e l d}$ is a $\sigma$-lattice which is also a field. A complete lattice is a lattice which is closed under arbitrary union and closed under arbitrary intersection. The family $\Sigma$ is said to be monotone class if whenever $\left\{A_{n}\right\}$ is a monotone sequence in $\Sigma$, both $\cup_{n=1}^{\infty} A_{n}$ and $\cap_{n=1}^{\infty} A_{n}$ are in $\Sigma$.

The collection of complements of a monotone class is a monotone class and the collection of complements of a $\sigma$-lattice (lattice, or complete lattice) $\Sigma$ is a $\sigma$-lattice (lattice or complete lattice); such a collection is denoted by $\Sigma^{c}$. The intersection of an arbitrary collection of $\sigma$-lattices is a $\sigma$-lattice. In particular, the intersection of a $\sigma$-lattice $\Sigma$ with its complement $\Sigma^{c}$ is a $\sigma$-field. Let $\Gamma$ be a family of subsets of $\Omega$. The $\sigma$-lattice generated by $\Gamma$ is the intersection of all $\sigma$-lattices which contain $\Gamma$. Similarly for the definition of $\underline{\sigma-f i e l d}$ generated by a family of subsets in $\Omega$.

Let $\leq$ be a quasi-ordering on $\Omega$ and let $\Sigma$ be the family of all upper sets. Then $\Sigma$ is a complete lattice, such a complete lattice is said to be induced by $\leq$. On the other hand, let $\Gamma$ be a family of subsets of $\Omega$ and let $\leq$ be the binary relation defined on $\Omega$ so that $\mu \leq \nu$ if $\mu \in U$ and $U \in \Gamma$ imply $\nu \in U$; if no set in $\Gamma$ contains $\mu$, then $\mu \leq \nu$ for every $v \in \Omega$. Then the binary relation $\leq$ is a quasi-ordering on $\Omega$; such an ordering is said to be induced by $\Gamma$.

Let $\Delta$ be the collection of all families of subsets of $\Omega$ each of which induces the same quasi-ordering $\leq$ Let $\Gamma \in \Delta$. For each $\omega \in \Omega$, let $U(\omega)=\cap\{U: \omega \in U, U \in \Gamma\}$ and let $L(\omega)=\cap\left\{L: \omega \in L, L^{c} \in \Gamma\right\}$; and if there is no set in $\Gamma$ which contains $\omega$, set $U(\omega)=\Omega$ and if every set in $\Gamma$ contains $\omega$, set $L(\omega)=\Omega$. It is clear from the definitions, for each pair $\mu$ and $\nu$ in $\Omega, \mu \leq \nu, \nu \in U(\mu)$ and $\mu \in L(\nu)$ are equivalent. Since $U(\mu)$ is the set of all elements $\nu$ such that $\mu \leq \nu$, it follows that for each $\mu \in \Omega, U(\mu)$ is the same for all families in $\Delta$.

Let $\Gamma \in \Delta$, and let $U \in \Gamma$. Then for each $\omega \in U$, we have $\omega \in U(\omega) \subset U$ and hence $U=\cup_{\omega \in U} U(\omega)$. Let $\quad \Sigma$ be the family of arbitrary unions of $U(\omega)$ 's. If $U \in \Sigma$, then for each $\omega \in U$ we have $U(\omega) \subset U$. Since $\Sigma$ contains every $U(\omega), \Sigma \in \Delta$ and if $\Gamma \in \Delta$ then we have $\Gamma \subset \Sigma$. The family $\Sigma$ is maximal in $\Delta$ in the sense of set inclusion. By the convention, a void union of sets is
the empty set. The family $\Sigma$ contains $\Omega$ and $\phi$, and it is closed under arbitrary union. We shall show that $\Sigma$ is also closed under arbitrary intersection. It will then follow that $\Sigma$ is a complete lattice. Let $\left\{U_{a}: a \in \Lambda\right\}$ be a subfamily of $\Sigma$ and let $A=\cap a \in \Lambda U_{a}$. If $\omega \in A$, then $\omega \in U_{a}$ for each $a \in \Lambda$ and hence $\mathrm{U}(\omega) \subset \mathrm{U}_{\mathrm{a}}$ for each $a \in \Lambda$. It follows that

$$
A=\cup_{\omega \in A}\{\omega\} \subset \cup_{\omega \in A} U(\omega) \subset A
$$

and hence $A \in \Sigma$. If $A=\phi$, then $A \in \Sigma$. If $\Sigma^{\prime}$ is a complete lattice in $\Delta$, then $\Sigma^{\prime}$ contains $\{U(\omega): \omega \in \Omega\}$. It follows that $\Sigma \subset \Sigma^{\prime}$. But we have just shown that $\Gamma \subset \Sigma$ for each $\Gamma \in \Delta$. Therefore, the collection $\Delta$ contains at most one complete lattice.

Let $\leq$ be a quasi-ordering in $\Omega$, let $\Sigma^{\prime}$ be the complete lattice induced by $\leq$ and let $\leq$ ' be the quasi-ordering induced by $\Sigma^{\prime}$. Let $\mu$ and $\nu$ be in $\Omega$. If $\mu \leq \nu$, then every upper set $U \in \Sigma^{\prime}$ which contains $\mu$, contains $\nu$. It follows that $\mu \leq s^{\prime} \nu$. On the other hand, if $\mu \nsubseteq v$ then there exists an upper set $U \in \Sigma^{\prime}$ such that $\mu \in U$ and $\nu \notin U$. It follows that $\mu \not \Phi^{\prime} v$ and therefore $\leq$ and $\leq^{\prime}$ are identical. The complete lattice $\Sigma^{\prime}$ is in $\Delta$, thus we have proved the following theorem.

Theorem 4.1. Let $\Omega$ be a given set. Quasi-orderings on $\Omega$ and complete lattices of $\Omega$ are in one-to-one correspondence such
that if $\leq$ corresponds to $\Sigma$, then $\leq$ induces $\Sigma$ and $\Sigma$ induces $\leq$.

Let $\Delta$ be the collection of all families of subsets of $\Omega$ each of which induces the same quasi-ordering $\leq$. We have shown that the maximal element in $\Delta$ is the complete lattice induced by the quasi-ordering $\leq$. However, minimal elements in $\Delta$ need not be unique.

Let $\leq_{1}$ and $\leq_{2}$ be two quasi-orderings on $\Omega . \leq_{1}$ is said to be finer than $\leq_{2}$ if for each pair $\mu$ and $\nu$ in $\Omega$, $\mu \leq 1_{1} \nu$ implies $\mu \leq_{2} \nu$. Let $\Gamma_{1}$ and $\Gamma_{2}$ be two families of subsets of $\Omega$ and let $\leq_{1}$ and $\leq_{2}$ be the quasi-orderings ind uced respectively by $\quad \Gamma_{1}$ and $\Gamma_{2}$. If $\Gamma_{2} \subset \Gamma_{1}$ and $\mu \leq{ }_{1} \nu$, then every $U \in \Gamma_{1}$ which contains $\mu$, contains $v$; so does every $U \in \Gamma_{2}$. It follows that $\leq_{1}$ is finer than $\leq_{2}$. But if $\leq_{1}$ is finer than $\leq_{2}$, it need not be true that $\Gamma_{2} \subset \Gamma_{1}$. On the other hand, let $\Sigma_{1}$ and $\Sigma_{2}$ be the complete lattices induced respectively by two quasi-orderings $\leq_{1}$ and $\leq_{2}$ on $\Omega$. Let $\leq_{1}$ be finer than $\leq_{2}$. For each $\omega \in \Omega$, let $U_{i}(\omega)$ be the set of $\nu^{\prime} s$ such that $\omega \leq_{i} \nu, i=1,2$. If $\mu \in U_{2}(\omega)$, then $U_{1}(\mu) \subset U_{2}(\omega)$ and hence $U_{2}(\omega)=\cup_{\mu \in U_{2}(\omega)} U_{1}(\mu)$. Since $\Sigma_{i}$ is the family of arbitrary unions of $U_{i}(\omega)$ 's for $i=1,2, \quad \Sigma_{2} \subset \Sigma_{1}$. Thus we have proved the following theorem.

Theorem 4.2. Let $\Omega$ be a given set, let $\leq_{1}$ and $\leq_{2}$ be two quasi-orderings on $\Omega$ and let $\Sigma_{1}$ and $\Sigma_{2}$ be the complete lattices induced respectively by $\leq_{1}$ and $\leq_{2}$. Then $\leq_{1}$ is finer than $\leq_{2}$ if an only if $\Sigma_{2} \subset \Sigma_{1}$.

The relation "finer than" on quasi-orderings of $\Omega$ is by itself a partial ordering. The finest quasi-ordering is the one induced by the power set of $\Omega$, and the least fine quasi-ordering is the one induced by $\{\Omega, \phi\}$. Let $\left\{\leq \frac{a}{a}: a \in \Lambda\right\}$ be a collection of quasiorderings on $\Omega$, and let $\Delta=\left\{\leq: \leq\right.$ is finer than $\leq_{a}$ for each $a \in \Lambda\}$. We claim that $\Delta$ has a unique minimal element $\leq_{m}$ in the sense that $\leq \in \Delta$ implies $\leq$ is finer than $\leq m$. For each $a \in \Lambda$, let $\Sigma_{a}$ be the complete lattice induced by $\leq_{a}$. Let $\Phi=\left\{\Sigma: \Sigma\right.$ is a complete lattice, $\Sigma \supset \Sigma_{a}$ for each $\left.a \in \Lambda\right\}$ and let $\Sigma_{m}=\cap_{\Phi} \Sigma$. Then $\Sigma_{m}$ is a complete lattice. The quasi-ordering $\leq_{m}$ induced by $\Sigma_{m}$ is finer than $\leq_{a}$ for each $a \in \Lambda$. If $\leq$ is finer than $\leq_{a}$ for each $a \in \Lambda$, then $\Sigma \supset \Sigma_{a}$ for each $a \in \Lambda$ where $\Sigma$ is the complete lattice induced by $\leq$ Since $\Sigma \supset \Sigma_{m}$, $\leq$ is finer than $\leq_{m}$. Similarly, if $\Delta=\left\{\leq: \leq_{a}\right.$ is finer than $\leq$ for each $a \in \Lambda\}$, then $\Delta$ has a unique maximal element $\leq_{m}$ in the sense that $\leq \in \Delta$ implies $\leq_{m}$ is finer than $\leq$. Thus, we have proved the following theorem.

Theorem 4.3. Let $\Omega$ be a given set. The relation "finer than" on quasi-orderings of $\Omega$ is a partial ordering such that every non-empty collection of quasi-orderings of $\Omega$ has a greatest lower bound and a least upper bound with respect to the partial ordering "finer than".

Let $\leq_{1}$ and $\leq_{2}$ be two quasi-orderings induced respectively by complete lattices $\Sigma$ and $\Sigma^{c}$. Then $\mu \leq_{1} \nu$ if and only if $\nu \leq L_{2} \mu$ for each pair $\mu$ and $v$ in $\Omega$. If $U \in \Sigma \cap \Sigma^{c}$, then $U$ and $U^{C}$ are unrelated with respect to $\leq_{1}$. A complete lattice $\Sigma$ is a field if and only if $U(\omega)$ is an equivalence class for each $\omega$, i.e., $\nu \in U(\mu)$ implies $\mu \in U(\nu)$ for each pair $\mu$ and $\nu$ where $U(\omega)$ is the set of all $v^{\prime}$ s such that $\omega \leq \nu$ with the quasiordering $\leq$ induced by $\Sigma$. A quasi-ordering $\leq$ is partially ordered if and only if $U(\mu)=U(\nu)$ implies $\mu=\nu$ for each pair $\mu$ and $v$. A partial ordering has a tree structure if and only if for each pair $\mu$ and $\nu$, we have $U(\mu) \cap U(\nu)=\phi, U(\mu) \subset U(\nu)$ or $U(\nu) \subset U(\mu)$. A tree structured partial ordering is linearly ordered if and only if $U(\mu) \cap U(\nu) \neq \phi$ for each pair $\mu$ and $\nu$.

When $\Omega$ is finite, every lattice is a complete lattice. Let $\Omega=\mathrm{R}$ and let $\leq$ be the natural ordering of real numbers. The complete lattice corresponding to $\leq$ is the $\sigma$-lattice generated by ( $a,+\infty$ ) for all real $a$. Let $\Omega=R^{2}$, let $\leq$ be the partial
ordering defined by $(a, \beta) \leq(a, b)$ if $a \leq a$ and $\beta \leq b$, let $\Sigma$ be the complete lattice corresponding to $\leq$, let $U \in \Sigma$ with $U \neq \varnothing$ and let $f$ be the extended real-valued function defined on $R$ such that $f(x)=\inf \{y:(x, y) \in U\}$. It is clear by the definition that if $(x, y) \in U$ and $(x, y) \leq(a, \beta)$ then $(a, \beta) \in U$. Let $x_{1}$ and $x_{2}$ be two real numbers such that $x_{1}<x_{2}$. Since $y>f\left(x_{1}\right)$ implies $\left(x_{1}, y\right) \in U$ and $\left(x_{2}, y\right) \in U, f\left(x_{1}\right) \geq f\left(x_{2}\right)$ and hence $f$ is monotone decreasing. It follows that $\{(\mathrm{x}, \mathrm{y}): \mathrm{y}>\mathrm{f}(\mathrm{x})\} \subset \mathrm{U} \subset\{(\mathrm{x}, \mathrm{y}): \mathrm{y} \geq \mathrm{f}(\mathrm{x})\}$. Let $\Sigma_{0}$ be the family of Borel measurable sets in $\Sigma$. Since the family of Borel measurable sets in $\mathrm{R}^{2}$ is a $\sigma$-lattice, $\Sigma_{0}$ is a $\sigma$-lattice. It is clear that the partial ordering induced by $\Sigma_{0}$ is the same as that induced by $\Sigma$. It follows that $\Sigma_{0}$ is the $\sigma$-lattice generated by $U(a, \beta)=\{(a, b): a \geq a, b \geq \beta\}$ for all $(a, \beta) \in R^{2}$.

As far as measurability is concerned, our interest will be in $\sigma$-lattices rather than complete lattices. An element of a $\sigma$-lattice $\Sigma$ is an upper set with respect to the quasi-ordering indueced by $\Sigma$. However, an upper set with respect to the quasi-ordering induced by a $\sigma$-lattice $\Sigma$ need not be in $\Sigma$.

A $\sigma$-lattice is a monotone class. By a proof analogous to that of the Monotone Class Theorem, one cah show that a monotone class which contains a lattice $\Sigma_{0}$ contains the $\sigma$-lattice generated by $\Sigma_{0}$. It follows that a monotone class which is a lattice is a $\sigma$-lattice.

Let $\Gamma$ be a family of subsets of $\Omega$. The $\sigma$-lattice $\Sigma$ generated by $\Gamma$ can be described as follows. Since $\Omega$ and $\phi$ are in $\Sigma$ and the $\sigma$-lattice generated by $\Gamma \cup\{\Omega, \phi\}$ is $\Sigma$, without loss of generality we may assume $\Omega$ and $\phi$ in $\Gamma$. Let $\Gamma_{0}=\Gamma$ and for each ordinal number $a>0$, let $\Gamma_{a}$ bedefined inductively by

$$
\Gamma_{a}=\left\{\left(\cup_{m=1}^{\infty} A_{m}\right) \cup\left(\cap_{n=1}^{\infty} B_{n}\right): A_{n}, B_{m} \in \cup_{\gamma<a} \Gamma_{\gamma}\right\}
$$

Let $\quad \Lambda=\cup_{a<\beta} \Gamma_{a}$ where $\beta$ is the first uncountable ordinal number. Since $\Gamma_{a} \subset \Sigma$ implies $\Gamma_{a+1} \subset \Sigma$ and since $\Gamma_{0} \subset \Sigma$, by the transfinite ind uction $\Lambda \subset \Sigma$. On the other hand, if $\left\{C_{n}\right\} \subset \Lambda$, then for each $n$ there is $a_{n}<\beta$ such that $C_{n} \in \Lambda_{a_{n}}$. Let $a=\sup _{n} a_{n}$. Then $a<\beta$ and $C_{n} \in \Lambda_{a}$ for each $n$. Therefore, $\cup_{n=1}^{\infty} C_{n}$ and $\cap_{n=1}^{\infty} C_{n}$ are in $\Gamma_{a+1}$ and hence they are in $\Lambda$. It follows that $\Lambda$ is a $\sigma$-lattice containing $\Gamma$ and thus $\Sigma=\Lambda$.

A $\sigma$-lattice $\Sigma$ is said to be linearly ordered if whenever $\mathrm{U}_{1}, \mathrm{U}_{2} \in \Sigma$ then either $\mathrm{U}_{1} \subset \mathrm{U}_{2}$ or $\mathrm{U}_{2} \subset \mathrm{U}_{1}$. If $\Sigma$ is a linearly ordered $\sigma$-lattice, then so is $\Sigma^{\mathbf{C}}$ and in such a situation $\Sigma \cap \Sigma^{\text {C }}$ is the trivial $\sigma$-field $\{\Omega, \varnothing\}$. If $\Omega \quad$ is a linearly ordered set, then the collection of all upper sets is a linearly ordered $\sigma$-lattice. However, the ordering induced by a linearly ordered $\sigma-1$ lattice need not be linear.

Let $(\Omega, F, P)$ be a probability space, i.e., $F$ is a $\sigma$-field of subsets of $\Omega$ and $P$ is a probability measure on $F$. A random variable is an equivalence class of extended real-valued F-measurable functions defined on $\Omega$ such that each pair of functions in the class differ by a set of $P$ measure zero. Therefore, the term "almost everywhere" will be omitted from the context. Whenever we say that a random variable satisfies a property, it means that one of its representations satisfies that property. It is implicit that if a random variable is integrable, then it is finite.

Let $\Sigma$ be a sub- $\sigma$-lattice of $F$. A random variable $X$ is said to be $\underline{\Sigma}$-measurable if $[X \geq a] \in \Sigma$ for each real number $a$ or equivalently $[X>a] \in \Sigma$ for each real number $a$. The family of $\Sigma$-measurable random variables is denoted by $R(\Sigma)$. Indicators of upper sets in $\Sigma$ are $\Sigma$-measurable. If $X$ is a random variable, then the family of all sets $[X \geq a]$ and $[X>a]$ for all extended real numbers $a$ is a $\sigma$-lattice. Such a $\sigma$-lattice, denoted by $\Sigma(X)$, is said to be induced by $X$. The $\sigma$-lattice $\Sigma(X)$ is linearly ordered. The family of all sets $[X \in B]$ for all Borel measurable sets in the extended real line is a $\sigma$-field. Such a $\sigma$-field, denoted by $F(X)$, is said to be induced by $X$. It is obvious that the intersection of all $\sigma$-fields containing $\Sigma(X)$ is the $\sigma$-field $F(X)$. Therefore $F(X)$ is the $\sigma$-field generated by $\quad \Sigma(\mathrm{X})$.

