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A. E. Livingston

It is well known that every Banach space with a Schauder basis is separable. However, whether the converse of the above statement is true is not known.

It is therefore the purpose of this thesis to investigate the question of under what conditions an arbitrary, separable, Banach space has a basis. Some interesting criteria are developed for the existence of a basis in a Banach space. Some conditions are also developed for the existence of a basis in conjugate spaces.

The Basis in Separable Banach Spaces

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James Richard Gepner

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Chairman of Department of Mathematics

[redacted]

Dean of Graduate School

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THE BASIS IN SEPARABLE BANACH SPACES

I. INTRODUCTION

A Banach space is defined to be a complete, normed, linear space. A separable Banach space is a complete, normed, linear space with a countable dense subset. Since separability is a necessary condition for the existence of a basis, only separable Banach spaces are considered in this connection. It should be noted that we are considering only Schauder bases as opposed to vector space bases.

We give some examples of Banach spaces, the third of which is not separable but nevertheless useful for some of our later discussion because it is the dual space of the space of Example 1.2.

Example 1.1. Let c_0 be the set of all sequences of complex numbers which are convergent to zero. Then c_0 is a Banach space under the norm $\|x\| = \sup_n |a_n|$ where $x = \{a_n\}_{n=1}^{\infty}$.

Example 1.2. Let $\ell = \ell_1$ be the set of all absolutely convergent sequences of complex numbers. Then ℓ is a Banach space under the norm $\|x\| = \sum_{n=1}^{\infty} |a_n|$ where $x = \{a_n\}_{n=1}^{\infty}$.

Example 1.3. Let m be the set of all bounded sequences of complex numbers. Then m is a Banach space under the same norm as that for c_0.

In this paper we will develop certain criteria for the existence of a basis in a Banach space. The purpose of this paper is not so much to develop new results concerning the basis as it is to bring together some of the known results on the subject. Of course there is much more literature available than can be contained in this paper. It was therefore decided to present some of the results which center around the concepts of weak convergence, convergence, and boundedness in the Banach space.

In the final chapter some criteria are developed for the existence of bases in conjugate spaces. These results center around the same ideas as mentioned above. An interesting condition for the reflexivity of a Banach space is also given.

Before proceeding to the body of the thesis, we list some notation. Any other notation used will be defined at the time it is introduced.

Notation

x, y, z	elements of arbitrary Banach spaces
f, g, h	elements of dual spaces
X, Y, Z	elements of bidual spaces
θ	the zero element of a Banach space
$\{x_n\}$	the infinite sequence $\{x_n\}_{n=1}^{\infty}$
$[x_1, x_2, \ldots, x_n]$	the span of $x_1, x_2, \ldots, x_n$

Σx_n	the infinite sum $\sum_{n=1}^{\infty} x_n$
a, b, c	elements of the scalar field F where F will be taken to be the complex numbers

II. PRELIMINARIES

The results which are developed in this chapter are essential to the rest of the paper. If the reader desires additional information on the results of this chapter, he may refer to either Berberian (1961) or Wilansky (1964) from which some of this material was taken.

Definition 2. 1. A sequence $\{x_n\}$ of points in a Banach space A is a basis if for any $x \in A$ there exists a unique sequence of scalars $\{a_n\}$ such that $x = \Sigma a_n x_n$ or equivalently that

$$\|x - \sum_{n=1}^{m} a_n x_n \| \to 0 \quad \text{as} \quad m \to \infty .$$

Since we are going to consider weak convergence in a Banach space, then it is only natural to introduce the concept of a weak basis in a Banach space.

Definition 2. 2. A sequence of points $\{x_n\}$ in a Banach space A is a weak basis for A if to each $x \in A$ there corresponds a unique sequence of scalars $\{a_n\}$ such that $\sum_{n=1}^{m} a_n x_n$ converges weakly to x. By weak convergence we mean $f(\sum_{n=1}^{m} a_n x_n) \to f(x)$ for all f in A^* (i. e., for all $f \in L_c(A, F)$ where $L_c(A, F)$ denotes the set of all continuous linear functions on A to F, the scalar field).

Definition 2.3. Let A and B be Banach spaces. The mapping t from A into B is said to be linear if for any $x, y \in A$ and scalars $a, b \in F$ we have $t(ax+by) = at(x) + bt(y)$.

Definition 2.4. A mapping $t : A \to B$ is said to be continuous at $x \in A$ if $x_n \to x$ implies $t(x_n) \to t(x)$. If t is continuous at each point in A, then t is said to be a continuous mapping.

In Banach spaces there are many useful reformulations of continuity of linear mappings. One of the most useful of these says that boundedness is equivalent to continuity. More explicitly, we say t is continuous on A if there exists a constant M such that $\|t(x)\| \leq M \cdot \|x\|$ for all $x \in A$. It is this notion of continuity which will most often be used to prove mappings continuous.

As mentioned before, A^* is just the set of all continuous linear functions from A to its scalar field. In a similar manner we define A^{**}. There is a natural correspondence between A and A^{**}. In order to more fully develop this correspondence, the following results are needed, the first of which will be used in the chapters to follow.

Theorem 2.1. If A_0 is a subspace of a normed linear space A, z a fixed point of $A \sim A_0$, and $d = \inf \{\|z-x\| : x \in A_0\} > 0$, then there exists an f in A^* such that $f(z) = 1$, $f(x) = 0$ for all $x \in A_0$, and $\|f\| = 1/d$.

Proof: If $x \in A_0 + [z]$, then we can write x uniquely as $x = y + az$ where $y \in A_0$ and a is a scalar. We then define the functional ϕ from $A_0 + [z]$ to F as $\phi(x) = a$ for all $x \in A_0 + [z]$. Clearly ϕ is linear, and furthermore $\phi(z) = 1$ and $\phi(x) = 0$ for all $x \in A_0$. Pick an arbitrary $x \in A_0 + [z]$ and let $x_n \to x$ where $\{x_n\} \in A_0 + [z]$. Then for each n there must exist a y_n and a_n such that $x_n = y_n + a_n z$. Also $x = y + az$. Agreeing that $(y-y_n)/(a_n-a)$ is θ when $a_n=a$, we have

$$\begin{aligned} \|x_n - x\| &= \|(y_n - y) + (a_n - a)z\| \\ &= |a_n - a| \; \|z - (y-y_n)/(a_n-a)\|. \end{aligned}$$

But y and y_n are in A_0. Thus $(y-y_n)/(a_n-a)$ must also be in A_0. Hence

$$\|z - (y-y_n)/(a_n-a)\| \geq d.$$

Thus we have

$$\|x_n - x\| \geq |a_n - a| \; d$$

or

$$|a_n - a| \leq (1/d)\|x_n - x\|.$$

Now by definition of ϕ we see

$$\|\phi(x_n) - \phi(x)\| = |a_n - a| \leq 1/d \|x_n - x\| \to 0.$$

But this implies $\phi(x_n) \to \phi(x)$. Thus ϕ is a continuous linear functional defined on $A_0 + [z]$. Now by the Hahn-Banach Theorem (which is stated for reference at the end of this chapter) we can extend ϕ to a continuous linear functional f defined on the whole space, for which $\|f\| = \|\phi\|$. Then $f \in A^*$ and f agrees with ϕ on $A_0 + [z]$. In particular $f(z) = 1$ and $f(x) = 0$ for all x in A_0. It remains now to show that $\|f\| = 1/d$. By the Hahn-Banach Theorem it is sufficient to show $\|\phi\| = 1/d$. Suppose $x \in A_0 + [z]$ and $x = y + az$ where $y \in A_0$ and $a \in F$. For $a \neq 0$, we have

$$|\phi(x)| = |a| \leq |a| \left(\frac{\|y/a + z\|}{d} \right)$$

$$\leq 1/d \|y + az\| = 1/d \|x\|,$$

which implies $\|\phi\| \leq 1/d$. To show the reverse inequality, observe that for any $y \in A_0$ we have

$$1 = |\phi(z-y)| \leq \|\phi\| \cdot \|z-y\|$$

or equivalently

$$1/\|\phi\| \leq \|z-y\|.$$

But since y was arbitrary in A_0, by taking infs in the above equation we have

$$1/\|\phi\| \le d$$

or

$$\|\phi\| \ge 1/d.$$

Combined with the previous inequality we have $\|\phi\| = 1/d$ which implies $\|f\| = 1/d$, the desired result.

From now on we define the distance from a point to a subspace as $d(z,A_0) = \inf\{\|z-x\| : x \in A_0\}$ where A_0 is an arbitrary subspace of a Banach space.

<u>Corollary 2.1.</u> Given an element $x \in A$ with $\|x\| \neq 0$ there exists an f in A^* with $f(x) = \|x\|$ and $\|f\| = 1$.

