AN ELEMENTARY APPROACH TO HYPERBOLIC GEOMETRY

by

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A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1950

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Date thesis presented: May 2, 1950

Typed by: Edith M. Smith

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Introduction

So much has been written on the history and importance of non-Euclidean geometry that little concerning it need be mentioned here. Suffice it to say that the papers of Lobatchewski and Bolyai stand as perhaps among the most important milestones in the history of mathematics, involving, as they do, ideas which were completely new and exceedingly fruitful. These led to an investigation of the postulates and tacit assumptions of Euclid in particular, and of postulate sets in general. Perhaps the culmination of this phase of development was the formulation by Hilbert of his postulate set for Euclidean geometry, and of corresponding sets for the two non-Euclidean geometries. The Hilbert postulates are now so standard that they are the obvious ones to follow in a paper of this sort, and such has been done.

Inasmuch as this paper is concerned with the hyperbolic geometry, it might be well at this point to state explicitly the characteristic parallel postulate chosen by Hilbert for the hyperbolic geometry. It is as follows: Given a line b and a point A not lying on b, then there exists, in the plane determined by b and A, an infinite number of lines which contain A but not any point of b.

Inevitably, an investigation of postulate sets leads to questions of consistency. The question might be asked, "Although no contradictions have ever been reached in non-Euclidean geometry, can we be sure that none will arise?" As every mathematician knows, the answer is, "Only relatively." In the field of analysis, postulate sets are shown to be as consistent as the famous postulate set of Peano, which, it is felt, is almost certainly consistent. Even in the field of geometry, we have elaborate treatments showing the Hilbert sets as consistent as Peano's. On the other hand, however, if we can devise a Euclidean model for hyperbolic geometry, (that is, a Euclidean geometrical system in which the primitive terms of hyperbolic geometry can be interpreted in such a manner as to satisfy, say, Hilbert's postulates for hyperbolic geometry), then this geometry must be as consistent as Euclidean geometry, whose consistence is taken for granted.

Now, many methods have been devised for deriving the theorems of non-Euclidean geometry, ranging from the pure synthetic methods of Lobatchewski and Bolyai to the more advanced viewpoint of modern projective geometry. "But," it might be asked,"why cannot these theorems be derived directly from a Euclidean model?" Obviously, it is theoretically possible, since the elements of the geometry of a model are in direct one-to-one correspondence with the elements of the geometry which the model represents. The beauty of this idea is that we would be working with Euclidean geometry, with which we are more familiar.

This is precisely the viewpoint of this paper. From the Poincaré model of the hyperbolic geometry, many of the theorems of hyperbolic geometry will be developed.

The simplicity of proofs and the rapidity of development in this approach are to be noticed. Proofs of theorems will be presented which, in other approaches, become difficult, often requiring many cases or complicated accompanying figures.

It must be pointed out, however, that unfortunately some of the proofs using the model become only reiterations of already existing proofs. Such are many of those theorems of a metric nature. These have not been included here since they may be found elsewhere. But since these proofs apply equally well in the model, and the model gives a more graphic illustration of them, it may be felt that the value of this approach is not diminished, and remains justified.

The Poincaré Model

In this model the points of the hyperbolic plane are represented by the points in the interior of a circle, called the fundamental circle, which we will refer to as F. The points on F correspond to the ideal points of the hyperbolic geometry. The points exterior to F are not considered as points of the geometry at all. The arcs of circles orthogonal to F and interior to F constitute the lines of the geometry. Since they are not actually lines, they will be referred to as nominal lines, after Carslaw, and any nominal line will be referred to by placing the letters indicating two of its points adjacent to one another and placing a bar over them, as $\overline{AB}$. In fact, in referring to a nominal triangle, or any other nominal figure, the bar will be used. When referring to the circle coinciding with any nominal line $\overline{AB}$, we will use the symbol $\odot\widehat{AB}$. The angle between two intersecting nominal lines is defined as the angle between the tangent lines, at the point of intersection, of the circles which coincide with these nominal lines. It is realized that there are three possible relationships two nominal lines may have. They may intersect inside F, in which case they are called intersecting; they may be tangent to one another on F, when they are said to be parallel; or they may not intersect at all, in which case they are designated as non-intersecting lines.