If $X_{1}$ and $X_{2}$ are $\Sigma$-measurable random variables, then so are $X_{1}+X_{2}, X_{1} \vee X_{2}, X_{1} \wedge X_{2}$ and $\delta X_{1}$ for each $\delta \geq 0$ (cf. Barlow and coworkers (1972)). Let $\left\{X_{n}\right\}$ be a monotone increasing sequence of $\Sigma$-measurable random variables and let $X$ be defined by $X(\omega)=\lim _{n \rightarrow \infty} X_{n}(\omega)$ for each $\omega$. Then $X=V_{n=1} X_{n}$ and $[x \leq a]=\cap_{n=1}^{\infty}\left[X_{n} \leq a\right]$ for each $a$. Therefore, $[\mathrm{X}>a]=\cup_{\mathrm{n}=1}^{\infty}\left[\mathrm{X}_{\mathrm{n}}>a\right] \in \Sigma$ for each $a$ and X is $\Sigma$-measurable. Similarly, the limit of a monotone decreasing sequence of $\Sigma$-measurable random variables is $\Sigma$-measurable. It follows that the upper limit and the lower limit of any sequence of $\Sigma$-measurable random variables are $\Sigma$-measurable. In particular, the limit of a pointwise convergent sequence of $\Sigma$-measurable random variables is $\Sigma$ measurable. Thus, we have shown that $R(\Sigma)$ is a convex cone which is closed under countable meet, closed under countable join and closed under pointwise convergence.

A random variable is simple if its range is finite. A simple random variable is $\Sigma$-measurable if and only if it is a finite linear combination of indicators of upper sets in $\Sigma$ such that the coefficients are non-negative except the one that corresponds to l. A non-negative $\Sigma$-measurable random variable is the limit of a nondecreasing sequence of simple $\Sigma$-measurable random variables. An arbitrary $\Sigma$-measurable random variable $\quad X$ is the limit of a sequence of some $\Sigma$-measurable random variables $\left\{X_{n}\right\}$ such that


#### Abstract

$\left|X_{n}\right| \leq|X|$ for each $n$. Let $X$ be a $\Sigma$-measurable random variable and let $f$ be a non-decreasing function defined on the extended real line. Then the random variable $f \circ X$ is $\Sigma$ measurable.


## IV. 2 Conditional Expectation as a Generalized Projection

Let $L_{1}(\Omega, F, P)$ and $L_{2}(\Omega, F, P)$ be the linear spaces of integrable random variables and of square-integrable random variables respectively. When there is no ambiguity, we shall use $\mathrm{L}_{1}$ and $L_{2}$. The linear space $L_{2}$ is a subspace of $L_{1}$. Let $E$ be the operator defined by

$$
\begin{equation*}
E X Y=\int X Y d P \tag{4.1}
\end{equation*}
$$

for each pair of random variables $X$ and $Y$, provided that the integral on the right-hand side exists. If the random variable $Y$ is the constant random variable 1 , then we shall use $E X$, the integral of $X$, instead of EXY when the latter exists. The operator E restricted to $L_{2} \times L_{2}$ is an inner product. The linear space $L_{2}$ with the inner product $E$ is known to be a Hilbert space. The norm of $X \in L_{2},\|X\|$, is the square root of $E X^{2}$.

The convex cones $L_{1} \cap R(\Sigma)$ and $L_{2} \cap R(\Sigma)$ are denoted
briefly by $L_{1}(\Sigma)$ and $L_{2}(\Sigma)$ respectively. Let $\left\{Z_{n}\right\}$ be a sequence in $L_{2}(\Sigma)$ which converges to $Z$ in $L_{2}$, i.e., $\left\|Z_{n}-Z\right\| \rightarrow 0$ as $n \rightarrow \infty$. Then $Z \in L_{2}$ and there exists a subsequence $\left\{Z_{n^{\prime}}\right\}$ which converges to $Z$ pointwise (cf. Ash (1972)). It follows that $Z \in R(\Sigma)$ and hence $L_{2}(\Sigma)$ is a closed convex cone in $L_{2}$. The uniqueness and existence of the projection $P\left(X \mid L_{2}(\Sigma)\right)$ for each $X \in L_{2}$ follows from Theorem 2.1. Theorem 2.7 shows that a random variable $X * \in L_{2}(\Sigma)$ is $P\left(X \mid L_{2}(\Sigma)\right)$ if and only if $E[(X-X *) X *]=0$ and $E[(X-X *) Z] \leq 0$ for each $Z \in L_{2}(\Sigma)$. Let $X *=P\left(X \mid L_{2}(\Sigma)\right) . \quad$ Since constant random variables are in $L_{2}(\Sigma)$, we have

$$
\begin{equation*}
E X *=E X \tag{4.2}
\end{equation*}
$$

For each $Z \in L_{2}(\Sigma)$, there is a sequence of simple random variaables $\left\{\mathrm{Z}_{\mathrm{n}}\right\} \subset \mathrm{L}_{2}(\Sigma)$ and $\left|\mathrm{Z}_{\mathrm{n}}\right| \leq|\mathrm{Z}|$ for each n such that $\mathrm{Z}_{\mathrm{n}}$ converges to $Z$ pointwise. For each $n,\left|(X-X *) Z_{n}\right| \leq|X-X *||Z|$. Since $|X-X *||Z|$ is integrable, by the Dominated Convergence Theorem $E[(X-X *) Z]=\lim _{n \rightarrow \infty} E\left[(X-X *) Z_{n}\right]$. It follows that the condition $E[(X-X *) Z] \leq 0$ for each $Z \in L_{2}(\Sigma)$ is equivalent to $E\left[(X-X *) l_{U}\right] \leq 0 \quad$ for each upper set $U \in \Sigma$. Brunk (1965) showed that if $X *=P\left(X \mid L_{2}(\Sigma)\right)$ and $g(X *) \in L_{2}$ for a real-valued function g, then $E[(X-X *) g(X *)]=0$. In particular, $E\left[(X-X *) 1_{B}(X *)\right]=0$ for each Borel set B Therefore, a necessary and sufficient
condition for $P\left(X \mid L_{2}(\Sigma)\right)$ can be stated as follows.

Theorem 4.4. A random variable $X * \in L_{2}(\Sigma)$ is $P\left(X \mid L_{2}(\Sigma)\right)$ if and only if $X^{*}$ satisfies

$$
\begin{equation*}
E\left[(X-X *) 1_{B}(X *)\right]=0 \quad \text { for each Borel set } B \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
E\left[(X-X *) 1_{U}\right] \leq 0 \quad \text { for each } \quad U \in \Sigma \tag{4.4}
\end{equation*}
$$

Theorem 2.3 shows that $E(X-Z)^{2} \geq E(X-X *)^{2}+E(X *-Z)^{2}$ for each $Z \in L_{2}(\Sigma)$. By (4.2), we have $E(X-X *)=0$ and hence $E(X-Z)=E(X *-Z)$. It follows that

$$
\operatorname{Var}(Z-Z) \geq \operatorname{Var}(X-X *)+\operatorname{Var}(X *-Z) \text { for each } Z \in L_{2}(\Sigma)
$$

In particular, $\operatorname{Var} X \geq \operatorname{Var} X *+\operatorname{Var}(X-X *)$ where the variance of a random variable $Z$ is defined by $\operatorname{Var} Z=E(Z-E Z)^{2}$. Theorem 2.7 and (4.2) show that

$$
\begin{equation*}
\operatorname{Cov}(X, X *)=\operatorname{Var} X * \tag{4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Cov}(X, Z) \leq \operatorname{Cov}(X *, Z) \text { for each } Z \in L_{2}(\Sigma) \tag{4.6}
\end{equation*}
$$

where the covariance of a pair of random variables $X$ and $Z$ is defined by $\operatorname{Cov}(X, Z)=E(X-E X)(Z-E Z)$.

If $\operatorname{Cov}(X, Z) \leq 0$ for each $Z \in L_{2}(\Sigma)$, then by Theorem 4.4
the projection $P\left(X \mid L_{2}(\Sigma)\right)$ is the constant random variable $E X$.
Let $X \in L_{2}$, let $Y \in L_{2}(\Sigma)$ and let $X *=P\left(X \mid L_{2}(\Sigma)\right)$. If $\left.E[(X-Y)]_{B}(Y)\right]=0$ for each Borel set $B$, then $E(X-Y) Y=0$ and $E X=E Y$. Since $E[(X-X *) Y] \leq 0$, we have $E Y^{2}=E X Y \leq E X * Y \leq\left(E X *^{2} E Y^{2}\right)^{1 / 2}$ and hence $E Y^{2} \leq E X *^{2}$. By the fact $E X *=E Y, \quad \operatorname{Var} Y \leq \operatorname{Var} X *$. Let $T \in L_{2}$. If $\left.E[(X-T)]_{U}\right] \leq 0$ for each $U \in \Sigma$, then $E[(X-T) Z] \leq 0$ for each $Z \in L_{2}(\Sigma)$. In particular, $E[(X-T) X *] \leq 0$. By the fact that $E X *^{2}=E X X *$, we have $E X *^{2}=E X X * \leq E T X * \leq\left(E X *^{2} E^{2}\right)^{1 / 2}$ and hence $E X *^{2} \leq E T^{2}$.

If $Z_{1}$ and $Z_{2}$ are in $L_{2}(\Sigma)$, then so are $Z_{1} \vee Z_{2}$ and $Z_{1} \wedge Z_{2}$. Since $X *+Z=X * \vee Z+X * \wedge Z$ for each $Z \in L_{2}$, by (4.5) we have $\operatorname{Cov}(X-X *, Z)=\operatorname{Cov}(X-X *, X * \vee Z)+\operatorname{Cov}(X-X, X * \wedge Z)$. For each $Z \in L_{2}(\Sigma)$, by (4.6) we have $\operatorname{Cov}(X-X *, Z) \leq 0$, $\operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{X} * \vee \mathrm{Z}) \leq 0$ and $\operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{X} * \wedge \mathrm{Z}) \leq 0$. It follows that $\operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{Z}) \leq \operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{X} * \vee \mathrm{Z})$ and
$\operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{Z}) \leq \operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{X} * \wedge \mathrm{Z})$. It is clear that $|\mathrm{X} *-\mathrm{Z} \vee \mathrm{X} *| \leq|\mathrm{X} *-\mathrm{Z}|$ and $|\mathrm{X} *-\mathrm{Z} \wedge \mathrm{X} *| \leq|\mathrm{X} *-\mathrm{Z}|$. By the identity $E(X-Z)^{2}=E(X-X *)^{2}+E(X *-Z)^{2}-2 E[(X-X *) Z]$ and the iden tities when $Z$ is replaced respectively by $Z \vee X *$ and $Z \wedge X *$, we have

$$
\begin{equation*}
E(X-Z \vee X *)^{2} \leq E(X-Z)^{2} \text { and } E(X-Z \wedge X *)^{2} \leq E(X-Z)^{2} \tag{4.7}
\end{equation*}
$$

for each $Z \in L_{2}(\Sigma)$.
By (4.2) and by an argument similar to that described above,
$\mathrm{C}(\mathrm{X}-\mathrm{X} *, \mathrm{Z}) \leq \operatorname{Cov}(\mathrm{X}-\mathrm{X} *, \mathrm{Z} \vee \gamma) \leq 0$ and
$\operatorname{Cov}(X-X *, Z) \leq \operatorname{Cov}(X-X *, Z \wedge \gamma) \leq 0$ for each $Z \in L_{2}(\Sigma)$ and for each real number $\gamma$. If $a \leq X \leq \beta$, then by the monotonicity in Theorem 4.5 we have $a \leq X^{*} \leq \beta$. Suppose $a \leq X^{*} \leq \beta$ for a pair of real number $a$ and $\beta$. It is obvious that $|X *-Z \vee a|$, $|X *-Z \wedge \beta|$ are bounded by $|X *-Z|$. By a proof analogous to that of (4.7), we have

$$
\begin{equation*}
E(X-Z \vee a)^{2} \leq E(X-Z)^{2} \text { and } E(X-Z \wedge \beta)^{2} \leq E(X-Z)^{2} \tag{4.8}
\end{equation*}
$$

for each $Z \in L_{2}(\Sigma)$. Cons equently, $E[X-(Z \vee a) \wedge \beta]^{2} \leq E(X-Z)^{2}$. (2.6) shows that $E\left(P\left(X \mid L_{2}(\Sigma)\right)-P\left(Y \mid L_{2}(\Sigma)\right)\right)^{2} \leq E(X-Y)^{2}$ and hence $\operatorname{Var}\left(\mathrm{P}\left(\mathrm{X} \mid \mathrm{L}_{2}(\Sigma)\right)-\mathrm{P}\left(\mathrm{Y} \mid \mathrm{L}_{2}(\Sigma)\right)\right) \leq \operatorname{Var}(\mathrm{X}-\mathrm{Y})$ for each pair X , $Y \in L_{2}$. Let $\Sigma_{0}$ be a sub- $\sigma$-lattice of $\Sigma$. Then $L_{2}\left(\Sigma_{0}\right)$ is a closed convex cone contained in $L_{2}(\Sigma)$. (2.7) shows that $E\left(P\left(X \mid L_{2}(\Sigma)\right)-P\left(X \mid L_{2}\left(\Sigma_{0}\right)\right)\right)^{2} \leq E\left(X-P\left(X \mid L_{2}\left(\Sigma_{0}\right)\right)\right)^{2}-E\left(X-P\left(X \mid L_{2}(\Sigma)\right)\right)^{2}$, (2.17) shows that $\operatorname{Var}\left(P\left(X \mid L_{2}(\Sigma)\right)-P\left(X \mid L_{2}\left(\Sigma_{0}\right)\right)\right) \leq \operatorname{Var}\left(P\left(X \mid L_{2}(\Sigma)\right)\right)$ $-\operatorname{Var}\left(P\left(X \mid L_{2}\left(\Sigma_{0}\right)\right)\right)$ and (2.16) shows that $\operatorname{Var} \mathrm{P}\left(\mathrm{X} \mid \mathrm{L}_{2}\left(\Sigma_{0}\right)\right) \leq \operatorname{Var} \mathrm{P}\left(\mathrm{P}\left(\mathrm{X} \mid \mathrm{L}_{2}(\Sigma)\right) \mid \mathrm{L}_{2}\left(\Sigma_{0}\right)\right) \leq \operatorname{Var} \mathrm{P}\left(\mathrm{X} \mid \mathrm{L}_{2}(\Sigma)\right) \leq \operatorname{Var} \mathrm{X}$. Some other properties of the gene ralized projection $P\left(X \mid L_{2}(\Sigma)\right)$ will be given in Section IV. 4.

## IV. 3 Conditional Expectation Given a $\sigma$-Lattice

Conditional expectation given a $\sigma$-lattice is an extension of generalized projection on the closed convex cone $L_{2}(\Sigma)$. Let $X$ be a random variable defined on the probability space $(\Omega, F, P)$ and let $\Sigma$ be a sub- $\sigma$-lattice of $F$. A random variable $X * \in R(\Sigma)$ is said to be the conditional expectation of $X$ given $\Sigma$ if $X^{*}$ satisfies (4.3) and (4.4). The conditional expectation of $X$ given $\Sigma$ need not exist. If it exists, then it is unique (cf. Theorem 4.6); such a random variable is denoted by $E(X \mid \Sigma)$.

If $X \in L_{1}$, then by Theorem 4.7, $E(X \mid \Sigma)$ exists. Let $X *=E(X \mid \Sigma)$ for an $X \in L_{1}$. The conditions (4.3) and (4.4) can be represented as

$$
\begin{equation*}
\int_{[X * \in B]} X d P=\int_{[X * \in B]} X * d P \text { for each Borel set } B \tag{4.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{U} X d P \leq \int_{U} X * d P \quad \text { for each } \quad U \in \Sigma \tag{4.10}
\end{equation*}
$$

It follows that $X * \in L_{1}$ and $E X=E X *$. (4.9) shows that $X *$ is the Radon-Nikodym derivative of the measure $\mu$ with respect to $P$ restricted to the $\sigma$-field $F(X *)$ where $\mu(A)=E\left(X 1_{A}\right)$ for each $A \in F$. (4.10) implies that $E X Z \leq E X * Z$ for every $Z \in R(\Sigma)$ provided that both integrals exist. If $\Sigma$ is by itself a $\sigma$-field, then
$E X 1_{U}=E X{ }^{*} 1_{U}$ for each $U \in \Sigma$ and hence $X *$ is the RadonNikodym derivative of the measure $\mu$ given above with respect to $P$ restricted to $\Sigma$. It follows that conditional expectation given a $\sigma$-field is a special case of conditional expectation given a $\sigma$-lattice.

Example 4.1. Let $\Omega=(-1 / 2,1 / 2)$, let $F$ be the family of all Borel subsets of $\Omega$, let $P$ be the Lebesgue measure on $F$ and let $\Sigma$ be the $\sigma$-lattice induced by $Z$ where $Z(\omega)=\omega$ for each $\omega \in \Omega$. Let $X$ be the random variable defined by $X(\omega)=1 /|\omega|$ if $\omega \neq 0$ and $X(0)=0$. By the definition, one can show that the conditional expectation of $X$ given $\Sigma$ does not exist. Another way to see the non-existence of $E(X \mid \Sigma)$ is by (4.13). Let $\left.Z_{s}=\inf \left\{(t-s)^{-1}\left(E X 1_{(s, t)}\right): s<t<1 / 2\right)\right\} 1_{(s, 1 / 2)}$ and let $Y_{s}=\sup \left\{(s-r)^{-1}\left(E X l_{(r, s)}\right):-1 / 2<r<s\right\} l_{(-1 / 2, s)}{ }^{+\infty} l_{[s, 1 / 2)}$ for each $s \in \Omega$. For $s>0, Z_{s}=(-\ln 2 s)^{-1} 1_{(s, 1 / 2)}$ and $Y_{s}=+\infty$ For $s<0, \quad Z_{s}=|s|^{-1} 1_{(s, 1 / 2)} \quad$ and $Y_{s}=|s|^{-1} 1_{(-1 / 2, s)}{ }^{+\infty} l_{(s, 1 / 2)}$. For $s=0, \quad Z_{0}=+\infty l_{(0,1 / 2)}$ and $Y_{0}=+\infty$. Let $Z=V_{S \in \Omega} Z_{S}$ and let $Y=\Lambda_{S \in \Omega} Y_{S}$. Then $Y=Z$ and $Z(\omega)=X(\omega)$ if $\omega<0$ and $Z(\omega)=+\infty$ if $\omega \geq 0$. If $E(X \mid \Sigma)$ exists, then it must be $Z$. However, $E(X-Z)=-\infty$. Therefore $E(X \mid \Sigma)$ does not exist.

For each $s \in(-1 / 2,0)$, let $Y_{s}$ be the random variable defined by $\quad Y_{s}(\omega)=X(\omega)$ if $\omega \leq s$ and $Y_{s}(\omega)=+\infty$ if $\omega>s$.