Proof. Let $A_0 = \{\theta\}$ in the previous theorem. Clearly $d = \inf\{\|x-z\| : z \in A_0\} = \|x-\theta\| = \|x\| > 0$. Hence there exists a g in A^* with $g(x) = 1$ and $\|g\| = 1/d$. Since $g \in A^*$ then certainly $\|x\|g$ is in A^*. Let $f = \|x\|g$. Then $f \in A^*$ and $f(x) = \|x\| \cdot g(x) = \|x\|$. Also $\|f\| = \|(\|x\|)g\| = \|x\|(1/d) = \|x\|(1/\|x\|) = 1$ as desired.

<u>Theorem 2.2.</u> For any $x \in A$ let $X : A^* \to F$ be defined by $X(f) = f(x)$ for any $f \in A^*$. Then $X \in L_c(A^*, F)$ and furthermore, $\|X\| = \|x\|$.

Proof: Since f is linear, it is easily seen that X is also.

We also have

$$|X(f)| = |f(x)| \leq \|f\| \cdot \|x\| = \|x\| \cdot \|f\|.$$

Hence X is continuous and $\|X\| \leq \|x\|$. Now by Corollary 2.1 we know there exists a $g \in A^*$ such that $\|g\| = 1$ and $g(x) = \|x\|$ for $x \in A$, $x \neq \theta$. Hence

$$|X(g)| = |g(x)| = \|x\| = \|x\| \cdot \|g\| \quad \text{or}$$

$$\|x\| \cdot \|g\| = |X(g)| \leq \|X\| \cdot \|g\|.$$

But since $\|g\| = 1$ we have $\|x\| \leq \|X\|$. Combining the two inequalities we get $\|x\| = \|X\|$, the desired result.

Suppose we define $T(x) = X$. Then T is a mapping from A to A^{**} and is in fact that natural correspondence which was mentioned earlier. We say that T is the evaluation of x at any $f \in A^*$. Now besides having the important properties of linearity and continuity, T has also the property of being one to one. For suppose $T(x) = T(y)$. Then for all $f \in A^*$, $X(f) = Y(f)$, which implies that $f(x) = f(y)$ or $f(x-y) = 0$. However, given $z \neq 0$ in A, there is an $f \in A^*$ such that $\|f\| = 1$ and $f(z) = \|z\|$. Since $f(x-y) = 0$ for each $f \in A^*$, $x-y = \theta$. Thus, T is one to one.

With the aid of the mapping T we can identify A with a subset of A^{**}. If T is onto (as well as one to one), then we identify A with all of A^{**} and in this case A is said to be reflexive.

Closely associated with the concept of the basis is that of a biorthogonal sequence. We make the following definition.

Definition 2.5. Suppose that $\{x_n\}$ and $\{f_m\}$ are sequences in A and A^* respectively. We say that $(\{x_n\}, \{f_m\})$ forms a biorthogonal pair in A, A^* if $f_m(x_n) = 1$ for $m = n$ and zero otherwise.

We state for future reference the following well known theorems.

The Hahn-Banach Theorem: If A is a normed linear space and f is a continuous linear functional defined on a subspace A_0 of A, then there is a continuous linear functional g, defined on all of A, such that $g(x) = f(x)$ for $x \in A_0$ and $\|g\| = \|f\|$.

It should be noted that the original theorem as given by Hahn and Banach was for the real case only. The extension to the complex case is due to Bohnenblust and Sobczyk (1938).

The Open Mapping Theorem: If t is a closed linear transformation of a Banach space A onto a Banach space B, then t is open.

The Principle of Uniform Boundedness: Suppose A is a

Banach space, B is a normed linear space, and $\mathcal{T}$ is a family of continuous linear mappings from A to B. If $\mathcal{T}$ is bounded at each point of A, then there exists an M such that $\|t\| \leq M$ for all t in $\mathcal{T}$.

III. NORM EQUIVALENCE

In this chapter we present several lemmas which are essential to the proofs of some later theorems. The lemmas given here are used to develop certain results about the norm in the Banach space.

<u>Lemma 3.1.</u> Let t be a linear transformation from a normed space A to a normed space B. Then t is one to one and t^{-1} is in L_c (range t, A) if and only if there is a real number $m > 0$ such that for all $x \in A$, $\|t(x)\| \geq m\|x\|$.

Proof: Suppose first that t is one to one and t^{-1} is in L_c (range t, A). The result is trivial if $A = \{\theta\}$. In the remaining case take $m = 1/\|t^{-1}\|$. Consider any $x \in A$ and let $y = t(x)$ where $y \in B$. Then $t^{-1}(y) = x$ and we have

$$\|t^{-1}(y)\| \leq \|t^{-1}\| \cdot \|y\| = (1/m)\|y\|.$$

Thus $\|x\| \leq (1/m)\|t(x)\|$ or $\|t(x)\| \geq m \cdot \|x\|$ as desired.

Conversely suppose there exists an $m > 0$ such that $\|t(x)\| \geq m\|x\|$ for all $x \in A$. Let x_1, x_2 be in A with $x_1 \neq x_2$. Then

$$\|t(x_1) - t(x_2)\| = \|t(x_1 - x_2)\| \geq m\|x_1 - x_2\| > 0$$

which implies that $t(x_1) \neq t(x_2)$ and so t is one to one. Now suppose y is in range t. Then for some $x \in A$

$$\|t^{-1}(y)\| = \|x\| \leq (1/m)\|t(x)\| = (1/m)\|y\|$$

which implies the continuity of t^{-1}. Thus $t^{-1} \in L_c$ (range t, A) as desired.

Suppose now that we have a Banach space A with a basis $\{x_n\}$ and further suppose $\|\cdot\|$ is the norm imposed on A. It is sometimes useful to define a new norm on A. With this new norm it may be possible to justify certain results in a much easier way. Of course if completeness of the space is necessary to the desired results, then the new norm will have to be shown to be complete. Given an arbitrary Banach space with a basis $\{x_n\}$ we define a new norm in terms of the old one. The resulting norm is complete as is now shown.

<u>Lemma 3.2.</u> Suppose that A is a Banach space with a basis $\{x_n\}$ and norm $\|\cdot\|$. If we define our new norm as

$$\|x\|_1 = \sup \{\|\sum_{k=1}^{n} a_k x_k\| : n = 1, 2, 3, \ldots, \}$$

where $x = \Sigma a_k x_k$, then A is also a Banach space under $\|\cdot\|_1$.

Proof: We must prove A is complete under $\|\cdot\|_1$. Let $\{z_n\}$ be a Cauchy sequence in the new norm. Since $\{x_k\}$ is a basis, we have for each n

$$z_n = \sum_{k=1}^{\infty} a_{nk} x_k$$

where the series converges in the old norm. Then for any k we have

$$\begin{aligned} \|a_{pk}x_k - a_{qk}x_k\| &= \|(a_{pk}-a_{qk})x_k\| \\ &= \left\|\sum_{m=1}^{k} (a_{pm}-a_{qm})x_m - \sum_{m=1}^{k-1} (a_{pm}-a_{qm})x_m\right\| \\ &\leq \left\|\sum_{m=1}^{k} (a_{pm}-a_{qm})x_m\right\| + \left\|\sum_{m=1}^{k-1} (a_{pm}-a_{qm})x_m\right\| \\ &\leq \|z_p - z_q\|_1 + \|z_p - z_q\|_1 \\ &\leq 2\|z_p - z_q\|_1 \end{aligned}$$

But since $\{z_n\}$ is Cauchy in the new norm, then clearly for each k, $\{a_{nk}x_k\}$ is Cauchy in the old norm. Since the old norm is complete, then for each k there must exist a y_k such that

$$y_k = \text{old} \lim_{n\to\infty} a_{nk}x_k. \qquad (1)$$

Now for some appropriate scalar a_k, we have $y_k = a_k x_k$ for each k and $\lim_{n\to\infty} a_{nk} = a_k$. For observe that $a_{nk}x_k$ is in $[x_k]$

for all n. Since $[x_k]$ is a finite dimensional subspace of A, then $[x_k]$ must be closed. But y_k is a limit of points in $[x_k]$ and thus $y_k = a_k x_k$ for some appropriate a_k. Also

$$\|a_k x_k - a_{nk} x_k\| = |a_k - a_{nk}| \cdot \|x_k\| \to 0 \quad \text{as} \quad n\to\infty.$$

Hence we also have

$$a_k = \lim_{n\to\infty} a_{nk} \quad \text{for all} \quad k. \tag{2}$$

It is now possible to show that $\Sigma\, y_k$ converges in the old norm. For observe

$$\|\sum_{k=p}^{q} y_k\| \leq \|\sum_{k=p}^{q} (y_k - a_{nk} x_k)\| + \|\sum_{k=p}^{q} (a_{nk} - a_{mk}) x_k\| + \|\sum_{k=p}^{q} a_{mk} x_k\|.$$