Suppose we have a nominal line $\overline{AB}$ intersecting F in points S and T, with B between T and A, as shown. The metric of the model may be defined as

$$\overline{AB} = \ln(TS, AB),$$

the natural logarithm of the cross-ratio of the indicated cyclic range. This can be shown to satisfy the usual requirements of a metric, provided we define displacement properly.

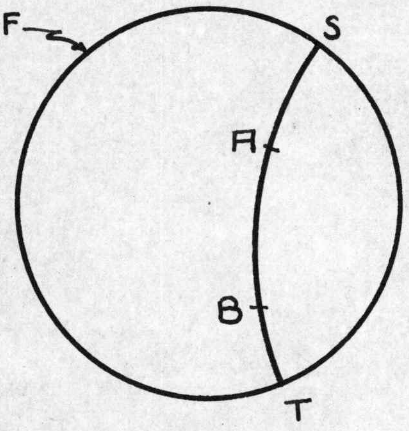

Fig. I

We find that in the model inversion takes the place of reflection. That is, a reflection in nominal line $\overline{AB}$ is represented by an inversion with respect to $\odot\widehat{AB}$. It can be shown that the metric is invariant under an inversion so defined. Since a displacement can always be factored into a product of reflections, it follows that the metric is invariant under the corresponding nominal displacement.

In three dimensions, which we will need to consider, we have instead of a fundamental circle, a fundamental sphere, S. Points within S are the points of the geometry; points on S represent the ideal points; lines are circles orthogonal to S; planes are spheres orthogonal to S. The metric is defined as in the plane, and reflections are represented by appropriate inversions. For a verification of the fact that three dimensional inversion is entirely analogous to the two dimensional case, cf. Court (1, p. 214).

Properties of Parallels and Non-intersecting Lines

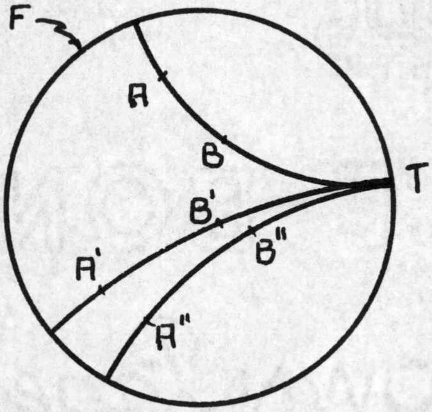

Fig. II

Suppose we have given $\overline{AB} \parallel \overline{A'B'} \parallel \overline{A''B''}$, as in the figure. Obviously, $\odot\widehat{AB}$ is tangent at T to $\odot\widehat{A'B'}$, which is in turn, tangent at T to $\odot\widehat{A''B''}$. Two immediate and obvious theorems are:

Th. 1: If one line is parallel to a second, then the second is parallel to the first.

Th. 2: If two lines are parallel to a third line in the same direction, they are parallel to each other.

Further, since $\odot\widehat{AB}$ is the only circle through either A or B, orthogonal to F, and tangent to $\odot\widehat{A'B'}$ at T, we have

Th. 3: If a line is the parallel through a given point to a given line in a given direction, it is the parallel at each of its points to the given line and in the given direction.

Th. 4: There is one and only one common perpendicular to two non-intersecting lines.

This is obvious, since there is one and only one circle orthogonal to three given circles, namely, the radical circle of the three given circles, and this radical circle is real in the case under consideration.

Th. 5: If two given lines have a common perpendicular, the given lines must be non-intersecting.

If the two given lines are parallel, i.e., if the circles corresponding to these two lines are tangent on F, the radical circle of the given circles and F degenerates to a point-circle, namely, the point of tangency. If the two given lines are intersecting, i.e., if the circles corresponding to these two lines intersect within F, the radical circle of the given circles and F becomes imaginary. Thus, if the two given lines (circles) have a common perpendicular (circle orthogonal to them and to F) they must be non-intersecting.

Properties of Triangles

Note: If we are given any point P in F, it is always easy to find an inversion in a circle orthogonal to F which will carry P into O, the center of F, and F into itself. This fact is important to this development, for it means that we can always place a triangle with one vertex, or any other point, at O, and be assured that our treatment of it is perfectly general.

Th. 1: In $\triangle \overline{OAB}$ an exterior angle is greater than the sum of the opposite interior angles.

