RECTANGULAR PLYWOOD PLATES WITH THE GRAIN OF THE FACE PLIES INCLINED TO THE EDGES

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RECTANGULAR PLYWOOD PLATES WITH THE GRAIN OF THE FACE PLIES INCLINED TO THE EDGES[1]

By

H. W. MARCH, Head Mathematician

The purpose of this report is to present: (1) material on the elastic behavior of plywood plates that has furnished the basis for mimeographed reports that have been issued by the Laboratory and also, (2) related material that can be used in solving other problems that arise in connection with such plates. Under the first heading, but given last in this report are the details of the analysis in the application of the energy method to the buckling of long flat plates, the results of which were published in mimeographs Nos. 1316, 1316-B, and 1316-C. In this connection, it has seemed worthwhile to give the differential equation of buckling for a plate having the orthotropic axes of the plywood inclined to the edges, for the purpose of comparison with the corresponding but simpler differential equation for the buckling of a plate having the orthotropic axes parallel to the edges.

Under the second heading are given the basic differential equations for the stress function for a plate in a state of plane stress under the action of forces in its plane and for the deflection of a plate under a load normal to its surface. The first of these differential equations is needed not only for studying the stress distribution in a plate with a given distribution of forces on its edges, including concentrated forces, but also for studying the effects of holes in the interior of large plates. The second differential equation furnishes the basis for determining the behavior of a plate under a load normal to its surface and subject to prescribed conditions of support on its edges. From one of the coefficients of this equation it is possible to calculate the flexural rigidity of a broad strip of plywood having the grain of the face plies inclined to the edges. The steps taken in deriving this differential equation are also useful in making approximate estimates of the behavior of plywood plates which are not of the typical construction assumed in this report.

Subject to certain assumptions that are made concerning the structure and elastic behavior of plywood, the analysis used in this report involves only the standard procedures of the theory of elasticity and no claim for novelty of treatment is made. However, it is hoped that it will be helpful

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[1]This mimeograph is one of a series of progress reports prepared by the Forest Products Laboratory to further the Nation's war effort. Results here reported are preliminary and may be revised as additional data become available.

to have this material made available in a form directly applicable to plywood plates. The assumptions are the same as those of previous mimeographed reports of the U. S. Forest Products Laboratory[2]. In particular, it will be assumed that the construction of the plywood plate is symmetrical with respect to its middle plane. The notation is chiefly that of U. S. Forest Products Laboratory mimeograph No. 1503. When notation is used that is not found in that publication, references to other publications will be given.

1. Plywood Plate under the Action of Forces in its Plane. No Buckling. Grain of Face Plies Inclined to the Edges

Consider a rectangular plywood plate in a state of plane stress under the action of a system of forces acting on its edges and in the plane of the plate. The grain in the face plies makes an angle θ with the direction $O\xi$ as shown in figure 1. The axes $O\xi$ and $O\eta$ are parallel to the edges of the plate while the axes OX and OY are parallel and perpendicular, respectively, to the grain of the face plies. The angle θ is the negative of the angle θ used in mimeograph No. 1503. As in mimeograph No. 1503, the stress components are denoted either by X_x, Y_y, X_y, or by t_{xx}, t_{yy}, t_{xy}, whichever notation seems most convenient. The state of stress being one of plane stress, the remaining stress components vanish. The strain components are denoted by e_{xx}, e_{yy}, e_{xy}.

The linear relation between the strain components $e_{\xi\xi}$, $e_{\eta\eta}$, $e_{\xi\eta}$, and the mean stress components $t_{\xi\xi}$, $t_{\eta\eta}$, $t_{\xi\eta}$ as referred to the axes $O\xi$, $O\eta$ will be written as in equation (40) of mimeograph No. 1503 in the form:

$$\begin{aligned} e_{\xi\xi} &= a_{11}\, t_{\xi\xi} + a_{12}\, t_{\eta\eta} + a_{13}\, t_{\xi\eta} \\ e_{\eta\eta} &= a_{21}\, t_{\xi\xi} + a_{22}\, t_{\eta\eta} + a_{23}\, t_{\xi\eta} \\ e_{\xi\eta} &= a_{31}\, t_{\xi\xi} + a_{32}\, t_{\eta\eta} + a_{33}\, t_{\xi\eta} \end{aligned} \tag{1}$$

The values of the coefficients a_{11}, a_{22}, ... , can be obtained from equations (42) - (47) of mimeograph No. 1503 by using instead of E_x, E_y, μ_{xy}, and the Poisson's ratios, the effective moduli and Poisson's ratios that were found in that report for plywood in a state of plane stress. The angle θ in figure 1 is the negative of the corresponding angle in mimeograph No. 1503. Consequently, terms containing odd powers of sin θ have signs opposite to those of the previous report.

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[2]Mimeographs Nos. 1312, pages 6, 7; 1316, pages 17, 18.

For plywood for which all plies are composed of the same species of wood, E_x, E_y, σ_{yx}, and σ_{xy} may be replaced with sufficient approximation by E_a, E_b, $\overline{\sigma}_{yx}$, $\overline{\sigma}_{xy}$, respectively, as defined in equations (53), (55), and (60) of the previous report. If the veneers are composed of wood of different species, the constants defined by equations (65), (70), (75), (76), and (83) of the previous report may be used in all ordinary cases.