Choose $\epsilon > 0$. Now fix m large enough so that $\|z_n - z_m\|_1 \leq \epsilon/6$ for all $n \geq m$. Thus for $n \geq m$

$$\|\sum_{k=p}^{q} (a_{nk} - a_{mk}) x_k\| \leq \|\sum_{k=1}^{q} (a_{nk} - a_{mk}) x_k\| + \|\sum_{k=1}^{p-1} (a_{nk} - a_{mk}) x_k\|$$

$$\leq \|z_n - z_m\|_1 + \|z_n - z_m\|_1$$

$$\leq 2\|z_n - z_m\|_1 = 2\epsilon/6 = \epsilon/3.$$

Now since $\sum_{k=1}^{\infty} a_{mk} x_k$ is convergent for each m, then with m fixed as above, we can choose N such that for $p, q \geq N$ we have

$$\|\sum_{k=p}^{q} a_{mk} x_k\| \leq \epsilon/3.$$

Now fix both p and q such that $p \geq N$ and $q \geq N$. Then using equation (1), it is possible to pick n large enough so that

$$\|\sum_{k=p}^{q} (y_k - a_{nk} x_k)\| \leq \epsilon/3.$$

Combining these results we have for arbitrary $p, q \geq N$ that

$$\|\sum_{k=p}^{q} y_k\| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Hence Σy_k must converge in the old norm. Now define z as $z = \Sigma y_k$. Clearly $z \in A$ since it has a representation in terms of the basis. We now need to show $\|z - z_n\|_1 \to 0$ as $n \to \infty$. Choose $\epsilon > 0$ and N such that $q \geq p \geq N$ implies $\|z_p - z_q\|_1 \leq \epsilon$. Then for any m and any p, q such that $q \geq p \geq N$ we have

$$\|\sum_{k=1}^{m} (a_{pk} - a_{qk}) x_k\| \leq \|z_p - z_q\|_1 \leq \epsilon.$$

Letting $q \to \infty$ we have by (2) that

$$\left\| \sum_{k=1}^{m} (a_{pk} - a_k) x_k \right\| \leq \epsilon .$$

Since the above relation is true for all m, we have by taking the sup over all m

$$\| z_p - z \|_1 \leq \epsilon \quad \text{for} \quad p \geq N.$$

Hence $\| z_p - z \|_1 \to 0$ as $p \to \infty$ and the completeness of $\| \cdot \|_1$ is proved.

Having introduced this new norm in terms of the old one, one might ask what the relationship is, if any, between the two norms. As it turns out, we are able to say that the norms are equivalent. The following definition makes precise the concept of norm equivalence.

Definition 3.1. For any two norms $\| \cdot \|_1$ and $\| \cdot \|_2$, we say that $\| \cdot \|_1$ is stronger than $\| \cdot \|_2$ and $\| \cdot \|_2$ is weaker than $\| \cdot \|_1$ if, whenever $\{x_n\}$ is a sequence such that $\|x_n\|_1 \to 0$ as $n \to \infty$, then also $\|x_n\|_2 \to 0$ as $n \to \infty$. If both $\| \cdot \|_1$ and $\| \cdot \|_2$ are stronger than each other, then the norms are said to be equivalent.

Depending on how the norms are defined, it may be rather difficult to show their equivalence. However, there is one special case in which the equivalence of norms may be proved quite easily. We appeal to the Open Mapping Theorem in the following result.

Lemma 3.3. Suppose that A is a normed linear space. Let A_1 be the space A with norm $\|\cdot\|_1$. Let A_2 be the space A with norm $\|\cdot\|_2$. Suppose both $\|\cdot\|_1$ and $\|\cdot\|_2$ are complete. If then one norm is stronger than the other, the norms are equivalent.

Proof: We may suppose without loss of generality that $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$. Suppose now $x_n \to x$ in A_1 and that $I(x_n) \to y$ in A_2 where $I: A_1 \to A_2$ such that $I(x) = x$ for $x \in A_1$. We thus have

$$\|x-y\|_2 \leq \|x_n-x\|_2 + \|y-x_n\|_2$$
$$\leq \|x_n-x\|_2 + \|y-I(x_n)\|_2.$$

Since $x_n \to x$ in A_1, then $\|x_n-x\|_1 \to 0$ which implies $\|x_n-x\|_2 \to 0$. Also since $I(x_n) \to y$, then $\|y-I(x_n)\|_2 \to 0$. Letting $n \to \infty$ on the right hand side of the above relation we have $\|x-y\|_2 = 0$. Thus $x = y$ and I must be a closed mapping. Since I is clearly an onto map, then by the Open Mapping Theorem we may conclude that I is also an open mapping. This implies I must take open sets in A_1 into open sets in A_2. But this just says

I^{-1} is continuous or that I^{-1} is in $L_c(A_2, A_1)$ where range of $I = A_2$. Since I is clearly one to one, we may conclude by Lemma 3.1 that there exists a real number $m > 0$ such that $\|Ix\|_2 \geq m\|x\|_1$, for all x. Now suppose that $\{x_n\}$ is such that $\|x_n\|_2 \to 0$ as $n \to \infty$. Then for each n we have

$$\|x_n\|_1 \leq 1/m\,\|Ix_n\|_2 = 1/m\,\|x_n\|_2$$

and clearly $\|x_n\|_1 \to 0$ as $n \to \infty$. Thus $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$ and combining this with our original assumption, we have that the two norms are equivalent.

IV. BASES IN BANACH SPACES

In some, perhaps in all, separable, infinite dimensional Banach spaces there exist sequences with special properties. These sequences, which have become known as bases, were first discussed by Schauder. The question of whether all separable Banach spaces have a basis remains unanswered.

The first result we prove gives a necessary condition for the existence of a basis in a Banach space. Later in the chapter, with one additional assumption, we will be able to prove the sufficiency of the stated condition. The theorem is due to M. M. Grynblyum.

Theorem 4.1. Suppose the sequence $\{x_n\}$ of points in the Banach space A is a basis for A. Then there exists a number M such that for every sequence $\{a_n\}$ of scalars and positive integers p and q with $q > p$ we have

$$\left\| \sum_{k=1}^{p} a_k x_k \right\| \leq M \left\| \sum_{k=1}^{q} a_k x_k \right\|.$$

Proof: For any $x \in A$ we define a new norm as

$$\|x\|_1 = \sup \left\{ \left\| \sum_{k=1}^{n} a_k x_k \right\| : n = 1, 2, \ldots \right\}$$

where $x = \Sigma a_k x_k$, the series converging in the old norm. By Lemma 3.2 this norm is also complete. Now for $x \in A$ we have

$$\|x\| = \|\sum a_k x_k\| \leq \sup\{\|\sum_{k=1}^{n} a_k x_k\| : n = 1, 2, \ldots\} = \|x\|_1$$

Now suppose that $\{y_n\}$ is such that $\|y_n\|_1 \to 0$ as $n \to \infty$. By the above relation this clearly implies that $\|y_n\| \to 0$. Hence $\|\cdot\|_1$ is stronger than $\|\cdot\|$ and by Lemma 3.3 the norms must be equivalent. We can now show that there exists an M such that for all $x \in A$

$$\|x\|_1 \leq M\|x\| .$$

Suppose there exists no such M. Then for each integer n there must exist a y_n such that

$$\|y_n\|_1 > n^2 \|y_n\|.$$

Now $\|y_n\| \neq 0$ for any n since if not this would imply $\|y_n\|_1 = 0$. Now define the sequence $\{z_n\}$ as follows

$$z_n = y_n / n \cdot \|y_n\|.$$

Then for each n

$$\|z_n\| = \|y_n / (n \cdot \|y_n\|)\| = 1/n.$$

Thus $\|z_n\| \to 0$ as $n \to \infty$. But for each n, $\|z_n\|_1 > n$ which contradicts the equivalence of the norms. Thus there exists an M such that

$$\|x\|_1 \leq M\|x\|$$

for all $x \in A$. Now let $\{a_n\}$, p, q be as in the statement of the theorem. We then have

$$\|\sum_{k=1}^{p} a_k x_k\| \leq \sup \{\|\sum_{k=1}^{n} a_k x_k\| : n = 1, 2, \ldots, q\}$$

$$\leq \|\sum_{k=1}^{q} a_k x_k\|_1$$

$$\leq M \|\sum_{k=1}^{q} a_k x_k\|$$

the desired relation.

In the next result we give three conditions which together are necessary and sufficient for the existence of a basis in a Banach space. This theorem is probably the most important of the paper in the sense that other results will be obtained by application of this theorem.