Then $Y_{S} \in R(\Sigma)$ and $Y_{S}$ satisfies (4.9) and (4.10) for each $s \in(-1 / 2,0)$. Therefore, conditions (4.3) and (4.4) for $E(X \mid \Sigma)$ can not be replaced by (4.9) and (4.10). []

Let $X *$ be the conditional expectation of $X$ given $\Sigma$. For each real number $a$, we have

$$
\int_{[X *>a]}(X-X *) d P=0
$$

and hence for each $L \in \Sigma^{c}$,

$$
\begin{aligned}
\int_{L_{\cap}[X *>a]}(X-X *) d P & =\int_{[X *>a]}(X-X *) d P-\int_{L^{c}}{ }_{\cap[X *>a]}(X-X *) d P \\
& =-\int_{L^{c}{ }_{\cap}[X *>a]}(X-X *) d P .
\end{aligned}
$$

Since $L^{c} \cap\left[X^{*}>a\right] \in \Sigma$, by (4.4) we have

$$
\begin{equation*}
\int_{L_{\cap}[X *>a]} X d P \geq \int_{L_{\cap}[X *>a]} X * d P . \tag{4.11}
\end{equation*}
$$

Similarly, if $U \in \Sigma$ then we have
(4. 12)

$$
\int_{U \cap[X *<a]} X d P \leq \int_{U \cap[X *<a]} X * d P
$$

for each real number $a$. The inequality (4.11) was introduced by

Brunk (1965).

Theorem 4.5. Let $X_{i}^{*}=E\left(X_{i} \mid \Sigma\right), \quad i=1,2$ for a pair of random variables $X_{1}$ and $X_{2}$ such that $X_{1} \leq X_{2}$. Then $X_{1}^{*} \leq X_{2}^{*}$.

Proof. We shall show that $P\left(X_{1}^{*}>X_{2}^{*}\right)=0$. The event $\left[X_{1}^{*}>X_{2}^{*}\right] \quad$ can be represented by $\left[X_{1}^{*}>X_{2}^{*}\right]=\cup_{\text {a rational }}\left[X_{1}^{*}>a>X_{2}^{*}\right]$. Therefore, we need only to show that $P\left(X_{1}^{*}>a>X_{2}^{*}\right)=0$ for each rational number a. By (4.11) and (4.12), we have

$$
\begin{aligned}
\int_{\left[x_{2}^{*}<a\right] \cap\left[x_{1}^{*}>a\right]} x_{1}^{*} d P & \leq \int_{\left[X_{2}^{*}<a\right] \cap\left[x_{1}^{*}>a\right]} X_{1} d P \\
& \leq \int_{\left[X_{2}^{*}<a\right] \cap\left[X_{1}^{*}>a\right]} X_{2} d P \\
& \leq \int_{\left[X_{2}^{*}<a\right] \cap\left[X_{1}^{*}>a\right]} X_{2}^{*} d P
\end{aligned}
$$

for each real number $a$. Since $X_{2}^{*}-X_{1}^{*}<0$ on $\left[X_{1}^{*}>a>X_{2}^{*}\right]$, it follows that $P\left(X_{2}^{*}<a<X_{1}^{*}\right)=0$. This completes the proof. []

The above property is called monotonicity, which appears in Barlow and coworkers (1972).

Theorem 4.6. Let $X$ be a random variable. The conditional
expectation of $X$ given $\Sigma$ is unique provided that it exists.

Proof. Suppose $Y_{1}$ and $Y_{2}$ are two $\Sigma$-measurable random variables such that each satisfies (4.3) and (4.4). The inequalities (4.11) and (4.12) hold when we replace $X *$ by $Y_{1}$ or $Y_{2}$. Since $X \leq X$, by a similar argument in the proof of Theorem 4.5 we have $P\left(Y_{1}>Y_{2}\right)=0$ and $P\left(Y_{2}>Y_{1}\right)=0$. Therefore, $Y_{1}=Y_{2} . \quad \square$

The uniqueness of $E(X \mid \Sigma)$ can also be obtained by (4.13). The proof of the theorem is through communication with Professor H.D. Brunk. The following existence theorem is given by Brunk (1963).

Theorem 4.7. If $X \in L_{1}$, then the conditional expectation of $X$ given $\Sigma$ exists.

Let $\left\{Z_{a}: a \in \Lambda\right\}$ be a collection of random variables. The random variable $Z$ is said to be the essential supremum of $\left\{Z_{a}: a \in \Lambda\right\} \quad$ if for any random variable $Y, \quad Z \leq Y \quad$ if and only if $Z_{a} \leq Y$ for each $a \in \Lambda$. Similarly for the definition of the essential infimum of $\left\{Z_{a}: a \in \Lambda\right\}$. They are denoted by $V_{a \in \Lambda} Z_{a}$ and $\Lambda_{a \in \Lambda} Z_{a}$ respectively. It is known that any family of random variables has an essential supremum and an essential infimum and
$\Lambda_{a \in \Lambda} Z_{a}=-V_{a \in \Lambda}^{(-Z)}$. The following identity derived from (4.11) and (4.12) was introduced by Brunk and Johansen (1970).

$$
\begin{align*}
& \left.E(X \mid \Sigma)=V_{U \in \Sigma^{\prime}} \inf _{L \in \Sigma^{c}}\left[P(L U)^{-l} E X l_{L U}\right] l_{U}{ }^{-\infty} l_{U} c^{c}\right\}  \tag{4.13}\\
& P(L U)>0 \\
& =\Lambda_{L \in \Sigma^{c}}\left\{\sup _{U \in \Sigma} \quad\left[P(L U)^{-1} E X l_{L U}\right]_{L_{L}}{ }^{+\infty} l_{L}^{c}\right\} \\
& P(L U)>0
\end{align*}
$$

provided that $E(X \mid \Sigma)$ exists.
Let $a \leq X \leq \beta, \quad$ let
and let

$$
\left.Y=\Lambda_{L \in \Sigma c^{c}\left\{\sup _{\substack{ \\P(L U)>0}}\left[P(L U)^{-1} E X l_{L U}\right] 1_{L}+\beta l_{L}^{c}\right\} .}\right\} .
$$

Then $Z \leq Y$ and (4.13) can be represented as $E(X \mid \Sigma)=Z=Y$. If $Z \neq Y$, then $E(X \mid \Sigma)$ does not exist. If $Z=Y, E(X \mid \Sigma)$ need not exist as shown in Example 4.1.

If $\Omega$ if finite, then every upper set with respect to the quasiordering induced by $\Sigma$ is in $\Sigma$. Suppose $P(\{\omega\})>0$ for each $\omega \in \Omega$. (4.13) may be represented by the following identity.
(4.14)

$$
\begin{aligned}
X *(\omega)= & \max _{\omega \in U} \min M(L U) \\
& =\min \quad \max M(L U) \\
& \omega \in L \quad \omega \in U
\end{aligned}
$$

where $X *$ is the isotonic regression of $X, \quad U$ is an upper set, L is a lower set, $M(L U)$ is the weighted average of $X$ over $L \cap U$ and the weight function $W$ is defined by $W(\omega)=P(\{\omega\})$ for each $\omega \in \Omega$.

The identity (4.14) appears in Barlow and coworkers (1972). Ayer and coworkers (1955) introduced (4.14) for the case that $\Omega$ is a linearly ordered set.

## IV. 4 Properties of Conditional Expectation

Our interest in this section and the following section is the case when random variables are integrable, although most of the properties hold in general. If $X \in L_{1}$, then $X$ is finite and $E(X \mid \Sigma)$ exists.

It is trivial that $E(X+a \mid \Sigma)=E(X \mid \Sigma)+a$ for each real $a$, $E(\delta X \mid \Sigma)=\delta E(X \mid \Sigma) \quad$ for each real $\quad \delta \geq 0$, and $E\left(X \mid \Sigma^{c}\right)=-E(-X \mid \Sigma)$. The monotonicity shows that if $X_{1} \leq X_{2}$, then $E\left(X_{1} \mid \Sigma\right) \leq E\left(X_{2} \mid \Sigma\right)$. It follows that $E(X \wedge Y \mid \Sigma) \leq E X \mid \Sigma) \leq E(X \vee Y \mid \Sigma)$ for each pair $X, \quad Y \in L_{1}$. By a proof analogous to Theorem 4.5, if $X_{1}<X_{2}$ then $E\left(X_{1} \mid \Sigma\right)<E\left(X_{2} \mid \Sigma\right)$.

The following three convergence theorems which appear in Barlow and coworkers (1972) are given below without proof.

Theorem 4.8. Let $X, X_{n} \in L_{l}$ and let $\left\{X_{n}\right\}$ be a monotone sequence such that it converges to $X$. Then $\left\{E\left(X_{n} \mid \Sigma\right)\right\}$ converges to $E(X \mid \Sigma)$.

Thecrem 4.9. If $\left|X_{n}\right| \leq Y$ for each $n$ with $Y \in L_{1}$ and $\left\{X_{n}\right\}$ converging to $X$, then $\left\{E\left(X_{n} \mid \Sigma\right)\right\}$ converges to $E(X \mid \Sigma)$.

Theorem 4.10. Let $\left\{\Sigma_{n}\right\}$ be a monotone sequence of sub-$\sigma$-lattices of $F$ and let $X \in L_{1}$. Then $\left\{E\left(X \mid \Sigma_{n}\right)\right\}$ converges to $E(X \mid \Sigma)$ where $\Sigma$ is the $\sigma$-lattice generated by $\cup_{n=1}^{\infty} \Sigma_{n}$ if $\left\{\Sigma_{n}\right\}$ is monotone increasing and $\Sigma=\cap_{n=1}^{\infty} \Sigma_{n}$ if $\left\{\Sigma_{n}\right\} \quad$ is montone decreasing.

A similar result to Corollary 2.11.1 can be applied to the operator of conditional expectation.

Theorem 4.11. Let $X \in L_{1}$ and let $\Sigma_{1}$ and $\Sigma_{2}$ be two sub- $\sigma$-lattices of $F$ with $\Sigma_{1} \subset \Sigma_{2}$. Then $E\left(E\left(X\left|\Sigma_{2}\right| \Sigma_{1}\right)=E\left(X \mid \Sigma_{1}\right) \quad\right.$ if and only if

$$
\int_{U} E\left(X \mid \Sigma_{2}\right) d P \leq \int_{U} E\left(X \mid \Sigma_{1}\right) d P \quad \text { for each } \quad U \in \Sigma_{1}
$$

Proof. Let $Y_{i}=E\left(X \mid \Sigma_{i}\right), i=1,2$. If $Y_{1}=E\left(Y_{2} \mid \Sigma_{1}\right)$,
then by (4.4) we have

$$
\int_{U} Y_{2} d P \leq \int_{U} Y_{1} d P \quad \text { for each } \quad U \in \Sigma_{1}
$$

Conversely, if the above inequality holds for each $U \in \Sigma_{1}$, then

$$
\int_{\left[Y_{1}>a\right]} Y_{2} d P \leq \int_{\left[Y_{1}>a\right]} Y_{1} d P \quad \text { for each real } \quad a
$$

By (4.3) and (4.4) we have

$$
\int_{\left[Y_{1}>a\right]} Y_{1} d P=\int_{\left[Y_{1}>a\right]} X d P \leq \int_{\left[Y_{1}>a\right]} Y_{2} d P \text { for each real } a
$$

since $\left[Y_{1}>a\right] \in \Sigma_{1} \subset \Sigma_{2}$. Combining the last two inequalities, we have

$$
\int_{\left[Y_{1}>a\right]} Y_{2} d P=\int_{\left[Y_{1}>a\right]} Y_{1} d P \quad \text { for each real } a .
$$

It follows that

$$
\int_{\left[Y_{1} \in B\right]} Y_{2} d P=\int_{\left[Y_{1} \in B\right]} Y_{1} d P \quad \text { for each Bored set } B .
$$

Since $\mathrm{Y}_{\mathrm{l}}$ satisfies (4.3) and (4.4) when we replace X and $\Sigma$ by $Y_{2}$ and $\Sigma_{1}$ respectively, $Y_{1}=E\left(Y_{2} \mid \Sigma\right)$. This completes the proof. []

Theorem 4.12. Let $X \in L_{1}$ and let $\Sigma_{1}$ and $\Sigma_{2}$ be two sub- $\sigma$-lattices of $F$ with $\Sigma_{1} \subset \Sigma_{2}$. If either $\Sigma_{1}$ or $\Sigma_{2}$ is a $\sigma$-field, then $E\left(E\left(X \mid \Sigma_{2}\right) \mid \Sigma_{1}\right)=E\left(X \mid \Sigma_{1}\right)$.

Proof. Let $Y_{i}=E\left(X \mid \Sigma_{i}\right), i=1,2$. If $\Sigma_{1}$ is a $\sigma$-field, then $U^{C} \in \Sigma_{1}$ if $U \in \Sigma_{1}$. Therefore, if we apply both $U$ and $U^{c}$ to (4.4) then we have

$$
\int_{U} Y_{1} d P=\int_{U} X d P=\int_{U} Y_{2} d P
$$

for each $U \in \Sigma_{1}$ because $\Sigma_{1} \subset \Sigma_{2}$. It follows from Theorem 4.11 that $Y_{1}=E\left(Y_{2} \mid \Sigma_{1}\right)$. If $\Sigma_{2}$ is a $\sigma$-field, then by a similar argument applying to $\Sigma_{2}$ we have

$$
\int_{U} Y_{2} d P=\int_{U} X d P \leq \int_{U} Y_{1} d P \quad \text { for each } \quad U \in \Sigma_{1}
$$

By Theorem 4.11, $\quad Y_{1}=E\left(Y_{2} \mid \Sigma_{1}\right) . \square$

The theorem which appears in Robertson (1968), is called the smoothing property. It is parallel to Corollary 2.4.1. If $F_{0}$ is the trivial $\sigma$-field $\{\Omega, \phi\}$, then $E\left(X \mid F_{0}\right)=E X$. Therefore, the operator $E$ can be regarded as conditional expectation given $F_{0}$. By the smoothing property with $\quad \Sigma_{1}$ replaced by $F_{0}$, we have (4.2), i.e., $\quad E(E(X \mid \Sigma))=E X$.

Let I be an open interval in the real line. The monotonicity implies that if $P(X \in I)=1$ then $P(E(X \mid \Sigma) \in I)=1$. Let $X *=E(X \mid \Sigma)$. Then by (4.3), $\quad X *=E(X \mid F(X *))$. Therefore the following version of the Jensen's inequality follows (cf. Ash (1972)).

Theorem 4.13. Let $X \in L_{1}$ and let $I$ be an open interval in the real line such that $P(X \in I)=1$. If $g$ is a real-valued convex function defined on $I$, then

$$
E(g(X) \mid F(X *)) \geq g(X *)
$$

where $\quad X *=E(X \mid \Sigma)$.

By the smoothing property, we have $E g(X) \geq E g(X *)$, and hence $E|X| \geq E|X *|$ and $E X^{2} \geq E X *{ }^{2}$ where $X *=E(X \mid \Sigma)$, provided $X \in L_{1}$.

The following version of the Jensen's inequality appears in Barlow and coworkers (1972).

Theorem 4.14. Under the same assumption as in Theorem 4.13, if $X^{\prime}(X *), X * g^{\prime}(X *)$ and $g(X *)$ are in $L_{1}$, then

$$
E(g(X) \mid \Sigma) \geq g(X *)
$$

where $g^{\prime}$ is a determination of the derivative of $g$.

Immediate results of the theorem are

$$
|E(X \mid \Sigma)| \leq E(|X| \mid \Sigma) \quad \text { and } \quad(E(X \mid \Sigma))^{2} \leq E\left(X^{2} \mid \Sigma\right)
$$

Let $\quad \Gamma \in F$ such that $P(\Gamma)>0$, let $F \mid \Gamma:=\{A \cap \Gamma: A \in F\}$
and let $P_{\Gamma}$ be defined by $P_{\Gamma}(A):=P(A \cap \Gamma) / P(\Gamma)$ for each $A \in F$. Then ( $\Gamma, F \mid \Gamma, P_{\Gamma}$ ) is a probability space. Let
$\Sigma \mid \Gamma:=\{\mathrm{U} \cap \Gamma: \mathrm{U} \in \Sigma\}$. Then $\Sigma \mid \Gamma$ is a sub- $\sigma$-lattice of $F \mid \Gamma$.
Let $R(\Sigma \mid \Gamma)$ be the family of $\Sigma \mid \Gamma$-measurable random variables defined on ( $\Gamma, F \mid \Gamma, P_{\Gamma}$ ). The restricted conditional expectation of X given $\Sigma$ to $\Gamma$ is the conditional expectation of $\mathrm{X} \mid \Gamma$ given $\Sigma \mid \Gamma \quad$ with respect to $\left(\Gamma, F \mid \Gamma, P_{\Gamma}\right)$. Such a random variable is denoted by $E(X \mid \Sigma, \Gamma)$ if it exists.

Let $X *=E(X \mid \Sigma)$ and let $\Gamma=[X *>a]$ for some real $a$
such that $P(\Gamma)>0$. It is obvious that

$$
\int_{[\mathrm{X} * \in \mathrm{~B}] \cap \Gamma} \mathrm{XdP}=\int_{[\mathrm{X} * \in \mathrm{~B}] \frown \Gamma} \mathrm{X} * \mathrm{dP} \text { for each Borel set } B .
$$

and

$$
\int_{U \cap \Gamma} \mathrm{XdP} \leq \int_{\mathrm{U} \cap \Gamma} \mathrm{X} * \mathrm{dP} \quad \text { for each } \quad \mathrm{U} \in \Sigma
$$

provided that $X \in L_{1}$. It follows that $X * \mid \Gamma=E(X \mid \Sigma, \Gamma)$.
Suppose $\quad \Gamma \in \Sigma \cap \Sigma^{c}$ such that $P(\Gamma) P\left(\Gamma^{c}\right)>0 . \quad$ By (4.11)
and (4.4), we have

$$
\int_{[X *>a] \cap \Gamma} X d P=\int_{[X *>a] \cap \Gamma} X * d P \text { for each real } a
$$

and hence

$$
\int_{[X * \in B] \cap \Gamma} X d P=\int_{[X * \in B] \cap \Gamma} \quad X * d P
$$

It is trivial that

$$
\int_{U \cap \Gamma} x d P \leq \int_{U \cap \Gamma} X * d P \quad \text { for each } \quad U \in \Sigma
$$

Therefore $X * \mid \Gamma=E(X \mid \Sigma, \Gamma)$ and similarly $\quad X * \mid \Gamma^{c}=E\left(X \mid \Sigma, \Gamma^{c}\right)$. It follows that for each $X \in L_{1}$ we have

$$
\begin{equation*}
\mathrm{E}(\mathrm{X} \mid \Sigma)=\mathrm{E}(\mathrm{X} \mid \Sigma, \Gamma) 1_{\Gamma}+\mathrm{E}\left(\mathrm{X} \mid \Sigma, \Gamma^{\mathrm{c}}\right) 1_{\Gamma^{c}} \tag{4.15}
\end{equation*}
$$

where $E(X \mid \Sigma, \Gamma) 1_{\Gamma}$ and $E\left(X \mid \Sigma, \Gamma^{c}\right){ }_{\Gamma}{ }_{\Gamma}^{c} \quad$ are extensions of $E(X \mid \Sigma, \Gamma)$ and $E\left(X \mid \Sigma, \Gamma^{c}\right)$ to $\Omega$.