The coefficients a_{11}, ... a_{33} of equations (1) have the following values:

$$a_{11} = \frac{1}{E_\xi} = \frac{\cos^4\theta}{E_a} + \frac{\sin^4\theta}{E_b} - 2\sin^2\theta\cos^2\theta\,\frac{\overline{\sigma}_{yx}}{E_b} + \frac{\sin^2\theta\cos^2\theta}{\overline{\mu}_{xy}} \qquad (2)$$

$$a_{22} = \frac{1}{E_\eta} = \frac{\sin^4\theta}{E_a} + \frac{\cos^4\theta}{E_b} - 2\sin^2\theta\cos^2\theta\,\frac{\overline{\sigma}_{yx}}{E_b} + \frac{\sin^2\theta\cos^2\theta}{\overline{\mu}_{xy}} \qquad (3)$$

$$a_{33} = \frac{1}{\mu_{\xi\eta}} = 4\left[\frac{\sin^2\theta\cos^2\theta}{E_a} + \frac{\sin^2\theta\cos^2\theta}{E_b} + 2\sin^2\theta\cos^2\theta\,\frac{\overline{\sigma}_{yx}}{E_b}\right] + \frac{(\cos^2\theta - \sin^2\theta)^2}{\overline{\mu}_{xy}} \qquad (4)$$

$$a_{12} = a_{21} = -\frac{\sigma_{\eta\xi}}{E_\eta} = -\frac{\sigma_{\xi\eta}}{E_\xi} \qquad (5)$$

$$= \frac{\sin^2\theta\cos^2\theta}{E_a} + \frac{\sin^2\theta\cos^2\theta}{E_b} - (\cos^4\theta + \sin^4\theta)\,\frac{\overline{\sigma}_{yx}}{E_b} - \frac{\sin^2\theta\cos^2\theta}{\overline{\mu}_{xy}} \qquad (6)$$

$$a_{13} = a_{31} = \frac{2\sin\theta\cos^3\theta}{E_a} - \frac{2\sin^3\theta\cos\theta}{E_b} + 2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)\,\frac{\overline{\sigma}_{yx}}{E_b} - \frac{\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)}{\overline{\mu}_{xy}} \qquad (7)$$

$$a_{23} = a_{32} = \frac{2 \sin^3 \theta \cos \theta}{E_a} - \frac{2 \sin \theta \cos^3 \theta}{E_b} - 2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) \frac{\overline{\sigma}_{yx}}{E_b}$$

$$+ \frac{\sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta)}{\overline{\mu}_{xy}} \tag{8}$$

The equations of equilibrium for the mean stress components are:

$$\begin{aligned} \frac{\partial t_{\xi\xi}}{\partial \xi} + \frac{\partial t_{\xi\eta}}{\partial \eta} &= 0 \\ \frac{\partial t_{\xi\eta}}{\partial \xi} + \frac{\partial t_{\eta\eta}}{\partial \eta} &= 0 \end{aligned} \tag{9}$$

where $t_{\xi\xi} = \overline{X}_x$, $t_{\eta\eta} = \overline{Y}_y$, $t_{\xi\eta} = \overline{X}_y$.

These equations are satisfied if the stress components are considered to be obtained from a stress function F in accordance with the following equations:

$$t_{\xi\xi} = \frac{\partial^2 F}{\partial \eta^2}, \quad t_{\eta\eta} = \frac{\partial^2 F}{\partial \xi^2}, \quad t_{\xi\eta} = - \frac{\partial^2 F}{\partial \xi \partial \eta} \tag{10}$$

The strain components must satisfy the following equation of compatibility[3]:

$$\frac{\partial^2 e_{\xi\xi}}{\partial \eta^2} + \frac{\partial^2 e_{\eta\eta}}{\partial \xi^2} = \frac{\partial^2 e_{\xi\eta}}{\partial \xi \partial \eta} \tag{11}$$

If the expressions (10) for the stress components in terms of the stress function F are introduced into (1) and the results are substituted in (11), the following differential equation for the function F is obtained:

$$a_{22} \frac{\partial^4 F}{\partial \xi^4} - 2a_{23} \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + (2a_{12} + a_{33}) \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - 2a_{13} \frac{\partial^4 F}{\partial \xi \partial \eta^3}$$

$$+ a_{11} \frac{\partial^4 F}{\partial \eta^4} = 0 \tag{12}$$

[3]Love, A.E.H., The Mathematical Theory of Elasticity, Art. 17; Timoshenko, S., Theory of Elasticity, Art. 12.

This equation is to be solved subject to suitable boundary conditions which for a given problem are satisfied by the stress components.

The differential equation (12) could also have been obtained by first finding its form when the strain components and the mean stress components are referred to the orthotropic axes OX and OY of figure 1 and then changing the independent variables from x and y to ξ and η by the linear transformation

$$\begin{aligned} \xi &= x \cos \theta - y \sin \theta \\ \eta &= x \sin \theta + y \cos \theta \end{aligned} \qquad (13)$$

The first steps in this procedure will be outlined.