Theorem 4.2. A sequence $\{x_n\}$ of points in a Banach space

A is a basis for A if and only if the following three conditions are satisfied;

1. there is a sequence $\{f_n\}$ in A^* such that $(\{x_n\}, \{f_n\})$ forms a biorthogonal pair in A, A^*,
2. the linear closure of $\{x_n\}$ is A, and
3. if we define $z_m : A \to A$ by $z_m(x) = \sum_{k=1}^{m} f_k(x)x_k$, then $\{z_m\}$ is pointwise bounded on A, that is, given $x \in A$ there exists a constant M_x such that $\|z_m(x)\| \leq M_x$ for all m.

Proof: Suppose first that $\{x_n\}$ is a basis for A. Clearly (2) holds and as for (3) we observe that since $\{x_n\}$ is a basis for A, then $z_m(x) \to x$ as $m \to \infty$. Hence $\{z_m(x)\}$ must be bounded as desired. Now for each n define f_n as follows

$$f_n(x) = f_n(\Sigma\, a_k x_k) = a_n$$

where $x = \Sigma\, a_k x_k$. For each n, f_n is clearly linear and $f_n(x_k) = 1$ if $n = k$ and zero otherwise. Now by Theorem 4.1 we have

$$\|a_n x_n\| = \|\sum_{k=1}^{n} a_k x_k - \sum_{k=1}^{n-1} a_k x_k\|$$

$$\leq 2M\|x\|.$$

But this implies

$$|f_n(x)| = |a_n| \leq (2M/\|x_n\|)\|x\|.$$

Thus $f_n \in A^*$ and condition (1) is proved.

Conversely, suppose that the three conditions of the theorem hold. Since A is a Banach space and $\{z_m\}$ is pointwise bounded on A, then by the Principle of Uniform Boundedness we have that $\{z_m\}$ is norm bounded on A, that is, there exists an M such that $\|z_m\| \leq M$ for all m. Now define $B = \{x \in A: z_m(x) \to x\}$. Clearly $B \subset A$. If we let $G_m(x) = z_m(x) - x$ for all x and all m, we then have

$$\begin{aligned} \|G_m(x)\| = \|z_m(x)-x\| &\leq \|z_m(x)\| + \|x\| \\ &\leq \|z_m\| \cdot \|x\| + \|x\| \\ &\leq (M+1)\|x\|. \end{aligned}$$

Thus $\|G_m\| \leq M + 1$ for all m. With this result we can now show that B is a closed subspace of A. It is clear enough that B is a subspace of A. Suppose that $\{y_n\}$ is in B and $y_n \to y$. We have

$$\|G_m(y)\| = \|G_m(y) - G_m(y_n) + G_m(y_n)\|$$
$$\leq \|G_m(y) - G_m(y_n)\| + \|G_m(y_n)\|$$
$$\leq \|G_m(y-y_n)\| + \|G_m(y_n)\|$$
$$\leq (M+1)\|y-y_n\| + \|G_m(y_n)\|.$$

Now choose $\epsilon > 0$ and N such that for all $n \geq N$ we have $\|y-y_n\| \leq \epsilon/2(M+1)$. Then

$$\|G_m(y)\| \leq (M+1)\|y-y_N\| + \|G_m(y_N)\|$$
$$\leq (M+1)\,\epsilon/(M+1)2\epsilon + \|G_m(y_N)\|$$
$$\leq \epsilon/2 + \|G_m(y_N)\|.$$

But since $y_N \in B$, then there must exist an N_0 such that for all $m \geq N_0$,

$$\|G_m(y_N)\| \leq \epsilon/2.$$

Hence for all $m \geq N_0$ we have

$$\|G_m(y)\| \leq \epsilon$$

which implies that $G_m(y) \to 0$ as $m \to \infty$. But this just says $z_m(y) \to y$ and so $y \in B$ and B is closed. By the biorthogonality of the sequence $\{f_n\}$ we have that $z_m(x_n) = x_n$ for $m \geq n$. Hence $G_m(x_n) = 0$ for $m \geq n$. But this implies that $x_n \in B$ for all n.

Now by the way each G_m is defined, it is clear that $[\{x_n\}]$ is also in B. Since A is the linear closure of $\{x_n\}$ then we know A is the smallest closed set containing $[\{x_n\}]$. Thus $A \subset B$ and combining this with the fact that $B \subset A$, then $A = B$. It has therefore been shown that every $x \in A$ has a representation as $\Sigma a_n x_n$ where $a_n = f_n(x)$. It only remains to show this representation is unique. Suppose not. Then the point $\theta \in A$ must have a nontrival representation, that is, $\theta = \Sigma a_n x_n$ where at least one a_n, suppose a_k, is nonzero. We then have

$$0 = f_k(\theta) = f_k(\sum a_n x_n) = \sum_{n=1}^{\infty} a_n f_k(x_n) = a_k \neq 0.$$

The representation must then be unique and $\{x_n\}$ is a basis for A.

As mentioned before, this theorem is quite important in the sense that many other results will be proved by its application. To facilitate further proofs we make the convention that reference to conditions (1), (2), and (3) will just mean those conditions of the previous theorem.

It is now possible to prove the sufficiency of the relation in Theorem 4.1. Note that we must add the extra condition that the sequence $\{x_n\}$ be fundamental in A (we take fundamental to mean the linear closure of $\{x_n\}$ is A). We state the entire theorem for completeness and further reference.

Theorem 4.3. Suppose the sequence of points in a Banach space A is fundamental in A. Then $\{x_n\}$ is a basis for A if and only if there exists a number M such that for every sequence $\{a_n\}$ of scalars and positive integers p, q with $q > p$ we have

$$\left\|\sum_{k=1}^{p} a_k x_k\right\| \leq M \left\|\sum_{k=1}^{q} a_k x_k\right\|. \tag{1}$$

Proof: We need only prove sufficiency as necessity has already been established. The method of proof is to verify conditions (1), (2), and (3) of Theorem 4.2. Condition (2) is given. To prove condition (1) we first observe that the sequence $\{x_n\}$ is linearly independent. For if it were not, then for some n and scalars $b_1, b_2, \ldots, b_{n-1}$ we would have $x_n = \sum_{k=1}^{n-1} b_k x_k$.

Now by the above relation this implies

$$\|x_n\| = \left\|\sum_{k=1}^{n-1} b_k x_k\right\| \leq M \left\|\sum_{k=1}^{n-1} b_k x_k - x_n\right\| = 0.$$

But this implies $x_n = \theta$ which is a contradiction. Thus $\{x_n\}$ is linearly independent. We now define the following subspaces of A. Let $B_n = [x_1, x_2, \ldots, x_n]$ for all n and for any integer $r < n$

define $B_{nr} = [x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n]$. Each of the above subspaces are closed since they are finite dimensional. Now with the aid of the assumed relation we can show that for $r = 2, 3, 4, \ldots$ and $n > r$ we have

$$d(x_r, B_{r-1}) \leq Md(x_r, B_{nr})$$

where M is the constant in (1). If $x \in B_{nr}$ then $x - x_r \in B_r$ and we have $x - x_r = \sum_{k=1}^{n} a_k x_k$ for some appropriate scalars $a_1, a_2, \ldots, a_n$ where $a_r = -1$. Thus

$$M\|x-x_r\| = M\|\sum_{k=1}^{n} a_k x_k\| \geq \|\sum_{k=1}^{r} a_k x_k\|$$

$$\geq \|\sum_{k=1}^{r-1} a_k x_k - x_r\|.$$

But $\sum_{k=1}^{r-1} a_k x_k$ is in B_{r-1}. Thus we certainly have

$$\|\sum_{k=1}^{r-1} a_k x_k - x_r\| \geq d(x_r, B_{r-1}).$$

Consequently for any $x \in B_{nr}$ we have

$$M\|x-x_r\| \geq d(x_r, B_{r-1}).$$

Now by taking the inf over all $x \in B_{nr}$ we get

$$d(x_r, B_{r-1}) \leq Md(x_r, B_{nr})$$

as desired. Now since $\{x_n\}$ is linearly independent then for any r, $(x_1, x_2, \ldots, x_r)$ is linearly independent. Hence $d(x_r, B_{r-1}) > 0$ since B_{r-1} is a closed subspace of A. We then have that

$$d(x_r, B_{nr}) \geq (1/M)d(x_r, B_{r-1}) \geq \delta > 0.$$

But δ is independent of n which implies that the distance from x_r to the span of $\{x_n : n \neq r\}$ is also greater than or equal to δ. But this implies that x_r is not in the linear closure of $\{x_n : n \neq r\}$. We can thus apply Theorem 2.1 and conclude there exists an f_r in A^* such that $f_r(x_r) = 1$ and $f_r(x_n) = 0$ for $n \neq r$. Since this is true for all r, then condition (1) must hold. Now define $\{z_m\}$ as in condition (3) of Theorem 4.2. Consider first any $x \in [\{x_n\}]$ and suppose that $x = \sum_{k=1}^{r} a_k x_k$ for appropriate scalars $a_1, a_2, \ldots, a_r$.