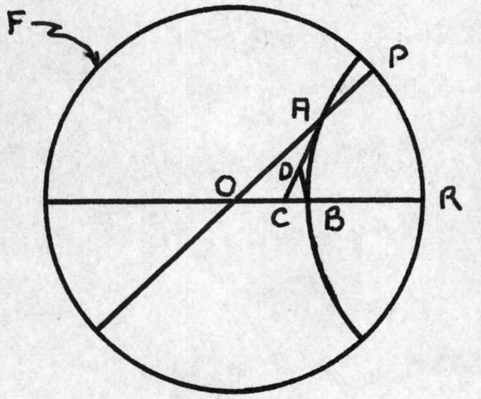

Fig. III

Let OB intersect F at R. Draw the tangent to $\odot \widehat{AB}$ at A. This must intersect OB at some point C. Draw the tangent to $\odot \widehat{AB}$ at B. This must intersect AC at some point D. Now,

$$\measuredangle \overline{ABR} = \measuredangle DBR > \measuredangle DCB,$$

since $\measuredangle DBR$ is an exterior angle of $\triangle DCB$. But, $\measuredangle DCB = \measuredangle COA + \measuredangle OAC$. Therefore

$$\measuredangle DBR = \measuredangle \overline{ABR} > \measuredangle \overline{BOA} + \measuredangle \overline{OAB}.$$

Similarly, if OA intersects F in P,

$$\measuredangle \overline{PAB} > \measuredangle \overline{AOB} + \measuredangle \overline{OBA}.$$

Further, we see that $\measuredangle \overline{ABR} = \pi - \measuredangle \overline{ABO}$. Therefore

$$\pi - \measuredangle \overline{ABO} > \measuredangle \overline{BOA} + \measuredangle \overline{OAB}$$

or, $$\measuredangle \overline{ABO} + \measuredangle \overline{BOA} + \measuredangle \overline{OAB} < \pi ,$$

and we have

Th. 2: The sum of the angles of a triangle is less than two right angles.

Cor.: The sum of the angles of a convex quadrilateral is less than two straight angles.

This is evident since any convex quadrilateral may be divided into two triangles by a line joining either pair of opposite vertices. Each of these triangles has an angle sum less than two right angles, and by summing angles, the corollary follows.

If, in Fig. III, A falls on F, $\overline{OA}$ and $\overline{BA}$ become nominal parallel lines, and $\overline{OAB}$ becomes what has been termed a singly-asymptotic triangle by Coxeter (2, p. 188). Obviously, theorems 1 and 2 above apply to this type of triangle as well as to one with ordinary vertices. But since $\measuredangle \overline{OAB} = 0$, then $\measuredangle \overline{ABR} > \measuredangle \overline{AOB}$, and $\measuredangle \overline{ABO} + \measuredangle \overline{BOA} < \pi$. Thus we have

Th. 3: In a singly-asymptotic triangle, an exterior angle is greater than the opposite interior angle.

Th. 4: In a singly-asymptotic triangle, the sum of the interior angles is less than two right angles.

It is seen that these two theorems might have been stated as properties of parallels, and indeed, theorem 4 may be taken as the characteristic postulate of hyperbolic geometry, in contradiction to Euclid's fifth postulate.

Now, in the singly-asymptotic triangle $\overline{AOB}$, let $\measuredangle\,\overline{ABO}$ be right, as in Fig. IV. Now $\measuredangle\,\overline{BOA}$ is the angle of parallelism corresponding to distance $\overline{OB}$. If we let $\odot\,\widehat{AB}$ shrink continuously, always remaining orthogonal to OR and F, it will assume the positions of members of a non-intersecting coaxal family of circles with limiting point R. At the same time, since OA remains tangent to $\odot\,\widehat{AB}$ at A, $\measuredangle\,\overline{BOA}$ will diminish continuously, and approach the value zero.