Theorem 4.15. Let $U \in \Sigma$ and let $L \in \Sigma^{c}$. Then

$$
E(X \mid \Sigma, U) \leq E(X \mid \Sigma) \mid U \quad \text { and } \quad E(X \mid \Sigma, L) \geq E(X \mid L) \mid L
$$

provided that $P(U) P\left(U^{c}\right) P(L) P\left(L^{c}\right)>0$. If any equality holds, then $E(X \mid \Sigma)$ can be represented by (4.15) with $\Gamma$ replaced by $U$ or $L$ according as $E(X \mid \Sigma, U)=E(X \mid \Sigma) \mid U$ or $E(X \mid \Sigma, L)=E(X \mid \Sigma) \mid L$.

Proof. This is a similar proof to that of Theorem 4.5. Let $X *=E(X \mid \Sigma), \quad Y_{1}=E(X \mid \Sigma, U)$ and $Y_{2}=X * \mid U$. We are going to show $P_{U}\left(Y_{1}>a>Y_{2}\right)=0$ for each real $a$. The first inequality will then follow. Similarly for the second inequality. The relation between $P$ and $P_{U}$ is that $d P_{U}=P(U)^{-1} 1_{U} d P$.

By (4.11) and (4.12), we have

$$
\begin{aligned}
\int_{\left[Y_{1}>a>Y_{2}\right]} Y_{1} d P_{U} & \leq \int_{\left[Y_{1}>a>Y_{2}\right]} X_{U} \\
& =P(U)^{-1} \int_{\left[Y_{1}>a>Y_{2}\right]} X d P \\
& =P(U)^{-1} \int_{\left[Y_{1}>a\right] \cap[X *<a]} X d P \\
& \leq P(U)^{-1} \int_{\left[Y_{1}>a\right] \cap[X *<a]} X_{1} * d P \\
& =\int_{\left[Y_{1}>a>Y_{2}\right]} Y_{2} d P_{U}
\end{aligned}
$$

for each real $a$, because $\left\{\omega: Y_{1}(\omega)>a\right\} \in \Sigma$ and $\left[Y_{1}>a\right] \cap[X *<a]=\left[Y_{1}>a>Y_{2}\right]$. Since $Y_{2}-Y_{1}<0$ on $\left[Y_{1}>a>Y_{2}\right], \quad P_{U}\left(Y_{1}>a>Y_{2}\right)=0$.

Suppose $\mathrm{Y}_{1}=\mathrm{Y}_{2}$. Then

$$
E(X \mid \Sigma)=X * 1_{U}+X_{U} * 1_{U}^{c} \leq E(X \mid \Sigma, U) 1_{U}+E\left(X \mid \Sigma, U^{c}\right) l_{U} c .
$$

Applying $E$ to both sides of the inequality, by the monotonicity of E we have

$$
\begin{aligned}
E X & \leq E\left\{E(X \mid \Sigma, U) l_{U}\right\}+E\left\{E\left(X \mid \Sigma, U^{c}\right) l_{U} c^{c}\right\} \\
& =E\left(X l_{U}\right)+E\left(X l_{U} c^{\prime}\right) \\
& =E X .
\end{aligned}
$$

It follows that $E(X \mid \Sigma)=E(X \mid \Sigma, U) l_{U}+E\left(X \mid \Sigma, U^{c}\right) l_{U^{c}} . \quad$ ]
Corollary 4.15.1. Let $U \in \Sigma, L \in \Sigma^{C}$ such that $P(U) P\left(U^{c}\right) P(L) P\left(L^{c}\right)>0$ and let $X \geq 0$. Then

$$
E\left(X l_{U} \mid \Sigma\right)=E(X \mid \Sigma, U) l_{U} \leq E(X \mid \Sigma) l_{U}
$$

and

$$
E\left(X l_{L} \mid \Sigma\right)=E(X \mid \Sigma, L) l_{L} \geq E(X \mid \Sigma) l_{L}
$$

Proof. It is trivial that $E\left(\mathrm{Xl}_{\mathrm{U}} \mid \Sigma, \mathrm{U}^{\mathrm{c}}\right)=0$. Let $Y=E\left(X l_{U} \mid \Sigma\right)$. Then

$$
E X l_{U}=\int_{U} X d P \leq \int_{U} Y d P \leq \int Y d P=E Y
$$

Since $E\left(X l_{U}\right)=E Y$ and $Y \geq 0, \quad Y \mid U^{c}=0$, i.e., $\quad E\left(X l_{U} \mid \Sigma\right) \mid U^{c}=0$. By Theorem 4. 15,

$$
E\left(X l_{U} \mid \Sigma\right)=E(X \mid \Sigma, U) l_{U}
$$

The monotonicity implies that $E\left(X_{l_{U}} \mid \Sigma\right) \leq E(X \mid \Sigma)$. The first statement will then follow. Similarly we have the second statement. []

Corollary 4.15.2. If $X \geq a$ on $U$ for a set $U \in \Sigma$, then $E(X \mid \Sigma) \geq a$ on U. If $X \leq \beta$ on $L$ for a set $L \in \Sigma^{c}$, then $E(X \mid \Sigma) \leq \beta$ on $L$.

Proof. If $X \mid U \geq a$, then by the monotonicity we have $E(X \mid \Sigma, U) \geq$ a. From Corollary 4.15.1, we have $E(X \mid \Sigma) \mid U \geq E(X \mid \Sigma, U)$. Therefore, $E(X \mid \Sigma) \mid U \geq a$. Similarly for the second statement. []

## IV. 5 On a Linearly Ordered $\sigma$-Lattice

Let $Y$ be a random variable. The $\sigma$-lattice $\Sigma(Y)$ induced by $Y$ is linearly ordered. Let $X$ be a random variable and let $X *=E(X \mid \Sigma(Y))$. Since $X *$ is $\Sigma(Y)$-measurable, $X *$ if $F(Y)-$ measurable. It has been shown that an $F(Y)$-measurable random variable is a function of $Y$, i.e., there is an extended real-valued function $f$ defined on the extended real line such that $X *=f \circ Y$ (cf. Ash (1972)).

Theorem 4. 16. Let $E(X \mid \Sigma(Y))=f \circ Y$. Then $f$ is monotone increasing on the range of $Y$.

Proof. Let $\omega_{1}$ and $\omega_{2}$ in $\Omega$ such that $Y\left(\omega_{1}\right)<Y\left(\omega_{2}\right)$. Since $X^{*} \in R(\Sigma(Y)), \quad\left[X^{*} \geq X *\left(\omega_{1}\right)\right] \in \Sigma(Y)$. Upper sets in $\Sigma(Y)$ are either $[Y>a]$ or $[Y \geq a]$ for each $a$. It follows that $\left[\mathrm{Y} \geq \mathrm{Y}\left(\omega_{1}\right)\right]$ is the smallest element in $\Sigma(\mathrm{Y})$ which contains $\omega_{1}$. Since $\quad \omega_{1} \in\left[X * \geq X *\left(\omega_{1}\right)\right], \quad\left[X * \geq X *\left(\omega_{1}\right)\right] \supset\left[Y \geq Y\left(\omega_{1}\right)\right]$. The element $\omega_{2}$ is in $\left[Y \geq Y\left(\omega_{1}\right)\right]$, so $\omega_{2} \in\left[X * \geq X *\left(\omega_{1}\right)\right]$ and hence $X *\left(\omega_{2}\right) \geq X *\left(\omega_{1}\right)$. In other words, $f\left(Y\left(\omega_{2}\right)\right) \geq f\left(Y\left(\omega_{1}\right)\right)$ whenever $Y\left(\omega_{2}\right)>Y\left(\omega_{1}\right)$. This completes the proof. []

Let $\Sigma$ be a sub- $\sigma$-lattice of $F$. Let a binary relation $\leq \Sigma$ be defined on the linear space of random variables by $X \leq \Sigma Y \quad$ if

$$
\int_{U} X d P \leq \int_{U} Y d P \quad \text { for each } \quad U \in \Sigma
$$

The binary relation $\leq_{\Sigma}$ is a quasi-ordering. For each $X$ we have $X \leq \sum_{\Sigma} E(X \mid \Sigma)$ provided that $X \in L_{1}$. If the $\sigma$-field generated by $\Sigma$ is $F$, then $\leq_{\Sigma}$ is a partial ordering.

Theorem 4.17. Let $\Sigma$ be a linearly ordered sub- $\sigma$-lattice of $F$ and let $\Sigma_{1}$ and $\Sigma_{2}$ be two sub- $\sigma$-lattices of $\Sigma$ such that $\Sigma_{1} \subset \Sigma_{2}$. For each $X \in L_{1}$, we have

$$
E\left(X \mid \Sigma_{1}\right) \leq_{\Sigma} E\left(E\left(X \mid \Sigma_{2}\right) \mid \Sigma_{1}\right) \leq_{\Sigma} E\left(X \mid \Sigma_{2}\right)
$$

Proof. Let $Y_{i}=E\left(X \mid \Sigma_{i}\right), \quad i=1,2$, and let $Y_{0}=E\left(Y_{2} \mid \Sigma_{1}\right)$. Suppose it is not true that $Y_{1} \leq \Sigma Y_{0}$. Then there exists a $U$ such that

$$
\int_{U} Y_{1} d P>\int_{U} Y_{0} d P
$$

Let $a=\inf \left\{Y_{1}(\omega): \omega \in U\right\}$ and let $A=\left[X_{1} \geq a\right]$ and $B=\left[Y_{1}>a\right]$. By the linear ordering property of $\Sigma$, we have $A \supset \cup \supset B$. For each $C \in \Sigma_{1} \cap F\left(Y_{1}\right)$,

$$
\int_{C} Y_{1} d P=\int_{C} X d P \leq \int_{C} Y_{2} d P \leq \int_{C} Y_{0} d P
$$

In particular for $C$ is $A$ or $B$. Therefore,

$$
\int_{U-B} Y_{1} d P=a P(U-B)>\int_{U-B} Y_{0} d P
$$

and hence $\mathrm{P}\left(\mathrm{U} \cap\left[\mathrm{Y}_{0}<a\right]\right)>0$. By the linear property of $\Sigma$ again, $\left[\mathrm{Y}_{0} \geq \mathrm{a}\right] \subset \mathrm{U}$. It follows that

$$
\begin{aligned}
\int_{A} Y_{0} d P & =\int_{U} Y_{0} d P+\int_{A-U} Y_{0} d P \\
& <\int_{U} Y_{1} d P+a P(A-U) \\
& =\int_{A} Y_{1} d P
\end{aligned}
$$

This contradicts that $\int_{A} Y_{1} d P \leq \int_{A} Y_{0} d P$. Thus $Y_{1} \leq Y_{0}$. Bya similar argument, we have $\mathrm{Y}_{0} \leq_{\Sigma} \mathrm{Y}_{2} . \square$

A sequence $\left\{X_{n}, \Sigma_{n}\right\}$ is said to be a submartingale with respect to $\leq_{\Sigma}$ if $\left\{\Sigma_{n}\right\}$ is a monotone increasing sequence of $\sigma$ lattices, $X_{n}$ is $\Sigma_{n}$-measurable for each $n$ and $X_{n} \leq{ }_{\Sigma} E\left(X_{n+k} \mid \Sigma_{n}\right)$ for each non-negative integer $k$. The following corollary is an immediate result of Theorem 4. 17.

Corollary 4.17.1. Let $\Sigma$ be linearly ordered and let $\left\{\Sigma_{n}\right\}$ be a monotone increasing sequence of sub- $\sigma$-lattices of $\Sigma$. For each $\mathrm{X} \in \mathrm{L}_{1}$, the sequence $\left\{\mathrm{X}_{\mathrm{n}}, \Sigma_{\mathrm{n}}\right\}$ is a submartingale with respect to $\leq_{\Sigma}$ where $X_{n}=E\left(X \mid \Sigma_{n}\right)$ for each $n$.

By a similar argument in the proof of Theorem 4.17, one may obtain the following result.

Corollary 4.17.2. If $\Sigma$ is linearly ordered and $X \in L_{1}$, then $E(X \mid \Sigma)$ is a minimal element in the class $\left\{Z \in L_{1}(\Sigma): X \leq{ }_{\Sigma} Z\right.$ with respect to $\leq_{\Sigma}$.

Corollary 4.17.3. Under the same assumptions as in Corollary 4.17.1, if $X \leq_{\Sigma} X_{n}$ for some $n$, then $X_{n+k}=E(X \mid \Sigma)$ for each positive integer $k$.

Proof. Apply Corollary 4.17.2 and Theorem 4.17. []

Theorem 4.18. Let $\Sigma$ be a linearly ordered sub- $\sigma$-lattice of F, let $\Sigma_{1}$ be a sub- $\sigma$-lattice of $\Sigma$ and let $X \in L_{1}(\Sigma)$. Then $E\left(X \mid \Sigma_{1}\right)=E\left(X \mid F_{1}\right)$ where $\quad F_{1}$ is the $\sigma$-field generated by $\Sigma_{1}$.

Proof. Let $X *=E\left(X \mid \Sigma_{1}\right)$. Then $X \leq \Sigma_{1} X *$. But on the other hand $X=E(X \mid \Sigma)$ and $\Sigma \supset \Sigma_{1}$. By Theorem 4.17, $X * \leq_{\Sigma_{1}} X$. Therefore $E\left(X-X^{*}\right) l_{U}=0$ for each $U \in \Sigma_{1}$ and hence $X^{*}=E\left(X \mid F_{1}\right) . \quad[]$

Corollary 4.18.1. Let $\Sigma$ be a linearly ordered sub- $\sigma$-lattice of $F$ and let $\left\{\Sigma_{n}\right\}$ be a monotone increasing sequence of sub- $\sigma$ lattices of $\Sigma$. For each $X \in L_{1}(\Sigma)$, the sequence $\left\{X_{n}, \Sigma \Sigma_{n}\right\}$ is a martingale where $X_{n}=E\left(X \mid \Sigma_{n}\right)$.

Proof. It is obvious that $\left\{E\left(X \mid F_{n}\right), F_{n}\right\}$ is a martingale where $F_{n}$ is the $\sigma$-field generated by $\Sigma_{n}$ for each $n$. By Theorem 4.18, $E\left(X \mid F_{n}\right)=E\left(X \mid \Sigma_{n}\right)$ for each $n$. It follows that $E\left(X_{n+k} \mid \Sigma_{n}\right)=E\left(E\left(X \mid \Sigma_{n+k}\right) \mid \Sigma_{n}\right)=E\left(E\left(X \mid F_{n+k}\right) \mid \Sigma_{n}\right)=E\left(X \mid \Sigma_{n}\right)=X_{n}$. This completes the proof. []

## V. MULTIVARIATE ISOTONIC REGRESSION

## V. 1 Introduction

Let $\Omega$ be a finite set and let $H$ be the linear space of vector-valued functions $Y: \Omega \rightarrow R^{m}$ for a fixed positive integer $m$. For convenience, let $\Omega=\{1,2, \ldots, n\}$ and for each $j \in \Omega$ let $Y(j)$ be the function value of $Y$ at $j$ which is an $m$-component column vector. The function $Y$ in this case is an $m \times n$ matrix $\left(y_{i j}\right)$. For each $i$, we denote the $n$-component row vector $\left(y_{i 1}, y_{i 2}, \ldots, y_{i n}\right)$ by $Y_{i}$. Therefore, $Y_{i}$ is a function from $\Omega$ to $R, i=1, \ldots, m$. For each $i$, let $\leq_{i}$ be a quasi-ordering defined on $\Omega$ and let $M_{i}$ be the family of real-valued isotonic functions with respect to the ordering $\leq_{i}$. Let $M$ be defined by $Y \in M$ if $Y_{i} \in M_{i}$ for each $i=1, \ldots, m$. The minimization problem we are interested in is to minimize

$$
\begin{equation*}
f(Z)=\Sigma_{j=1}^{n} w_{j}(X(j)-Z(j))^{t} V(X(j)-Z(j)) \tag{5.1}
\end{equation*}
$$

subject to $Z \in M$ where $X$ is a given $m \times n$ matrix, $V$ is a given $m \times m$ positive definite matrix and $\quad w_{1}, w_{2}, \ldots, w_{n}$ are positive real numbers.

Let us define a bilinear functional (.,.) on $H$ by

$$
(Y, Z)=\sum_{j=1}^{n} w_{j} Y(j)^{t} V Z(j)
$$

for each $Y$ and $Z$ in $H$. Then the bilinear functional (.,.) is an inner product and the linear space $H$ with $(\cdot, \cdot)$ is a Hilbert space. Since $M_{i}$ is finitely generated as described in Chapter III for $i=1, \ldots, m$, so is $M$ and hence $M$ is a closed convex cone in $H$. Existence and uniqueness of the optimal solution to (5.1) follow from Theorem 2.1. Such an optimal solution is called the multivariate isotonic regression of $X$ and is denoted by $P(X \mid M)$. For convenience, we denote $P\left(X_{i} \mid M_{i}\right)$ as the isotonic regression of $X_{i}, \quad i=1, \ldots, m, \quad$ i.e., $P\left(X_{i} \mid M_{i}\right) \operatorname{minimizes} \quad \sum_{j=1}^{n}\left(x_{i j}-z_{i j}\right)^{2} w_{j}$ subject to $\quad Z_{i} \in M_{i}$.

A necessary and sufficient condition for an $m \times n$ matrix $X *$ to be the multivariate isotonic regression of $X$ is given by Theorem 2.7. Let $J^{i}$ be an $m \times n$ matrix such that each entry at the ith row has value one with zeros elsewhere, $i=1, \ldots, m$. Then $J^{i}$ and $-J^{i}$ are in $M, i=1, \ldots, m$. If $X *=P(X \mid M)$, then by (2.11) we have

$$
\Sigma_{j=1}^{n} w_{j}(X(j)-X *(j))^{t} V J^{i}(j)=0 \quad i=1, \ldots, m
$$

The matrix $V$ is positive definite. Solving the above equations, we shall obtain

$$
\begin{equation*}
\Sigma_{j=1}^{n} w_{j} x_{i j}=\Sigma_{j=1}^{n} w_{j} x_{i j}^{*} \quad i=1, \ldots, m \tag{5.2}
\end{equation*}
$$

If $V$ is a diagonal matrix, i.e., $v_{i j}=0$ if $i \neq j$, then (5.1) can be written as

$$
\begin{aligned}
f(Z) & =\Sigma_{j=1}^{n} w_{j} \Sigma_{i=1}^{m} v_{i i}\left(x_{i j}-z_{i j}\right)^{2} \\
& =\Sigma_{i=1}^{m} v_{i i}\left\{\Sigma_{j=1}^{n}\left(x_{i j}-z_{i j}\right)^{2} w_{j}\right\} .
\end{aligned}
$$

Therefore, $P(X \mid M)_{i}=P\left(X_{i} \mid M_{i}\right)$ for $i=1, \ldots, m$. If $V$ is not a diagonal matrix, the minimization problem could be very complicated.