We then have

$$\|z_m(x)\| = \|\sum_{n=1}^{m} (\sum_{k=1}^{r} a_k f_n(x_k)) x_n\|$$

$$= \|\sum_{n=1}^{r} a_n x_n\| = \|x\|$$

if $m \geq r$. If $m < r$ then we have

$$\|z_m(x)\| = \left\|\sum_{k=1}^{m} a_k x_k\right\| \leq M \left\|\sum_{k=1}^{r} a_k x_k\right\| = M\|x\|.$$

In any case we have that for all m, $1 \leq \|z_m\| \leq N$ where $N = \max(M, 1)$ and the norm is taken over all x in $[\{x_n\}]$. We now want to extend this result so that $\|z_m\|$ may be taken over all $x \in A$. Choose an arbitrary $x \in A$ with $\|x\| = 1$ and any $\epsilon > 0$. Then there must be a $y \in [\{x_n\}]$ such that $\|x-y\| < \epsilon / \|z_m\|$ for any m. We then have

$$\begin{aligned}\|z_m(x)\| &= \|z_m(x-y) + z_m(y)\| \\ &\leq \|z_m(x-y)\| + \|z_m(y)\| \\ &\leq \|z_m\| \cdot \|x-y\| + N\|y\| \\ &< \epsilon + N(1+\epsilon/\|z_m\|) \leq \epsilon + N(1+\epsilon).\end{aligned}$$

But since ϵ was arbitrary we have for any m

$$\|z_m(x)\| \leq N = N \cdot \|x\|.$$

Now taking the sup over all $x \in A$ with $\|x\| = 1$ we have $\|z_m\| \leq N$ for any m. But now the norm is taken over all of A and so condition (3) is proved. Thus $\{x_n\}$ is a basis for A.

In many cases, when trying to prove that a sequence $\{x_n\}$ is a basis in a Banach space, certain properties of the sequence are needed before the proof can be carried through. In particular we very often need to assume the sequence is fundamental before we can prove it is a basis. Of course if we were not to assume that the sequence $\{x_n\}$ is fundamental we could still prove a lesser result, namely that the sequence $\{x_n\}$ is a basis for its own linear closure. Sometimes this lesser result may be all that is necessary for one's purposes and the added restriction of $\{x_n\}$ being fundamental is unessential.

One of the most natural questions which seems to arise when investigating bases in Banach spaces is what relationships, if any, exist between bases in the conjugate space and the original space. One side of this question will be analyzed in more detail in the next chapter. However, the question of when a Banach space A has a basis, given that A^* has a basis, can be answered now with positive results.

<u>Theorem 4.4.</u> Suppose that $(\{x_n\}, \{f_n\})$ is a biorthogonal pair in A, A^*. If $\{f_n\}$ is a basis for A^*, then $\{x_n\}$ is a basis for A.

Proof: The proof simply involves the verification of conditions (1), (2), and (3) in Theorem 4.2. Condition (1) is assumed. To verify (2) we assume there exists a $y \in A$ such that $\|y-x\| \geq \delta > 0$ for all x in the linear closure of $\{x_n\}$. Then by Theorem 2.1 there must exist an $f \in A^*$ such that $f(y) = 1$ and $f(x_n) = 0$ for all n. Since f is not the zero functional, there must exist scalars $\{a_n\}$ not all zero such that $f = \Sigma a_n f_n$. Suppose that $a_k \neq 0$. We then have

$$0 = f(x_k) = \sum_{n=1}^{\infty} a_n f_n(x_k) = a_k f_k(x_k) = a_k \neq 0$$

which is a contradiction and conditon (2) must hold. To prove (3) we let B be the canonical image of A under the mapping T described in the introduction. If we now consider $T : A \to B$ then T is an isometric, onto mapping. To prove condition (3) it will then suffice to show for each X in B that $\{Y_n(X)\}$ is pointwise bounded on B where

$$Y_n(X) = \sum_{k=1}^{n} X(f_k) X_k$$

and $X_k = T(x_k)$ for each k. Since each $Y_n(X)$ is in B then for any $f \in A^*$ we have

$$(Y_n(X))(f) = \sum_{k=1}^{n} X(f_k) X_k(f) = X(\sum_{k=1}^{n} X_k(f) f_k) \to X(f)$$

as $n \to \infty$. Hence $\{(Y_n(X)(f))\}$ is pointwise bounded on A^*. By the Uniform Boundedness Principle we have $\{Y_n(X)\}$ is pointwise bounded on B, the desired conclusion. The three conditions have been verified and we conclude that $\{x_n\}$ is a basis for A.

In the next chapter we prove a partial converse of this theorem. That the converse does not hold can be seen by applying Example 5.2 of the next chapter.

The most serious drawback to Theorem 4.4 is that given a Banach space A, the conjugate space A^* may not have the desired properties with which to apply the theorem. We give two examples of this considering A to be the spaces ℓ and c_0 respectively.

Example 4.1. Suppose we let $A = \ell = \ell_1$ as was defined in Chapter I. It can easily be verified that $A^* = m$ and that for any $f \in A^*$ and $x \in m$ we have $f(x) = \Sigma a_k b_k$ where $f = \{a_k\}$ and $x = \{b_k\}$. It is clear that ℓ has a basis. However we cannot use the previous theorem to show this since m does not have a basis. This follows from the fact that m is not separable.

Example 4.2. Suppose now we let $A = c_0$ as was defined in Chapter I. It can be verified that $A^* = \ell$. Now for each n let z_n be a sequence of zeros with a one in the $n^{\underline{th}}$ place. Suppose now we define $f_1 = z_1$ and $f_n = z_n - (-1)^{n-1} z_1$ for $n \geq 2$. Now as in Example 4.1 $f(x) = \Sigma a_k b_k$ for any $x \in A$ and $f \in A^*$ where $f = \{a_k\}$ and $x = \{b_k\}$. From this relation it is clear that there can be no sequence $\{x_n\}$ in A such that $(\{x_n\}, \{f_n\})$ forms a biorthogonal pair in A, A^*. For suppose we try and find an x_1 such that $f_1(x_1) = 1$ and $f_n(x_1) = 0$ for $n \geq 2$. Then by the nature of f_1 we see that $a_1 = 1$ where $x_1 = \{a_n\}$. Similarly we must have $a_2 = -1$ if $f_2(x_1) = 0$ is to hold. In general we must have $a_n = (-1)^{n-1}$ for all n. Now by the nature of the norm in c_0 it is clear $x_1 = \{a_n\}$ cannot be in c_0. We therefore conclude that no such x_1 can exist in c_0 with the desired properties. Hence there can exist no biorthogonal sequence $\{x_n\}$ in c_0 and again Theorem 4.4 is not applicable even though c_0 does have a basis.

Recall the definition of a weak basis in a Banach space A. It is clear that given a basis for A, this basis is also a weak basis for A. That the converse of this statement is also true is proved in the next result.

Theorem 4.5. A weak basis for a Banach space is a basis for the space.

Proof: Suppose $\{x_n\}$ is a weak basis for a Banach space A. Again we use Theorem 4.2 to prove the result. Let x be arbitrary in A. Then there exists a unique sequence $\{a_n\}$ of scalars where

$\sum_{n=1}^{k} a_k x_k$ converges weakly to x. For each n we define the functional f_n as $f_n(x) = a_n$ for all $x \in A$. Clearly $f_n(x_m) = 1$ for $m = n$ and zero otherwise. We now must show that $f_n \in A^*$ for each n, that is, that f_n is linear and continuous. The linearity is clear. To prove the continuity we first define a new norm on A as

$$\|x\|_1 = \sup \{\|\sum_{n=1}^{k} a_n x_n\|: k = 1, 2, \ldots\}.$$

We must show that $\|\cdot\|_1$ is complete. The proof proceeds as in Lemma 3.2. Suppose $\{z_n\}$ is Cauchy in A. For each n, let $\{a_{nk}\}_{n=1}^{\infty}$ be the sequence of scalars associated with z_n. Then as in Lemma 3.2 we have that $\{a_{nk} x_k\}_{n=1}^{\infty}$ is Cauchy in the old norm for each k. By the completeness of the old norm we have

$$y_k = \text{old} \lim_{n\to\infty} a_{nk} x_k.$$

for appropriate y_k. We also have for all k

$$a_k = \lim_{n\to\infty} a_{nk}$$

for appropriate a_k. It is now possible to show that $\sum_{k=1}^{n} y_k$ converges weakly. Consider any f in A^*. We then have

$$|f(\sum_{k=p}^{q} y_k)| \leq |f(\sum_{k=p}^{q} (y_k - a_{nk}x_k))| + |f(\sum_{k=p}^{q} (a_{nk}-a_{mk})x_k)| + |f(\sum_{k=p}^{q} a_{mk}x_k)| .$$