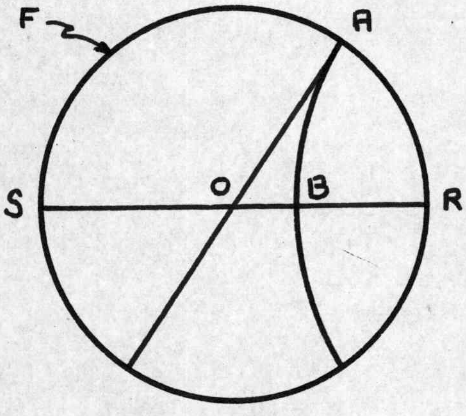

Fig. IV

Also $\overline{OB}$, which is given by $\ln(SR,BO) = \ln(SB/BR)$ will increase without bound, for $BR \rightarrow 0$ as $\odot\,\widehat{AB} \rightarrow R$, SB remaining finite. On the other hand, as $\odot\,\widehat{AB}$ expands, always remaining orthogonal to both F and OR, $\measuredangle\,\overline{BOA}$ increases to $\pi/2$ and $\overline{OB} = \ln(SB/BR)$ decreases to zero since $SB \rightarrow SO$, $BR \rightarrow OR$, $SB/BR \rightarrow SO/OR = 1$. Thus we have

<u>Th. 5</u>: The angle of parallelism H, corresponding to a distance h, is acute, and $H \rightarrow 0$ as $h \rightarrow \infty$, and $H \rightarrow \pi/2$ as $h \rightarrow 0$.

For an actual derivation of the functional relation between an angle and its corresponding distance of parallelism, from the model, see Wolfe (5, p. 214).

<u>Th. 6</u>: The perpendicular bisectors of the sides of a triangle are concurrent.

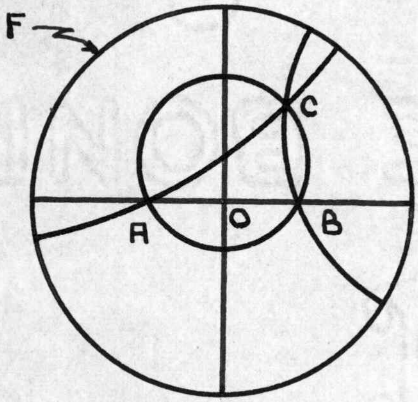

Fig. V

Let us place our triangle as in Fig. V, where $\overline{ABC}$ is the triangle; $\overline{AB}$ lies along a diameter of F, and O, the center of F, is the midpoint of $\overline{AB}$ and thus the midpoint also of AB. Draw the nominal perpendicular bisector of $\overline{AB}$. Since this line is also the perpendicular bisector of AB, it is certainly orthogonal to the circle determined by points A, B, and C, i.e., the circumcircle of $\triangle\overline{ABC}$. Since this argument is perfectly general, the nominal perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ must also be orthogonal to the circumcircle. Since these perpendicular bisectors are orthogonal to both the circumcircle and F, they are members of a coaxal family of circles. If the circumcircle lies entirely within F, the three nominal perpendicular bisectors are concurrent in a point within F. If the circumcircle is internally tangent to F, the three nominal perpendicular bisectors are parallel, and thus have an ideal point in common. If the circumcircle intersects F in two distinct points, the nominal perpendicular bisectors belong to a non-intersecting coaxal family of circles. But in this case, we may draw a circle through the two points of intersection of the circumcircle and F which is orthogonal to F and of necessity orthogonal to the three nominal perpendicular bisectors. Thus, there exists a nominal line perpendicular to all three nominal bisectors, and they are said to have an ultra-ideal point in common.

Th. 7: The internal angle bisectors of a triangle are concurrent. The external angle bisectors are concurrent in pairs with the internal angle bisectors at the opposite vertices.

The concurrence of the internal angle bisectors may be established as in Euclidean geometry. Now, in nominal ΔABC we may escribe circles E_a, E_b, E_c externally to ΔABC such that E_a, E_b, E_c lie within nominal angles BAC, ABC and ACB, respectively. Now by inverting first B and then C to the center of F, it is easy to see that the external angle bisectors at B and C are both orthogonal to E_a. By inverting A to the center of F we see that the internal angle bisector at A is also orthogonal to E_a. It follows that these angle bisectors are concurrent in an ordinary point, an ideal point, or an ultra-ideal point according as E_a lies entirely within F, is internally tangent to F, or intersects F in two distinct points.

Note: For a derivation of the functional relations among the parts of a triangle, and hyperbolic trigonometry in general, from the Poincaré model, see Hoggatt (3).