In this thesis, we shall treat $m=2$ and $w_{1}=w_{2}=\ldots=w_{n}=1$.
A $2 \times 2$ positive matrix $V$ can be represented by $v_{11}=v_{1}^{2}$, $v_{22}=v_{2}^{2}$ and $v_{12}=v_{21}=-\rho v_{1} v_{2}$ where $v_{1}$ and $v_{2}$ arepositive and $-1<\rho<1$. The objective function is

$$
f(Z)=\Sigma_{j=1}^{n} \Sigma_{i=1}^{2}\left\{v_{i}^{2}\left(x_{i j}-z_{i j}\right)^{2}-\rho v_{1} v_{2}\left(x_{1 j}-z_{1 j}\right)\left(x_{2 j}-z_{2 j}\right)\right\}
$$

Let $Y$ be the $2 \times n$ matrix such that $Y_{i}=v_{i} X_{i}, i=1,2$ and let $Y^{*}$ be the multivariate isotonic regression of $Y$ with respect to the positive definite matrix $T$ such that $t_{11}=t_{22}=1$ and $t_{12}=t_{21}=-\rho$. Let $X *$ be defined by $X_{i}^{*}=v_{i}^{-1} Y_{i}^{*}, i=1,2$. Then $X *$ is the multivariate isotonic regression of $X$ with respect to the positive definite matrix V. Without loss of generality, we may
as some that $v_{1}=v_{2}=1$.
Let $X *$ be the multivariate isotonic regression of $X$. If $M_{2}$ is the family of constant functions, i.e., the quasi-ordering $\leq_{2}$ is such that $j \leq_{2} k$ and $k \leq_{2} j$ for each $j$ and $k$ between one and $n$, then by (5.2) we have $X_{2}^{*}=\bar{x}_{2}$ where $\bar{x}_{2}=\sum_{j=1}^{n} x_{2 j} / n$. For each $Z$ such that $Z_{2}=\bar{x}_{2}$, we have

$$
f(Z)=\sum_{j=1}^{n}\left\{\left[x_{1 j}-\rho\left(x_{2 j}-\bar{x}_{2}\right)\right]-z{ }_{1 j}\right\}^{2}+\left(1-\rho^{2}\right) \Sigma_{j=1}^{n}\left(x_{2 j}-\bar{x}_{2}\right)^{2}
$$

It follows that $X_{1}^{*}=P\left(X_{1}-\rho\left(X_{2}-\bar{x}_{2}\right) \mid M_{1}\right)$.
Let $M_{2}$ be the family of all functions, i.e., the quasiordering $\leq_{2}$ is such that neither $j \leq_{2} k$ nor $k \leq_{2} j$ for each $j$ and $k$ between one and $n$ with $j \neq k$. Let $U^{j}$ be a $2 \times n$ matrix such that the $(2, j)$ entry has value one with zeros elsewhere. Then $U^{j}$ and $-U^{j}$ are in $M$ for $i=1, \ldots, n$. By (2.11), we have $\rho\left(x_{1 j}-x_{1 j}^{*}\right)=x_{2 j}-x_{2 j}^{*}, j=1, \ldots, n$. For each $Z \quad$ such that $\rho\left(x_{1 j}-z_{1 j}\right)=x_{2 j}-z_{2 j}, j=1, \ldots, n, \quad$ we have

$$
f(Z)=\left(1-\rho^{2}\right) \Sigma_{j=1}^{n}\left(x_{l j}-z{ }_{l j}\right)^{2}
$$

It follows that $X_{1}^{*}=P\left(X_{1} \mid M_{1}\right)$ and $X_{2}^{*}=X_{2}-\rho\left(X_{1}-X_{1}^{*}\right)$.

Let $H$ be the linear space of $2 \times n$ matrices, let $M$ be the family of matrices such that $Z \in M \quad$ if $z_{i j} \leq z_{i j}+1$ for $i=1,2$ and for $j=1, \ldots, n-1$. The problem in the remainder of the chapter is to minimize

$$
\begin{equation*}
f(Z)=\Sigma_{j=1}^{n} \Sigma_{i=1}^{2}\left\{\left(x_{i j}-z_{i j}\right)^{2}-\rho\left(x_{1 j}^{-z_{1 j}}\right)\left(x_{2 j}-z_{2 j}\right)\right\} \tag{5.3}
\end{equation*}
$$

subject to $Z \in M$ where $X$ is a given $2 \times n$ matrixand $\rho$ is a given real number, $-1<\rho<1$. The optimal solution to the problem is called the bivariate isotonic regression of $x$ with respect to $\rho$ and is denoted by $P(X \mid M, \rho)$.

Let $\quad M_{i d}:=\left\{Z: Z_{1} \in M_{1},-Z_{2} \in M_{2}\right\}, \quad$ i.e., $Z \in M_{i d}$ if the first row of $Z$ is monotone increasing and the second row of $Z$ is monotone decreasing. Let $Y$ be such that $Y_{1}=X_{1}$ and $Y_{2}=-X_{2}$, let $\quad Y *=P(Y \mid M,-\rho)$ and let $X *$ be defined by $X_{1}^{*}=Y_{1}$ and $X_{2}^{*}=-Y_{2}^{*}$. Then $X *=P\left(X \mid M_{i d}, \rho\right)$. Similarly for the situations $M_{d i}$ and $M_{d d}$, where $M_{d i}=\left\{Z:-Z_{1} \in M_{1}, Z_{2} \in M_{2}\right\} \quad$ and $\quad M_{d d}=\left\{Z:-Z_{1} \in M_{1},-Z_{2} \in M_{2}\right\}$. The sign of $\rho$ will play the most important role in analyzing the properties of $P(X \mid M, \rho)$ as we shall soon see. Therefore $P(X \mid M, \rho), \quad P\left(X \mid M_{i d},-\rho\right), P\left(X \mid M_{d i},-\rho\right) \quad$ and $\quad P\left(X \mid M_{d d}, \rho\right) \quad$ will have the same properties.

Let $l_{i j}$ be the $2 \times n$ matrix such that the $j$ th, ..., nth entries at the ith row have values one with zeros elsewhere, $\mathrm{i}=1,2, \mathrm{j}=1,2, \ldots, \mathrm{n}$. The set $\left\{1_{\mathrm{ij}} ; \mathrm{i}=1,2, \mathrm{j}=1,2, \ldots, \mathrm{n}\right\}$ is linearly independent, and the family $M$ is the cone generated by the set such that $Z \in M \quad$ if and only if $Z=\sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{i j} l_{i j}$ with $\beta_{i j} \geq 0$ if $j>1$. Let $\quad X *=\sum_{i=1}^{2} \Sigma_{j=1}^{n} a_{i j}^{*} 1_{i j}$ with $\quad a_{i j}^{*} \geq 0$ if $j>1$. Theorem 2. 12 and Theorem 2.13 show that $X *=P(X \mid M, \rho)$ if and only if $X *$ satisfies

$$
\begin{align*}
& \Sigma_{h=j}^{n}\left(x_{1 h}-\rho x_{2 h}\right) \leq \Sigma_{h=j}^{n}\left(x_{l h}^{*}-\rho x_{2 h}^{*}\right) \quad j=1, \ldots, n  \tag{5.4}\\
& \Sigma_{h=j}^{n}\left(x_{2 h}-\rho x_{l h}\right) \leq \Sigma_{h=j}^{n}\left(x_{2 h}^{*}-\rho x_{l h}^{*}\right) \quad j=1, \ldots, n
\end{align*}
$$

$$
\begin{array}{ll}
\Sigma_{h=j}^{n}\left(x_{l h}-\rho x_{2 h}\right)=\Sigma_{h=j}^{n}\left(x_{l h}^{*}-\rho x_{2 h}^{*}\right) & \text { if } \quad a_{l j}^{*}>0  \tag{5.5}\\
\Sigma_{h=j}^{n}\left(x_{2 h}-\rho x_{l h}\right)=\Sigma_{h=j}^{n}\left(x_{2 h}^{*}-\rho x_{l h}^{*}\right) & \text { if } \quad a_{2 j}^{*}>0
\end{array}
$$

and (5.2), i.e., $\quad \sum_{h=1}^{n} x_{i j}=\Sigma_{h=1}^{n} x_{i h}^{*} \quad i=1,2$.

Theorem 5.1. If $\rho \geq 0$ and $X_{1}-\rho X_{2}$ is monotone increasing then $X_{2}^{*}=P\left(X_{2} \mid M_{2}\right)$ and $X_{1}^{*}=X_{1}+\rho\left(X_{2}^{*}-X_{2}\right)$ where $X *=P(X \mid M, \rho) . \quad$ Similarly if we interchange indices 1 and 2.

Proof. Let $Y$ be defined by $Y_{2}=P\left(X_{2} \mid M_{2}\right)$ and $Y_{1}=X_{1}+\rho\left(Y_{2}-X_{2}\right)$. We are going to show that $Y=P(X \mid M, \rho) . \quad B y$
the assumption that $X_{1}-\rho X_{2}$ is monotone increasing and $\rho \geq 0$, we have $Y \in M$. Note that $X_{1}-\rho X_{2}=Y_{1}-\rho Y_{2}$. Therefore $Y$ satisfies (5.4) and (5.5). By (4.2), $\sum_{j=1}^{n} y_{2 j}=\sum_{j=1}^{n} x_{2 j}$ and it follows that $\sum_{j=1}^{n} y_{l j}=\sum_{j=1}^{n} x_{l j}$. This completes the proof. $]$

Theorem 5.2. Let $\rho \leq 0$. If $X_{1}-\rho X_{2}$ is monotone increasing and $P\left(X_{2} \mid M_{2}\right)$ is constant, then $X_{2}^{*}=P\left(X_{2} \mid M_{2}\right)$ and $X_{1}^{*}=X_{1}+\rho\left(X_{2}^{*}-X_{2}\right)$ where $X *=P(X \mid M, \rho)$. Similarly if we inter change indices 1 and 2. If $P\left(X_{1} \mid M_{1}\right)$ and $P\left(X_{2} \mid M_{2}\right)$ are constant, then $X_{i}^{*}=P\left(X_{i} \mid M_{i}\right), \quad i=1,2$.

Proof. The first statement is similar to Theorem 5.1 except that we need $P\left(X_{2} \mid M_{2}\right)$ be constant to ensure that $X_{1}+\rho\left(X_{2}^{*}-X_{2}\right)$ is isotonic. Let us consider the second statement. Let $P\left(X_{1} \mid M_{1}\right)$ and $P\left(X_{2} \mid M_{2}\right)$ be constant with value $\gamma_{1}$ and $\gamma_{2}$ respectively. Then $\gamma_{i}=\Sigma_{k=1}^{n} x_{i j} / n$ and $\Sigma_{k=j}^{n} x_{i j} \leq(n-j+1) \gamma_{i}$ for $i=1,2$ and for $j=1, \ldots, n$. Let $Y=\gamma_{1} l_{11}+\gamma_{2} l_{21}$. By the assumption $\rho \leq 0$, $Y$ satisfies (5.2), (5.4) and (5.5). Therefore $Y=P(X \mid M, \rho)$. $]$

The average property which plays the most important role in the isotonic regression and its algorithms, is the one we are interested in. Let $X *$ be the bivariate isotonic regression and let $x_{i j}^{*}<x_{i j+1}^{*}$ and $x_{i k}^{*}<x_{i k+1}^{*}$ for some $j$ and $k$ with $j<k$, $i=1,2$. By (5.5) and (5.2), we have

$$
\begin{aligned}
& \Sigma_{h=1}^{j}\left(x_{1 h}-\rho x_{2 h}\right)=\Sigma_{h=1}^{j}\left(x_{1 h}^{*}-\rho x_{2 h}^{*}\right) \\
& \Sigma_{h=1}^{j}\left(x_{2 h}-\rho x_{1 h}\right)=\Sigma_{h=1}^{j}\left(x_{2 h}^{*}-\rho x_{1 h}^{*}\right) \\
& \Sigma_{h=j+1}^{k}\left(x_{1 h}-\rho x_{2 h}\right)=\Sigma_{h=j+1}^{k}\left(x_{1 h}^{*}-\rho x_{2 h}^{*}\right) \\
& \Sigma_{h=j+1}^{k}\left(x_{2 h}-\rho x_{1 h}\right)=\Sigma_{h=j+1}^{k}\left(x_{2 h}^{*}-\rho x_{l h}^{*}\right) \\
& \Sigma_{h=k+1}^{n}\left(x_{1 h}-\rho x_{2 h}\right)=\Sigma_{h=k+1}^{n}\left(x_{l h}^{*}-\rho x_{2 h}^{*}\right)
\end{aligned}
$$

and

$$
\Sigma_{h=k+1}^{n}\left(x_{2 h}-\rho x_{1 h}\right)=\Sigma_{h=k+1}^{n}\left(x_{2 h}^{*}-\rho x_{1 h}^{*}\right)
$$

The first two equations show that

$$
\Sigma_{h=1}^{j} x_{i h}=\Sigma_{h=1}^{j} x_{i h}^{*}, \quad i=1,2,
$$

the next two equations show that

$$
\Sigma_{h=j+1}^{k} x_{i h}=\Sigma_{h=j+1}^{k} x_{i h}^{*}, \quad i=1,2
$$

and the last two equations show that

$$
\Sigma_{h=k+1}^{n} x_{i h}=\Sigma_{h=k+1}^{n} x_{i h}^{*}, \quad i=1,2 .
$$

If $x_{l j}^{*}<x_{l j+1}^{*}$ and $x_{1 k}^{*}<x_{l k+1}^{*}$ for a pair of $j$ and $k$ with $j<k$, then by (5.5) and (5.2) we have

$$
\begin{aligned}
& \sum_{h=1}^{j}\left(x_{1 h}-\rho x_{2 h}\right)=\Sigma_{h=1}^{j}\left(x_{1 h}^{*}-\rho x_{2 h}^{*}\right) \\
& \sum_{h=j+1}^{k}\left(x_{1 h}-\rho x_{2 h}\right)=\Sigma_{h=j+1}^{k}\left(x_{1 h}^{*}-\rho x_{2 h}^{*}\right)
\end{aligned}
$$

and

$$
\Sigma_{h=k+1}^{n}\left(x_{1 h}-\rho x_{2 h}\right)=\Sigma_{h=k+1}^{n}\left(x_{1 h}^{*}-\rho x_{2 h}^{*}\right)
$$

Similarly if we interchange indices 1 and 2.

Theorem 5.3. If $\rho \geq 0$ and $V X \leq V Y$, then $P(X \mid M, \rho) \leq P(Y \mid M, \rho)$.

Proof. Let $U^{l k}$ be the $2 \times n$ matrix such that the $(1, k)$ entry has value one, the $(2, k)$ entry has value $\rho$ with zeros elsewhere, $k=1, \ldots, n$. Similarly for the $U^{21}, U^{22}, \ldots, U^{2 n}$. We shall show $P\left(X+\delta U^{i j} \mid M, \rho\right) \geq P(X \mid M, \rho) \quad$ for any $\delta>0$ and for each $i$ and $j$. Since $Y-X$ can be represented as a non-negative linear combination of $U^{i j}, i=1,2 ; j=1, \ldots, n$, it will then follow $P(X \mid M, \rho) \leq P(Y \mid M, \rho)$.

Let $\quad X *=P(X \mid M, \rho)$ and let $Y *=P(X+U \mid M, \rho) \quad$ where $\mathrm{U}=\delta \mathrm{U}^{\mathrm{lk}}$ for a fixed positive real number $\delta$. Then by (2.3)

$$
(\mathrm{X}+\mathrm{U}-\mathrm{Y} *, \mathrm{Y} *-\mathrm{X} *) \geq 0
$$

and

$$
(\mathrm{X}-\mathrm{X} *, \mathrm{Y} *-\mathrm{X} *) \leq 0 .
$$

The difference of the above two inequalities is

$$
(U-Y *+X *, Y *-X *) \geq 0
$$

and hence

$$
(U, Y *-X *) \geq(Y *-X *, Y *-X *)
$$

The left-hand side of the above inequality is $\delta\left(1-\rho^{2}\right)\left(y_{1 k}^{*}-x_{1 k}^{*}\right)$ and hence $y_{1 k}^{*} \geq x_{1 k}^{*}$.

It is trivial that

$$
\|X+U-Z\|^{2}=\|x-Z\|^{2}+g(Z)
$$

where $g(Z)=2 \delta\left(1-\rho^{2}\right)\left(x_{1 k}-z_{1 k}+\delta / 2\right)$. Since $y_{1 k}^{*} \geq x_{1 k}^{*}$, $g(X * \wedge Y *)=g(X *)$ where the $(i, j)$ entry of $X * \wedge Y *$ has the value $x_{i j}^{*} \wedge y_{i j}^{*}$. By a similar argument as in (4.7), we have

$$
(X+U-Y *, Y *-X *) \geq(X+U-Y *, Y *-X * \wedge Y *)
$$

Since

$$
\begin{aligned}
\|Y *-X *\|^{2}= & \Sigma_{j=1}^{n}(Y *(j)-X *(j))^{t} V(Y *(j)-X *(j)) \\
= & \Sigma_{j=1}^{n}\left\{\left(y_{1 j}^{*}-x_{1 j}^{*}\right)^{2}+\left(y_{2 j}^{*}-x_{2 j}^{*}\right)^{2}\right. \\
& \left.-2 \rho\left(y_{1 j}^{*}-x_{i j}^{*}\right)\left(y_{2 j}^{*}-x_{2 j}^{*}\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& (Y *(j)-X *(j))^{t} V(Y *(j)-X *(j)) \geq(Y *(j)-X *(j) \wedge Y *(j))^{t} V(Y *(j)-X *(j) \wedge Y *(j)), \\
& j=1, \ldots, n \text { provided that } \rho \geq 0, \text { we have } \\
& \|Y *-X *\|^{2} \geq\|Y *-X * \wedge Y *\|^{2} \text {. Therefore, }
\end{aligned}
$$

$$
\begin{aligned}
\|\mathrm{X}+\mathrm{U}-\mathrm{X} *\|^{2}= & \|\mathrm{X}+\mathrm{U}-\mathrm{Y} *\|^{2}+\|\mathrm{Y} *-\mathrm{X} *\|^{2}+2(\mathrm{X}+\mathrm{U}-\mathrm{Y} *, \mathrm{Y} *-\mathrm{X} *) \\
\geq & \|\mathrm{X}+\mathrm{U}-\mathrm{Y} *\|^{2}+\|\mathrm{Y} *-\mathrm{X} * \wedge \mathrm{Y} *\|^{2} \\
& +2(\mathrm{X}+\mathrm{U}-\mathrm{Y} *, \mathrm{Y} *-\mathrm{X} * \wedge \mathrm{Y} *) \\
= & \|\mathrm{X}+\mathrm{U}-\mathrm{X} * \wedge \mathrm{Y} *\|^{2}
\end{aligned}
$$

and it follows

$$
\begin{aligned}
\|\mathrm{X}-\mathrm{X} *\|^{2} & =\|\mathrm{X}+\mathrm{U}-\mathrm{X} *\|^{2}-\mathrm{g}(\mathrm{X} *) \\
& \geq\|\mathrm{X}+\mathrm{U}-\mathrm{X} * \wedge \mathrm{Y} *\|^{2}-\mathrm{g}(\mathrm{X} * \wedge \mathrm{Y} *) \\
& =\|\mathrm{X}-\mathrm{X} * \wedge \mathrm{Y} *\|^{2} .
\end{aligned}
$$

By the fact that $\|X-X *\| \leq\|X-Z\|$ for each $Z \in M$ and $Z=X^{*}$ if the equality holds, we have $\mathrm{X}^{*} \wedge \mathrm{Y} *=\mathrm{X} *$ and hence $\mathrm{X} * \leq \mathrm{Y}$. This completes the proof. []

Corollary 5.3.1. If $\rho \geq 0$ and $a \leq V X \leq b$, then $\mathrm{V}^{-1} \mathrm{a} \leq \mathrm{P}(\mathrm{X} \mid \mathrm{M}, \mathrm{\rho}) \leq \mathrm{V}^{-1} \mathrm{~b}$ where $a$ and b are $2 \times 1$ column vectors.