The following relations connect the strain components and the mean stress components when referred to the axes OX and OY:

$$e_{xx} = \frac{1}{E_a} \overline{X}_x - \frac{\overline{\sigma}_{yx}}{E_b} \overline{Y}_y$$

$$e_{yy} = \frac{1}{E_b} \overline{Y}_y - \frac{\overline{\sigma}_{xy}}{E_a} \overline{X}_x$$

$$e_{xy} = \frac{1}{\overline{\mu}_{xy}} X_y$$

On expressing the mean stress components in terms of a stress function F and using the equation of compatibility (11) as written for the strain components referred to the axes OX and OY, it is readily found that the stress function F satisfies the differential equation:

$$\frac{1}{E_b} \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{\overline{\mu}_{xy}} - \frac{2\overline{\sigma}_{yx}}{E_b}\right) \frac{\partial^4 F}{\partial x^2 \, \partial y^2} + \frac{1}{E_a} \frac{\partial^4 F}{\partial y^4} = 0$$

Equation (12) is then obtained by changing the independent variables from x and y to ξ and η in accordance with the linear transformation (13).

2. Plywood Plate under a Load Normal to its Faces. Grain of Face Plies Inclined to the Edges

A plate such as that shown in figure 1 is supported along its edge in a prescribed manner and is subjected to a distribution of load normal to its faces. The procedure to be followed in obtaining the differential equation satisfied by the deflection w is identical with that used in mimeograph No. 1312 in obtaining the corresponding differential equation for the case in which the edges of plate are parallel or perpendicular to the grain. The notation of mimeograph No. 1312 will be used. Reference is made to appendix 2 of that report for an explanation of the significance of the steps taken in the derivation of the differential equation.

For small deflections, the strain components have the values:

$$e_{\xi\xi} = -z\frac{\partial^2 w}{\partial \xi^2}, \qquad e_{\eta\eta} = -z\frac{\partial^2 w}{\partial \eta^2}$$

$$e_{\xi\eta} = -2z\frac{\partial^2 w}{\partial \xi \partial \eta} \tag{14}$$

In each ply, equations (1) connect the strain components at a given point in that ply with the stress components at that point. In the coefficients a_{11}, a_{22}, ---- a_{33}, the constants E_a, E_b, $\overline{\sigma}_{xy}$, $\overline{\sigma}_{yx}$, and $\overline{\mu}_{xy}$ are to be replaced by the constants E_x, E_y, σ_{xy}, σ_{yx}, and μ_{xy}, respectively, of the ply in question. On solving equations (1), interpreted as just described, the following equations are obtained:

$$\begin{aligned} t_{\xi\xi} &= b_{11}\, e_{\xi\xi} + b_{21}\, e_{\eta\eta} + b_{31}\, e_{\xi\eta} \\ t_{\eta\eta} &= b_{12}\, e_{\xi\xi} + b_{22}\, e_{\eta\eta} + b_{32}\, e_{\xi\eta} \\ t_{\xi\eta} &= b_{13}\, e_{\xi\xi} + b_{23}\, e_{\eta\eta} + b_{33}\, e_{\xi\eta} \end{aligned} \tag{15}$$

where $$b_{ij} = A_{ij}/\Delta \tag{16}$$

and Δ is the determinant of the coefficients of (1) and A_{ij} is the cofactor of the element a_{ij} of Δ.

On substituting (14) in (15) the following expressions are obtained for the stress components at a given point in a given ply:

$$t_{\xi\xi} = -z\left(b_{11}\frac{\partial^2 w}{\partial\xi^2} + b_{21}\frac{\partial^2 w}{\partial\eta^2} + 2b_{31}\frac{\partial^2 w}{\partial\xi\,\partial\eta}\right)$$

$$t_{\eta\eta} = -z\left(b_{12}\frac{\partial^2 w}{\partial\xi^2} + b_{22}\frac{\partial^2 w}{\partial\eta^2} + 2b_{32}\frac{\partial^2 w}{\partial\xi\,\partial\eta}\right) \qquad (17)$$

$$t_{\xi\eta} = -z\left(b_{13}\frac{\partial^2 w}{\partial\xi^2} + b_{23}\frac{\partial^2 w}{\partial\eta^2} + 2b_{33}\frac{\partial^2 w}{\partial\xi\,\partial\eta}\right)$$

The bending and twisting moments[4] are obtained by integrating the products of z and the various stress components over the thickness of the plate.

Thus,

$$m_\xi = \int_{-h/2}^{h/2} t_{\xi\xi}\, z\, dz\ , \qquad m_\eta = \int_{-h/2}^{h/2} t_{\eta\eta}\, z\, dz\ ,$$

$$m_{\xi\eta} = \int_{-h/2}^{h/2} t_{\xi\eta}\, z\, dz\ .$$

Then

$$m_\xi = -B_{11}\frac{\partial^2 w}{\partial\xi^2} - B_{21}\frac{\partial^2 w}{\partial\eta^2} - 2B_{31}\frac{\partial^2 w}{\partial\xi\,\partial\eta}$$

$$m_\eta = -B_{12}\frac{\partial^2 w}{\partial\xi^2} - B_{22}\frac{\partial^2 w}{\partial\eta^2} - 2B_{32}\frac{\partial^2 w}{\partial\xi\,\partial\eta} \qquad (18)$$

$$m_{\xi\eta} = -B_{13}\frac{\partial^2 w}{\partial\xi^2} - B_{23}\frac{\partial^2 w}{\partial\eta^2} - 2B_{33}\frac{\partial^2 w}{\partial\xi\,\partial\eta}$$

where

$$B_{ij} = B_{ji} = \int_{-h/2}^{h/2} b_{ij}\, z^2\, dz \qquad (19)$$

[4] Mimeograph No. 1312, page 36 and figure 29.