The Lambert and Saccheri Quadrilaterals

Def.: A Lambert quadrilateral is one in which three of the interior angles are right.

Th. 1: In a Lambert quadrilateral, the fourth angle is acute.

This is obvious, since the sum of all four angles must be less than four right angles, and the sum of three of these is three right angles.

Def.: A Saccheri quadrilateral is formed by drawing equal perpendiculars to the ends of a line segment on the same side of it, and joining the extremities. The given line segment is called the base, the side opposite to the base is called the summit, and the two interior angles at the ends of the summit are called the summit angles.

Th.2: The summit angles of a Saccheri quadrilateral are equal and acute.

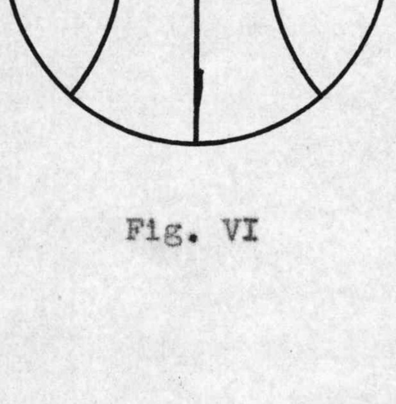

Fig. VI

Place the quadrilateral as shown, with O at the midpoint of AB. Draw OE $\perp$ AB, intersecting $\overline{DC}$ in P. It is not difficult to prove the congruence of figures OBCP and OADP by reflection in OE, Therefore $\measuredangle \overline{BCP} = \measuredangle \overline{ADP}$, and each must be acute since

OBCP is a nominal Lambert quadrilateral.

Th. 3: The line joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both.

In Fig. VI, it is easy to show that if OP $\perp$ AB, then the center of $\odot \overset{\frown}{DC}$ lies on OP, due to the symmetry of the figure. Thus, $\overline{OP} \perp \overline{DC}$, and OP bisects $\overset{\frown}{DC}$, thus bisecting $\overline{DC}$.

Th. 4: The summit and base of a Saccheri quadrilateral are non-intersecting lines.

From theorem 3, $\overline{AB}$ and $\overline{DC}$ have a common perpendicular, ergo, they are non-intersecting lines.

The Equidistant Curve

<u>Th.</u>: The locus of points equally distant from a given line is the orthogonal trajectory of the perpendiculars to the given line and conversely. This locus is called an equidistant curve, and the given line is called the base line.

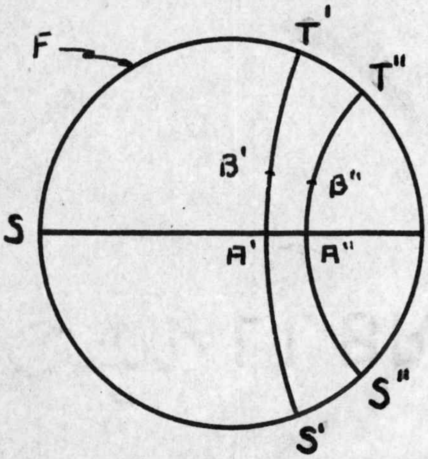

Fig. VII (a)

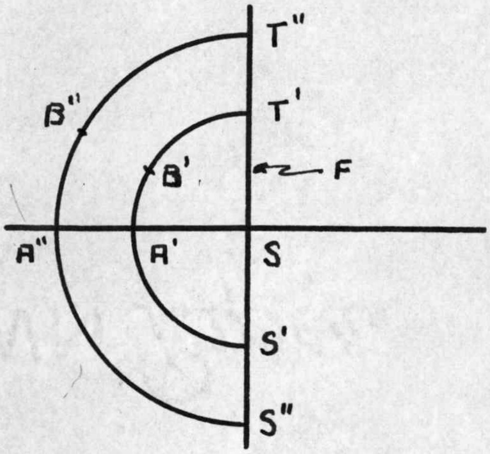

Fig. VII (b)

Suppose that $\overline{ST}$ is any given nominal line placed along a diameter of F as in Fig. VII (a). Let $\overline{S'T'}$ and $\overline{S''T''}$ be two nominal perpendiculars to $\overline{ST}$ and B', B" be two points of the required locus. If we invert the whole figure with respect to T as a center of inversion we get Fig. VII (b), where F and $\overline{ST}$ become perpendicular straight lines, and $\odot\,\widehat{S'T'}$ and $\odot\,\widehat{S''T''}$ become concentric circles with S as their common center. Now, assume first that $\overline{A'B'} = \overline{A''B''}$. Then, since cross-ratio is invariant under inversion, we have, in Fig. VII (b).