An upper bound and a lower bound of $P(X \mid M, \rho)$ for $\rho \geq 0$ can be determined by the above corollary. Let $Y$ and $Z$ be $2 \times k$ and $2 \times(n-k)$ matrices defined by $Y(j)=X(j), j=1, \ldots, k$ and $Z(j)=X(k+j), j=1, \ldots, n-k \quad$ respectively. Let $M_{k}:=\left\{U \in R^{2 \times k}: u_{i j} \leq u_{i j+1}, i=1,2, j=1, \ldots, k-1\right\}$ for each
positive integer $k$. If there exists a $2 \times 1$ vector $c$ such that $P\left(Y \mid M_{k}, \rho\right) \leq c$ and $P\left(X \mid M_{n-k}, \rho\right) \geq c$, then by (5.2), (5.4) and (5.5), $P(X \mid M, \rho)$ can be obtained by considering the first $k$ components and the last $n-k$ components independently for each $\rho$ between -1 and 1 . From (2.2) $P(X \mid-M, \rho)=-P(-X \mid M, \rho)$, so if $\rho \geq 0$ and $a \leq V X \leq b$ then $V^{-1} a \leq P(X \mid-M, \rho) \leq V^{-1} b$. But it is not true in general that $X \geq 0$ implies $P(X \mid M, \rho) \geq 0$ for $\rho \geq 0$.

Theorem 5.4. If $\rho \leq 0$ and $a \leq X \leq b$, then $\mathrm{a} \leq \mathrm{P}(\mathrm{X} \mid \mathrm{M}, \rho) \leq \mathrm{b}$ where a and b are $2 \times 1$ vectors.

Proof. We shall show that if $X \geq 0$ then $P(X \mid M, \rho) \geq 0$ and $P(X \mid-M, \rho) \geq 0$. It will then follow that $X \geq c$ implies $P(X-c \mid M, \rho) \geq 0$ and $P(X-c \mid-M, \rho) \geq 0$ and hence $P(X \mid M, \rho) \geq c$ and $P(X \mid-M, \rho) \geq c$. By the identity $P(X \mid M, \rho)=-P(-X \mid-M, \rho), \quad$ if $X \leq b$, then $-X \geq-b$ and hence $-P(X \mid M, \rho)=P(-X \mid-M, \rho) \geq-b$. Therefore, if $a \leq X \leq b$ then $a \leq P(X \mid M, \rho) \leq b$ provided that $\rho \leq 0$.

Let $a$ and $\beta$ be non-negative real numbers and let $x$ and $y$ be real numbers. If $x \leq 0$, then

$$
\begin{aligned}
(a-x)^{2}+(\beta-y)^{2}-2 \rho(a-x)(\beta-y)= & \left(1-\rho^{2}\right)(a-x)^{2}+(\beta-y-a \rho+\rho x)^{2} \\
\geq & \left(1-\rho^{2}\right) a^{2}+(\beta-y-a \rho+\rho x)^{2} \\
= & \left(1-\rho^{2}\right) a^{2}+(\beta-y+\rho x)^{2}+\rho^{2} a^{2} \\
& -2 \rho a(\beta-y+\rho x) \\
= & a^{2}+(\beta-y+\rho x)^{2}-2 \rho a(\beta-y+\rho x)
\end{aligned}
$$

If also $y-\rho x<0$, then

$$
a^{2}+(\beta-y+\rho x)^{2}-2 \rho a(\beta-y+\rho x) \geq a^{2}+\beta^{2}-2 \rho a \beta
$$

Therefore

$$
(a-x, \beta-y) V(a-x, \beta-y)^{t} \geq(a, \beta-(y-\rho x) \vee 0) V(a, \beta-(y-\rho x) \vee 0)^{t}
$$

Similarly, if $y \leq 0$ then

$$
(a-x, \beta-y) V(a-x, \beta-y)^{t} \geq(a-(x-\rho y) \vee 0, \beta) V(a-(x-\rho Y) \vee 0, \beta)^{t}
$$

Let $Z \in M$ and let $k$ be the largest index between $l$ and $n$ such that $Z(k)$ is not non-negative. Let $Y$ be defined by $Y(j)=Z(j)$ for each $j>k$ and

$$
\begin{aligned}
& y_{l j}= \begin{cases}\left(z_{1 j}-\rho z_{2 j}\right) \vee 0 & \text { if } \\
z_{l k} \geq 0 \\
0 & \text { if } \\
z_{1 k}<0\end{cases} \\
& y_{2 j}= \begin{cases}0 & \text { if } \\
z_{l k} \geq 0 \\
\left(z_{2 j}-\rho z_{l j}\right) \vee 0 & \text { if } \\
z_{1 k}<0\end{cases}
\end{aligned}
$$

for $j=1, \ldots, k$. It is trivial that $Y \geq 0$ and $Y \in M$.
If $z_{1 k}<0$, then $z_{1 j}<0$ for $j=1, \ldots, k$. By the
inequality described above, if $X \geq 0$ then

$$
(X(j)-Z(j))^{t} V(X(j)-Z(j)) \geq(Z(j)-Y(j))^{t} V(X(j)-Y(j))
$$

$j=1, \ldots, n$. Therefore $\|X-Z\| \geq\|X-Y\|$. If $z_{1 k} \geq 0$, then
$\mathrm{z}_{2 \mathrm{k}}<0$ and similarly $\|\mathrm{X}-\mathrm{Z}\| \geq\|\mathrm{Z}-\mathrm{Y}\|$. It follows that
$P(X \mid M, \rho) \geq 0$. By a symmetric argument in the sense of reversal we have $P(X \mid-M, \rho) \geq 0$. $]$

Theorem 5.5. If $\rho \geq 0, x_{1 k-1} \geq x_{1 k}$ and $x_{1 k-1}{ }^{-x_{1 k}} \geq \rho\left(x_{2 k-1}-x_{2 k}\right)$ then $x_{1 k}^{*}=x_{1 k-1}^{*}$ where $X^{*}=P(X \mid M, \rho)$. Similarly if we interchange indices 1 and 2.

Proof. Suppose it were not true, i.e., $\quad x_{1 k}^{*}>x_{1 k-1}^{*}$. From (5.4) and (5.5), we have

$$
\begin{aligned}
& \Sigma_{h=k-1}^{n}\left(x_{l j}-\rho x_{2 j}\right) \leq \Sigma_{h=k-1}^{n}\left(x_{l j}^{*}-\rho x_{2 j}^{*}\right) \\
& \Sigma_{h=k}^{n}\left(x_{l j}-\rho x_{2 j}\right)=\Sigma_{h=k}^{n}\left(x_{l j}^{*}-\rho x_{2 j}^{*}\right)
\end{aligned}
$$

and

$$
\Sigma_{h=k+1}^{n}\left(x_{1 j}-\rho x_{2 j}\right) \leq \Sigma_{h=k+1}^{n}\left(x_{l j}^{*}-\rho x_{2 j}^{*}\right) \quad \text { if } \quad k+1 \leq n .
$$

Therefore $\mathrm{x}_{1 \mathrm{k}-1}-\rho \mathrm{x}_{2 \mathrm{k}-1} \leq \mathrm{x}_{1 \mathrm{k}-1}^{*}-\rho \mathrm{x}_{2 \mathrm{k}-1}^{*}$ and $x_{1 k}-\rho x_{2 k} \geq x_{1 k}^{*}-\rho x_{2 k}^{*}$.

$$
\begin{aligned}
& \text { Consider the case } \begin{aligned}
& x_{2 k}^{*}=x_{2 k-1}^{*} \text {. We have } \\
x_{1 k}-\rho x_{2 k} & \geq x_{l k}^{*}-\rho x_{2 k}^{*} \\
& >x_{l k-1}^{*}-\rho x_{2 k-1}^{*} \\
\geq & x_{1 k-1}-\rho x_{2 k-1}
\end{aligned}
\end{aligned}
$$

and hence $\rho\left(x_{2 k-1}-x_{2 k}\right)>x_{1 k-1}-x_{1 k}$. This contradicts the assumption that $x_{1 k-1}-x_{1 k} \geq \rho\left(x_{2 k-1}-x_{2 k}\right)$.

Consider the alternative $x_{2 k}^{*}>x_{2 k-1}^{*}$. Then we have $x_{2 k-1}-\rho x_{l k-1} \leq x_{2 k-1}^{*}-\rho x_{1 k-1}^{*}$ and $\quad x_{2 k}-\rho x_{1 k} \geq x_{2 k}^{*}-\rho x_{1 k}^{*}$. Since $\quad x_{1 k-1}-\rho x_{2 k-1} \leq x_{1 k-1}^{*}-\rho x_{2 k-1}^{*}$ and $x_{1 k}-\rho x_{2 k} \geq x_{1 k}^{*}-\rho x_{2 k}^{*}, \quad$ it follows that

$$
\mathrm{VX}(\mathrm{k}-1) \leq \mathrm{VX} *(\mathrm{k}-1) \quad \text { and } \quad \mathrm{VX} *(\mathrm{k}) \leq \mathrm{VX}(\mathrm{k})
$$

and hence $\quad x_{l k} \geq x_{l k}^{*}>x_{l k-1}^{*} \geq x_{l k-1}$. This contradicts the assumption that $\quad x_{1 k-1} \geq x_{1 k}$.

If $\rho \geq 0$, the index $(1, k)$ or $(2, k)$ with $k>1$ satisfying both inequalities described in the above theorem is said to be pivotal. An algorithm based on the pivotal indices will be given in the next section.

Theorem 5.6. If $\rho \leq 0$ and $x_{1 k-1}-x_{1 k} \geq \rho\left(x_{2 k-1}-x_{2 k}\right)$ then $x_{l k}^{*}=x_{l k-1}^{*}$ where $\quad X *=P(X \mid M, \rho)$. Similarly if we
interchange indices 1 and 2.

Proof. By the same argument as in the proof of Theorem 5.5, if $x^{*}{ }_{1 k}>x_{1 k-1}^{*}$, then we have $x_{1 k-1}-\rho x_{2 k-1} \leq x_{1 k-1}^{*}-\rho x_{2 k-1}^{*}$ and $x_{1 k}-\rho x_{2 k} \geq x_{1 k}^{*}-\rho x_{2 k}^{*}$. Therefore,

$$
\begin{aligned}
x_{1 k}-\rho x_{2 k} & \geq x_{1 k}^{*}-\rho x_{2 k}^{*} \\
& >x_{1 k-1}^{*}-\rho x_{2 k-1}^{*} \\
& \geq x_{1 k-1}-\rho x_{2 k-1}
\end{aligned}
$$

and hence $\rho\left(x_{2 k-1}-x_{2 k}\right)>x_{1 k-1}-x_{1 k}$. This contradicts the assumption. $]$

If $\rho \leq 0$, the index $(1, k)$ or $(2, k)$ with $k>1$ satisfying the inequality described in the above theorem is said to be pivotal. Let $\rho \geq 0$. If $x_{1 k-1} \geq x_{1 k}$ or $x_{2 k-1} \geq x_{2 k}$, then at least one of $(1, k)$ and $(2, k)$ is pivotal. But if $x_{1 k-1} \geq x_{1 k}$ and $x_{2 k-1} \geq x_{2 k}$, it is not necessary that both $(1, k)$ and $(2, k)$ are pivotal. Therefore, whenever there is a violation, i.e., $x_{i k-1} \geq x_{i k}$, there is a pivotal element. Let $\rho \leq 0$. If $x_{1 k-1} \geq x_{1 k}$ or $x_{2 k-1} \geq x_{2 k}$, it is not necessary that $(1, k)$ or $(2, k)$ is pivotal. But is $x_{1 k-1} \geq x_{1 k}$ and $x_{2 k-1} \geq x_{2 k}$, then both $(1, k)$ and $(2, k)$ are pivotal. In some cases, we may have violations but there is no pivotal element.

## V. 3 Simplified Projection

The bivariate isotonic regression of $X$ is the generalized projection of $X$ to the closed convex cone $M$. By Theorem 2.13, $P(X \mid M, \rho)$ can be obtained as the projection to a linear space which is generated by a subset of $\left\{1_{i j}: i=1,2 ; j=1, \ldots, n\right\}$. Therefore, how to obtain the projection to such a linear space in our present structure is our primary work.

Let $\Gamma$ be a subindex set of $\{(i, j): i=1,2 ; j=1, \ldots, n\}$, let $\Gamma_{i}=\{j:(i, j) \in \Gamma\}, i=1,2$ and let $S_{\Gamma}$ be the linear space generated by $\left\{1_{i j}:(i, j) \in \Gamma\right\}$, i.e.,

$$
S_{\Gamma}=\left\{Y: Y=\Sigma_{j \in \Gamma_{1}} \beta_{1 j}{ }^{1} 1 j+\Sigma_{j \in \Gamma_{2}} \beta_{2 j} l_{2 j}, \quad \beta_{i j} \text { real }\right\} .
$$

For convenience, in the following three paragraphs we denote $X=\left(X_{1}, X_{2}\right)$ as a $2 n$-component row vector and similarly for $Y, Z$ and $l_{i j}$ 's. Let $T$ be a $2 n \times 2 n$ positive definite matrix which is the tensor product of $V$ and $I_{n}$, i.e., $t_{i i}=1, t_{\text {in }+i}=-\rho$, $t_{i j}=t_{i n+j}=0$ for $i, j=1, \ldots, n$ with $j \neq i$ and $t_{n+i, i}=-\rho$, $t_{n+i, n+i}=1, t_{n+i, n+j}=t_{n+i, j}=0$ for $i, j=1, \ldots, n$ with $j \neq i$. Let $\Gamma_{1}$ have $p$ elements, $\Gamma_{2}$ have $q$ elements and let $A$ be the $(\mathrm{p}+\mathrm{q}) \times 2 \mathrm{n}$ matrix which is composed of $1_{\mathrm{ij}},(\mathrm{i}, \mathrm{j}) \in \Gamma$, such that the first $p$ rows are ${ }^{1}{ }_{1 j_{1}}, l_{1 j_{2}}, \ldots, l_{1 j_{p}}$ with
$\mathrm{j}_{1}<\mathrm{j}_{2}<\ldots<\mathrm{j}_{\mathrm{p}}$ and the last q rows are $\mathrm{l}_{2 \mathrm{k}_{1}}, \mathrm{l}_{2 \mathrm{k}_{2}}, \ldots, \mathrm{l}_{2 \mathrm{k}_{\mathrm{q}}}$ with $k_{1}<k_{2}<\ldots<k_{q}$. Every row vector $Y$ in $S_{\Gamma}$ is of the form $Y=U A$ where $U$ is a $p+q$-component row vector.

Let us define an inner product by

$$
(X, Z)=X T Z^{t}
$$

for each pair of row vectors $X$ and $Z$. The quantity $\|X-Z\|^{2}=(X-Z, X-Z)$ is the same as $f(Z)$ in (5.3). The projection of $X$ to the linear space $S_{\Gamma}$ is the product of $X$ and the $2 n \times 2 n$ matrix $P$ where $P=T A^{t}\left(A T A^{t}\right)^{-1} A$. To verify the above result, we shall show $(X-X P, Y)=0$ for each $Y \in S_{\Gamma}$. Let $U$ be the $p+q$ row vector such that $U A=Y$. It is trivial that the matrix A has rank $p+q$ and $A T A^{t}$ is invertible. Therefore

$$
\begin{aligned}
(X-X P, Y) & =(X-X P) T Y^{t} \\
& =X T A^{t} U^{t}-X T A^{t}\left(A T A^{t}\right)^{-1} A T A^{t} U^{t} \\
& =X T A^{t} U^{t}-X T A^{t} U^{t} \\
& =0 .
\end{aligned}
$$

The $(p+q) \times 2 n$ matrix $A$ can be written as

$$
A=\left[\begin{array}{ll}
C_{1} & 0 \\
0 & C_{2}
\end{array}\right]
$$

where $C_{1}$ is a $p \times n$ matrix with rank $p$ and $C_{2}$ is a $q \times n$ matrix with rank $q$. The $(p+q) x(p+q)$ matrix $A T A^{t}$ is

$$
A T A^{t}=\left[\begin{array}{cc}
C_{1} C_{1}^{t} & -\rho C_{1} C_{2}^{t} \\
-\rho C_{2} C_{1}^{t} & C_{2} C_{2}^{t}
\end{array}\right]
$$

Let $U_{1}$ and $U_{2}$ be $p$-component and $q$-component row vectors respectively auth that

$$
\left[\begin{array}{ll}
\mathrm{U}_{1} & \mathrm{U}_{2}
\end{array}\right]=\mathrm{XTA}^{\mathrm{t}}\left(\mathrm{ATA}^{\mathrm{t}}\right)^{-1}
$$

Then $\left[\begin{array}{lll}U_{1} C & U_{2} & C_{2}\end{array}\right]=X P$. Since $\left[\begin{array}{ll}U_{1} & U_{2}\end{array}\right] A T A^{t}=X T A{ }^{t}$, we have

$$
\left[\begin{array}{ll}
U_{1} & U_{2}
\end{array}\right]\left[\begin{array}{cc}
I_{p} & -\rho C_{1} C_{2}^{t} \\
-\rho C_{2} C_{1}^{t}\left(C_{1} C_{1}^{t}\right)^{-1} & C_{2} C_{2}^{t}
\end{array}\right]=X T A^{t}\left[\begin{array}{cc}
\left(C_{1} C_{1}^{t}\right)^{-1} & 0 \\
0 & I_{q}
\end{array}\right]
$$

and

$$
\begin{aligned}
& {\left[\begin{array}{cc}
U_{1} & \left.U_{2}\right]\left[\begin{array}{cc}
I_{p} & 0 \\
-\rho C_{2} C_{1}^{t}\left(C_{1} C_{1}^{t}\right)^{-1} & C_{2} C_{2}^{t}-\rho^{2} C_{2} C_{1}^{t}\left(C_{1} C_{1}^{t}\right)^{-1} C_{1} C_{2}^{t}
\end{array}\right] \\
= & X T A^{t}\left[\begin{array}{cc}
\left(C_{1} C_{1}^{t}\right)^{-1} & 0 \\
0 & I_{q}
\end{array}\right]\left[\begin{array}{cc}
I_{p} & \rho C_{1} C_{2}^{t} \\
0 & I_{q}
\end{array}\right] \\
= & {\left[\left(X_{1}-\rho X_{2}\right) C_{1}^{t}\right.} \\
\left.\left(X_{2}-\rho X_{1}\right) C_{2}^{t}\right]\left[\begin{array}{cc}
\left(C_{1} C_{1}^{t}\right)^{-1} & \rho\left(C_{1} C_{1}^{t}\right)^{-1} C_{1} C_{2}^{t} \\
0 & I_{q}
\end{array}\right]
\end{array} .\left\{\begin{array}{cc}
\end{array}\right]\right.}
\end{aligned}
$$

Let $B_{1}=C_{1}^{t}\left(C_{1} C_{1}^{t}\right)^{-1} C_{1}$. Then

$$
\begin{equation*}
U_{2}\left(C_{2} C_{2}^{t}-\rho^{2} C_{2} B_{1} C_{2}^{t}\right)=\left(X_{2}-\rho X_{1}\right) C_{2}^{t}+\rho\left(X_{1}-\rho X_{2}\right) B_{1} C_{2}^{t} \tag{5.6}
\end{equation*}
$$

and

$$
U_{1}-\rho U_{2} C_{2} C_{1}^{t}\left(C_{1} C_{1}^{t}\right)^{-1}=\left(X_{1}-\rho X_{2}\right) C_{1}^{t}\left(C_{1} C_{1}^{t}\right)^{-1}
$$

It follows that

$$
\begin{equation*}
(\mathrm{XP})_{2}=\mathrm{U}_{2} \mathrm{C}_{2} \tag{5.7}
\end{equation*}
$$

and

$$
\begin{equation*}
(X P)_{1}=\left(X_{1}-\rho X_{2}+\rho(X P)_{2}\right) B_{1} \tag{5.8}
\end{equation*}
$$

Similarly if we interchange indices 1 and 2.
Let $\Gamma_{1}=\left\{j_{1}, j_{2}, \ldots, j_{p}\right\}$ and let $Y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ be a row vector. Then $\mathrm{YC}_{1}^{t}=\left(a_{1}, \ldots, a_{p}\right)$ such that $a_{k}=\Sigma_{h=j_{k}}^{n} y_{h}$, $k=1, \ldots, p$, and $Y B_{1}=\left(\beta_{1}, \ldots, \beta_{n}\right)$ such that $\beta_{j}=\sum_{h=j_{k}}^{j_{k+1}^{-1}} y_{h} /\left(j_{k+1}^{-j_{k}}\right), \quad k=0,1, \ldots, p+1 \quad$ where $\quad j_{k} \leq j<j_{k+1}$,
$j_{0}=1$ and $j_{p+1}=n+1$. Similarly if we interchange indices 1 and 2.