The components of vertical shear[5] are given by the equations:

$$p_\xi = \frac{\partial m_\xi}{\partial \xi} + \frac{\partial m_{\xi\eta}}{\partial \eta}, \qquad p_\eta = \frac{\partial m_\eta}{\partial \eta} + \frac{\partial m_{\xi\eta}}{\partial \xi} \tag{20}$$

Then

$$p_\xi = -\left[B_{11}\frac{\partial^3 w}{\partial \xi^3} + (B_{12} + 2B_{33})\frac{\partial^3 w}{\partial \xi \partial \eta^2} + 3B_{31}\frac{\partial^3 w}{\partial \xi^2 \partial \eta} + B_{23}\frac{\partial^3 w}{\partial \eta^3}\right]$$

$$p_\eta = -\left[(B_{12} + 2B_{33})\frac{\partial^3 w}{\partial \xi^2 \partial \eta} + B_{22}\frac{\partial^3 w}{\partial \eta^3} + B_{13}\frac{\partial^3 w}{\partial \xi^3} + 3B_{23}\frac{\partial^3 w}{\partial \xi \partial \eta^2}\right] \tag{21}$$

The condition for equilibrium of forces normal to the plate leads to the equation:

$$\frac{\partial p_\xi}{\partial \xi} + \frac{\partial p_\eta}{\partial \eta} = -p \tag{22}$$

where p is the normal load per unit area.

The substitution of (21) in (22) yields the final form of the differential equation for the deflection of the plate:

$$B_{11}\frac{\partial^4 w}{\partial \xi^4} + 4B_{31}\frac{\partial^4 w}{\partial \xi^3 \partial \eta} + (2B_{12} + 4B_{33})\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + 4B_{23}\frac{\partial^4 w}{\partial \xi \partial \eta^3}$$

$$+ B_{22}\frac{\partial^4 w}{\partial \eta^4} = p \tag{23}$$

A more direct drivation of equation (23) consists in starting with the differential equation of the plate as referred to the axes OX and OY. This equation[6] is

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[5]Mimeograph No. 1312, page 37 and figure 29.

[6]Mimeograph No. 1312, Appendix 2, where the differential equation is derived and the coefficients D_1, D_2, and K are defined.

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2K \frac{\partial^4 w}{\partial x^2 \, \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = p \tag{24}$$

If the independent variables are changed to ξ and η by the transformation (13), the following differential equation is obtained:

$$\left[D_1 \cos^4 \theta + 2K \sin^2 \theta \cos^2 \theta + D_2 \sin^4 \theta\right] \frac{\partial^4 w}{\partial \xi^4}$$

$$+ \left[4D_1 \sin \theta \cos^3 \theta + 4K \sin \theta \cos \theta \, (\sin^2 \theta - \cos^2 \theta) - 4D_2 \sin^3 \theta \cos \theta\right] \frac{\partial^4 w}{\partial \xi^3 \partial \eta}$$

$$+ \left[6D_1 \sin^2 \theta \cos^2 \theta + 2K (\sin^4 \theta + \cos^4 \theta - 4 \sin^2 \theta \cos^2 \theta) \right.$$

$$\left. + 6D_2 \sin^2 \theta \cos^2 \theta \right] \frac{\partial^4 w}{\partial \xi^2 \, \partial \eta^2}$$

$$+ \left[4D_1 \sin^3 \theta \cos \theta + 4K \sin \theta \cos \theta \, (\cos^2 \theta - \sin^2 \theta) \right.$$

$$\left. -4D_2 \sin \theta \cos^3 \theta \right] \frac{\partial^4 w}{\partial \xi \, \partial \eta^3}$$

$$+ \left[D_1 \sin^4 \theta + 2K \sin^2 \theta \cos^2 \theta + D_2 \cos^4 \theta\right] \frac{\partial^4 w}{\partial \eta^4} = p \tag{25}$$

It seems desirable to have both forms (23) and (25) of the differential equation. Without doubt the form (25) is the easier one to use as the coefficients are readily determined from D_1, D_2, K, and θ. It is, however, useful to have available the expressions for the bending and twisting moments given in (18) and those for the components of vertical shear given in (21).

3. Buckling of a Plywood Plate under Forces in Its Plane. Grain of Face Plies Inclined to the Edges

Consider a plate of thickness h, having the grain of the face plies inclined to the edges as in figure 1. The plate is under the action of compressive or tensile forces and shear forces in its plane. Certain conditions along the edges of the plate are prescribed. For example, the edges may be simply supported or they may be clamped. It is desired to determine the conditions under which the plate will become unstable and buckle from its plane. The differential equation for the deflection of the middle surface of the plate from its plane can be written, when referred to the axes OX and OY of figure 1, in the form:

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2K \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = h(t_{xx} \frac{\partial^2 w}{\partial x^2} + t_{yy} \frac{\partial^2 w}{\partial y^2} + 2t_{xy} \frac{\partial^2 w}{\partial x \partial y}) \qquad (26)$$

When t_{xx} and t_{yy} are positive, they denote tensile stresses.