$$(T'A'/S'A')/(T'B'/S'B') = (T''A''/S''A'')/(T''B''/S''B'').$$

But $T'A' = A'S'$, $T''A'' = A''S''$, and so $T'B'/S'B' = T''B''/S''B''$. Therefore, in Fig. VII (b), B' and B'' lie on a straight line through S and when inverted back into Fig. VII (a), this line becomes a circle through S and T. But this is a member of the family of circles orthogonal to $\odot \widehat{A'B'}$ and $\odot \widehat{A''B''}$, and the direct theorem is established. The converse is established by a reverse argument.

The Variation in Distance Between Two Lines

Th. 1: Given two intersecting lines, the perpendicular distance from one to the other increases continuously and without bound as the line along which the distance is measured moves away from the point of intersection, and decreases to zero as the line moves toward the point of intersection.

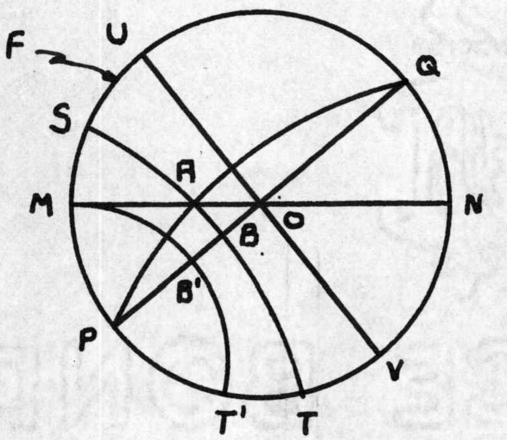

Fig. VIII

Let $\overline{PQ}$ and $\overline{MN}$ be nominal intersecting lines, their point of intersection being placed at O, the center of F. Let $\widehat{PAQ}$ be a nominal equidistant curve with base line $\overline{PQ}$ and intersecting $\overline{MN}$ at A. Let $\overline{AB}$ be perpendicular to $\overline{PQ}$ at B. Thus, the distance from $\widehat{PAQ}$ to $\overline{PQ}$ is $\overline{AB}$. Now let A move continuously toward M. Obviously, as it does so, $S \to M$, $B \to B'$, $T \to T'$, where $MB' = B'T'$, $MP = PT'$, and

$\overline{AB} = \ln(TS,AB) = \ln(TA/SA)(SB/TB) \to \ln(T'M/MM)(MB'/T'B')$
$= \ln(T'M/0)(-1) = \infty.$

On the other hand, as $A \to O$, $B \to O$, $S \to U$, and $T \to V$. Therefore

$$\overline{AB} \to \ln(VU,OO) = \ln 1 = 0.$$

Th. 2: Given two parallel lines, the perpendicular distance from one to the other increases continuously and without bound as the line along which the distance is measured moves opposite to the direction of parallelism, and decreases to a limit of zero as the line moves in the direction of parallelism.

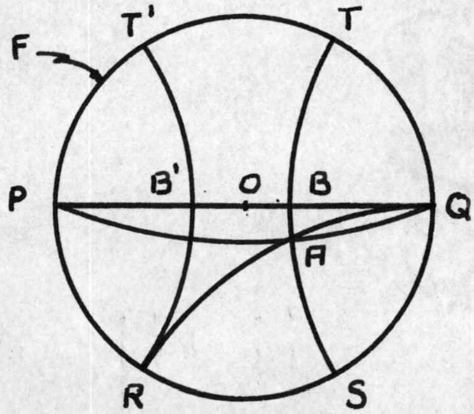

Fig. IX

Let $\overline{PQ}$ through O, and $\overline{RQ}$ be the parallel lines, $\widehat{PAQ}$ an equidistant curve with base line $\overline{PQ}$ and distance $\overline{AB}$ where $\overline{AB}$ is perpendicular to $\overline{PQ}$, and A is the intersection of $\widehat{PAQ}$ and $\overline{RQ}$. Now as $A \to R$, we see that $T \to T'$, $B \to B'$, where $T'P = PR$, $T'B' = B'R$. Therefore $\overline{AB} = \ln(TS, AB)$ $= \ln(TA/SA)(SB/TB) \to \ln(T'R/RR)(RB'/TB') = \ln(T'R/0)(-1)$ $= \infty$. As $A \to Q$, it is evident that AT and AS approach equality, and $\overline{AB} \to \ln 1 = 0$.