Suppose $k \in \Gamma_{1} \frown \Gamma_{2}$ and $k>1$. Let us define

$$
S_{1}=\left\{Y: Y=\Sigma_{i=1}^{2} \Sigma_{j \in \Gamma_{i}} \beta_{i j}\left(1_{i j}^{-1} l_{i k}\right), \quad \beta_{i j} \text { real }\right\}
$$

and

$$
S_{2}=\left\{Y: Y=\Sigma_{i=1}^{2} \Sigma_{j \in \Gamma_{i}} \beta_{i j}{ }^{1}{ }_{i j}, \quad \beta_{i j} \text { real }\right\}
$$

Then $S_{1}$ and $S_{2}$ are orthogonal and $S_{1}+S_{2}=S_{\Gamma}$. It follows that

$$
\begin{equation*}
P\left(X \mid S_{\Gamma}, \rho\right)=P\left(X \mid S_{1}, \rho\right)+P\left(X \mid S_{2}, \rho\right) \tag{5.9}
\end{equation*}
$$

In other words, $P\left(X \mid S_{\Gamma}, \rho\right)$ can be obtained by considering the first $\mathrm{k}-1$ components and the last $\mathrm{n}-\mathrm{k}+\mathrm{l}$ components independently.

Suppose $\Gamma_{2}=\{1\}$ and $1 \in \Gamma_{1}$. Then $C_{2}$ is a row vector with each entry having value one, $B_{1} C_{2}^{t}=C_{2}^{t}, C_{2} C_{2}^{t}=n$ and $U_{2}$ is a real number. Thus (5.6) is an $=\Sigma_{j=1}^{n} x_{2 j}$ and $X_{2}^{*}=a C_{2}$, i.e., $x_{21}^{*}=x_{22}^{*}=\ldots=x_{2 n}^{*}=\Sigma_{j=1}^{n} x_{2 j} / n$.

Let $\quad X=\Sigma_{i=1}^{2} \Sigma_{j=1}^{n} a_{i j}{ }_{i j}, \quad X *=\Sigma_{i=1}^{2} \Sigma_{j=1}^{n} a_{i j}^{*}{ }_{i j} \quad$ and let $\Lambda=\left\{(\mathrm{i}, \mathrm{j}): \mathrm{a}_{\mathrm{ij}}^{*}>0\right\} \cup\{(1,1),(2,1)\}$ where $\quad X *=P(X \mid M, \rho)$. Theorem 5.5 shows that if $\rho \geq 0, a_{1 k} \leq 0$ and $a_{1 k} \leq \rho a_{2 k}$ for some $k>1$, then $a_{l k}^{*}=0,(1, k) \nmid \Lambda$ and $(1, k)$ is pivotal. Similarly if we interchange indices 1 and 2 . Whenever $X \notin M$, there is at least one pivotal element. Theorem 5.6 shows that if $\rho<0$ and $a_{1 k} \leq \rho a_{2 k}$ for some $k>1$, then $a_{1 k}^{*}=0,(1, k) \nmid \Lambda$ and $(1, k)$ is pivotal. Similarly if we interchange indices 1 and 2. If $X \notin M$, we may or may not have pivotal elements.

Let $\Lambda^{0}$ be the collection of all non-pivotal elements. Then $\Lambda^{0} \supset \Lambda$. Since $P(X \mid M, \rho)=P\left(X \mid S_{\Lambda}, \rho\right)$ as indicated in Theorem 2.13 and (5.2), by the smoothing property we have
$P(X \mid M, \rho)=P\left(P\left(X \mid S_{\Lambda 0}, \rho\right) \mid S_{\Lambda}, \rho\right)$. Let $\quad X^{1}=P\left(X \mid S_{\Lambda^{0}}, \rho\right)$. If $X^{1} \in M$, then by Theorem 2.8 and Theorem 2.11 we have $X^{1}=P(X \mid M, \rho)$. Otherwise, write $\quad X^{1}=\Sigma_{i=1}^{2} \Sigma_{j=1}^{n} a_{i j}^{1}{ }_{1}{ }_{i j}$ and let $\Lambda^{1}$ be the set of non-pivotal elements in $\Lambda$ with respect to $X^{1}$. Since $P\left(X^{1} \mid M_{\Lambda}, \rho\right)=X *, \quad \Lambda^{0} \supset \Lambda^{1} \supset \Lambda$. If $\rho \geq 0$, then $\Lambda^{0} \neq \Lambda^{1}$. Applying the above procedure inductively, we shall terminate at a positive integer $k$ such that $X^{k} \in M$ and $X^{k}=X *$. The projection $P\left(X^{h} \mid S_{\Lambda h}, \rho\right)$ for $h=0,1, \ldots, k-1$ can be obtained by (5.6), (5.7), (5.8) and (5.9). Such an algorithm for $\rho>0$ is called the Simplified Projection (cf. Appendix V).

The monotone decreasing sequence $\left\{\Lambda^{h}\right\}$ can be replaced by a monotone decreasing sequence $\left\{\Gamma^{h}\right\}$ such that $\Gamma^{h} \supset \Lambda^{h}$. The purpose for the presence of $\Gamma^{h}$ is that we may use (5.9) more efficiently and hence the order of the simultaneous linear equations (5.6) can be reduced significantly. Such a device may be helpful when we use desk calculators.

If $\rho<0$, we may possibly obtain the bivariate isotonic regression of $X$ by the Simplified Projection algorithm. However, if there exists an $k$ such that $x^{k} \notin M$ and $\Lambda^{k-1}=\Lambda^{k}$, then the Simplified Projection algorithm will fail to yield the bivariate isotonic regression of $X$. At this stage, we may choose any one of the 2 xn matrices $\mathrm{X}, \mathrm{X}^{1}, \mathrm{X}^{2}, \ldots, \mathrm{X}^{k}$ as our given data. For each h
between one and $k$, the problem of obtaining $P\left(X^{h} \mid M_{\Lambda^{h}}, \rho\right)$ is the same problem of obtaining $P(X \mid M, \rho)$ for $P\left(\left.X^{h}\right|_{\Lambda^{h}}, \rho\right)=P(X \mid M, \rho)$.

Example 5.1. The data $\mathbf{X}$ given below is a portion of that from Bhattacharyya and Kotz (1966).

$$
\mathbf{X}=\left[\begin{array}{cccccc}
25 & 13 & 2 & 15 & 14 & 21 \\
57 & 36 & 77 & 89 & 76 & 62
\end{array}\right]
$$

Let $\rho=-0.1$. By Theorem 5.6, the set of non-pivotal indices $\Lambda^{0}$ is $\{(1,1),(1,4),(1,6),(2,1),(2,3),(2,4)\}$. Since $4 \in \Lambda_{1}^{0} \cap \Lambda_{2}^{0}$ where $\Lambda_{i}^{0}=\left\{j:(i, j) \in \Lambda^{0}\right\}$, we may consider the first three elements and the last three elements independently as shown in (5.9). Let $\Gamma=\left\{(i, j):(i, j) \in \Lambda^{0}, j \leq 3\right\}$. Then $\Gamma_{1}=\{1\}$ and $l \in \Gamma_{2}$. So $x_{11}^{1}=x_{12}^{1}=x_{13}^{1}=(25+13+2) / 3=13.333$. $X_{2}-\rho X_{1}+\rho(X P)_{2}=(58.167,35.967,75.867)$ for the first three elements. By (5.8) we take the average for the first two elements of $X_{2}-\rho X_{1}+\rho(X P)_{2}$ and the first three elements of $X^{l}$ are

$$
\left[\begin{array}{lll}
13.333 & 13.333 & 13.333 \\
47.067 & 47.067 & 75.867
\end{array}\right]
$$

Similarly, we have the last three elements of $\mathrm{X}^{1}$. Therefore

$$
X^{1}=\left[\begin{array}{llllll}
13.333 & 13.333 & 13.333 & 15.183 & 15.183 & 19.633 \\
47.067 & 47.067 & 75.867 & 75.667 & 75.667 & 75.667
\end{array}\right]
$$

At this stage, $\Lambda^{1}$ is $\{(1,1),(1,4),(1,6),(2,1),(2,3)\}$ and the equation (5.6) is

$$
\left[\begin{array}{ll}
5.94 & 3.96 \\
3.96 & 3.9667
\end{array}\right]\left[\begin{array}{l}
\beta_{1} \\
\beta_{2}
\end{array}\right]=\left[\begin{array}{l}
393.03 \\
300.03
\end{array}\right]
$$

The solution to the above linear equations is $\beta_{1}=47.066$ and $\beta_{2}=28.651$. By (5.7) and (5.8), we have

$$
x^{2}=\left[\begin{array}{llllll}
13.338 & 13.338 & 13.338 & 15.178 & 15.178 & 19.628 \\
47.066 & 47.066 & 75.717 & 75.717 & 75.717 & 75.717
\end{array}\right]
$$

where $\quad X^{2}=P\left(X^{1} \mid S_{\Lambda}, \rho\right)$. Since $X^{2} \in M, X^{2}$ is $P(X \mid M, \rho)$ because

$$
\begin{aligned}
P(X \mid M, \rho) & =P\left(X \mid M_{\Lambda^{1}}, \rho\right)=P\left(P\left(P\left(X \mid S_{\Lambda^{0}}, \rho\right) \mid S_{\Lambda^{1}, p}\right) \mid M_{\Lambda^{1}}, p\right) \\
& =P\left(P\left(X^{1} \mid S_{\Lambda^{1}}, \rho\right) \mid M_{\Lambda^{1}}, \rho\right)=P\left(X^{2} \mid M_{\Lambda^{1}}, \rho\right)=X^{2}
\end{aligned}
$$

The projection $P\left(X^{1} \mid M_{\Lambda^{1}}, \rho\right)$ is $X^{2}$, but generally $P\left(X^{1} \mid M, \rho\right)$ is not $X^{2}$. Consider $X^{1}$. We have

$$
X^{1}(1)=X^{1}(2)=\left[\begin{array}{l}
13.333 \\
47.067
\end{array}\right] \leq X^{1}(j) \quad j=3,4,5,6
$$

By Theorem 5.4, $P\left(X^{1} \mid M, \rho\right)$ can be obtained by considering the first two elements of $X^{l}$ and the last four elements of $X^{l}$ independently. Consider the last four elements of $X^{1}$. By the Pool-Adjacent-Violators algorithm, we have $P\left(X_{2}^{1} \mid M_{2}\right)$ is the constant 75.717. The 4-component row vector $X_{1}^{1}-\rho X_{2}^{1}$ is monotone increasing. By Theorem 5.2, $\quad Y_{2}=P\left(X_{2}^{l} \mid M_{2}\right)$ and $Y_{1}=X_{1}^{l}+\rho\left(Y_{2}-Y_{2}^{l}\right)$ where $Y=P\left(X^{1} \mid M, \rho\right)$. Therefore,

$$
P\left(X^{1} \mid M, \rho\right)=\left[\begin{array}{llllll}
13.333 & 13.333 & 13.348 & 15.178 & 15.178 & 19.628 \\
47.067 & 47.067 & 75.717 & 75.717 & 75.717 & 75.717
\end{array}\right]
$$

## V. 4 Approximation

Let $X$ be a given $2 \times n$ matrix and let $X *=P(X \mid M, \rho)$. The process introduced here is that we may obtain a region $A(j)$ such that $X *(j) \in A(j), j=1, \ldots, n$ by the Pool-AdjacentViolators algorithm. But the process can be only applied to the case $\rho \geq 0$.

$$
\begin{aligned}
& \quad \text { Let } a_{1}=\min \left\{x_{1 j}-\rho x_{2 j}: j=1, \ldots, n\right\}, \\
& a_{2}=\max \left\{x_{1 j}-\rho x_{2 j}: j=1, \ldots, n\right\}, \beta_{1}=\min \left\{x_{2 j}-\rho x_{l j}: j=1, \ldots, n\right\}, \\
& \beta_{2}=\max \left\{x_{2 j}-\rho x_{l j}: j=1, \ldots, n\right\} \text { and let } 2 x n \text { matrices } x, Y, T \\
& \text { and } U \text { be defined by }
\end{aligned}
$$

(5. 10)

$$
\begin{aligned}
& \left\{\begin{array}{l}
z_{l j}-\rho z_{2 j}=x_{l j}-\rho x_{2 j} \\
z_{2 j}-\rho z_{l j}=\beta_{l}
\end{array}\right. \\
& \left\{\begin{array}{l}
y_{l j}-\rho y_{2 j}=x_{l j}-\rho x_{2 j} \\
y_{2 j}-\rho y_{l j}=\beta_{2}
\end{array}\right. \\
& \left\{\begin{array}{l}
t_{l j}-\rho t_{2 j}=a_{l} \\
t_{2 j}-\rho t_{l j}=x_{2 j}-\rho x_{l j}
\end{array}\right.
\end{aligned}
$$

and

$$
\left\{\begin{array}{l}
u_{1 j}-\rho u_{2 j}=a_{2} \\
u_{2 j}-\rho u_{l j}=x_{2 j}-\rho x_{2 j}
\end{array}\right.
$$

for $j=1, \ldots, n$. By the structure, $\quad Z_{2}-\rho Z_{1}, Y_{2}-\rho Y_{1}, T_{1}-\rho T_{2}$
and $U_{1}-\rho U_{2}$ are constant and hence monotone increasing. Theorem 5. 1 shows that if $\rho \geq 0$, then their bivariate isotonic regressions $\mathrm{Z} *, \mathrm{Y} *, \mathrm{~T} *$ and $\mathrm{U} *$ can be obtained by the Pool-Adjacent-Violators algorithm.

$$
\text { For each } j, \quad V Z(j) \leq V X(j) \leq V Y(j) \text { and }
$$

$V T(j) \leq V X(j) \leq V U(j)$. If $\rho \geq 0$, then by Theorem 5.3 we have

$$
\mathrm{Z} *(\mathrm{j}) \leq \mathrm{X} *(\mathrm{j}) \leq \mathrm{Y} *(\mathrm{j})
$$

and

$$
T *(j) \leq X *(j) \leq U *(j)
$$

where $X *$ is the bivariate isotonic regression of $X$. Therefore

$$
\begin{equation*}
Z *(j) \vee T *(j) \leq X *(j) \leq Y *(j) \wedge U *(j) . \tag{5.11}
\end{equation*}
$$

When $\rho=0$, Theorem 5.1 shows that $Z_{1}^{*}=Y_{1}^{*}=P\left(X_{1} \mid M_{1}\right)$,
$Z_{2}^{*}=\beta_{1}, \quad Y_{2}^{*}=\beta_{2}, T_{1}^{*}=a_{1}, U_{1}^{*}=a_{2} \quad$ and $\quad T_{2}^{*}=U_{2}^{*}=P\left(X_{2} \mid M_{2}\right)$. It follows that $Z *(j) \vee T *(j)=Y *(j) \wedge U *(j)$ for $j=1, \ldots, n$. If $\rho>0$ and it is small enough, then the region $A(j)$ determined by (5.11) is small for each $j=1, \ldots, n$. In such a situation, a good approximation can be obtained.
(5.3) shows that $x_{1 n}-\rho x_{2 n} \leq x_{1 n}^{*}-\rho x_{2 n}^{*}$ and $x_{2 n}-\rho x_{1 n} \leq x_{2 n}^{*}-\rho x_{1 n}^{*}$. Combining (5.2) and (5.3), we have $x_{11}-\rho x_{21} \geq x_{11}^{*}-\rho x_{21}^{*}$ and $x_{21}-\rho x_{11} \geq x_{21}^{*}-\rho x_{11}^{*}$. Therefore, no matter whether $\rho \geq 0$ or $\rho<0, V X(1) \geq V X *(1)$ and $\operatorname{VX}(n) \leq V X *(n)$. For each $i, x_{i 1}^{*} \leq x_{i n}^{*}$ and $\Sigma_{j=1}^{n} x_{i j}=\sum_{j=1}^{n} x_{i j}^{*}$, it follows that $\mathrm{x}_{\mathrm{i} 1}^{*} \leq \overline{\mathrm{x}}_{\mathrm{i}} \leq \mathrm{x}_{\mathrm{in}}^{*}$ where $\overline{\mathrm{x}}_{\mathrm{i}}=\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{ij}} / \mathrm{n}$.