Choose $\epsilon > 0$. Now fix m large enough so that $\|z_n - z_m\|_1 \leq \epsilon/6\|f\|$ for $n \geq m$. By the continuity of f, we then have

$$|f(\sum_{k=p}^{q} (a_{nk}-a_{mk})x_k)| \leq 2\|f\| \cdot \|z_n - z_m\|_1 = \epsilon/3$$

for all $n \geq m$. Now since $\sum_{k=1}^{n} a_{mk}x_k$ converges weakly for each m, then with m fixed as above, we can choose N such that for $p, q \geq N$ we have

$$|f(\sum_{k=p}^{q} a_{mk}x_k)| \leq \epsilon/3.$$

Now fix p and q such that $q \geq p \geq N$. We can then pick n large enough so that

$$|f(\sum_{k=p}^{q} (y_k - a_{nk}x_k))| \leq \epsilon/3.$$

Thus $\sum_{k=1}^{n} y_k$ must converge weakly. Then define z to be that element of A to which $\sum_{k=1}^{n} y_k$ converges weakly. Proceeding exactly as in Lemma 3.2 we then have that

$$\|z_n - z\|_1 \to 0 \quad \text{as} \quad n \to \infty$$

which shows $\|\bullet\|_1$ to be complete.

We would now like to show the two norms are equivalent. This will follow by Lemma 3.3 if we can show one of the norms to be stronger than the other. Let x be arbitrary in A. We may assume $x \neq \theta$. Now by Corollary 2.1 there exists an $g \in A^*$ such that $g(x) = \|x\|$ and $\|g\| = 1$. Now suppose $\{a_n\}$ is the sequence of scalars associated with x. We then have

$$\begin{aligned}
\|x\| = |g(x)| &= \lim_{n\to\infty} |g(\sum_{k=1}^{n} a_k x_k)| \\
&= |g(\sum a_k x_k)| \\
&\leq \|g\| \cdot \| \sum a_n x_k \| \\
&\leq \|g\| \cdot \|x\|_1 \\
&\leq \|x\|_1 .
\end{aligned}$$

Thus we have

$$\|x\| \leq \|x\|_1$$

for all $x \in A$. Hence $\|\bullet\|_1$ must be stronger than $\|\bullet\|$ and by

Lemma 3.3 the norms are equivalent. But each of the linear functionals f_n is continuous with respect to the new norm since for $x \in A$ we have

$$|f_n(x)| = \|\sum_{k=1}^{n} f_k(x)x_k - \sum_{k=1}^{n-1} f_k(x)x_k\| / \|x_n\|$$

$$\leq 2\|x\|_1 / \|x_n\|$$

$$\leq (2/\|x_n\|)\|x\|_1.$$

But by the equivalence of the norms, f_n must also be continuous with respect to the old norm. Condition (1) of Theorem 4.2 is thus proved. To verify condition (2), we suppose $\{x_n\}$ is not fundamental. We can then find a $y \in A$ and $f \in A^*$ such that $f(y) = 1$ and $f(x) = 0$ for all x in the linear closure of $\{x_n\}$. Since $y \in A$, there exists a sequence of scalars $\{b_n\}$ such that $\sum_{k=1}^{n} b_n x_n$ converges weakly to y. But f is in A^* and so

$$1 = f(y) = f(\Sigma\, b_n x_n) = \Sigma\, b_n f(x_n) = 0$$

which implies $\{x_n\}$ must be fundamental. Now define the sequence $\{z_n(x)\}$ as in condition (3). Since $z_n(x)$ is in A for each n, then let Z_n be the element of A^{**} which is identified with $z_n(x)$ under the mapping T. Then for any $f \in A^*$ we have

$$|Z_n(f)| = |f(\sum_{k=1}^{n} f_k(x)x_k)|.$$

But by the weak convergence of $\sum_{k=1}^{n} f_k(x)x_k$ we have that $Z_n(f)$ converges to $Z(f)$ for some Z in A^{**}. This implies that the sequence $\{Z_n(f)\}$ is pointwise bounded on A^{**}. By the Uniform Boundedness Principle the sequence $\{Z_n\}$ is bounded in A^{**}. But this implies that the sequence $\{z_n(x)\}$ is pointwise bounded in A. Condition (3) is thus proved and $\{x_n\}$ must be a basis for A.

As a corollary to the above theorem we have the following result.

Corollary 4.1. Suppose $(\{x_n\}, \{f_n\})$ is a biorthogonal pair in A, A^*. If $\sum_{n=1}^{m} f(x_n)f_n$ converges weakly to f for all f in A^*, then $\{x_n\}$ is a basis for A.

Proof: Consider any $x \in A$ and let $X = T(x)$ be in A^{**}. Then for any f in A^* we have

$$X(\sum_{n=1}^{m} f(x_n)f_n) \rightarrow X(f).$$

But by the way X was defined we have for all m

$$X\left(\sum_{n=1}^{m} f(x_n)f_n\right) = \sum_{n=1}^{m} f(x_n)f_n(x)$$

$$= f\left(\sum_{n=1}^{m} f_n(x)x_n\right)$$

Hence $f\left(\sum_{n=1}^{m} f_n(x)x_n\right) \to X(f) = f(x)$ as $m \to \infty$.

Since x was arbitrary and the above convergence holds for all f, then $\{x_n\}$ is a weak basis and hence a basis by Theorem 4.5.

V. BASES IN CONJUGATE SPACES

Suppose now we consider the problem of determining when A^* has a basis for an arbitrary Banach space A. Of course all the theorems of the previous chapter go through since we can let A^* take the place of A, A^{**} of A^*, and A^{***} of A^{**}. However a more interesting problem arises if we assume the original space A has a basis. For example consider again the spaces c_0, ℓ and m. We know that both c_0 and ℓ have bases while m does not. Thus both ℓ and m are conjugate spaces of a space that has a basis, and yet only ℓ has a basis. The question we want to consider then is what type of conditions can we impose on the space A^* to insure it has a basis whenever A has a basis.

In this chapter we will make the following standard assumption.

<u>Standard Assumption.</u> Assume A is a Banach space with basis $\{x_n\}$ and $\{f_n\}$ is a sequence in A^* such that $(\{x_n\}, \{f_n\})$ forms a biorthogonal pair in A, A^*.

It will then be possible to give several conditions which, when imposed on A^*, are necessary and sufficient for $\{f_n\}$ to be a basis for A^*. It will also be shown that the choice of basis in A plays an important role in what can be said about a basis in A^*.

Perhaps one of the simplest conditions we add to insure that

$\{f_n\}$ is a basis for A^* is given in the following result.

Theorem 5.1. If $\{f_n\}$ is fundamental, then $\{f_n\}$ is a basis for A^*

Proof: Theorem 4.2 again lends itself to the proof. By assumption condition (2) is met. If we let $X_n = T(x_n)$ for each n, then $(\{f_n\}, \{X_n\})$ form a biorthogonal pair in A^*, A^{**} and condition (1) is met. Now suppose we define the functionals $g_n : A^* \to A^*$ by $g_n(f) = \sum_{k=1}^{n} X_k(f)f_k$. We must show the sequence $\{g_n\}$ is pointwise bounded on A^*. Since $g_n(f)$ is in A^* for all n, then for any $x \in A$ we have

$$(g_n(f))(x) = \sum_{k=1}^{n} X_k(f)f_k(x)$$

$$= \sum_{k=1}^{n} f(x_k)f_k(x)$$

$$= f(\sum_{k=1}^{n} f_k(x)x_k) \to f(x) \quad \text{as} \quad n \to \infty.$$

Hence $\{(g_n(f))(x)\}$ must be bounded on A. By the Uniform Boundedness Principle we have that $\{g_n\}$ is pointwise bounded on A^*, the desired conclusion. Thus $\{f_n\}$ is a basis for A^*.

The following theorem, due to James (1950), gives a more interesting criterion for $\{f_n\}$ to be a basis for A^*.

<u>Theorem 5.2.</u> The sequence $\{f_n\}$ is a basis for A^* if and only if for any $g \in A^*$ we have $\lim_{n\to\infty} \|g\|_n = 0$ where $\|g\|_n$ is the norm of g on the linear closure of $(x_n, x_{n+1}, \ldots)$.