Th. 3: Given two non-intersecting lines, the perpendicular distance from one to the other increases without bound as the distance is measured along a line moving away from their common perpendicular, and decreases as the line moves toward their common perpendicular.

Let $\overline{PQ}$, through O, and $\overline{MN}$ be two non-intersecting lines (Fig. X), $\widehat{MBN}$ an equidistant curve with base line

$\overline{MN}$, and $\overline{UV}$ the common perpendicular to $\overline{MN}$ and $\overline{PQ}$. Certainly, in the figure, if $\overline{A'B'}$ and $\overline{AB}$ are both perpendicular to $\overline{MN}$, then $\overline{A'B'} = \overline{AB}$. Now, as $B \to Q$, so that $\widehat{MBN}$ approaches $\widehat{MUN}$, then $T \to Q$, $S \to S''$, and $A \to A''$. Therefore,

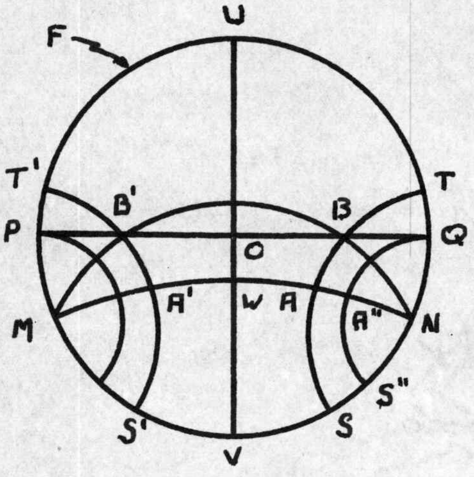

Fig. X

$$\overline{AB} = \overline{A'B'} = \ln(TS,AB)$$
$$= \ln(TA/SA)(SB/TB)$$
$$\to \ln(QA''/S''A'')(S''Q/QQ)$$
$= \infty$. As $B \to O$, $T, T' \to U$, $S, S' \to V$, and $A, A' \to W$. Therefore, $\overline{AB} = \overline{A'B'}$ $= \ln(TS,AB)$ approaches $\ln(UV,WO) = \overline{WO}$, the distance from $\overline{MN}$ to $\overline{PQ}$ along their common perpendicular.

Properties of Horocycles and Equidistant Curves

Note: A horocycle may be defined as the orthogonal trajectory of a sheaf of parallel lines. In the model, a horocycle is represented by a circle tangent to the fundamental circle, F, for a sheaf of parallel lines is represented by a pencil of nominal lines tangent to one another, orthogonal to F, and with the vertex on F, and every orthogonal trajectory of this pencil is a circle tangent to F at the vertex of the pencil.

Th. 1: Every horocycle is congruent to every other horocycle.

If we have given any two nominal horocycles, we can always invert one into the other using their external center of similitude for center of inversion, and a circle orthogonal to F as the circle of inversion. Thus F will invert into itself.

Th. 2: Equal chords on the same or different horocycles subtend equal arcs, and conversely.

We can always invert one horocycle into another, and we can find a sequence of inversions in nominal lines which will invert the horocycle and F into themselves, and invert either equal arcs or chords one into another. The theorem is thus proved by congruence.

Th. 3: A straight line cannot cut a horocycle in more than two points.

Two circles intersect in no more than two points.

Th. 4: If a straight line cuts a horocycle in one point, and is not a radius, it will in general cut it in a second point.

Since the nominal horocycle, H, lies entirely within F, if H and a nominal straight line, C, intersect in one point within F, they must intersect in a second point within F unless C intersects H in its point of tangency with F, in which case C is a radius of H.

Th. 5: Three points of a horocycle determine it.

Obviously, three points of a circle determine it.

Th. 6: Any two equidistant curves with the same distance are congruent.