Equation (26) has been used in a number of papers dealing with the buckling of plates of orthotropic material. It is readily obtained from the corresponding equation for plates of isotropic materials[7] by replacing the expression

$$D\, (\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4})$$

by the left-hand member of equation (24). If the independent variables in (26) are transformed in accordance with equation (13) so that the new independent variables are the variables ξ and η referred to axes parallel to the edges of the plate as in figure 1, the left-hand member of equation (26) becomes the left-hand member of equation (25) while the right-hand member becomes

$$h\ (t_{\xi\xi} \frac{\partial^2 w}{\partial \xi^2} + t_{\eta\eta} \frac{\partial^2 w}{\partial \eta^2} + 2t_{\xi\eta} \frac{\partial^2 w}{\partial \xi \partial \eta})$$

If the left-hand member of (25) is denoted by L(w), equation (26) after transformation to the new axes becomes:

$$L(w) = h\ (t_{\xi\xi} \frac{\partial^2 w}{\partial \xi^2} + t_{\eta\eta} \frac{\partial^2 w}{\partial \eta^2} + 2t_{\xi\eta} \frac{\partial^2 w}{\partial \xi \partial \eta}) \qquad (27)$$

In the cases to be considered in this report, certain of the stress components $t_{\xi\xi}$, $t_{\eta\eta}$, and $t_{\xi\eta}$ have constant values while the remaining component or components vanish. The problem of determining the critical buckling stress from equation (27) and appropriate boundary conditions is

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[7] See, for example, Timoshenko, S., Theory of Elastic Stability. Equation (197), page 305 and equation (209), page 324.

much more difficult than the corresponding problem for equation (26) because of the presence of the terms containing $\frac{\partial^4 w}{\partial \xi^3 \partial \eta}$ and $\frac{\partial^4 w}{\partial \xi \partial \eta^3}$ in $L(w)$ in equation (27). Consequently, as in numerous problems of elastic stability, an energy method is used to obtain approximate solutions. This consists in assuming a reasonable form for the buckled surface and equating the strain energy of bending of the deformed plate to the work done by the applied forces acting in the middle plane of the plate[8] in producing the deformation of the plate.

The method will be applied to an infinitely long plywood plate under uniform compression, uniform shear, or combined uniform compression and shear in order to give some of the details of the derivation of the formulas published in Mimeographs 1316 and 1316C. For definiteness, all plies will be taken to be rotary-cut. The notation is that of figure 2 where p denotes the uniform compressive stress and q the uniform shearing stress. For plates with simply-supported edges, the form of the buckled surface is taken to be represented by the equation

$$w = H \sin \frac{\pi \xi}{a} \sin \frac{\pi}{b} (\eta - \gamma \xi) \tag{28}$$

where w denotes the deflection of the middle surface from the plane of the plate; a the width of the plate; b the half-wave length of the buckled surface, that is, one-half the distance within which the phenomenon repeats itself; and $\gamma = \tan \phi$, the angle ϕ being the inclination of the buckling wrinkles to the axis $O\xi$. (See page 17.)

First, axes of reference OX and OY are chosen parallel and perpendicular, respectively, to the grain of the face plies. With respect to these axes, the strain energy of bending is represented by the integral[9]

$$V = \frac{h^3}{24\lambda} \int \int \left[E_1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + E_2 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\sigma_{TL} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 4\lambda \mu_{LT} \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 \right] dy\, dx \tag{29}$$

where E_1 and E_2 are defined as in mimeograph No. 1312 and $\lambda = 1 - \sigma_{LT}\sigma_{TL}$. The integration is to be taken over an area of width a and of length b

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[8] See Timoshenko, S., Theory of Elastic Stability, pages 78, 325.

[9] Mimeograph No. 1312, page 46, equation (3.21).

parallel to the axis 0η. As written, the constants in (29) are those for a plywood plate made of rotary-cut veneers of wood of the same species throughout. Suitable modifications can be made for plates of other constructions.

On changing the axes of reference from OX and OY to 0ξ and 0η by the transformation (13) the integral for the strain energy of bending over one half-wave length becomes:

$$V = \frac{h^3}{24\lambda} \int_0^a \int_0^b \left[K_1 \left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 + K_2 \left(\frac{\partial^2 w}{\partial \eta^2}\right)^2 + K_3 \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} \right.$$

$$\left. + K_4 \left(\frac{\partial^2 w}{\partial \xi \, \partial \eta}\right)^2 + K_5 \frac{\partial^2 w}{\partial \eta^2} \frac{\partial^2 w}{\partial \xi \, \partial \eta} + K_6 \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \xi \, \partial \eta} \right] d\eta \, d\xi \tag{30}$$

where

$$K_1 = E_1 \cos^4\theta + E_2 \sin^4\theta + 2A \sin^2\theta \cos^2\theta$$

$$K_2 = E_1 \sin^4\theta + E_2 \cos^4\theta + 2A \sin^2\theta \cos^2\theta$$

$$K_3 = 2E_1 \sin^2\theta \cos^2\theta + 2E_2 \sin^2\theta \cos^2\theta + 2(\sin^4\theta + \cos^4\theta) E_L \sigma_{TL}$$

$$- 8 \sin^2\theta \cos^2\theta \, \lambda \mu_{LT} \tag{31}$$

$$K_4 = 4E_1 \sin^2\theta \cos^2\theta + 4E_2 \sin^2\theta \cos^2\theta - 8 E_L \sigma_{TL} \sin^2\theta \cos^2\theta$$

$$+ 4\lambda \, \mu_{LT} (\sin^2\theta - \cos^2\theta)^2$$

$$K_5 = 4(E_1 - A) \sin^3\theta \cos\theta - 4(E_2 - A) \sin\theta \cos^3\theta$$