Proof: Suppose that $\{f_n\}$ is a basis for A^*. Let g be any point of A^*. Then there exists scalars $\{a_n\}$ such that $g = \Sigma a_n f_n$. For notational convenience we define A_n to be the linear closure of $(x_n, x_{n+1}, \ldots)$ for all n. We have

$$\|g\|_n = \sup \{|g(x)| : x \in A_n \text{ and } \|x\| \leq 1\}.$$

Since $\|g\|_n$ is finite for all n, then given $\epsilon > 0$ there must exist a $y \in A_n$ with $\|y\| = 1$ and

$$|g(y)| > \|g\|_n - \epsilon .$$

But since $g = \Sigma a_n f_n$ then $g(y) = \Sigma a_n f_n(y)$. Since $\{x_n\}$ is a basis for A we have that $y = \sum_{m=n}^{\infty} b_m x_m$ for some appropriate scalars $(b_m, b_{m+1}, \ldots)$. We then have

$$g(y) = \sum_{k=1}^{\infty} a_k f_k \Big(\sum_{m=n}^{\infty} b_m x_m\Big)$$

$$= \sum_{k=1}^{\infty} a_k \Big(\sum_{m=n}^{\infty} b_m f_k(x_m)\Big)$$

$$= \sum_{k=n}^{\infty} a_k \Big(\sum_{k=n}^{\infty} b_k f_k(x_k)\Big)$$

$$= \sum_{k=n}^{\infty} a_k f_k \Big(\sum_{k=n}^{\infty} b_k x_k\Big).$$

But this implies

$$|g(y)| = \Big|\sum_{k=n}^{\infty} a_k f_k \Big(\sum_{k=n}^{\infty} b_k x_k\Big)\Big|$$

$$\leq \Big\|\sum_{k=n}^{\infty} a_k f_k\Big\| \cdot \|y\|$$

$$\leq \Big\|\sum_{k=n}^{\infty} a_k f_k\Big\|.$$

Putting the inequalities together we then have

$$\|g\|_n < |g(y)| + \epsilon \leq \Big\|\sum_{k=n}^{\infty} a_k f_k\Big\| + \epsilon .$$

Since ϵ was arbitrary we have that

$$\|g\|_n \leq \|\sum_{k=n}^{\infty} a_k f_k\|.$$

But $\sum a_k f_k$ is a convergent series in A^* and thus $\|\sum_{k=n}^{\infty} a_k f_k\| \to 0$ as $n \to \infty$. But this implies $\|g\|_n \to 0$ as $n \to \infty$, the desired result.

Conversely, suppose that $\lim_{n\to\infty} \|g\|_n = 0$ for all g in A^*. Let g be arbitrary in A^*. We want to show the existence of a unique sequence $\{a_n\}$ of scalars such that $\|g - \sum_{k=1}^{n} a_k f_k\| \to 0$ as $n \to \infty$. Choose $\epsilon > 0$. Since $\{x_n\}$ is a basis for A, then by Theorem 4.3 there exists an M such that

$$\|\sum_{n=1}^{p} a_n x_n\| \leq M \|\sum_{n=1}^{q} a_n x_n\|$$

for any scalars $\{a_n\}$ and integers p and q with $q > p$. Now choose N such that $\|g\|_N \leq \epsilon/(M+1)$. Then for $x \in A$ and any integer n we have

$$|g(x) - \sum_{k=1}^{n} g(x_k)f_k(x)| = |g(\sum_{k=n+1}^{\infty} a_k x_k)|$$

$$\leq \|g\|_N \|\sum_{k=n+1}^{\infty} a_k x_k\|$$

$$\leq \|g\|_N \|\sum_{k=1}^{\infty} a_k x_k - \sum_{k=1}^{n} a_k x_k\|$$

$$\leq \|g\|_N (\|x\| + M\|x\|)$$

$$\leq \|g\|_N (\|x\|)(M+1)$$

$$\leq \epsilon \|x\|$$

But this implies that

$$\|g - \sum_{k=1}^{n} g(x_k)f_k\| \leq \epsilon \quad \text{for all} \quad n \geq N.$$

Thus g must certainly have a representation as $\Sigma a_n f_n$ for appropriate a_n. In fact if we let $a_n = g(x_n)$ for all n then the uniqueness of the representation certainly follows from the biorthogonality relationships and we see that $\{f_n\}$ is a basis for A^*.

We are now in a position to show that whether or not $\{f_n\}$ is a basis for A^* is dependent upon the choice of basis in A. Consider the following two examples.

Example 5.1. Let $A = c_0$ and $A^* = \ell$. Suppose we choose as a basis for c_0 the sequence $\{x_n\}$ where $x_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, the one in the $n^{\underline{th}}$ position, for all n. Then define the sequence $\{f_n\} \epsilon A^*$ by $f_n = x_n$ for all n. Clearly $\{f_n\}$ is biorthogonal with respect to $\{x_n\}$. Since $\{f_n\}$ is also a basis for ℓ, then our choice of basis in A gives a corresponding basis in A^*.

Example 5.2. Of course $A = c_0$ and $A^* = \ell$. However this time let us define $\{x_n\}$ as $x_1 = (1, 0, 0, \ldots)$ and $x_n = ((-1)^{n+1}, (-1)^{n+2}, \ldots, (-1)^{2n}, 0, 0, \ldots)$ for $n \geq 2$. To see that $\{x_n\}$ is a basis for A first define $y_n = x_n$ for all n where the x_n is of Example 5.1. Since $\{y_n\}$ is a basis for c_0, then for any x we have $x = \Sigma a_n y_n$ for an appropriate sequence $\{a_n\}$. Now for all n define the sequence $\{b_n\}$ as $b_n = a_n + a_{n+1}$. If we define x_0 as $x_0 = (0, 0, \ldots)$ then we have

$$\left\|x - \sum_{n=1}^{m} a_n y_n\right\| = \left\| x - \sum_{n=1}^{m} a_n(x_n + x_{n-1})\right\|$$

$$= \left\|x - \sum_{n=1}^{m} a_n x_n - \sum_{n=1}^{m} a_n x_{n-1}\right\|$$

$$= \left\|x - \sum_{n=1}^{m} a_n x_n - \sum_{n=1}^{m-1} a_{n+1} x_n\right\|$$

$$= \left\|x - \sum_{n=1}^{m-1} (a_n + a_{n+1}) x_n - a_n x_n\right\|$$

$$\geq \left\|x - \sum_{n=1}^{m-1} b_n x_n\right\| - \left\|a_n x_n\right\|.$$

Thus we have

$$\left\|x - \sum_{n=1}^{m-1} b_n x_n\right\| \leq \left\|x - \sum_{n=1}^{m} a_n y_n\right\| + |a_m|.$$

But since $x \in c_0$ then $|a_m| \to 0$ as $m \to \infty$. Hence

$$\left\|x - \sum_{n=1}^{m-1} b_n x_n\right\| \to 0 \quad \text{as} \quad m \to \infty.$$

Since x was arbitrary we have that $\{x_n\}$ is also a basis.

Now for this particular basis the biorthogonal sequence, as may be expected, is defined by $f_n = (0, 0, \ldots, 0, 1, 1, 0, 0, \ldots)$ where the two ones are in the n and $n+1$ positions. We now want to show that the sequence $\{f_n\}$ is not a basis for A^*. To do this we apply the previous theorem and show the existence of a g in ℓ such that $\|g\|_n$ does not approach zero as $n \to \infty$. Define g as $g = (1, 0, 0, \ldots)$. Then g is in ℓ. Now consider any n. Then $\|x_n\| = 1$. Hence $\|g\|_n \geq |g(x_n)| = 1$. Thus $\|g\|_n \geq 1$ for all n and clearly $\|g\|_n$ does not approach zero as $n \to \infty$. Applying the previous theorem we conclude that $\{f_n\}$ is not a basis for A^*.

As in the previous chapter, weak convergence plays a role in the existence of a basis. In this particular case two different, but very similar conditions, are both necessary and sufficient for $\{f_n\}$ to be a basis in A^*.

Theorem 5.3. The sequence $\{f_n\}$ is a basis for A^* if and only if one and hence both of the following conditions hold.

1. If a bounded sequence $\{y_k\} \epsilon A$ must converge weakly to θ whenever it is true that $\lim_{k\to\infty} f_n(y_k) = 0$ for each n.
2. If a bounded sequence $\{y_k\} \epsilon A$ must converge weakly to θ whenever y_k is in the linear closure of $(x_k, x_{k+1}, \ldots)$ for each k.

Proof: Sufficiency is easy. First of all observe that $(1) \to (2)$

and thus we need only prove condition (2) sufficient. Consider any $g \in A^*$. Choose a sequence $\{y_n\}$ such that $\|y_n\| = 1$ for all n, $|g(y_n)| \geq 1/2 \|g\|_n$, and y_n is in A_n where A_n is defined as in the proof of the previous theorem. Now $\{y_n\}$ satisfies the hypothesis of (2) above and so $y_n \to \theta$ weakly. In particular $g(y_n) \to g(\theta) = 0$. But this implies that $\|g\|_n \to 0$ as $n \to \infty$. Since g was arbitrary in A^*, then by Theorem 5.2 we have that $\{f_n\}$ is a basis for A^*.