Example 5.2. Let $X$ be the $2 \times 5$ matrix given below and let $\rho=1 / 2$.

$$
X=\left[\begin{array}{lllll}
1 & 1 & 2 & 2 & 4 \\
0 & 1 & 3 & 2 & 1
\end{array}\right]
$$

Let $X *$ be the bivariate isotonic regression of $X$. We would guess that $X *(1)=X(1)$. Consider the last four elements of $X$. If the regions $A(2), A(3), A(4)$ and $A(5)$ determined by (5.11) for the last four elements are such that $a \in A(j)$ implies $a \geq X(1)$,
$j=2,3,4,5$, then our conjecture $X *(1)=X(1)$ has been verified.
Let $Z, Y, T$ and $U$ be the $2 \times 4$ matrices defined by (5.10) with respect to the last four components of $X$. They are

$$
\begin{aligned}
& Z=1 / 3\left[\begin{array}{cccc}
0 & 0 & 2 & 12 \\
-3 & -3 & -2 & 3
\end{array}\right] \\
& Y=1 / 3\left[\begin{array}{cccc}
6 & 6 & 8 & 18 \\
9 & 9 & 10 & 15
\end{array}\right] \\
& T=1 / 3\left[\begin{array}{cccc}
3 & 6 & 4 & 0 \\
3 & 9 & 5 & -3
\end{array}\right]
\end{aligned}
$$

and

$$
U=1 / 3\left[\begin{array}{rrrr}
15 & 18 & 16 & 12 \\
5 & 15 & 11 & 3
\end{array}\right]
$$

Let $Z *, Y *, T *$ and $U *$ be the bivariate isotonic regressions of $\mathrm{X}, \mathrm{Y}, \mathrm{T}$ and U respectively. Then $\mathrm{Z} *=\mathrm{Z}, \mathrm{Y} *=\mathrm{Y}$,

$$
T *=1 / 9\left[\begin{array}{llll}
9 & 10 & 10 & 10 \\
9 & 11 & 11 & 11
\end{array}\right]
$$

and

$$
\mathrm{U} *=1 / 9\left[\begin{array}{llll}
45 & 46 & 46 & 46 \\
27 & 29 & 29 & 29
\end{array}\right] .
$$

Therefore, $\mathrm{A}(2)=\left\{(a, \beta)^{t}: 1 \leq a \leq 2,1 \leq \beta \leq 3\right\}$,
$A(3)=\left\{(a, \beta)^{t}: 10 / 9 \leq a \leq 2,11 / 9 \leq \beta \leq 3\right\}$,
$A(4)=\left\{(a, \beta)^{t}: 10 / 9 \leq a \leq 8 / 3,11 / 9 \leq \beta \leq 29 / 9\right\}$ and
$A(5)=\left\{(a, \beta)^{t}: 4 \leq a \leq 46 / 9,11 / 9 \leq \beta \leq 29 / 9\right\}$. It follows that
$X *(1)=X(1)$.
For the remaining four elements, we have $V X *(2) \leq \operatorname{VX}(2)$. In other words $X *(2) \leq X(2)$. Since $X *(2) \in A(2)$ and $X(2) \leq a$ for each $a \in A(2)$, we have $X *(2)=X(2)$. In the remaining three elements of $X, X_{1}-\rho X_{2}$ is monotone increasing. By Theorem 5.1, we can obtain the bivariate isotonic regression for the last three elements. Therefore,

$$
X *=\left[\begin{array}{llccc}
1 & 1 & 3 / 2 & 2 & 9 / 2 \\
0 & 1 & 2 & 2 & 2
\end{array}\right] \cdot \square
$$

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APPENDICES

## APPENDIX I

(Daily Maximum) Temperatures Measured (in Fahrenheit) at Oregon State University in 1974
(cf. Section I. 2)


## APPENDIX II

Combined Explosive Rates (cf. Section I. 2)
$r_{j}$ explosive rate of ten samples dropped at the $j$ th height. $r_{1}=0.30, r_{2}=0.20, r_{3}=0.70, r_{4}=0.80$ and $r_{5}=0.50$. $Q(\mathrm{i}, \mathrm{j})$ explosive rate if $10 \mathrm{x}|\mathrm{j}-\mathrm{i}+\mathrm{l}|$ samples were dropped at the jth height which were dropped at heights between the ith height and the jth height.


Table of $q_{i j}{ }^{\prime} s$ (and $Q(i, j)$ 's)

|  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | height | 1 | 2 | 3 | 4 | 5 |
| 1 | 0.30 | $0.25-$ | $0.40-$ | $0.50-$ | $0.50-$ |  |
|  | 2 | $0.25+$ | 0.20 | $0.45-$ | $0.57-$ | $0.55-$ |
|  | 3 | $0.40+$ | $0.45+$ | 0.70 | $0.75-$ | $0.67-$ |
| 4 | $0.50+$ | $0.57+$ | $0.75+$ | 0.80 | 0 | $65-$ |
|  | 5 | $0.50+$ | $0.55+$ | $0.67+$ | $0.65+$ | 0.50 |

$Q(i, j) \geq q_{i j} \quad$ if $\quad i \leq j, \quad Q(i, j) \leq q_{i j} \quad$ otherwise.

## APPENDIX III

## MLE of Variance Components

 (cf. Section I. 2)$$
y_{i j k}=\mu+a_{i}+b_{j}+c_{i j}+e_{i j k} \quad i=1, \ldots, r ; j=1, \ldots, s ;
$$

$k=1, \ldots, t$ where $\left\{a_{i}\right\},\left\{b_{j}\right\},\left\{c_{i j}\right\}$ and $\left\{e_{i j k}\right\}$ are mutually independent sets of normal variates with variances $\sigma_{A}^{2}, \sigma_{B}^{2}, \sigma_{A B}^{2}$ and $\sigma^{2}$ and with means zero. Let $y$ be the rst-component column vector which is composed of $\left\{y_{i j k}\right\}$. Let

$$
\begin{aligned}
L & =\text { likelihood function } \\
& =(2 \pi)^{-\frac{1}{2} r s t}(\operatorname{det}(\Delta))^{-\frac{1}{2}} \exp \left\{-\frac{1}{2}(y-\mu)^{t} \Delta^{-1}(y-\mu)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda & =-2 \log (L) \\
& =(y-\mu)^{t} \Delta^{-1}(y-\mu)+\log \operatorname{det}(\Delta)+\text { constant }
\end{aligned}
$$

where $\Delta=\operatorname{Cov}(y, y)$ Let $z$ be the rst-component column vector which is composed of

$$
\begin{aligned}
& \text { y... = a. }+\mathrm{b} .+\mathrm{c} .+\mathrm{e} \ldots \\
& y_{i} \ldots-y \ldots=a_{i}-a+c_{i .}-c \ldots+e_{i .}-e \ldots \quad i=1, \ldots, r-1
\end{aligned}
$$

$$
\begin{aligned}
& y_{i j .}-y_{i \ldots} \ldots,{ }^{-y} .{ }^{-y} \ldots=c_{i j}-c_{i .}-c_{. j}+c_{\ldots}+e_{i j .}-e_{i .}-e_{. j}+e_{\ldots} \\
& \mathrm{i}=1, \ldots, \mathrm{r}-1 ; \mathrm{j}=1, \ldots, \mathrm{~s}-1 \\
& y_{i j k}-y_{i j .}=e_{i j k}-e_{i j} . \\
& i=1, \ldots, r ; j=1, \ldots, s ; k=1, \ldots, t-1
\end{aligned}
$$

 $y_{i j}$. and $y_{\text {... are averages. Consider the transformation }}$ $y=T z$ where $T$ is constructed by

$$
\begin{aligned}
& y_{i j k}=y^{y} \ldots{ }^{+}\left(y_{i .}{ }^{-y} \ldots\right)+\left(y_{. j} .{ }^{-y} \ldots\right)+\left(y_{i j} .{ }^{-y_{i .}}{ }^{-y} . j .{ }^{+y} \ldots\right) \\
& +\left(y_{i j k}-y_{i j} .\right)
\end{aligned}
$$

if $i<r, j<s, t<k$. For $i=r$, the $y_{r . .}{ }^{-y} . .$. term is replaced by $\Sigma_{i=1}^{r-1}-\left(y_{i .}-y \ldots\right)$ and the $y_{r j}{ }^{-y} y_{r . .}-y_{. j}+y^{y} \ldots$ term is replaced by $\Sigma_{i=1}^{r-1}-\left(y_{i j} . y_{i .}{ }^{-y} . j .{ }^{+y} \ldots\right)$. Similarly for the cases $j=s$ and $t=k$. Therefore

$$
\lambda=\Sigma_{a} f_{a}\left(\log \theta_{a}+M S_{a} / \theta_{a}\right)+\text { constant }
$$

where $f_{a}$ is the degree of freedom associated with $M S_{a}$ and $\theta_{a}$ is the expectation of $\mathrm{MS}_{a}$ for each $a=A, B, A B, e$.
(I) $\max L$
subject to $\theta_{\mathrm{e}} \leq \theta_{\mathrm{AB}} \leq \theta_{\mathrm{A}}, \quad{ }_{\mathrm{AB}} \leq \theta_{\mathrm{B}}$
(II) $\min \lambda$
subject to $\theta_{\mathrm{e}} \leq \theta_{\mathrm{AB}} \leq \theta_{\mathrm{A}},{ }^{\prime}{ }_{\mathrm{AB}} \leq \theta_{\mathrm{B}}$
(III)
$\min \Sigma_{a} f_{a}\left(\log \theta_{a}-\log M S_{a}-1+M S_{a} / \theta_{a}\right)$
subject to $\theta_{\mathrm{e}} \leq \theta_{\mathrm{AB}} \leq \theta_{\mathrm{A}}, \theta_{\mathrm{AB}} \leq \theta_{\mathrm{B}}$

The problems (I), (II) and (III) are equivalent. Let $\Phi(x)=-\log x$. Then $\Phi$ is convex. Let

$$
\Delta\left(\mathrm{MS}_{\mathrm{a}}, \theta_{a}\right)=\log \theta_{a}-\log \left(\mathrm{MS}_{a}\right)+\left(\mathrm{MS}-\theta_{a}\right) / \theta_{a} .
$$

By Theorem 1.10 of Barlow and coworkers (1972), (III) is equivalent to (IV).
(IV) $\quad \min \Sigma_{a} f_{a}\left(M S_{a}-\theta_{a}\right)^{2}$
subject to $\theta_{\mathrm{e}} \leq \theta_{\mathrm{AB}} \leq \theta_{\mathrm{A}},{ }^{\theta} \mathrm{AB} \leq \theta_{\mathrm{B}}$

## APPENDIX IV

Bound ed Isotonic Regression (cf. Example 3.6)
$\min f(Z)=\sum_{i=1}^{10}\left(z_{i}-x_{i}\right)^{2}$
subject to $\quad g_{i}(Z)=10+1.5(i-1)-z_{i} \leq 0 \quad i=1, \ldots, 10$

$$
g_{10+i}(Z)=z_{i}-13-1.5(i-1) \leq 0 \quad i=1, \ldots, 10
$$

and

$$
g_{20+i}(Z)=z_{i}-z_{i+1} \leq 0, \quad i=1, \ldots, 9
$$

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{x}_{\mathrm{i}}$ | 25 | 13 | 2 | 15 | 14 | 21 | 9 | 33 | 25 | 15 |
| $\mathrm{z}_{1 \mathrm{i}}$ | 10 | 11.5 | 13 | 14.5 | 16 | 17.5 | 19 | 20.5 | 22 | 23.5 |
| $\mathrm{z}_{2 \mathrm{i}}$ | 13 | 14.5 | 16 | 17.5 | 19 | 20.5 | 22 | 23.5 | 25 | 26.5 |
| $\mathrm{x}_{\mathrm{i}}$ | 13.33 | 13.33 | 13.33 | 14.5 | 14.5 | 15 | 15 | 24.33 | 24.33 | 24.33 |
| $\hat{\mathrm{x}}_{\mathrm{i}}$ | 13 | 13 | 13 | 15 | 16 | 19 | 19 | 23.5 | 23.5 | 23.5 |
| $\lambda_{\mathrm{i}}$ | 0 | 0 | 22 | 0 | 4 | 0 | 16 | 0 | 0 | 14 |
| $\lambda_{10+\mathrm{i}}$ | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 19 | 0 | 0 |
| $\lambda_{20+\mathrm{i}}$ | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 3 |  |

$$
\lambda_{i}, i=1, \ldots, 29 \text { are Lagrangian multipliers }
$$

$$
\begin{aligned}
& \nabla f(\hat{X})=\left(\begin{array}{llllllllll}
-24 & 0 & 22 & 0 & 4 & -4 & .20 & -19 & -3 & 17
\end{array}\right) \\
& \sum_{i=1}^{10} \lambda_{i} \nabla g_{i}(\hat{X})=\left(\begin{array}{llllllllll}
0 & 0 & -22 & 0 & -4 & 0 & -16 & 0 & 0 & -14
\end{array}\right) \\
& \Sigma_{i=11}^{20} \lambda_{i} \nabla g_{i}(\hat{X})=\left(\begin{array}{llllllllll}
24 & 0 & 0 & 0 & 0 & 0 & 0 & 19 & 0 & 0
\end{array}\right) \\
& \Sigma_{i=21}^{29} \lambda_{i} \nabla g_{i}(\hat{X})=\left(\begin{array}{lllllllllll} 
& 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 3 & -3
\end{array}\right)
\end{aligned}
$$

Kuhn-Tucker condition

$$
\begin{gathered}
g_{i}(\hat{X}) \leq 0, \quad i=1, \ldots, 29 \\
\lambda_{i} g_{i}(\hat{X})=0, \quad i=1, \ldots, 29 \\
\nabla f(\hat{X})+\Sigma_{i=1}^{29} \lambda_{i} \nabla g_{i}(\hat{X})=0
\end{gathered}
$$



$$
\begin{aligned}
& \hat{x}_{1}, \hat{x}_{5}, \hat{x}_{7} \text { and } \hat{x}_{8} \text { are determined by Lemma } 3.1 \\
& x--X \text { value } \\
& *--X * \text { value, } X * \text { is the isotonic regression of } X \\
& 0--\hat{X} \text { value, } \widehat{X} \text { is the bounded isotonic regression of } X
\end{aligned}
$$

```
                                    APPENDIX VSimplified Projection Algorithm
            pEgr,RAM LEET (cf. Section V.3)
        SIMPLIFIED ? ?JJECIION ALGORITHM
        MIN (5. Z) F(?)=TRACF(V*(X-Z)*TRANS(X-7))
        OY TG WH=RE X IS A GIUEN IS INCRFASING OR OECPEASIMGGAS OETEONINET
        SUCH THAT V (1, 1)=V1*V1, V(?,2)=V2*V? AND V(1, 2)=V (2,1)=-V1*V ***
```





```
    13 EFAT(5), 14)N,又,V,TO
```



```
    PE A!)(5),13)(x(1, J), l=1,N)
    pram(5j,{R)(x(?,j),j=1,N)
    19 FCRMGT(ITF3.3)
    N1='1+1
```



```
    IFIID(T).07.?HINIGO TO 25
    q=-*
    V(T)=-V(I)
```



```
    px(l,j)=v(I)*x(T,J)
    Bor(I2J)=1+13
    <a.covilvue
```



```
        IFRD.LT.`.!r, T= q6
    34 INIT=.5
        01=Fx(1, J; -ox(1, J-1)
        \Gamma?=F\times(2,J)-2\times(2,J-1)
        IL(J)=12(J)=?
        IF(3P(1,j).E0.1HP)GO TO 4?
```




```
        \F(2,j)=i4j"
        IMr,=1
        47 COVTINUE.
    IF NO INPRFASINS [N NUMIES OF PIVOTAL INUICES, DX IS IHF
    OOTIMAL SOLJTTNN.
        IFIINC.CO. IICJ rO 87
        O= 了=0
        II(I)=T2(1)=NL=1
        1-N吹 Nマ=2,N
        J=NP-NL+1
        IF(TP(1,NQ).FD.1HP)GO TN 5n
        I= (J)=1
5?
    Y1(J)=P\times(1,NR)-R*DX(?,NP)
    I?(J)=1
54 Y< (J)= OX{2,NR)-2*OX(1,NP)
```



```
    Y1(1)=rx(1,NL)-2*PX(?,NL
    Yว(1)=0x(2,NL)-又*P)(1,NL)
    NO=NQ-NL
C
OFCOMPOSITTON TECHNIGUE.
    IF(0.LTGO)COSTO 75
    75 CALL PROJ_(V),R,IC,IL,Y?,Y1, ))
75
    M=J+& & L L-1
    Px(1,k)=Y1(J)
    Px(?,k)=r?(j)
ag Covityuf
```





```
GO 1034
37 0n \(91 \quad I=1\) ?
```




```
91 COVIITUE
WOITE( \(51, ~+3)(J, \times(1, J), P \times(1, J), \times(2, J), P \times(2, J), J=1, V)\)
```




```
    (o) Tn \(13^{\circ}\)
37
```



```
77
GחTO 13
    EMO
```

```
            SUGROUTINE OROJ(NT,?,J1,J2,71,G2,M)
        1CJ(2[E).2(1JN)
            NO1=N!?+1
            IL=1
            00 119 I=?, v?1
            IF(J1(I).E2.0)GJ TU 119
            IR=I-1
            \DeltaV=?
            QOM=A 13 J=IL,I?
            AV=AV+91(J)
    113 CONTTNUF
        AV=(AV*Q)/FLJ\DeltaI(K)
        Q? 11? J=IL,I?
    CovTIVUE
    Ik=T
    119 CONTIMUE
        K=9
        IF(Jट(I).Fう.?)G) TC 127
        K=K+4
        Q(<)=?.
        \cap\cap 127 J=I,N]
        F(K)=3(K)+3?(J)
    127 CONTINUF
    (5.5) r.Z=3 MxY STMULTANEJUS LINËAR EGUATIONS.
    L二?
    OO 1E4 JJ=1,NO
    *O, TO 164
    0?137 I=1;jJ
        Coj(I)=7.
        O\cap 140 I=JJ. 4D
        CIII=1.
    140 COvTINUE
    IL=1
    IFIJI(II,GQ.T)r, TO 155
        K=I-IL
        AV=[.
        O\ i 4O J=IL,I?
        Av=\hat{A}v+CJ(j)
    149
    AV=(AV*2S21/FLDAT(K)
    0\capj53 J=IG:IR
    153
        coNTI:!jE
    155 CCNTIM!UE
        K=?
        nOL+164 I=1, NT
        IF(J\sum(I).Ex.J)GO TC 164
        K=k+1
        C(K,L)=?
        C(K,L)= (K,L)+C, (J)
    154 CONTIMUE
```



```
    THOSE ON THE CAHE COLJMH.
    M1=">-1
        Du,17? J=1,41
        JF=J+1
        IF(JP.GT.1)T0 T\ 17%
        00 177 I=J0,4
```



```
            7(I)=?(I)-E*?
        C(I,k)=C(I,\langle)-F*C(J,K)
    179 CONTIMUE
        T(H)=3(M)/C,(M,N)
```


$S_{3}=E$
$K 1=\bar{K}+1$

$2(k)=(a(k)-S 3) / C(K, k)$
${ }^{1}$
003007

SUM $=J=3$
$002 n \mathrm{I}=1$, N ?

$J=j+1$
S! $4=$ SU4 + Z1ノ
3) F ? (I) = Siju

201 CONTINUE
IL=1
ก9 ?15I=2.ソ 1
If(J1 (I).E2.3) 50 Tา 215
$I R=I-1$
$K=I-I L$
$\Delta V=2$.
$A V=1 ? J=I L ; T R$
$A V=A V+31(J i+2 * ?(J)$
210 COWTMNUE
$\Delta V=\triangle V / F L \cap \cap T(\langle )$
$07214 \mathrm{~J}=\mathrm{IL}, \mathrm{I}$
81(J)=AV
? 14 CONIINUE
21 h ILNT CUNE
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