$$K_6 = 4(E_1 - A) \sin\theta \cos^3\theta - 4(E_2 - A) \sin^3\theta \cos\theta$$

$$A = E_L \sigma_{TL} + 2\lambda \mu_{LT}$$

On substituting (28) in (30), using the abbreviations

$$\alpha = \frac{\pi}{a}, \qquad \beta = \frac{\pi}{b} \tag{32}$$

and performing the integration, the value of V is found to be:

$$V = \frac{H^2\ abh^3}{96\lambda} \Big[K_1\ (\alpha^4 + 6\,\alpha^2\beta^2\gamma^2 + \beta^4\gamma^4) + K_2\,\beta^4 + R_1\,\beta^2\,(\alpha^2 + \beta^2\gamma^2) - K_5\,\beta^4\gamma - K_6\,\beta^2\gamma\,(3\,\alpha^2 + \beta^2\gamma^2) \Big] \tag{33}$$

where

$$R_1 = K_3 + K_4 = 6(E_1 + E_2)\ \sin^2\theta\cos^2\theta + 2A(\sin^4\theta + \cos^4\theta - 4\,\sin^2\theta\cos^2\theta) \tag{34}$$

Under the uniform compressive stress p the work done by the external forces during buckling is given by the integral:

$$T = p\,\frac{h}{2}\int_0^a\int_0^b \left(\frac{\partial w}{\partial \xi}\right)^2 \cdot d\eta\, d\xi \tag{35}$$

On substituting (28) in (35) and performing the integration it is found that:

$$T = p\ h\ H^2\,\beta^2\ ab\ /\ 8 \tag{36}$$

Equating T and V, solving for p, and using (32), it follows that;

$$p = \frac{\pi^2}{12\lambda} \left[K_1 \left(z^2 + 6\gamma^2 + \frac{\gamma^4}{z^2}\right) + \frac{K_2}{z^2} + R_1 \left(1 + \frac{\gamma^2}{z^2}\right) - K_5 \frac{\gamma}{z^2} - K_6 \left(3\gamma + \frac{\gamma^3}{z^2}\right) \right] \frac{h^2}{a^2} \tag{37}$$

where

$$z = \frac{b}{a}$$

Now γ and z must be determined to make p a minimum.

From the relations $\partial p/\partial z = 0$ and $\partial p/\partial \gamma = 0$, the following equations are obtained:

$$z^4 = \gamma^4 + \frac{K_2 + R_1\gamma^2 - K_5\gamma - K_6\gamma^3}{K_1} \tag{38}$$

$$z^2 = \frac{3K_6\gamma^2 + K_5 - 2R_1\gamma - 4K_1\gamma^3}{12K_1\gamma - 3K_6} \tag{39}$$

Values of z and γ are to be found by solving the simultaneous equations (38) and (39). These values are then to be used in equation (37). Before making the substitution, it is advisable to collect the terms in (37) that contain the factor $1/z^2$ and make use of (38). Then (37) becomes

$$p = \frac{\pi^2}{12\lambda} \left[2K_1 z^2 + 6K_1\gamma^2 + R_1 - 3K_6\gamma \right] \frac{h^2}{a^2} \tag{40}$$

Under uniform shearing stress q, the work done by the external forces during buckling of the plate to the form (28) is given by the integral

$$T = -q\,h \int_0^a \int_0^b \frac{\partial w}{\partial \xi} \frac{\partial w}{\partial \eta}\, d\eta\, d\xi \; . \tag{41}$$

Using (28), it is found that

$$T = q\ h\ H^2\ \beta^2 \gamma\ \frac{ab}{4} \tag{42}$$

The expression (33) for the strain energy of bending, V, is unchanged. On solving for q the result of equating T and V, it is found that

$$q = \frac{\pi^2}{24\lambda\gamma} \left[K_1\ (z^2 + 6\gamma^2 + \frac{\gamma^4}{z^2}) + \frac{K_2}{z^2} + R_1\ (1 + \frac{\gamma^2}{z^2}) \right.$$

$$\left. - K_5 \frac{\gamma}{z^2} - K_6 \gamma (3 + \frac{\gamma^2}{z^2}) \right] \frac{h^2}{a^2} \tag{43}$$

The quantities γ and z are to be determined to make q a minimum. From the relations $\partial q/\partial z = 0$ and $\partial q/\partial \gamma = 0$, the following equations are obtained:

$$z^4 = \gamma^4 + \frac{K_2 + R_1 \gamma^2 - K_5\gamma - K_6\gamma^3}{K_1} \tag{44}$$

$$z^2 = \frac{2K_2 - 2K_1 \gamma^4 - K_5 \gamma + K_6 \gamma^3}{6K_1 \gamma^2 - R_1} \tag{45}$$

The minimum value of q is obtained by substituting in (43), the values of z and γ obtained by solving the simultaneous equations (44) and (45). Before the substitution is made, equation (43) should be simplified by collecting the terms containing the factor $1/z^2$ and making use of (44). The simplified form of (43) is:

$$q = \frac{\pi^2}{24\lambda\gamma} \left[2K_1\ z^2 + 6K_1 \gamma^2 + R_1 - 3K_6 \gamma \right] \frac{h^2}{a^2} \tag{46}$$

In the case of combined uniform compressive stress p and uniform shear stress q̂, the work during buckling of the combined system of external forces will be the sum of the expressions (36) and (42). Then

$$T = \frac{H^2 \; h \; ab}{8} \; \beta^2 \quad \left[p + 2q\gamma \right] \tag{47}$$

Let $\quad q = fp \qquad (48)$

Then
$$T = \frac{H^2 \; h \; ab \; \beta^2 \; p}{8} \; (1 + 2f\gamma) \tag{49}$$

The expression (33) for the strain energy of bending, V, is unchanged.