Since (1) $\to$ (2) it is enough to prove necessity for (1). Suppose then that the sequence $\{f_n\}$ is a basis for A^* and $\{y_k\}$ is any bounded sequence in A such that $f_n(y_k) \to 0$ as $k \to \infty$ for each n. Choose an arbitrary $f \in A^*$ and suppose $f = \Sigma a_n f_n$ for appropriate scalars $\{a_n\}$. This implies that given $\epsilon > 0$ there must exist an M independent of k such that for all $m \geq M$

$$\|f(y_k) - \sum_{n=1}^{m} a_n f_n(y_k)\| \leq \epsilon/2 \quad \text{for all} \quad n.$$

Now since $\lim_{k\to\infty} f_n(y_k) = 0$ for all n then there exist integers $N_1, N_2, \ldots, N_M$ such that

$$|a_n f_n(y_k)| \leq \epsilon/2M$$

for $n = 1, 2, \ldots, M$ and all $k \geq N_1, N_2, \ldots, N_M$ respectively.

Taking $N = \max(N_1, N_2, \ldots, N_M)$ we have for all $k \geq N$,

$$\left|\sum_{n=1}^{M} a_n f_n(y_k)\right| \leq \sum_{n=1}^{M} |a_n f_n(y_k)| \leq \frac{M\epsilon}{2M} = \epsilon/2.$$

Thus we have

$$|f(y_k)| - \left|\sum_{n=1}^{M} a_n f_n(y_k)\right| \leq \left|f(y_k) - \sum_{n=1}^{M} a_n f_n(y_k)\right| \leq \epsilon/2.$$

But this implies

$$|f(y_k)| < \epsilon/2 + \left|\sum_{n=1}^{M} a_n f_n(y_k)\right| < \epsilon/2 + \epsilon/2 = \epsilon .$$

Hence $f(y_k) \to f(\theta) = 0$ as $k \to \infty$ and $\{y_k\}$ must be weakly convergent to θ as desired.

The next result shows that if boundedness in A^* implies convergence in A^*, then we have that $\{f_n\}$ will be a basis for A^*. We sometimes denote this condition by saying that A^* is boundedly complete.

<u>Theorem 5.4.</u> The sequence $\{f_n\}$ is a basis for A^* if and only if $\sum a_n f_n$ converges whenever $\sup_k \left\|\sum_{n=1}^{k} a_n f_n\right\| < \infty$.

Proof: Suppose first that $\{f_n\}$ is a basis for A^* and that $\|\sum_{n=1}^{k} a_n f_n\| \leq M < \infty$ for all k. Since the sequence $\{x_n\}$ is a basis, then $\{x_n\}$ is linearly independent. We can then define a linear functional on the span of $\{x_n\}$ by

$$f(x_n) = a_n \qquad n = 1, 2, \ldots$$

Clearly f is continuous on the span of $\{x_n\}$. For suppose $x = \sum_{n=1}^{m} b_n x_n$. We then have

$$|f(x)| = |f(\sum_{n=1}^{m} b_n x_n)|$$

$$= |\sum_{n=1}^{m} b_n f(x_n)|$$

$$= |\sum_{n=1}^{m} b_n a_n|$$

But since $\{x_n\}$ is a basis, then $b_n = f_n(x)$ for each n. Hence we have

$$|f(x)| = \left|\sum_{n=1}^{m} b_n a_n\right| = \left|\sum_{n=1}^{m} f_n(x) a_n\right|$$

$$= \left|\left(\sum_{n=1}^{m} a_n f_n\right)(x)\right|$$

$$\leq \left\|\sum_{n=1}^{m} a_n f_n\right\| \cdot \|x\|$$

$$\leq M \cdot \|x\|$$

and f is continuous. Now by the Hahn-Banach Theorem we can extend f so as to be defined and continuous on all of A. Hence f is in A^* and $f(x_n) = a_n$ for all n. But since $\{f_n\}$ is a basis for A^* then

$$f = \Sigma f(x_n) f_n = \Sigma a_n f_n.$$

Hence $\Sigma a_n f_n$ must be convergent.

Conversely suppose that $\Sigma a_n f_n$ is convergent whenever $\sup_k \left\|\sum_{n=1}^{k} a_n f_n\right\| < \infty$. Let f be arbitrary in A^*. We first want to show that there exists an M such that for all k,

$$\left\|\sum_{n=1}^{k} f(x_n) f_n\right\| \leq M. \tag{1}$$

We first observe that if x is arbitrary in A and

$$\left|\sum_{n=1}^{k} f(x_n)f_n(x)\right| \leq M$$

for all k, then by the Uniform Boundedness Principle the relation in (1) will hold. But since f is continuous and

$$\left|\sum_{n=1}^{k} f(x_n)f_n(x)\right| = \left|f\left(\sum_{n=1}^{k} f_n(x)x_n\right)\right| \leq \|f\|\cdot\left\|\sum_{n=1}^{k} f_n(x)x_n\right\|,$$

then it is sufficient to prove

$$\left\|\sum_{n=1}^{k} f_n(x)x_n\right\|$$

is bounded for all k. But

$$\sum_{n=1}^{k} f_n(x)x_n \to x \quad \text{as} \quad k \to \infty$$

and so the relation in (1) must hold. By hypothesis we then have that $\Sigma f(x_n)f_n$ is convergent. Suppose now that

$$\sum_{n=1}^{k} f(x_n)f_n \to g \quad \text{as} \quad k \to \infty$$

in norm. Now for any m we have that

$$\sum_{n=1}^{k} f(x_n)f_n(x_m) = f(x_m)$$

for $n \geq m$. Thus we have

$$f(x_m) = \lim_{k \to \infty} \sum_{n=1}^{k} f(x_n)f_n(x_m) = g(x_m).$$

Thus $f(x_m) = g(x_m)$ for any m. Hence we have $f(x) = g(x)$ for any x in the span of $\{x_n\}$. But since the span of $\{x_n\}$ is dense in A then $f(x) = g(x)$ for all $x \in A$. Now let x be arbitrary in A and $\|x\| \leq 1$. Then

$$\begin{aligned} \|f-g\| &= \sup \{|(f-g)(x)|: \|x\| \leq 1\} \\ &= \sup \{|f(x)-g(x)|: \|x\| \leq 1\}. \end{aligned}$$

But since $f(x) = g(x)$ for all $x \in A$, then

$$\|f-g\| = 0.$$

Hence we have

$$\sum_{n=1}^{k} f(x_n)f_n \to f \quad \text{as} \quad k \to \infty.$$

Since f was arbitrary then every f must have a representation as above. By the biorthogonality relationships the representation must be unique and so $\{f_n\}$ is a basis for A^* as desired.

In the last two results we show a relation between the reflexivity of A and the sequence $\{f_n\}$ in A^*.

Theorem 5.5. Suppose A is reflexive. Then the sequence $\{f_n\}$ is a basis for A^*.

Proof: This is a simple consequence of Theorem 4.4 of the previous chapter. Let $X_n = T(x_n)$ for all n. Clearly $(\{f_n\}, \{X_n\})$ forms a biorthogonal pair in A^*, A^{**}. But since A is reflexive then $\{X_n\}$ is a basis for A^{**}. By Theorem 4.4 $\{f_n\}$ is a basis for A^* as desired.

One might ask if the converse of the above theorem holds. The answer is of course no. All we have to do is consider the spaces c_0, ℓ, and m with c_0 and ℓ having their natural bases. Clearly c_0 is not relfexive. However if we add the extra condition that A be boundedly complete, that is, that $\Sigma a_n x_n$ be convergent whenever $\sup_k \| \sum_{n=1}^{k} a_n x_n \| < \infty$, then A is reflexive. In fact it can be shown that A is reflexive if and only if $\{f_n\}$ is a basis for A^* and A is boundedly complete. This result was first given by James (1950).

<u>Theorem 5.6.</u> Let A be a Banach space. Then A is reflexive if and only if $\{f_n\}$ is a basis for A^* and A is boundedly complete.

Proof: Suppose first A is reflexive. By the previous theorem $\{f_n\}$ is a basis for A^*. Suppose now that $\| \sum_{n=1}^{k} a_n x_n \| \leq M$ for all k. Since the sequence $\{f_n\}$ is linearly independent, we can define a functional X on the span of $\{f_n\}$ such that $X(f_n) = a_n$ for each n. Clearly X is linear. Also X is continuous on $[\{f_n\}]$ for suppose $f = \sum_{k=1}^{n} b_k f_k$. We then have