Given two nominal equidistant curves with the same distance, we may invert one into the other using the external center of similitude of the base lines for center of inversion, and a circle orthogonal to F for the circle of inversion.

Th. 7: For the same or congruent equidistant curves, equal chords subtend equal arcs, and conversely.

The proof of this is analogous to the proof of the corresponding theorem for horocycles, i.e., theorem 2 of this section.

Th. 8: A straight line cannot cut one branch of an equidistant curve in more than two points.

Two circles intersect in no more than two points.

<u>Th. 9</u>: If a straight line cuts an equidistant curve in one point, it will in general cut it in a second unless it is parallel to the base line.

Since the two branches of the nominal equidistant curve form a closed curve lying entirely within F, a circle intersecting it in one point must intersect it in a second point within F, unless the circle intersects the equidistant curve in a point on F, in which case it is parallel to the base line of the equidistant curve.

<u>Th. 10</u>: Three points of an equidistant curve determine it, provided they lie on the same branch.

Three points of a circle determine it.

<u>Note</u>: A nominal circle is actually a circle inside F, since a circle may be defined as the orthogonal trajectory of a pencil of lines. But in the model, this becomes a pencil of circles all orthogonal to F and as such constitute a coaxal family all of whose orthogonal trajectories are circles.

<u>Th. 11</u>: A horocycle is the limiting case of a circle whose radius increases without bound.

Given a nominal horocycle H, internally tangent to F at T, draw the line through T and O, the center of F. TO intersects H in some point P. Draw any circle with center on PT which passes through P. If we let the nominal center of this circle move out toward T along PT, we see that the nominal radius of the circle increases without bound, and that the limiting position of the circle is actually the horocycle H.

Th. 12: A horocycle is the limiting case of an equidistant curve whose distance increases without bound.

Given a nominal horocycle H, internally tangent to F at T, draw the line through T and O, the center of F. TO intersects H in some point P. Let $\overline{AB}$ be a nominal line perpendicular to PT at some point between P and T. Draw the equidistant curve APB. Now, if we let A and B approach T along F, we see that the nominal distance of equidistant curve APB increases without bound, and that the limiting case of APB is the horocycle H.

The Geometry on a Horosphere

In hyperbolic geometry, a horosphere is defined as that surface generated by revolving a horocycle about one of its axes. Both Lobatchewski and Bolyai made use of the horosphere in their developments. They defined an axis of a horosphere to be any line through the horosphere parallel to the axis of revolution. Thus all the axes of the horosphere are parallel to one another, and it can be proved that each is orthogonal to the horosphere. The intersection of a horosphere and a plane through an axis is a horocycle.

Lobatchewski showed that if we call such horocycles "limit sphere lines", and the triangle formed by the intersections of three of these limit sphere lines taken two by two a "limit sphere triangle", then the sum of the angles of a limit sphere triangle is equal to two right angles. Bolyai showed, in addition, that if a limit sphere transversal falls across two limit sphere parallels, then the sum of the interior angles on the same side is equal to two right angles. (See 4, p. 364 and 384.) Thus we are led to suspect that the geometry on the horosphere defined in this manner is Euclidean in nature. Rigorous proofs of this have been presented, but perhaps none quite so simple as the proof which can be obtained from the model.

In the three dimensional Poincaré model, it can be seen that a horosphere is represented by a sphere, H, internally tangent to the fundamental sphere, S. The axes are represented by circles orthogonal to both S and H, and all going through the point of tangency, T. Planes through these axes are spheres orthogonal to S and H, and all going through T. The intersections of these spheres with H are circles on H through T. Thus the geometry we must consider is the geometry of the pencil of circles on a sphere going through one point, T.

To prove that this geometry is isomorphic to plane Euclidean geometry, let us invert the figure with respect to a sphere with any radius and with T as center. If the geometry on the sphere is Euclidean, then certainly the inverted geometry is also, and vice versa, for inversion preserves angles, and there is a one-to-one correspondence between the elements of the two geometries. But in the process of inversion, the sphere becomes a plane, the pencil of circles on the sphere becomes all the ordinary lines of the plane, and the point T itself becomes the ideal point. Since this leads us to ordinary Euclidean geometry, then certainly the geometry on H, and hence on a horosphere, is Euclidean in nature.

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