On solving the result of equating the expressions for T in (49) and V in (33), it is readily found that

$$p = \frac{\pi^2}{12\lambda \; (1 + 2f\gamma)} \left[K_1 \; (z^2 + 6\gamma^2 + \frac{\gamma^4}{z^2}) + \frac{K_2}{z^2} + R_1 \; (1 + \frac{\gamma^2}{z^2}) - K_5 \frac{\gamma}{z^2} - K_6\gamma \; (3 + \frac{\gamma^2}{z^2}) \right] \frac{h^2}{a^2} \tag{50}$$

The quantities γ and z are to be determined to make p a minimum. From the relations $\partial p/\partial z = 0$ and $\partial p/\partial\gamma = 0$, the following equations are obtained:

$$z^4 = \gamma^4 + \frac{K_2 + R_1\gamma^2 - K_5\gamma - K_6\gamma^3}{K_1} \tag{51}$$

$$z^2 = \frac{4K_2 \; f - 4K_1(\gamma^3 + f\gamma^4) - 2R_1\gamma + K_5(1 - 2f\gamma) + K_6(3\gamma^2 + 2f\gamma^3)}{12K_1(\gamma + f\gamma^2) - 2R_1 \; f - 3K_6} \tag{52}$$

The values of z and γ obtained from the simultaneous equations (51) and (52) are to be substituted in (50). Before this substitution (50) should be reduced to the following simpler form by collecting the terms containing the factor $1/z^2$ and using (51):

$$p = \frac{\pi^2}{12\lambda \; (1 + 2f\gamma)} \left[2K_1 \; z^2 + 6K_1 \; \gamma^2 + R_1 - 3K_6\gamma \right] \frac{h^2}{a^2} \tag{53}$$

The critical value of q is found from (48) and (53) since the particular combination of shearing and compressive stresses given by (48) has been assumed.

In the cases that have been considered, those for which the grain of the face plies is inclined at an angle θ to the direction of the width of the plate, it is to be carefully noted that the constants E_1 and E_2 which enter into the definitions of K_1, K_2, etc. represent the mean Young's moduli in bending of strips parallel and perpendicular, respectively, to the grain of the face plies.

Attention is again called to the fact that the formulas for the buckling of plates with inclined grain were obtained by an energy method and are consequently to be considered approximate. In particular, the surface assumed in equation (28) which forms the basis for the derivation of these formulas, appears to represent quite well the form of the buckled surface although it does not satisfy one of the boundary conditions for simply supported edges, namely, that the bending moment shall vanish along the edges of the infinitely long plate. However, in the case of an infinitely long isotropic plate having simply supported edges and buckling under uniform shear for which case an exact solution is available, the assumption of this form for the buckled surface leads to a buckling stress that is only about 6-1/2 percent higher than the exactly determined value.[10] It may reasonably be assumed that the effects of the edge moments that are associated with a buckled surface of this form will be of the same order of magnitude for the problems that were considered in Mimeographs 1316 and 1316-C and again in the latter part of the present report. The small effect of these edge moments as found for isotropic plates under shear and as expected in the problems under consideration is undoubtedly associated with the fact that the moments in question vanish at points where the crests and troughs of the waves of the surface described by equation (28) meet the edges and that their average value is zero over a segment of an edge between points on two consecutive nodal lines and over any full wave length.

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[10] Timoshenko, S., Theory of Elastic Stability, pages 360, 361.

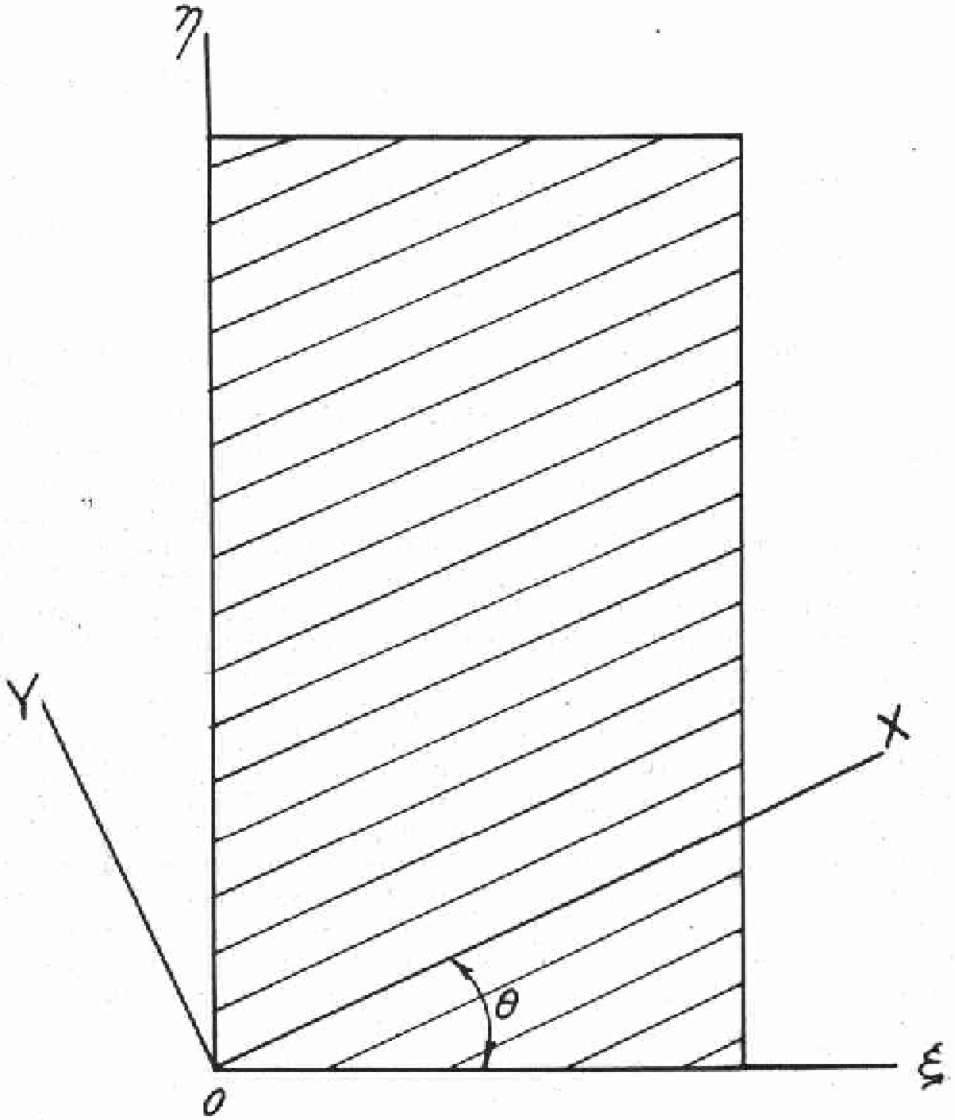

FIG. I
RECTANGULAR PLYWOOD PLATE
WITH GRAIN OF FACE PLIES
INCLINED TO THE EDGES.

Z M 52800 F

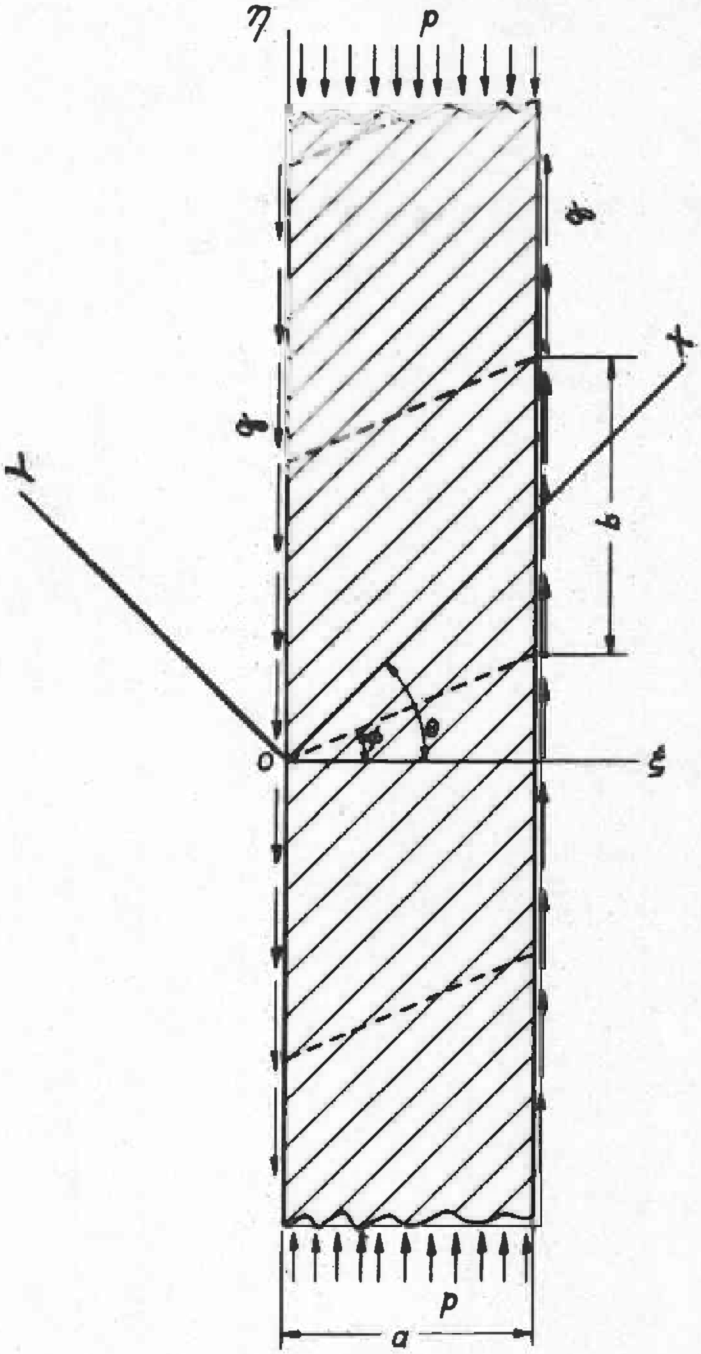

FIG. 2
LONG PLATE UNDER COMPRESSION AND SHEAR.

Z M 41114 F